Forms of reductive algebraic groups.

Kevin Buzzard

September 30, 2013

Last modified 30/09/2013.

Introduction

This began with an attempt to understand Borel's paper on L-groups in Corvallis; I had to go back and read Springer's Corvallis paper first to understand the details of inner/outer forms and Galois groups acting on Dynkin diagrams etc. I learnt/worked out some of the general theory and thought lots about the explicit example of SL_n over the reals; I'm writing it down before I forget.

1 Automorphisms of connected reductive groups: the absolute theory.

Let G be a connected reductive algebraic group over an algebraically closed field K. Then G is classified up to isomorphism by its "based root datum", which is a purely combinatorial gadget. Moreover there is a relationship between the automorphisms of the algebraic group G and the automorphisms of the based root datum, which is the following. Let Ψ_0 be the based root datum associated to G (or, more precisely, associated to G and a choice of Borel and torus). In the semisimple case the based root datum carries, I think, essentially the same information as the Dynkin diagram of the group (this is not quite true because the based root datum can see the group itself whereas the Dynkin diagram only sees it up to isogeny).

One can explicitly write down the automorphisms of G when one has all this data. The key construction is the following, for which we have to introduce some notation. Given G, choose once and for all B and T. This gives us a set of simple roots Δ. For each simple root $\alpha \in \Delta$ we make the following rather involved-looking construction: there is a copy G_α of SL_2 or PSL_2 in G, namely the derived subgroup of the centralizer of the identity component of the kernel of α, and there's a map $\mathbf{G}_a \to G_\alpha$ as in 2.3 of Springer, which "corresponds to α" in a precise sense: the moral is that U_α, the image of $\mathbf{G}_a$ under this map, is an algebraic group whose tangent space corresponds to the 1-dimensional subspace of the Lie algebra of G corresponding to the simple root α. One chooses $u_\alpha \neq 1$ in U_α for each simple root α. If ρ is an automorphism of an object X, and Y is a subset of X, then say "ρ preserves Y" if $\rho(Y) \subseteq Y$ (but we don't demand $\rho(y) = y$ for all $y \in Y$). Now if ρ is an automorphism of G which preserves B, T and the set $\{u_\alpha : \alpha \in \Delta\}$, then ρ clearly induces an automorphism of the associated based root datum Ψ_0. One useful result is Proposition 2.13 of Springer, which asserts that the induced map $\mathrm{Aut}(G, B, T, \{u_\alpha\}_{\alpha\in\Delta}) \to \mathrm{Aut}(\Psi_0)$ is an isomorphism.

Example: Let A be the antidiagonal matrix in $\mathrm{GL}_n(\mathbf{C})$ with alternating 1s and -1s going from the bottom left to the top right. Then (imagining a matrix as sitting on a chessboard with the leading diagonal consisting of white squares) conjugation by A "rotates a matrix by 180 degrees and then changes the signs of all the entries of A on black squares". So if $G = \mathrm{SL}_n$ and $\rho(g) = A.g^{-t}.A^{-1}$ (here g^{-t} means g inverse transpose) then ρ preserves the Borel of upper triangular matrices, and the torus, and the set of u_α if we define u_α to be the identity matrix plus the appropriate elementary matrix (so u_α has 1s on the diagonal and one more 1 in the appropriate place). The associated automorphism of the based root datum is trivial if $n = 1$ or $n = 2$, and non-trivial for $n \geq 3$.

Now let $\mathrm{Aut}(G)$ denote the group of automorphisms of the algebraic group G. We can now see two subgroups of this group, namely the normal subgroup $\mathrm{Inn}(G)$ of inner automorphisms (which is a subgroup isomorphic to $G(K)/Z(K)$ of course) and and the not-necessarily normal (but I think it's finite if G is semisimple), subgroup $\mathrm{Aut}(G, B, T, \{u_\alpha\})$ of automorphisms which preserve all this extra stuff too. The theorem is that $\mathrm{Aut}(G)$ is in fact the semidirect product of these two groups, with the inner automorphisms being the normal subgroup, and the action of $\mathrm{Aut}(G, B, T, \{u_\alpha\})$ on $G(K)/Z(K)$ being the obvious one.

Example: $G = \mathrm{SL}_n$ and $K = \mathbf{C}$. For $n = 1$ and $n = 2$ we have $\mathrm{Aut}(G) = \mathrm{PSL}_n(\mathbf{C}) = \mathrm{PGL}_n(\mathbf{C})$. For $n \geq 3$ however the automorphism group of the based root datum has order 2, the non-trivial element being represented by the automorphism ρ above, which induces the obvious automorphism of order 2 of the Dynkin diagram (a chain of $n-1$ dots), and we deduce that every automorphism is either inner, or an inner automorphism composed with this outer one.

2 Forms of connected reductive groups.

Now let G be a connected reductive group defined over a field k and let K/k is a separable closure of k, with $\Gamma = \mathrm{Gal}(K/k)$. For convenience we will fix once and for all a Borel B and a torus T in G_K, which we shall use later on (but not immediately).

Now abstract nonsense tells us that the k-forms of G, that is, the groups H/k which become isomorphic to G over K, are parametrised up to isomorphism by the cohomology set $H^1(\Gamma, \mathrm{Aut}(G_K))$. Let me explicitly write down some definitions and the bijection, so we get normalisations straight. Firstly $\Gamma = \mathrm{Gal}(K/k)$ acts on $\mathrm{Aut}(G_K)$ thus: if $\gamma \in \Gamma$ then γ sends $\sigma : G_K \to G_K$ to the pullback of this map along the isomorphism $K \to K$ induced by γ. Explicitly, Γ acts on K and hence on $G(K)$, and the action of Γ on $\mathrm{Aut}(G_K)$ is given on points by the following recipe: if $\gamma \in \Gamma$ and $\sigma \in \mathrm{Aut}(G_K)$ then $\gamma * \sigma$ is the map sending $g \in G(K)$ to $\gamma(\sigma(\gamma^{-1}(g)))$.

Now for nonabelian cohomology. If a group Γ acts on a group M then a 1-cocycle is $c : \Gamma \to M$ such that $c(st) = c(s).(s * c(t))$ with . the group law on M and $*$ the action of Γ on M. An example of a 1-cocycle is a 1-coboundary, which is: choose $m \in M$ and define $c(s) = m^{-1}.(s * m)$. Then $c(s).(sc(t)) = m^{-1}.sm.sm^{-1}.stm$ (I'm dropping the $*$s) so it's a cocycle. More generally c is equivalent to d iff there is $m \in M$ such that $c(s) = m^{-1}.d(s).sm$. The nonabelian $H^1(\Gamma, M)$ is the pointed set of cocycles modulo equivalence, with the point being the coboundaries.

Now back to the situation we're interested in. If G is a connected reductive algebraic group over k and K is a separable closure of k with $\Gamma = \mathrm{Gal}(K/k)$ then the forms of G biject with the set $H^1(\Gamma, \mathrm{Aut}(G_K))$, with G being the point of this set. Note that G_K only depends on G over K but the Γ-action depends on G/k. Here's the dictionary. If H is a form of G then choose an isomorphism $i : G(K) \to H(K)$ and given $\gamma \in \Gamma$ we get an automorphism of $G(K)$ thus: γ acts on both $G(K)$ and $H(K)$ so we start with an element of $G(K)$, do γ^{-1}, then i, then γ, then i^{-1}. Explicitly $c(t) = i^{-1}.t.i.t^{-1}$. Now $sc(t) = s.i^{-1}.t.i.t^{-1}.s^{-1}$ so $c(s).sc(t) = i^{-1}.st.i.(st)^{-1} = c(st)$ and we have a cocycle. Conversely, given the cocycle and G only, we can compute $H(k)$ by considering the fixed points of Γ on $H(K) = G(K)$ which one can do by considering $c(t).t$ on $G(K)$; this is semilinear and its fixed points as t varies is $H(k)$. Explicitly: the dictionary between the Galois actions is that the t-action on $H(K)$ is $c(t).t*$ where $*$ is the t-action on $G(K)$.

It turns out that Γ preserves the subgroup $\mathrm{Inn}(G_K)$ of $\mathrm{Aut}(G_K)$; one does the exercise and sees that the induced action of γ on an element of $G(K)/Z(K)$ is just the obvious one (explicitly one has to check that the maps $\gamma.\mathrm{Inn}(h)$ and $\mathrm{Inn}(\gamma.h)$ coincide). This observation means that γ will act on the quotient $\mathrm{Aut}(\Psi_0)$ too but I can believe that in general Γ may not preserve the explicit subgroup $\mathrm{Aut}(G_K, B, T, \{u_\alpha\})$, if some of the u_α are not defined over k. On the other hand we don't care because we can regard $\mathrm{Aut}(\Psi_0)$ as a Γ-invariant quotient and leave it at that.

Let us however explicitly work out the action of Γ on $\mathrm{Aut}(\Psi_0)$. If $\gamma \in \Gamma$ and we have an automorphism ρ of Ψ_0 then let us lift it to an automorphism ρ of G_K which preserves B, T and $\{u_\alpha\}_{\alpha \in \Delta}$. Now we know how Γ acts on this automorphism; $\gamma \in \Gamma$ sends ρ to the map

$G(K) \to G(K)$ sending x to $\gamma(\rho(\gamma^{-1}(x)))$. Now we know that this map (which is a morphism of algebraic groups $G_K \to G_K$) will send our fixed Borel B into some other Borel, so we can conjugate this other Borel back to B, and in fact if we choose our conjugating element appropriately then the resulting morphism of algebraic groups will preserve $\{u_\alpha\}_{\alpha\in\Delta}$ too, and hence give us another automorphism of Ψ_0. That's the action of Γ on $\mathrm{Aut}(\Psi_0)$: $\gamma * \rho = \mathrm{Inn}(g)\gamma\rho\gamma^{-1}$. We'll use this in a sec.

The short exact sequence $0 \to \mathrm{Inn}(G_K) \to \mathrm{Aut}(G_K) \to \mathrm{Aut}(\Psi_0) \to 0$ is a short exact sequence of Γ-modules, and hence we get an associated long exact sequence of cohomology (pointed) sets

$$0 \to \mathrm{Inn}(G_K)^\Gamma \to \mathrm{Aut}(G_K)^\Gamma \to \mathrm{Aut}(\Psi_0)^\Gamma \to H^1(\Gamma, \mathrm{Inn}(G_K)) \to H^1(\Gamma, \mathrm{Aut}(G_K)) \to H^1(\Gamma, \mathrm{Aut}(\Psi_0)).$$

Note that all of these terms depend *heavily* on the actual form G we chose; the if G and H are forms of the same group then $G_K = H_K$ as groups over K but the action of Γ depends on the form itself (and indeed if G is a form of SL_n over the reals then the size of the (finite) image of $H^1(\Gamma, \mathrm{Inn}(G_K))$ in $H^1(\Gamma, \mathrm{Aut}(G_K))$ will depend strongly, in general, on whether G is an inner form of $\mathrm{SL}_n/\mathbf{R}$ or not. Oh—inner forms. We say that a form of G is *inner* if the associated element of $H^1(\Gamma, \mathrm{Aut}(G_K))$ lies in the image of $H^1(\Gamma, \mathrm{Inn}(G_K))$.

This long exact sequence of cohomology sets must surely give us some clue as to how to compute the inner and outer forms of a group. I've worked out how this works to a certain extent—enough to get the hang of it in several cases. Before I explain it, I have to explain the amazing map $\mu_G : \Gamma \to \mathrm{Aut}(\Psi_0)$.

Given a group G/k as above we currently have (after some choices) a based root datum Ψ_0 and an action of Γ on the group $\mathrm{Aut}(\Psi_0)$. But we can also get a natural homomorphism $\mu_G : \Gamma \to \mathrm{Aut}(\Psi_0)$. The definition of μ_G is slightly more delicate because it does not come from a K-linear automorphism of T.

So here's μ_G. Choose a Borel B of G_K and let T be its maximal torus. If $\gamma \in \Gamma$ then $\gamma(B)$ is a Borel of G_K so we can conjugate it back to B via an inner automorphism. We can even choose this inner automorphism so that it sends $\gamma(T)$ to T. This map (γ and then conjugation) is a non-K-linear map of $G(K)$ sending B to B and T to T. Call this map r_γ. Now say $\sigma : T \to \mathbf{G}_m$ is a character. We define $\mu_G(\gamma)$ by saying that it sends σ to the map $\gamma^{-1}\sigma r_\gamma$; this is K-linear again because the characters of T are all defined over K.

[Added April 2011: here's what Brian Conrad had to say about this. I'll paraphrase so any errors are due to me etc etc. Brian doesn't like looking at non-K-linear maps. Given $\sigma : T \to \mathrm{GL}_1$ defined over K, and $\gamma \in \Gamma$, we pull σ back along $\gamma^* : \mathrm{Spec}(K) \to \mathrm{Spec}(K)$ and get $\gamma^*(\sigma) : \gamma^*(T) \to \mathrm{GL}_1$. Now compose this with the K-linear isomorphism $T \to \gamma^*(T)$ given by conjugation as above (the one sending B to $\gamma^*(B)$), and we get a new K-linear map $T \to \mathrm{GL}_1$ and this is going to be $\mu_G(\gamma)(\sigma)$.]

Claim: If H is a form of G, then H is an inner form of G iff $\mu_G = \mu_H$. This is not so hard to check. We choose an isomorphism $G_K = H_K$ and choose a Borel and torus in this group. Now $\mu_G(\gamma) = \mu_H(\gamma)$ iff r_γ is the same for G and H. Now if c is the cocycle representing H as a form of G then the action $*_H$ of Galois on $H(K) = G(K)$ is related to the action $*_G$ by $\gamma *_H g = c(\gamma)(\gamma *_G g)$ and unravelling we see that r_γ is the same for G and H iff $c(\gamma)$ is an inner automorphism.

Claim: the action of Γ on $\mathrm{Aut}(\Psi_0)$ is given thus: γ sends ρ to $\mu_G(\gamma)\rho\mu_G(\gamma^{-1})$. Again this is just unravelling the definitions.

Claim: if G is split then $\mu_G = 1$, so Γ acts trivially on $\mathrm{Aut}(\Psi_0)$. Proof: Clear. It suffices to check $\mu_G = 1$ and for this we just observe that if T is a torus that is split over k then the Galois action on T induces the trivial action on the character group, as all characters of T are also defined over k so Galois commutes itself away. Note: if G is quasi-split then μ_G isn't 1 in general; $mu_G = 1$ iff G is an inner form of a split group.

Claim: If G is split then the map $H^1(\Gamma, \mathrm{Aut}(G_K)) \to H^1(\Gamma, \mathrm{Aut}(\Psi_0)) = \mathrm{Hom}(\Gamma, \mathrm{Aut}(\Psi_0))$ modulo inner automorphisms, sends a form H of G to the map μ_H. Proof: unravel.

Remark: Payman Kassaei pointed out to me that one can't deduce that two forms of a split group are equal iff they have the same image in $H^1(\Gamma, \mathrm{Aut}(\Psi_0))$, because $\mathrm{Aut}(\Psi_0)$ might be non-abelian.

3 Examples.

Here I shall compute the inner and outer forms for a few classes of connected reductive groups.

3.1 A torus.

In this case G/k is abelian so all forms are outer, and if G_0 is the split torus of rank n then the root datum associated to G_0 is essentially the free abelian group of rank n, the Galois action is trivial, and we recover the result that all the rank n tori biject naturally with $\mathrm{Hom}(\Gamma_k, \mathrm{GL}_n(\mathbf{Z}))$.

Now we'll stick to forms over $\mathbf{R}$, the reals.

3.2 Forms of $\mathrm{SL}_2/\mathbf{R}$.

Well the root datum is $\mathbf{Z}$ and the based root datum is $2 \in \mathbf{Z}$ so there are no automorphisms and all forms are inner. To compute the number of inner forms we have to compute $H^1(\Gamma, \mathrm{PSL}_2(\mathbf{C}))$ and we do this via a digression.

3.3 Digression on $H^1(\Gamma, \mathrm{PSL}_n(\mathbf{C}))$.

Well $\mathrm{PSL}_n(\mathbf{C}) = \mathrm{PGL}_n(\mathbf{C})$ (the obvious inclusion is an isomorphism) with its obvious Galois action, and the exact sequence $0 \to \mathbf{C}^\times \to \mathrm{GL}_n(\mathbf{C}) \to \mathrm{PGL}_n(\mathbf{C}) \to 0$ and Hilbert 90 shows that $H^1(\Gamma, \mathrm{PSL}_n(\mathbf{C}))$ injects into $H^2(\Gamma, \mathbf{C}^\times)$ which has order 2. Note that the (true) statement "$H^1(\Gamma, \mathrm{GL}_n(\mathbf{C})) = 0$" means that every cocycle is equivalent to the trivial cocycle, and because a cocycle is determined by its values on complex conjugation we deduce that what this says is that if $D \in \mathrm{GL}_n(\mathbf{C})$ and $D\overline{D} = 1$ then there is $B \in \mathrm{GL}_n(\mathbf{C})$ such that $D = B^{-1}\overline{B}$.

Now we can compute $H^1(\Gamma, \mathrm{PGL}_n(\mathbf{C}))$. We already know that it has order at most 2, so all we have to do is to decide whether given $D \in \mathrm{GL}_n(\mathbf{C})$ with $D\overline{D}$ a scalar, we can find B and λ such that $D = \lambda B^{-1}\overline{B}$. The key observation is that if $D\overline{D} = \mu$ is a scalar then $\mu = \overline{\mu}$ is real and non-zero, but if $D = \lambda B^{-1}\overline{B}$ then $D\overline{D}$ is always a positive real. So if we can find D such that $D\overline{D}$ is negative, the group has order 2. And if n is even we can do this, because $D = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ works for $n = 2$ and lots of these running down the diagonal work for n even.

But if n is odd then computing determinants gives $\det(D)\det(\overline{D}) = \lambda^n$ so λ^n is a norm and hence positive, so $\lambda > 0$ and changing D to $D/\sqrt{\lambda}$ reduces us to the GL_2 case, so the group has order 1.

3.4 Back to $\mathrm{SL}_2/\mathbf{R}$.

We now see that for $n = 2$ there are two forms of SL_2, itself and one other, which one could actually see explicitly: its real points are the subgroup of $\mathrm{SL}_2(\mathbf{C})$ fixed by $g \mapsto D\overline{g}D^{-1}$ with $D = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and for $g \in \mathrm{SL}_2(\mathbf{C})$ we have $D\overline{g}D^{-1} = \overline{g}^{-it}$ so we're trying to solve $g.\overline{g}^t = 1$ in $\mathrm{SL}_2(\mathbf{C})$ and we get $SU(2)$.

The funny thing is that I can think of two more forms: $\mathrm{SU}(1,1)$ and $\mathbf{H}^{N=1}$, the unit quaternions, but the usual embedding of the quaternions into $M_2(\mathbf{C})$ sends the unit quaternions isomorphically onto $SU(2)$, and Richard Hill assures me that $\mathrm{SU}(1,1) = \mathrm{SL}_2$ as groups over the reals, although the isomorphism isn't the identity in this case, you have to use the fact that the upper half plane is the open unit disc, so he assured me.

3.5 $\mathrm{GL}_2/\mathbf{R}$.

Here the based root datum is $\{\mathbf{Z}^2, \{(1,-1)\}\}$ plus the dual of this, and an automorphism must preserve $\mathbf{Z}^2$, fix $(1,-1)$ and fix the dual and in particular preserve the line annihiliated by the dual root, which is $\mathbf{Z}(1,1)$, so it has order 2 (the non-trivial element sends (x,y) to (y,x)) and there will exist outer automorphisms. The short exact sequence is

$$0 \to \mathrm{PGL}_2(\mathbf{C}) \to \mathrm{Aut}(\mathrm{GL}_2/\mathbf{C}) \to \pm 1 \to 0$$

and an example of an outer automorphism is $g \mapsto g^{-t}$ (this must be outer because it doesn't preserve determinant). The sequence has an action of Γ (the obvious one; complex conjugation acts as complex conjugation on $\mathrm{PGL}_2(\mathbf{C})$ and as conjugation by complex conjugation on $\mathrm{Aut}(\mathrm{GL}_2/\mathbf{C})$, and $g \mapsto g^{-t}$ commutes with this action, so the sequence is a semidirect product as a Γ-module.

3.6 Digression on the relation between inner automorphisms and $g \mapsto g^{-t}$.

Let G be $\mathrm{GL}_n(\mathbf{C})$ for $n \geq 1$ or $\mathrm{SL}_n(\mathbf{C})$ for $n \geq 3$. Then $g \mapsto g^{-t}$ is an automorphism of G and it's not inner (it moves the based root datum). So we have a natural subgroup of $\mathrm{Aut}(G)$, which contains $\mathrm{Inn}(G)$ with index 2, and whose elements are either of the form $\mathrm{Inn}(x)$ (for $x \in \mathrm{PGL}_n(\mathbf{C})$), which sends g to xgx^{-1}, or $\mathrm{Out}(x)$, which sends g to $xg^{-t}x^{-1}$. What is the group law?

$$\mathrm{Inn}(x)\,\mathrm{Inn}(y) = \mathrm{Inn}(xy)$$

$$\mathrm{Inn}(x)\,\mathrm{Out}(y) = \mathrm{Out}(xy)$$

$$\mathrm{Out}(x)\,\mathrm{Inn}(y) = \mathrm{Out}(xy^{-t})$$

$$\mathrm{Out}(x)\,\mathrm{Out}(y) = \mathrm{Out}(xy^{-t})$$

$$\mathrm{Out}(x) = \mathrm{Inn}(x)\,\mathrm{Out}(1)$$

$$\mathrm{Out}(x)^{-1} = \mathrm{Out}(x^t)$$

$$\overline{\mathrm{Inn}(x)} = \mathrm{Inn}(\overline{x})$$

$$\overline{\mathrm{Out}(x)} = \mathrm{Out}(\overline{x})$$

Here the bar is complex conjugation.

3.7 Back to $\mathrm{GL}_2/\mathbf{R}$.

Every element of $\mathrm{Aut}(\mathrm{GL}_2(\mathbf{C}))$ is either of the form $\mathrm{Inn}(x)$ or $\mathrm{Out}(x)$ with notation as above. The short exact sequence

$$0 \to \mathrm{PGL}_2(\mathbf{C}) \to \mathrm{Aut}(\mathrm{GL}_2(\mathbf{C})) \to \pm 1 \to 0$$

is still exact when you take Galois invariants (one only needs to check surjectivity on the right, and $\mathrm{Out}(1)$ does this), so we deduce the existence of a short exact sequence

$$0 \to H^1(\mathrm{PGL}_2(\mathbf{C})) \to H^1(\mathrm{Aut}(\mathrm{GL}_2(\mathbf{C}))) \to \pm 1.$$

In particular we checked above that $H^1(\mathrm{PGL}_2(\mathbf{C}))$ had order 2 and hence we have two inner forms of $\mathrm{GL}_2(\mathbf{C})$ and the real points of the non-split one is the $g \in \mathrm{GL}_2(\mathbf{C})$ such that (as before)

$g = D\overline{g}D^{-1}$ and we see that this is the non-zero quaternions under the usual embedding. So $\mathbf{H}^\times$ is the unique non-split inner form of GL_2.

We know that $U(2)$ and $U(1,1)$ are also forms of GL_2 so they must be outer forms. In fact they are the only other two forms of $\mathrm{GL}_2/\mathbf{R}$ and they are inner forms of each other. The calculation showing this is "the same as" the SL_n case so I'll do that case instead and leave the above statement as an exercise (hint for one bit of it: all forms of GL_2 which are not inner forms of $\mathrm{GL}_2/\mathbf{R}$ must be inner forms of each other as they must have μ the unique non-trivial map!).

3.8 Forms of $\mathrm{SL}_n/\mathbf{R}$, $n \geq 3$.

The map $g \mapsto g^{-t}$ is outer (as $n \geq 3$; it moves roots) and fixed by Galois, so taking Galois invariants of $0 \to \mathrm{PGL}_n(\mathbf{C}) \to \mathrm{Aut}(\mathrm{SL}_n/\mathbf{C}) \to \pm 1 \to 0$ keeps it exact and hence we have

$$0 \to H^1(\mathrm{PGL}_n(\mathbf{C})) \to H^1(\mathrm{Aut}(\mathrm{SL}_n(\mathbf{C}))) \to \pm 1$$

and so there are one or two inner forms depending on whether n is odd or even, the non-split inner one in the even case being something like the norm 1 elements in $M_{n/2}(\mathbf{H})$ I guess.

The outer forms are more interesting! The cocycle must send complex conjugation to $\mathrm{Out}(x)$ for some x such that $\mathrm{Out}(x)\,\mathrm{Out}(\overline{x}) = 1$, that is $x\overline{x}^{-t} = 1$ in $\mathrm{PGL}_n(\mathbf{C})$, and $\mathrm{Out}(x)$ is equivalent to everything of the form $\mathrm{Inn}(y)^{-1}\,\mathrm{Out}(x)\,\mathrm{Inn}(\overline{y}) = \mathrm{Inn}(y^{-1})\,\mathrm{Out}(x\overline{y}^{-t}) = \mathrm{Out}(y^{-1}x\overline{y}^{-t})$ and to $\mathrm{Out}(1)\,\mathrm{Out}(x)\,\mathrm{Out}(1) = \mathrm{Out}(x^{-t})$.

So we're trying to work out the equivalence classes of $x \in \mathrm{GL}_n(\mathbf{C})$ with $x\overline{x}^{-t} = \lambda \in \mathbf{C}^\times$, subject to the equivalence relation defined by $x \sim \mu x$, $x \sim Bx\overline{B}^t$ and $x \sim x^{-t}$.

The first thing to observe is that $x\overline{x}^{-t} = \lambda$ implies $\overline{x}x^{-t} = x^{-t}\overline{x} = \lambda^{-1}$ so $\overline{\lambda} = \lambda^{-1}$ and λ is in $U(1)$. Now changing x by μ changes λ by $\mu\overline{\mu}^{-1}$ so if $\mu \in U(1)$ then it changes it by μ^2 and we can assume $x\overline{x}^{-t} = 1$, now subject to $x \sim \mu x$ with $\mu \in \mathbf{R}^\times$ and the other two relations. In fact if $\mu > 0$ then we can use a B to swallow it up, so we're looking at isomorphism classes of nondegenerate Hermitian sesquilinear forms x (that is $x = \overline{x}^t$) modulo $x \sim -x$ and $x \sim x^{-t} = \overline{x}^{-1}$. The fact that $x \mapsto x^{-t}$ normalises $x \mapsto Bx\overline{B}^t$ means that we're simply looking at signatures of Hermitian forms modulo $x \sim -x$ and we deduce that the outer forms of $\mathrm{SL}_n/\mathbf{R}$ naturally biject with the special unitary groups $SU(a,b)$ with $a + b = n$.

To finish the job we have to do the tedious exercise of checking that if J is a bunch of 1s and -1s down the leading diagonal then the twist of SL_n by $\mathrm{Out}(J)$ corresponds to the special unitary group $\mathrm{SU}(J)$. Perhaps the easiest way to do this is to go the other way. I'll be brief and vague. If we consider the special unitary group $\mathrm{SU}(J)$ defined by J then $\mathrm{SU}(J)(\mathbf{C})$ is a subgroup of $\mathrm{SL}_n(\mathbf{C})^2$ consisting of (g,h) with $gJh^t = J$, and the associated cocycle for SL_n is this: it sends $g = (g, Jg^{-t}J)$ first to $(\overline{g}, J\overline{g}^{-t}J)$, then over to $\mathrm{SU}(J)(\mathbf{C})$ and complex conjugation on this is can be worked out explicitly; the point is that to get from SU to SL we're using the map $\mathbf{C}\otimes\mathbf{C} \to \mathbf{C}\oplus\mathbf{C}$ sending $x\otimes y$ to $(xy, x\overline{y})$ (the second variable in the tensor is where the I which defines the unitary group is, the first variable is the coefficients) so complex conjugation sends $(xy, x\overline{y})$ to $(\overline{x}y, \overline{xy})$ so it sends (z,w) to $(\overline{w},\overline{z})$ so (back to the calculation) we go to $(Jg^{-t}J, g)$ and hence the associated cocycle sends complex conjugation to $\mathrm{Out}(J)$ and this completes the calculation.

3.9 Inner forms of special unitary groups.

By the previous section we know the answer: all the special unitary groups (at least for $n \geq 3$) must have $\mu \neq 1$ so must all be inner forms of one another. One can see this explicitly. Let's let J be the matrix with 1s up the antidiagonal and let's work with $SU(J)$, the $g \in \mathrm{SL}_n(\mathbf{C})$ with $gJ\overline{g}^t = J$. This group is quasi-split, the upper triangular matrices giving a Borel. The map $g \mapsto Jg^{-t}J$ is an outer automorphism because it fixes the Borel but moves the simple roots, and complex conjugation fixes this, so the Γ-invariants of $0 \to \mathrm{Inn}(G_\mathbf{C}) \to \mathrm{Aut}(G_\mathbf{C}) \to \pm 1$ are still exact and we deduce that $H^1(\Gamma, \mathrm{Inn}(G_\mathbf{C}))$ injects into $H^1(\Gamma, \mathrm{Aut}(G_\mathbf{C}))$.

To compute $H^1(\Gamma, \mathrm{Inn}(G_\mathbf{C}))$ we use the explicit definition of H^1. Complex conjugation on $\mathrm{PGL}_n(\mathbf{C})$ is supposed to have $PSU(J)$ as its fixed points and one checks that it sends g to $J\overline{g}^{-t}J$.

So we're looking for $M \in \mathrm{PGL}_n(\mathbf{C})$ with $MJ\overline{M}^{-t}J = 1$ with $M\tilde{B}^{-1}MJ\overline{B}^{-t}J$. Setting $x = MJ$ we want to solve $x\overline{x}^{-t} = 1$ in $\mathrm{PGL}_n(\mathbf{C})$ with $x\tilde{C}x\overline{C}^t$ (with $C = B^{-1}$). Now we already did this calculation when doing outer forms of SL_n; the answer is that we can assume $x = \overline{x}^t$ and then the relation is $x \sim Cx\overline{C}^t$ and $x \sim -x$ and again we get special unitary groups.

Note that here we have an explicit example (when $n \geq 4$) of distinct outer forms of a group having a different number of inner forms.

3.10 $\mathrm{SL}_2 \times \mathrm{SL}_2 / \mathbf{R}$

The point about this is that Ψ_0 is again $\mathbf{Z}^2$ with basically two basis elements, so the automorphism group is ± 1 again. The outer automorphism sending (g, h) to (h, g) (it's outer because it acts non-trivially on the centre! or because it moves the normal subgroup $\mathrm{SL}_2 \times 1$) is fixed by Galois, so the exact sequence $0 \to \mathrm{Inn}(G_{\mathbf{C}}) \to \mathrm{Out}(G_{\mathbf{C}}) \to \pm 1$ remains exact when you take Galois invariants. Hence the inner forms of $G = \mathrm{SL}_2 \times \mathrm{SL}_2$ biject with $H^1(\Gamma, \mathrm{PGL}_2(\mathbf{C})^2) = H^1(\Gamma, \mathrm{PGL}_2(\mathbf{C}))^2$ and they are $SU(2) \times \mathrm{SL}_2$ and so on, the 4 possibilities.

The outer forms are more interesting. Complex conjugation will be sent to the automorphism of the form $(g, h) \mapsto \mathrm{Out}(x, y)(g, h) := (xhx^{-1}, ygy^{-1})$, the complex conjugate of $\mathrm{Out}(x, y)$ is $\mathrm{Out}(\overline{x}, \overline{y})$, the inverse of $\mathrm{Out}(x, y)$ is $\mathrm{Out}(y^{-1}, x^{-1})$, the product $\mathrm{Out}(a, b)\,\mathrm{Out}(c, d) = \mathrm{Inn}(ad, bc)$, and the cocycle condition is that $x\overline{y} = 1$ and the equivalence relation is (using inner automorphisms only) that $\mathrm{Out}(x, y) \sim \mathrm{Inn}(a^{-1}, b^{-1})\,\mathrm{Out}(x, y)\,\mathrm{Inn}(\overline{a}, \overline{b}) = \mathrm{Out}(a^{-1}x\overline{b}, b^{-1}y\overline{a})$ so they are all equivalent to $\mathrm{Out}(1, 1)$!

An easy calculation shows that $\mathrm{Out}(1, 1)$ has as corresponding group $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}\,\mathrm{SL}_2$.

3.11 $\mathrm{SL}_2^3 / \mathbf{R}$.

This is an interesting case but I've run out of time and enthusiasm for it. The reason it's interesting is that the automorphisms of the based root datum are non-abelian (it's the symmetric group on three symbols) so the fibres of the map $H^1(\mathrm{Aut}(G)) \to H^1(\mathrm{Aut}(\Psi_0))$ might not be the groups which are inner forms of one another. I never checked this carefully though.

3.12 $\mathrm{GL}_n / \mathbf{R}$.

This is also interesting because it would show that the answers to some questions are not isogeny-invariant. For example a question Toby just asked me has alerted me to the fact that if $G = \mathrm{SL}_3 \times \mathrm{GL}_1$ then $\mathrm{Aut}(\Psi_0)$ will be $C_2 \times C_2$ but if $G = \mathrm{GL}_3$ then the automorphism group may well be smaller. I haven't chased this up but it seems plausible: the root datum gets replaced by a sublattice and there may be some automorphisms of the lattice that don't preserve the sublattice. In particular is it the case that for $G = \mathrm{GL}_n$ with $n \geq 1$ the automorphisms of the based root data are always cyclic of order 2? Again I have no time to sort this out.

3.13 $\mathrm{SU}(2, 1)$.

Anyway, let me explain $\mathrm{SU}(2, 1)$, not because it tells me anything new but because I seem to have typed it up when I was trying to understand μ (the above notes on μ are a much more coherent explanation of this homomorphism, by the way! I just can't face deleting this nonsense below). I guess it's nice to see a non-trivial example of something that's quasi-split but not split, at least.

Now $\mathrm{SU}(2, 1)$ is the matrices $g \in \mathrm{SL}_3(\mathbf{C})$ such that $gJ\overline{g}^t = J$, where J is the antidiagonal matrix with 1s up the antidiagonal; this choice of J is quite convenient to use because $SU(2, 1)$ is quasi-split and the upper triangular matrices are a Borel in this representation. One checks that the maximal torus in $SU(2, 1)$ is $\mathrm{diag}(\lambda, \overline{\lambda}/\lambda, 1/\overline{\lambda})$ with $\lambda \in \mathbf{C}^\times$ so the maximal torus is isomorphic to Deligne's $\mathbf{S}$, the restriction of scalars from $\mathbf{C}$ to $\mathbf{R}$ of $\mathbf{G}_m$.

Now the torus isn't split and one can see this because the simple roots can't be defined over the ground field, I don't think. Tedious calculations show that the matrices in $SU(2, 1)(\mathbf{R})$ which are upper triangular, have 1s down the leading diagonal, and the only other non-zero terms are just above the leading diagonal, have as those non-zero terms entries of the form $(x+iy, x-iy)$, so one is zero iff the other is. In particular the corresponding 2-dimensional real subspace of the Lie algebra contains neither simple root, and if α and β are the simple roots over $\mathbf{C}$ then Galois is going to have to move them, and of course it swaps them. One can write down an explicit parametrisation of the torus over $\mathbf{C}$; now $\mathrm{SU}(2,1)(\mathbf{C})$ is again $\mathrm{diag}(\lambda, \overline{\lambda}/\lambda, 1/\overline{\lambda})$ with $\lambda = x + Iy$ and x, y complex, $\overline{I} = -I$ and so on, and this group can be identified with $(\mathbf{C}^\times)^2$ by sending $x+Iy$ to $(w, z) = (x+iy, x-iy)$. The free abelian groups involved in the definitions of root space and so on are now $\mathbf{Z}^2$, if λ corresponds to (w, z) then $\overline{\lambda} = (z, w)$ and $c(\lambda) = \overline{x} + I\overline{y}$ (c being complex conjugation on the coefficients, i.e. the fixed points of $\mathrm{SU}(2,1)(\mathbf{C})$ are $\mathrm{SU}(2,1)(\mathbf{R})$ corresponds to $(\overline{z}, \overline{w})$, the torus in $\mathrm{SU}(2,1)(\mathbf{R})$ corresponds to the fixed points of this namely (w, z) with $w = \overline{z}$ so it's $\mathbf{C}^\times$, and (after choosing the obvious isomorphism $\mathrm{SU}(2,1)(\mathbf{C}) = \mathrm{SL}_3(\mathbf{C})$ sending I to i) the simple roots are $\lambda^2/\overline{\lambda} = w^2/z$ and $\overline{\lambda}^2/\lambda = z^2/w$ so in (w, z) coordinates they are $(2, -1)$ and $(-1, 2)$. The coroots: the first sends $t \in \mathbf{G}_m$ to $\mathrm{diag}(t, t^{-1}, 1)$ so $\lambda = t$ and $\overline{\lambda}/\lambda = t$ so after using the usual isomorphism sending λ to w we get that $w = t$ and $z/w = t^{-1}$ so $z = 1$ and in the dual coordinates the first coroot is $(1, 0)$ and similarly the second is $(0, 1)$. Moreover Galois is switching both the roots and the coroots.

Anyway, that's μ.

4 Odds and ends.

I never worked out why every connected reductive group had a quasi-split inner form. It's something to do with the exact sequence splitting but I'm not sure I ever got my head around it.