AN INTRODUCTION TO QUANTUM COHOMOLOGY

TOM COATES

Abstract. Lecture notes for the LMS course "Symplectic Geometry: From Dynamics to Quantum Cohomology" to be held at the University of Aberdeen in July 2011.

Contents

1. Pseudoholomorphic Curves in Symplectic Manifolds

1.1. **Introduction and Outline.** Overall goal: a gentle introduction to quantum cohomology and Gromov–Witten invariants, emphasizing concrete examples. It will not be possible to give, or even to sketch, proofs of many results. Nothing here is original. References:

- McDuff–Salamon, *J-holomorphic curves and quantum cohomology.* American Mathematical Society, University Lecture Series, 6. 1994.
- McDuff–Salamon, *J-holomorphic curves and symplectic topology.* American Mathematical Society Colloquium Publications 52. 2004.
- Fulton–Pandharipande, *Notes on stable maps and quantum cohomology.* Algebraic geometry—Santa Cruz 1995, 45–96, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., 1997.
- Givental, *A tutorial on quantum cohomology.* Symplectic geometry and topology (Park City, UT, 1997), 231–264, IAS/Park City Math. Ser., 7, Amer. Math. Soc., 1999.
- Kock–Vainsencher, *An invitation to quantum cohomology.* Progress in Mathematics, 249. Birkhuser, 2007.

Darboux: symplectic manifolds have no local invariants. So how are we to distinguish symplectic manifolds? Gromov: look at moduli spaces of holomorphic curves in (M, ω).

1.2. **Pseudoholomorphic curves.**

Definition. An *almost-complex structure* on X is $J \in \mathrm{End}(TX)$ such that $J^2 = -\,\mathrm{id}$.

This makes TX into a complex vector bundle.

Key words and phrases. Quantum cohomology, Gromov–Witten invariants, symplectic geometry.

Aside for later. If X admits an almost-complex structure then the Chern classes of the complex vector bundle TX are independent of the choice of J.

Definition. Let (M, ω) be a symplectic manifold. An almost-complex structure J is *compatible* with ω if and only if:

(1) $\omega(v, Jv) > 0$ for all non-zero $v \in TX$;
(2) $\omega(Jv, Jw) = \omega(v, w)$ for all $v, w \in TX$.

c.f. Kähler geometry. Imposing only the condition (1) is J being *tamed* by ω.

Exercise 1. Show that:

(1) an almost-complex structure J is compatible with ω if and only if $g(v, w) := \omega(v, Jw)$ is a Riemannian metric on M;
(2) with g as above, $g + \mathtt{i}\omega$ is a Hermitian form on TM;
(3) if h is a Hermitian form on TM such that the imaginary part of h is ω then the real part of h determines a compatible almost complex structure J

Let (M, ω) be a symplectic manifold. The space of almost-complex structures compatible with (or indeed tamed by) ω is non-empty and contractible.

Typically almost-complex structures on M will not be integrable, i.e. will not arise from a complex structure on M. Example: $\mathbb{P}^n$.

Definition. Let Σ be a Riemann surface with complex structure $j \in \mathrm{End}(T\Sigma)$. Let J be an almost-complex structure on M. A map $f : \Sigma \to M$ is called (j, J)*-holomorphic* or J*-holomorphic* or *pseudo-holomorphic* if and only if the derivative df_x is complex-linear for all $x \in \Sigma$, i.e. if and only if:

$$df_x \circ j_x = J_{f(x)} \circ df_x \qquad \text{for all } x \in \Sigma. \tag{1}$$

These are the *Cauchy–Riemann equations.*

Exercise 2. Take $\Sigma = M = \mathbb{C}$. Show that (1) becomes the familiar Cauchy–Riemann equations.

Locally, pseudoholomorphic curves exist in abundance because the equations (1) are elliptic PDEs. But moduli spaces of *global* solutions carry a lot of information about the geometry of M, c.f. gauge theory on 4-manifolds.

Exercise 3. Let (M, ω) be a symplectic manifold and let J be an almost-complex structure tamed by ω. Let $f : \Sigma \to M$ be a J-holomorphic curve. Show that

$$\int_\Sigma f^\star \omega \geq 0$$

with equality if and only if f is a constant map.

1.3. **Moduli spaces of pseudoholomorphic maps from a fixed curve.** Let (M, ω) be a symplectic manifold. Fix a Riemann surface Σ and an element $d \in H_2(M; \mathbb{Z})$. Let $\mathcal{M}(\Sigma, J, d)$ denote the moduli space of pseudoholomorphic maps $f : \Sigma \to M$ such that $f_\star[\Sigma] = d$, and for later convenience let us delete from $\mathcal{M}(\Sigma, J, d)$ any points that represent multiply-covered holomorphic maps. (These would typically be singular points of $\mathcal{M}(\Sigma, J, d)$.) What does $\mathcal{M}(\Sigma, J, d)$ look like?

Non-compactness 1: reparametrization. The group of automorphisms of Σ acts in the obvious way on $\mathcal{M}(\Sigma, J, d)$.

Exercise 4. If $\Sigma = \mathbb{P}^1$ then $\mathrm{Aut}(\Sigma) = \mathrm{PSL}(2, \mathbb{C})$

Since $\mathrm{PSL}(2, \mathbb{C})$ is non-compact, we can never hope for $\mathcal{M}(\Sigma, J, d)$ to be compact in general: the best we could hope for is that $\mathcal{M}(\Sigma, J, d)/\operatorname{Aut}(\Sigma)$ be compact.

Non-compactness 2: bubbling. But in general this isn't true. For example, let $M = \mathbb{P}^2$. Let $t \in \mathbb{R}$ be non-zero, and consider the map $\mathbb{P}^1 \to \mathbb{P}^2$ defined by $[x : y] \mapsto [x^2 : ty^2 : xy]$. On the main chart: $[z : 1] \mapsto [z : tz^{-1} : 1]$, so this has image the graph of $y = tx^{-1}$. Now let $t \to 0$. The limit is the nodal curve $xy = 0$. So in general $\mathcal{M}(\Sigma, J, d)/\operatorname{Aut}(\Sigma)$ cannot be compact.

Very very roughly, Gromov compactness says that $\mathcal{M}(\Sigma, J, d)$ fails to be compact only for these two reasons. If you believe that, then you should also believe:

Theorem 5 (Gromov compactness, very special case)**.** *Let (M, ω) be a symplectic manifold and J an almost-complex structure tamed by ω. Let $d \in H_2(M)$ be such that any non-constant J-holomorphic sphere $f : \mathbb{P}^1 \to M$ has $\omega(d) \leq \omega(e)$, where $e = f_\star[\Sigma]$. Then the moduli space $\mathcal{M}(\mathbb{P}^1, J, d)/\operatorname{Aut}(\mathbb{P}^1)$ is compact.*

We will see in the next section that this has remarkable geometric consequences.

Dependence on J. How does $\mathcal{M}(\Sigma, J, d)$ depend on J? Let g be the genus of Σ.

- for generic J tamed by ω (indeed for a set of such J of second category), $\mathcal{M}(\Sigma, J, d)$ is a smooth oriented manifold of dimension:
$$(1-g)\dim M + 2c_1(TX)\cdot d$$
- Let J_0, J_1 be almost complex structures tamed by ω and let J_t, $t \in [0,1]$ be a path between them. Set:
$$\mathcal{M}(\Sigma, \{J_t\}, d) = \big\{(t,f) : t \in [0,1], f \in \mathcal{M}(\Sigma, J_t, d)\big\}$$
In general it will not be possible to choose a path such that $\mathcal{M}(\Sigma, J_t, d)$ is a smooth manifold for all t, but a generic path $\{J_t\}$ will be such that $\mathcal{M}(\Sigma, \{J_t\}, d)$ is a smooth manifold. In this case $\mathcal{M}(\Sigma, \{J_t\}, d)$ gives a cobordism between $\mathcal{M}(\Sigma, J_0, d)$ and $\mathcal{M}(\Sigma, J_1, d)$. (Think: irregular almost-complex structures are "codimension 1" in the space of all ω-tame almost complex structures.)

Note that this is not so useful without compactness:

Exercise 6. Let X and Y be any two manifolds. Exhibit a non-compact cobordism between X and Y.

1.4. **Gromov's Non-Squeezing Theorem.** Consider the evaluation map:
$$\begin{aligned} \mathrm{ev} : \mathcal{M}(\Sigma, J, d) \times \Sigma &\to M \\ (f,x) &\mapsto f(x) \end{aligned}$$
This respects the action of $\mathrm{Aut}(\Sigma)$ on $\mathcal{M}(\Sigma, J, d) \times \Sigma$, where $g \in \mathrm{Aut}(\Sigma)$ maps (f,x) to $(f \circ g^{-1}, g(x))$. So we get an evaluation map out of the quotient:
$$\begin{aligned} \mathrm{ev} : \mathcal{M}(\Sigma, J, d) \times_{\mathrm{Aut}(\Sigma)} \Sigma &\to M \\ (f,x) &\mapsto f(x) \end{aligned}$$
For the remainder of this lecture, take $\Sigma = \mathbb{P}^1$ and take (M,ω) to be a symplectic manifold with $\pi_2(M) = 0$. Consider now the product $\mathbb{P}^1 \times M$, with the product symplectic form, and choose a generic almost-complex structure J that is tamed by this product symplectic form.

Claim. *There is a J-holomorphic curve through every point of $\mathbb{P}^1 \times M$.*

Sketch of proof. □

Theorem 7 (Gromov's Non-Squeezing Theorem)**.** *Suppose that (M,ω) is a symplectic manifold of dimension $2n-2$ with $\pi_2(M) = 0$. If there is a symplectic embedding from the ball $B^{2n}(r)$ into the cylinder $B^2(\lambda) \times M$ then $r \le \lambda$.*

Sketch of proof. □

2. Moduli spaces of stable maps

Last time: non-compactness of $\mathcal{M}(\Sigma, J, d)$ from reparametrization and bubbling. This time: build a compact moduli space of equivalence classes of J-holomorphic maps to (M,ω), where equivalence means reparametrization.

2.1. Stable Maps.

Definition. Let (M,ω) be a symplectic manifold and J be an almost-complex structure compatible with ω. A genus-g n-pointed *stable map* to (M,ω) of degree $d \in H_2(M)$ is:

- a possibly-nodal Riemann-surface Σ of genus g;
- n distinct marked smooth points $x_1, \ldots, x_n$;
- a J-holomorphic map $f : \Sigma \to M$ (which means…) such that $f_\star[\Sigma] = d$;

satisfying a stability condition defined as follows. For each component C of Σ, write S_C for the set of special points on C, i.e. marked points or nodes. Then stability is the conditions:

- for each component C of genus zero such that $f(C) = \mathrm{pt}$, $|S_C| \ge 3$;
- for each component C of genus one such that $f(C) = \mathrm{pt}$, $|S_C| \ge 1$.

Definition. Two stable maps $(\Sigma; x_1, \ldots, x_n; f)$ and $(\Sigma'; x'_1, \ldots, x'_n; f')$ are equivalent if and only if there exists a biholomorphism $\varphi : \Sigma \to \Sigma'$ such that $\varphi(x_i) = x'_i$ for all i and $f = f' \circ \varphi$.

Note that stability means "no infinitesimal automorphisms". Thus we might hope that the set of equivalence classes of stable maps is something like a smooth orbifold.

Definition. The *moduli space of stable maps* $\mathcal{M}_{g,n}(M,J,d)$ is the set of equivalence classes of genus-g n-pointed stable maps to M of degree d.

The space $\mathcal{M}_{g,n}(M,J,d)$ admits a topology, called the *Gromov topology*, which makes it into a compact Hausdorff metrizable space. Later in the lecture we will describe the convergent sequences in this topology.

Why Compactness? Let (M,ω) be a symplectic manifold and J an almost-complex structure compatible with ω. Recall that ω and J together define a Riemannian metric g on M. The essential point in the proof of compactness is that the *energy*:

$$E(f) := \frac{1}{2}\int_\Sigma |df(z)|^2 \, d\mathrm{vol}_\Sigma$$

of a J-holomorphic map $f : \Sigma \to M$ is a topological invariant:

$$E(f) = \int_\Sigma f^\star \omega$$

This gives uniform bounds on the first derivatives of all J-holomorphic curves of the same degree $d = f_\star[\Sigma]$. (That said, the analysis involved is not easy. e.g. McDuff–Salamon is roughly 650 pages long.)

2.2. **Warm-up: Deligne–Mumford space.** Suppose that the symplectic manifold M is a point. Then the moduli space of stable maps becomes *Deligne–Mumford space* $\overline{\mathcal{M}}_{g,n}$. This consists of equivalence classes of:

- possibly-nodal Riemann surfaces Σ of genus g; together with
- distinct smooth marked points $x_1, \ldots, x_n$ in Σ

satisfying a stability condition as above. Two marked Riemann surfaces $(\Sigma; x_1, \ldots, x_n)$ and $(\Sigma'; x_1', \ldots, x_n')$ are equivalent if and only if there exists a biholomorphism $\varphi : \Sigma \to \Sigma'$ such that $\varphi(x_i) = x_i'$ for all i. Deligne and Mumford showed that $\overline{\mathcal{M}}_{g,n}$ is a compact orbifold of dimension $6g-6+2n$. Examples: $\overline{\mathcal{M}}_{0,n}$ is empty for $n<3$, $\overline{\mathcal{M}}_{1,0}$ is empty, $\overline{\mathcal{M}}_{0,3} = \mathrm{pt}$, $\overline{\mathcal{M}}_{0,4} = \mathbb{P}^1$.

Exercise 8. $\overline{\mathcal{M}}_{0,5}$ is the blow-up of $\mathbb{P}^2$ at 4 points.

The orbifold $\overline{\mathcal{M}}_{g,n}$ is stratified by the combinatorial type of Σ. In the same way, for general M the moduli space of stable maps $\mathcal{M}_{g,n}(M,J,d)$ is stratified by the combinatorial type of the source curve of the stable map.

Definition. There is a map $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ defined by forgetting the last marked point and then contracting any unstable components.

In fact this is the universal family for the moduli problem that defines $\overline{\mathcal{M}}_{g,n}$.

Definition. There are sections $\sigma_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$ of π, defined by the marked points.

2.3. **Gromov Convergence.** For simplicity we will discuss only Gromov convergence for J-holomorphic *spheres* and for fixed J. There are similar notions for J-holomorphic curves of higher genus, for varying J, and for stable maps[1].

Definition (Energy)**.** Let (M,ω) be a symplectic manifold and let J be an almost-complex structure compatible with ω. Let $f : \Sigma \to M$ be a stable map and let B be a subset of Σ. Write:

$$E(f,B) := \frac{1}{2}\int_B |df(z)|^2 \, d\mathrm{vol}_\Sigma$$

for the energy of the restriction of f to B.

Definition (Gromov Convergence for J-holomorphic spheres)**.** Let (M,ω) be a symplectic manifold and let J be an almost-complex structure compatible with ω.

- Let $f : \Sigma \to M$ be an n-pointed stable map of genus zero, let T be the set of components of Σ, and let $f_\alpha : \mathbb{P}^1 \to M$, $\alpha \in T$, be the restrictions of f to components of Σ. Let $x_1, \ldots, x_n \in \Sigma$ be the marked points. Let Z_α be the set of nodes on the component α of Σ.
- Let $f^\nu : \mathbb{P}^1 \to M$ be a sequence of J-holomorphic spheres in M, and let $(x_1^\nu, \ldots, x_n^\nu)$ be marked points on $\mathbb{P}^1$.

We say that $(\mathbb{P}^1; x_1^\nu, \ldots, x_n^\nu; f^\nu)$ *Gromov converges* to $(\Sigma; x_1, \ldots, x_n; f)$ if and only if there exist a collection of Möbius maps $\{\phi_\alpha^\nu\}_{\alpha\in T}^{\nu\in\mathbb{N}}$ such that:

[1]See McDuff–Salamon, *J*-holomorphic curves and symplectic topology, chapter 5.

(1) for each $\alpha \in T$, the sequence $f_\alpha^\nu := f^\nu \circ \phi_\alpha$ converges to f_α uniformly on compact subsets of $\mathbb{P}^1 - Z_\alpha$.
(2) for each pair α, β of components of Σ that meet at a node $z_{\alpha\beta} \in \Sigma$, the limits:
$$\lim_{\epsilon\to 0}\lim_{\nu\to\infty} E(f_\alpha^\nu, B_\epsilon(z_{\alpha\beta}))$$
$$\lim_{\epsilon\to 0}\lim_{\nu\to\infty} E(f_\beta^\nu, B_\epsilon(z_{\alpha\beta}))$$
exist and concide with the energies of f restricted to the two "halves" of Σ that meet at $z_{\alpha\beta}$. Here $B_\epsilon(z_{\alpha\beta}))$ denotes a ball of radius ϵ about $z_{\alpha\beta}$ in respectively the component α and the component β of Σ.
(3) for each pair α, β of components of Σ that meet at a node $z_{\alpha\beta} \in \Sigma$, the sequence $(\phi_\alpha^\nu)^{-1} \circ \phi_\beta^\nu$ converges to $z_{\alpha\beta}$ uniformly on compact subsets of $\mathbb{P}^1 - \{z_{\alpha\beta}\}$.
(4) for each i such that the marked point x_i lies on the component α of Σ, the limit:
$$\lim_{\nu\to\infty} (\phi_\alpha^\nu)^{-1}(x_i^\nu)$$
exists and equals x_i.

Exercise 9. Show that the example from Lecture 1 (the graphs of $y = (\nu x)^{-1}$ in $\mathbb{P}^2$, $\nu = 1, 2, \ldots$) defines a Gromov convergence sequence of stable maps.

The *Gromov topology* on the moduli space of stable maps $\mathcal{M}_{g,n}(M, J, d)$ is a Hausdorff topology such that the convergent sequences are the Gromov convergent sequences. With this topology, $\mathcal{M}_{g,n}(M, J, d)$ is compact and metrizable. This is proved for *semipositive* symplectic manifolds in McDuff–Salamon.

2.4. **Evaluation and Collapsing Maps.**

Definition. The *evaluation maps* ev_i, $1 \leq i \leq n$ are:
$$\begin{aligned} \mathrm{ev}_i : \mathcal{M}_{g,n}(M, J, d) &\to M \\ (\Sigma; x_1, \ldots, x_n; d) &\mapsto f(x_i) \end{aligned}$$

The maps ev_i are continuous with respect to the Gromov topology.

Definition. There is a *forgetful map* or *collapsing map*:
$$\mathrm{ct} : \mathcal{M}_{g,n}(M, J, d) \to \overline{\mathcal{M}}_{g,n}$$
that sends $(\Sigma; x_1, \ldots, x_n; d)$ to the stable map defined by Σ with any unstable components contracted.

Note that the stability conditions for stable maps and stable curves are different, so in general one will need to contract some components of Σ. The map ct is continuous with respect to the Gromov topology.

2.5. **Expected Dimension and Virtual Class.** Fix a stable map $(\Sigma; x_1, \ldots, n_n; f)$ of genus g and degree d. The expected dimension of $\mathcal{M}_{g,n}(M, J, d)$ is:
$$\underbrace{6g - 6 + 2n}_{\text{dim. of moduli space of complex structures on } \Sigma} \qquad + \qquad \underbrace{\text{index of } \overline{\partial}\text{-operator}}_{\text{dim. of moduli space for fixed } \Sigma}$$
i.e.
$$6g - 6 + 2n + (1 - g)\dim M + 2c_1(TX) \cdot d$$
Henceforth we will assume that $\mathcal{M}_{g,n}(M, J, d)$ is a compact smooth orbifold of the expected dimension. For example this holds, for generic J, when $g = 0$ and M is a homogeneous space.

Exercise 10. Let (M, ω) be an arbitrary symplectic manifold and J an almost-complex structure compatible with ω. Compute the expected and actual dimensions of $\mathcal{M}_{g,1}(M, J, 0)$.

In general things are significantly worse than this, and one needs the notions of Kuranishi structure and virtual fundamental class; see e.g.:

- Fukaya–Ono, *Arnold conjecture and Gromov-Witten invariant for general symplectic manifolds.* The Arnoldfest (Toronto, ON, 1997), 173–190, Fields Inst. Commun., 24, Amer. Math. Soc., Providence, RI, 1999.
- Li–Tian, *Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds.* Topics in symplectic 4-manifolds (Irvine, CA, 1996), 47–83, First Int. Press Lect. Ser., I, Int. Press, Cambridge, MA, 1998.
- Ruan, *Virtual neighborhoods and pseudo-holomorphic curves.* Proceedings of 6th Gökova Geometry-Topology Conference. Turkish J. Math. 23 (1999), no. 1, 161–231.

- Siebert, *Symplectic Gromov-Witten invariants.* New trends in algebraic geometry (Warwick, 1996), 375–424, London Math. Soc. Lecture Note Ser., 264, Cambridge Univ. Press, Cambridge, 1999.
- Joyce, *Kuranishi homology and Kuranishi cohomology: a User's Guide.* `arXiv:0710.5634`

But there is always a canonically-defined class in $H_k(\mathcal{M}_{g,n}(M,J,d))$, where k is the virtual dimension, called the *virtual fundamental class*. When $\mathcal{M}_{g,n}(M,J,d)$ is a smooth orbifold of the expected dimension, the virtual fundamental class coincides with the ordinary fundamental class.

2.6. **Gromov–Witten Invariants.**

Definition. Let (M,ω) be a symplectic manifold and let J be a generic almost-complex structure compatible with ω. Let $\alpha_1,\ldots,\alpha_n \in H^\bullet(M)$, and write:

$$\langle \alpha_1,\ldots,\alpha_n\rangle^M_{g,n,d} = \int_{[\mathcal{M}_{g,n}(M,J,d)]^{vir}} \mathrm{ev}_1^\star \alpha_1 \cup \cdots \cup \mathrm{ev}_n^\star(\alpha_n) \tag{2}$$

where the integral denotes cap product against the virtual fundamental class. This is a *Gromov–Witten invariant* of M.

Gromov–Witten invariants are independent of the choice of J; they are invariants of the symplectic manifold (M,ω). In good cases they have enumerative meaning, as follows...

Note however that, because $\mathcal{M}_{g,n}(M,J,d)$ may have orbifold singularities, the Gromov–Witten invariant $\langle \alpha_1,\ldots,\alpha_n\rangle^M_{g,n,d}$ will in general be a rational number even if $\alpha_1,\ldots,\alpha_n$ are integral cohomology classes on M.

3. Quantum Cohomology

3.1. **Recap: Gromov–Witten invariants.** Definition and enumerative meaning.

3.2. **Basic properties of Gromov–Witten invariants.** Behaviour under permutations of $\alpha_1,\ldots,\alpha_n$.

Lemma 11 (The String Equation)**.** *Let (M,ω) be a symplectic manifold. Let g, n, and d be such that either $d \neq 0$ or $2g-2+n>0$. For any cohomology classes $\alpha_1,\ldots,\alpha_n \in H^\bullet(M)$, we have:*

$$\langle \alpha_1,\ldots,\alpha_n,1\rangle^M_{g,n+1,d} = 0$$

Proof. □

Exercise 12 (The Divisor Equation)**.** Let (M,ω) be a symplectic manifold. Let g, n, and d be such that either $d \neq 0$ or $2g-2+n>0$. For any cohomology classes $\alpha_1,\ldots,\alpha_n \in H^\bullet(M)$, and any class $\beta \in H^2(M)$, we have:

$$\langle \alpha_1,\ldots,\alpha_n,\beta\rangle^M_{g,n+1,d} = (\beta\cdot d)\,\langle \alpha_1,\ldots,\alpha_n\rangle^M_{g,n,d}$$

What does this mean in terms of enumerative geometry?

3.3. **Big Quantum Cohomology and Small Quantum Cohomology.** Quantum cohomology is a family of algebra structures on $H^\bullet(M)$ defined in terms of genus-zero Gromov–Witten invariants of M. This family is either parametrized by $t \in H^\bullet(M)$, in the case of *big quantum cohomology*, or by $t \in H^2(M)$, in the case of *small quantum cohomology*. First we introduce a new coefficient ring for cohomology, the *Novikov ring* of M, which will allow us to define the quantum product in terms of an infinite sum without worrying about questions of convergence. Depending on your point of view, this is either:

- essential; or
- completely unimportant.

3.3.1. *The Novikov Ring.*

Definition. The *Novikov ring* $\Lambda(M)$ of M is the completion of the group ring $\mathbb{Q}[H_2(M;\mathbb{Z})]$ with respect to the valuation v such that:

$$v(d) = \omega \cdot d \qquad\qquad d \in H_2(M)$$

Concretely: suppose that M is a Kähler manifold, choose a basis $\phi_1,\ldots,\phi_r$ for the free part of $H^2(M;\mathbb{Z})$ such that the cone spanned by $\phi_1,\ldots,\phi_r$ contains the Kähler cone of M. Let $\beta_1,\ldots,\beta_r$ be the dual basis for the free part of $H_2(M;\mathbb{Z})$ and write $d \in H_2(M)$ as $d = d_1\beta_1+\cdots+d_r\beta_r$. Then the Novikov ring consists of elements:

$$\sum_{d \in H_2(M;\mathbb{Z})} \alpha_d\, q_1^{d_1}\cdots q_r^{d_r}$$

such that for each $\lambda \in \mathbb{R}$, the set

$$\big\{d \in H_2(M;\mathbb{Z}) : d \cdot \omega \leq r \text{ and } \alpha_d \neq 0\big\}$$

is finite.

3.3.2. *The Gromov–Witten Potential.* The Gromov–Witten potential F^0_M is a generating function for all genus-zero Gromov–Witten invariants of M.

Definition. Let (M,ω) be a symplectic manifold. The *Gromov–Witten potential* or *genus-zero Gromov–Witten potential* is a function of $t \in H^\bullet(M)$ taking values in $\Lambda(M)$, defined by:

$$F^0_M(t) = \sum_{d \in H_2(M;\mathbb{Z})} \sum_{n \geq 0} \frac{q^d}{n!} \langle t, \ldots, t \rangle^M_{0,n,d}$$

It is conventional in the physics literature, and also in mathematics in situations where the series defining the Gromov–Witten potential converges (examples. . .) to omit the Novikov variables, replacing the factor q^d by 1. If M is Kähler and we have set things up as above, then $q^d = q_1^{d_1} \cdots q_r^{d_r}$ and no negative powers of the variables q_i occur.

Exercise 13 (The Gromov–Witten potential of $\mathbb{P}^2$)**.** Let $M = \mathbb{P}^2$ with the standard symplectic form and complex structure. Let $p \in H^2(M)$ be the first Chern class of $\mathcal{O}(1)$, and write $x \in H^\bullet(M)$ as $x = s1 + tp + up^2$. Set:

$$N_d = \Big\langle \underbrace{p^2, \ldots, p^2}_{3d-1} \Big\rangle^M_{0,3d-1,d} \qquad\qquad d = 1, 2, 3, \ldots$$

(1) Give an enumerative interpretation of the Gromov–Witten invariant N_d.
(2) Use the String Equation and the Divisor Equation to express the Gromov–Witten potential $F^0_M(x)$ in terms of s, t, u, and N_d.

3.3.3. *The Big Quantum Product.*

Definition. Let (M,ω) be a symplectic manifold, and let $t \in M$. The (big) quantum product $*_t$ on $H^\bullet(M;\Lambda(M))$ is defined by:

$$\big(u *_t v, w\big) = \nabla_u \nabla_v \nabla_w F^0_M(t) \qquad\qquad \text{for all } u, v, w \in H^\bullet(M)$$

where $(\cdot,\cdot)$ denotes the Poincaré pairing on $H^\bullet(M)$ and ∇_u denotes directional derivative along $u \in H^\bullet(M)$.

Note the Novikov variables hiding in the definition of F^0_M. More concretely:

$$\phi_\alpha *_t \phi_\beta = \sum_{d \in H_2(M)} \sum_{n \geq 0} \frac{q^d}{n!} \langle \phi_\alpha, \phi_\beta, t, t, \ldots, t, \phi^\gamma \rangle^M_{0,n+3,d} \, \phi_\gamma$$

where $\{\phi_\alpha\}$ and $\{\phi^\beta\}$ are bases for $H^\bullet(M)$ that are dual with respect to the Poincaré pairing and we use the summation convention, summing over repeated indices. Big quantum cohomology is the family of algebras $\big(H^\bullet(M;\Lambda(M)), *_t\big)$. This family of algebras encodes, via its structure constants, genus zero Gromov–Witten invariants of M. Properties:

- (super)commutative
- associative
- unital, with unit equal to $1 \in H^\bullet(M)$
- a deformation of the usual cup product on $H^\bullet(M)$

From the point of view of symplectic geometry, this is rather depressing: Gromov–Witten invariants satisfy many universal identities (i.e. identities which hold independent of the geometry of M), and so provide fewer tools for distinguishing symplectic manifolds than one might hope. (For a maximal example of this, see Teleman's proof of Givental's conjectural formula for the higher-genus Gromov–Witten invariants of manifolds with semisimple quantum cohomology.) But from the point of view of algebraic geometry, this is good news: there is a remarkable amount of hidden structure to enumerative geometry. (See e.g. Example 16 below.)

3.3.4. *The Small Quantum Product.* In practice we will concentrate attention on a subfamily of big quantum cohomology.

Definition. The *small quantum product* $\star$ on $H^\bullet(M;\Lambda(M))$ is $*_t|_{t=0}$.

Remark. I previously described small quantum cohomology as being parametrized by $H^2(M)$, whereas in the definition we set $t=0$. This was not a mistake, but rather has to do with the presence of the Novikov variables and with the Divisor Equation. Let $\phi_1,\ldots,\phi_r$ be basis elements which span $H^2(M)$; let $t^1,\ldots,t^t$ be the co-ordinates on $H^2(M)$ corresponding to $\phi_1,\ldots,\phi_r$; and let $q_1,\ldots,q_r$ be the corresponding Novikov variables. Then the Divisor Equation implies that the quantum product $*_t$ depends on $t^1,\ldots,t^r$ and $q_1,\ldots,q_r$ only through the products $q_i e^{t^i}$, $1\le i\le r$. Thus the products $*_t|_{t=0}$ and $*_t|_{t\in H^2(M)}$ carry the same information.

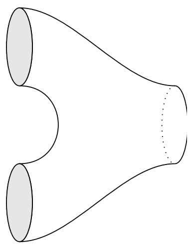
The small quantum product, concretely:

$$\phi_\alpha \star \phi_\beta = \sum_{d\in H_2(M)} q^d \left\langle \phi_\alpha,\phi_\beta,\phi^\gamma\right\rangle^M_{0,3,d}\,\phi_\gamma$$

The small quantum product, geometrically:

$$\phi_\alpha \star \phi_\beta = \sum_{d\in H_2(M)} q^d \mathrm{ev}_3^\star\big(\mathrm{ev}_1^\star\phi_\alpha \cup \mathrm{ev}_2^\star\phi_\beta\big)$$

This should remind you of Topological Field Theories:



Properties:

- (super)commutative
- associative
- unital
- graded[2], where $\deg q^d = d\cdot\omega$
- a deformation of the usual cup product on $H^\bullet(M)$

The grading is one reason why the completion involved in the definition of Novikov ring is often not so important in practice.

Theorem 14 (The Small Quantum Cohomology of Projective Space)**.** *Let $M=\mathbb{P}^n$ with the standard symplectic form. Let $p\in H^2(M)$ be the first Chern class of $\mathcal{O}(1)$, let $\beta\in H_2(M)$ be dual to p, and let q be the Novikov variable that corresponds to β in the sense of §3.3.1. Then the small quantum cohomology algebra $\big(H^\bullet(M;\Lambda(M)),\star\big)$ is:*

$$\mathbb{Q}[p,q]/(p^{n+1}-q)$$

It is a graded algebra with $\deg q = 2n+2$.

Proof. □

4. Associativity and the WDVV Equations

Theorem 15. *Let (M,ω) be a symplectic manifold. The big quantum cohomology algebra:*

$$\big(H^\bullet(M;\Lambda(M)),*_t\big)$$

is associative. Equivalently, let $\{\phi_\alpha\}$ be a basis for $H^\bullet(M)$, let $t=t^\alpha\phi_\alpha$ (summation convention!), let $g_{\alpha\beta}=(\phi_\alpha,\phi_\beta)$, and let $g^{\alpha\beta}$ be such that $g^{\alpha\epsilon}g_{\epsilon\beta}=\delta^\alpha{}_\beta$. Then:

$$\frac{\partial^3 F^0}{\partial t^\alpha\partial t^\beta\partial t^\mu}(t)g^{\mu\nu}\frac{\partial^3 F^0}{\partial t^\nu\partial t^\gamma\partial t^\delta}(t)$$

is symmetric in indices α, β, γ, δ.

[2]Big quantum cohomology is also graded, but it is a little more complicated.

So associativity of quantum cohomology is equivalent to a system of non-linear PDEs for the Gromov–Witten potential. These are the WDVV equations.

Sketch of Proof. □

In some cases, the WDVV equations are strong constraints on F^0_M.

Exercise 16. (after Kontsevich) Recall from Exercise 13 that one can write the Gromov–Witten potential for $\mathbb{P}^2$ in terms of the Gromov–Witten invariants:

$$N_d = \langle p^2, \ldots, p^2 \rangle^M_{0,3d-1,d} \qquad d = 1, 2, 3, \ldots$$

Show that the WDVV equations give the following recursive formula for N_d:

$$N(d) = \sum_{k+l=d} N(k)N(l)k^2 l \left[l \binom{3d-4}{3k-2} - k \binom{3d-4}{3k-1} \right] \qquad d \geq 2$$

Together with the initial condition $N_1 = 1$, this recursively determines the number of degree-d rational curves in the plane through $3d-1$ points in general position.

But in some cases, the WDVV equations give no information at all.

Exercise 17. (the quintic threefold) Let $Q \subset \mathbb{P}^4$ be the hypersurface defined by a generic degree-5 polynomial in 5 variables. Let:

$$n_{0,d} = \langle \ \rangle^Q_{0,0,d} \qquad \text{(the empty correlator)}$$

- Give an enumerative interpretation of the Gromov–Witten invariant n_d.
- Express the genus-zero Gromov–Witten potential of Q in terms of n_d.
- Show that the WDVV equations give no constraints on n_d.

5. A Word About Mirror Symmetry

Example 18 (The Landau–Ginzburg Model that is mirror to $\mathbb{P}^n$)**.** Recall that $\mathbb{P}^n = (\mathbb{C}^{n+1}) /\!\!/ \mathbb{C}^\times$, where $\mathbb{C}^\times$ acts on $\mathbb{C}^{n+1}$ via the embedding:

$$\begin{aligned} \mathbb{C}^\times &\to (\mathbb{C}^\times)^{n+1} \\ t &\mapsto (t, \ldots, t) \end{aligned}$$

The dual to this embedding is:

$$\begin{aligned} p : (\mathbb{C}^\times)^{n+1} &\to \mathbb{C}^\times \\ (x_0, \ldots, x_n) &\mapsto x_0 x_1 \cdots x_n \end{aligned}$$

There is a function $W : (\mathbb{C}^\times)^{n+1} \to \mathbb{C}$, called the superpotential, defined by:

$$W(x_0, \ldots, x_n) = x_0 + x_1 + \cdots + x_n$$

The fiber Y_y of $p : (\mathbb{C}^\times)^{n+1} \to \mathbb{C}^\times$ over $y \in \mathbb{C}^\times$ is the torus $x_0 x_1 \ldots x_n = y$. The family

$$\big(p\colon (\mathbb{C}^\times)^{n+1} \to \mathbb{C}^\times, W\big)$$

is a family of open Calabi–Yau manifolds[3] Y_y, $y \in \mathbb{C}^\times$, equipped with a family of functions $W : Y_y \to \mathbb{C}$. This is an example of a *Landau–Ginzburg model.* The algebra of functions on the critical set of $W|_{Y_y}$ is isomorphic to:

$$\mathbb{C}\{x, y\}/(x^{n+1} - y)$$

i.e. to the small quantum cohomology algebra of $\mathbb{P}^n$.

In fact all of the symplectic topology of $\mathbb{P}^n$ (Gromov–Witten invariants of all genera, the derived Fukaya category, etc.) can be recovered from the Landau–Ginzburg model. There is a similar story for other toric varieties, and for a broad class of complete intersections in toric varieties.

Department of Mathematics, Imperial College London, London SW7 2AZ, United Kingdom
E-mail address: tomc@imperial.ac.uk

[3]In fact to define a family of Calabi–Yau manifolds I need to specify also a holomorphic volume form on each fiber Y_y; this is:

$$\omega_y = \frac{d \log x_0 \wedge \cdots d \log x_n}{d \log y}$$