Simplicial Groupoids as Models for Homotopy Type

By Philip John Ehlers

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In the introduction, we set out some of the notation we will use, and follow with a brief history of the origins of higher homotopy, some of the milestones in its development, mentioning in fuller detail more recent developments that have taken place.

Chapter one describes in detail the Joyal-Tierney loop-groupoid functor, and its adjoint. Chapter two gives some technical results on the nature of simplicial groupoids and their classifying spaces. In chapter three we construct the codiagonal functor from bisimplicial sets to simplicial sets as the adjoint of a cotriple resolution. In chapter four we construct a left adjoint to the nerve functor from simplicial groupoids to bisimplicial sets. In chapter five we describe Kan's right adjoint to the Joyal-Tierney loop groupoid functor as the composition of the nerve and the codiagonal, while in chapter six we explain how the composition of the cotriple resolution and the left adjoint to nerve fails to be the loop groupoid functor. Chapter seven is concerned with the Moore complex of a simplicial groupoid, its homology, and its failure to be a crossed complex. In chapter eight we describe the semidirect decomposition of a simplicial groupoid. In chapter nine we reexamine Ashley's work on simplicial-T-complexes, crossed complexes and group-T-complexes, extending some of his results from the group to the groupoid case. In chapter ten we descibe the functor from simplicial groupoids to crossed complexes.

Keywords: Groupoids, simplicial sets, bisimplicial sets, simplicial groupoids, crossed complexes, higher homotopy Joyal-Tierney loop-groupoid functor, simplicial-T-complexes, group-T-complexes.

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Summary

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0. Introduction

Notation

We will use the following notation:

$\underline{\mathcal{SS}}$, the category of simplicial sets and simplicial maps;

$\underline{\mathcal{SS}}_*$, the category of reduced simplicial sets;

$\underline{\mathcal{SG}}$, the category of simplicial groups and simplicial group homomorphisms;

$\underline{\mathcal{SG}pds}$, the category of simplicial groupoids and simplicial groupoid morphisms;

$\underline{\mathcal{B}i\mathcal{SS}}$ the category of bisimplicial sets and bisimplicial maps;

Then, we have the following subcategories:

$\underline{\mathcal{SS}/O} \subset \underline{\mathcal{SS}}$: $X \in \underline{\mathcal{SS}/O} \Longleftrightarrow X_0 = O$, and $f \in \underline{\mathcal{SS}/O}(X,Y) \Longleftrightarrow f_0 = id_O$

$\underline{\mathcal{SG}pds}_* \subset \underline{\mathcal{SG}pds}$: $G \in \underline{\mathcal{SG}pds}_*$ if and only if the simplicial set of objects is constant.

$\underline{\mathcal{SG}pds/O} \subset \underline{\mathcal{SG}pds}_*$: $G \in \underline{\mathcal{SG}pds/O} \Longleftrightarrow ob(G) = O$, and

$$f \in \underline{\mathcal{SG}pds/O}(G,H) \Longleftrightarrow f|_{obG} = id$$

$\underline{\mathcal{B}i\mathcal{SS}}_* \subset \underline{\mathcal{B}i\mathcal{SS}}$: $Y_{*,*} \in \underline{\mathcal{B}i\mathcal{SS}}_*$ iff the simplicial set $Y_{0,*}$ is constant.

$\underline{\mathcal{B}i\mathcal{SS}/O} \subset \underline{\mathcal{B}i\mathcal{SS}}_*$: $Y_{*,*} \in \underline{\mathcal{B}i\mathcal{SS}/O} \Longleftrightarrow Y_{0,*} = O$ and

$$f \in \underline{\mathcal{B}i\mathcal{SS}/O}(X,Y) \Longleftrightarrow f_{0,*} = id$$

By "a k-box in X_n" we shall mean a set of elements of X_n, $x_0,\cdots,x_{k-1},x_{k+1},\cdots,x_{n+1}$, such that $d_i x_j = d_{j-1} x_i$ for $i < j$, $i,j \neq k$: we shall sometimes refer to this in general as "a compatible set in X_n".

History

The fundamental group as an invariant of homotopy type was already well known when, in 1932, Čech prepared a paper on higher homotopy to present to the International Congress of Mathematicians. It was an idea he attributed to Dehn, who never published it (see [Br.2] for references).

However, the paper was never presented, and only a small note appeared in the proceedings; Alexander and Hopf persuaded him to withdraw the paper, on the grounds that higher homotopy groups were abelian, and therefore must be homology - also already well known to topologists. This caused at least Hopf some embarrassment, as he commented much later to Dyer. Hopf had shown as early as 1929 that there was a nontrivial morphism from S^3 to S^2, thus making $\pi_3(S^2)$ nontrivial, whereas $H_m(S^n)$ is trivial unless $m = n$ or 0.

Work on higher homotopy was taken up by Hurewicz, although he never published proofs for his four notes on homotopy groups (1935,1936). Feldbau worked on homotopy with Ehresmann in the early 1940's, and is said to have provided proofs for some of Hurewicz' results, but if he did, these (along with much of his work) were never published. He was deported to Germany where he died. Certainly, the first published proof of the Homotopy Addition Lemma was in [Hu] - as late as 1953.

The study of abstract homotopy theory began with Kan. He showed (in [K.1] and [K.2]) that homotopy could be defined purely simplicially. In [K.1], Kan first defined the loop-group functor, G, from reduced simplicial spaces to simplicial groups, and then extended this to simplicial spaces in general, by taking a maximal tree. He also defined the right adjoint to G, the classifying space functor $\overline{W}$.

J.Milnor, in an unpublished paper called "On the Construction of FK" (lecture notes from Princetown, 1956 - see [A]) commented that the loop-group functor gives a simplicial group which is homotopically equivalent to the loop space of the original simplicial set, so long as it is a Kan complex. For the singular simplicial set of a topological space, this holds true.

It was not until 1984 that Dwyer and Kan proved that there was an adjunction between simplicial spaces and simplicial groupoids (see [D-K]) generalising the loop-group functor and the classifying space functor constructed so long before by Kan.

This is surprising, as one of the original objections to higher homotopy groups was that they were abelian, and so (in some sense) *less* complex than the fundamental group, rather than more. Working with groupoids rather than groups gets round this, as higher homotopy groupoids do have the non-commutativity that mathematicians had expected in higher homotopy.

It was clear from an early stage that groups were insufficient to model all homotopy types, and there resulted two distinct approaches. One was to study the homotopy groups of specific types of space. For example, Eilenberg and Maclane found that the n^{th} homotopy group functor, π_n, is an equivalence between the category of groups and the category of connected CW-complexes with only the n^{th} homotopy group non-trivial.

Further, the group of homotopy classes of continuous maps from an aspherical space, X, to a connected CW-complex, Y, with only the n^{th} homotopy group non-trivial is modeled by the n^{th} cohomology group, that is $[X,Y] \cong H^n(\pi_1 X, \pi_n Y)$. (A space is said to be aspherical if it has all homotopy groups trivial except the fundamental group.)

The second was to develop algebras to model the homotopy of larger classes of spaces: crossed modules, (which Whitehead defined in the early forties) form a model for the homotopy type of spaces with all but the first two homotopy groups trivial.

A more general structure was the n-fold groupoid (see [Eh]). This is a set with n compatible groupoid structures, where compatibility means that two distinct compositions, $+_i$ and $+_j$, satisfy the usual interchange law:-

$$(a +_i b) +_j (c +_i d) = (a +_j b) +_i (c +_j d)$$

whenever this is defined.

In 1948, (see [Bl]) Blakers used what we now call reduced crossed complexes to study how the homotopy and homology of pairs of spaces related. A reduced crossed complex is a chain of groups:-

$$\cdots \longrightarrow C_n \xrightarrow{\partial_{n-1}} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial_0} C_0$$

where C_i is abelian for $i \geq 2$, there is an action of C_0 on C_i ($\forall\ i \geq 1$) which the ∂_i respect, and $C_1 \xrightarrow{\partial_0} C_0$ is a crossed module. Blakers called them "Group Systems" and credited Eilenberg with suggesting he use them. He constructed a functor from reduced crossed complexes to simplicial sets; much later it was shown that the functor was to a subcategory of simplicial sets - namely, simplicial-T-complexes, (see [As]).

Whitehead picked up reduced crossed complexes (see [W.1] and [W.2]), considering those with certain freeness conditions in each dimension, called them "homotopy systems" and linked them to chain complexes with operators - his prime example of a homotopy system was the fundamental reduced crossed complex of the skeletal filtration of a CW-complex.

Brown and Higgins generalised these ideas to define a crossed complex, as a model for homotopy: it is from this definition that the older concept gains the name reduced crossed complex.

A crossed complex is a chain

$$\cdots \longrightarrow C_n \xrightarrow{\partial_{n-1}} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial_0} C_0,$$

where C_0 is a groupoid; for $i \geq 1$, C_i is a family of groups indexed by obC_0; for $i \geq 2$ and $p \in obC_0$, each $C_i(p)$ is abelian; there is an action of C_0 on C_i (for $i \geq 1$), such that for $g \in C_0(p,q)$, $h \in C_1(p)$ and $c \in C_i(q)$, then ${}^{g}c \in C_i(p)$, ${}^{\partial_0 h}c = c$ for $i \geq 2$, and ${}^{\partial_0 h}c = hch^{-1}$ for $i = 1$. Further, the ∂ respect the action.

Brown and Higgins defined the fundamental crossed complex of a filtered space, $X = \{X_n\}$, to be $\pi_1(X_1, X_0)$ at the base, and to be $\pi_n(X_n, X_{n-1}, p)$ $(p \in X_0)$ for the families of groups at higher levels.

They further proved that the fundamental crossed complex functor, π, from filtered CW-complexes, $\underline{\mathfrak{CW}}$, to crossed complexes, $\underline{\mathfrak{CC}}$, had a right adjoint, B, called the classifying space functor. However, for $X \in ob(\underline{\mathfrak{CW}})$, $B\pi X$ is not (in general) homotopically equivalent to X.

Rather than using simplices as the basis for homotopy, they used cubical complexes with connections; connections are essentially additional degeneracies (see [B-H.1] and [B-H.2]).
With this extra structure, they defined ω-groupoids as functors from a canonical cubical complex with connections to the category of groupoids. These have an infinite number of compatible groupoid structures, and so generalise n-fold groupoids.

There is an equivalence between the categories of crossed modules, ω-groupoids and ∞-groupoids (in an ∞-groupoid, there is a hierarchy of groupoid structures).

Alongside and underlying this work is a large body of techniques and abstract theory. The main milestone in abstract homotopy was Quillen's "Homotopical Algebra" ([Q]). He showed that many of the constructions of topological homotopy theory could be mimicked in any category with a certain structure: these are the closed model categories.

The axioms for a model category require finite completeness and cocompleteness, the existence of three specified collections of maps - called weak equivalencies, fibrations and cofibrations (in analogy to such structures in topological spaces) - and interrelations between them. The model category becomes closed when any two of the specified collections of maps will give the third by certain specified means.

Quillen showed that many categories of simplicial algebras (for example simplicial groups and simplicial Lie algebras) supported this structure.

The case for simplicial groups was already well known, having been studied by Moore, as well as various students of Kan. They worked with absolute homotopy groups, calculating them via the homology of the Moore complex.

In order to prove that the homotopy of simplicial sets was equivalent to that of simplicial groupoids, Dwyer and Kan proved that $\underline{\mathcal{S}\mathcal{G}pds}$ was a closed model category in the sense of Quillen, and that there was a pair of adjoint functors between the two categories. This adjunction was found independently by both Joyal and Tierney, but they never published it. We construct it in detail later in the paper.

Working with relative homotopy groups/groupoids and with n-skeletons is significantly easier than dealing with complete simplicial sets and absolute homotopy; Carrasco (see [Ca]) used skeleta to obtain a crossed complex from a simplicial group, and Whitehead's original construction of a crossed module was as $\partial:\pi_2(X,A,x) \longrightarrow \pi_1(A,x)$ where ∂ is the boundary map.

For a pair of spaces, X and A, with $x \in A \subset X$, the n^{th} relative homotopy group is homotopy classes of maps from an n-simplex to X, where the boundary is mapped to A, and all but one face of the boundary is mapped to the base point and the homotopies are $relA$.

Illusie (in [I.2]) defined the functors Dec and Total Dec. The functor "Dec" takes a simplicial set, X_*, to the simplicial set Y_*, where $Y_i = X_{i+1}$, and d_i, s_j on Y_n are d_{i+1}, s_{i+1} on X_{n+1}; this is the functor we refer to as P later in the paper. "Total Dec" is then the bisimplicial set with n^{th} column $\mathrm{Dec}^n X$; we construct this as P^* later in the paper. The origins of Illusie's work go back to Deligne and Verdier, among others.

The nerve of a small category is a simplicial set, where the 0-simplices are the objects, the 1-simplices are the morphisms, and the n-simplices are chains of length n of composible morphisms. It is a well known result that $\mathrm{Ner}(\mathbf{C})$ is a Kan complex if and only if $\mathbf{C}$ is a groupoid. However, the nerve of a groupoid is more than a Kan complex.

In 1975 Dakin, working with the nerves of groupoids, defined the concept of a T-complex. This is a simplicial complex, X_*, with specified subsets, T_n, for each n (called the thin elements), such that:-

1) every box in X_n has a unique filler in T_{n+1},

2) all degenerate elements are in T_*, and

3) the thin filler of a thin box has a thin lid.

Nerves of groupoids are T-complexes where the thin elements are generated by the degenerate elements, and the n-simplices for $n \geq 2$ are all thin.

Ashley noted that if a simplicial group was a group-T-complex, then the thin elements were precisely the ones generated by degenerate elements, and that a necessary and sufficient condition for a group complex to be a T-complex was that the intersection of the degenerate subgroups with the Moore complex should be trivial.

The two different approaches to homotopy (cubical or simplicial) are linked by the concept of a T-complex, since both the categories of T-complexes and of ω-groupoids are equivalent to the category of crossed complexes, see [As] and [B-H.1] and [B-H.2]).

For a simplicial group, G, Conduché has used a decomposition of G_n as $\mathrm{Ker}d_n \rtimes s_{n-1}G_{n-1}$, and from this G_n may be decomposed into semidirect products of the Moore complex terms, NG_i, and their degeneracies (for $i \leq n$). He went on to define a concept of 2-crossed modules, which are equivalent to simplicial groups whose Moore complex is trivial above the second term, (that is, has three non-trivial terms).

Carrasco (in [Ca]) has developed a concept of hypercrossed complex, which is a non-abelian chain complex with the additional information required (in the form of structure maps) to rebuild a simplicial group from it. In this case, the simplicial group will be a T-complex if and only if certain of the structure maps are trivial.

Using hypercrossed modules and the definition of the fundamental crossed complex, Carrasco and Cegarra have shown that any simplicial group will yield a crossed complex if you take as the n^{th} term the quotient $\frac{(NG)_n}{(NG_n \cap D_n)\partial(NG_{n+1} \cap D_{n+1})}$, define the ∂ maps to be the restrictions of d_0 to this quotient, and the action to be based on conjugation by elements of $(s_0)^n G_0$. A direct proof that this is indeed a crossed complex has been given by Porter ([P]).

Since $NG_0 = G_0$, and $\mathrm{Ker} d_1 \cap D_1 = 1$, the term in the crossed complex in dimension 0 will be G_0, while the first will be $\frac{NG_1}{d_0(NG_2 \cap D_2)}$.

It is well known that for a graph in *Groups*, $\left[C_1 \;\substack{\xrightarrow{s} \\ \xleftarrow{i} \\ \xrightarrow{t}}\; C_0\right]$, to form a crossed module $\mathrm{Ker} s \xrightarrow{t} C_0$, the necessary and sufficient condition is that $[\mathrm{Ker} s, \mathrm{Ker} t]$ is trivial. Brown and Loday showed that for simplicial groups $[\mathrm{Ker} d_1^1, \mathrm{Ker} d_0^1] = d_0(\mathrm{Ker} d_1^2 \cap \mathrm{Ker} d_2^2 \cap D_2)$. Porter (in [P]) noted that $(NG_n \cap D_n)\partial(NG_{n+1} \cap D_{n+1}) = [NG_n, K_n]$ where $K_n = \mathrm{Ker}(G_n \longrightarrow \pi_0 G_*)$. He was working with simplicial groups over a fixed group, commenting that Quillen worked with simplicial algebras over a fixed algebra.

In 1982, in [L], Loday first used the concept of Cat^n-group as an algebraic model for n-truncated homotopy types: a gap in the proof of his main theorem was filled by Steiner. Gilbert (in [G]) redescribed the functor between n-cubes of spaces and Cat^n-groups, while Ellis (see [E], also [E-S]), defined crossed-n-cubes, and showed that the category of crossed-n-cubes is equivalent to the category of Cat^n-groups.

The history and survey goes well beyond the scope of the dissertation: it will not touch on Cat^n-groups and crossed-n-cubes. Instead, it is restricted to generalising results known for reduced simplicial sets and simplicial groups to simplicial sets and simplicial groupoids.

1. The G, $\overline{W}$ adjunction

We construct the functor $G:\underline{\mathcal{SS}/O} \longrightarrow \underline{\mathcal{S}\mathfrak{G}pds/O}$ (usually called the Joyal-Tierney loop groupoid functor, which generalises Kan's G-functor $(G:\underline{\mathcal{SS}}_* \longrightarrow \underline{\mathcal{S}\mathfrak{G}})$) and its right adjoint $\overline{W}:\underline{\mathcal{S}\mathfrak{G}pds/O} \longrightarrow \underline{\mathcal{SS}/O}$. This construction was first written in a paper of Dwyer and Kan, [D-K], but we have taken the liberty of correcting what we believe to be typing errors in their text, as well as changing certain of the conventions to give, what is for us, a more natural construction (which is of course a matter of taste!).

A simplicial groupoid, $\mathfrak{G}$, with fixed object set O - i.e. an object in $\underline{\mathcal{S}\mathfrak{G}pds/O}$ - is a set of groupoids $\{\mathfrak{G}_i\}_{i \in \mathbb{N}}$, together with face and degeneracy morphisms, $\delta_i:\mathfrak{G}_n \longrightarrow \mathfrak{G}_{n-1}$, $\sigma_i:\mathfrak{G}_n \longrightarrow \mathfrak{G}_{n+1}$ $(0 \le i \le n)$, (which obey the usual simplicial identities). Further, $ob\mathfrak{G}_n = O$ is the same for all n, and the face and degeneracy morphisms are the identity on the object set.

Let K be a simplicial set, with vertex set K_0. Then, construct a simplicial groupoid as follows:- let $(GK)_n$ be a groupoid with object set $\{\overline{x}: x \in K_0\}$ and morphisms generated by $\overline{y}:\overline{d_1 d_2 .. d_{n+1} y} \longrightarrow \overline{d_0 d_2 .. d_{n+1} y}$, for all $y \in K_{n+1}$, with relations $\overline{s_0 z} = id_{\overline{d_1 .. d_n z}}$, for all $z \in K_n$.

We see immediately that the groupoids so defined are free, as the relations precisely kill off the 0^{th}-degeneracy elements, which are generators in the free groupoid.

We then have face and degeneracy operators in the usual manner, defined as follows:-

$$\delta_0\bar{x} = (\overline{d_1x})(\overline{d_0x})^{-1}$$

$$\delta_i\bar{x} = \overline{d_{i+1}x} \quad \text{for} \quad i > 0$$

$$\sigma_i\bar{x} = \overline{s_{i+1}x} \quad \text{for} \quad i \geq 0$$

These definitions are the same as for the reduced simplicial set/simplicial group case. The definition of δ_0 models a twisting function

Lemma 1.1

The face and degeneracy morphisms defined above are the identity on objects.

Proof:

Since $\overline{d_1d_2..d_nd_{i+1}x} = \overline{d_1d_2..d_{i+1}d_{i+1}d_{i+3}..d_{n+1}x} = \overline{d_1d_2..d_{n+1}x}$, and

$$\overline{d_0d_2..d_nd_{i+1}x} = \overline{d_0d_2..d_{i+1}d_{i+1}d_{i+3}..d_{n+1}x} = \overline{d_0d_2..d_{n+1}x},$$

δ_i will not change the source or target of any map for $i > 0$.

Further, the source of $\overline{d_1x}$ is $\overline{d_1d_2..d_nd_1x}$ which is

$$\overline{d_1d_2..d_{n-1}d_1d_{n+1}x} = \overline{d_1d_1d_3..d_{n+1}x} = \overline{d_1d_2d_3..d_{n+1}x}$$

and the target of $\overline{d_1x}$ is $\overline{d_0d_2..d_nd_1x}$ which is

$$\overline{d_0d_2..d_{n-1}d_1d_{n+1}x} = \overline{d_0d_1d_3..d_{n+1}x} \quad ;$$

then $(\overline{d_0x})^{-1}$ has source $\overline{d_0d_2..d_nd_0x}$ which is

$$\overline{d_0d_2..d_{n-1}d_0d_{n+1}x} = \overline{d_0^2d_3..d_{n+1}x} = \overline{d_0d_1d_3..d_{n+1}x}$$

and target $\overline{d_1d_2..d_nd_0x}$ which is

$$\overline{d_1d_2..d_{n-1}d_0d_{n+1}x} = \overline{d_0d_2..d_{n+1}x} \quad .$$

Thus the composite $(\overline{d_1x})(\overline{d_0x})^{-1}$ has source $\overline{d_1d_2..d_{n+1}x}$ and target $\overline{d_0d_2..d_{n+1}x}$, which means that δ_0 does not change the source or target (as for δ_i, $i \geq 1$).

For σ_i: if $i < n-1$, $\overline{s_{i+1}x}: \overline{d_1..d_{n+1}s_{i+1}x} \longrightarrow \overline{d_0d_2..d_{n+1}s_{i+1}x}$:

Then
$$\overline{d_1..d_{n+1}s_{i+1}x} = \overline{d_1..d_ns_{i+1}d_nx}$$
$$= \overline{d_1..d_{i+1}d_{i+2}s_{i+1}d_{i+2}..d_nx} = \overline{d_1..d_nx},$$

and similarly $\quad \overline{d_0 d_2 \ldots d_{n+1} s_{i+1} x} = \overline{d_0 d_2 \ldots d_n x}$.

We mention briefly the cases σ_{n-1} and σ_n (σ_0 has been dealt with by the preceding equations); it is clear that σ_n and σ_{n-1} are degenerate cases, as $d_{n+1}s_n = d_{n+1}s_{n+1} = id$. Thus, σ_i does not change the object set either. ∎

Lemma 1.2

σ_i and δ_j obey the simplicial identities.

Proof:

$$\sigma_i\sigma_j\overline{x} = \overline{s_{i+1}s_{j+1}x} = \overline{s_{j+2}s_{i+1}x} = \sigma_{j+1}\sigma_i\overline{x}, \qquad (i \le j),$$

$$\delta_i\delta_j\overline{x} = \overline{d_{i+1}d_{j+1}x} = \overline{d_j d_{i+1}x} = \delta_{j-1}\delta_i\overline{x} \qquad (0 < i < j)$$

$$\delta_i\sigma_j\overline{x} = \overline{d_{i+1}s_{j+1}x} = \begin{cases} \overline{s_j d_{i+1} x} = \sigma_{j-1}\delta_i\overline{x} & (0 < i < j) \\ \overline{idx} = \overline{x} & (i = j, j+1,\ i > 0) \\ \overline{s_{j+1}d_i x} = \sigma_j\delta_{i-1}\overline{x} & (j+1 < i) \end{cases}$$

$$\delta_0\sigma_j\overline{x} = (\overline{d_1 s_{j+1} x})(\overline{d_0 s_{j+1} x})^{-1} = (\overline{s_j d_1 x})(\overline{s_j d_0 x})^{-1} = \sigma_{j-1}\delta_0\overline{x} \qquad (j > 0)$$

$$\delta_0\sigma_0\overline{x} = (\overline{d_1 s_1 x})(\overline{d_0 s_1 x})^{-1} = (\overline{x})(\overline{s_0 d_0 x})^{-1} = \overline{x}.e = \overline{x}$$

$$\delta_0\delta_j\overline{x} = (\overline{d_1 d_{j+1} x})(\overline{d_0 d_{j+1} x})^{-1} = (\overline{d_j d_1 x})(\overline{d_j d_0 x})^{-1} = \delta_{j-1}\delta_0\overline{x} \qquad (j > 1)$$

$$\begin{aligned} \delta_0\delta_0\overline{x} &= \delta_0((\overline{d_1 x})(\overline{d_0 x})^{-1}) = \delta_0(\overline{d_1 x})(\delta_0(\overline{d_0 x}))^{-1} \\ &= (\overline{d_1 d_1 x})(\overline{d_0 d_1 x})^{-1}((\overline{d_1 d_0 x})(\overline{d_0 d_0 x})^{-1})^{-1} \\ &= (\overline{d_1 d_1 x})(\overline{d_0 d_0 x})^{-1}(\overline{d_0 d_0})(\overline{d_1 d_0 x})^{-1} = (\overline{d_1 d_1 x})(\overline{d_1 d_0 x})^{-1} \\ &= (\overline{d_1 d_2 x})(\overline{d_0 d_2 x})^{-1} = \delta_0(\overline{d_2 x}) = \delta_0\delta_1\overline{x} \end{aligned}$$ ∎

Theorem 1.3 *GX* is a simplicial groupoid. ∎

If, further, $f{:}R \longrightarrow T$ is a map of simplicial sets, then we define Gf as the simplicial groupoid morphism which naturally extends f; that is $Gf(r_1^{\varepsilon_1}r_2^{\varepsilon_2}\ldots r_n^{\varepsilon_n}) = (fr_1)^{\varepsilon_1}(fr_2)^{\varepsilon_2}\ldots(fr_n)^{\varepsilon_n}$. This is of course possible as *GR* is a free groupoid. Thus *G* is a well defined functor.

We now proceed to the generalisation of the $\bar{W}$ construction for simplicial groups. We construct $\bar{W}$ explicitly here (working from [D-K]), but $\bar{W}$ may be given as a composite functor, as we show in subsequent sections.

Let $X \in \underline{\mathcal{S}\mathcal{G}pd\Delta/O}$; we construct a simplicial set $\bar{W}X$ as follows:-

$(\bar{W}X)_0 = O$,

$(\bar{W}X)_n = \{(g_{n-1},..g_0): g_i \in arrX_i,\ domg_i = codg_{i+1}\}$ for $n > 0$,

defining $\delta_i: (\bar{W}G)_n \longrightarrow (\bar{W}G)_{n-1}$ by:-

$$\delta_n(g_{n-1},..g_0) = (d_{n-1}g_{n-1},..d_1g_1)$$

$$\delta_0(g_{n-1},..g_0) = (g_{n-2},..g_0),$$ and for $0 < i < n$,

$$\delta_i(g_{n-1},..g_0) = (d_{i-1}g_{n-1},..d_0g_{n-i}\cdot g_{n-i-1},..g_0).$$

and $\sigma_j: (\bar{W}G)_{n-1} \longrightarrow (\bar{W}G)_n$ by:-

$$\sigma_0(g_{n-1},..g_0) = (id_{x_n}, g_{n-1},..g_0),$$ and for $i > 0$,

$$\sigma_i(g_{n-1},..g_0) = (s_{i-1}g_{n-1},..s_0g_{n-i}, id_{x_{n-i}}, g_{n-i-1},..g_0),$$

where $x_i = codg_i$, and where $x_n = domg_{n-1}$.

We can consider this construction in a more pictorial way: each n-simplex of $(\bar{W}X)$ is a string of maps:-

$$x_n \xrightarrow{g_{n-1}} x_{n-1} \xrightarrow{g_{n-2}} x_{n-2}\cdots\cdots x_1 \xrightarrow{g_0} x_0,$$ where $g_i \in X_i$, $x_i = O$.

So $$\delta_i(x_n \xrightarrow{g_{n-1}} x_{n-1} \xrightarrow{g_{n-2}} x_{n-2}\cdots x_1 \xrightarrow{g_0} x_0)$$

$$= x_n \xrightarrow{d_{i-1}g_{n-1}} x_{n-1}\cdots x_{n-i+1} \xrightarrow{d_0g_{n-i}\cdot g_{n-i-1}} x_{n-i-1}\cdots x_1 \xrightarrow{g_0} x_0$$

$$\delta_0(x_n \xrightarrow{g_{n-1}} x_{n-1} \xrightarrow{g_{n-2}} x_{n-2}\cdots x_1 \xrightarrow{g_0} x_0) = x_{n-1} \xrightarrow{g_{n-2}} x_{n-2}\cdots x_1 \xrightarrow{g_0} x_0,$$

$$\delta_n(x_n \xrightarrow{g_{n-1}} x_{n-1} \xrightarrow{g_{n-2}} x_{n-2}\cdots x_1 \xrightarrow{g_0} x_0) = x_n \xrightarrow{d_{n-1}g_{n-1}} x_{n-1}\cdots x_2 \xrightarrow{d_1g_1} x_1.$$

$$\sigma_i(x_n \xrightarrow{g_{n-1}} x_{n-1} \xrightarrow{g_{n-2}} x_{n-2}\cdots x_1 \xrightarrow{g_0} x_0)$$ is

$$x_n \xrightarrow{s_{i-1}g_{n-1}} x_{n-1}\cdots x_{n-i+1} \xrightarrow{s_0g_{n-i}} x_{n-i} \xrightarrow{id_{x_{n-i}}} x_{n-i} \xrightarrow{g_{n-i-1}} x_{n-i-1}\cdots x_1 \xrightarrow{g_0} x_0).$$

Thus, if we call the vertex x_{n-i} the i^{th} vertex, the i^{th} face map may be considered as deleting the i^{th} vertex, and the i^{th} degeneracy may be considered as doubling up the i^{th} vertex.

We must check that the δ_i and σ_j obey the usual simplicial identities. Firstly, $\delta_i\delta_j$:-

Consider $1 < i+1 < j < n$,

$\delta_i\delta_j(g_{n-1},..g_0) = \delta_i(d_{j-1}g_{n-1},..d_0g_{n-j}.g_{n-j-1},..g_0)$

$= (d_{i-1}d_{j-1}g_{n-1},..(d_0d_{j-i}g_{n-i}).(d_{j-i-1}g_{n-i-1}),..(d_0g_{n-j}).g_{n-j-1},..g_0)$

$= (d_{j-2}d_{i-1}(g_{n-1}),..d_{j-i-1}(d_0(g_{n-i}).g_{n-i-1}),..d_0(g_{n-j}).g_{n-j-1},..g_0)$

$= \delta_{j-1}(d_{i-1}g_{n-1},..d_0g_{n-i}.g_{n-i-1},..g_{n-j},g_{n-j-1},..g_0)$

$= \delta_{j-1}\delta_i(g_{n-1},...g_0)$

Then for $0 < i < n-1$,

$\delta_i\delta_{i+1}(g_{n-1},\cdots,g_0) = \delta_i(d_ig_{n-1},\cdots,d_0g_{n-i-1}.g_{n-i-2},\cdots,g_0)$

$= (d_{i-1}d_ig_{n-1},\cdots d_0d_1g_{n-i}.d_0g_{n-i-1}.g_{n-i-2},\cdots,g_0)$

$= (d^2_{i-1}g_{n-1},\cdots,d_0(d_0g_{n-i}.g_{n-i-1}).g_{n-i-2},\cdots,g_0)$

$= \delta_i(d_{i-1}g_{n-1},\cdots,d_0g_{n-i}.g_{n-i-1},g_{n-i-2},\cdots,g_0) = \delta^2_i(g_{n-1},\cdots,g_0)$

Further, for $n > i > 0$

$\delta_0\delta_i(g_{n-1},\cdots,g_0) = \delta_0(d_{i-1}g_{n-1},\cdots,d_0g_{n-i-1}.g_{n-i-2},\cdots,g_0)$

$= (d_{i-2}g_{n-2},\cdots,d_0g_{n-i-1}.g_{n-i-2},\cdots,g_0) = \delta_{i-1}(g_{n-2},\cdots,g_0)$

$= \delta_{i-1}\delta_0(g_{n-1},\cdots,g_0)$

and

$\delta_i\delta_n(g_{n-1},\cdots,g_0) = \delta_i(d_{n-1}g_{n-1},\cdots,d_1g_1)$

$= (d_{i-1}d_{n-1}g_{n-1},\cdots,d_0d_{n-i}g_{n-i}.d_{n-i-1}g_{n-i-1},\cdots,d_1g_1)$

$= (d_{n-2}d_{i-1}g_{n-1},\cdots,d_{n-i-1}(d_0g_{n-i}.g_{n-i-1}),\cdots,d_1g_1)$

$= \delta_{n-1}(d_{i-1}g_{n-1},\cdots,d_0g_{n-i}.g_{n-i-1},\cdots,g_0)$

$= \delta_{i-1}\delta_0(g_{n-1},\cdots,g_0)$

finally

$\delta_0\delta_n(g_{n-1},\cdots,g_0) = \delta_0(d_{n-1}g_{n-1},\cdots,d_1g_1) = (d_{n-2}g_{n-2},\cdots,d_1g_1)$

$= \delta_{n-1}(g_{n-2},\cdots,g_0) = \delta_{n-1}\delta_0(g_{n-1},\cdots,g_0).$

The special cases involving δ_0 and δ_n are far simpler than the general cases, and so from now on we leave these (and cases involving σ_0) to the reader.

For $\sigma_i\sigma_j$, consider $1 \le i \le j$,

$\sigma_i\sigma_j(g_{n-1},\cdots,g_0) = \sigma_i(s_{j-1}g_{n-1},\cdots,s_0g_{n-j},id_{x_{n-j}},g_{n-j-1},\cdots,g_0)$

$= (s_{i-1}s_{j-1}g_{n-1},\cdots,s_0s_{j-i}g_{n-i},id_{x_{n-i}},\cdots,s_0g_{n-j},id_{x_{n-j}},\cdots,g_0)$

$= (s_js_{i-1}g_{n-1},\cdots,s_{j-i+1}s_0g_{n-i},id_{x_{n-i}},\cdots,s_0g_{n-j},id_{x_{n-j}},\cdots,g_0)$

$= \sigma_{j+1}(s_{i-1}g_{n-1},\cdots,s_0g_{n-i},id_{x_{n-i}},g_{n-i-1},\cdots,g_0) = \sigma_{j+1}\sigma_i(g_{n-1},\cdots,g_0)$

For $\delta_i\sigma_j$, consider $0 < i < j$:

$\delta_i\sigma_j(g_{n-1},\cdots,g_0) = \delta_i(s_{j-1}g_{n-1},\cdots,s_0g_{n-j},id_{x_{n-j}},g_{n-j-1},\cdots,g_0)$

$= (d_{i-1}s_{j-1}g_{n-1},\cdots,d_0s_{j-i}g_{n-i}\cdot s_{j-i-1}g_{n-i-1},\cdots,s_0g_{n-j},id_{x_{n-j}},\cdots,g_0)$

$= (s_{j-2}d_{i-1}g_{n-1},\cdots,s_{j-i-1}(d_0g_{n-i}\cdot g_{n-i-1}),\cdots,s_0g_{n-j},id_{x_{n-j}},\cdots,g_0)$

$= \sigma_{j-1}\delta_i(g_{n-1},\cdots,g_0)$

then $n > i > j+1$:

$\delta_i\sigma_j(g_{n-1},\cdots,g_0) = \delta_i(s_{j-1}g_{n-1},\cdots,s_0g_{n-j},id_{x_{n-j}},g_{n-j-1},\cdots,g_0)$

$= (d_{i-1}s_{j-1}g_{n-1},\cdots,d_{i-j}s_0g_{n-j},id_{x_{n-j}},\cdots,d_0g_{n-i+1}\cdot g_{n-i},\cdots,g_0)$

$= (s_{j-1}d_{i-2}g_{n-1},\cdots,s_0d_{i-j-1}g_{n-j},id_{x_{n-j}},\cdots,d_0g_{n-i+1}\cdot g_{n-i},\cdots,g_0)$

$= \sigma_j(d_{i-2}g_{n-1},\cdots,d_{i-j-1}g_{n-j},d_{i-j-2}g_{n-j-1},\cdots,d_0g_{n-i+1}\cdot g_{n-i},\cdots,g_0)$

$= \sigma_j\delta_{i-1}(g_{n-1},\cdots,g_0)$

Finally, consider $0 < i < n$:

$\delta_i\sigma_i(g_{n-1},\cdots,g_0) = \delta_i(s_{i-1}g_{n-1},\cdots,s_0g_{n-i},id_{x_{n-i}},g_{n-i-1},\cdots,g_0)$

$= (d_{i-1}s_{i-1}g_{n-1},\cdots,d_0s_0g_{n-i}\cdot id_{x_{n-i}},g_{n-i-1},\cdots,g_0) = (g_{n-1},\cdots,g_0)$

and $0 < i < n-1$:

$\delta_{i+1}\sigma_i(g_{n-1},\cdots,g_0) = \delta_{i+1}(s_{i-1}g_{n-1},\cdots,s_0g_{n-i},id_{x_{n-i}},g_{n-i-1},\cdots,g_0)$

$= (d_is_{i-1}g_{n-1},\cdots,d_1s_0g_{n-i},id_{x_{n-i}}\cdot g_{n-i-1},\cdots,g_0) = (g_{n-1},\cdots,g_0)$

If we have a morphism of simplicial groupoids $\phi: X \longrightarrow Y$, then $\overline{W}\phi$ from $\overline{W}X$ to $\overline{W}Y$ is defined in the obvious way, remembering that morphisms in any simplicial category commute with the face and degeneracy maps. We now check that the two functors we have defined are adjoint.

For $R \in ob(\underline{\mathcal{SS}/O})$ define $\eta_S: R \longrightarrow \overline{W}GR$ by $\eta_S(x) = (\overline{x}, \overline{d_0 x}, \cdots, \overline{d_0^{n-1} x})$ for $x \in R_n$. This has the required property that the vertices of x are the domains and codomains of the morphisms $\overline{x}, \overline{d_0 x}, \cdots, \overline{d_0^{n-1} x}$ and it is straightforward to check that $dom\overline{d_0^i x} = cod\overline{d_0^{i-1} x}$. It is further clear that $\{\eta_S\}$ is a natural transformation from $1 \longrightarrow \overline{W}G$, since face operators commute with simplicial maps. The claim is that η_S is the unit of the adjunction $\overline{W} \vdash G$.

We make the further claim, that the counit of the adjunction, $\varepsilon_X: G\overline{W}X \longrightarrow X$, is given by $\varepsilon_X: \overline{(x_{n-1}, \cdots, x_0)} \mapsto x_{n-1}$; again, this is clearly a natural transformation from $G\overline{W} \longrightarrow 1$.

To prove these claims, we need only check that:-

(i) $\overline{W}(\varepsilon_X).\eta_{\overline{W}X} = 1_{\overline{W}X}$, and (ii) $\varepsilon_{GS}.G(\eta_S) = 1_{GS}$.

(i) Consider $(g_{n-1}, \cdots, g_0) \in \overline{W}X$; then

$$
\begin{aligned}
&\overline{W}(\varepsilon_X)\eta_{\overline{W}X}(g_{n-1}, \cdots, g_0) \\
&= \overline{W}(\varepsilon_X)(\overline{(g_{n-1}, \cdots, g_0)}, \overline{(g_{n-2}, \cdots, g_0)}, \cdots, \overline{(g_0)}) \\
&= (g_{n-1}, \cdots, g_0)
\end{aligned}
$$

as required.

(i) Consider $r \in R_n$; then

$$\varepsilon_{GS}.G(\eta_S)(\overline{r}) = \varepsilon_{GS}\overline{(\overline{r}, \overline{d_0 r}, \cdots \overline{d_0^{n-1} r})} = \overline{r},$$

as required.

From these we can form the bijection $\phi: \underline{\mathcal{SG}pd\Delta/O}(GR, X) \longrightarrow \underline{\mathcal{SS}/O}(R, WX)$, as for $f: GR \longrightarrow X$, and $g: R \longrightarrow \overline{W}X$, $\phi(f) = \overline{W}(f).\eta_S$, and $\phi^{-1}(g) = \varepsilon_X.G(g)$.

2. Properties of a simplicial groupoid and of its classifying space

It is well known that a simplicial group is a Kan complex, and it is fairly trivial to show that the same is true of a simplicial groupoid: the proofs are the same, except that in the groupoid case, we need to check that various composites are well defined. While this proof naturally comes here, it is also important in section 9. For this reason, we will restate it there as Lemma 9.1. The proof is essentially from Ashley [As] (though it is doubtless much older).

First some notation; D_n is the subgroupoid of G_n generated by the degenerate elements.

Proposition 2.1

A simplicial groupoid is a Kan complex, and furthermore, for any k-box in G_{n-1} there is a filler in D_n.

Proof

We construct degenerate elements as follows:-

If $0 < k < n$, then define $w_0 = s_0x_0$ and $w_i = w_{i-1}(s_id_iw_{i-1})^{-1}s_iy_i$ for $0 < i \leq k-1$, then $w_n = w_{k-1}(s_{n-1}d_nw_{k-1})^{-1}s_{n-1}y_n$ and $w_j = w_{j+1}(s_{j-1}d_jw_{j+1})^{-1}s_{j-1}y_j$ for $k+1 \leq j < n$

The w_j are well defined, as all the y_j have the same source and target (as the d_j preserve objects), and so the w_j all have the same source and target. They are also clearly degenerate.

Considering the original box again, then for $j < k$,

$d_jw_j = d_j(w_{j-1}(s_jd_jw_{j+1})^{-1}s_jy_j) = d_jw_{j-1}(d_jw_{j+1})^{-1}y_j = y_j$ and for $k < j$, $d_jw_j = d_j(w_{j+1}(s_{j-1}d_jw_{j+1})^{-1}s_{j-1}y_j) = d_jw_{j+1}(d_jw_j)^{-1}y_j = y_j$.

Further, if we assume that w_i has been defined, and that for all j "up to" i, $d_j w_i = y_j$, then:-

for $j < i < k$, $d_j w_{i+1} = y_j(s_{i-1}d_{i-1}y_j)^{-1}s_{i-1}d_j y_i = y_j$, (since $d_{i-1}y_j = d_j y_i$),

for $j > i > k$, $d_j w_{i-1} = y_j(s_{i-1}d_i y_j)^{-1}s_{i-1}d_{j-1}y_i = y_j$ (as before),

and lastly, for $j < k < i$, $d_j w_{i-1} = y_j(s_{i-2}d_{i-1}y_j)^{-1}s_{i-2}d_j y_i = y_j$.

Thus, if we have a box missing the k^{th} face, then w_{k+1} has faces which match up with the elements of the box, and thus the w_{k+1} is a filler. Further, w_{k+1} is degenerate.

The cases $k = 0$ and $k = n$ are similar:-

If $k = 0$, then define $w_n = s_{n-1}y_n$, and $w_j = w_{j+1}(s_{j-1}d_j w_{j+1})^{-1}s_{j-1}y_j$ for $1 \le j < n$. In this case, w_1 fills the box.

If $k = n$, then define $w_0 = s_0 x_0$ and $w_i = w_{i-1}(s_i d_i w_{i-1})^{-1}s_i y_i$ for $0 < i \le n-1$. In this case, w_{n-1} fills the box. ∎

It is also true that $\overline{W}G$ is a Kan complex (a property it obtains from the groupoid G). This, too, is fairly straightforward to check, with the exception of one case, where the calculations are slightly messy.

Proposition 2.2

Let G be a simplicial groupoid, then $\overline{W}G$ is a Kan complex.

Proof

We proceed by induction: for $(\overline{W}G)_0$ a compatible set is a single element (an object of G_0), and the filler is the identity at that object; for $(\overline{W}G)_1$ we have three cases:-

i) $d_0x_0 = d_0x_1$, the filler here is $(s_0(x_1.x_0^{-1}),x_0) \in (\overline{W}G)_1$

ii) $d_1x_0 = d_0x_2$, the filler here is $(s_0x_2,x_0) \in (\overline{W}G)_1$ and

iii) $d_1x_1 = d_1x_2$, the filler here is $(s_0x_2, x_2^{-1}.x_1) \in (\overline{W}G)_1$

We remark that in all cases the x_i's are in the same hom-set, and so the elements are well defined elements of $(\overline{W}G)_1$.

We now assume that we have the extension condition satisfied for all compatible sets in $(\overline{W}G)_i$ for $i < n$. Let $w_0,\cdots,w_{k-1},w_{k+1},\cdots,w_{n+1}$ be a compatible set in $(\overline{W}G)_n$. We shall consider three cases.

<u>The case $k = 1$.</u> Let $w_0,w_2,\cdots,w_{n+1}$ be a compatible set in $(\overline{W}G)_n$. Then, if v is a filler, $\delta_0 v = w_0$, and so v must have the form (y,w_0) for $y \in G_n$. If we have such a v, $\delta_i v = (d_{i-1}y, \delta_{i-1}w_0) = (d_{i-1}y, \delta_0 w_i)$ for $i \geq 2$. For v to be a filler, we require $\delta_i v = w_i$. Since δ_0 deletes the first term, this will be satisfied if $d_{i-1}y = p_1 w_i$.

Define $x_{i-1} = p_1 w_i$ for $i \geq 2$. Then, for $0 < i < j$, $d_i x_j = d_i p_1 w_{j+1} = p_1 \delta_{i+1} w_{j+1} = p_1 \delta_j w_{i+1} = d_{j-1} p_1 w_{i+1} = d_{j-1} x_i$, and so the x_i form a compatible set in G_{n-1} for $i \geq 2$. Thus there is a filler, which we will call y, such that $d_i y = x_i$ for $n \geq i \geq 1$. This is clearly the y we require.

<u>The case $k \geq 2$.</u> We have $w_0,w_1,\cdots,w_{k-1},w_{k+1},\cdots,w_{n+1}$ a compatible set in $(\overline{W}G)_n$. As before, v must be of the form (y,w_0) to be a filler and we define $x_i = p_1 w_{i+1}$, $i \neq k-1$, and $x = p_1 w_0$.

Now, $\delta_1 v = (d_0 y.x, \delta_0 w_0) = (d_0 y.x, \delta_0 w_1)$, while for $i \geq 2$, $\delta_i v = (d_{i-1}y, \delta_{i-1}w_0) = (d_{i-1}y, \delta_0 w_i)$, thus for v to be a filler, we require y to be a filler for the set $\{x_0.x^{-1}\} \cup \{x_i \text{ s.t. } i \geq 1\}$ where we consider $x_0.x^{-1}$ to be the zero face of the box. All we require for such a y to exist, is for the set to be a $(k-1)$-box.

For $0 < i < j$, compatibility follows from the last section, so we need only check that $d_0 x_i = d_{i-1}(x_0 . x^{-1})$.

So $$d_0 x_i = (p_1\delta_1 w_{i+1})(p_2 w_{i+1})^{-1} = p_1((\delta_1 w_{i+1})(\delta_0 w_{i+1})^{-1})$$
$$= p_1((\delta_i w_1)(\delta_i w_0)^{-1}) = d_{i-1}(x_0 . x^{-1}) \qquad \text{as required.}$$

<u>The case $k = 0$.</u> We have a set of compatible elements in $(\overline{W}G)_n$, $w_1, \cdots, w_{n+1}$. We write $w_1 = (x_0, y, v)$ and $w_i = (x_{i-1}, y_{i-2}, v_{i-2})$ (for $i \geq 2$) where $x_j \in G_{n-1}$, $y, y_j \in G_{n-2}$ and $v, v_j \in (WG)_{n-1}$. Since $\delta_{i-1} w_1 = \delta_1 w_i$ for $i > 1$, we have $d_0 x_0 . y = d_0 x_1 . y_0$ and $d_{i-2} x_0 = d_0 x_{i-1} . y_{i-2}$ for $3 \leq i \leq n-1$, by equating the first projections, and thus $y_0 . y^{-1} = d_0(x_1{}^{-1} . x_0)$ and $y_i = d_0 x_{i+1}^{-1} . d_i x_0$ $1 \leq i \leq n-1$. Further, we note that $d_i x_j = d_{j-1} x_i$ for $0 < i < j$ from the compatibility conditions on the w_j, $j > 2$.

Now, if we have a filler $w \in (\overline{W}G)_{n+1}$, then $\delta_0{}^2 w = \delta_0 \delta_1 w = \delta_0 w_1$, so we may write $w = (a, b, y, v)$ where $a \in G_n$, $b \in G_{n-1}$ (and y and v are already defined). Then, $\delta_1 w = (d_0 a . b, y, v)$ so we require $d_0 a . b = x_0$; $\delta_2 w = (d_1 a, d_0 b . y, v)$ so we require $v = v_0$, $d_0 b . y = y_0$, and $d_1 a = x_1$; and for $3 \leq i \leq n+1$, $\delta_i w = (d_{i-1} a, d_{i-2} b, \delta_{i-2}(y, v))$ so we require $\delta_{i-2}(y, v) = v_{i-2}$, $d_{i-2} b = y_{i-2}$ and $d_{i-1} a = x_{i-1}$.

Firstly, $\delta_{i-2}(y, v) = \delta_{i-2}\delta_0 w_1 = \delta_0 \delta_{i-1} w_1 = \delta_0 \delta_1 w_i = \delta_0{}^2 w_i = v_{i-2}$, for $2 \leq i \leq n+1$. Then, we recall that $y_0 . y^{-1} = d_0(x_1{}^{-1} . x_0)$ and $y_i = d_0 x_{i+1} . d_i x_0$ $1 \leq i \leq n-1$, so we wish to define $b \in G_{n-1}$ with $d_i b = d_0 x_{i+1}^{-1} . d_i x_0$ $1 \leq i \leq n-1$, and $d_0 b = y_0 . y^{-1} = d_0(x_1{}^{-1} . x_0)$, and of course for such a b, $cod b = cod(d_i b) = cod(d_i x_0) = cod x_0 = dom y$.

We define functions (*not* morphisms) on each G_n as follows:-

$g_i x = x^{-1}(s_i d_i x)$ $(0 \leq i \leq n)$, $\qquad f_0 x = (s_0 d_0 x)^{-1} x$ and $f_i x = (s_i d_i x) x^{-1}$ $(1 \leq i \leq n)$. We note the following properties of

f_i and g_i:-

1) $d_i f_j(x) = f_{j-1}(d_i x) \quad 0 \le i < j$ 2) $d_0 f_1 x = f_0(d_0 x)^{-1}$

3) $d_{i+1} f_i(x) = d_i x . d_{i+1} x^{-1} \quad 0 < i$ 4) $d_1 f_0 x = (d_0 x)^{-1}(d_1 x)$

5) $d_i f_j(x) = f_j(d_i x) \quad i > j+1 > 0$ 6) $d_i f_i(x) = 1 = d_i g_i(x)$

7) $d_i g_j(x) = g_{j-1}(d_i x) \quad i < j$ 8) $d_{i+1} g_i(x) = d_{i+1} x^{-1} . d_i x$

9) $d_i g_j(x) = g_j(d_i x) \quad i > j+1$

10) $g_j x$ and $f_0 x$ are loops at $codx$ $\forall\ j$, and

11) $f_i x$ is a loop at $domx$ $\forall\ i > 0$

Define $b = (f_{n-2} \cdots (f_0 x_n)).(f_{n-3} \cdots (f_0 x_{n-1})). \cdots .(f_0 x_2).x_1^{-1}.x_0$ Since $d_0 x_0 . y = d_0 x_1 . y_0$ and $d_i x_j = d_{j-1} x_i$ for $0 < i < j$, we have $domx_i = domx_0$ and $codx_i = codx_j$ $\forall\ i, j \ge 1$ and so b is well defined in G_{n-1}.

Now $d_i b = d_i((f_{i-1} \cdots (f_0 x_{i+1}))).(f_{i-2} \cdots (f_0 d_i x_i)). \cdots .d_i x_1^{-1}.d_i x_0$ and $d_i(f_{i-1} \cdots (f_0 x_{i+1})) = d_{i-1}((f_{i-2} \cdots (f_0 x_{i+1}))).(f_{i-2} \cdots (f_0 d_i x_{i+1}))^{-1}$.

This process continues, so that we get a term:-

$(d_0 x_{i+1})^{-1} d_1 x_{i+1} . (f_0 d_2 x_{i+1})^{-1} . (f_1 f_0 d_3 x_{i+1})^{-1} . \cdots . (f_{i-2} \cdot\cdot (f_0 d_i x_{i+1}))^{-1}$

and since we have $d_i x_j = d_{j-1} x_i$ for $0 < i < j$ this becomes

$(d_0 x_{i+1})^{-1} d_i x_1 . (f_0 d_i x_2)^{-1} . (f_1 f_0 d_i x_3)^{-1} . \cdots . (f_{i-2} \cdot\cdot (f_0 d_i x_i))^{-1}$ and so $d_i b = (d_0 x_{i+1})^{-1}(d_i x_0)$ as required.

Now, we require to construct $a \in G_n$, remembering that we need $d_i a = x_i$ and $d_0 a . b = x_0$, so $d_0 a = x_0 . b^{-1}$. Note, as b is defined in terms of x_i's terminating in x_0, that $d_0 a$ is defined in terms of x_i's for $i > 0$, and so we need a to be dependant only on these x_i's. Define $a = (s_0 x_1).g_1(s_0 x_2). \cdots .g_{n-1}(\cdot\cdot g_1(s_0 x_n))$, well defined in G_n as $codx_i = codx_j$ $\forall\ i, j \ge 1$, and $g_i x$ is a loop at $codx$.

For $i \ge 2$, $d_i a = (s_0 d_{i-1} x_1).g_1(s_0 d_{i-1} x_2). \cdots .d_i(g_{i-1}(\cdot\cdot g_1(s_0 x_i)))$ and as before we can rewrite $d_i(g_{i-1}(\cdot\cdot g_1(s_0 x_i)))$ as:-

$(g_{i-2}(\cdot\cdot g_1(s_0 d_{i-1} x_{i-1})))^{-1}.(g_{i-3}(\cdot\cdot g_1(s_0 d_{i-1} x_{i-2})))^{-1}. \cdots .(s_0 d_{i-1} x_1)^{-1}.x_i$

and so $d_i a = x_i$ for $i \ge 2$.

The case $i = 1$ is simpler still; $d_1 a = d_1 s_0 x_1 = x_1$. Lastly, $d_0 a = x_1.(g_0 x_2).\cdots.(g_{n-2}(\cdot\cdot(g_0 x_n)).$ Now, $(f_0 x)^{-1} = ((s_0 d_0 x)^{-1} x)^{-1} = x^{-1}(s_0 d_0 x) = g_0 x$, and for $i > 0$, $(f_i x^{-1})^{-1} = ((s_i d_i x^{-1})x)^{-1} = x^{-1}(s_i d_i x) = g_i x$, and so $d_0 a = x_0.b^{-1}$ as required. This completes the proof. ∎

These two together show that if we consider the full subcategory of Kan complexes, then the adjunction $\overline{W} \vdash G$ that we constructed in §1 restricts to an adjunction between the category of Kan complexes and the category of simplicial groupoids. If we are considering the singular simplicial set of a topological space, then we will always be working within this subcategory of $\underline{\mathcal{SS}}$.

3. The P^*, ∇ adjunction

The work in the next four sections is based on (and would have been impossible without) the notes which Prof. J. Duskin and Prof. D. Van Osdol kindly sent to me and for which I am extremely grateful. The notation used here is largely theirs.

We define $\underline{\mathcal{Del}}$ to be the category of finite ordinals, (that is $[n] = \{0, 1, \cdots, n\}$) with morphisms the order preserving maps, and $\underline{\mathcal{Del}_0}$ to be the wide subcategory of $\underline{\mathcal{Del}}$ where the morphisms fix the zero. Then, $in: \underline{\mathcal{Del}_0} \longrightarrow \underline{\mathcal{Del}}$ is the inclusion functor, and $b: \underline{\mathcal{Del}} \longrightarrow \underline{\mathcal{Del}_0}$ is defined by:-

$b: [n] \mapsto [n+1]$ on objects,

$bf(0) = 0$, and $bf(i) = f(i-1)+1$ for $i > 0$ where $f: [n] \longrightarrow [m]$.

Now, $b \dashv in$, as there is a bijection $\theta: \underline{\mathcal{Del}_0}([n+1], [m]) \longrightarrow \underline{\mathcal{Del}}([n], [m])$ defined by:-

$$\theta(f)(i) = f(i+1), \qquad \text{with inverse}$$

$$\theta^{-1}(f)(i) = \begin{cases} f(i-1) & \text{for} \quad i > 0 \\ 0 \quad \text{for} & i = 0 \end{cases}$$

Let $\underline{\mathcal{S}}$ be a complete and cocomplete category (which we will often refer to as a category of "spaces"). We define $\underline{\mathcal{SS}}$ to be the category of contravariant functors from $\underline{\mathcal{Del}}$ to $\underline{\mathcal{S}}$: this is usually known as the category of simplicial spaces; and we define $\underline{\mathcal{CSS}}$ to be the category of contravariant functors from $\underline{\mathcal{Del}_0}$ to $\underline{\mathcal{S}}$: this category is usually known as the category of contractible simplicial spaces. Then, for any $X \in \underline{\mathcal{SS}}$, there is a composite functor $Xin \in \underline{\mathcal{CSS}}$ and so we can define a functor $in^*: \underline{\mathcal{SS}} \longrightarrow \underline{\mathcal{CSS}}$, which is composition with in. Similarly, there is a functor $b^*: \underline{\mathcal{CSS}} \longrightarrow \underline{\mathcal{SS}}$ which is composition with b. Then, we can define a functor $P = b^* in^*: \underline{\mathcal{SS}} \longrightarrow \underline{\mathcal{SS}}$.

Proposition 3.1

If X_* is a Kan complex, then so is PX, and further, $\pi_0(PX) \cong X_0$.

Proof

Let $w_0, \cdots, w_{k-1}, w_{k+1}, \cdots, w_{n+1}$ be a k-box in $(PX)_n$. Then, $w_i \in X_{n+1}$ $\forall$ $i \neq k$. We have $d_i w_j = d_{j-1} w_i$ for $i < j$, $i, j \neq k$ for d_i in $(PX)_n$. This means that in X_{n+1}, $d_{i+1} w_j = d_j w_i$, $i+1 < j$, $i+1, j \neq k$.

So, rename $w_i = v_{i+1}$, and we have $v_1, \cdots, v_k, v_{k+2}, \cdots, v_{n+1} \in X_{n+1}$, and $d_i v_j = d_{j-1} v_i$ for $0 < i < j$, $i, j \neq k+1$. Define $x_{i-1} = d_0 v_i$, $0 < i < n$, $i \neq k+1$, then x_i, $i \neq k$ form a k-box in X_n since $d_i x_j = d_i d_0 v_{j+1} = d_0 d_{i+1} v_{j+1} = d_0 d_j v_{i+1} = d_{j-1} d_0 v_{i+1} = d_{j-1} x_i$. Then call the filler $v_0 \in X_{n+1}$, and we have $d_i v_0 = x_i = d_0 v_{i+1}$ $i = k$. Thus, we can extend our set v_i, $i = 0, k$ by an element v_0, and the set is still compatible in X_{n+1}, and so has a filler in X_{n+2}. This element is therefore a filler in $(PX)_{n+1}$ for the original set, as required.

Now, the morphism $d_1: (PX)_0 \longrightarrow X_0$ weakly coequalises the two maps $d_1, d_0: (PX)_1 \longrightarrow (PX)_0$, that is $d_2, d_1: X_2 \longrightarrow X_1$. Thus, by the universal property of coequalisers, there is a unique map $f: \pi_0(PX_*) \longrightarrow X_0$, which takes $[x] \longrightarrow d_1 x$.

This is well defined, as $x \simeq y$ in $(PX)_0$ iff $\exists$ $z \in X_2$, and $d_2 z = x$, $d_1 z = y$, and thus $d_1 x = d_1 d_2 z = d_1{}^2 z = d_1 y$. However, since X is Kan, if $x, y \in X_1$, and $d_1 x = d_1 y$, then there is a filler, z s.t. $d_1 z = y$ and $d_2 z = x$, and so $[x] = [y]$. Thus, f is a bijection, and $\pi_0(PX_*) \cong X_0$. ∎

Since $\underline{\mathcal{D}el}$ and $\underline{\mathcal{D}el_0}$ are small, and (as we have required) $\underline{\mathcal{S}}$ is both complete and cocomplete (as in the case of where $\underline{\mathcal{S}}$ is $\underline{\mathcal{S}ets}$ or $\underline{\mathcal{G}pds}$), then each functor $T: \underline{\mathcal{D}el_0} \longrightarrow \underline{\mathcal{S}}$ (i.e. $T \in \underline{\mathcal{CSS}}$), has both right and left Kan extensions along in, and every functor $T:\underline{\mathcal{D}el} \longrightarrow \underline{\mathcal{S}}$ (i.e. $T \in \underline{\mathcal{SS}}$) has right and left Kan extensions along b. That is, both b^* and in^* have both left and right adjoints. Since $b^* \dashv in^*$, then we have C and W such that $C \dashv b^* \dashv in^* \dashv W$. We will not make use of C and W here, and mention them only for the sake of completeness.

Consider the adjunction $b^* \dashv in^*$. It has unit s_0 and counit d_0 (where s_0 is in $\underline{\mathcal{CSS}}$ and d_0 is in $\underline{\mathcal{SS}}$). This means that the cotriple $P = b^*in^*$ has counit d_0, and comultiplication s_0. Thus for a simplicial space X, we have a simplicial object P^*X, where $d_0:P^2X \longrightarrow PX$ is d_0, and $d_1:P^2X \longrightarrow PX$ is $Pd_0 = d_1$. Similarly, $s_0:PX \longrightarrow P^2X$ is s_0, and since $Ps_0 = s_1$ we will be able to obtain s_i for higher levels of the simplicial object. Thus, $(P^*X)_n = P^{n+1}X$ and the simplicial maps are precisely the simplicial maps of X.

Since each $P^{n+1}X$ is itself a simplicial space, we have, in P^*X, a bisimplicial set. This bisimplicial set is usually called the cotriple resolution of $P(X)$. We have drawn a section of it below:-

$$\begin{array}{ccccc}
X_1 & \begin{smallmatrix}\longleftarrow d_0 - \\ \longleftarrow d_1 -\end{smallmatrix} & X_2 & \begin{smallmatrix}\longleftarrow d_0 - \\ \longleftarrow d_1 - \\ \longleftarrow d_2 -\end{smallmatrix} & X_1 \longleftarrow \\
\uparrow\ \uparrow & & \uparrow\ \uparrow & & \uparrow\ \uparrow \\
d_1\ d_2 & & d_1\ d_2 & & d_1\ d_2 \\
|\ \ | & & |\ \ | & & |\ \ | \\
X_2 & \begin{smallmatrix}\longleftarrow d_0 - \\ \longleftarrow d_1 -\end{smallmatrix} & X_3 & \begin{smallmatrix}\longleftarrow d_0 - \\ \longleftarrow d_1 - \\ \longleftarrow d_2 -\end{smallmatrix} & X_2 \longleftarrow \\
\uparrow\ \uparrow\ \uparrow & & \uparrow\ \uparrow\ \uparrow & & \uparrow\ \uparrow\ \uparrow \\
d_1\ d_2\ d_3 & & d_1 d_2 d_3 & & d_1 d_2 d_3 \\
|\ \ |\ \ | & & |\ \ |\ \ | & & |\ \ |\ \ | \\
X_3 & \begin{smallmatrix}\longleftarrow d_0 - \\ \longleftarrow d_1 -\end{smallmatrix} & X_4 & \begin{smallmatrix}\longleftarrow d_0 - \\ \longleftarrow d_1 - \\ \longleftarrow d_2 -\end{smallmatrix} & X_3 \longleftarrow \\
\uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots
\end{array}$$

We clearly have the object part of a functor $P^*:\underline{\mathcal{SS}} \longrightarrow \underline{\mathcal{B}i\mathcal{SS}}$, and P^* can easily be extended to morphisms by $P^*(f)_{n,m} = f_{n+m+1}$.

We now consider a different functor, $or^*: \underline{\mathcal{SS}} \longrightarrow \underline{\mathcal{BiSS}}$, which we will prove to be equal to P^*. Firstly, we define $or: \underline{\mathcal{D}el} \times \underline{\mathcal{D}el} \longrightarrow \underline{\mathcal{D}el}$. This is the functor ordinal sum, defined on objects by:-

$$or: ([n],[m]) \mapsto [n+m+1],$$

and on arrows by:-

$$or: (f,g) \mapsto (f+g),$$

where for $\quad (f,g): ([n],[m]) \longrightarrow ([n'],[m']),$

$$(f+g): [n+m+1] \longrightarrow [n'+m'+1],$$

and $(f+g)$ is defined as follows:-

$$(f+g)(i) = \begin{cases} f(i) & 0 \leq i \leq n \\ g(i-n-1)+n'+1 & n+1 \leq i \leq n+m+1 \end{cases}$$

Now, consider $i^-: [n-1] \longrightarrow [n]$ and $i^+: [n+1] \longrightarrow [n]$ where

$$i^-(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases} \quad \text{and} \quad i^+(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}.$$

These are the morphisms which become the i^{th} face and degeneracy maps (respectively) of a simplicial set. For a bisimplicial set, $X: (\underline{\mathcal{D}el} \times \underline{\mathcal{D}el})^{op} \longrightarrow \underline{\mathcal{S}ets}$, the maps $X(i^-,1)$ and $X(i^+,1)$ are d_i^h and s_i^h (the i^{th} horizontal face and degeneracy maps), while $X(1,i^-)$ and $X(1,i^+)$ are d_i^v and s_i^v (the i^{th} vertical face and degeneracy maps).

Then, $\quad (i^+ +1)(j) = \left\{\begin{matrix} j & \text{if } j \leq i \\ j-1 & \text{if } i < j \leq n+m+2 \end{matrix}\right\} = i^+: [n+m+2] \longrightarrow [n+m+1],$

and $\quad (i^- +1)(j) = \left\{\begin{matrix} j & \text{if } j < i \\ j+1 & \text{if } i \leq j \leq n+m \end{matrix}\right\} = i^-: [n+m] \longrightarrow [n+m+1].$

Also, $\quad (1+i^+)(j) = \left\{\begin{matrix} j & \text{if } j \leq n+i+1 \\ j-1 & \text{if } n+i+1 < j \leq n+m+2 \end{matrix}\right\}$

$$= (n+i+1)^+: [n+m+2] \longrightarrow [n+m+1]$$

and $\quad (1+i^-)(j) = \left\{\begin{matrix} j & \text{if } j < n+i+1 \\ j+1 & \text{if } n+i+1 \leq j \leq n+m \end{matrix}\right\} = (n+i)^-: [n+m] \longrightarrow [n+m+1].$

Thus, $(or^*X)_{n,m} = X_{n+m+1}$,

$d_i^h\colon (or^*X)_{n,m} \longrightarrow (or^*X)_{n-1,m}$ is $d_i\colon X_{n+m+1} \longrightarrow X_{n+m}$,

$s_i^h\colon (or^*X)_{n,m} \longrightarrow (or^*X)_{n+1,m}$ is $s_i\colon X_{n+m+1} \longrightarrow X_{n+m+2}$,

$d_i^v\colon (or^*X)_{n,m} \longrightarrow (or^*X)_{n,m-1}$ is $d_{i+n+1}\colon X_{n+m} \longrightarrow X_{n+m+1}$ and

$s_i^v\colon (or^*X)_{n,m} \longrightarrow (or^*X)_{n,m+1}$ is $s_{i+n+1}\colon X_{n+m+1} \longrightarrow X_{n+m+2}$.

Thus, as claimed, $P^* = or^*$, and since $\underline{\mathcal{Del}} \times \underline{\mathcal{Del}}$ is a small category, we have left and right adjoints for P^*, (the Kan extensions along or).

From now on, we will consider $\underline{\mathcal{S}}$ to be the category $\underline{\mathcal{Sets}}$.

We now proceed to construct a right adjoint to the functor $or^* = P^*$, which we will call $\nabla\colon \underline{\mathcal{BiSS}} \longrightarrow \underline{\mathcal{SS}}$. Recall that for a simplicial set X, the cotriple resolution of $P(X)$ is the bisimplicial set $P^*(X)$. We redraw it below, with its augmentation (that is $d_0\colon P^*X \longrightarrow X$) bracketed in the left hand column. There is a similar augmentation which is the row above the diagram, where the map is $d_{last}\colon P^*X \longrightarrow X$.

$$\left[\begin{array}{c}
X_0 \xleftarrow{\;d_0\;} \\
\uparrow\ \uparrow \\
d_0\ d_1 \\
X_1 \xleftarrow{\;d_0\;} \\
\uparrow\ \uparrow\ \uparrow \\
d_0\ d_1\ d_2 \\
X_2 \xleftarrow{\;d_0\;} \\
\uparrow \\
\vdots
\end{array}\right]
\begin{array}{ccc}
X_1 & \overset{\xleftarrow{\;d_0\;}}{\xleftarrow[\;d_1\;]{}} & X_2 \leftarrow\cdots \\
\uparrow\ \uparrow & & \uparrow\ \uparrow \\
d_1\ d_2 & & d_2\ d_3 \\
X_2 & \overset{\xleftarrow{\;d_0\;}}{\xleftarrow[\;d_1\;]{}} & X_3 \leftarrow\cdots \\
\uparrow\ \uparrow\ \uparrow & & \uparrow\ \uparrow\ \uparrow \\
d_1\ d_2\ d_3 & & d_2\ d_3\ d_4 \\
X_3 & \overset{\xleftarrow{\;d_0\;}}{\xleftarrow[\;d_1\;]{}} & X_4 \leftarrow\cdots \\
\uparrow & & \uparrow \\
\vdots & & \vdots
\end{array}$$

If we have a bisimplicial set, $Y_{*,*}$, and a simplicial set, X_*, then we want $\underline{\mathcal{BiSS}}(P^*X, Y) \cong \underline{\mathcal{SS}}(X, \nabla Y)$. Now, $(P^*X)_{p,q} = X_{p+q+1}$, so a morphism $f\colon (P^*X) \longrightarrow Y$ is a family of morphisms, $f_{p,q}\colon X_{p+q+1} \longrightarrow Y_{p,q}$, *with the property that:-*

$$d_i^h f_{p+1,q} = f_{p,q} d_i^h = f_{p,q} d_i \quad (\text{for } 0 \le i \le p+1),$$

$$d^v_j f_{p,q+1} = f_{p,q} d^v_j = f_{p,q} d_{j+p+1} \quad (\text{for } 0 \le j \le q+1)$$
$$s^h_i f_{p-1,q} = f_{p,q} s^h_i = f_{p,q} s_i \quad (\text{for } 0 \le i \le p-1)$$
$$s^v_j f_{p,q-1} = f_{p,q} s^v_j = f_{p,q} s_{j+p} \quad (\text{for } 0 \le j \le q-1)$$

Thus, we have $(f_{0,n}, f_{1,n-1}, \cdots, f_{n-1,1}, f_{n,0}): X_{n+1} \longrightarrow Y_{0,n} \times \cdots \times Y_{n,0}$, a family of maps with the above conditions. The only constraints they impose is $d^h_i f_{p,q} = f_{p-1,q} d_i = d^v_{i-p} f_{p-1,q+1}$ when $0 \le i \le p$ and $0 \le i-p \le q+1$ and so this only holds when $i = p$, so $d^h_p f_{p,q} = d^v_0 f_{p-1,q+1}$. So we define $(\nabla Y)_n$ to be the subobject of $(Y_{0,n} \times \cdots \times Y_{n,0})$ for which these relations hold, thus it is the equaliser of the two maps:-

$(d^v_0 p_1, d^v_0 p_2, \cdots, d^v_0 p_n): Y_{0,n} \times \cdots \times Y_{n,0} \longrightarrow Y_{0,n-1} \times \cdots \times Y_{n-1,0}$ and $(d^h_1 p_2, d^h_2 p_3, \cdots, d^h_n p_{n+1}): Y_{0,n} \times \cdots \times Y_{n,0} \longrightarrow Y_{0,n-1} \times \cdots \times Y_{n-1,0}$, where p_i is the i^{th} projection map.

There is a special case, namely $n = 0$. In this case we take $(\nabla Y)_n = Y_{0,0}$. Now, ∇Y is a simplicial set, and so we must find it's faces and degeneracies. For clarity, let us call them σ_i and δ_i.

As we have already said, for every map $f_{*,*}: P^*X \longrightarrow Y$, there must be a unique map $\phi_*: X \longrightarrow \nabla Y$. We will construct this at the same time as the faces and degeneracies.

For the 0-simplices, we require $\phi_0: X_0 \longrightarrow Y_{0,0}$. Now $f_{0,0}: X_1 \longrightarrow Y_{0,0}$, so if we set $\phi_0 = f_{0,0} s_0$, we will have a well defined map $\phi_0: X_0 \longrightarrow Y_{0,0}$. There is, in fact, no sensible alternative! Since simplicial maps commute with the face and degeneracy maps, we require that $\delta_i \phi_j = \phi_{j-1} d_i$, and $\sigma_0 \phi_{j-1} = \phi_j s_0$. At the level $j = 1$, this means that $\delta_i \phi_1 = \phi_0 d_i$, and $\sigma_0 \phi_0 = \phi_1 s_0$.

This means that $f_{0,0}s_0d_0 = f_{0,0}d_0s_1 = d_0^h f_{1,0}s_1$, and $f_{0,0}s_0d_1 = f_{0,0}d_2s_0 = d_1^v f_{0,1}s_0$. Thus, $\phi_1 : X_1 \longrightarrow (\nabla Y)_1$ must be $(f_{0,1}s_0, f_{1,0}s_1)$ and so $\delta_0 : (\nabla Y)_1 \longrightarrow (\nabla Y)_0$ must be $d_0^h p_2$ and δ_1 must be $d_1^v p_1$. Then, $(f_{0,1}s_0, f_{1,0}s_1)s_0 = (f_{0,1}s_0^2, f_{1,0}s_1s_0)$ and $s_1s_0 = s_0^2$, and $f_{0,1}s_0^2 = f_{0,1}s_1s_0 = s_0^v f_{0,0}s_0$, while $f_{1,0}s_1s_0 = f_{1,0}s_0^2 = s_0^h f_{0,0}s_0$, so that $\sigma_0 = (s_0^v, s_0^h) : (\nabla Y)_0 \longrightarrow (\nabla Y)_1$.

For $n = 2$, we have $\delta_1\phi_2 = \phi_1 d_1$, and $\sigma_1\phi_1 = \phi_2 s_1$. So

$$\phi_1 d_0 = (f_{0,1}s_0d_0, f_{1,0}s_1d_0) = (f_{0,1}d_0s_1, f_{1,0}d_0s_2) = (d_0^h f_{1,1}s_1, d_0^h f_{2,0}s_2)$$

$$\phi_1 d_1 = (f_{0,1}s_0d_1, f_{1,0}s_1d_1) = (f_{0,1}d_2s_0, f_{1,0}d_1s_2) = (d_1^v f_{0,2}s_0, d_1^h f_{2,0}s_2)$$

and

$$\phi_1 d_2 = (f_{0,1}s_0d_2, f_{1,0}s_1d_2) = (f_{0,1}d_3s_0, f_{1,0}d_2s_0) = (d_2^v f_{0,2}s_0, d_1^v f_{1,1}s_1)$$

This gives us $\delta_0 = (d_0^h p_2, d_0^h p_3)$, $\delta_1 = (d_1^v p_1, d_1^h p_3)$ and $\delta_2 = (d_2^v p_1, d_1^v p_2)$, while $\phi_2 = (f_{0,2}s_0, f_{1,1}s_1, f_{2,0}s_2)$.

Further, $\phi_2 s_0 = (f_{0,2}s_0, f_{1,1}s_1, f_{2,0}s_2)s_0$

$= (f_{0,2}s_1s_0, f_{1,1}s_0^2, f_{2,0}s_0s_1) = (s_0^v f_{0,1}s_0, s_0^h f_{0,1}s_0, s_0^h f_{1,0}s_1)$ and,

$\phi_2 s_1 = (f_{0,2}s_0, f_{1,1}s_1, f_{2,0}s_2)s_1 = (f_{0,2}s_2s_0, f_{1,1}s_2s_1, f_{2,0}s_1^2)$

$= (s_1^v f_{0,1}s_0, s_0^v f_{1,0}s_0, s_1^h f_{1,0}s_1)$, so,

$\sigma_0 = (s_0p_1, s_0p_1, s_0p_2)$ and $\sigma_1 = (s_1p_1, s_0p_2, s_0p_2)$.

Then, in general, $\phi_n = (f_{0,n}s_0, f_{1,n-1}s_1, \cdots, f_{r,n-r}s_r, \cdots, f_{n,0}s_n)$, $\sigma_i = (s_i^v p_1, s_{i-1}^v p_2, \cdots, s_0^v p_{i+1}, s_i^h p_{i+1}, \cdots, s_i^h p_{n+1})$, for $0 < i < n$ $\delta_i = (d_i^v p_1, d_{i-1}^v p_2, \cdots, d_i^v p_i, d_i^h p_{i+2}, \cdots, d_i^h p_{n+1})$, $\delta_0 = (d_0^h p_2, d_0^h p_3 \cdots, d_0^h p_{n+1})$ and $\delta_n = (d_n^v p_1, d_{n-1}^v p_2, \cdots, d_1^v p_n)$. This follows by induction!

If $f_{*,*}:Y_{*,*} \longrightarrow Z_{*,*}$, then $(\nabla f)_n:(\nabla Y)_n \longrightarrow (\nabla Z)_n$ is defined by the restriction to the equaliser of the two maps above of the map $(f_{0,n}, f_{1,n-1}, \cdots, f_{n-1,1}, f_{n,0})$, as might reasonably be expected! This map has domain the corresponding equaliser in (∇Z), because the face and degeneracy maps of a (bi)simplicial set commute with (bi)simplicial maps.

We should also check the simplicial identities. For $1 < i+1 < j < n$,

$$(d_i^v p_1, d_{i-1}^v p_2, \cdot\cdot, d_1^v p_i, d_i^h p_{i+2}, \cdot\cdot, d_i^h p_n)(d_j^v p_1, d_{j-1}^v p_2, \cdot\cdot, d_1^v p_j, d_i^h p_{j+2}, \cdot\cdot, d_j^h p_{n+1})$$
$$= (d_i^v d_j^v p_1, \cdots, d_1^v d_{j-i+1}^v p_i, d_i^h d_{j-i-1}^v p_{i+2}, \cdots, d_i^h d_1^v p_j, d_i^h d_j^h p_{j+2}, \cdots, d_i^h d_j^h p_{n+1})$$
$$= (d_{j-1}^v d_i^v p_1, \cdot\cdot, d_{j-i}^v d_1^v p_i, d_{j-i-1}^v d_i^h p_{i+2}, \cdot\cdot, d_1^v d_i^h p_j, d_{j-1}^h d_i^h p_{j+2}, \cdot\cdot, d_{j-1}^h d_i^h p_{n+1})$$
$$= (d_{j-1}^v p_1, \cdots, d_1^v p_{j-1}, d_{j-1}^h p_{j+1}, \cdots, d_{j-1}^h p_n)(d_i^v p_1, \cdots, d_1^v p_i, d_i^h p_{i+2}, \cdots, d_i^h p_{n+1})$$

and so $\delta_i \delta_j = \delta_{j-1} \delta_i$.

In the case $1 < i+1 = j < n$, $\delta_i \delta_{i+1} = \delta_i \delta_i$, because

$$(d_i^v p_1, \cdots, d_1^v p_i, d_i^h p_{i+2}, \cdots, d_i^h p_n)(d_{i+1}^v p_1, \cdots, d_1^v p_{i+1}, d_{i+1}^h p_{i+3}, \cdots, d_{i+1}^h p_{n+1})$$
$$= (d_i^v d_{i+1}^v p_1, \cdots, d_1^v d_2^v p_i, d_i^h d_{i+1}^h p_{i+3}, \cdots, d_i^h d_{i+1}^h p_{n+1})$$
$$= (d_i^v d_i^v p_1, \cdots, d_1^v d_1^v p_i, d_i^h d_i^h p_{i+3}, \cdots, d_i^h d_i^h p_{n+1})$$
$$= (d_i^v p_1, \cdots, d_1^v p_i, d_i^h p_{i+2}, \cdots, d_i^h p_n)(d_i^v p_1, \cdots, d_1^v p_i, d_i^h p_{i+2}, \cdots, d_i^h p_{n+1})$$

Further, $\delta_0 \delta_i = (d_0^h d_{i-1}^v p_2, \cdots, d_0^h d_1^v p_i, d_0^h d_i^h p_{i+2}, \cdots, d_0^h d_i^h p_{n+1})$

$$= (d_{i-1}^v d_0^h p_2, \cdots, d_1^v d_0^h p_i, d_{i-1}^h d_0^h p_{i+2}, \cdots, d_{i-1}^h d_0^h p_{n+1}) = \delta_{i-1}\delta_0,$$

and $\delta_i \delta_n = (d_i^v d_n^v p_1, d_{i-1}^v d_{n-1}^v p_2, \cdots, d_1^v d_{n-i+1}^v p_i, d_i^h d_{n-i-1}^v p_{i+2}, \cdots, d_i^h d_1^v p_n)$

$$= (d_{n-1}^v d_i^v p_1, \cdots, d_{n-i}^v d_1^v p_i, d_{n-i-1}^v d_i^h p_{i+2}, \cdots, d_1^v d_i^h p_n) = \delta_{n-1}\delta_i$$

Since $(s_i^v p_1, \cdot\cdot, s_0^v p_{i+1}, s_i^h p_{i+1}, \cdot\cdot, s_i^h p_{n+1})(s_j^v p_1, \cdot\cdot, s_0^v p_{j+1}, s_j^h p_{j+1}, \cdot\cdot, s_j^h p_n)$

$$= (s_i^v s_j^v p_1, \cdots, s_0^v s_{j-i}^v p_{i+1}, s_i^h s_{j-i}^v p_{i+1}, \cdots, s_i^h s_j^v p_{j+1}, s_i^h s_j^h p_{j+1}, \cdots, s_i^h s_j^h p_n)$$
$$= (s_{j+1}^v s_i^v p_1, \cdot, s_{j-i+1}^v s_0^v p_{i+1}, s_{j-i+1}^h s_i^v p_{i+1}, \cdot, s_{j+1}^h s_i^v p_{j+1}, s_{j+1}^h s_i^h p_{j+1}, \cdot, s_{j+1}^h s_i^h p_n)$$
$$= (s_{j+1}^v p_1, \cdots, s_0^v p_{j+2}, s_{j+1}^h p_{j+2}, \cdots, s_{j+1}^h p_{n+1})(s_i^v p_1, \cdots, s_0^v p_{i+1}, s_i^h p_{i+1}, \cdots, s_i^h p_n)$$

then $\sigma_i\sigma_j = \sigma_{j+1}\sigma_i$.

Further, if $0 \le i \le j$, then $\delta_i\sigma_j = \sigma_{j-1}\delta_i$ because
$(d_i^v s_j^v p_1, \cdots, d_i^v s_{j-i+1}^v p_i, d_i^h s_{j-i-1}^v p_{i+2}, \cdots, d_i^h s_0^v p_{j+1}, d_i^h s_j^h p_{j+1}, \cdots, d_i^h s_j^h p_n)$
$= (s_{j-1}^v d_i^v p_1, \cdots, s_{j-i}^v d_i^v p_i, s_{j-i-1}^v d_i^h p_{i+2}, \cdots, s_0^v d_i^h p_{j+1}, s_{j-1}^h d_i^h p_{j+1}, \cdots, s_{j-1}^h d_i^h p_n)$.

Also, for $j+1 \le i \le n$, we have $\delta_i\sigma_j = \sigma_j\delta_{i-1}$, because
$(d_i^v s_j^v p_1, \cdots, d_{i-j}^v s_0^v p_{j+1}, d_{i-j-1}^v s_j^h p_{j+1}, \cdots, d_1^v s_j^h p_{i-1}, d_i^h s_j^h p_{i+1}, \cdots, d_i^h s_j^h p_n)$
$= (s_j^v d_{i-1}^v p_1, \cdots, s_0^v d_{i-j-1}^v p_{j+1}, s_j^h d_{i-j-1}^v p_{j+1}, \cdots, s_j^h d_1^v p_{i-1}, s_j^h d_{i-1}^h p_{i+1}, \cdots, s_j^h d_{i-1}^h p_n)$

Lastly, $(d_i^v p_1, \cdots, d_i^v p_i, d_i^h p_{i+2}, \cdots, d_i^h p_{n+1})(s_i^v p_1, \cdots, s_0^v p_{i+1}, s_i^h p_{i+1}, \cdots, s_i^h p_{n+1})$
$= (d_i^v s_i^v p_1, \cdots, d_i^v s_i^v p_i, d_i^h s_i^h p_{i+1}, \cdots, d_i^h s_i^h p_n) = id = \delta_i\sigma_i$. There are more cases to check, but they are fairly simple to write down from those above.

So, the simplicial identities hold, and thus we have a well defined functor $\nabla: \underline{\mathcal{B}i\mathcal{S}\mathcal{S}} \longrightarrow \underline{\mathcal{S}\mathcal{S}}$.

Now we have $\nabla: \underline{\mathcal{B}i\mathcal{S}\mathcal{S}} \longrightarrow \underline{\mathcal{S}\mathcal{S}}$, we need to construct the adjunction, $P^* \dashv \nabla$. We have already seen, that given $f \in \underline{\mathcal{B}i\mathcal{S}\mathcal{S}}(P^*X, Y)$ we can construct a morphism $\phi: X \longrightarrow \nabla Y$, where $\phi_n = (f_{0,n}s_0, \cdots, f_{n,0}s_n)$.

For $0 \le r \le n$:-
$f_{r,n-r} = f_{r,n-r}d_{r+1}s_{r+1} = d_{r+1}^h f_{r+1,n-r}s_{r+1} = d_{r+1}^h p_{r+2}\phi_{n+1}$, and also $f_{r,n-r} = f_{r,n-r}d_r s_r = d_0^v f_{r,n+1-r}s_r = d_0^v p_{r+1}\phi_{n+1}$. Now, if we start with a morphism $\phi: X \longrightarrow \nabla Y$, we can construct a morphism $f_{*,*}$ by letting $f_{n,m}: (P^*X)_{n,m} \longrightarrow y_{n,m}$ be $d_0^v p_{n+1}\phi_{n+m+1} = d_{n+1}^h p_{n+2}\phi_{n+m+1}$, (as ∇Y is the coequaliser of $(d_0^v p_1, d_0^v p_2, \cdots, d_0^v p_n)$ and $(d_1^h p_2, d_2^h p_3, \cdots, d_n^h p_{n+1})$).

By considering the identity map $1: P^*X \longrightarrow P^*X$, we see the unit of the adjunction, $\eta_X: X \longrightarrow \nabla P^*X$, is $(s_0, s_1, \cdots, s_n): X_n \longrightarrow \prod_{i=1}^{n+1} X_{n+1}$, for each n. Considering $1: \nabla Y \longrightarrow \nabla Y$, we find the $(n,m)^{th}$ component of the counit of the adjunction, $(\varepsilon_Y)_{n,m}: (P^*\nabla Y)_{n,m} \longrightarrow Y_{n,m}$, is $d_0^v p_{n+1} = d_{n+1}^h p_{n+2}$.

4. The *Ner*, $\tilde{F}$ adjunction

To save possible confusion, we state here that $Y_{m,n}$ is the term of a bisimplicial set in the m^{th} column and n^{th} row.

Now, consider $Ner: \mathcal{SGpds} \to \mathcal{BiSS}$ - this takes a simplicial groupoid, X_*, to the bisimplicial set with the n^{th} row the nerve of X_n. That is it takes X_n to the simplicial set of composible strings - i.e. $Ner(X_n)_m = \{(x_1, x_2, \cdots x_m): dom x_i = cod x_{i-1}, \quad x_i \in X_n\}$, and $Ner(X_n)_0 = obX_n$.

Then the horizontal face maps are:-

for $m = 1$, $d_1^h(x) = dom x$ and $d_0^h(x) = cod x$,

for $m \geq 2$, $d_0^h(x_1, x_2, \cdots x_m) = (x_2, x_3, \cdots x_m)$,

$d_m^h(x_1, x_2, .. x_m) = (x_1, x_2, \cdots x_{m-1})$, and

$d_i^h(x_1, x_2, \cdots x_m) = (x_1, x_2, \cdots, x_i . x_{i+1}, \cdots, x_m)$ for $0 < i < m$.

The horizontal degeneracy maps are:-

for $m = 0$, $s_0^h t = 1_t$ (for $t = obX_n$), and if $m \geq 1$,

$s_i^h(x_1, x_2, \cdots, x_m) = (x_1, \cdots, x_i, 1_t, x_{i+1}, \cdots, x_m)$ where $t = cod x_i$.

The vertical face and degeneracy maps for the m^{th} column are the m-fold products Πd_i and Πs_i.

The simplicial identities trivially follow from this.

If *Ner* is to have a left adjoint, $\tilde{F}$, then for $G \in \mathcal{SGpds}$ and $Y \in \mathcal{BiSS}$, $g_{m,n}: Y_{m,n} \longrightarrow (NerG)_{m,n} \subseteq G_n^m$, we must be able to construct a unique $\phi(g)_n: (\tilde{F}Y)_n \to G_n$.

Let G be a groupoid. Consider $(x_1, x_2, \cdots, x_n) \in (NerG)_n$. This element can be written as an element of G^n as follows:-

$$(x_1, x_2, \cdots x_n) = (x_1, 1, \cdots, 1)(1, x_2, 1, \cdots, 1) \cdots (1, \cdots, 1, x_n)$$

This is then a word in degenerate elements of $(NerG)_n$, as $(1, \cdots, 1, x_i, 1, \cdots, 1) = (s_n \cdots s_{i+1})(s_0)^{i-1} x_i$.

It is then reasonable to consider that a left adjoint to nerve would be based on just the 1-skeleton of a simplicial set and thus the analogous result for bisimplicial sets would be based on the horizontal 1-skeletons of the bisimplicial set.

With this in mind, the most natural construction for $\tilde{F}$ would be a free construction on the graph $Y_{1,n} \to Y_{0,n}$, where the identity is s_0^h, and the source and target maps are d_1^h and d_0^h respectively.

The face and degeneracy maps would then be the extensions of the vertical face and degeneracy maps.

In fact, this is insufficient, as the composition in G has been neglected, and when considering morphisms $f:\tilde{F}Y \longrightarrow G$ (whatever $\tilde{F}Y$ turns out to be) we need the relations $f(xy) = f(x)f(y)$. This turns out to be all the relations we need, as we now show.

Define $(\tilde{F}Y)_n$ as the groupoid generated by $\left[\begin{matrix} & \xrightarrow{\ d_1\ } & \\ Y_{1,n} & \xleftarrow{\ s_0\ } & Y_{0,n} \\ & \xrightarrow{\ d_0\ } & \end{matrix}\right]$

with relations generated by $\{(d_1^h z) = (d_2^h z)(d_0^h z):\ z \in Y_{2,n}\}$.

Note that for $y_1, y_2 \in Y_{1,n}$, that if y_1 is homotopic to y_2, (that is $d_i^h y_1 = d_i^h y_2$ $(i = 0,1)$ and $\exists\ z \in Y_{2,n}$ with $d_0 z = s_0 d_0 y_1$, $d_1 z = y_1$ and $d_2 z = y_2$), then $y_1 = y_2$ in the specified quotient groupoid (since s_0 is the identity operator in the groupoid).

The face and degeneracy maps are the natural extensions of the vertical face and degeneracy maps. Then, if $w = \prod_{i=1}^{m} x_i^{\pm 1} \in (\tilde{F}Y)_n$, $d_j w = \prod_i (d_j^v x_i)^{\pm 1}$ which has source $d_1^h d_j^v x_1 = d_j^h d_1^v x_1$ and target $d_0^h d_j^v x_m$. Similarly $s_j w = {}_i\prod s_j \overset{\pm 1}{x_i}$.

This construction will extend to a functor $\tilde{F}: \underline{\mathcal{B}i\mathcal{S}\mathcal{S}} \longrightarrow \underline{\mathcal{S}\mathcal{G}pd\alpha}$ where $\tilde{F}(f)_n$ is the extension of $f_{1,n}$ on arrows, is $f_{0,n}$ on objects and the extension of $f_{2,n}$ on relations.

Let $g_{*,*}:Y \longrightarrow NerG$. We define $\phi: \underline{\mathcal{B}i\mathcal{S}\mathcal{S}}(Y, NerG) \longrightarrow \underline{\mathcal{S}\mathcal{G}pds}(\tilde{F}Y, G)$ by $\phi(g)_n = \bar{g}_{1,n}: (\tilde{F}Y)_n \longrightarrow arrG_n$, where $\bar{g}_{1,n}$ is the extension of $g_{1,n}$ to words in $Y_{1,n}$. This restricts on object sets to $g_{0,n}$. So, we must consider how $g_{m,n}$ relates to $\phi(g)_n$.

Since morphisms of bisimplicial spaces commute with the bisimplicial maps, we have that:-

$$g_{1,n}(d_0^h)^{j-1}d_{j+1}^h \cdots d_m^h = (d_0^h)^{j-1}d_{j+1}^h \cdots d_m^h g_{m,n}: Y_{m,n} \longrightarrow (NerG_n)_1 = G_n$$

but $(d_0^h)^{j-1}d_{j+1}^h \cdots d_m^h: (NerG_n)_m \longrightarrow (NerG_n)_1$ picks out the j^{th} term of $(g_1, g_2, \cdots, g_m)$.

Thus, $g_{m,n} = (g_{1,n}d_2^h \cdots d_m^h, g_{1,n}d_0^h d_3^h \cdots d_m^h, \cdots, g_{1,n}(d_0^h)^{m-1})$.

Consider $m = 2$. Let $g_{2,n}y = (x_1, x_2)$, then $g_{1,n}d_2^h y = d_2^h g_{2,n}y = x_1$, $g_{1,n}d_0^h y = d_0^h g_{2,n}y = x_2$, and $g_{1,n}d_1^h y = d_1^h g_{2,n}y = x_1x_2$, that is $g_{1,n}d_1^h y = (g_{1,n}d_2^h y)(g_{1,n}d_0^h y) = \bar{g}_{1,n}((d_2^h y)(d_0^h y))$, because of the relations we imposed on $\tilde{F}(Y)_n$. Now for $m > 2$, let $y \in Y_{m,n}$ and consider:-

$$g_{m,n}y = (g_{1,n}d_2^h \cdot\cdot d_m^h y, g_{1,n}d_0^h d_3^h \cdot\cdot d_m^h y, \cdots, g_{1,n}(d_0^h)^{m-1}y).$$

Now, $d_i^h g_{m,n}y = g_{m-1,n}d_i^h$. This will follow naturally from simplicial identities except for the i^{th} component where (using the identity above) we get

$$g_{1,n}(d_0^h)^{i-1}d_{i+1}^h \cdots d_{m-1}^h d_i y = (g_{1,n}(d_0^h)^{i-1}d_{i+1}^h \cdot\cdot d_m^h y)(g_{1,n}(d_0^h)^i d_{i+2}^h \cdot\cdot d_m^h y)$$

Using simplicial identities, we obtain

$$(d_0^h)^{i-1}d_{i+1}^h \cdot\cdot d_{m-1}^h d_i y = d_1(d_0^h)^{i-1}d_{i+2}^h \cdot\cdot d_m^h y$$

and $(d_0^h)^{i-1}d_{i+1}^h \cdots d_m^h y = d_2^h(d_0^h)^{i-1}d_{i+2}^h \cdots d_m^h y$, so if we write $(d_0^h)^{i-1}d_{i+2}^h \cdots d_m^h y = z$, the equation is $g_{1,n}d_1^h z = (g_{1,n}d_2 z)(g_{1,n}d_0^h z)$ which is already satisfied because of the relations from $Y_{2,n}$. Thus $\phi: \underline{\mathcal{B}i\mathcal{S}\mathcal{S}}(Y, NerG) \longrightarrow \underline{\mathcal{S}\mathcal{G}pds}(\tilde{F}Y, G)$ is a well defined function.

Further, if we have $f_n: (\tilde{F}Y)_n \longrightarrow G_n$, we can construct $\theta(f): Y \longrightarrow NerG$ by $\theta(f)_{n,m} = (f_n d_2^h \cdots d_m^h, f_n d_0^h d_3^h \cdots d_m^h, \cdots, f_n (d_0^h)^{m-1})$. Which is clearly well defined. That θ and ϕ are mutually inverse bijections follow trivially from our earlier analysis of $g_{*,*}$, and thus we have an adjunction $\tilde{F} \dashv Ner$.

5. $\overline{W}$ as ∇Ner

Let G be a simplicial groupoid, and consider $NerG$. Then, we have a two dimensional array, which has at the $(p,q)^{th}$ lattice point, the set G^p_q - that is the set of chains of composible arrows in G_q of length p. Now, we consider a square in the lattice:-

$$\begin{array}{ccc} G^{p+1}_{q+1} & \xrightarrow{d^h_i} & G^p_{q+1} \\ \downarrow d^v_i & & \downarrow d^v_i \\ G^{p+1}_q & \xrightarrow{d^h_i} & G^p_q \end{array}$$

this diagram commutes because the d^v_i commute with the $d^h_i : G^{p+1}_* \longrightarrow G^p_*$ for all i, as the d^h_i form a simplicial set map for all i and as the d^v_i are just the face maps of G_* and as such are groupoid morphisms - so $(d_i g_j)(d_i g_{j+1}) = d_i(g_j g_{j+1})$.

We now restrict our attention to $\underline{\mathcal{SG}pds_*}$. Certainly Ner is still well defined on this subcategory, but it is significant that we are in the subcategory and so we will use Ner_* to denote the restriction of Ner to the subcategory. So, let $G \in \underline{\mathcal{SG}pds_*}$ and consider $\nabla(Ner_*G)$. The product $Y_{0,n} \times \cdots \times Y_{n,0}$ becomes the product $O \times G^1_{n-1} \times \cdots \times G^{n-1}_1 \times G^n_0$, where O is the object set (of all the groupoids).

Because we must consider the equaliser of the maps

$$(d^v_0 p_1, d^v_0 p_2, \cdots, d^v_0 p_n) : Y_{0,n} \times \cdots \times Y_{n,0} \longrightarrow Y_{0,n-1} \times \cdots \times Y_{n-1,0}$$

and $(d^h_1 p_2, d^h_2 p_3, \cdots, d^h_n p_{n+1}) : Y_{0,n} \times \cdots \times Y_{n,0} \longrightarrow Y_{0,n-1} \times \cdots \times Y_{n-1,0}$

we find that if $(g_1, g_2, \cdots, g_{p-1}) \in G^{p-1}_{n-p+1}$ and $(h_1, h_2, \cdots, h_p) \in G^p_{n-p}$ are two consecutive components of an element in $O \times G^1_{n-1} \times \cdots \times G^{n-1}_1 \times G^n_0$, then as p_p picks out $(g_1, g_2, \cdots, g_{p-1})$ and p_{p+1} picks out $(h_1, h_2, \cdots, h_p)$ we have $d^v_0(g_1, g_2, \cdots, g_{p-1}) = d^h_p(h_1, h_2, \cdots, h_p)$.

Using the definitions of d^v and d^h this yields $(d_0g_1, d_0g_2, \cdots, d_0g_{p-1}) = (h_1, h_2, \cdots, h_{p-1})$.

What this tells us is that much of the information in $O \times G^1_{n-1} \times \cdots \times G^{n-1}_1 \times G^n_0$ becomes redundant when we pass to the equaliser we want, and that the only component of $(h_1, h_2, \cdots, h_p) \in G^p_{n-p}$ that is <u>not</u> determined by $(g_1, g_2, \cdots, g_{p-1}) \in G^{p-1}_{n-p+1}$ is the last, namely h_p. When $p = 1$, this translates that the O-component in the element is the domain of the G_{n-1}-component of the element - and as such, the O-component is also redundant.

Discarding redundant information, and noting that $dom(h_p) = cod(h_{p-1}) = cod(d_0g_{p-1}) = cod(g_{p-1})$, we find that we have, in effect, an element $(g_{n-1}, \cdots, g_1, g_0)$ of the set $G_{n-1} \times \cdots \times G_1 \times G_0$, where the source of g_i is the codomain of g_{i+1}. This is precisely the definition of $\overline{W}(G)_n$.

Let us consider the simplicial maps of ∇Ner_*G. Consider an element of $(\nabla Ner_*G)_n$, that is $(O, g_{n-1}, (g_{n-2_1}, g_{n-2_2}), \cdots, (g_{0_1}, g_{0_2}, \cdots g_{0_n}))$ which we have shown can be reduced to $(g_{n-1}, g_{n-2_2}, \cdots, g_{0_n})$.

First, we deal with the case n. For ∇, $\delta_n = (d^v_n p_1, d^v_{n-1} p_2, \cdots, d^v_1 p_n)$. Now, d^v_i sends each term of each component to its i^{th} face, and so combining this with the projection, and throwing away redundant information (as we did above) we find that

$$\delta_n(g_{n-1}, g_{n-2_2}, \cdots, g_{0_n}) \longmapsto (d_{n-1}g_{n-1}, d_{n-2}g_{n-2_2}, \cdots, d_1g_{1_n}).$$

Next, we consider $\delta_0 = (d_0^h p_2, d_0^h p_3, \cdots, d_0^h p_{n+1})$. Now, d_0^h deletes the first term of each component, excepting the G_{n-1}-component, which it takes to its codomain. Thus,

$\delta_0(O, g_{n-1}, (g_{n-2_1}, g_{n-2_2}), \cdots, (g_{0_1}, g_{0_2}, \cdots g_{0_n}))$

$= (cod g_{n-1}, g_{n-2_2}, \cdot\cdot, (g_{0_2}, \cdot\cdot, g_{0_n})) = (dom g_{n-2_2}, g_{n-2_2}, \cdot\cdot, (g_{0_2}, \cdot\cdot, g_{0_n}))$

(as $dom g_{n-2_2} = cod g_{n-2_1} = cod(d_0 g_{n-1}) = cod(d_0 g_{n-1})$), which reduces to $(g_{n-2_2}, g_{n-3_3}, \cdots, g_{0_n})$ as we would expect.

Lastly, for the general case, $\delta_i : (\nabla Ner_* G)_n \longrightarrow (\nabla Ner_* G)_{n-1}$ is $\delta_i = (d_i^v p_1, d_{i-1}^v p_2, \cdots, d_1^v p_i, d_i^h p_{i+2}, \cdots, d_i^h p_{n+1})$ where d_i^h is "compose the i^{th} and $(i+1)^{th}$ components" and d_i^v takes each component to its i^{th} face.

In the first i components, we see that, after reduction, we will get $(d_{i-1} g_{n-1}, d_{i-2} g_{n-2_2}, \cdots, d_1 g_{n-i+1_{i-1}}, \cdots\cdots)$. In the $(i+2)^{th}$ component of $(\nabla Ner_* G)_n$ we have a string of $i+1$ elements. Now, d_i^h composes the last two elements of this, so in the $(i+1)^{th}$ component, we will reduce to $g_{n-i-1_i} \cdot g_{n-i-1_{i+1}}$. However, $g_{n-i-1_i} = d_0 g_{n-i_i}$ and so the $(i+1)^{th}$ component will become $d_0 g_{n-i_i} \cdot g_{n-i-1_{i+1}}$. Then, the d_i^h for the subsequent components will combine the i^{th} and $(i+1)^{th}$ terms, which will not effect the last term. Thus, when we reduce, we will simply have g_{n-j_j} and gathering all this information together, we have $\delta_i(g_{n-1}, g_{n-2_2}, \cdots, g_{0_n})$

$= (d_{i-1} g_{n-1}, d_{i-2} g_{n-2_2}, \cdots, d_1 g_{n-i+1_{i-1}}, d_0 g_{n-i_i} \cdot g_{n-i-1}, g_{n-i-2}, \cdots, g_{0_n})$.

This is precisely what we would get from $\overline{W}$. It is also transparent that $\nabla Ner_* = \overline{W}$ on morphisms, and thus $\nabla(Ner_* G)_n \cong \overline{W}(G)_n$.

It is important to note that $\overline{W}$ is defined only on $\underline{\mathcal{SG}pd_{\mathcal{O}}}_*$ and so while ∇Ner exists, we have only that $\nabla Ner_* = \overline{W}$.

6. $G \neq \tilde{F}P^*$

Now, we look to P^* and compose with $\tilde{F}$ and see if we get the Joyal-Tierney loop-groupoid functor G. The functor P^* gives us an array

$$\begin{tikzcd}
X_1 & X_2 \arrow[l, shift right, "d_0"'] \arrow[l, shift left, "d_1"] & \cdots \arrow[l, dotted] \\
X_2 \arrow[u, shift right, "d_1"] \arrow[u, shift left, "d_2"'] & X_3 \arrow[l, shift right, "d_0"'] \arrow[l, shift left, "d_1"] \arrow[u, shift right, "d_2"] \arrow[u, shift left, "d_3"'] & \cdots \arrow[l, dotted] \\
X_3 \arrow[u, shift right=2, "d_1"] \arrow[u, "d_2"] \arrow[u, shift left=2, "d_3"'] & X_4 \arrow[l, shift right, "d_0"'] \arrow[l, shift left, "d_1"] \arrow[u, shift right=2, "d_2"] \arrow[u, "d_3"] \arrow[u, shift left=2, "d_4"'] & \cdots \arrow[l, dotted] \\
\vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

What we want for $(GX)_{i-1}$ is the free groupoid generated on the set of arrows X_i, with objects X_0, with $s_0^i X_0$ as the identities, and $d_1 d_2 \cdots d_i$ and $d_0 d_2 \cdots d_i$ as the source and target maps.

Clearly P^* will not yield this, because we have no X_0.

Consider the following diagram:-

$$\begin{tikzcd}
\underline{\mathcal{SG}pds}_* \arrow[r, "i"] \arrow[d, shift right, "W"'] & \underline{\mathcal{SG}pds} \arrow[d, shift right, "Ner"'] & \underline{\mathcal{SG}pds}_* \arrow[l, "i"'] \arrow[d, shift right, "Ner_*"'] \\
\underline{\mathcal{SS}} \arrow[u, shift right, "G"'] \arrow[r, shift left, "P^*"] & \underline{\mathcal{B}i\mathcal{SS}} \arrow[u, shift right, "\tilde{F}"'] \arrow[l, shift left, "\nabla"] & \underline{\mathcal{B}i\mathcal{SS}}_* \arrow[l, "i"'] \arrow[u, shift right, "\tilde{F}_*"']
\end{tikzcd}$$

We use $(-)_*$ to denote the restriction of the functor $(-)$ to the relevant subcategory. The failure of $\tilde{F}P^*$ to be G is precisely here. We require a left adjoint to the inclusion $\underline{\mathcal{SG}pds}_* \xrightarrow{i} \underline{\mathcal{SG}pds}$.

Let $X \in \underline{\mathcal{SS}}$ then:-

$$\underline{\mathcal{SG}pds}_*(GX, GX) \cong \underline{\mathcal{SS}}(X, \overline{W}GX) \cong \underline{\mathcal{SS}}(X, \nabla Ner\, iGX)$$

$$\cong \underline{\mathcal{B}i\mathcal{SS}}(P^*X, Ner\, iGX) \cong \underline{\mathcal{SG}pds}(\tilde{F}P^*X, iGX).$$

Thus, there is a canonical morphism in $\underline{\mathscr{S}\mathcal{G}pd\sigma}$ between FP^*X and iGX which is the result of mapping 1_{GX} by the composite of the natural bijections of the adjunctions in the sequence above. This is as good as we can get.

7. The Moore Complex of a simplicial groupoid

The following definition is due to Brown and Higgins, and generalises a similar definition due to Whitehead. We place it here, as is needed in this section, but properly it comes in section 9.

A crossed complex over a groupoid, C, is a sequence

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_n} C_n \longrightarrow \cdots\cdots \longrightarrow C_2 \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0$$

where (i) C_0 is a groupoid, (we write $C_0(a)$ for $C_0(a,a)$, $a \in obC_0$)

(ii) C_n is a family of groups $\{C_n(a)\}_{a \in obC_0}$ for $n \geq 1$,

(iii) $C_n(a)$ is abelian for $n \geq 2$, $a \in obC_0$

(iv) C_0 acts on C_n on the left for all $n \geq 1$, by $(h,g) \mapsto {}^h g$

where $g \in C_n(a)$, $h \in C_0(b,a)$, ${}^h g \in C_n(b)$ and the usual action laws hold

(v) the ∂_n are all groupoid morphisms which preserve the action

(vi) if $x,y \in C_1(a)$, then ${}^{\partial_0 x} y = xyx^{-1}$, and $\partial_0 C_1(a)$ acts trivially on $C_n(b)$ for $n \geq 2$ for all $a,b \in obC_0$

(vii) $\partial_n \partial_{n-1}$ is trivial for $n \geq 1$

Now let G be a simplicial groupoid. Define $(NG)_n := \bigcap_{i=1}^{n} \mathrm{Ker} d_i^n$ and $\partial_n : (NG)_n \longrightarrow (NG)_{n-1}$ by $\partial_n := d_0^n|_{(NG)_n}$. This is of course the standard definition of the Moore Complex of a simplicial group applied to a simplicial groupoid.

Since we are working with simplicial groupoids over a fixed object set, we have that the face and degeneracy maps are the identity on objects. So, for $n \geq 1$, each $\mathrm{Ker} d_i^n$ (and hence $(NG)_n$) will be a totally disconnected, normal, wide subgroupoid of G_n.

Further $\partial_{n-1}\partial_n = d_0^{n-1}d_0^n = d_0^{n-1}d_1^n$. Thus $\partial_{n-1}\partial_n$ maps $(NG)_n$ to the discrete groupoid on obG_n. This gives us a chain complex of groupoids, over a fixed object set. Since each of the groupoids $(NG)_n$ is totally disconnected and the morphisms ∂ are the identity on objects, what we have, to be more precise, is a family of chain complexes of groups, together with a groupoid at the base.

We check that $\partial_n(NG)_n$ is normal in $(NG)_{n-1}$. Let $g \in (NG)_n$ for $n \geq 1$ with $domg = codg = x$, and $h \in G_{n-1}(x,y)$. Then, $d_0^n s_0^{n-1}h = h$. So $h(\partial_n g)h^{-1} = (d_0^n s_0^{n-1}h)(d_0^n g)(d_0^n s_0^{n-1}h)^{-1} = d_0^n((s_0^{n-1}h)g(s_0^{n-1}h)^{-1})$. So $\partial_n(NG)_n$ is normal in G_{n-1}, and hence in $(NG)_{n-1}$ and $\bigcap_{i=0}^{n} \mathrm{Ker} d_i^n$.

Now, we want to investigate the properties of the chain:-

$\cdots \longrightarrow NG_n \xrightarrow{\partial_n} NG_{n-1} \longrightarrow \cdots\cdots \longrightarrow NG_1 \xrightarrow{\partial_1} G_0$. We define an action of G_0 on the chain by ${}^h g = ((s_0)^n h)g((s_0)^n h)^{-1}$ (for $h \in NG_0$, $g \in NG_n$). This action is a well defined action, which the ∂ respect (i.e $\partial({}^h g) = {}^h(\partial g)$).

The chain with this action is not a crossed complex (after Brown-Higgins), however, it does have some of the properties. By definition, G_0 is a groupoid, and all the NG_i are families of groups, indexed by the objects of G_0. The ∂ are all groupoid morphisms (and the identity on objects), and the composite $\partial_n\partial_{n+1}$ is trivial for all n.
However, for $g,h \in NG_1(x)$, ${}^{\partial_1 g}h \neq ghg^{-1}$ in general; the NG_i are not families of abelian groups for $i \geq 2$, and the action of $\partial_1(NG)_1$ is not trivial on $(NG)_i$ for $i \geq 2$. We will discuss this further later, but for now we move on to consider other properties of this chain complex.

Consider the homology of the Moore complex of the groupoid, that is:-

$$H_n(NG) = \frac{\mathrm{Ker}\partial_n}{\partial_{n+1}(NG)_{n+1}} = \bigcap_{i=0}^{n} \mathrm{Ker}d_i^n \Big/ \partial_n\left[\bigcap_{i=1}^{n+1} \mathrm{Ker}d_i^n\right]$$

Let us consider what this represents. The groupoid used to quotient $\mathrm{Ker}\partial_n$ is in fact the subgroupoid of G_n consisting of those elements which are homotopic to an identity. $\mathrm{Ker}\partial_n$ itself is the subgroupoid of G_n whose elements have trivial boundary. This, means that the homology of the Moore complex of a simplicial groupoid describes the relative homotopy groupoids of the simplicial groupoid:-

$$\pi_n(G, obG_0) \cong \frac{\mathrm{Ker}\partial_n}{\partial_{n+1}(NG)_{n+1}}.$$

This is, of course, analogous to the homotopy groups of a simplicial group, which are also described by the homology of the Moore complex.

The case $n = 0$ is worth special mention. For a simplicial object X_*, $\pi_0 X_*$ is the coequaliser of the pair $d_1, d_0 : X_1 \longrightarrow X_0$, and so $\pi_0(G, obG)$ will be the coequaliser of the pair $d_1, d_0 : G_1 \longrightarrow G_0$.

Define $q : G_0 \longrightarrow \pi_0(G_0, obG)$ to be the canonical quotient map, and so $qd_1x = qd_0x \quad \forall\ x \in G_0$. Thus, $q((d_1x)(d_0x)^{-1}) = 1$. We would therefore want that $d_0(\mathrm{Ker}d_1) = \{(d_0x)(d_1x)^{-1} \ :\ x \in G_0\}$. For convenience, we call this latter set Q.

Certainly, $(d_1x)(d_0x)^{-1} = d_0((s_0d_1x)x^{-1})$ and $(s_0d_1x)x^{-1} \in \mathrm{Ker}d_1$, so that $Q \subseteq d_0\mathrm{Ker}d_1$. Further, if $x \in \mathrm{Ker}d_1$, then setting $y = x^{-1}(s_0d_0x)$, we obtain $d_0y = 1$, $d_1y = d_0x$, and so $(d_1y)(d_0y)^{-1} = d_0x$ and $d_0\mathrm{Ker}d_1 \subseteq Q$. Therefore $Q = d_0\mathrm{Ker}d_1$, and so $H_0(NG) \cong \pi_0(G_0, obG)$.

8. The semidirect decomposition of a simplicial groupoid

The Dold-Kan theorem states that the Moore complex functor from simplicial abelian groups to chain complexes of abelian groups is an equivalence of categories. The quasi-inverse to the Moore functor involves (at the n^{th} level) the direct sum of terms of the chain complex up to the n^{th} level. We refer the reader to [P], for a description.

We now consider the decomposition of a simplicial groupoid in terms of the semidirect product, which is due (in the group case) to Conduché. He, among others (Ashley, Carrasco and Cegarra) have investigated non-abelian generalisations of the Dold-Kan Theorem. Ashley (see [As]) shows an equivalence between simplicial-T-complexes and crossed complexes, while Conduché (see [Co]) considers simplicial groups whose Moore complex is trivial in dimensions 2 and allies them with a construct he calls a "2-crossed module". Carrasco considers the multiplication in the simplicial group under this decomposition, and makes precise the structure needed to reconstruct the simplicial group (up to isomorphism) from the Moore complex.

We wish to show that $G_n \cong \mathrm{Ker}d_n^n \rtimes s_{n-1}^{n-1}G_{n-1}$, where the action of $s_{n-1}^{n-1}G_{n-1}$ on $\mathrm{Ker}d_n^n$ is ${}^{g}k = gkg^{-1}$ for $g \in s_{n-1}^{n-1}G_{n-1}(b,a)$ $k \in \mathrm{Ker}d_n^n(a)$ and trivial otherwise.

This result was proved for simplicial groups by Conduché, and the isomorphism in that case is:-

$\theta : g \longmapsto (gs_{n-1}^{n-1}d_n^n g^{-1}, s_{n-1}^{n-1}d_n^n g)$, with inverse $\phi : (k,g) \longmapsto kg$.

That θ and ϕ are inverse functions is clear, and both θ and ϕ are homomorphisms since:-

$$\begin{aligned}\theta(g)\theta(h) &= (gs_{n-1}d_ng^{-1}, s_{n-1}d_ng)(hs_{n-1}d_nh^{-1}, s_{n-1}d_nh)\\ &= (gs_{n-1}d_ng^{-1}(s_{n-1}d_ng)hs_{n-1}d_nh^{-1}(s_{n-1}d_ng)^{-1}, s_{n-1}d_ngs_{n-1}d_nh)\\ &= (ghs_{n-1}d_n(gh)^{-1}, s_{n-1}d_ngh) = \theta(gh)\end{aligned}$$

and:- $\phi((k,g)(l,h)) = \phi(kglg^{-1}, gh) = kglh = \phi(k,g)\phi(l,h)$

To generalise this to groupoids, we require some concept of a semidirect product for groupoids. In groups, we have that $G \rtimes H = \{(g,h): g \in G, h \in H\}$ and $(g_1,h_1)(g_2,h_2) = (g_1{}^{h_1}g_2, h_1h_2)$ where ${}^hg$ is the action of H on G (usually conjugation). Normally, the semidirect product will be between two subgroups of a group, where one of them (G here) is normal.

The following construction is essentially that of [Br.1] chapter 9: the differences between the definitions of action and semidirect product there and those here are merely matters of convention and convenience.

Suppose we have a morphism of groupoids (where obG is considered as a discrete groupoid) $\omega: \Gamma \longrightarrow obG$, then define an *action of* G *on* Γ *via* ω as follows. For each $g \in G(x,y)$, and $\gamma \in \omega^{-1}[y] \subset \Gamma$ there is an element $g\cdot\gamma \in \omega^{-1}[x]$; further, if $h \in G(z,x)$, then $h\cdot(g\cdot\gamma) = (hg\cdot\gamma)$, $1_y\gamma = \gamma$ and if $\gamma\delta$ is defined for some $\delta \in \omega^{-1}[y]$ then $(g\cdot\gamma)(g\cdot\delta)$ is defined in $\omega^{-1}[x]$, and $g\cdot(\gamma\delta) = (g\cdot\gamma)(g\cdot\delta)$. Clearly, the action takes identities to identities (or objects to objects).

Now, suppose we define $\Gamma \rtimes G$ as having object set $ob\Gamma$, and arrows $(\gamma,g):x \longrightarrow y$ where $g \in G(\omega x,\omega y)$ and $\gamma \in \Gamma(x,g\cdot y)$. Further, if we take $(\eta,h):y \longrightarrow z$, where $h \in G(\omega y,\omega z)$ and $\eta \in \Gamma(y,h\cdot z)$ then we have $(\gamma,g)(\eta,h) = (\gamma(g\cdot\eta),gh):x \longrightarrow z$.

Note that this requires that $\eta \in \omega^{-1}[\omega y]$, therefore that $g\cdot\eta \in \omega^{-1}[\omega x]$ and therefore that $\gamma \in \omega^{-1}[\omega x]$. Further, it requires that $dom(g\cdot\eta) = cod\gamma = g\cdot y$, that $dom(\gamma(g\cdot\eta)) = x$ and $cod(\gamma(g\cdot\eta)) = gh\cdot z$. Observation will show that these requirements are met. Similarly, observation will show that the identities are $(1_x, 1_{wx})$, and that the inverse to (γ,g) is $((g^{-1}\cdot\gamma^{-1}),g^{-1})$.

Now consider a split epimorphism in $\underline{\mathfrak{Gpds}/O}$, $X \underset{\longleftarrow c \,-}{\overset{-\, q \longrightarrow}{}} H$, where $qc = 1_H$. Since $cq:\mathrm{Ker}q \longrightarrow c(obH) = O$, if we set $\omega = cq|_{\mathrm{Ker}q}$, $\mathrm{Ker}q = \Gamma$, and $cH = G$, we have $\omega:\Gamma \longrightarrow G$ as above, with an action $ch\cdot k = (ch)k(ch)^{-1}$ for $h \in H(x,y)$, $k \in q^{-1}[y]$.

Thus, we have the groupoid $\mathrm{Ker}q \rtimes cH$, which has objects O, and arrows (k,ch) where $k \in X(x,x)$ and $h \in H(x,y)$ for $x,y \in O$. We note that ω is the identity on objects (as are both c and q). Obviously, (k_1,ch_1) and (k_2,ch_2) will be composible if and only if $domh_2 = codh_1$, and then $(k_1,ch_1)(k_2,ch_2) = (k_1(ch_1)k_2(ch_1)^{-1},c(h_1h_2))$

Now, let us define a morphism of groupoids $\phi:\mathrm{Ker}q \rtimes cH \longrightarrow X$ by $\phi:(k,h) \mapsto kh$ where $k \in \mathrm{Ker}q(x,x)$ and $h \in cH(x,y)$. This is well defined as

$\phi((k_1,h_1)(k_2,h_2)) = \phi(k_1h_1k_2h_1^{-1},h_1h_2) = k_1h_1k_2h_2 = \phi(k_1,h_1)\phi(k_2,h_2)$.

Clearly, ϕ is identity on objects.

We define a groupoid morphism $\theta: X \longrightarrow \mathrm{Ker}q \rtimes cH$ by:-

$\theta: g \mapsto (g(cqg)^{-1}, cqg)$.

Now, since $qc = 1_H$, $q(g(cqg)^{-1}) = (qg)(qcqg)^{-1} = (qc)(qc)^{-1} = 1$, so $g(cqg)^{-1} \in \mathrm{Ker}q$ and clearly, $cqg \in cH$. Further, θ is identity on objects, and is well defined since:-

$$\begin{aligned}\theta(g_1)\theta(g_2) &= (g_1(cqg_1)^{-1}, cqg_1)(g_2(cqg_2)^{-1}, cqg_2)\\ &= (g_1(cqg_1)^{-1}(cqg_1)(g_2(cqg_2)^{-1})(cqg_1)^{-1}, cqg_1g_2)\\ &= (g_1g_2(cq(g_1g_2))^{-1}, g_1g_2) = \theta(g_1g_2).\end{aligned}$$

Since $qc = 1_H$, if $h \in cH$, then $h = ch_1$ for some $h_1 \in H$, and so $cqh = cqch_1 = ch_1 = h$. Also, if $k \in \mathrm{Ker}q$, then $cqk = 1_x$ for $domk = x$. Then, $\theta\phi(k,h) = \theta(kh) = (kh(cq(kh)^{-1}, cq(kh)) = (k,h)$.

Further, $\phi\theta(g) = \phi(g(cqg)^{-1}, cqg) = g$.

Thus, θ and ϕ are mutually inverse, and $\mathrm{Ker}q \rtimes cH \cong X$.

Now, since $G_n \underset{\longleftarrow s^{n-1}_{n-1} \text{---}}{\text{---} d^n_n \longrightarrow} G_{n-1}$ is a split epimorphism, we have that

$$\mathrm{Kerd}^n_n \rtimes s^{n-1}_{n-1}G_{n-1} \cong G_n.$$

Now, we repeat this process as often as necessary to obtain each of the G_n as a multiple semidirect product of degeneracies of terms in the Moore complex. To do this, we define K_*, a simplicial subgroupoid of G_*, by $K_{n-1} = \mathrm{Kerd}^n_n$, $\delta^{n-1}_i = d^n_i|_{\mathrm{Kerd}^n_n}$ and $\sigma^{n-1}_i = s^n_i|_{\mathrm{Kerd}^n_n}$.

Since $d^{n-1}_{n-1}d^n_i = d^{n-1}_i d^n_n \ \forall\ i \leq n\text{-}1$, Kerd^n_n is mapped to $\mathrm{Kerd}^{n-1}_{n-1}$ by all the morphisms δ^n_i $i \leq n\text{-}1$. Further, $d^{n+1}_{n+1}s^n_i = s^{n-1}_i d^n_n$, $i \leq n\text{-}2$, thus we have that σ^{n+1}_i maps Kerd^n_n to $\mathrm{Kerd}^{n+1}_{n+1}$. We can generalise this to form a simplicial subgroupoid $K_*[m]$ where $K_{n-m-1}[m] = \bigcap\limits_{j=n-m}^{n} \mathrm{Kerd}^n_j$ - in this notation, K_* becomes $K_*[0]$. Clearly, $K_0[m] = \bigcap\limits_{j=1}^{m+1} \mathrm{Kerd}^{m+1}_j = NG_{m+1}$.

We can then decompose G_n as follows:-

$G_n \cong NG_n \rtimes s_0NG_{n-1} \rtimes s_1NG_{n-1} \rtimes s_1s_0NG_{n-2} \rtimes s_2NG_{n-1} \rtimes \cdots \rtimes s_{n-1}\cdots s_0G_0$,

where the products of s_i are in lexicographic order -

that is $\emptyset$; 0; 1; 1,0; 2; 2,0; 2,1; 2,1,0; etc. The number of terms will be 2^n, and the bracketing of terms is firstly into pairs, then in pairs of pairs, etc.

To give some idea of this, we will explicitly give the first few levels of the decomposition. Firstly, we know that $G_0 = NG_0$. Further

$G_1 \cong NG_1 \rtimes s_0G_0$,

$G_2 \cong (NG_2 \rtimes s_0NG_1) \rtimes s_1(NG_1 \rtimes s_0G_0)$,

$$G_3 \cong ((NG_3 \rtimes s_0NG_2) \rtimes s_1(NG_2 \rtimes s_0NG_1)) \rtimes \\ \rtimes s_2((NG_2 \rtimes s_0NG_1) \rtimes s_1(NG_1 \rtimes s_0G_0))$$

Now, NG_i (for all $i > 0$) is totally disconnected (it is a family of groups indexed by the object set of the G_0) and the elements of one vertex group have no effect on the elements other vertex groups, (either by multiplication or by action in the semidirect product). Thus G_* is determined by the simplicial groups $NG(p)$ for $p \in obG_*$ and the action of G_0 on them.

9. Groupoid-T-complexes and Crossed complexes over groupoids

This chapter stems from the work of Ashley (see [As]). He goes into great detail about simplicial-T-complexes, and as we are only interested in groupoid-T-complexes here, we have (in places) taken different (and somewhat shorter) paths to the results we generalise. However, the three results (Lemmas 9.1, 9.2 and theorem 9.3) are essentially his (the only difference being that we extend them to cover groupoids).

Now recall the definition of a T-complex (due to Dakin [D]). A T-complex, (K,T), is a Kan complex, K, with special elements in each dimension, T_n, called thin elements, together with the following axioms:-

T.1. Every degenerate element is thin

T.2. Every "box" has a unique thin filler

T.3. The thin filler of a thin box has a thin "lid"

We recall that:-

a "box" in K_n, we mean a set of elements $x_0, x_1, \cdots, x_{i-1}, x_{i+1}, x_n$ such that $x_j \in K_{n-1}$ (for $j \neq i$) and $d_k x_j = d_{j-1} x_k$ for ($j > k$, $j,k \neq i$).

a "filler" is an element $y = K_n$ s.t. $d_j = x_j$ for $j \neq i$, and the "lid" is $d_i y$.

We recall the definition of a crossed complex from section 7, and remark that T-complexes <u>are</u> equivalent to crossed complexes over groupoids (see [As]).

A groupoid-T-complex, (G,T) is a T-complex where G_n is a groupoid and T_n is a subgroupoid for all n, and the simplicial maps are groupoid morphisms. In a group-T-complex (with the obvious definition!) the thin elements of a group-T-complex are precisely the elements of the subgroups generated by the degenerate elements at each level (again, see [As]). The groupoid case is similar.

We define $D = \{D_n\}_{n>0}$ to be the graded subgroupoid of G generated by the degenerate elements in each level, $A_i^n = \bigcap_{j \neq i} \mathrm{Ker} d_j^n$, $A^n = \bigcup_{i=0}^{n} A_i^n$ and $A = \{A_n\}_{n>0}$

Lemma 9.1

Let G be a simplicial groupoid with constant object of objects. Let $x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n \in G_{n-1}$, such that $d_j x_k = d_{k-1} x_j$ for $j < k$ and $j,k \neq i$, (i.e. a box in G_n), then there is a filler for the box in D_n.

This is a restatement of Proposition 2.1, and we refer the reader to it for the proof. ∎

Lemma 9.2

If (G,T) is a groupoid-T-complex, then $T = D$.

Proof

Certainly, $D \subseteq T$, by axiom T1 for a T-complex. Then, if $t = T_n$, the box $d_1 t, d_2 t, \cdots, d_n t$ has a filler in D_n: call this d. Thus, as $d \in D$, it is thin, so it is the unique thin filler for the box. But, t also clearly fills the box, and is thin, so $d = t$. Thus, all thin elements are in D, and so $T = D$. ∎

Theorem 9.3

If G is a simplicial groupoid, then (G,D) is a groupoid-T-complex if and only if $D \cap A$ is trivial.

Proof

If (G,D) is a groupoid-T-complex, then for $x \in A_i^n \cap D_n$, there is a box $d_0x,\cdots,d_{i-1}x,d_{i+1}x,\cdots,d_nx$ - all of which are an identity - so the box is filled by an identity in D_n, but it is also filled by x, so by uniqueness, $x = 1$. Since this holds for all i and n, we have $D \cap A$ is trivial.

Conversely, (G,D) satisfies T1, and the existence part of T2. If $D \cap A$ is trivial, then consider a box $y_0,\cdots,y_{i-1},y_{i+1},\cdots,y_n$, with filler $y \in D$. Define $v_{i+1} = y^{-1}w_{i+1}$ where $w_{i+1} \in D$ is the filler constructed in lemma 9.1. In the case $i < n$, v_{i+1} fills the trivial box (the box with all faces an identity), so $v_{i+1} \in D_n \cap A_i^n$, and thus $v_{i+1} = 1$, that is $y = w_{i+1}$, so the D-filler is unique. The case $i = n$ is similar.

Lastly, the D-filler of a box in D will have two levels of degeneracy by construction, and so its lid will also be in D. ∎

Recall from section 7 that the Moore complex of a simplicial groupoid failed to be a crossed complex in three ways - that for $n \geq 2$ and $a \in obG_0$, $NG_n(a)$ was not abelian, the action of $\partial_1 NG_1(a)$ on it was not trivial, and the action of $\partial_1 NG_1(a)$ on $NG_1(a)$ was not conjugation. In the case that G is a simplicial-T-groupoid, these three stumbling blocks are removed by the following lemma, which is really <u>three</u> technical lemmas.

Lemma 9.4

Let (G,D) is a groupoid-T-complex, then the following hold for any $a \in ob(G)$:-

(i) Elements of $NG_n(a)$ commute with elements of $\mathrm{Ker}d_n \cap G_n(a)$, for $n \geq 2$.

(ii) $\partial_1 x_y = ((s_1)^{n-1}x)y((s_1)^{n-1}x)^{-1}]$ for any $x \in NG_1(a)$ and $y \in NG_n(a)$.

(iii) For $n \geq i$, $\forall\ k$ s.t. $i > k \geq 1$, for any $x \in G_{n-i+1}(a)$ and $y \in NG_n(a)$,

$$(s_0^{k-1}s_1^{i-k}x)y(s_0^{k-1}s_1^{i-k}x)^{-1} = (s_0^k s_1^{i-k-1}x)y(s_0^k s_1^{i-k-1}x)^{-1}$$

Proof:

(i) Let $n \geq 2$, $x \in \mathrm{Ker}d_n^n \cap G_n(a)$ and $y \in NG_n(a)$, and consider $g = (s_1x)(s_ny)(s_1x)^{-1}(s_ny)^{-1}$.

Then, $g \in D_{n+1}$, and hence is thin.

Also, $d_jg = 1$ for $n-1 \geq j \geq 1$ and for $j = n+1$, and

$$d_0g = (s_0d_0x)(s_{n-1}d_0y)(s_0d_0x)^{-1}(s_{n-1}d_0y)^{-1} \in D_n.$$

Thus $\{d_0g, 1, \cdots, 1, -, 1\}$ is a thin box (which is missing the n^{th} face), and so it has a thin lid. But the lid is $d_ng = xyx^{-1}y^{-1} \in D_n \cap NG_n$, and as (G,D) is a groupoid-T-complex, $D_n \cap NG_n = 1$. Hence $xyx^{-1}y^{-1} = 1$.

So elements of $NG_n(a)$ commute with elements of $\mathrm{Ker}d_n^n \cap G_n(a)$, for $n \geq 2$, . □

(ii) Now let $x \in NG_1(a)$ and $y \in NG_n(a)$, and consider

$$((s_0)^n d_0x)y((s_0)^n d_0x)^{-1}((s_1)^{n-1}x)y^{-1}((s_1)^{n-1}x)^{-1}.$$

We can rewrite it as

$$d_0\big[((s_1)^n x)s_0y((s_1)^n x)^{-1}(s_0(s_1)^{n-1}x)(s_0y)^{-1}(s_0(s_1)^{n-1}x)^{-1}\big].$$

Set $g = \big[((s_1)^n x)s_0y((s_1)^n x)^{-1}(s_0(s_1)^{n-1}x)(s_0y)^{-1}(s_0(s_1)^{n-1}x)^{-1}\big]$

and note $g \in D_{n+1}$.

Then, $d_1 g = \left[((s_1)^{n-1}x)y((s_1)^{n-1}x)^{-1}((s_1)^{n-1}x)y^{-1}((s_1)^{n-1}x)^{-1}\right] = 1$, and (for $i \geq 2$) $d_i s_0 y = s_0 d_{i-1} y = 1$, and therefore $d_i g = 1$ for $i \geq 1$. Then, g is the unique degenerate filler for the trivial box (where the 0^{th} face is missing), and so $g = 1$.

Thus $d_0 g = ((s_0)^n d_0 x)y((s_0)^n d_0 x)^{-1}((s_1)^{n-1}x)y^{-1}((s_1)^{n-1}x)^{-1} = 1$, and so $((s_0)^n d_0 x)y((s_0)^n d_0 x)^{-1} = {}^{\partial_1 x}y = ((s_1)^{n-1}x)y((s_1)^{n-1}x)^{-1}\big)$. □

(iii) Let $x \in G_{n-i+1}(a)$ and $y \in NG_n(a)$, where $n \geq i \geq 1$. Consider

$$g = (s_0^k s_1^{i-k}x)(s_k y)(s_0^k s_1^{i-k}x)^{-1}(s_0^{k+1}s_1^{i-k-1}x)(s_k y)^{-1}(s_0^{k+1}s_1^{i-k-1}x)^{-1},$$

where $(i-1) \geq k \geq 1$ and so $g \in D_{n+1}$.

Now, for $0 < j < k$ $\quad d_j s_k y = s_{k-1} d_j y = 1$,

and for $k+1 < j \leq n+1$ $\quad d_j s_k y = s_k d_{j-1} y = 1$,

so $d_j g = 1$ for $j \neq 0, k, k+1$.

We note at this point, since $1 \leq k$, that these three cases are always distinct, and further, that $d_0 g$ is clearly degenerate (always).

Now, $d_{k+1} s_0^{k+1} s_1^{i-k-1} x = d_{k+1} s_0^k s_1^{i-k} x$ since $d_{k+1} s_0^k = s_0^k d_1$, and thus $d_{k+1} g = 1$. Thus we have a box in G_n, with the k^{th} face missing, and all other faces the identity, except the 0^{th} which is degenerate. Thus, g is the unique degenerate filler, and the lid is thin, that is $d_k g \in D_n$.

But $d_k g = (s_0^{k-1}s_1^{i-k}x)y(s_0^{k-1}s_1^{i-k}x)^{-1}(s_0^k s_1^{i-k-1}x)y^{-1}(s_0^k s_1^{i-k-1}x)^{-1} \in NG_n$ and so $d_k g \in NG_n \cap D_n$ and must be the identity. Thus, as required $(s_0^{k-1}s_1^{i-k}x)y(s_0^{k-1}s_1^{i-k}x)^{-1} = (s_0^k s_1^{i-k-1}x)y(s_0^k s_1^{i-k-1}x)^{-1}$. ■

Theorem 9.5

If G is a simplicial-T-groupoid, then its Moore complex, NG_*, is a crossed complex.

Proof:

We need only show that the three axioms which fail to be satisfied in general are satisfied for the T-groupoid case.

(i) From 9.4 (i), elements of $NG_n(a)$ commute with elements of $\mathrm{Ker}d_n^n \cap G_n(a)$ for $n \geq 2$. But, $NG_n(a) \leq \mathrm{Ker}d_n^n \cap G_n(a)$, and so $NG(a)$ is abelian for $n \geq 2$. □

(ii) From 9.4 (ii), ${}^{\partial_1 x}y = ((s_1)^{n-1}x)y((s_1)^{n-1}x)^{-1})$ for any $x \in NG_1(a)$ and $y \in NG_n(a)$. For the case $n = 1$, this yields that ${}^{\partial_1 x}y = xyx^{-1}$, that is, $\partial_1 NG_1$ acts on NG_1 by conjugation. □

(iii) From 9.4 (ii), ${}^{\partial_1 x}y = ((s_1)^{n-1}x)y((s_1)^{n-1}x)^{-1})$ for any $x \in NG_1(a)$ and $y \in NG_n(a)$, and so ${}^{\partial_1 x}y = (s_1^{n-1}x)y(s_1^{n-1}x)^{-1}$. From, 9.4 (iii), for $n \geq 2$, $\forall\, k \quad n > k \geq 1$, for $x \in G_1(a)$ and for $y \in NG_n(a)$, $(s_0^{k-1}s_1^{n-k}x)y(s_0^{k-1}s_1^{n-k}x)^{-1} = (s_0^k s_1^{n-k-1}x)y(s_0^k s_1^{n-k-1}x)^{-1}$.

Thus,
$$
\begin{aligned}
{}^{\partial_1 x}y &= (s_1^{n-1}x)y(s_1^{n-1}x)^{-1} = (s_0^1 s_1^{n-2}x)y(s_0^1 s_1^{n-2}x)^{-1}\\
&= (s_0^2 s_1^{n-3}x)y(s_0^2 s_1^{n-3}x)^{-1} = \cdots = (s_0^{n-2}s_1 x)y(s_0^{n-2}s_1 x)^{-1}\\
&= (s_0^{n-1}x)y(s_0^{n-1}x)^{-1}.
\end{aligned}
$$

Define $h = (s_0^n x)(s_n y)(s_0^n x)^{-1}(s_n y)^{-1} \in D_{n+1}$. As $d_{n+1}s_0^n = s_0^n d_1$, and $x \in NG_1$, then $d_{n+1}h = 1$. For $0 < i < n$, as $d_i s_n y = s_{n-1}d_i y = 1$, then $d_i h = 1$ and $d_0 h = (s_0^{n-1}x)(s_{n-1}d_0 y)(s_0^{n-1}x)^{-1}(s_{n-1}d_0 y)^{-1} \in D_n$. Thus, we have a box with n^{th}-face missing, and all other faces identity or degenerate. Thus h is the thin filler of a thin box, and $d_n h \in D_n$. But, $d_n h = (s_0^{n-1}x)y(s_0^{n-1}x)^{-1}y^{-1} = {}^{\partial_1 x}y.y^{-1} \in NG_n$, thus $d_n h \in NG_n \cap D_n = 1$, and so ${}^{\partial_1 x}y.y^{-1}$, or ${}^{\partial_1 x}y = y$. Thus the action of $\partial_1 NG_1$ on the groups $NG_n(a)$ for $n \geq 2$ is trivial.

Therefore, if (G,D) is a groupoid-T-complex, the Moore complex of G, NG, is a crossed complex. ■

10. The Crossed complex associated to a Simplicial Groupoid

As we have seen, NG is a crossed complex when (G,D) is a groupoid-T-complex , and the necessary and sufficient condition for this to hold is $NG_n \cap D_n$ is trivial. Thus, if we are to obtain a crossed complex from NG we will have to quotient it by some subgroupoid containing $NG_n \cap D_n$. However, the naive approach of taking the quotient $\dfrac{NG_n}{NG_n \cap D_n}$ for the n^{th} term will not suffice, since $NG_n \cap D_n$ is not (in general) normal in NG_n, and further, $d_0(NG_{n+1} \cap D_{n+1})$ is not necessarily contained in $(NG_n \cap D_n)$.

Now, if G is a simplicial group, then we can form a reduced crossed complex by taking $C_n = \dfrac{(NG)_n}{(NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})}$. This was proved by Carrasco and Cegarra [Ca], using the theory of hypercrossed complexes. There is also a direct proof (by Porter [P]).

We shall do the same with the groupoid case, and show that C_* is a crossed complex, where $C_n = \dfrac{(NG)_n}{(NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})}$.

We first note that $NG_1 \cap D_1 = 1$. Any element of D_1 has the form s_0x for $x \in G_0$, and so if $y \in NG_1 \cap D_1$, then $y = s_0x$ for some $x \in G_0$. Then, $1_p = d_1y = d_1s_0x = x$, and so $x = 1_p$ and therefore $y \in 1_p$ for some $a \in obG$.

Secondly, since the NG_i are totally disconnected, so will $(NG_n \cap D_n)\partial(NG_{n+1} \cap D_{n+1})$ be, and so C_n will be

$$\left\{\frac{(NG)_n(a)}{(NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})(a)}\right\}_{a \in obG}$$

So, for each a, we are in the reduced case: that is, C_* is a family of reduced crossed complexes together with an action of G_0. Thus, we have already that C_n is a family of groups for $n \geq 1$, and that for $n \geq 2$, these are abelian. Further, we have that the action of NG_1 on NG_i is trivial for $i \geq 2$, and conjugation for $i = 1$.

Thus we require only that there be an action of G_0 on each $C_n(a)$ such that, for $x \in G_0(b,a)$ and $y \in C_n(a)$, ${}^x y \in C_n(b)$, and that $\partial_n({}^x y) = {}^x(\partial_n y)$.

We have the action of G_0 on NG_n, so define ${}^x[y] := [{}^x y]$ for $x \in G_0(b,a)$ and $[y] \in \dfrac{(NG)_n(a)}{(NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})(a)}$ $(y \in NG_n)$.

Let $g \in (NG_n \cap D_n)(a)$, $h \in (NG_{n+1} \cap D_{n+1})(a)$, and consider $(s_0^n x)gd_0h(s_0^n x)^{-1}$ (clearly $gd_0h \in (NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})(a)$).

Now, $(s_0^n x)gd_0h(s_0^n x)^{-1} = (s_0^n x)g(s_0^n x)^{-1}d_0((s_0^{n+1}x)h(s_0^{n+1}x)^{-1})$, and so $(s_0^n x)gd_0h(s_0^n x)^{-1} \in (NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})(b)$ and thus ${}^x[gd_0h]$ is trivial.

So if $y' \in [y]$, then $[1_b] = [(s_0^n x)y(y')^{-1}(s_0^n x)^{-1}] = {}^x[y]\,{}^x[y'^{-1}]$ and since ${}^x[y^{-1}] = [(s_0^n x)y^{-1}(s_0^n x)^{-1}] = [((s_0^n x)y(s_0^n x)^{-1})^{-1}] = ({}^x[y])^{-1}$, we have $({}^x[y'])^{-1} = {}^x[y'^{-1}] = ({}^x[y])^{-1}$, so ${}^x[y]$ is well defined.

Since the action of G_0 on NG_n has the property $\partial_n({}^x y) = {}^x(\partial_n y)$, then this will pass to the quotient groups as well. This completes the proof. ∎

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