Algebraic Homotopy in Simplicially Enriched Groupoids

Philip John Ehlers

January 27, 1993

for Nicky

Declaration

This thesis is my own work, except where stated in the text, and has not been submitted for any other degree in this or any other institution.

P. J. Ehlers:___________________________________

Dr. T. Porter:___________________________________
(PhD supervisor)

Acknowledgements

I would like to thank my supervisor, Dr. Tim Porter, for his support, his guidance and (particularly) his tolerance over the past three and a half years. I am also grateful to Prof. Ronnie Brown, Dr. Mark Lawson, Dr. Marcus Pfenniger and Andy Tonks for helpful discussions and comments on mathematics, and advice on how to approach a PhD.
Many thanks to my family and friends for their support, particularly while I was "writing up", and to all the staff and students at the School of Mathematics, with whom I have had the pleasure of working.
Finally, thanks are due to the University of Wales who paid my fees and supported me with a University Studentship.

Thanks be to God.

This Thesis was typeset using LaTeX.

Contents

Summary

The first two chapters describe and develop some of the basics of Simplicial set theory. They include a study of the category of finite ordinals and monotonic functions, together with some related categories (particular note being given to the monoidal structure), many of the basic technical results, and a description of the functorial relation between simplicial sets and topological spaces and small categories. There is also a section outlining those results in the author's MSc thesis which are to be used in later chapters.

Chapter three outlines a monoidal structure for simplicial sets based on the monoidal structure of the category of finite ordinals, and obtains a tensor product ($\otimes$) for augmented simplicial sets which models the operation of join on topological spaces. Using this, a simplicial model for the n-sphere, ($\mathbf{S}^n$), is defined, which have the property that $\mathbf{S}^p \otimes \mathbf{S}^q \cong \mathbf{S}^{p+q+1}$.

Chapters four and five study the subdivision functor of Porter and Cordier (Sd), prove that it may be described as the composite functor $diagDEC$, and recall some of the theory of anodyne extensions. A similar concept, called weak anodyne extension is defined. Most space is taken in technical lemmas and combinatorial descriptions culminating in a proof that the unit of "Nerve/Categorisation" adjunction $\eta_{Sd\triangle[n]}$ is a weak anodyne extension for each n, and that the filling scheme respects the cosimplicial structure.

The last chapter uses this, and the adjunctions, "Nerve/categorisation" and "loopgroupoid/classifying-space", to define a retraction from $GNer\Pi Sd\triangle[n]$ to $GSd\triangle[n]$ (where G is the loopgroupoid functor). Finally, this retraction and some topological morphisms are used to describe a Van Kampen type theorem for a functor which is a quotient of the loop groupoid functor of the singular complex of a space.

Introduction

It has always been a problem of homotopy theory to construct functors which are easy to calculate with, yet still preserve sufficient homotopy information to be interesting. The (Seifert-) Van Kampen Theorem is one of the earliest theorems about the calculational use of such a functor. It states that the fundamental group of a pointed space $(\mathcal{W}, *)$, $\pi_1(\mathcal{W}, *)$, may be described as the quotient of the free product of $\pi_1(\mathcal{U}, *)$ and $\pi_1(\mathcal{V}, *)$ by information in $\pi_1(\mathcal{U} \cap \mathcal{V}, *)$, where $\mathcal{W} = \mathcal{U} \cup \mathcal{W}$, where $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{U} \cap \mathcal{V}$ are all open path connected subspaces of $\mathcal{W}$ and where $* \in \mathcal{U} \cap \mathcal{V}$. It is this theorem which provides the main motivation for this thesis.

To work with topology, it also made sense to work with simplicial sets (originally called Semi Simplicial Complexes), rather than directly with topological spaces. In this area, there is a lot of material: there are the two fundamental papers of D. Kan ([30] and [31]) defining the homotopical structure of simplicial sets: the extension condition, Kan complexes, homotopy groups of simplicial sets, the loop group functor and its right adjoint. J. Milnor (in an unpublished paper, printed as part of [1]) comments that the loop group of a simplicial set is homotopically equivalent to the loop space, provided the simplicial set is a Kan complex. There is the paper of Moore ([37]) studying simplicial sets and Postnikov systems based on them. All these can in part be seen as aspects of Whitehead's view of Combinatorial Homotopy (see [43] and [44]). Following the work of Quillen, in "Homotopical Algebra" ([39]), in which he outlined the basic properties required of a category for it to support a homotopy theory, it became possible to study homotopy in a variety of different categories.

Following more directly from Whitehead's view, much work has been done on the category of Crossed Complexes by Brown et al. There are a variety of related algebraic categories which model $n+1$-types (Crossed n-cubes, Cat^n groups, see the work of Porter, Loday, Ellis & Steiner, Gilbert et al.) which generalise the crossed modules of Whitehead, and the crossed 2-modules of Conduché (models for 2-types and 3-types respectively).

It made sense to try to find Van Kampen Type Theorems for the new algebraic models, and so there are now Van Kampen Type Theorems for the fundamental groupoid of a space (see [9]) and the fundamental crossed complex of a filtered topological space (see [11]). The idea also became broader, so that a Van Kampen Type Theorem now represents the preservation properties of the functor on more general pushouts.

The purpose of this thesis is to examine some of the phenomena arising in the category of simplicial sets, and to see how they relate to operations in other categories, (including topological spaces). The emphasis is on the simplicial structure, rather than the topological. The aim is to describe a functor from topological spaces to simplicial groupoids which is a quotient of the loop groupoid functor, and describe a Van Kampen Type theorem for this functor. The point is that the (non-abelian) homology of the Moore complex of the loop groupoid of a simplicial set yields the absolute homotopy groups for the simplicial set. Thus, if a quotient of the loop groupoid functor can be shown to satisfy a Van Kampen Theorem, and if the quotient has the same homotopy type as the loop groupoid, it should be possible to calculate all homotopy types of the union of two spaces for which all absolute homotopy groups are already known.

Chapter 1

Models for Simplices

1.1 Definitions

The basis for using simplicial sets to calculate topological invariants is the link between the model for simplicial sets (that is, the category of finite ordinals and monotonic functions) and the affine simplices in the category of topological spaces. The theory reproduced below is well known. There is one other model of simplicial structure which will be useful in this thesis, and it is dealt with also. One important note is that the labelling conventions are different from those used by Mac Lane in [35].

Definition 1.1 (i)

The category of finite ordinals and monotonic functions will be denoted by Δ. The objects are the ordered sets $\{0, 1, \cdots, n\}$ for $n \geq -1$, which will be written as $[n]$. The object denoted by $[-1]$ is the empty set. There is precisely one morphism from $[-1]$ to $[n]$ for all n and no morphism from $[n]$

to $[-1]$ for any $n \geq 0$. Thus $[-1]$ is initial in the category Δ.

The morphisms of Δ are generated (under composition) by functions

$$\delta_i^{n-1} : [n-1] \to [n] \qquad \text{and} \qquad \sigma_j^{n+1} : [n+1] \to [n]$$

where

$$\delta_i^{n-1}(k) = \begin{cases} k & \text{if } 0 \leq k < i \\ k+1 & \text{if } n-1 \geq k \geq i \end{cases}$$
$$\sigma_j^{n+1}(k) = \begin{cases} k & \text{if } 0 \leq k \leq i \\ k-1 & \text{if } n+1 \geq k > i \end{cases}$$

These morphisms obey the following identities:

$$\delta_j^n \delta_i^{n-1} = \delta_i^n \delta_{j-1}^{n-1} \text{ if } 0 \leq i < j \leq n$$

$$\sigma_j^n \sigma_i^{n+1} = \sigma_i^n \sigma_{j+1}^{n+1} \text{ if } 0 \leq i \leq j \leq n$$

$$\sigma_j^{n+1} \delta_i^n = \begin{cases} \delta_i^{n-1} \sigma_{j-1}^n & \text{if } i < j \\ id & \text{if } i = j, j+1 \\ \delta_{i-1}^{n-1} \sigma_j^n & \text{if } i > j+1 \end{cases}$$

These equations are standard and may be found in [24], [30], [36] and [35] (among others!). It is clear from them that $[0]$ is terminal in Δ.

There is a full subcategory of Δ, denoted Δ^+ which does not have the empty set as an object, but otherwise has all the objects of Δ.

Definition 1.1 (ii)

The affine n-simplex, which will be denoted by $\triangle^n$, is the subset of $\mathbb{R}^{n+1}$ defined by

$$\{\, (x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 1 \, x_i \geq 0 \forall 0 \leq i \leq n \,\}$$

There are continuous functions between the affine simplices generated (under composition) by the continuous functions:

$$\delta_i^{n-1} : \triangle^{n-1} \to \triangle^n \text{ and } \sigma_j^{n+1} : \triangle^{n+1} \to \triangle^n$$

where $\delta_i^{n-1}(x_0, \cdots, x_{n-1}) = (x_0, \cdots, x_i, 0, x_{i+1}, \cdots, x_{n-1})$

$$\sigma_i^{n+1}(x_0, \cdots, x_{n+1}) = (x_0, \cdots, x_i + x_{i+1}, \cdots, x_{n+1}).$$

It is sometimes useful to consider the empty topological space as an affine simplex. In such cases it will be denoted $\triangle^{-1}$. Since the empty set is initial in the category of topological spaces, it is initial in the subcategory of affine simplices.

Definition 1.1 (iii)

The category of small categories and functors will be denoted by $\underline{Cat}$. The category, $\mathbf{N}^+$, is the full subcategory of $\underline{Cat}$ defined by the objects $ob\mathbf{N}^+ = \{\, [n] \,|\, n \in \mathbb{N}\}$, where $[n]$ is the category with objects the elements of the set $\{0, 1, \cdots, n\}$ and a unique morphism from i to j *iff* $i \leq j$. A functor from $[n]$ to $[m]$ is then an order preserving map on the ordered set $[n]$. As in the cases of finite ordinals and affine simplices, there are canonical maps $\delta_i^{n-1} : [n-1] \longrightarrow [n]$ and $\sigma_j^{n+1} : [n+1] \longrightarrow [n]$. These are defined as follows:

$$\delta_i^{n-1}(!_{k,l} : k \to l) = \begin{cases} (!_{k,l} : k \to l) & \text{if } \ 0 \leq k \leq l < i < n \\ (!_{k,l+1} : k \to l+1) & \text{if } \ 0 \leq k < i \leq l < n \\ (!_{k+1,l+1} : k+1 \to l+1) & \text{if } \ 0 \leq i \leq k \leq l < n \end{cases}$$

$$\sigma_j^{n+1}(!_{k,l} : k \to l) = \begin{cases} (!_{k,l} : k \to l) & \text{if } \ 0 \leq k \leq l \leq j \leq n+1 \\ (!_{k,l-1} : k \to l-1) & \text{if } \ 0 \leq k \leq j < l \leq n+1 \\ (!_{k-1,l-1} : k-1 \to l-1) & \text{if } \ 0 \leq i < k \leq l \leq n+1 \end{cases}$$

(Here, $!_{k,l}$ is the unique map from k to l, for $k \leq l$.) These definitions make it clear that $[0]$ is the terminal object in $\mathbf{N}^+$.

The category $\mathbf{N}^+$ together with the empty category is denoted $\mathbf{N}$; the empty category (which shall be denoted $[-1]$) is initial in $\mathbf{N}$.

Both the categories $\mathbf{N}^+$ and $\mathbf{N}$ are themselves small categories. It is clear that they may be described as categories of directed sets, where $[n]$ is now the directed set $\{0 < 1 < \cdots < n\}$, and morphisms are those set functions which respect the ordering on the set.

In all cases, the categories Δ^+, $\mathbf{N}^+$ and the set of affine simplices, the superscripts on the δ_i and σ_j will generally be omitted.

There are obvious inclusion functors, $\Delta^+ \to \Delta$ and $\mathbf{N}^+ \to \mathbf{N}$.

The comment made earlier about Mac Lane's notation may now be expanded. Although the notation for the categories Δ^+ and Δ is Mac Lane's notation, the notation for the objects of all the categories described is not. Where n is used here, Mac Lane uses $n+1$; thus Mac Lane denotes the empty set by 0, $\mathbf{0}$ and $\triangle^0$, respectively. Note that the category described here by $[n]$ may be better known to category theorists as $\mathbf{n+1}$. (This is Mac Lane's notation: see [35]). There should not be too much confusion arising from this: the reason for maintaining a different notation for the objects themselves is to stay in keeping with the standard definitions of simplicial objects.

There is also a category Δ_0 which is a wide subcategory of Δ^+. It has morphisms all monotonic functions which fix 0. This effectively means that there are no δ_0 morphisms and the morphisms are generated by the δ_i and σ_j where $i \neq 0$. This category is the model for contractible simplicial sets. It has the property that $[0]$ is both the terminal object (as it is for Δ^+ and Δ) and the initial object (as there is a unique morphism $[0] \to [n]$ for every n, which takes the 0 to 0). There is, of course, a natural embedding $\Delta_0 \to \Delta^+$ and a natural embedding $\Delta_0 \to \Delta$.

Two further comments will be made here, but dealt with later.
First, there is an inversion function on the category of finite ordinals which reverses the order, that is, sends $\{0 < 1 < \cdots < n\}$ to $\{0 > 1 > \cdots > n\}$. This clearly yields a category which is equivalent to Δ^+, but the isomorphism between them is a set function which is not monotonic.
Secondly, it has already been noted that $\mathbf{N}$ is a small category, and the same is clearly true of the category Δ (which is essentially isomorphic to $\mathbf{N}$). This implies that all finite powers of the category Δ (or of the category $\mathbf{N}$) are also small. In particular, this is true of the binary product $\Delta \times \Delta$.

1.2 Functors on the Models

Between the categories Δ and Δ^n (that is, the categorical product of n copies of Δ) there are two important functors. These are the diagonal functor, ∂, and the join functor, which is linked to the monoidal structure of Δ.

1.2.1 The Diagonal

The diagonal functor $\partial^n : \Delta \longrightarrow (\Delta)^{n+1}$ is given by

$$\partial^n : ([p] \xrightarrow{f} [q]) \mapsto \Big(([p], \cdots, [p]) \xrightarrow{f^{n+1}} ([q], \cdots, [q]) \Big),$$ where $([p], \cdots, [p])$ is the product of $n+1$ copies of $[p]$, and $([q], \cdots, [q])$ is the product of $n+1$ copies of $[q]$. The superscript on ∂ will be dropped when there is no ambiguity, and will always be dropped in the case $n = 1$.

Proposition 1.2.1.1

The functor ∂^n has neither left nor right adjoints.

Proof

From the theory of adjunctions, (see [35]), the functor ∂^n has a left adjoint if Δ has $(n+1)$-fold coproducts and a right adjoint if Δ has $(n+1)$-fold products.

Consider the case of binary products: let $[1] \times [1] \cong [n]$.

Then, $\Delta([1],[1]) \times \Delta([1],[1]) \cong \Delta([1],[n])$ and specifically,

$$|\,\Delta([1],[1])\,| \;\times\; |\,\Delta([1],[1])\,| \;=\; |\,\Delta([1],[n])\,|$$

Now, $|\,\Delta([1],[1])\,| \;=\; 3$ and $|\,\Delta([1],[n])\,| \;=\; \frac{(n+1)(n+2)}{2}$. Thus $9 = \frac{(n+1)(n+2)}{2}$, so $(n+1)(n+2) = 18$. Since $3 \times 4 = 12$ and $4 \times 5 = 20$, there is no $n \in \mathbb{N}$ such that $(n+1)(n+2) = 18$ and so $[1] \times [1]$ does not exist in Δ. Since binary products do not exist in this case, then finite products do not exist in general.

Now consider the case of binary coproducts. Let $[0] \sqcup [0] \cong [m]$. Therefore $\Delta([m],[r]) \cong \Delta([0],[r]) \times \Delta([0],[r])$ and specifically,

$$|\,\Delta([m],[r])\,| \;=\; |\,\Delta([0],[r])\,| \;\times\; |\,\Delta([0],[r])\,|$$

Now $|\,\Delta([0],[2])\,| = 3$ and $|\,\Delta([m],[2])\,| = \frac{(m+2)(m+3)}{2}$. Thus $9 = \frac{(m+2)(m+3)}{2}$, that is $18 = (m+2)(m+3)$. As has already been shown, there is no integer m with this property, and hence there is no binary coproduct of $[0]$ and $[0]$ in Δ, and so finite coproducts do not exist in general in Δ. Therefore the functor ∂^n has neither left or right adjoints, for all $n \geq 1$. ■

1.2.2 The Monoidal Structure

Definition 1.2.2 (i)

Let $f_i : [p_i] \to [q_i]$ for $0 \leq i \leq n$. Define the "ordinal sum" functor, $or^n : \Delta^{n+1} \longrightarrow \Delta$, as follows:-

$$or^n([p_0], \cdots, [p_n]) = [\sum_{i=0}^{n} p_i + n]$$

$$or^n(f_0, \cdot\cdot, f_n) = \begin{cases} f_0(k) & \text{if} \quad 0 \leq k \leq p_0 \\ f_1(k - p_0 - 1) + q_0 + 1 & \text{if} \quad p_0 + 1 \leq k \\ \vdots & \vdots \quad \vdots \qquad\qquad \leq p_0 + p_1 + 1 \\ \vdots & \vdots \quad \vdots \\ f_r(k - \sum_{i=0}^{r-1} p_i - r) & \text{if} \quad \sum_{i=0}^{r-1} p_i + r \leq k \\ \vdots \qquad\quad + \sum_{i=0}^{r-1} q_i + r & \vdots \quad \vdots \qquad\qquad \leq \sum_{i=0}^{r} p_i + r \\ \vdots & \vdots \quad \vdots \\ f_n(k - \sum_{i=0}^{n-1} p_i - n) & \text{if} \quad \sum_{i=0}^{n-1} p_i + n \leq k \\ \qquad\quad + \sum_{i=0}^{n-1} q_i + n & \qquad\qquad\quad \leq \sum_{i=0}^{n} p_i + n \end{cases}$$

Note that any number of copies of $[-1]$ (the empty set) may be added into the sum without affecting it, as would be expected. As with ∂^n, the superscript on *or* will usually be dropped, and will always be dropped in the case $n = 1$.

For the rest of this section it will be assumed that $n = 1$. The ordinal sum is the join of two finite ordinals, and generates the monoidal structure on Δ. The unit of the monoid is the unique map $! : [-1] \to [0]$; the multiplication is the unique arrow $\sigma_0 : [1] \to [0]$.

For a study of this, see Mac Lane [35]. He describes the links between the various categories defined in section 1.1 , shows that Δ is a strict monoidal category. He further proves that $< \Delta, !, \sigma_0 >$ is universal in the sense that for any monoid, $< c, \mu, \eta >$, in a strict monoidal category, $< B, \otimes, e >$, there is a unique functor

$$F :< \Delta, or, \sigma_0 > \longrightarrow < B, \otimes, e >$$

where $F([-1]) = c, F(\sigma_0) = \mu$ and $F(!) = \eta$.

Definition 1.2.2 (ii)

For two directed sets X and Y, $X \vee Y$ is defined as the disjoint union of the elements, with $a < b$ iff $a < b \in X$ or $a < b \in Y$ or $a \in X$ and $b \in Y$. In the case that $X = [n], Y = [m]$, it is clear that $[n] \vee [m] \cong [n + m + 1]$.

There is also a join defined on affine simplices. This is the join on subsets of a normed vector space. The join for topological spaces (which generalises that on subsets of a vector space and will be looked at later) is described in some detail in [9].

Definition 1.2.2 (iii)

Let X and Y be subsets of a normed vector, V. Consider $U \subset V$, where $U = \{r\mathbf{x}+s\mathbf{y} \,|\, r+s=1,\ r,s \geq 0\mathbf{x} \in X,\ \mathbf{y} \in Y\}$. If X and Y are placed in V in such a way that no two lines in the set U cross (that is, they meet only at endpoints) then the join of X and Y, written $X*Y$, is defined by $X*Y = U$.

Proposition 1.2.2.1

$$\triangle^p * \triangle^q \cong \triangle^{p+q+1}$$

Proof

Consider the vector space $\mathbb{R}^{p+q+1}$ and the two compact convex subsets:

$$X = \{\,(x_0, x_1, \cdots, x_p, 0, \cdots, 0) \,|\, \sum_{i=0}^{p} x_i = 1\,\}$$

$$Y = \{\,(0, \cdots, 0, y_0, y_1, \cdots, y_q) \,|\, \sum_{j=0}^{q} y_j = 1\,\}$$

First note that $X \cong \triangle^p$ and $Y \cong \triangle^q$. Furthermore, it is clear that no two lines in the set $U = \{\,r\mathbf{x} + (1-r)\mathbf{y} \,|\, 0 \leq r \leq 1, \mathbf{x} \in X, \mathbf{y} \in Y\,\}$ intersect except at endpoints. Thus $X * Y = U$. However, U is the subset of $\mathbb{R}^{p+q+1}$ given by

$$\{\,(rx_0, \cdots, rx_p, (1-r)y_0, \cdots, (1-r)y_q) \,|\, \sum_{i=0}^{p} rx_i + \sum_{j=0}^{q}(1-r)y_j = 1\,\}.$$

That is, U is the affine $(p+q+1)$-simplex. Therefore $\triangle^p * \triangle^q \cong \triangle^{p+q+1}$. ■

Comment

The ordinal sum on the category Δ models the join operation, $*$, on the affine simplices in the same way that the finite ordinals model the affine simplices. Since it has been noted already that Δ and $\mathbf{N}$ are isomorphic categories, and it is clear from the definitions above that the "isomorphism " takes *or* to $\vee$, then the same may be said of $\vee$ on the category $\mathbf{N}$.

It has been mentioned already that reversing the order on an ordered set, X gives an ordered set which is bijectively equivalent to X, but that there is no monotonic function to express this fact. There is a similar problem with *or*. Despite the fact that $[n]or[m] \cong [m]or[n]$ in <u>*Sets*</u>, there is no way of obtaining the isomorphism from the morphisms of Δ^+. The link between these two "problems" is shown by the equation $R([p]or[q]) = (R[q])or(R[p])$, where RX is the ordered set obtained by reversing the ordering on X. Given this problem with ordinal sum, it is important to be aware and draw the distinction between those cases when $[p]or[q]$ is just the set $\{0, 1, \cdots, p+q+1\}$ and when the cosimplicial structure of $[p]or[q]$ is being used.

The functor *or* is also connected to the inclusion functor $in : \Delta_0 \longrightarrow \Delta$, (this comes from [18], and has also been covered in [20]). The functor *in* has a left adjoint $b : \Delta \longrightarrow \Delta_0$ which is defined on objects by:-

$$b([n]) = [n+1]$$

and on morphisms by

$$b(f)(i) = \begin{cases} 0 & \text{if} \quad i = 0 \\ f(i-1)+1 & \text{if} \quad i \geq 1 \end{cases}$$

This may be rewritten as:-

$b : (f : [n] \to [m]) \mapsto ((id)or(f) : [0]or[n] \to [0]or[m])$

The composite functor $inb : \Delta \longrightarrow \Delta$ forms a monad in Δ with unit the unit of the adjunction (that is $\delta_0^n : [n] \mapsto [n+1]$) and multiplication $in(\sigma_0^{n+2} : [n+2] \mapsto [n+1]) = \sigma_0^{n+2}$ (since σ_0^* is the counit of the adjunction). The algebras are simply the arrows $\sigma_0^{n+1} : [n+1] \mapsto [n]$ where $n \geq 0$.

It is possible to embed the category Δ_0 in the category Δ^+. The adjoint has the same description, the difference being that while $[0] \in b(\Delta)$, $[0] \notin b(\Delta^+)$. Further, the functor $or : \Delta^+ \times \Delta^+ \to \Delta^+$ is well defined. This is useful to the extent that it is sometimes convenient to be able to move between Δ and Δ^+ (and hence between augmented simplicial categories and simplicial categories) and know that the adjunctions and associated monad and comonad structures extend naturally.

Chapter 2

Simplicial Categories

2.1 Preliminaries

Let $\mathcal{C}$ be a category. The following definitions are standard, and are noted for completeness.

Definition 2.1 (i)

The category of *simplicial objects in* $\mathcal{C}$ is defined to be $\mathcal{C}^{\Delta^{+op}}$.

The category of *augmented simplicial objects in* $\mathcal{C}$ is defined to be $\mathcal{C}^{\Delta^{op}}$.

The category of *contractible simplicial objects in* $\mathcal{C}$ is defined to be $\mathcal{C}^{\Delta_0^{op}}$.

In a similar way, categories of multiple simplicial objects in $\mathcal{C}$ may be defined. The most common of these are the bisimplicial categories.

Definition 2.1 (ii)

The category of *bisimplicial objects in* $\mathcal{C}$ is defined to be $\mathcal{C}^{(\Delta^{+}\times\Delta^{+})^{op}}$.

Categories of augmented bisimplicial objects will be left for the moment. Note that the opposite of a product of categories is the product of their opposites.

When the above functor categories take values in the category of Sets and functions, (denoted $\underline{Sets}$) then their names and notations are somewhat different.

Definition 2.1 (iii)

A *simplicial set* is a functor in $\underline{Sets}^{\Delta^+}$. The category of simplicial sets is usually denoted SS.

Similarly there are contractible simplicial sets, augmented simplicial sets and bisimplicial sets, and the respective categories are denoted CSS, ASS, and $BiSS$.

The category of augmented bisimplicial sets is a little more complex: the following three categories of contravariant, set-valued functors might all be described as augmented bisimplicial sets:

$$\underline{Sets}^{(\Delta\times\Delta^+)^{op}},\ \underline{Sets}^{(\Delta^+\times\Delta)^{op}} \text{ and } \underline{Sets}^{(\Delta\times\Delta)^{op}}.$$

This will be discussed in more detail in section 2.5 .

A simplicial set, X, has face and degeneracy maps based on the morphisms δ_i and σ_j respectively (which were defined in the first chapter): for $x \in X_n$, $d_i^n(x) = X(\delta_i^{n-1})(x)$ (in geometric terms, this is the face opposite the i^{th} vertex of x); similarly, $s_i^n(x) = X(\sigma_i^{n+1})(x)$ (in geometric terms, this is the degenerate simplex obtained by "doubling" the i^{th} vertex). Normally, the superscripts will be left off the face and degeneracy maps.

The simplicial set which is called the n-simplex, $\triangle[n]$, is the representable functor, $\Delta(-, [n])$. The simplicial set $\triangle[n]$ will be referred to as the *standard n-simplex.*

Definition 2.1 (iv)

A non-degenerate n-simplex of a simplicial set X is one which cannot be written as $s_i x$ for some $x \in X_{n-1}$. Note that this implies the definition of degenerate simplex (as one which can be written in the form $s_i x$ for some x).

A non-degenerate n-simplex will be called *maximal* if it cannot be written as $d_j y$ for any nondegenerate $n+1$-simplex $y \in X$. A simplicial set is said to be *generated* by a set of simplices $\{x_i\}_{i \in I}$ if each simplex of X may be expressed as $s_{i_1} \cdots s_{i_k} d_{j_1} \cdots d_{j_l} x_i$ for some $i \in I$, for some $k \geq 0$, $l \geq 0$.

In this event, X may be written as $< x_i : i \in I >$.

Definition 2.1 (v)

If X is a simplicial set, then the n-skeleton of X, $sk_n X$, is the simplicial set generated by the m-simplices of X, for all $m \leq n$.

It follows that if $X \cong sk_n X$ for some $n \in \mathbb{N}$, then X is generated by its maximal elements. Note that the standard n-simplex is generated by the unique non-degenerate n-simplex of $\triangle[n]$ which is the identity morphism $\iota_n : [n] \longrightarrow [n] \in \Delta$.

Definition 2.1 (vi)

A *simplicial complex* is a simplicial set, X, with the property that any nondegenerate simplex is completely defined by its vertices; that is, for a set of $n+1$ distinct vertices in X_0, there is at most one n-simplex with those vertices.

Definition 2.1 (vii)

Let $0 \leq k \leq n$. Consider $\{x_j \in X_{n-1}\, s.t.\, 0 \leq j \leq n,\, j \neq k\}$ where the x_j satisfy the property $d_i x_j = d_{j-1} x_i$ for $i < j$ and $i, j \neq k$.

If, for all n, for all $0 \leq k \leq n$ and for all such sets, there exists $x \in X_n$ such that $d_i x = x_i$, then X is called a *Kan Complex*.

If, for all n, for all $0 < k < n$ and for all such sets, there exists $x \in X_n$ such that $d_i x = x_i$, then X is called a *Weak Kan Complex*.

2.2 Nerves

Definition 2.2 (i)

Recall the definition of the *nerve* of a small category. Let $\mathbf{C}$ be a small category, and define a simplicial set $Ner\mathbf{C}$ as follows:-

$$(Ner\mathbf{C})_0 = ob(\mathbf{C})$$

$$(Ner\mathbf{C})_1 = arr(\mathbf{C});$$

$$(Ner\mathbf{C})_n = \left\{ \begin{array}{rr} (x_1, \cdots, x_n) \mid & x_i \in arr(\mathbf{C}),\, dom(x_{i+1}) = cod(x_i) \\ & 1 \leq i \leq (n-1) \end{array} \right\}$$

For $x \in (Ner\mathbf{C})_1$, $\quad d_1(x) = dom(x), \quad d_0(x) = cod(x)$

and for $y \in (Ner\mathbf{C})_0,$, $\quad s_0(y) = id_y$.

For $(x_1, \cdots, x_n) \in (Ner\mathbf{C})_n$,

$d_0(x_1, \cdots, x_n) = (x_2, \cdots, x_n);$

$d_n(x_1, \cdots, x_n) = (x_1, \cdots, x_{n-1});$

$d_i(x_1, x_2, \cdots, x_n) = (x_1, \cdots, x_i x_{i+1}, \cdots, x_n)$ if $0 < i < n$

$s_0(x_1, \cdots, x_n) = (id_{dom(x_1)}, x_1, \cdots, x_n);$

$s_n(x_1, \cdots, x_n) = (x_1, \cdots, x_n, id_{cod(x_n)})$

$s_i(x_1, \cdots, x_n) = (x_1, \cdots, x_i, id_{cod(x_i)}, x_{i+1}, \cdots, x_n)$ if $0 < i < n$.

This may be described more compactly by saying $(Ner\mathbf{C})_n = \underline{Cat}([n], \mathbf{C})$, and letting the natural functors between the objects of $\mathbf{N}^+$ describe the face and degeneracy morphisms. This construction extends to a functor from $\underline{Cat}$ to SS in the obvious way.

Proposition 2.2.1

$$\triangle[n] \cong Ner[n]$$

Proof

The Yoneda lemma states that $Ner\mathbf{C}_n \cong SS(\triangle[n], Ner\mathbf{C})$. Therefore

$$\underline{Cat}([n], \mathbf{C}) \cong SS(\triangle[n], Ner\mathbf{C})$$

If $\mathbf{C}$ is the category $[n]$, this gives

$$SS(\triangle[n], Ner[n]) \cong \underline{Cat}([n], [n])$$

Therefore the standard n-simplex, $\triangle[n]$, may be described as $Ner[n]$. ■

The functor Ner has a left adjoint Π: this is the process of "categorisation". The classical description of this is constructive, however the definition that will be used here is a coend, from which the constructive definition will be derived. The coend description of Π arises directly from it being the left adjoint to Ner.

As a notational convenience, $\int^{[n]}$ and $\int_{[n]}$ shall be written as $\int^{n}$ and $\int_{n}$ respectively. Similarly (when occasion arises) the $\int^{[p],[q]}$ and $\int_{[p],[q]}$ shall be written $\int^{p,q}$ and $\int_{p,q}$ respectively.

Proposition 2.2.2

$$\Pi X \cong \int^{n} X_n \cdot [n]$$

Proof

Let X be a simplicial set, and let $\mathbf{C}$ be a small category, then:-

$$\underline{Cat}(\Pi X, \mathbf{C}) \cong SS(X, Ner\mathbf{C}) \cong \int_n \underline{Sets}(X_n, (Ner\mathbf{C})_n)$$

$$\cong \int_n \underline{Sets}(X_n, \underline{Cat}([n], \mathbf{C})) \cong \int_n \underline{Cat}(X_n \cdot [n], \mathbf{C}) \cong \underline{Cat}(\int^n X_n \cdot [n], \mathbf{C})$$

As this is true for any small category $\mathbf{C}$, it follows that $\Pi X \cong \int^n X_n \cdot [n]$. ∎

Recall that $[n]$ is the category with the $(n+1)$ objects $\{0, 1, \cdots, n\}$ and a unique arrow for every $i \leq j$. A functor from $[n]$ to a small category $\mathbf{C}$ is then precisely a chain of n composible maps in $\mathbf{C}$. The δ_i morphisms in $\mathbf{N}$ induce composition of the chain across the codomain of the i^{th} morphism in the chain, and the σ_j morphisms induce insertion of an identity morphism between the j^{th} and $(j+1)^{th}$ morphisms in the chain.

Thus, constructively, the functor Π takes a simplicial set X and constructs the category which has objects X_0, arrows chains of 1-simplices, and relations induced by the δ and σ maps in $\mathbf{N}$: this is the free category on the graph $X_1 \; \begin{matrix} \xrightarrow{d_1} \\ \xleftarrow{s_0} \\ \xrightarrow{d_0} \end{matrix} \; X_0$ with relations induced by $d_2 x d_0 x \sim d_1 x$ for $x \in X_2$, $(d_2d_3x)(d_0d_3x)(d_0d_0x) \sim d_1d_2x$ for $x \in X_3$ and in general, $(d_2d_3 \cdots d_nx)(d_0d_3 \cdots d_nx) \cdots (d_0^j d_{j+2} \cdots d_nx) \cdots (d_0^{n-1}x) \sim d_1d_2 \cdots d_{n-1}x$, for $x \in X_n$. This higher order information is essentially associativity information.

Proposition 2.2.3

The relations given by the n-simplices, for $n \geq 3$, are obtainable from the relations given by the 2-simplices.

Proof

Let $x \in X_3$. Since $d_3x \in X_2$, there is a relation

$$(d_2d_3x)(d_0d_3x) \sim (d_1d_3x) = (d_2d_1x).$$

Since the relation must be preserved by composition, it follows that

$$\begin{aligned} &(d_2d_3)(d_0d_3x)(d_0d_0x) = (d_2d_3)(d_0d_3x)(d_0d_1x) \\ &\sim (d_2d_1x)(d_0d_1x) \sim (d_1d_1x) = (d_1d_2x). \end{aligned}$$

Thus the relations given by the 3-simplices are obtainable from the relations given by the 2-simplices.

Next, assume the relations given by the $(n-1)$-simplices are obtainable from the relations on the 2-simplices. Then, for $x \in X_n$, d_nx yields the relation $(d_2 d_3 \cdots d_{n-1} d_n x)(d_0 d_3 \cdots d_{n-1} d_n x) \cdots (d_0^{n-2} d_n x) \sim d_1 d_2 \cdots d_{n-2} d_n x$, which is obtainable from the relations on the 2-simplices.

So, $(d_2 d_3 \cdots d_n x) \cdot \cdot (d_0^j d_{j+2} \cdots d_n x) \cdot \cdot (d_0^{n-1} x) \sim (d_1 \cdots d_{n-2} d_n x)(d_0^{n-1} x)$. As $d_0^{n-1} = d_0 d_1 \cdots d_{n-2}$ and $d_1 d_2 \cdots d_{n-2} d_n = d_2 d_1 d_2 \cdots d_{n-2}$, then

$$
\begin{aligned}
(d_2 d_3 \cdots d_n x) \cdots (d_0^j d_{j+2} \cdots d_n x) \cdots (d_0^{n-1} x) \\
\sim (d_2 d_1 d_2 \cdots d_{n-2} x)(d_0 d_1 d_2 \cdots d_{n-2} x) \\
\sim d_1 d_1 \cdots d_{n-2} x = d_1 d_2 \cdots d_{n-1} x.
\end{aligned}
$$

Thus, by induction, all the relations are obtainable from the relations on the 2-simplices. ■

Definition 2.2 (ii)

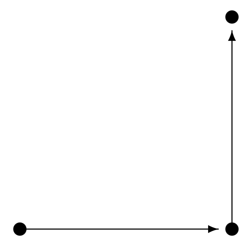
For any $0 \leq k \leq n$, $\wedge^k[n]$ is defined to be the subsimplicial set of $\triangle[n]$ generated by the simplices $d_i \iota_n$ for $0 \leq i \leq n$ and $i \neq k$. (Recall that ι_n is the unique nondegenerate n-simplex of $\triangle[n]$).

Proposition 2.2.4

$\Pi(\wedge^k[n]) \cong [n]$ *for* $0 < k < n$, *and if* $4 \leq n$, *for* $0 \leq k \leq n$.

Proof

Consider $\wedge^1[2]$. It has two non-degenerate 1-simplices, $d_0 i_2$ and $d_2 i_2$, and no non-degenerate 2-simplices. So $\Pi(\wedge^1[2])$ has no relations, and is the free category on the graph



Clearly, this is [2].

Further, note that $sk_{n-2} \wedge^k [n] \cong sk_{n-2}\triangle[n]$. Therefore, for $n \geq 4$, $sk_1 \wedge^k [n] \cong sk_1\triangle[n]$ and $sk_2 \wedge^k [n] \cong sk_2\triangle[n]$, and so $\Pi \wedge^k [n] \cong \Pi\triangle[n] \cong [n]$. So, consider the case $n = 3$.

Now, $sk_1 \wedge^k [3] \cong sk_1\triangle[3]$, so $\Pi(\wedge^k[3])$ is generated by the same elements as $\Pi(\triangle[3])$. As $|sk_2 \wedge^k [3]| = |sk_2\triangle[3]| - 1$, there is potentially one relator missing, namely $d_k i_3$. Had it been present, the 2-simplex $d_k i_3$ would have given the relation $(d_2 d_k i_3)(d_0 d_k i_3) = (d_1 d_k i_3)$. If $k = 1$ or 2, this may be obtained from the other relations as follows:

$k = 1$

$$(d_2 d_1 i_3)(d_0 d_1 i_3) = (d_1 d_3 i_3)(d_0 d_0 i_3)$$

$$= (d_2 d_3 i_3)(d_0 d_3 i_3)(d_0 d_0 i_3) = (d_2 d_2 i_3)(d_2 d_0 i_3)(d_0 d_0 i_3)$$

$$= (d_2 d_2 i_3)(d_1 d_0 i_3) = (d_2 d_2 i_3)(d_0 d_2 i_3)$$

$$= (d_1 d_2 i_3) = (d_1 d_1 i_3) \qquad \text{as required.}$$

$k = 2$

$$(d_2 d_2 i_3)(d_0 d_2 i_3) = (d_2 d_3 i_3)(d_1 d_0 i_3)$$

$$= (d_2 d_3 i_3)(d_2 d_0 i_3)(d_0 d_0 i_3) = (d_2 d_3 i_3)(d_0 d_3 i_3)(d_0 d_1 i_3)$$

$$= (d_1 d_3 i_3)(d_0 d_1 i_3) = (d_2 d_1 i_3)(d_0 d_1 i_3)$$

$$= (d_1 d_1 i_3) = (d_1 d_2 i_3) \qquad \text{as required.}$$

Thus, if $k = 1$ or 2, then $\Pi(\wedge^k[3]) \cong \Pi(\triangle[3]) \cong [3]$. ∎

Consider now the unit of the adjunction. Obviously the unit is the identity on 0-simplices, and the unit takes each 1-simplex to the element of $arr(\Pi X)$ which is the equivalence class containing it. In general, the unit takes $x \in X_n$ to $([d_2 \cdots d_n x], [d_0 d_3 \cdots d_n x], \cdots, [d_0^{n-2} x])$, where $[w]$ is the element of $arr(\Pi X)$ which is the equivalence class of $w \in X_1$.

Lemma 2.2.5 *Let X be a weak Kan complex, and consider $x_1, \cdots, x_n \in X_n$ where $d_1 x_i = d_0 x_{i-1}$ for $2 \leq i \leq n$. Then, there is a $z \in X_n$ with the property $d_0^{i-1} d_{i+1}^{n-i} z = x_i$ for $1 \leq i \leq n$.*

Proof

This is a simple case of lemma 5.3.3: the machinery is more easily dealt with in chapter 5 than here.

Proposition 2.2.6

If X is a weak Kan complex, the unit of the adjunction η_X is epic.

Proof

The category ΠX is a quotient of the free category on the graph which is the 1-skeleton of X. Thus, it is a quotient of the set

$$\{X_1 \sqcup (X_{1d_0} \times_{d_1} X_1) \sqcup (X_{1d_0} \times_{d_1} X_{1d_0} \times_{d_1} X_1) \cdots\},$$

(where $(X_{1d_0} \times_{d_1} X_1)$ is the set of words in X_1 of length two, xy, with $d_0x = d_1y$).

If X is a weak Kan complex, then any for any word $xy \in (X_{1d_0} \times_{d_1} X_1)$ there exists $z \in X_2$ such that $d_2z = x, d_0z = y$. But the relations give that $d_2zd_0z \sim d_1z$ and so $xy \sim d_1z$. Thus, by induction, the equivalence class of any finite chain $x_1x_2 \cdots x_n$ contains an element of X_1, so any arrow in ΠX has a representative which is a 1-simplex of X. Therefore $(\eta_X)_1$ is epic.

Given $(f, g) \in (Ner\Pi X)_2$ (so $f, g \in arr(\Pi X)$ and $domg = codf$) then there are $x, y \in X_1$, with $[x] = f$ and $[y] = g$, and $d_0x = d_1y$ and hence $z \in X_2$ with $d_2z = x$, $d_0z = y$. Thus $(f, g) = (\eta_X)_2(z)$.

Now, consider $(f_1, \cdots, f_n) \in (Ner\Pi X)_n$. For each f_i, there is an $x_i \in X_1$ with $[x_i] = f_i$, and thus $d_1x_i = d_0x_{i-1}$ (for $2 \leq i \leq n$). Lemma 2.2.5 states that given such a chain of simplices in a weak Kan complex, there is an element $z \in X_n$ with $d_0^{i-1}d_{i+1}^{n-i}z = x_i$ (for $1 \leq i \leq n$), and so $(f_1, \cdots, f_n) = (\eta_X)_n(z)$ and so $(\eta_X)_n$ is epic.

Since SS is a presheaf topos, the fact that each of the set morphisms $(\eta_X)_n$ is epic, implies that (η_X) is epic. ■

Theorem 2.2.7

If X is a Kan complex, then ΠX is a groupoid, all fillers in $Ner\Pi X$ are unique, and η_X is a Kan fibration.

Proof

If X is a Kan complex, then X is a weak Kan complex, and so by Proposition 2.2.6 η_X is epic. Therefore, every arrow of ΠX is representable by a 1-simplex of X. Consider an arrow $f = [x] \in arr(\Pi X)$. Since $d_1 x = d_1(s_0 d_1 x)$, x and $s_0 d_1 x$ form a 0-horn in X_1, with $s_0 d_1 x$ thought of as the 1-face. Then, there exists $y \in X_2$ with $d_1 y = s_0 d_1 x$ and $d_2 y = x$. Then in ΠX, $[x][d_0 y] = [s_0 d_1 y]$. Similarly, $d_0 x = d_0 s_0 d_0 x$, and so there exists $z \in X_2$, with $d_1 z = s_0 d_0 x$ and $d_0 z = x$, and so in ΠX, $[d_2 z][x] = [s_0 d_0 y]$. Since $[s_0 p]$ is an identity, for any $p \in X_0$, it follows that $[d_2 z] = [d_2 z][x][d_0 y] = [d_0 y]$, and this is a two sided inverse for $f = [x]$. It is therefore unique. Thus, every arrow of ΠX has a unique inverse, and so ΠX is a groupoid.

Therefore, $Ner\Pi X$ is a Kan complex, since ΠX is a groupoid, and the nerve of a small category is a Kan complex if and only if it is a groupoid.

Let $([a_1], [a_2], \cdots, [a_n])$ and $([a'_1], [a'_2], \cdots, [a'_n])$ be two distinct fillers for $\{\, \eta_X(x_i)\,\}_{i \neq k}$ in $(Ner\Pi X)_n$, where $\{\, \eta_X(x_i)\,\}_{i \neq k}$ is a k-horn in $(Ner\Pi X)_n$. Let If $0 < k < n$, then

$$([a_2], \cdots, [a_n]) = ([a'_2], \cdots, [a'_n]) = \eta_X(x_0)$$

and
$$([a_1], \cdots, [a_{n-1}]) = ([a'_1], \cdots, [a'_{n-1}]) = \eta_X(x_n),$$

and so $([a_1], \cdots, [a_n]) = ([a'_1], \cdots, [a'_n])$ and the two fillers are equal.
If $k = n$, then $k \neq 0$ and so $([a_2], \cdots, [a_n]) = ([a'_2], \cdots, [a'_n]) = \eta_X(x_0)$. Further, $k \neq 1$ and so $[a_1][a_2] = [a'_1][a'_2]$ (from $\eta_X(x_1)$). Therefore, since $[a_2] = [a'_2]$, and all elements have inverses, $[a_1] = [a'_1]$ and so the two fillers are (again) equal. A similar argument applies if $k = 0$.

Since X is a Kan complex, any k-horn in X_n has a filler. Further, the image of the k-horn in $(Ner\Pi X)_n$ has a unique filler which must be the image of any filler in X_n. Therefore for any commuting diagram of the form

$$\begin{array}{ccc} \wedge^k[n] & \xrightarrow{f} & X \\ i \downarrow & & \eta_X \downarrow \\ \triangle[n] & \xrightarrow{g} & Ner\Pi X \end{array}$$

the arrow g is uniquely defined, and so for any extension of f, $\overline{f}$ say, $\eta_X \overline{f} = g$ by the uniqueness property. Thus, η_X is a Kan fibration.

Proposition 2.2.8

If X is a simplicial complex, the unit of the adjunction η_X is monic.

Proof

The unit of the adjunction is the identity on the 0-simplices of X. Consider $z, z' \in X_n$ with $\eta_n(z) = \eta_n(z')$. Then the vertices of z and z' are the same, and since X is a simplicial complex, this implies that $z = z'$. ∎

Since Ner is a right adjoint, it preserves all small limits (specifically, products) and so $Ner\mathbf{C} \times Ner\mathbf{D} \cong Ner(\mathbf{C} \times \mathbf{D})$.

Theorem 2.2.9

The composite ΠNer is the identity functor on <u>Cat</u>, $\epsilon_{\mathbf{C}} : \Pi Ner\mathbf{C} \longrightarrow \mathbf{C}$ is the identity natural transformation (where ϵ is the counit of the adjunction), $\eta_{Ner\mathbf{C}} : Ner\mathbf{C} \longrightarrow Ner\Pi Ner\mathbf{C}$ is the identity, and $\Pi(\eta_X) = id_{\Pi X}$, (where η is the unit of the adjunction).

Proof

Consider a small category $\mathbf{C}$. $Ner\mathbf{C}$ is the simplicial set with 0-simplices the objects of $\mathbf{C}$ and 1-simplices the arrows of $\mathbf{C}$. The higher order simplices are chains of composible arrows. The category $\Pi Ner\mathbf{C}$ is then a quotient of the free category on the graph $U\mathbf{C}$ (where $U : \underline{Cat} \longrightarrow \underline{Graphs}$ is the forgetful functor). The relations which specify the quotient are precisely the composition information of the category $\mathbf{C}$. Therefore, the functor, ΠNer is the identity on $\underline{Cat}$.

The counit of the adjunction takes an equivalence class in $\Pi Ner\mathbf{C}$ and maps it to the composite of all its representative elements, and since the composite is also a representative element, the counit is the identity transformation.

Since the counit and unit satisfy the equation $Ner(\epsilon_{(-)})\eta_{Ner(-)} = id_{Ner(-)}$, it follows that $\eta_{Ner\mathbf{C}} : Ner\mathbf{C} \longrightarrow Ner\Pi Ner\mathbf{C}$ is also the identity.

Lastly, since $(\epsilon_{\Pi X})\Pi(\eta_X) = id_{\Pi X}$, and $\epsilon_{\Pi X} = id_{\Pi X}$, then $\Pi(\eta_X) = id_{\Pi X}$. ■

It is immediate that Π is full and Ner is faithful.

Theorem 2.2.10

The category $\underline{Cat}$ is monadic over SS.

Proof

Let $SS^{Ner\Pi}$ be the category of $Ner\Pi$-algebras. Beck's monadicity theorem states that the unique comparison functor, $K : \underline{Cat} \longrightarrow SS^{Ner\Pi}$ is an isomorphism of categories if and only if the functor Ner creates coequalisers for those parallel pairs of arrows f, g for which $Nerf$, $Nerg$ have an absolute coequaliser in SS.

Let $f, g : \mathbf{C} \to \mathbf{D}$ and let $h : Ner\mathbf{D} \to X$ be an absolute coequaliser for $Nerf$, $Nerg$. Thus $\Pi(h)$ is a coequaliser for the pair $\Pi Ner(f)$, $\Pi Ner(g)$. But, since ΠNer is the identity functor, Πh is the coequaliser for f, g and so K is an isomorphism of categories. ■

Corollary 2.2.11

Every $Ner\Pi$-algebra has the identity as structure map.

Proof

The comparison functor $K : \underline{Cat} \longrightarrow SS^{Ner\Pi}$ is defined on objects as $K(\mathbf{C}) =< Ner\mathbf{C}, Ner\epsilon_{\mathbf{C}} >$.

Proposition 2.2.12

Let $\mathbf{C}$ be a small category and X a simplicial set. For any simplicial morphism $f : X \longrightarrow Ner\mathbf{C}$, there exists uniquely $\overline{f} : Ner\Pi X \longrightarrow Ner\mathbf{C}$ such that $\overline{f}\eta_X = f$.

Proof

From theorem 2.2.9, $\epsilon_{\mathbf{C}} = id_{\mathbf{C}}$, $\eta_{Ner\mathbf{C}} = id_{Ner\mathbf{C}}$ and $\Pi(\eta_X) = id_{\Pi X}$. Let $\phi_{X,\mathbf{C}} : \underline{Cat}(\Pi X, \mathbf{C}) \longrightarrow SS(X, Ner\mathbf{C})$ be the bijection (natural in X and $\mathbf{C}$) of the adjunction $\Pi \dashv Ner$. Let $\overline{f} : Ner\Pi X \longrightarrow Ner\mathbf{C}$ be such that $\overline{f}\eta_X = f$. First,

$$\Pi(\overline{f}) = \Pi(\overline{f})\Pi(\eta_X) = \Pi(\overline{f}\eta_X)) = \Pi(f).$$

Then

$$\overline{f} = \phi(\phi^{-1}(\overline{f})) = \phi(\epsilon_{\mathbf{C}}\Pi(\overline{f})) = \phi(\Pi(\overline{f}))$$

$$= Ner\Pi(\overline{f})\eta_{Ner\Pi X} = Ner\Pi(\overline{f}) = Ner\Pi(f).$$

Thus, $\overline{f} = Ner\Pi(f)$ and so exists uniquely as claimed. ■

Corollary 2.2.13

Consider a k-horn in $(Ner\mathbf{C})_n$. *It has a unique filler if either* $2 \leq n \leq 3$ *and* $0 < k < n$ *or* $n \geq 4$ *and* $0 \leq k \leq n$.

Proof

Consider $X = \wedge^k[n]$. Then proposition 2.2.4 proves that under the conditions of the corollary $Ner\Pi(\wedge^k[n]) \cong \triangle[n]$.

There is an important caveat about $Ner\Pi$ as a functor: $Ner\Pi$ does not necessarily preserve homotopy type, and that specifically, for a simplicial set, X, with non-trivial homotopy groups, it is possible for $Ner\Pi X$ to be contractible, that is to have all homotopy groups trivial. However, if X is contractible, then the simplicial sets X and $Ner\Pi X$ will have the same homotopy type.

No proof of these comments will be given here: for a study of the category of small categories as a homotopy category, and for homotopy inverses for Ner, the reader is referred to the work of Fritsch, Latch, Quillen, Segal, Thomason (see [23], [32], [40], [41], [42].)

2.3 Simplicial Groupoids

This section is a brief resumé of some of the results of the author's MSc Thesis (see [20]). The work relies heavily on a variety of sources, chiefly [19], [18], and [31]; reader's are referred to [20] for full acknowledgements. The results are quoted without proof.

Definition 2.3 (i)

The category of *simplicial groupoids*, $\underline{SGpds}$, is the category of simplicial objects in the category of groupoids.

Definition 2.3 (ii)

The category of *simplicially enriched groupoids* or *simplicial groupoids with a constant object of objects*, $\underline{SGpds_*}$, is the full subcategory of $\underline{SGpds}$ whose objects are the simplicial groupoids with a constant object of objects (that is, $ob(G_n)$ is the same for all n, and the simplicial face and degeneracy maps are the identity on the objects).

Definition 2.3 (iii)

The *loop groupoid functor* is a functor $G : SS \longrightarrow \underline{SGpds_*}$ which takes the simplicial set X to the simplicially enriched groupoid GX where $(GX)_n$ is the free groupoid on the graph $X_{n+1} \overset{s,t}{\longrightarrow} X_0$ where $s = (d_1)^{n+1}$ and $t = d_0(d_2)^n$, with relations $s_0x = id$ for $x \in X_n$. The degeneracy maps (usually denoted σ) are given on the generators by $\sigma_i(x) = s_{i+1}(x)$ for $x \in X_{n+1}$. The face maps (usually denoted δ) are given on the generators by $\delta_i(x) = d_{i+1}$ for $x \in X_{n+1}$ and $\delta_0(x) = (d_1x)(d_0x)^{-1}$. It is clear that this is indeed a simplicially enriched groupoid, and that the groupoids at each level are free.

Definition 2.3 (iv)

The *classifying space functor* $\overline{W} : \underline{SGpds_*} \longrightarrow SS$ takes a simplicially enriched groupoid H to the simplicial set described by:

$(\overline{W}H)_0 = ob(H_0)$, $\quad (\overline{W}H)_1 = arr(H_0)$ and for $n \geq 2$

$$(\overline{W}H)_n = \left\{ \begin{array}{ll} (h_{n-1}, \cdots, h_0) & | \, h_i \in arr(H_i) \\ & \text{and } dom(h_{i-1}) = cod(h_i),\ 0 < i < n \end{array} \right\}.$$

The face and degeneracy maps between $(\overline{W}H)_1$ and $(\overline{W}H)_0$ are the source and target maps and identity maps of H_0, and the face and degeneracy maps at higher levels are given as follows (where δ and σ denote face and degeneracy maps in H):-

$d_0(h_{n-1}, \cdots, h_0) = (h_{n-2}, \cdots, h_0)$, $\quad d_n(h_{n-1}, \cdots, h_0) = (\delta_{n-1}h_{n-1}, \cdots, \delta_1 h_1)$,

and for $0 < i < n$,

$d_i(h_{n-1}, \cdots, h_0) = (\delta_{i-1}h_{n-1}, \delta_{i-2}h_{n-2}, \cdots, \delta_0 h_{n-i}h_{n-i-1}, h_{n-i-2}, \cdots, h_0)$

and

$s_0(h_{n-1}, \cdots, h_0) = (id_{dom(h_{n-1})}, h_{n-1}, \cdots, h_0)$, and for $n \geq i > 0$,

$s_i(h_{n-1}, \cdots, h_0) = (\sigma_{i-1}h_{n-1}, \cdots, \sigma_0 h_{n-i}, id_{cod(h_{n-i})}, h_{n-i-1}, \cdots, h_0)$.

Definition 2.3 (v)

The *Moore complex* of a simplicial groupoid, H, is the chain complex of groupoids (NH, ∂) where $(NH)_0 := H_0$, $(NH)_n := \cap_{i=1}^{n} Ker\delta_i$ for $n > 0$ and $\partial_n : (NH)_n \longrightarrow (NH)_{n-1}$ is the restriction of δ_0 to the subgroupoid $(NH)_n$. If H is a simplicially enriched groupoid, then $(NH)_n$ is a totally disconnected wide subgroupoid of H_n for $n > 0$.

Definition 2.3 (vi)

The functor $DEC : SS \longrightarrow BiSS$ takes a functor $X \in \underline{Sets}^{\Delta^{+op}}$ to the composite functor $Xor \in \underline{Sets^{\Delta^+ \times \Delta^+}}$. It is dealt with in more detail in section 2.7.

Proposition 2.3.1

The loop groupoid functor is left adjoint to the classifying space functor; the unit and counit are given as follows:

For X a simplicial set, and for $x \in X_n$

$$\eta_X(x) = (\overline{x}, \overline{d_0 x}, \cdots, \overline{d_0^{n-1} x})$$

where $\overline{x}$ is x considered as an element of $(GX)_{n-1}$.

For H a simplicially enriched groupoid, and for $\overline{(h_n, \cdots, h_0)} \in (G\overline{W}H)_n$

$$\epsilon_H(\overline{(h_n, \cdots, h_0)}) = h_n$$

Proposition 2.3.2

A simplicial groupoid is a Kan complex.

Given a simplicially enriched groupoid, H, $\overline{W}H$ is a Kan complex.

Proposition 2.3.3

The classifying space functor from simplicially enriched groupoids to simplicial sets may be expressed as the composite

$$\overline{W} = \nabla NER$$

where $NER : \underline{SGpds} \longrightarrow BiSS$ takes a simplicial groupoid to the bisimplicial set whose n^{th} row is $Ner(G_n)$, and where $\nabla : BiSS \longrightarrow SS$ is the right adjoint to the functor DEC.

Proposition 2.3.4

The n^{th} homology groupoid of the Moore complex of the loop groupoid of a simplicial set gives the $(n+1)^{th}$ homotopy groupoid relative to the vertices: that is,

$$H_n(NGX) \cong \pi_{n+1}(X_*, X_0)$$

Specifically, the fundamental groupoid of X relative to the vertices X_0 is given by $\pi_1(X_*, X_0) = (GX)_0/\delta_0(Ker\delta_1)$.

Here $H_n(NGX)$ is the n^{th} homology group of the nonabelian chain complex of groupoids, NGX; NGX is the Moore complex of the loop groupoid of X, and $\pi_n(X_*, X_0)$ is the n^{th} homotopy groupoid of X_* (where homotopy is "rel the vertices, X_0").

Proposition 2.3.5

Given a simplicially enriched groupoid, H a crossed complex, CH , may be defined in the following way:

$$(CH)_n := (NH)_n \,/\, ((NH)_n \cap D_n)\partial((NH)_{n+1} \cap D_{n+1})$$

where D_n is the subgroupoid of H_n generated by the degenerate elements. The boundary maps are induced by the chain maps of (NH, ∂).

2.4 Conjugation

Mention has been made of the set function reversing the order of the finite ordinals in Δ^+. The effect of this on simplicial sets is "Conjugation".

Definition 2.4 (i)

Given a simplicial set, X, $ConjX$ (the conjugate of X) is defined by

$$(ConjX)_n = X_n,$$

$$d_i^n : (ConjX)_n \to (ConjX)_{n-1} = d_{n-i}^n : X_n \to X_{n-1} \text{ and}$$

$$s_i^n : (ConjX)_n \to (ConjX)_{n-1} = s_{n-i}^n : X_n \to X_{n-1}.$$

To check this is well defined is simple (if somewhat laborious): it is also clear that $(Conj)^2 X = X$. However, there is in general no isomorphism $X \to ConjX$. In fact, in general, the only morphisms between a simplicial set and its conjugate are the trivial morphisms taking X to some point in $ConjX$ (where a point is the subsimplicial set generated by a single vertex).

Clearly the two simplicial sets are geometrically equivalent in some sense, just as the category Δ^+ with the order inverted is essentially the same category as Δ^+; however, there is no functor between them to reflect this fact.

2.5 Augmentations

It has already been mentioned that it is desirable to be able to move between Δ and Δ^+ and so between augmented simplicial categories and simplicial categories. While it is easy to see that in the model categories there is little difficulty, more care should be taken with functor categories.

For the purposes of this thesis, an *augmentation* of a simplicial set X, will be defined as a morphism q_X with domain X_0, which is a weak coequaliser for the pair $d_0, d_1 : X_1 \longrightarrow X_0$. The codomain of the augmentation will be denoted by X_{-1}. It is obvious that a simplicial set, together with an augmentation and the codomain of the augmentation, form an augmented simplicial set, that is a contravariant set valued functor from the category Δ.

Any simplicial set, X, has two natural augmentations. These arise as left and right adjoints to the forgetful functor from augmented simplicial sets to simplicial sets. The forgetful functor, U is the composition of the augmented simplicial set with the inclusion functor $\Delta^+ \rightarrow \Delta$ (mentioned earlier).

Proposition 2.5.1

The left adjoint to the forgetful functor is obtained by augmenting a simplicial set, X, by the coequaliser of $d_0, d_1 : X_1 \longrightarrow X_0$. The codomain of the augmentation is called $\pi_0 X$.

The right adjoint to the forgetful functor is obtained by augmenting a simplicial set, X, by the unique function with domain X_0 and codomain $*$, the one point set.

Proof

Let X be a simplicial set and Y be an augmented simplicial set, and denote the left and right adjoints to U by L_U and R_U respectively.

An augmented simplicial set, Y, is a cocone under the diagram UY. In other words, any augmentation factors uniquely through the colimit of the diagram UY, which is $Y_0 \longrightarrow \pi_0 UY$. So the left adjoint to U must be augmentation by the colimit of the diagram X, with $(L_U X)_{-1} \cong \pi_0 X$. This is indeed the case: given a morphism $f_* : X \to UY$, then

$$q_Y f_0 d_0^X = q_Y d_0^Y f_1 = q_Y d_1^Y f_1 = q_Y f_0 d_1^X.$$

Since $\pi_0 X$ is the domain of the coequaliser for $X_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} X_0$ there is a unique arrow $f_{-1} : X_{-1} \to Y_{-1}$ with $f_{-1} q_X = q_Y f_0$ as required. This augmentation will be referred to as the canonical augmentation.

In the case of the right adjoint, the only possible augmentation is the unique map from X_0 to the one point set, so that $X_{-1} = *$. Clearly, given $f_* : UY \to X$, there is a unique arrow $f_{-1} : Y_{-1} \to X_{-1}$ since X_{-1} is terminal in $\underline{Sets}$; this unique arrow extends $f_* : UY \to X$ to a morphism in ASS. This augmentation will be referred to as the trivial augmentation. ∎

The set $\pi_0 X$ (which was implicitly defined by the last proposition) is the set of path connected components, and it is well known that a simplicial set may be written as the disjoint union of its path connected components. Further, since any augmentation is a weak coequaliser of d_0^1 and d_1^1, any two 0-simplices which map down to different elements of X_{-1} must be in distinct path components. So, any augmentation must be a partition of the set of path connected components. Thus any augmented simplicial set may be considered as the disjoint union of a set of trivially augmented simplicial sets, $\{X_x \,|\, x \in X_{-1}\}$, where X_x is the disjoint union of those path connected subsimplicial sets of X which are augmented over x. Clearly, each X_x has the trivial augmentation.

The trivial augmentation does not have so clear a geometric description as the canonical augmentation. It may be thought of as a label for the space, and as will be seen later, the singular complex functor from topological spaces to *augmented* simplicial sets has the trivial augmentation. The algebraic effect of the trivial augmentation is to force and indeed enable the simplicial set to be dealt with as a single entity, rather than being split into path connected components. This comment will be made clearer towards the end of chapter 3.

A bisimplicial set, Y, may be augmented horizontally (so that it is extended from a functor $Y \in \underline{Sets}^{(\Delta^+ \times \Delta^+)^{op}}$ to a functor $Y_h \in \underline{Sets}^{(\Delta^+ \times \Delta)^{op}}$, or it may be augmented vertically (so that it is extended to a functor $Y_v \in \underline{Sets}^{(\Delta \times \Delta^+)^{op}}$).

As in the simplicial case, there are two natural augmentations, the canonical and the trivial, which are right and left adjoint to the forgetful functors. The forgetful functors are induced by the embeddings:

$$(\Delta^+ \times \Delta^+) \longrightarrow (\Delta \times \Delta^+)$$
$$(\Delta^+ \times \Delta^+) \longrightarrow (\Delta^+ \times \Delta)$$

In the horizontal case, the augmentations are obtained by augmenting each row in turn with the respective augmentation, and in the vertical case they are obtained by augmenting the columns in turn. The nature of the two simplicial adjoints ensures that the codomain of the augmentation is (in all cases) a simplicial set, and that it is left or right adjoint to the respective forgetful functor.

Once a bisimplicial set has been augmented, it may be further augmented. The codomain of the augmentation will then be an augmented simplicial set. Again, the canonical and trivial augmentations are defined by right and left adjoints to the forgetful functors, which are in turn induced by the embeddings:

$$(\Delta \times \Delta^{+}) \longrightarrow (\Delta \times \Delta)$$
$$(\Delta^{+} \times \Delta) \longrightarrow (\Delta \times \Delta)$$

Given a bisimplicial set Y it is possible therefore to augment both horizontally and vertically. The set $Y_{-1,-1}$ which results will be the trivial set in all cases except when both augmentations are canonical, when the set will be $\pi_0^h \pi_0^v Y_{*,*}$. This is clearly isomorphic to $\pi_0^h \pi_0^v Y_{*,*}$ since the horizontal and vertical morphisms of a bisimplicial set commute. Generally, a functor in the category $\underline{Sets}^{(\Delta\times\Delta)^{op}}$ will be referred to as a bi-augmented bisimplicial set.

2.6 Topology

Definition 2.6 (i)

The singular complex is defined for a topological space $\mathcal{U}$ by

$$(Sing\mathcal{U})_n = \mathcal{T}op(\triangle^n, \mathcal{U})$$

where the simplicial structure comes from the structure maps on the affine simplices (see section 1.1).

Note that for $X \in ob(\underline{Sets})$ and $Y \in ob(\mathcal{C})$, for some category $\mathcal{C}$, the X-indexed copower of Y shall be denoted by $X \cdot Y$ (if it exists). If $\mathcal{C} = \underline{Sets}$ then $X \cdot Y \cong X \times Y$.

Geometric realisation is the left adjoint to the singular complex functor. It is written $|-| : SS \longrightarrow \mathcal{T}op$. Thus for any simplicial set, X, and for all topological spaces, $\mathcal{U}$, $\mathcal{T}op(|X|, \mathcal{U}) \cong SS(X, Sing\mathcal{U})$.

Proposition 2.6.1

$$|X| \cong \int^n X_n \cdot \triangle^n$$

Proof

Let X be a simplicial set, and $\mathcal{U}$ be a topological space.

$$\mathcal{T}op(|X|, \mathcal{U}) \cong SS(X, Sing\mathcal{U})$$

$$\cong \int_n \underline{Sets}(X_n, (Sing\mathcal{U})_n) \cong \int_n \underline{Sets}(X_n, \mathcal{T}op(\triangle^n, \mathcal{U}))$$

$$\int_n \mathcal{T}op(X_n \cdot \triangle^n, \mathcal{U}) \cong \mathcal{T}op(\int^n X_n \cdot \triangle^n, \mathcal{U})$$

Therefore geometric realisation has the coend description

$$|X| = \int^n X_n \cdot \triangle^n$$

■

There are a number of different constructive definitions of geometric realisation. The process is essentially the following: take one copy of $\triangle^n$ for each non-degenerate n-simplex of X and glue them all together using the face and degeneracy maps of the simplicial set X (see [35]). The following constructive definition is from Curtis ([16]).

Definition 2.6 (ii)

Let X be a simplicial set. Define RX by:

$$RX = \sqcup_{n \in \mathbb{N}} \sqcup_{x \in X_n} \triangle^n_x$$

Define an equivalence relation on RX as generated by the following relation: writing $(\mathbf{p}, x)$ for $(p_0, \cdots, p_m) \in \triangle^m_x$ and $(\mathbf{q}, y)$ for $(q_0, \cdots, q_n) \in \triangle^n_y$ then $(\mathbf{p}, x) \sim (\mathbf{q}, y)$ if either

$$d_i x = y \text{ and } \delta_i(q_0, \cdots, q_n) = (p_0, \cdots p_m) \text{ or}$$
$$s_i x = y \text{ and } \sigma_i(q_0, \cdots, q_n) = (p_0, \cdots p_m).$$

Then $|X| \cong RX/\sim$ where $RX/\sim$ has the identification topology. No proof of this claim will be given.

In section 1.1, it was noted that the set of affine simplices $\{\triangle^n \,|\, n \in \mathbb{N}\}$ could be extended to include the empty set $\triangle^{-1}$. The singular complex of a topological space, U, may then be augmented where the codomain of the augmentation is $Sing\mathcal{U}_{-1} = \mathcal{T}op(\triangle^{-1}, \mathcal{U})$. Since the empty set is initial in the category of sets, there is a unique function from the empty set as a topological space to any topological space. Thus, $Sing\mathcal{U}_{-1} = \{*\}$, so $Sing$ as a functor to augmented simplicial sets has the trivial augmentation.

Thus, $Sing$ to augmented simplicial sets is equivalent to taking $Sing$ to simplicial sets and augmenting trivially. The geometric realisation functor (as the left adjoint to this $Sing$ functor) is equivalent to taking the forgetful functor on augmented simplicial sets, composed with the geometric realisation on simplicial sets.

This comes out of the coend formulation, since for any X_{-1}, $X_{-1} \cdot \triangle^{-1} = \triangle^{-1}$, that is, the empty set. Thus $\int^n X_n \cdot \triangle^n$ (where $n \in \mathbb{N} \cup \{-1\}$) is precisely $|UX|$ (i.e. the same coend taken over $n \in \mathbb{N}$).

2.7 Dec and Total Dec

This section is based on work of Duskin and Van Osdol (see [18]), and on work in the author's MSc Dissertation (see [20]). The adjunction $b \dashv in$ described in subsection 1.2.2 gives rise to an adjunction between the two functor categories $\underline{Sets}^{\Delta^{op}}$ (which is ASS) and $\underline{Sets}^{\Delta_0^{op}}$ (which is CSS).

The functor $in^* : ASS \longrightarrow CSS$ (which is obtained by composing $X : \Delta^{op} \longrightarrow \underline{Sets}$ with in) takes an augmented simplicial set and strips away the d_0 morphisms from each level, and "forgets" the augmentation, q_X, and its codomain, X_{-1}.

The functor $b^* : CSS \longrightarrow ASS$ (obtained by composing $Y : \Delta_0^{op} \longrightarrow \underline{Sets}$ with b) takes a contractible simplicial set, strips away the s_0 morphisms from each level.

Since $b \dashv in$, it follows automatically that $b^* \dashv in^*$.

If the category Δ is replaced by Δ^+, there is a similar adjunction: $in^* : SS \longrightarrow CSS$ is again obtained by composing the contravariant functor X with $in : \Delta_0 \longrightarrow \Delta^+$, but the effect is simply to strip away the d_0 morphisms at each level, as there is no augmentation to throw away. Similarly, b^* is obtained by composing the contravariant functor $Y \in obCSS$ with b; the effect is to strip away the s_0 morphisms from each level, and to discard X_0 and the morphism $d_1 : X_1 \longrightarrow X_0$.

The category SS is monadic over CSS via the functor in^*: the triple is $\mathbf{T} = (T, \eta, \mu)$ where $T = in^* b^*$, the unit $\eta_Y : Y \longrightarrow in^* b^* Y$ is formed by the s_0 at every level and the multiplication $\mu_Y : T^2 Y \longrightarrow TY$ is formed by d_1 at each level (for Y a contractible simplicial set). The T-algebras are then precisely the simplicial sets, since the conditions that a map be a T-algebra structure map are fulfilled precisely by maps consisting of d_0 at each level, and the conditions for T-algebra morphisms are satisfied precisely by simplicial set morphisms (see [18]).

Returning to augmented simplicial sets, the composite $b^* in^*$ is often called Dec ; the definitions imply that for an augmented simplicial set X, $(DecX)_n = X_{n+1}$, $d_i : (DecX)_n \to (DecX)_{n-1} = d_{i+1} : X_{n+1} \to X_n$ and $s_i : (DecX)_n \to (DecX)_{n+1} = s_{i+1} : X_{n+2} \to X_{n+1}$; further, $q_{DecX} = d_1$. There is a comonad structure on ASS, defined by Dec. This has counit d_0 at each level (with $q_x : DecX_{-1} \to X_{-1}$) , and comultiplication s_0 at each level.

Given an augmented simplicial set, X, not only is $DecX_{-1} = X_0$, but in fact $X_0 \cong \pi_0 DecX$ and the quotient map is $d_1 : X_1 \longrightarrow X_0$ (see [20] for proof). It is clear that this is a split epic (the right inverse provided by s_0).

The monad structure on Δ (described earlier) lifts to the comonad structure on ASS defined by Dec. However, the algebras do not necessarily lift to coalgebras; specifically, $\sigma_0 : [1] \to [0]$ is an algebra for inb, but $s_0^* : X \to DecX$ is not a coalgebra for Dec. In fact an algebra $f : [n] \to [n-1]$ will only lift to a coalgebra if f is the n^{th} component of a natural transformation from $inb \to Id$. This means that the algebras $\sigma_0 : (inb)^2[n] \to inb[n]$ go to form a coalgebra $s_0^* : DecX \to (Dec)^2 X$.

The cotriple resolution of the comonad on SS formed by Dec is the bisimplicial array, called Total Dec (and denoted DEC). This was defined by Illusie (see [29]). This array has X_{p+q+1} in the $(p,q)^{th}$ position, the horizontal face and degeneracies are $d_0, \cdots, d_p$ and $s_0, \cdots, s_p$ and the vertical face and degeneracies are $d_{p+1}, \cdots, d_{p+q+1}$ and $s_{p+1} \cdots, s_{p+q+1}$. Thus the array has two natural augmentations; one of the augmentations is made up of the d_0 morphisms at each level, the other of d_{n+1} at the n^{th} level; the codomains are the same, namely the simplicial set X. It is clear from the description that DEC is a functor from simplicial sets to bisimplicial sets. Therefore DEC is well defined on ASS (the category of augmented simplicial sets) as a functor with values in $BiASS$ (the category of bi-augmented bisimplicial sets).

Given the description of b earlier, it will come as no surprise that the ordinal sum, which is essentially a functor $\Delta \times \Delta \longrightarrow \Delta$ gives rise to the functor $DEC : ASS \longrightarrow BiASS$. Observe that the p^{th} column of the array $DECX$ is the simplicial set $X([p]or[-])$ with face and degeneracies defined by $X(id_{[p]}or\delta_i)$ and $X(id_{[p]}or\sigma_i)$, respectively. The q^{th} row is (similarly) the simplicial set $X([-]or[q])$ with face and degeneracies defined by $X(\delta_i orid_{[q]})$ and $X(\sigma_i orid_{[q]})$, respectively. Recall the comments made in section 1.1, that $R([p]or[q]) = R([q])orR([p])$. This means that the q^{th} row of $DECX$ may be described as $Conj(Dec^{q+1}(ConjX))$.

Thus $DECX = Xor$ (for X either a simplicial set or an augmented simplicial set).

2.8 The Diagonal on *ASS*

The functor ∂ (considered as a functor on either Δ^{+op} or Δ^{op}) may be composed with any functor $Y \in \underline{Sets}^{(\Delta^+\times\Delta^+)^{op}}$, (resulting in a functor $Y\partial \in \underline{Sets}^{\Delta^{+op}}$). Composition with ∂ is, then, a functor $diag$:$BiSS \longrightarrow SS$. Given a bisimplicial set Y, $diagY$ is the simplicial set given by:

$$(\text{diag}Y)_n = Y_{n,n}$$
$$d_i : (\text{diag}Y)_n \longrightarrow (\text{diag}Y)_{n-1} \;=\; d_i^v d_i^h : Y_{n,n} \longrightarrow Y_{n-1,n-1}$$
$$s_j : (\text{diag}Y)_n \longrightarrow (\text{diag}Y)_{n+1} \;=\; s_j^v s_j^h : Y_{n,n} \longrightarrow Y_{n+1,n+1}$$

where d_i^h, s_j^h are the horizontal face and degeneracy maps, and d_i^v, s_j^v are the vertical face and degeneracy maps of Y. Clearly it does not matter whether the horizontal or the vertical map is taken first, as the horizontal and vertical maps of a bisimplicial set commute.

2.9 Right and Left Adjoints

It was noted (in section 1.1) that the category Δ^{n+1} was small, and it is well known that the category of sets is complete and cocomplete. Thus, by the theory of Kan extensions, the functors *Dec*, *DEC* and *diag* have left and right adjoints. These may be written in terms of ends and coends.

Since this method of obtaining adjoints will be used frequently, the relevant equations are quoted here (the proofs are in [35]).

For categories $\mathcal{C}, \mathcal{M}$ and $\mathcal{A}$ and functors $T : \mathcal{M} \longrightarrow \mathcal{A}$ and $K : \mathcal{M} \longrightarrow \mathcal{C}$, if T has a left Kan extension along K, then it is given on objects by:-

$$(Lan_K T)(c) \cong \int^m \mathcal{C}(Km, c) \cdot Tm$$

and if T has a right Kan extension along K, then it is given on objects by:-

$$(Ran_K T)(c) \cong \int_m Tm^{\mathcal{C}(c,Km)}$$

It is important to remember that simplicial (and bisimplicial) objects are contravariant functor categories on Δ^+ (or $\Delta^+ \times \Delta^+$). Thus when dealing with Kan extensions along *or* the categories $\mathcal{C}$ and $\mathcal{M}$ will be $(\Delta^+ \times \Delta^+)^{op}$ and Δ^{+op} respectively, and when dealing with Kan extensions along ∂, $\mathcal{C}$ and $\mathcal{M}$ will be Δ^{+op} and $(\Delta^+ \times \Delta^+)^{op}$ respectively.

First, consider the functor *Dec*. For a simplicial set X, the left adjoint to *Dec* (which is described combinatorially in [18]) is the *cone over the connected components of* X: it will be denoted CX. It is described by the coend $\int^{[p]} X_p \cdot \triangle([0]or[p])$: although $\triangle([0]or[p]) = \triangle[p+1]$, $\triangle([0]or[p])$ highlights the fact that the structure of the simplicial set depends on $\triangle([0]or[p])$.

Combinatorially, CX is defined as follows:-

$(CX)_n = \sqcup_{p=-1}^{n} X_p$

For $x \in X_p$, $\mathbf{s}_i^n(x) = \begin{cases} s_i^p(x) & \text{if } 0 \leq i \leq p \\ x & \text{if } p < i \leq n \end{cases}$

For $x \in X_p$, $\mathbf{d}_i^n(x) = \begin{cases} d_i^p(x) & \text{if } 0 \leq i \leq p \\ x & \text{if } p < i \leq n \end{cases}$ where d_0^0 is written for the augmentation, q_X.

Even when the codomain of the augmentation is not $\pi_0 X$, then the definition above still works, and the construction is a left adjoint in the category of augmented simplicial sets. It is particularly useful to consider X augmented over a single point, as then CX has the structure that might be naively expected of a cone in simplicial sets.

Second, consider the functor DEC. The right adjoint to DEC (which is described in detail in a number of places, for example [18] and [20]) is called ∇ and, for a bisimplicial set Y, is given by the end:-

$$(\nabla Y)_n = \int_{p,q} Y_{p,q}^{\Delta^+([p]or[q],[n])}$$

The left adjoint (which will be of some use later on) is given for Y, by:-

$$(\triangle Y)_n = \int^{p,q} \Delta^+([n],[p]or[q]) \cdot Y_{p,q}$$

Now consider the functor *diag*. The right adjoint, in particular, has an elegant description: given a simplicial set X, the set of n-simplices, X_n, is given by $SS(\triangle[n], X)$; the right adjoint to the diagonal functor (which will be called R for the moment) is then given by

$$(RX)_{p,q} = SS(\triangle[p] \times \triangle[q], X)$$

The calculation is as follows:-

$$(RX)_{p,q} = \int_n X_n^{(\Delta^+ \times \Delta^+)(([n],[n]),([p],[q]))}$$

$$\cong \int_n \underline{Sets}(\Delta^+([n],[p]) \times \Delta^+([n],[q])), X_n)$$

$$\cong \int_n \underline{Sets}((\triangle[p] \times \triangle[q])_n, X_n)$$

$$\cong SS(\triangle[p] \times \triangle[q], X)$$

Thus, $(RX)_{p,*}$ (that is the p^{th}-column of RX) is the simplicial set $X^{\triangle[p]}$, and $(RX)_{*,q}$ (that is the q^{th}-row of RX) is the simplicial set $X^{\triangle[q]}$.

Using this right adjoint, there is a coend definition of *diag*.

$$SS(diagX, Y) \cong \int_{[p],[q]} \underline{Sets}(X_{p,q}, SS(\triangle[p] \times \triangle[q], Y)$$

$$\cong \int_{[p],[q]} SS(X_{p,q} \cdot (\triangle[p] \times \triangle[q]), Y)$$

Thus $diagX \cong \int^{[p],[q]} X_{p,q} \cdot (\triangle[p] \times \triangle[q])$. This will be needed later.

Also, *diag* has a left adjoint (called for the moment L) given on objects by the coend formula:-

$$(LX)_{p,q} = \int^{[n]} (\Delta^+ \times \Delta^+)(([p],[q]),([n],[n])) \cdot X_n$$

$$\cong \int^{[n]} \Delta^+([p],[n]) \times \Delta^+([q],[n]) \times X_n$$

$$\cong \int^{[n]} \triangle[n]_p \times \triangle[n]_q \times X_n$$

2.10 Product

There is another (bi-augmented) bisimplicial set which is needed in this thesis. Given two (augmented) simplicial sets, then there is a (bi-augmented) bisimplicial set which has p^{th} column $X_p \times Y_*$ (where X_p is thought of as the constant simplicial set at X_p) and q^{th} row $X_* \times Y_q$ (where Y_q is thought of as the constant simplicial set at Y_q). This construction will be called $P(X, Y)$.

Chapter 3

Tensor product

The category of simplicial sets is Cartesian closed (indeed, it is a presheaf topos), but the topos structure is based on the structure of the category of sets. This chapter sets out a monoidal closed structure on augmented simplicial sets, based on the monoidal structure of the category of finite ordinals and monotonic functions.

3.1 Definitions

Definition 3.1 (i)

For augmented simplicial sets X and Y, define an internal-hom, $[X, Y] \in ob(ASS)$ by $[X, Y]_{n-1} := ASS(X, Dec^n Y)$. The face and degeneracy maps (and the quotient map to $[X, Y]_{-1}$) are all induced by the structure of $Dec^* Y$ as an augmented simplicial object in the category of augmented simplicial sets.

Thus, given a simplicial morphism $\{f_m\}_{m \in \mathbb{N}} : X_m \longrightarrow (Dec^{n+1} Y)_m$ (that is an n-simplex of $[X, Y]$),

$((d_i(f))_m : X_m \longrightarrow (Dec^n Y)_m) = (d_i f_m : X_m \longrightarrow Y_{m+n})$ where $0 \leq i \leq n$; and

$((s_i(f))_m : X_m \longrightarrow (Dec^{n+2} Y)_m) = (s_i f_m : X_m \longrightarrow Y_{m+n+2})$ where $0 \leq i \leq n$.

The codomain of the augmentation of $[X, Y]$ is the set of simplicial morphisms from X to Y, and two 0-simplices $f, g : X \longrightarrow DecY$ map to the same element of $ASS(X, Y)$ when $q_Y f_{-1} = q_Y g_{-1}$ and $d_0 f_n = d_0 g_n$ for all $n \in \mathbb{N}$.

Definition 3.1 (ii)

The tensor product, $\otimes$, is formally defined so that together with the internal-hom described above, ASS becomes a monoidal closed category. Thus for each Y, the endofunctor $(-) \otimes Y$ is left adjoint to the endofunctor $[Y, -]$ which arises from the internal-hom. Thus for any three augmented simplicial sets X, Y and Z, there is a bijection $ASS(X \otimes Y, Z) \cong ASS(X, [Y, Z])$ which is natural in X and Z, and dinatural in Y.

The set of n-simplices of a simplicial set X (that is $X([n])$) is usually denoted by X_n. Although $X([m]or[n]) \cong X_{m+n+1}$ this notation does not indicate how the simplicial structure varies with m and n. Therefore, define $X_{m \vee n} := X([m]or[n])$. Using the bijection which arises from the monoidal closed structure, a more explicit description of $\otimes$ is obtained, by the following calculation.

$ASS(X \otimes Y, Z) \cong ASS(X, [Y, Z])$

$$\cong \int_m \underline{Sets}(X_m, ASS(Y, Dec^{m+1}Z))$$
$$\cong \int_m \underline{Sets}\Big(X_m, \int_n \underline{Sets}(Y_n, (Dec^{m+1}Z)_n)\Big)$$
$$\cong \int_m \int_n \underline{Sets}(X_m \times Y_n, (Dec^{m+1}Z)_n)$$
$$\cong \int_m \int_n \underline{Sets}(X_m \times Y_n, Z_{m\vee n})$$
$$\cong \int_{m,n} \underline{Sets}(X_m \times Y_n, Z_{m\vee n})$$
$$\cong \underline{Sets}\Big(X_m \times Y_n, ASS(\triangle([m]or[n]), Z)\Big)$$
$$\cong \int_{m,n} ASS(X_m \times Y_n \cdot \triangle([m]or[n]), Z)$$
$$\cong ASS(\int^{m,n} X_m \times Y_n \cdot \triangle([m]or[n]), Z)$$

This gives a coend definition for $\otimes$:

$$X \otimes Y \cong \int^{p,q} (X_p \times Y_q) \cdot \triangle([p]or[q])$$

There is a further characterisation of $\otimes$:

$$X \otimes Y \cong \triangle P(X, Y)$$

This is clear from the definitions already given.

The combinatorial definition is as follows. The set of n-simplices is:-

$$\coprod_{i=-1}^{n} X_{n-1-i} \times Y_i$$

the face maps are given by:-

$$d_i^n(x,y) = \begin{cases} (d_i^p x, y) & \text{if } 0 \leq i \leq p \\ (x, d_{i-p-1}^{n-p-1} y) \text{if } p < i \leq n \end{cases}$$

where $(x, y) \in X_p \times Y_{n-p-1}$, and d_0^0 is the augmentation (of X or Y);

lastly, the degeneracies are:-

$$s_i^{n-1}(x,y) = \begin{cases} (s_i^p x, y) & \text{if } 0 \leq i \leq p \\ (x, s_{i-p-1}^{n-p-2} y) \text{if } p < i \leq n-1 \end{cases}$$

where $(x, y) \in X_p \times Y_{n-p-2}$.

This definition makes the connection between $\otimes$ and C very clear (and in fact, if $X_{-1} = *$ then $X \otimes \triangle[0]$ is the cone over X) more generally, $X \otimes \triangle[0]$ is the cone over the codomain of the augmentation. In the case that X and Y are standard simplices, there is a characterisation, which will allow a further description of $\otimes$ on all (augmented) simplicial sets.

Proposition 3.1.1

$$\triangle[m] \otimes \triangle[n] \cong \triangle([m] or [n])$$

Proof

$$ASS(\triangle[m] \otimes \triangle[n], X) \cong ASS(\triangle[m], [\triangle[n], X])$$

$$\cong [\triangle[n], X]_m \cong ASS([n], Dec^{m+1} X)$$

$$\cong (Dec^{m+1} X)_n \cong X([m] or [n]) \cong ASS(\triangle([m] or [n]), X)$$

This is the required result. ∎

Definition 3.1 (iii)

A functor from Δ^+ to a category $\mathcal{C}$, that is an object in $\mathcal{C}^{\Delta^+}$, is called a *cosimplicial object* of $\mathcal{C}$.

Similarly, a functor from Δ to a category $\mathcal{C}$, that is an object in $\mathcal{C}^{\Delta}$, is called an *augmented cosimplicial object* of $\mathcal{C}$.

In particular functors from $\Delta^{+op} \times \Delta^+$ to $\underline{Sets}$ are called cosimplicial simplicial sets. As with bisimplicial sets, the two separate structures commute. The particular example of a cosimplicial simplicial set is the representable functor of $\underline{Sets}^{\Delta^{+op} \times \Delta^+}$, that is, the cosimplicial simplicial set where the standard n-simplices are the simplicial structure, and the natural morphisms between them forms the cosimplicial structure. Consider the functor $or : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$. Just as in the simplicial set case, where $or^* = DEC$, there is a functor or^* on cosimplicial simplicial sets which yields a bicosimplicial simplicial set. This functor may be composed with the functor $\triangle[-] = \Delta(-,-)$. The functor $\triangle[-]or : \Delta \times \Delta \longrightarrow SS$ is then the bicosimplicial simplicial set which has (in the $(p,q)^{th}$ position of the bicosimplicial array) the simplicial set $\triangle[p] \otimes \triangle[q] \cong \triangle([p]or[q])$.

This, together with proposition 3.1.1, means that

$$X \otimes Y \cong \int^{p,q} (X_p \times Y_q) \cdot (\triangle[p] \otimes \triangle[q]).$$

Note also, that if the functor $\triangle[-]$ were composed with the composite functor $or\delta$ then the result would be the diagonal of the bicosimplicial simplicial category $\triangle[-] \otimes \triangle[-]$: that is, $\{\triangle[n] \otimes \triangle[n]\}_{n \in \mathbb{N}}$. This comment may seem somewhat obtuse, but it will be useful in the next chapter.

Proposition 3.1.2

$$[X, [Y, Z]] \cong [X \otimes Y, Z]$$

Proof

$$[X,[Y,Z]] \cong \int^m ASS(X, Dec^{m+1}[Y,Z]) \cdot \triangle[m]$$
$$\cong \int^m \left(\int_n \underline{Sets}(X_n, [Y,Z]_{m\vee n})\right) \cdot \triangle[m]$$
$$\cong \int^m \left(\int_n \underline{Sets}(X_n, \int_p \underline{Sets}(Y_p, Z_{m\vee n\vee p}))\right) \cdot \triangle[m]$$
$$\cong \int^m \left(\int_n \int_p \underline{Sets}(X_n \times Y_p, Z_{m\vee n\vee p})\right) \cdot \triangle[m]$$
$$\cong \int^m \left(\int_{n,p} \underline{Sets}(X_n \times Y_p, ASS(\triangle([n]or[p]), Dec^{m+1}Z))\right)$$
$$\cong \int^m ASS(\int^{n,p}(X_n \times Y_p) \cdot \triangle([n]or[p]), Dec^{m+1}Z) \cdot \triangle[m]$$
$$\int^m [X \otimes Y, Z]_m \cdot \triangle[m] \cong [X \otimes Y, Z]$$

■

Corollary 3.1.3

$$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$$

Proof

Let W,X,Y and Z be augmented simplicial sets.

$$ASS((W \otimes X) \otimes Y, Z) \cong ASS(W \otimes X, [Y,Z])$$
$$\cong ASS(W, [X,[Y,Z]]) \cong ASS(W, [X \otimes Y, Z])$$
$$\cong ASS(W \otimes (X \otimes Y), Z)$$

Proposition 3.1.4

If X , Y and Z are augmented simplicial sets, then

$$(X \sqcup Y) \otimes Z \cong (X \otimes Z) \sqcup (Y \otimes Z)$$

and
$$X \otimes (Y \sqcup Z) \cong (X \otimes Y) \sqcup (X \otimes Z)$$

Proof

It suffices to comment that colimits commute. However, a more explicit proof will clarify things. Let W, X, Y and Z be augmented simplicial sets. Then:

$$ASS((X \sqcup Y) \otimes Z, W) \cong ASS((X \sqcup Y), [Z, W])$$

$$\cong ASS(X, [Z, W]) \times ASS(Y, [Z, W])$$

$$\cong ASS(X \otimes Z, W) \times ASS(Y \otimes Z, W)$$

$$\cong ASS((X \otimes Z) \sqcup (Y \otimes Z), W)$$

and

$$ASS(X \otimes (Y \sqcup Z), W) \cong ASS(X, [Y \sqcup Z, W])$$

$$\cong \int_n \underline{Sets}(X_n, ASS(Y \sqcup Z, Dec^{n+1}W))$$

$$\cong \int_n \underline{Sets}(X_n, ASS(Y, Dec^{n+1}W) \times ASS(Z, Dec^{n+1}))$$

$$\cong \int_n \underline{Sets}(X_n, [Y, W]_n) \times \int_n \underline{Sets}(X_n, [Z, W]_n)$$

$$\cong ASS(X, [Y, W]) \times ASS(X, [Z, W])$$

$$\cong ASS(X \otimes Y, W) \times ASS(X \otimes Z, W)$$

$$\cong ASS((X \otimes Y) \sqcup (X \otimes Z), W)$$

This concludes the proof. ∎

Recall from subsection 2.5 that an augmented simplicial set, X, may be written as the disjoint union of subsimplicial sets, $\{X_x \,|\, x \in X_{-1}\}$, where $y \in (X_x)_n$ *iff* $q_X(d_0)^n y = x$.

Corollary 3.1.5

If X and Y are augmented simplicial sets, then

$$X \otimes Y \cong \coprod_{x \in X_{-1},\, y \in Y_{-1}} X_x \otimes Y_y$$

Proof

This proposition highlights an interesting point: the cone in topological spaces is always a path connected space, whereas the construction CX in simplicial sets (section 2.9) is the cone over the augmentation. This means that there is an extra "degree of freedom" when considering the cone in simplicial sets. From the view of the "non-basepointed, non-connected homotopy theorist" this is philosophically very nice. From another point of view, it is another caveat to bear in mind. A similar situation arises with the construction of topological join.

3.2 Topological Join

The *topological join* is discussed in some detail in chapter 5, section 7 of [9]. Results proved there will be used here without proof: the notation for this section is largely taken from there.

Definition 3.2 (i)

This definition is a generalisation of the concept of join for two suitable subspaces of a vector space. Consider two topological spaces $\mathcal{U}$ and $\mathcal{V}$, and construct a set of 4-tuples (r, u, s, v), where $u \in \mathcal{U},\ v \in \mathcal{V},\ r, s \in [0, 1]$ and $r + s = 1$: in the case that $r = 0$, the u will be ignored, and in the case that $s = 0$, the v will be ignored. This set will be suggestively called $\mathcal{U} * \mathcal{V}$.

There are obvious projections from this set of 4-tuples:

$p_{\mathcal{U}} : \mathcal{U} * \mathcal{V} \to \mathcal{U}, \quad p_{\mathcal{V}} : \mathcal{U} * \mathcal{V} \to \mathcal{V}, \quad p_r : \mathcal{U} * \mathcal{V} \to (0, 1]$ and $p_s : \mathcal{U} * \mathcal{V} \to (0, 1]$ which are termed the *coordinate functions* of $\mathcal{U}*\mathcal{V}$. The first two are obviously defined, the last two take a point $(r, u, s, v) \in \mathcal{U} * \mathcal{V}$ to r and s respectively.

Then the *topological join of* $\mathcal{U}$ *and* $\mathcal{V}$ is defined as the set $\mathcal{U} * \mathcal{V}$ together with the initial topology with respect to the *coordinate functions*. Thus a function with codomain $\mathcal{U} * \mathcal{V}$ is a continuous function if and only if its composite with each of the coordinate functions is continuous. (For definitions of initial, final, and other topologies see [9]). The topological join of $\mathcal{U}$ and $\mathcal{V}$ is written $\mathcal{U} * \mathcal{V}$.

Since $r + s = 1$, the pair (r, s) defines the unit interval, and so $\mathcal{U} * \mathcal{V}$ consists of one unit interval for every pair of points $u \in \mathcal{U}$ and $v \in \mathcal{V}$, and this unit interval joins the point in $\mathcal{U}$ to the point in $\mathcal{V}$ in such a way that it does not intersect with any of the other unit intervals so defined: this is the vector space definition of join, and the "suitability" mentioned earlier is precisely that the lines between the two subspaces do not intersect; (recall proposition 1.2.2.1).

The set of points $\mathcal{U} * \mathcal{V}$ may be given the *identification* topology with respect to the function $\quad \mathcal{U} \times \mathcal{V} \times \mathbf{I} \longrightarrow \mathcal{U} * \mathcal{V}$,

given by $\quad (u, v, x) \mapsto xu + (1 - x)v$

where $0 \leq x \leq 1$, $u \in \mathcal{U}$ and $v \in \mathcal{V}$.

Brown discusses the idea that topological join should be defined in this way, but notes a major problem: in general, join defined with this topology is not an associative operation. With respect to the initial topology, the join operation *is* associative up to isomorphism.

However, in the case that $\mathcal{U}$ and $\mathcal{V}$ are compact Hausdorff spaces, the two topologies coincide. In fact, if the category of topological spaces is replaced with the category of k-spaces, then the two topologies on the join coincide; however, the definition of product in the category of k-spaces is distinct from that on the category of topological spaces, (see [9]). A full discussion of k-spaces is not appropriate here, nor will more comments be made on the results mentioned in this paragraph. Results proved by Brown in [9] will be used without further proof, but will be quoted.

Now that it has been shown that the ordinal sum on finite ordinals models $\otimes$ on augmented simplicial sets, it is possible to prove a lemma which formalises the comment following proposition 1.2.2.1.

Lemma 3.2.1

$$|\triangle[p]| * |\triangle[q]| \cong |\triangle[p] \otimes \triangle[q]|$$

Proof

Recall $|\triangle[m]| := \triangle^m$. Since $\triangle[p] \otimes \triangle[q] \cong \triangle([p]or[q]) = \triangle[p+q+1]$, the isomorphism exists for each pair (p,q).

The isomorphism is natural in p and q if it commutes with the bicosimplicial structure of $\triangle[-] \otimes \triangle[-]$ and $\triangle^* * \triangle^*$. Considering the structure on $\triangle^*$ and Δ outlined in the first chapter, this is clear.

Thus, the tensor product on the representable functors in ASS models the topological join on the affine simplices in the same way as the ordinal sum on Δ.

The aim now is to extend this to simplicial sets X and Y to obtain a result $|X \otimes Y| \cong |X| * |Y|$. In general this will not be true:

$$|(\triangle[0] \sqcup \triangle\triangle[0]) \otimes \triangle[0]| \cong |\triangle[1] \sqcup \triangle[1]| \cong \triangle^1 \sqcup \triangle^1$$

is not path connected whereas

$$|\triangle[0] \sqcup \triangle[0]| * |\triangle[0]| \cong (\triangle^0 \sqcup \triangle^0) * \triangle^0$$

is path connected. However, under certain conditions, the theorem is true.

Theorem 3.2.1.1

Let X and Y be trivially augmented simplicial sets. Then

$$|X \otimes Y| \cong |X| * |Y|$$

Proof

The definition of geometric realisation which is most useful here is definition 2.6 (iii). Then

$$|X|{*}|Y| := \left\{ \begin{array}{lll} r[(p_0, \cdots, p_m)_x] & s.t. & \sum_{i=0}^{m} p_i = 1 \ , \ \sum_{i=0}^{n} q_i = 1 \ , \\ \quad + s[(q_0, \cdots q_n)_y] & & x \in X_m \ , \ y \in Y_n \ , \ r + s = 1 \\ & & p_i, q_i, r, s \geq 0 \\ & & \text{and } [-] \text{ denotes equivalence class} \end{array} \right\}$$

It should also be noted that if $r = 0$ that the point from $|X|$ is ignored and similarly if $s = 0$ the point from $|Y|$ is ignored.

Define a map $f : |X| * |Y| \longrightarrow |X \otimes Y|$ as follows:

$$f(r[(p_0, \cdots, p_m)_x] + s[(q_0, \cdots, q_n)_y]) \mapsto [(rp_0, \cdots, rp_m, sq_0, \cdots, sq_n)_{x,y}]$$

The problem of f being well defined revolves around the fact that if $r = 0$, the point x is ignored. This means that for any y, it must be true that $(0, \cdots 0, q_0, \cdots, q_n)_{(x,y)} \sim (0, \cdots 0, q_0, \cdots q_n)_{(x',y)}$ for all $x, x' \in X$. This can only be true if the codomain of the augmentation of X is trivial. Similarly the augmentation of Y must be trivial for the case $s = 0$. It is for this reason that the simplicial sets X and Y must be trivially augmented, since the equivalence relation given in definition 2.6 (iii) does not allow simplices which map down to distinct elements of X_{-1} to be identified. Other than these two extreme cases, a moment's thought will show that the function f respects the relation and so is well defined. Continuity is also trivial. The obvious inverse function is also continuous under the definition of the topology on $|X| * |Y|$. Thus the two spaces are homeomorphic. ∎

Comment

It is not true that $Sing\mathcal{U} \otimes Sing\mathcal{V} \cong Sing(\mathcal{U} * \mathcal{V})$. Consider the case where $\mathcal{U}$ and $\mathcal{V}$ are both the one point set. Then $Sing\mathcal{U} \cong Sing\mathcal{V} \cong \triangle[0]$ but $Sing\triangle^1$ is not isomorphic to $\triangle[1]$.

The join has particular uses. First, the join of a space X with a point P is the cone under the space. There is a difference between the spaces $X * P$ and $P * X$ as the first is (categorically) a cone under P and the second a cocone over P; in fact, if X is a compact Hausdorff space, then $CX \cong X * P$.

Similarly there is a continuous bijection from SX (the suspension of X) to $X * \mathbf{S}^0$ which is a homeomorphism if X is a compact Hausdorff space.

Using these ideas, it is a simple step to the following result (proved in [9])

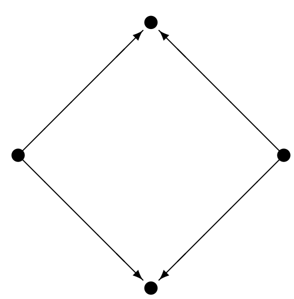
$$\mathbf{S}^p * \mathbf{S}^q \cong \mathbf{S}^{p+q+1}.$$

This essentially says that the n-sphere in the category of topological spaces is the join of $n+1$ copies of the 0-sphere.

The usual model for the n-sphere in simplicial sets is the simplicial set with one nondegenerate n-simplex and one non-degenerate 0-simplex and all other simplices degenerate (see [7]). The tensor product of a p-sphere with a q-sphere would then have one nondegenerate simplex in each of the dimensions 1, $p+1$, $q+1$ and $p+q+1$ and two nondegenerate simplices in dimension 0. Thus tensor product will not preserve this set of models for the spheres. However, there are models for the spheres in simplicial sets which are respected by tensor.

The models are defined inductively. Clearly there is only one possible description in simplicial sets for the 0-sphere, and that is $\mathbf{S}^0 := \triangle[0] \sqcup \triangle[0]$. However, in augmented simplicial sets there are two possible models; one has the canonical augmentation which has two points in the codomain, and the other has the trivial augmentation, with the singleton set as the codomain. If the former model is chosen, then a simple calculation shows that $\mathbf{S}^0 \otimes \mathbf{S}^0 \cong \triangle[1] \sqcup \triangle[1] \sqcup \triangle[1] \sqcup \triangle[1]$.

However, if the trivially augmented model is chosen, $\mathbf{S}^0 \otimes \mathbf{S}^0$ has four non-degenerate 1-simplices connected to each other in a "diamond" as in the following picture:-



The tensor of product of three copies of $\mathbf{S}^0$ has eight non-degenerate 2-simplices, twelve non-degenerate 1-simplices and six 0-simplices which join together as an octahedron.

So, $\mathbf{S}^0 \in obASS$ shall denote the simplicial set $\triangle[0] \sqcup \triangle[0]$ together with the trivial augmentation. It will be referred to as the simplicial 0-sphere. Define the simplicial n-sphere, $\mathbf{S}^n \in obASS$, as follows:-

$$\mathbf{S}^n := \underbrace{\mathbf{S}^0 \otimes \cdots \otimes \mathbf{S}^0}_{n+1}$$

It is clear from the definition of tensor product and of the simplicial 0-sphere that the simplicial n-sphere is a triangulation of the topological n-sphere. In fact, theorem 3.2.1.1 gives explicitly that

$$|\mathbf{S}^n| \cong \mathbf{S}^n$$

Chapter 4

Subdivision

Any classical attempt to prove a Van Kampen would require a concept of subdivision. This chapter will expound such a theory for simplicial sets.

4.1 Definitions

Definition 4.1 (i)

The *ordinal subdivision* of $\triangle[n]$ (the standard n-simplex in simplicial sets) is denoted by $Sd(\triangle[n])$, and is defined as follows:-

$$Sd(\triangle[n]) := \int^{p,q} \Delta([p]or[q],[n]) \cdot (\triangle[p] \times \triangle[q])$$

Definition 4.1 (ii)

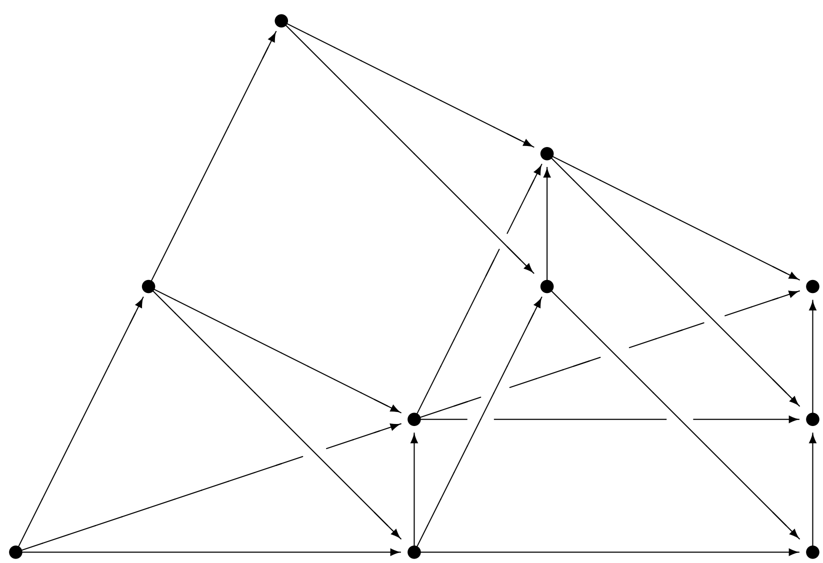
The *ordinal subdivision* of a simplicial set X is denoted by SdX and is defined as follows:-

$$SdX := \int^{n} X_n \cdot Sd\triangle[n]$$

This expands to

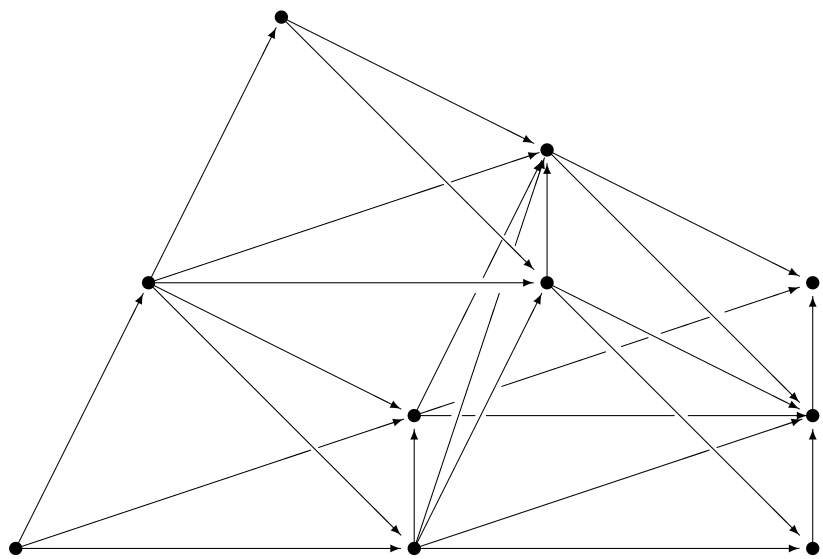
$$SdX := \int^{n} X_n \cdot \left(\int^{p,q} \Delta([p]or[q],[n]) \cdot (\triangle[p] \times \triangle[q]) \right)$$

These definitions are due to Cordier and Porter. Intuitively, each n-simplex of X is replaced by a subdivided n-simplex, which is made up of a set $\{\triangle[p] \times \triangle[n-p]\}_{0\leq p\leq n}$ where $\triangle[p] \times \triangle[n-p]$ has a face in common with $\triangle[p+1] \times \triangle[n-p-1]$. The following picture shows the case of $n = 3$; it has two triangular prisms which meet up in a rectangular face, and two tetrahedra - each one meeting one of the prisms at a triangular face.



There is an elegant geometric definition. Cut the affine n-simplex, $\triangle^n$ (defined by $\{(x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} x_i = 1, x_i \geq 0\}$) by the family of affine hyperplanes defined by $\{\sum_{i=0}^{r} x_i = \frac{1}{2} \,|\, 0 \leq r \leq n-1\}$. Note that this is also the set of affine hyperplanes defined by $\{\sum_{i=r}^{n} x_i = \frac{1}{2} \,|\, 1 \leq r \leq n\}$. This gives n affine hyperplanes.

Since the simplicial sets $\triangle[p] \times \triangle[q]$ split naturally into a set of $\binom{n}{p}$ n-simplices, Sd naturally splits the n-simplex into 2^n n-simplices.



It should be noted that there are important differences between this subdivision and the barycentric subdivision - the latter adds a vertex "in the middle" of every nondegenerate n-simplex, for all $n \in \mathbb{N}$. The former adds a vertex "in the middle" of every nondegenerate 1-simplex, but nothing more. Further, the right adjoint to barycentric subdivision is the functor Ex^∞, which has the property that $Ex^\infty X$ is a Kan complex (whatever the properties of X). This is not the case with the right adjoint to Sd (which is derived later).

It is possible that definition 4.1 (ii) could give rise to a contradictory definition of $Sd\triangle[n]$; it is important to check, therefore, that

$$Sd(\triangle[n]) \cong \int^m \triangle[n]_m \cdot Sd(\triangle[m]),$$

that is, that the two definitions agree on the simplicial set $\triangle[n]$. The following proposition goes somewhat further.

Proposition 4.1.1

$$SdX = \int^{n} X_n \cdot \Big(\int^{p,q} \Delta([p]or[q],[n]) \cdot (\triangle[p] \times \triangle[q]) \Big)$$
$$\cong \int^{p,q} DECX_{p,q} \cdot (\triangle[p] \times \triangle[q])$$

Proof

Consider $SS(SdX, Y)$ for any simplicial set Y. Then,

$$SS(SdX,Y) \cong SS\left(\int^{n} X_n \cdot \Big(\int^{p,q} \Delta([p]or[q],[n]) \cdot (\triangle[p] \times \triangle[q]) \Big), Y \right)$$
$$\cong \int_{n} \underline{Sets}\left(X_n, SS\Big(\int^{p,q} \Delta([p]or[q],[n]) \cdot (\triangle[p] \times \triangle[q]), Y \Big) \right)$$
$$\cong \int_{n} \underline{Sets}\left(X_n, \int_{p,q} \underline{Sets}\Big(\Delta([p]or[q],[n]), SS((\triangle[p] \times \triangle[q]), Y) \Big) \right)$$
$$\cong \int_{p,q} \underline{Sets}\Big(\int^{n} (SS(\triangle[n],X) \times SS(\triangle[p] \otimes \triangle[q], \triangle[n])), SS((\triangle[p] \times \triangle[q]), Y) \Big)$$
$$\cong \int_{p,q} \underline{Sets}\Big(SS(\triangle[p] \otimes \triangle[q], X), SS((\triangle[p] \times \triangle[q]), Y) \Big)$$
$$\cong SS(\int^{p,q} DECX_{p,q} \cdot (\triangle[p] \times \triangle[q]), Y)$$

Since this is true for any $Y \in SS$, the proposition follows. ∎

Corollary 4.1.2

$$(SdX) \cong diagDECX$$

Proof

It was proved in the second chapter that for a bisimplicial set Y,

$$diagY \cong \int^{p,q} Y_{p,q} \cdot (\triangle[q] \times \triangle[q]).$$

Replacing Y with $DECX$ gives the result.

Since $SdX = diagDECX$ the n-simplices of SdX are given by the set X_{2n+1}; the i^{th} face map is given by:-

$$\delta_i : (SdX)_n \longrightarrow (SdX)_{n-1} = d_i d_{n+1+i} : X_{2n+1} \longrightarrow X_{2n-1}$$

and the i^{th} degeneracy map by:-

$$\sigma_i : (SdX)_n \longrightarrow (SdX)_{n+1} = s_i s_{n+1+i} : X_{2n+1} \longrightarrow X_{2n+3}\ .$$

It is trivial that these satisfy the simplicial identities.

To emphasise the (philosophical) difference between $(Sd\triangle[n])_m$ and $\triangle[n]_{2m+1}$, denote m-simplices of $Sd\triangle[n]$ by $(2 \times m)$-matrices,

$$\begin{pmatrix} i_0 & \cdots & i_m \\ i_{m+1} & \cdots & i_{2m+1} \end{pmatrix}.$$

The conditions on $\triangle[n]$ imply that $i_k \leq i_{k+1}$.

Since $0 \leq i_k \leq n$, the nondegenerate n-simplices of $Sd\triangle[n]$ are given by matrices $\begin{pmatrix} i_0 & \cdots & i_n \\ i_{n+1} & \cdots & i_{2n+1} \end{pmatrix}$ where $i_0 = 0$, $i_{2n+1} = n$ and for $0 < k \leq n$, $i_{k-1} = i_k$ *iff* $i_{n+k} = i_{n+k+1} - 1$ and $i_{k-1} = i_k - 1$ *iff* $i_{n+k} = i_{n+k+1}$ and there are no other possibilities. Clearly there are no nondegenerate r-simplices for $r > n$.

Using this it is possible to describe the nondegenerate n-simplices of

$Sd\triangle[n]$ as degeneracies of the identity morphism $\iota_n : [n] \to [n]$.

Proposition 4.1.3

The nondegenerate n-simplices of $Sd\triangle[n]$ are given precisely by the elements of $\triangle_{2n+1}$ of the form $s_{j_n} s_{j_{n-1}} \cdots s_{j_0} \iota_n$, where $j_k < j_{k+1}$ for $0 \leq k \leq n-1$, there is some r such that $j_r = n$, and if there is some t ($0 \leq t \leq n$) such that $j_t = p$, then there is no k such that $j_k = n + 1 + p$ (for $0 \leq p \leq n$).

Proof

Note first that any element of $\triangle[n]_{2n+1}$ must be of the form

$$s_{j_{n+l+1}} \cdots s_{j_0} d_{k_l} \cdots d_{k_0} \iota_n.$$

Clearly, the nondegenerate ones will have no face maps in them, thus they must be of the form

$$s_{j_n} s_{j_{n-1}} \cdots s_{j_0} \iota_n.$$

Secondly, by use of the simplicial identities, the degeneracies may be reordered so that $j_k < j_{k+1}$ (see [7]). When the composite degeneracy is in this form, the suffices on each degeneracy denote those vertices in the final $2n+2$-tuple which are the same as the subsequent vertex. Thus, if there are r and r' such that $j_r + n + 1 = j'_r$, this means that both $i_{j_r} = i_{j_r+1}$ and $i_{j_r+n+1} = i_{j_r+n+2}$. Thus, putting the $2n+2$-tuple in the matrix form

$$\begin{pmatrix} i_0 & \cdots & i_{j_r} & i_{j_r+1} & \cdots & i_n \\ i_{n+1} & \cdots & i_{j_r+n+1} & i_{j_r+n+2} & \cdots & i_{2n+1} \end{pmatrix}$$

it is evident that the j_r^{th} vertex is the same as the j_{r+1}^{th}. Thus the matrix represents a degenerate simplex.

Lastly, assume that there is no r such that $j_r = n$; this means that the last entry on the top row of the matrix representing the element is not equal to the first entry of the second row of the matrix. Thus, for some $0 \leq r < n$, the matrix has the form

$$\begin{pmatrix} 0 & \cdots & r \\ r+1 & \cdots & n \end{pmatrix}.$$

This means that there are r points in the top row where the entry changes, and so $n - r$ points of no change. Therefore, for the matrix to represent a nondegenerate simplex, there must be at least $n - r$ points of change in the bottom row. However, since the first entry of the bottom row is $r + 1$, there can only be $n - r - 1$ points of change in the bottom row, thus there will be some point in the matrix where both the top and bottom rows remain the same, which is means it represents a degenerate element.

Since geometrically there are 2^n nondegenerate n-simplices in $Sd\triangle[n]$, they must be the 2^n simplices not excluded by the above conditions.

This concludes the proof. ■

Consider the set of n-simplices of X, X_n; by the Yoneda Lemma it is possible to describe this set as $SS(\triangle[n], X)$. Using the fact that $Sd = diagDEC$, it is then possible to describe the set $(SdX)_n$ as $SS(\triangle[n] \otimes \triangle[n], X)$.

Although the definitions and calculations in this section have dealt with simplicial sets, they clearly extend to augmented simplicial sets. It clear that $SdX_{-1} = X_{-1}$, since $(DECX)_{-1,-1} = X_{-1}$.

Proposition 4.1.4

Let X be a simplicial set, then $\pi_0(SdX) = \pi_0 X$.

Proof

Let $q : X_0 \longrightarrow \pi_0 X$ and $p : (SdX)_0 \longrightarrow \pi_0(SdX)$ be the canonical quotient maps. Then, $qd_0 = qd_1$ is a coequalising map for the pair d_0d_2 , d_1d_3. Thus, there is a unique map $f : \pi_0(SdX) \longrightarrow \pi_0 X$ such that $fp = qd_0$.

Consider $x \in X_1$; $p(s_0d_1x) = p(x)$ since $d_0d_2(s_0)^2x = x$ and $d_1d_3(s_0)^2x = s_0d_1x$; and $p(x) = p(s_0d_0x)$ since $d_0d_2(s_1)^2x = s_0d_0x$ and $d_1d_3(s_1)^2 = x$.

Define a function $g : \pi_0 X \longrightarrow \pi_0(SdX)$ for $y \in X_0$ by $g : q(y) \mapsto ps_0(y)$. To show this is well defined, take $y, z \in X_0$ *s.t.* $q(z) = q(y)$. This implies $\exists\ n \in \mathbb{N}$ and $w_i \in X_1, \quad 0 \leq i \leq n$ *s.t.* $y = d_{\epsilon_0}w_0,\ d_{1-\epsilon_{i-1}}w_{i-1} = d_{\epsilon_i}w_i$ $1 \leq i \leq n, \ \ \epsilon_i \in \{0, 1\}$ and $d_{1-\epsilon_n}w_n = z$.
Then $p(s_0y) = p(w_0) = p(w_1) = \cdots = p(w_n) = p(s_0z)$. Thus g is well defined, and $gq = ps_0$

Now $ps_0d_0 = p$, thus $p = ps_0d_0 = gqd_0 = gfp$. Further,
$fgq = fps_0 = qd_0s_0 = q$. Since p and q are both epic, these two equations imply that $fg = id$ and $gf = id$. Therefore, $\pi_0(SdX) = \pi_0 X$. This is the required result. ■

4.2 *Sd* and *Sing*

Let $\mathcal{W}$ be a topological space, and consider $SdSing\mathcal{W}$. For every affine n-simplex of $\mathcal{W}$, Sd produces 2^n affine n-simplices of $\mathcal{W}$; since $(Sing\mathcal{W})_n$ is by definition the set of all possible affine n-simplices of $\mathcal{W}$, it is tempting to assume that $(SdSing\mathcal{W})_n$ is a subset of $(Sing\mathcal{W})_n$.

Consider the specific example of $(SdSing\mathcal{W})_0$. By definition of Sd, this is the set of affine 1-simplices in $\mathcal{W}$. Each 1-simplex of $Sing\mathcal{W}$, as a 0-simplex of $SdSing\mathcal{W}$, represents the vertex which is the "midpoint" of the affine 1-simplex in $\mathcal{W}$; since $Sing\mathcal{W}$ already contains every point of the space W, it is tempting to assume that $(SdSing\mathcal{W})_0 \cong (Sing\mathcal{W})_0$, and so $(Sing\mathcal{W})_1 \cong (Sing\mathcal{W})_0$.

In fact this is not the case. It is possible for many of the 1-simplices of $Sing\mathcal{W}$ to have the same midpoint, but the functor Sd as a combinatorial device will consider them as separate points. To bring the functor back under some kind of topological control, it is necessary to form a quotient of $SdSing\mathcal{W}$.

Consider the cosimplicial topological spaces $\triangle^*$ - that is $\{\triangle^n\}_{n\in\mathbb{N}}$ - and $\triangle^* * \triangle^*$ - that is $\{\triangle^n * \triangle^n\}_{n\in\mathbb{N}}$. The cosimplicial structure of $\triangle^* * \triangle^*$ is the obvious one.

Define a cosimplicial morphism $\tau_* : \triangle^* \longrightarrow \triangle^* * \triangle^*$; $\tau_n : \triangle^n \longrightarrow \triangle^{2n+1}$ is given by:-

$$\tau_n(t_0, t_1, \cdots, t_n) \mapsto (\frac{t_0}{2}, \frac{t_1}{2}, \cdots, \frac{t_n}{2}, \frac{t_0}{2}, \frac{t_1}{2}, \cdots, \frac{t_n}{2})$$

(Note that although it is normal to denote the order of cosimplicial objects by superscripts, the topological map on the affine n-simplex is denoted τ_n.)

Consider $\tau_*^* : SdSing\mathcal{W} \longrightarrow Sing\mathcal{W}$. This takes $f : \triangle^{2n+1} \longrightarrow \mathcal{W}$ to

$f\tau_n$. Under this map, two affine $2n+1$-simplices, $f \neq g$ in $\mathcal{W}$ are mapped to the same affine n-simplex if they have the same "midpoint". So if

$$f(\frac{t_0}{2}, \frac{t_1}{2}, \cdots, \frac{t_n}{2}, \frac{t_0}{2}, \frac{t_1}{2}, \cdots, \frac{t_n}{2}) = g(\frac{t_0}{2}, \frac{t_1}{2}, \cdots, \frac{t_n}{2}, \frac{t_0}{2}, \frac{t_1}{2}, \cdots, \frac{t_n}{2})$$

then they are mapped to the same n-simplex of $Im(\tau_*^*)$. $Im(\tau_*^*)$ is a sub-simplicial set of $Sing\mathcal{W}$, and is what might be expected geometrically of subdivision in the singular complex.

To be more explicit, consider $\sigma : \triangle^n \longrightarrow \mathcal{W}$, an n-simplex of $Sing\mathcal{W}$: by an abuse of notation, σ will be allowed to denote both the topological map given, and its image under the bijection of the adjunction $|-| \dashv Sing$.

Now, $Sd\sigma : Sd\triangle[n] \longrightarrow SdSing\mathcal{W}$ is $\sigma : \triangle[n] \otimes \triangle[n] \longrightarrow Sing\mathcal{W}$, which is $\sigma : \triangle^n * \triangle^n \longrightarrow \mathcal{W}$. Recall the description of the nondegenerate n-simplices of $Sd\triangle[n]$, from proposition 4.1.3. They are of the form $s_{j_n} s_{j_{n-1}} \cdots s_{j_0} i_n$, where $j_k < j_{k+1}$ for $0 \leq k \leq n-1$, and where there is some r such that $j_r = n$, and if there is some t, $(0 \leq t \leq n)$ such that $j_t = p$, then there is no k such that $j_k = n+1+p$ (for $0 \leq p \leq n$).

For such a simplex, consider the subset of $\{0, \cdots, 2n+1\}$ whose elements are not equal to j_k for $0 \leq k \leq n$. Label them ϕ_l for $0 \leq l \leq n$, so that $\phi_l < \phi_{l+1}$. Note that $\phi_n = 2n+1$. For convenience, define $\phi_{-1} := -1$.

The map $Sd\sigma$ is defined by the action of σ on each of these nondegenerate simplices, and therefore it is given by the maps

$$\sigma(s_{j_0} s_{j_1} \cdots s_{j_n}) : \triangle^n * \triangle^n \longrightarrow \mathcal{W},$$

(which is considered an n-simplex of $SdSing\mathcal{W}$: here the s_j are the topological maps which induce the degeneracies in $Sing\mathcal{W}$).

Consider, therefore $\sigma(s_{j_0}s_{j_1}\cdots s_{j_n})\tau_n : \triangle^n \longrightarrow \mathcal{W}$.

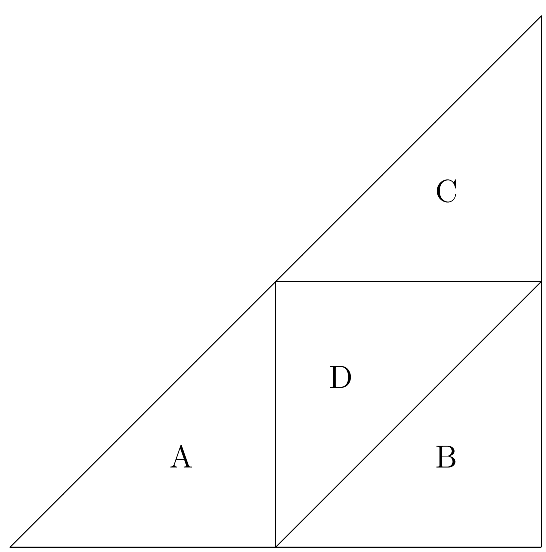
It takes the point $(t_0, t_1, \cdots, t_n)$ of $\triangle^n$ to the point $(u_0, u_1, \cdots, u_n)$, where

$$u_l = \frac{1}{2}\sum_{i=\phi_{l-1}+1}^{\phi_l} t_i \quad \text{for} \quad \phi_l < n$$

$$u_l = \frac{1}{2}\left(\sum_{i=\phi_{l-1}+1}^{n} t_i + \sum_{i=0}^{\phi_l - n} t_i\right) \quad \text{for} \quad \phi_l < n \,\&\, \phi_{l+1} > n$$

$$u_l = \frac{1}{2}\sum_{\phi_{l-1}+1-n}^{\phi_l - n} t_i \quad \text{for} \quad \phi_l > n$$

As an example, consider the four nondegenerate 2-simplices of $Sd\triangle[2]$. These are $s_2s_1s_0i_2$, $s_3s_2s_1$, $s_4s_3s_2$ and $s_4s_2s_0$. The simplices they produce in $\triangle^2$ are



$$A = (\tfrac{t_0+t_1+t_2+t_0}{2}, \tfrac{t_1}{2}, \tfrac{t_2}{2}),$$

$$B = (\tfrac{t_0}{2}, \tfrac{t_1+t_2+t_0+t_1}{2}, \tfrac{t_2}{2}),$$

$$C = (\tfrac{t_0}{2}, \tfrac{t_1}{2}, \tfrac{t_2+t_0+t_1+t_2}{2})$$

$$\text{and} \quad D = (\tfrac{t_0+t_1}{2}, \tfrac{t_2+t_0}{2}, \tfrac{t_1+t_2}{2}).$$

Thus $\tau^*(Sd\sigma)$ maps the 2^n copies of the affine n-simplex which go to make up $|Sd\triangle[n]|$ into the affine n-simplex in precisely the way that would be expected geometrically.

Definition 4.2 (i)

Let X be a simplicial complex, considered as a subsimplicial complex of $Sing(|X|)$. Clearly, $SdX \subset SdSing|X|$. If $\tau^*_*(SdX) \subset X$ and X is a Kan complex, then X shall be called a *Subdivision Complex*.

4.3 Adjoints to Subdivision

It is now known that $SdX = diagDECX$. Since both DEC and $diag$ have left and right adjoints, it is immediate that Sd has both left and right adjoints. These, described earlier in the thesis, are given by left and right Kan extensions: the object part of the composites of these adjoints (which are the adjoints to Sd) are given below as the object parts of left and right Kan extensions along the composite functor $or\partial$ (the notation is Mac Lane's, see [35]).

The left adjoint:-

$$Lan_{or\partial}X([n]) = Lan_{or}(\int^{m} \triangle[m]_p \times \triangle[m]_q \times X_m)([n])$$

$$\cong \int^{p,q} \triangle[p+q+1]_n \times (\int^{m} \triangle[m]_p \times \triangle[m]_q \times X_m)$$

$$\cong \int^{m} \Delta([n],[2m+1]) \cdot X_m$$

and the right adjoint:-

$$Ran_{or\partial}X([n]) = Ran_{or}(SS(\triangle[p] \times \triangle[q], X))([n])$$

$$\cong \int_{p,q} \underline{Sets}(\triangle[n]_{p+q+1}, \int_m \underline{Sets}(\triangle[p]_m \times \triangle[q]_m, X_m))$$

$$\cong \int_{p,q}\int_m \underline{Sets}(\triangle[n]_{p+q+1} \times \triangle[p]_m \times \triangle[q]_m, X_m)$$

$$\cong \int_m \underline{Sets}(\int^{p,q}(\triangle[n]_{p+q+1} \times \triangle[p]_m \times \triangle[q]_m), X_m)$$

$$\cong \int_m \underline{Sets}(\triangle[n]_{2m+1}, X_m)$$

There is an important observation to be made at this point. Let X be a simplicial set, then $SdX_n \cong SS(\triangle[n], SdX) \cong SS(\triangle[n] \otimes \triangle[n], X)$. Since adjoints are unique up to isomorphism, this implies that

$$Lan_{or\partial}(\triangle[n]) \cong \triangle[n] \otimes \triangle[n].$$

This means that

$$(\triangle[n] \otimes \triangle[n])_m \cong \int^{p} \Delta([m],[2p+1]) \times \Delta([p],[n])$$

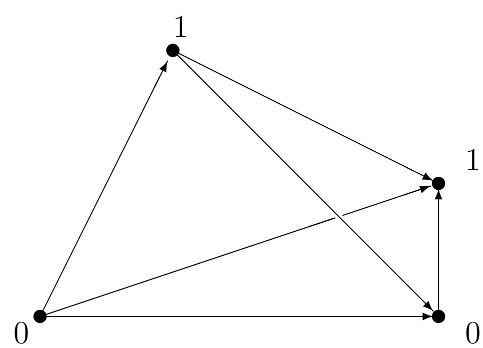
$$\cong \int^{p,q} \Delta([p],[n]) \times \Delta([q],[n]) \times \Delta([m],[p+q+1]).$$

It is not true in general, that $Lan_{or\partial}X \cong X \otimes X$. It is true in the particular case of $\triangle[n]$ because $\triangle[n]$ is generated by a single n-simplex. It does however give the following description of $Lan_{or\partial}X$:

$$Lan_{or\partial}X \cong \int^{n} X_n \cdot (\triangle[n] \otimes \triangle[n])$$

4.4 $Sd\triangle[1]$

Consider $\triangle[1]$ with vertices labelled 0 and 1. Then $Sd(\triangle[1])$ has vertices $\triangle[1]_1$ - thus labelled by pairs ij where $0 \leq i \leq j \leq 1$. The one simplices are then labelled $ijkl$ where $0 \leq i \leq j \leq k \leq l \leq 1$ - that is $\triangle[1]_3$. Then, given δ_0 and δ_1 as described above, the 1-simplex $ijkl$ runs from ik to jl. Now, $\triangle[1]$ has one nondegenerate 1-simplex running from 0 to 1. For this simplex to still exist after the subdivision has been taken, there would have to be $x \in \triangle[1]_3$ with $d_0d_2x = s_00$ and $d_1d_3x = s_01$ - that is a three simplex 0101. (See the picture below).



Clearly this is not possible.

Corollary 4.4.1

The subdivision of a weak Kan complex is not necessarily weak Kan.

Proof

Although $\triangle[1]$ is a weak Kan complex, $Sd\triangle[1]$ is not weak Kan, since there is no filler for the 1-horn $x_2 = 0001, x_0 = 0111$.

Although there is no cosimplicial simplicial morphism $\triangle[-] \longrightarrow Sd\triangle[-]$ which models the topological cosimplicial morphism τ_*^*, (as has been seen by the last example), there is a cosimplicial simplicial morphism $\triangle[-]$ to $Sd(\triangle[-] \otimes \triangle[-])$ which does. This is the unit of the adjunction $Lan_{or\partial} \dashv Sd$. The identity morphism $id : \triangle([n]or[n]) \longrightarrow \triangle([n]or[n])$ also denotes an n-simplex of $Sd(\triangle[n] \otimes \triangle[n])$. This n-simplex is therefore the unit of the adjunction, and further, since the unit of the adjunction is a natural transformation, it commutes with the simplicial structure of $\triangle[n]$ to form a cosimplicial simplicial morphism from $\triangle[-]$ to $Sd\triangle([-]or[-])$.

4.5 Sd in ***Cat***

In the first chapter, it was noted that the standard n-simplices had a categorical model - namely the category $\mathbf{N}$. Subdivision may be as easily defined in *Cat* as in SS. This is more than just a categorical exercise - once a subdivision of $[n]$ is defined, it would be possible to take its nerve, and so obtain a simplicial set which modelled the subdivision, but which was also a weak Kan complex (it is a standard fact that the nerve of a small category is a weak Kan complex, and that the nerve of a small category is a Kan complex if and only if the small category is a groupoid).

Definition 4.5 (i) If $\mathbf{C}$ is a small category, then define

$$Sd(\mathbf{C}) := \int^{p,q} \underline{Cat}(([p] \vee [q]), \mathbf{C}) \cdot ([p] \times [q])$$

This is instantly expressible as

$$\int^{p,q} (Ner\mathbf{C})_{p+q+1} \cdot ([p] \times [q])$$

Consider the case when $\mathbf{C} = [n]$ for some $n \in \mathbb{N}$.

Noting that $\underline{Cat}([p] \vee [q], [n]) \cong \Delta([p]or[q], [n])$,

$$Sd[n] = \int^{p,q} \Delta([p]or[q], [n]) \cdot ([p] \times [q])$$

Since this a category, then its nerve

$$Ner(\int^{p,q} \Delta([p]or[q], [n]) \cdot ([p] \times [q]))$$

will be a weak Kan complex.

Compare this with $Sd\triangle[n]$, that is $SdNer[n]$:

$$\int^{p,q} \Delta([p]or[q], [n]) \cdot (\triangle[p] \times \triangle[q])$$

Since $\triangle[n] \cong Ner[n]$, and since Ner is a right adjoint, this may be rewritten as

$$\int^{p,q} \Delta([p]or[q], [n]) \cdot Ner([p] \times [q])$$

There should be a link between these two definitions, and indeed, it arises as the unit of the adjunction $\Pi \dashv Ner$. Specifically,

$$\eta_n : SdNer[n] \to NerSd[n]$$

Proposition 4.5.1

$$\Pi(Sd\triangle[n]) \cong Sd[n]$$

Proof

Let $\mathbf{C}$ be any small category.

$$\underline{Cat}(\Pi Sd(\triangle[n]), \mathbf{C}) \cong SS(Sd(\triangle[n]), Ner\mathbf{C})$$

$$= SS(\int^{p,q} \Delta([p]or[q],[n]) \cdot (\triangle[p] \times \triangle[q]), Ner\mathbf{C}))$$

$$\cong \int_{p,q} \underline{Sets}(\Delta([p]or[q],[n]), SS(Ner([p] \times [q]), Ner\mathbf{C}))$$

$$\cong \int p, q\underline{Sets}(\Delta([p]or[q],[n]), \underline{Cat}(([p] \times [q]), \mathbf{C}))$$

$$\cong \underline{Cat}(\int^{p,q} \Delta([p]or[q],[n]) \cdot ([p] \times [q]), \mathbf{C})$$

$$= \underline{Cat}(Sd[n], \mathbf{C}) \qquad \blacksquare$$

Note that the proof relies on the fact that $\Pi Ner = Id_{Cat}$: see subsection 2.2

.

Corollary 4.5.2

Let $\mathbf{C}$ *be any small category. For any simplicial morphism*

$f : Sd\triangle[n] \longrightarrow Ner\mathbf{C}$ $\exists! \ \overline{f} : NerSd[n] \longrightarrow Ner\mathbf{C}$ *s.t.* $\overline{f}\eta_{Sd\triangle[n]} = f$.

Proof

The result follows from Proposition 4.5.1 and Proposition 2.2.12.

Chapter 5

Extensions

5.1 Anodyne Extensions

Before addressing the main matter of this chapter, recall some of the basic theory of Anodyne extensions, Kan complexes and Weak Kan complexes.

Definition 5.1 (i)

The simplicial set $\wedge^k[n]$ is defined to be the subsimplicial set of $\triangle[n]$ generated by the $(n-1)$-simplices $d_0\iota_n, d_1\iota_n, \cdots, d_{k-1}\iota_n, d_{k+1}\iota_n, \cdots d_n\iota_n$ where ι_n is the unique non-degenerate n-simplex in $\triangle[n]$. For any $n \in \mathbb{N}$, $\wedge^k[n]$ is commonly known as a "k-horn". There is a natural embedding $i : \wedge^k[n] \longrightarrow \triangle[n]$.

Definition 5.1 (ii)

A simplicial set, X, is a Kan complex if every morphism $f : \wedge^k[n] \longrightarrow X$ (for all n, for all $0 \leq k \leq n$) extends to a morphism $\overline{f} : \triangle[n] \longrightarrow X$, with $\overline{f}i = f$.

This definition of a Kan complex is equivalent to definition 2.1 (vi).

Definition 5.1 (iii)

An anodyne extension is an element in the saturated set of morphisms in SS which is generated by the family of inclusions

$$\{\wedge^k[n] \to \triangle[n] |\, n \in \mathbb{N},\, 0 \le k \le n\}$$

For a description of saturated sets and a study of anodyne extensions see [25] . It is a property of anodyne extensions that if $i : A \to B$ is an anodyne extension, and $f : A \to X$ is any simplicial morphism whose codomain, X, is a Kan complex, then $\exists \overline{f} : B \to X$ with $\overline{f} i = f$. Thus anodyne extensions are the class of simplicial morphisms along which any other simplicial morphism (with codomain a Kan complex) may be extended; the exposition of this in [25] is particularly elegant.

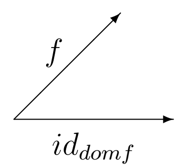
However, the class of Kan complexes does not include the standard n-simplices (for $n > 0$). A wider class of complexes which satisfies some extension conditions (and contains the standard n-simplices) is the class of *Weak Kan Complexes*.

Definition 5.1 (iv)

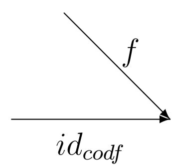
A simplicial set Y is called a *weak Kan complex* if for any $n \in \mathbb{N}$, and $0 < k < n$, any simplicial morphism $f : \wedge^k[n] \to Y$ extends to a morphism $\overline{f} : \triangle[n] \to Y$ with $\overline{f} i = f$.

This definition of Weak Kan complex is equivalent to definition 2.1 (vi).

Let $\mathbf{C}$ be a small category. It is well known that $Ner\mathbf{C}$ is a Kan complex if and only if $\mathbf{C}$ is a groupoid; as a quick demonstration of this fact, consider $f \in arr\mathbf{C}$. The pair $x_2 = f$, $x_1 = id_{domf}$ form a 0-horn in $Ner\mathbf{C}_1$, while the pair $x_0 = f$, $x_1 = id_{codf}$, forms a 2-horn in $Ner\mathbf{C}_1$:

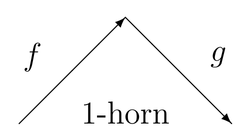


0-horn



2-horn

If $x = (f, g)$ is a filler for the 0-horn, then $d_2 x = x_2 = f$ and $d_1 x = f \cdot g = id_{domf}$. Similarly, if $x = (h, f)$ is a filler, then $d_0 x = x_2 = f$ and $d_1 x = h \cdot f = id_{codf}$. Thus for $Ner\mathbf{C}$ to be a Kan complex, every morphism must have a left and right inverse, which must therefore be unique and a two sided inverse. The converse argument (that if $\mathbf{C}$ is a groupoid then $Ner\mathbf{C}$ is a Kan complex) is equally simple. The concept of Kan complexes is therefore connected to the idea of the existence of inverses and composition. If $\mathbf{C}$ is not a groupoid, then $Ner\mathbf{C}$ is a weak Kan complex (a 1-horn in $Ner\mathbf{C}_1$ is simply a pair of composible maps, $f, g \in arr\mathbf{C}$



and hence by definition of Ner there is a canonical filler. This idea generalises to higher dimensions). Therefore the concept of weak Kan complex is connected with composition without inverses.

5.2 Extending $Sd\triangle[n]$

The aim of the whole of this chapter is to give a proof of the following result: given a cosimplicial simplicial set X where the simplicial set X^n_* is weak Kan for all n, and given a cosimplicial simplicial morphism $f : Sd\triangle[-] \longrightarrow X$, then there is a cosimplicial morphism $\overline{f} : NerSd[-] \longrightarrow X$ extending f.

There is a sketch proof of the result in unpublished work of Porter. Essentially, this claims that the morphism $\eta_{Sd\triangle[n]} : Sd\triangle[n] \longrightarrow NerSd[n]$ is a weak anodyne extension (see definition 5.3 i) for each n, and it is possible for the filling to be compatible with the cosimplicial structure.

In fact, in Porter's description, the subdiagonal of $\triangle[-] \times \triangle[-]$ is used in place of $NerSd[-]$ but as the description of $NerSd[-]$ below shows, they are the same cosimplicial simplicial complex.

Recall (see [39]) that a fibration in the category of simplicial sets is a Kan fibration, a cofibration is a monic, and a weak homotopy equivalence is a map which induces an isomorphism of homotopy groups. It follows that $\eta_{Sd\triangle[n]}$ is monic (see proposition 2.2.8) and a weak homotopy equivalence (since both $Sd\triangle[n]$ and $NerSd[n]$ are contractible, and so have trivial homotopy groups).

Therefore, if X^n is a Kan complex for each n, then for each n

$\eta_{X^n} : X^n \longrightarrow Ner\Pi X^n$ is a Kan fibration (see theorem 2.2.7) and so if there is a cosimplicial simplicial morphism $f : Sd\triangle[-] \longrightarrow X$, then each of the simplicial morphisms $f^n : Sd\triangle[n] \longrightarrow X^n$ may be extended to a simplicial morphism $\overline{f^n} : NerSd[n] \longrightarrow X^n$, with $f^n = \overline{f^n}\eta_{Sd\triangle[n]}$ and

$\eta_{X^n}\overline{f^n} = Ner\Pi(f^n)$, by the Quillen model category structure of SS.

This result is not the main theorem with the weak Kan condition replaced by a Kan condition, as the extensions are at each level, not over the whole structure.

It should be stressed that the main reason for attempting to prove the theorem as stated (with the weak Kan condition rather than the Kan condition) is that it is possible, and since it is more general, it is a preferable result. It is also constructive, in that the inductive method of proof gives an indication of how to build such fillers explicitly.

Before starting, consider the structure of the two simplicial sets, $Sd\triangle[n]$ and $NerSd[n]$. Notice first, that $\triangle[n]$ is a simplicial complex - that is, the non-degenerate m-simplices are defined as $(m+1)$-element subsets of the vertex set. Recall the notation for an m-simplex of $Sd\triangle[n]$ is

$$\begin{pmatrix} i_0 & \cdots & i_m \\ i_{m+1} & \cdots & i_{2m+1} \end{pmatrix}$$

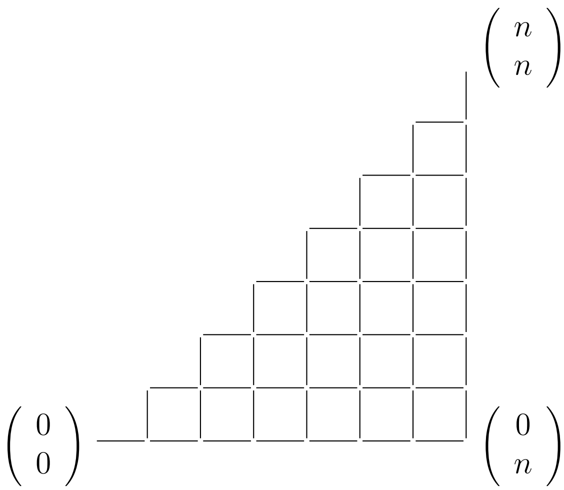
where $(i_0, \cdots, i_m, i_{m+1}, \cdots, i_{2m+1})$ is a $(2m+1)$-simplex of $\triangle[n]$, and that the p^{th} vertex of this m-simplex is $\begin{pmatrix} i_p \\ i_{m+p+1} \end{pmatrix}$.

The m-simplices of $NerSd[n]$ may be described by the matrix

$$\begin{pmatrix} i_0 & \cdots & i_m \\ j_0 & \cdots & j_m \end{pmatrix}$$

where $(i_0, \cdots, i_m), (j_0, \cdots, j_m)$ are both m-simplices of $\triangle[n]$, and $i_k \leq j_k$ for all k. The p^{th} vertex of this m-simplex is $\begin{pmatrix} i_p \\ j_p \end{pmatrix}$.

It is clear that the vertices of $Sd\triangle[n]$ and $NerSd[n]$ are the same. Therefore, to describe both $Sd\triangle[n]$ and $NerSd[n]$, consider the diagram:-



where the vertex $\begin{pmatrix} i \\ j \end{pmatrix}$ is on the i^{th} row, in the j^{th} column. The vertices have a natural partial order on them, given by $\begin{pmatrix} i_0 \\ j_0 \end{pmatrix} \leq \begin{pmatrix} i_1 \\ j_1 \end{pmatrix}$ when $i_0 \leq i_1$ and $j_0 \leq j_1$ and further $\begin{pmatrix} i_0 \\ j_0 \end{pmatrix} < \begin{pmatrix} i_1 \\ j_1 \end{pmatrix}$ when $i_0 \leq i_1$ and $j_0 < j_1$ or $i_0 < i_1$ and $j_0 \leq j_1$. A "path" in the diagram is defined to be a sequence of vertices which is strictly increasing.

Note first, that $Sd\triangle[n]$ is a simplicial complex, so that a particular set of vertices uniquely defines a non-degenerate simplex, and second that both $Sd\triangle[n]$ and $NerSd[n]$ are contractible, so that in this case the problem of $Ner\Pi$ not preserving homotopy type is avoided.

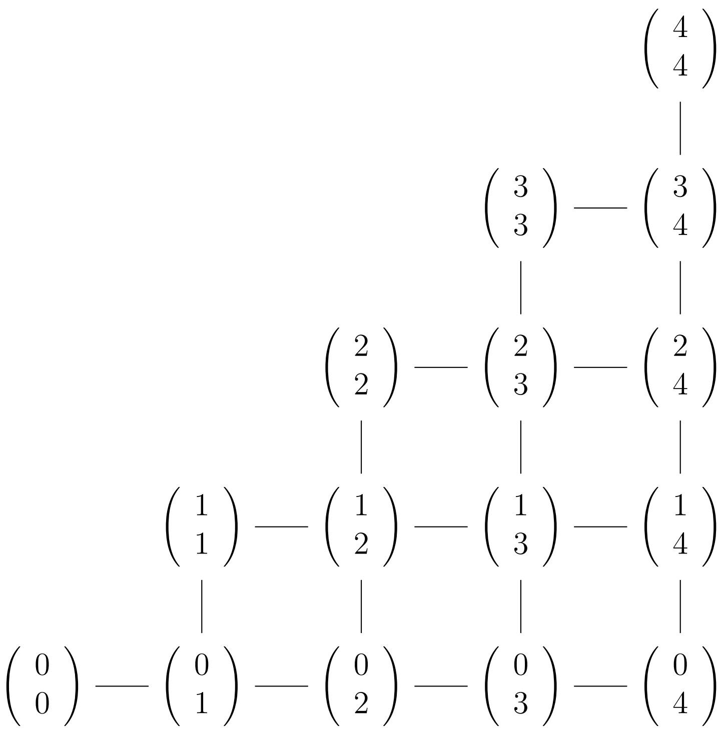
Then, the non-degenerate simplices of $Sd\triangle[n]$ are given by paths which lie entirely in a rectangle, and the n-simplices are the maximal paths in any

$(p \times q)$-rectangle, where $p + q = n$. Thus a non-degenerate m-simplex is described by a $(2 \times m)$ -matrix $\begin{pmatrix} i_0 & \cdots & i_m \\ i_{m+1} & \cdots & i_{2m+1} \end{pmatrix}$ where $(i_0, \cdots, i_{2m+1})$ is a $(2m+1)$-simplex of $\triangle[n]$, and $\begin{pmatrix} i_k \\ i_{m+k+1} \end{pmatrix} < \begin{pmatrix} i_{k+1} \\ i_{m+k+2} \end{pmatrix}$ for all $0 \leq k < m$.

Similarly, the non-degenerate simplices of $NerSd[n]$ are all paths in the diagram. Thus a non-degenerate m-simplex is described by a $(2 \times m)$-matrix $\begin{pmatrix} i_0 & \cdots & i_m \\ j_0 & \cdots & j_m \end{pmatrix}$ where $\begin{pmatrix} i_k \\ j_k \end{pmatrix} < \begin{pmatrix} i_{k+1} \\ j_{k+1} \end{pmatrix}$.

Note that the n-simplex $\begin{pmatrix} 0 & 1 & \cdots & n \\ 0 & 1 & \cdots & n \end{pmatrix}$ is in $NerSd[n]$ but not in $Sd\triangle[n]$, and also, that $NerSd[n]$ has $c(n+1)$ $2n$-simplices - where $c(n)$ is the n^{th} Catalan number. The n^{th} Catalan number is usually defined as the number of different bracketings of the word $a_1 a_2 \cdots a_n$, but there are other descriptions, one of which is the number of maximal paths under the diagonal of an $(n-1) \times (n-1)$-grid (that is a grid with n vertices on each side). For a description of Catalan numbers, and some of their properties and uses see [8] and [27].

As an example, consider $Sd\triangle[4]$. It may be described by the diagram:



The simplices of $Sd\triangle[4]$ are those paths which lie entirely in a particular rectangle - specifically, the non-degenerate 4-simplices are

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 4 & 4 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 2 & 3 & 3 & 4 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 2 \\ 2 & 3 & 4 & 4 & 4 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 2 & 3 & 3 & 4 & 4 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 4 & 4 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 2 & 2 \\ 2 & 3 & 3 & 3 & 4 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 2 & 3 \\ 3 & 4 & 4 & 4 & 4 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 1 & 2 & 3 \\ 3 & 3 & 4 & 4 & 4 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 & 2 & 3 \\ 3 & 3 & 3 & 4 & 4 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 & 3 & 3 \\ 3 & 3 & 3 & 3 & 4 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 & 4 \end{pmatrix},$$

The simplices of $NerSd[4]$ are all the paths of the diagram, and specifically include the following 8-simplices:

$$\begin{pmatrix} 0&0&0&0&0&1&2&3&4\\ 0&1&2&3&4&4&4&4&4 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&0&1&1&2&3&4\\ 0&1&2&3&3&4&4&4&4 \end{pmatrix},$$
$$\begin{pmatrix} 0&0&0&1&1&1&2&3&4\\ 0&1&2&2&3&4&4&4&4 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&0&1&2&2&3&4\\ 0&1&2&3&3&3&4&4&4 \end{pmatrix},$$
$$\begin{pmatrix} 0&0&1&1&1&1&2&3&4\\ 0&1&1&2&3&4&4&4&4 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&1&1&2&2&3&4\\ 0&1&2&2&3&3&4&4&4 \end{pmatrix},$$
$$\begin{pmatrix} 0&0&0&0&1&2&3&3&4\\ 0&1&2&3&3&3&3&4&4 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&1&2&2&2&3&4\\ 0&1&2&2&2&3&4&4&4 \end{pmatrix},$$
$$\begin{pmatrix} 0&0&1&1&1&2&2&3&4\\ 0&1&1&2&3&3&4&4&4 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&1&1&2&3&3&4\\ 0&1&2&2&3&3&3&4&4 \end{pmatrix},$$
$$\begin{pmatrix} 0&0&1&1&2&2&2&3&4\\ 0&1&1&2&2&3&4&4&4 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&1&2&2&3&3&4\\ 0&1&2&2&2&3&3&4&4 \end{pmatrix},$$
$$\begin{pmatrix} 0&0&1&1&1&2&3&3&4\\ 0&1&1&2&3&3&3&4&4 \end{pmatrix}, \quad \begin{pmatrix} 0&0&1&1&2&2&3&3&4\\ 0&1&1&2&2&3&3&4&4 \end{pmatrix},$$

To help keep control of the extension, certain definitions will be necessary.

Definition 5.2 (i)

The *weight* of a vertex,

$$w : \{\ \textit{vertices of } Sd\triangle[n]\} \longrightarrow \mathbb{N}, \text{ is defined by}$$

$$w\begin{pmatrix} i_r \\ j_r \end{pmatrix} = \begin{cases} i_r & \text{if } r \leq n \\ n - j_r & \text{if } r \geq n. \end{cases}$$

Definition 5.2 (ii)

The *depth* of a $2n$-simplex of $NerSd[n]$,

$$d : \{2n - \text{simplices of } NerSd[n]\} \longrightarrow \mathbb{N}, \text{ is defined by}$$

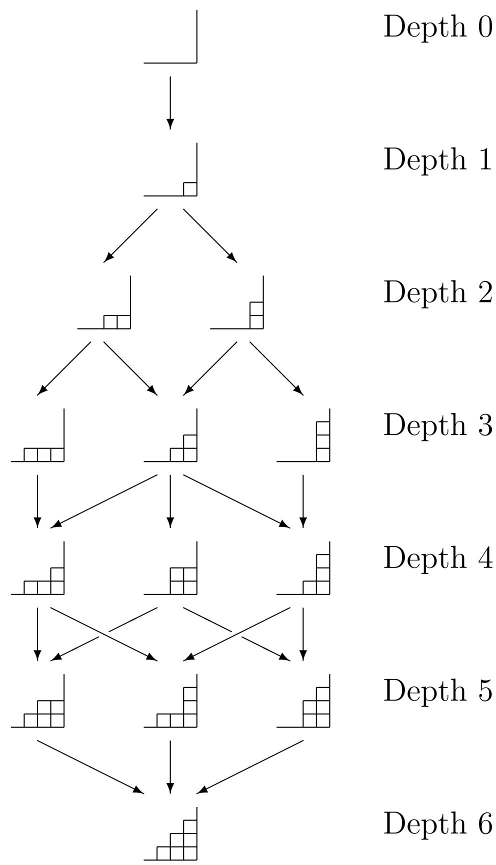
$$d : \begin{pmatrix} i_0 & \cdots & i_m \\ j_0 & \cdots & j_m \end{pmatrix} = \sum_{r=0}^{2n} w\begin{pmatrix} i_r \\ j_r \end{pmatrix}.$$

This depth function counts the number of squares in the grid (pictured above) which lie under the path described by the $2n$-simplex in question.

Definition 5.2 (iii)

Let x be a $2n$-simplex in $NerSd[n]$ of depth r (for $1 \leq r \leq n(n-1)/2$), a $2n$-simplex of depth $(r-1)$ will be called a *predecessor of x* if it differs from x at one vertex - that is, if they share a common $(2n-1)$-simplex. The number of predecessors of x will be denoted $p(x)$. The number of predecessors a simplex has is the number of "steps" the path representing it has in the diagram (irrespective of the height of the steps).

Thus, a path in the diagram may be uniquely described by the vertices at which the path turns from the vertical to the horizontal - the number of these vertices being the number of steps, and therefore the number of predecessors of the $2n$-simplex described by the path. It may be helpful to consider the *Hasse* diagram of the set of $2n$-simplices, where the relation is "is a predecessor of". The following is such a diagram for the case $n = 4$, with the $2n$-simplices denoted pictorially by the paths which represent them, and each arrow is to a $2n$-simplex from one of its predecessors.



More will be said on this in the proof of the main theorem.

If use is to be made of the cosimplicial structure of $NerSd[-]$ and $Sd\triangle[-]$, then some idea of it is needed. Consider the diagram used to describe the two simplicial sets $NerSd[n]$ and $Sd\triangle[n]$. The image of the cosimplicial morphism induced by the morphism $\delta_i \in \Delta$ is the subdiagram on all those vertices $\begin{pmatrix} j \\ k \end{pmatrix}$ for which $j \neq i$ and $k \neq i$. Before attempting to prove the main general result, consider the first few cases.

Lemma 5.2.1

Let X be a cosimplicial simplicial set where X^n is a weak Kan complex for each n, and let $f : Sd\triangle[-] \longrightarrow X$ be a cosimplicial simplicial morphism. Then, there exists a 2-truncated cosimplicial simplicial morphism $\overline{f} : tr^2(NerSd[-]) \longrightarrow tr^2(X)$ which extends $tr^2(f)$. Here tr^2 is the truncation of the cosimplicial structure at the 2-cosimplices.

The lemma essentially states that the general result holds up to the 2-cosimplices.

Proof

In this proof, reference will be made to lemma 5.3.1, which will not be proved until the next section. The lemma, which is not difficult, states the following: If $Y \subset \triangle[n+1]$ is generated by a set of n-faces of ι_{n+1}, which includes the 0^{th} and $(n+1)^{th}$ faces, and does not include all the faces, then for any morphism, f, from Y to a weak Kan complex, there is a morphism from $\triangle[n+1]$ with the same codomain, which extends f.

Note that $NerSd[0] \cong Sd\triangle[0] \cong \triangle[0]$, and that $NerSd[1] \cong \triangle[2]$ and $Sd\triangle[1] \cong \wedge^1[2]$. Thus, $\overline{f}^0 = f^0$ and $\overline{f}^1$ is defined by the weak Kan property of X^1, and the two must match up over the cosimplicial structure since $\overline{f}^0 = f^0$.

The case $n = 2$. The morphism $\overline{f}^2$ is defined on the four non-degenerate 2-simplices of $Sd\triangle[2]$ by f^2, that is
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 2 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 & 2 \\ 2 & 2 & 2 \end{pmatrix}$, and further, it is defined on three more non-degenerate 2-simplices of $NerSd[2]$ by inducting up the coskeleton of X; these are

$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}$.

Consider the 4-simplex of $NerSd[2]$: $\begin{pmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 & 2 \end{pmatrix}$.

As already shown, $\overline{f}^2$ has been defined over four of its 2-simplices (those not containing $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$). Using lemma 5.3.1, it is possible to extend $\overline{f}$ over the 3-simplices $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 2 & 2 & 2 \end{pmatrix}$. If it were not for the 2-simplex $\begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 2 \end{pmatrix}$ the extension would now be easy to define over $\begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 2 \end{pmatrix}$ by using lemma 5.3.1 once again. Instead, a little care must be taken: $\overline{f}$ may be extended over the three simplex $\begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 1 & 2 & 2 \end{pmatrix}$ since $\overline{f}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$, $\overline{f}\begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 2 \end{pmatrix}$ and $\overline{f}\begin{pmatrix} 0 & 0 & 2 \\ 1 & 2 & 2 \end{pmatrix}$ have all been defined, but $\overline{f}\begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}$ has not, and so the first three simplices form a simplicial 2-horn in X^2.

Thus $\overline{f}$ has been defined over three of the 3-simplices which are faces of $\begin{pmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 & 2 \end{pmatrix}$, including the 0^{th} and 4^{th}. Thus lemma 5.3.1 allows $\overline{f}$ to be extended to cover the 4-simplex itself.

Now consider the simplices of $\begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 2 \end{pmatrix}$ over which $\overline{f}$ has already been defined. Certainly it covers the 3-simplex $\begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 \end{pmatrix}$ since this is the only 3-simplex present in both 4-simplices. It also covers the three simplices $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, and $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}$. Therefore it is possible to extend $\overline{f}$ to cover $\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{pmatrix}$ since for each of these 3-simplices $\overline{f}$ has been defined over three faces (including the 0^{th} and 3^{rd} in each case). This means that $\overline{f}$ may be extended to cover the three of the 3-simplices which are faces of $\begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 2 \end{pmatrix}$, namely the 0^{th}, 2^{nd} and 4^{th}. Therefore the $\overline{f}$ may be defined as stated. ∎

As may be seen by this part of the theorem, the problem is not in having too few simplices over which $\overline{f}$ is already defined, but in having too many. It is also clear that to attempt to prove the main result using the "bare hands" approach of lemma 5.2.1 would be impractical.

It is necessary to define and develop a concept of weak anodyne extension, which will have the same property with respect to weak Kan complexes that anodyne extensions have with respect to Kan complexes. The next section does precisely that.

5.3 Weak Anodyne Extensions

Definition 5.3 (i)

A *Weak Anodyne Extension* is a morphism $i : Y \to X$ which is obtainable by a finite sequence $Y \xrightarrow{i_1} Y_1 \xrightarrow{i_2} Y_2 \cdots Y_{n-1} \xrightarrow{i_n} Y_n = X$ where there is a pushout diagram

$$\begin{array}{ccc} \wedge^k[m] & \hookrightarrow & \triangle[m] \\ \downarrow & \ulcorner & \downarrow \\ Y_j & \longrightarrow & Y_{j+1} \end{array}$$

for all $0 < j < n$, for some $0 < k < m$.

This definition lacks the categorical elegance of the definition of anodyne extension ([25]), and it may be possible to redefine the definition of weak anodyne extension along the same lines. However, the definition given here is sufficient for the purposes of this thesis.

The following technical lemmas prove that the embedding $Y \longrightarrow \triangle[n+1]$ is a weak anodyne extension, for certain subsimplicial sets, Y, of a standard $n+1$-simplex.

Lemma 5.3.1

Let $Y \subset \triangle[n{+}1]$ be generated by the n-simplices, $d_{\gamma_j}\iota_{n+1}$ for $0 \leq j \leq r$, where $1 \leq r \leq n$, $\gamma_j < \gamma_{j+1}$ for $0 \leq j \leq r-1$. Then the inclusion $Y \longrightarrow \triangle[n+1]$ is an anodyne extension, and if $\gamma_0 = 0,\ \ \gamma_r = n+1$, then the inclusion is a weak anodyne extension.

Proof

The proof is by induction. First consider the case $n = 1$. In this case, $r = 1$ and so $(\gamma_0, \gamma_1) \in \{(0,1), (0,2), (1,2)\}$. Therefore Y is a k-horn $\wedge^k[2]$, (for $0 \leq k \leq 2$), and so the result is trivial (as in all three cases, the embedding of Y in $\triangle[2]$ is an anodyne extension, and it is a weak anodyne extension in precisely the case $k = 1$).

Next, assume that the result is true for all $m < n$ and for all $1 \leq r \leq m$. Consider a set as described in the statement of the lemma. If $r = n$ then the simplices constitute a k-horn $\wedge^k[n+1]$ for some k (since there is precisely one k with $k \neq \gamma_j$ for all $0 \leq j \leq n$) and so the result is (again) trivial.

Then assume $r < n$. Let s be the smallest integer not equal to any of the

γ_j. Consider the following set of $(n-1)$-simplices:

$d_{\gamma_j} d_s \iota_{n+1}$ for $\gamma_j < s$ and $d_{\gamma_j - 1} d_s \iota_{n+1} =$ for $\gamma_j > s$.

These simplices are a set of faces of the n-simplex $< d_s \iota_{n+1} >$. Note that the face $d_0 d_s \iota_{n+1}$ is present *iff* $\gamma_0 = 0$ and the face $d_n d_s \iota_{n+1}$ is present *iff* $\gamma_r = n+1$. Further, these simplices generate $Y \cap < d_s \iota_{n+1} > \subset < d_s \iota_{n+1} >$. Thus, by induction, the embedding is an anodyne extension, and is weak anodyne if the 0^{th} and n^{th} faces are present in the set.

So, $Y \longrightarrow Y \cup < d_s \iota_{n+1} > := Y_1$ is an anodyne extension, and is a weak anodyne extension when the faces $d_0 \iota_{n+1}$ and $d_{n+1} \iota_{n+1}$ are both present in the set of generators of Y.

If $r + 1 = n$, then $Y_1 \cong \wedge^k[n+1]$ for some k; if not, repeat the process with a new s. Thus, it is possible to obtain a chain of anodyne extensions $Y \to Y_1 \to \cdots \to Y_{n-r} \cong \wedge^k[n+1]$ for some k, and these extensions are all weak anodyne extensions if the generators of Y include the faces $d_0 \iota_{n+1}$ and $d_{n+1} \iota_{n+1}$. Thus the lemma is proved. ∎

Lemma 5.3.2

Let Y be the subsimplicial set of $\triangle[n+1]$ generated by two simplices $x = d_0^{\alpha} \iota_{n+1}$ and $y = d_{n+2-\beta}^{\beta} \iota_{n+1}$, where $1 \leq \alpha, \beta \leq n$ and $\alpha + \beta \leq n+1$. Then the inclusion $i : Y \longrightarrow \triangle[n+1]$ is a weak anodyne extension.

Proof

Let $n = 1$. Then $\alpha = \beta = 1$ and the subsimplicial set generated by x and y

is $\wedge^1[2]$. The embedding of this into $\triangle[2]$ is weak anodyne. Then, assume the lemma has been proved for all cases up to $n-1$. The case $\alpha = \beta = 1$ has been dealt with by lemma 5.3.1.

Consider the case $\alpha \geq \beta \geq 1$. Then $d_0 y$ and x are in the standard n-simplex generated by $d_0 \iota_{n+1}$. The subsimplicial set generated by x and $d_0 y$ embeds into the standard n-simplex generated by $d_0 \iota_{n+1}$, and since $d_0^{\alpha-1}(d_0 \iota_{n+1}) = x$, and $d_0 d_{n+2-\beta}^{\beta} y = d_{n+1-\beta}^{\beta} d_0 y$, the embedding is a weak anodyne extension.

Then, $d_0 \iota_{n+1}$ and y are n-simplices with $d_0 y = d_n d_0 \iota_{n+1}$ and so the subsimplicial set generated by them embeds into $\triangle[n+1]$ by a weak anodyne extension. (The case $\beta \geq \alpha \geq 1$ is conjugate to this). ∎

Lemma 5.3.3

Let Y be the subsimplicial set of $\triangle[n+1]$ generated by the simplices $x_0, \cdots, x_k$, where x_i is an m_i-simplex, $d_0^{\alpha_{i-1}} x_{i-1} = d_{m_i-\beta_i+1}^{\beta_i} x_i$ for $1 \leq i \leq k-1$, $\beta_i, \alpha_i > 0$ for all i and $m_0 + \sum_{i=1}^{k} \beta_i = \sum_{i=0}^{k-1} \alpha_i + m_k = n+1$. Then the inclusion $Y \longrightarrow \triangle[n+1]$ is a weak anodyne extension.

Proof

The case $k = 1$ has been dealt with by lemma 5.3.2. Assume that the lemma has been dealt with for all k for all $m < n+1$, and for $m = n+1$ for all cases up to $k-1$.

Then, given Y generated by $x_0, \cdots, x_k$ as described in the statement of the theorem, the subsimplicial set generated by $x_0, \cdots, x_{k-1}$ embeds into the subsimplicial set generated by $d^{\beta_k}_{m_k-\beta_k+1}\iota_{n+1}$, and by the inductive assumption this embedding is a weak anodyne extension. Then, the simplicial set generated by x_k and $d^{\beta_k}_{m_k-\beta_k+1}\iota_{n+1}$ satisfy the conditions of lemma 5.3.2. ∎

Corollary 5.3.4

Let X be the simplicial complex with vertices $0 < 1 < \cdots < 2n$, generated by the n-simplices $(s, s+1, \cdots, s+n)$ for $0 \leq s \leq n$, and let X' be the $2n$-simplex $(0, 1, \cdots, 2n)$. Then the natural inclusion $i : X \longrightarrow X'$ is a weak anodyne extension.

Proof

Consider the pair of simplices $(s, s+1, \cdots, s+n)$, $(s+1, s+2, \cdots, s+n+1)$ for any $0 \leq s \leq n-1$. The pair satisfies the conditions of lemma 5.3.1, with $r = 1$ and $\gamma_0 = 0$, $\gamma_1 = n$. Thus the embedding of the pair into the $n+1$-simplex $(s, s+1, \cdots, s+n+1)$ is a weak anodyne extension. Thus there is a weak anodyne extension from X to the simplicial complex generated by the $n+1$-simplices $(s, s+1, \cdots, s+n+1)$ for $0 \leq s \leq n-1$. Repeating the process inductively extends the simplicial complex X to X' as required. ∎

Let X be the simplicial set which is the $2n$-simplex on the vertices $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ n \end{pmatrix}, \begin{pmatrix} 1 \\ n \end{pmatrix}, \cdots \begin{pmatrix} n \\ n \end{pmatrix}$. The corollary effectively proves that $Sd\triangle[n] \longrightarrow Sd\triangle[n] \cup X$ is a weak anodyne extension.

Lemma 5.3.5

Let $Y \subset \triangle[n+1]$ be generated by the n-simplices $d_{\gamma_j}\iota_{n+1}$, for $0 \leq j \leq r$, where $1 \leq r \leq n$, $\gamma_j \leq \gamma_{j+1}$ for $0 \leq j \leq r-1$. The embedding $Y \longrightarrow \triangle[n+1]$ is a weak anodyne extension if there is some c, $\gamma_0 < c < \gamma_r$, with $c \neq \gamma_j$ for all $0 \leq j \leq r$.

Proof

Consider the case $n = 1$. In this case, the conditions require that Y be generated by precisely two 1-simplices. If one of them is $d_0\iota_2$ and the other $d_2\iota_2$, then the conditions are satisfied for weak anodyne extension: if not, then the conditions fail (but anodyne extension is possible).

Now, assume that the lemma is true for all $n < m$. Let Y be generated by the m-simplices $d_{\gamma_j}\iota_{m+1}$ for $0 \leq j \leq r$, where $1 \leq r \leq m$, and assume that there is some c where $\gamma_0 < c < \gamma_r$ and $c \neq \gamma_j$ for all $0 \leq j \leq r$.

If $\gamma_0 \neq 0$, then consider the simplices $d_0 d_{\gamma_j}\iota_{m+1} = d_{\gamma_j - 1} d_0 \iota_{m+1}$. These $(m-1)$-simplices form a set of generators for $Y \cap < d_0\iota_{m+1} > \subset < d_0\iota_{m+1} >$, and since $c - 1 \neq \gamma_j - 1$ and $\gamma_0 - 1 < c - 1 < \gamma_j - 1$, they satisfy the conditions of the lemma, and so by induction, $Y \cap < d_0\iota_{m+1} > \longrightarrow < d_0\iota_{m+1} >$ is a weak anodyne extension, and so $Y \longrightarrow Y \cup < d_0\iota_{m+1} >$ is a weak anodyne extension. Thus, it is possible to extend Y by weak anodyne extensions to a simplicial subcomplex $Y' \subset \triangle[m+1]$ where Y' is generated by $d_0\iota_{m+1}, d_{\gamma_0}\iota_{m+1}, d_{\gamma_1}\iota_{m+1}, \cdots, d_{\gamma_r}\iota_{m+1}$.

If $\gamma_r \neq m+1$, then consider the simplex $d_m d_0 \iota_{m+1} = d_0 d_{m+1} \iota_{m+1}$ together with the simplices $d_m d_{\gamma_j}\iota_{m+1} = d_{\gamma_j} d_{m+1}\iota_{m+1}$. These $(m-1)$-simplices form the generators of $Y' \cap < d_{m+1}\iota_{m+1} >$ and they satisfy the conditions of the lemma (since $0 \leq \gamma_0 < c < \gamma_r$ and $c \neq 0$ and $c \neq \gamma_j$ for $0 \leq j \leq r$). Thus it is possible to extend Y' by a weak anodyne extension to a simplicial set $Y'' \subset \triangle[m+1]$, where Y'' is generated by the m-simplices $d_0\iota_{m+1}, d_{m+1}\iota_{m+1}$ and $d_{\gamma_j}\iota_{m+1}$ for $0 \leq j \leq r$, $1 \leq r \leq m$.

Since $0 \neq c$ and $m+1 \neq c$, it is still true that none of the generators is $d_c\iota_{m+1}$, and so by lemma 5.3.1 $Y'' \longrightarrow \triangle[m+1]$ is a weak anodyne extension, and so $Y \longrightarrow \triangle[m+1]$ is a weak anodyne extension. ■

Lemma 5.3.6

Let $Y \subset \triangle[n+1]$ be generated by the following simplices:

$d_0^{\alpha}\iota_{n+1}$ (for $1 \leq \alpha \leq n$), $d_{n+2-\beta}^{\beta}\iota_{n+1}$ (for $1 \leq \beta \leq n$), $d_{\gamma_j}\iota_{n+1}$ (for $1 \leq j \leq p$, $0 \leq p \leq n$ and $0 < \gamma_j < \gamma_{j+1} < n+1$) and $d_{\rho_k}d_{\kappa_k}\iota_{n+1}$ (for $1 \leq k \leq q$, $0 \leq q \leq \frac{n}{2}$, $0 < \rho_k < \rho_{k+1} < n+1$, $1 < \rho_k + 1 < \kappa_k < n+1$),

where $p = 0$ implies the set of n-simplices is empty, $q = 0$ implies the set of $n-1$-simplices is empty, all the ρ_k,κ_k and γ_j are distinct, $2q + p < n$ and the $(n-1)$ and n-simplices are maximal in Y. If there is a c, $0 < c < n+1$ such that the c^{th} vertex of ι_{n+1} is a vertex of all the generators of Y, then $Y \longrightarrow \triangle[n+1]$ is a weak anodyne extension.

Proof

If both $p = 0$ and $q = 0$, then the lemma reduces to lemma 5.3.2, and so is proved. If $n = 1$, then $Y \cong \wedge^1[2]$, and so the lemma is trivial. Note, the conditions on ρ_k and κ_k imply that $q = 0$ unless $n \geq 4$.

Assume the lemma is true for $1 \leq n < m$ and consider $Y \subset \triangle[n+1]$ as described. Consider $Y \cap d_{\kappa_q}\iota_{n+1}$. The conditions on the vertex, c, imply that the simplices $d_0^{\alpha}\iota_{n+1}$ and $d_{n+2-\beta}^{\beta}\iota_{n+1}$ span the vertices of $\triangle[n+1]$. Thus, either $\alpha \geq \kappa_q$ or $n+2-\beta \leq \kappa_q$.

If $\alpha \geq \kappa_q$ and $\kappa_q < n+2-\beta$, then $Y \cap < d_{\rho_1}\iota_{n+1} >$ is generated by $d_0^{\alpha}\iota_{n+1} = d_0^{\alpha-1}d_{\kappa_q}\iota_{n+1}$, $d_{n+1-\beta}^{\beta}d_{\kappa_q}\iota_{n+1}$, the $n-1$-simplices $d_{\gamma_j}d_{\kappa_q}\iota_{n+1}$ for $\gamma_j < \kappa_q$, $d_{\gamma_j-1}d_{\kappa_q}\iota_{n+1}$ for $\gamma_j > \kappa_q$, $d_{\rho_q}d_{\kappa_q}\iota_{n+1}$ and the $n-2$-simplices $d_{\rho_k}d_{\kappa_k}d_{\kappa_q}\iota_{n+1}$ for $1 \leq k \leq q-1$. If $c < \kappa_q$ then the c^{th} vertex of $d_{\kappa_q}\iota_{n+1}$ is in all the simplices described, and if $c > \kappa_q$, then the $(c-1)^{th}$ vertex of $d_{s_q}\iota_{n+1}$ is in all the simplices described.

In neither case is it an end vertex (i.e. $0 < c < n$ or $0 < c-1 < n$ which ever is applicable), and further, $\beta < n$ because if $\beta = n$, then $d_{n+1-\beta}d_{\kappa_q}\iota_{n+1}$ would be a vertex, whereas the existence of c implies it is at least a 1-simplex. Lastly, the number of $n-3$-simplices is $q-1$, and the number of $n-1$-simplices is $p+1$ and $2(q-1)+(p+1) = 2q+p-1 < n-1$ as required by the conditions of the Lemma.

Thus, $Y \cap < d_{\kappa_q}\iota_{n+1} > \longrightarrow < d_{\kappa_q}\iota_{n+1} >$ is a weak anodyne extension (by induction) and so $Y \longrightarrow Y \cup < d_{s_q}\iota_{n+1} > := Y_1$ is a weak anodyne extension.

If $\kappa_q > \alpha$ and $\kappa_q \geq n+2-\beta$, then the generators of $Y \cap < d_{\kappa_q}\iota_{n+1} >$ are the same as before, except that $d_0^{\alpha}\iota_{n+1}$ is replaced with $d_0^{\alpha}d_{\kappa_q}\iota_{n+1}$ and $d_{n+1-\beta}^{\beta}d_{\kappa_q}\iota_{n+1}$ is replaced with $d_{n+2-\beta}^{\beta}\iota_{n+1} = d_{n+2-\beta}^{\beta-1}d_{\kappa_q}\iota_{n+1}$. The problem now is the possibility that $\alpha = n$, but just as before, the existence of the vertex c means that this is *not* a problem. Thus, Y_1 may be defined as above, and $Y \longrightarrow Y_1$ is a weak anodyne extension.

Lastly, if $\kappa_q \leq \alpha$ and $\kappa_q \geq n+2-\beta$, then the first two generators of $Y \cap < d_{\kappa_q}\iota_{n+1} >$ become

$$d_0^{\alpha}\iota_{n+1} = d_0^{\alpha-1}d_{\kappa_q}\iota_{n+1} \text{ and } d_{n+2-\beta}^{\beta}\iota_{n+1} = d_{n+2-\beta}^{\beta-1}d_{\kappa_q}\iota_{n+1},$$

and so Y_1 may be defined as before.

Then, Y_1 is generated by a set of simplices as described by the initial conditions, except that there is one more n-simplex, and one less $n-1$-simplex. Continuing this process, it is possible to define a set chain of weak anodyne extensions $Y \to Y_1 \to \cdots \to Y_q$, where Y_q is generated by the $p+q$ n-simplices $d_{\gamma_j}\iota_{n+1}$ (for $1 \leq j \leq p$) $d_{\rho_k}\iota_{n+1}$ (for $1 \leq k \leq q$) and the two simplices $d_0^{\alpha}\iota_{n+1}$, and $d_{n+2-\beta}^{\beta}\iota_{n+1}$.

If $c = 1$, then $d_0^{\alpha}\iota_{n+1}$ is maximal, and further must be $d_0\iota_{n+1}$. Consider $Y_q \cap < d_{n+3-\beta}^{\beta-1}\iota_{n+1} >$. It is generated by a set of $(n+2-\beta)$-simplices including the 0^{th} and $(n+3-\beta)^{th}$ faces, and so
$Y_q \cap < d_{n+3-\beta}^{\beta-1}\iota_{n+1} > \longrightarrow < d_{n+3-\beta}^{\beta-1}\iota_{n+1} >$ is a weak anodyne extension.
So $Y_q \longrightarrow Y_q \cup < d_{n+3-\beta}^{\beta-1}\iota_{n+1} >$ is a weak anodyne extension. Clearly this process may be continued, extending the system to Y' generated by the n-simplices $d_0\iota_{n+1}$, $d_{n+1}\iota_{n+1}$, $d_{\gamma_j}\iota_{n+1}$ and $d_{\rho_k}\iota_{n+1}$ and specifically missing the face
$d_1\iota_{n+1}$. Thus, by lemma 5.3.1 $Y' \longrightarrow \triangle[n+1]$ is a weak anodyne extension, so $Y \longrightarrow Y_q \longrightarrow Y' \longrightarrow \triangle[n+1]$ is a chain of weak anodyne extensions.

If, rather $c = n$, the conjugate argument works, that is $d_{n+2-\beta}^{\beta}\iota_{n+1}$ must be $d_{n+1}\iota_{n+1}$, and the induction is defined on $d_0^{\alpha}\iota_{n+1}$.

In the case $1 < c < n$, then $Y_q \cap < d_0\iota_{n+1} >$ is generated by a suitable system of simplices and the $(c-1)^{th}$ vertex of $d_0\iota_{n+1}$ $(0 < c-1 < n)$ is in all the simplices generating $Y_q \cap < d_0\iota_{n+1} >$.

Thus, $Y_q \longrightarrow Y_q \cup Y_q \cap < d_0\iota_{n+1} >$ is a weak anodyne extension, and similarly, $(Y_q \cup \cap < d_0\iota_{n+1} >) \cap < d_{n+1}\iota_{n+1} >$ is generated by a suitable set of simplices, and the c^{th} vertex of $d_{n+1}\iota_{n+1}$ $(0 < c < n)$ is in all the simplices generating $(Y_q \cup \cap < d_0\iota_{n+1} >) \cap < d_{n+1}\iota_{n+1} >$ and so

$Y_q \longrightarrow Y' \longrightarrow \triangle[n+1]$ is a weak anodyne extension.

This completes the proof of the lemma. ■

One more lemma of this form is needed before moving on to the next section, where they are put into practice.

Lemma 5.3.7

Let $Y \subset \triangle[n+1]$ be generated by the following simplices:

$x_0, x_1 \cdots, x_l$ *(for $l \geq 1$),*

$d_{\gamma_j}\iota_{n+1}$ *(for $1 \leq j \leq p$, $0 \leq p \leq n$ and $0 < \gamma_j < \gamma_{j+1} < n+1$)*

and $d_{\rho_k}d_{\kappa_k}\iota_{n+1}$ *(for $1 \leq k \leq q$ $0 \leq q \leq \frac{n}{2}$, $0 < \rho_k < \rho_{k+1} < n+1$,*

$1 < \rho_k + 1 < \kappa_k < n+1$),

where $p = 0$ implies the set of n-simplices is empty,

$q = 0$ implies the set of $n-1$-simplices is empty,

all the ρ_k, κ_k and γ_j are distinct,

x_i is an m_i-simplex (where $m_i \geq 1$ for $0 \leq i \leq l$),

$$x_0 = d_{m_0+1}^{n+1-m_0}\iota_{n+1},\ x_l = d_0^{n+1-m_l},\ d_0^{\alpha_{i-1}}x_{i-1} = d_{m_i+1-\beta_i}^{\beta_i}x_i$$

(for $1 \le i \le l$, where $\alpha_i \ge 1$, $\beta_i \ge 1$, $\sum_{i=0}^{l-1}\alpha_i + m_l = m_0 + \sum_{i=1}^{l}\beta_i = n+1$ and $m_0 + m_l \ge n+1$), where all these generators with the exception of x_0 and x_r are maximal in Y and $2q+p<n$.

If for each pair $x_\eta, x_{\eta+1}$ there is a c_η, $\sum_{i=0}^{\eta-1}\alpha_i + 1 < c_\eta < \sum_{i=0}^{\eta}\alpha_i + m_{i+1}$ such that the c_η^{th} vertex of ι_{n+1} is a vertex of $d_{\gamma_j}\iota_{n+1}$ for $1 \le j \le p$ and $d_{\rho_k}d_{\kappa_k}$ for $1 \le k \le q$, then $Y \longrightarrow \triangle[n+1]$ is a weak anodyne extension.

Proof

If $l = 1$, the proposition reduces to lemma 5.3.6 . Consider a system as described, and assume that the proposition holds for any similar system with either a smaller n or a smaller l.

Consider x_0 and x_l. They have at least one common vertex. If ρ_1 is not the only common vertex of x_0 and x_l, then define $Y' := Y \cap < d_{\rho_1}\iota_{n+1} >$.

Then, Y' is generated by $(x_i \cap d_{\rho_1}\iota_{n+1})$ (for $0 \le i \le l$), $d_{\gamma_j}d_{\rho_1}\iota_{n+1}$ (for $\gamma_j < \rho_1$), $d_{\gamma_j-1}d_{\rho_1}\iota_{n+1}$ (for $\gamma_j > \rho_1$), $d_{\kappa_1-1}d_{\rho_1}\iota_{n+1}$ and $d_{\rho_k-1}d_{\kappa_k-1}d_{\rho_1}$ (for $2 \le k \le q$).

These generators satisfy the conditions of the proposition: in particular, the x_i still intersect as before, because it is assumed that ρ_1 is not the only connecting vertex for x_0 and x_l; there are now $q-1$ $(n-2)$-simplex generators and $p+1$ $(n-1)$-simplex generators (and $2(q-1)+p+1 < n-1$ as required). Thus, $Y' \longrightarrow < d_{\rho_1}\iota_{n+1} >$ is a weak anodyne extension, and therefore $Y \longrightarrow Y \cup < d_{\rho_1}\iota_{n+1} >$ is a weak anodyne extension.

If ρ_1 is the only vertex common to both x_0 and x_l, then define

$Y' := Y \cap < d_{\kappa_1}\iota_{n+1} >$. The definition of $(n-1)$-simplex and n-simplex generators is a little more involved, the proof is essentially the same.

Define $Y_1 := Y \cup < d_{\rho_1}\iota_{n+1} >$ in the first case, and
$Y_1 := Y \cup < d_{\kappa_1}\iota_{n+1} >$ in the second. The generators x_i may be altered by this, in particular, the generators $x_2, \cdots x_{l-1}$ may become redundant, and the generators x_0 and x_l may be subfaces of larger simplices. However, they will still meet (in fact, the overlap may be increased) and so the Y_1 will still be generated by a set of simplices which satisfies the conditions.

This process may then be continued, so that there is a chain of weak anodyne extensions, $Y \longrightarrow Y_1 \cdots \rightarrow Y_q$, where $Y_k := Y_{k-1} \cup < d_{\rho_k}\iota_{n+1} >$ if ρ_k is not the only vertex common to x_0 and x_l (where these are taken to be the modified x_0 and x_l of Y_{k-1}), and where $Y_k := Y_{k-1} \cup < s_{\kappa_k}\iota_{n+1} >$ otherwise.

Thus Y_q will be generated by a set of n-simplices, $d_{\gamma_j}\iota_{n+1}$ (for $1 \leq j \leq p$), (for each $1 \leq k \leq q$) *either* $d_{\rho_k}\iota_{n+1}$ *or* $d_{\kappa_k}\iota_{n+1}$, and a set of overlapping simplices, $x_0, \cdots, x_{l'}$ (where $l' \leq l$). We will assume that the notation for Y_q alters from that of Y, so that all the n-simplex generators will be $d_{\gamma_j}\iota_{n+1}$ for $1 \leq j \leq p+q$, and the overlaps of the x_i's (which will be described as m_i-simplices) will be described by the α_i's and β_i's as before.

Then, consider $Y'' := Y_q \cap < d_0^{\alpha_0}\iota_{n+1} >$. If $\alpha_0 \geq \gamma_1$, then x_l overlaps all the x_i for $1 \leq i \leq l'-1$, which means that $l' = 1$, and so the theorem is reduced to lemma 5.3.6.

Otherwise, if $\alpha_0 + 1 \neq \gamma_1$, then Y'' will be the simplicial subset of

$< \ d_0^{\alpha_0}\iota_{n+1} \ >$ generated by the simplices $x_1, \cdots, x_l$, together with the $(n-\alpha_0)$-simplices $d_{\gamma_j-\alpha_0}d_0^{\alpha_0}\iota_{n+1}$. This set of generators satisfies the conditions of the theorem, and so by induction, $Y_q \longrightarrow Y_q\cup < \ d_0^{\alpha_0}\iota_{n+1} \ >$ is a weak anodyne extension. But $Y_q\cup < \ d_0^{\alpha_0}\iota_{n+1} \ >$ is generated by the n-simplices $d_{\gamma_j}\iota_{n+1}$ together with x_0 and $d_0^{\alpha_0}\iota_{n+1}$, and so there is a weak anodyne extension $Y_q\cup < \ d_0^{\alpha_0} \ >\longrightarrow \triangle[n+1]$ by lemma 5.3.1, again.

If $\gamma_1 = \alpha_0+1$, then Y'' is generated by the $n-\alpha_0$-simplices $d_{\gamma_j-\alpha_0}d_0^{\alpha_0}\iota_{n+1}$ (for $2 \leq j \leq p+q$) and by the simplices x_2 and $d_0^{\alpha_0+1}\iota_{n+1} = d_0^{\alpha_0}d_{\gamma_1}\iota_{n+1}$. Therefore, again, the extension is reduced to lemma 5.3.6, and so $Y'' \longrightarrow< \ d_0^{\alpha_0}\iota_{n+1} \ >$ is a weak anodyne extension, and so $Y_q\cup < \ d_0^{\alpha_0} \ >\longrightarrow \triangle[n+1]$ is a weak anodyne extension.

This completes the proof of the proposition. ∎

It is now possible to prove the main result. It must be said that while it lacks both the "bare hands" approach of lemma 5.2.1, and the technical messiness of the lemmas from the last section, it could in no way be described as elegant.

5.4 The Main Result

Theorem 5.4.1 *Let X be a cosimplicial simplicial set where X^n is a weak Kan complex for each n, and let $f : Sd\triangle[-] \longrightarrow X$ be a cosimplicial simplicial morphism. Then, there exists a cosimplicial simplicial morphism*

$\overline{f} : (NerSd[-]) \longrightarrow X$ *which extends* f.

Proof

Lemma 5.2.1 proved that it was possible to define the extension up to the second level. Therefore, assume that $\overline{f}$ has been defined over $tr^{n-1}NerSd[-]$ and consider level n (where $n \geq 3$).

By means of the $n-1$ coskeleton (the images of $NerSd(\delta_i) : NerSd[n-1] \longrightarrow NerSd[n]$) $\overline{f}^n$ may be defined for a large amount of the structure of $NerSd[n]$.

In lemma 5.2.1, $\overline{f}^2$ was defined over the $2n$-simplex of depth 0 before it was defined over the $2n$-simplex of depth 1. This will be a principle for the general case, that $\overline{f}$ must be defined on the $2n$-simplices of depth less than r before they are defined on those of depth r. This will ensure that before $\overline{f}$ is defined on a $2n$-simplex, it is defined on all its predecessors. Thus, the process must start with the $2n$-simplex of depth 0.

Therefore, consider the simplices of $\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & \cdots & n \\ 0 & 1 & \cdots & n & n & \cdots & n \end{pmatrix}$ over which $\overline{f}^n$ has been defined. These are all the n-simplices of $Sd\triangle[n]$, (of which there are $n+1$), together with the $n-1$ $2n-2$-simplices which come from extending over the coskeleton, that is:
$\begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & i-1 & i+1 & \cdots & n \\ 0 & \cdots & i-1 & i+1 & \cdots & n & \cdots & n & n & \cdots & n \end{pmatrix}$ for $0 < i < n$.

Consider this in the notation of proposition 5.3.7: n is $2n-1$, the x_i are the n-simplices arising from $Sd\triangle[n]$, and the simplices $d_{\rho_k} d_{\kappa_k} \iota_{2n}$ are those arising from $NerSd[n-1]$, where $\rho_k = k$, $\kappa_k = n+k$ and $1 \leq k \leq n-1$. Since the n^{th} vertex is common to all these generators, the embedding of the subsimplicial set generated by these simplices into the $2n$-simplex of depth 0 is a weak anodyne extension, and so $\overline{f}$ may be extended over it.

Next, consider the $2n$-simplex of depth 1. In this case, $\overline{f}$ is already defined on the n-simplices $\begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & n-1 & n-1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & \cdots & n \\ n-1 & n & \cdots & n \end{pmatrix}$ and $\begin{pmatrix} 0 & \cdots & 0 & 1 & 1 & \cdots & i \\ i & \cdots & n-1 & n-1 & n & \cdots & n \end{pmatrix}$ (for $1 \leq i \leq n-1$), on the $(2n-2)$-simplices $\begin{pmatrix} 0\cdots & 0 & 0 & \cdots & 0 & 1 & 1\cdots & j-1 & j+1 & \cdots n \\ 0\cdots & j-1 & j+1 & \cdots & n-1 & n-1 & n\cdots & n & n & \cdots n \end{pmatrix}$ (for $2 \leq j \leq n-2$), and on the $2n-1$-simplex $\begin{pmatrix} 0 & \cdots & 0 & 1 & \cdots & n \\ 0 & \cdots & n-1 & n & \cdots & n \end{pmatrix}$.

Again, the generators satisfy the conditions of proposition 5.3.7, although this time there is no vertex common to all the simplices. In fact, the $(n-1)^{th}$ vertex of ι_{2n} is common to all the simplices except $\begin{pmatrix} 1 & 1 & \cdots & n \\ n-1 & n & \cdots & n \end{pmatrix}$, and the $(n+1)^{th}$ vertex of ι_{2n} is common to all the simplices except $\begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & n-1 & n-1 \end{pmatrix}$.
Thus the embedding of the subsimplicial set generated by these simplices into the $2n$-simplex of depth 1 is a weak anodyne extension, and so $\overline{f}$ may be further extended, so that is defined on it.

Now, consider x, a general non-degenerate $2n$-simplex of $NerSd[n]$, and assume that $\overline{f}$ has been defined over all the simplices of less depth than x: in particular, over all its predecessors. Note, that it does not matter in which order the simplices of a certain depth are dealt with, as the simplices common to two $2n$-simplices of the same depth are contained in a $2n$-simplex of less depth.

Let $Y \subset < x >$ be generated by the simplices of $< x >$ on which $\overline{f}$ has been defined up to that point: if it can be shown that $Y \longrightarrow < x >$ is a weak anodyne extension (by proposition 5.3.7), then $\overline{f}$ can be extended over x and so over the whole of $NerSd[n]$.

As has been noted earlier, x is determined by the vertices of $NerSd[n]$ where the path describing x turns from the vertical to the horizontal. Assume that x turns at the vertices: $\left\{ \begin{pmatrix} \gamma'_j \\ \gamma''_j \end{pmatrix} \mid 0 \leq \gamma'_j \leq \gamma''_j \leq n, 1 \leq j \leq p \right\}$.

Note that each of x's predecessors intersects x in a $(2n-1)$-simplex, and the missing vertex in each of these faces is one of the turning points of x, so that in the notation of proposition 5.3.7, $\gamma_j = \gamma'_j + \gamma''_j$.

Then, $\delta_k(NerSd[n-1])$ intersects x in a $(2n-2)$-simplex for $0 < k < n$, precisely when $k \neq \gamma'_j, \gamma''_j$ for all $1 \leq j \leq p$.

Further, $\delta_0(NerSd[n-1])$ and $\delta_n(NerSd[n-1])$ give x_0 and x_l (so long as x has depth greater than 1), and $m_0, m_l \geq n$ (and so x_0 and x_l will always connect with each other).

Lastly, $Sd\triangle[n]$ intersects with x in the n-simplices $\psi_{m,n}$, for $\gamma'_p \leq m \leq \gamma''_1$, where $\psi_{m,n}$ is the nondegenerate n-simplex of x which has initial vertex the m^{th} vertex of x, and final vertex the $(m+n)^{th}$ vertex of x. Note that these simplices only occur when $\gamma'_p \leq \gamma''_1$. Otherwise, the simplices common to x and $Sd\triangle[n]$ are contained in the simplices already described.

It only remains to show that there are suitable common vertices for these simplices so that proposition 5.3.7 may be used. A moment of thought will show that $\begin{pmatrix} 0 \\ \gamma''_1 \end{pmatrix}$ is common to all except x_0, which is $\begin{pmatrix} 0 & \cdots & \gamma'_p \\ 0 & \cdots & n-1 \end{pmatrix}$, and that $\begin{pmatrix} \gamma'_p \\ n \end{pmatrix}$ is common to all except x_l, which is $\begin{pmatrix} 1 & \cdots & n \\ \gamma''_1 & \cdots & n \end{pmatrix}$.

Therefore, so long as $\gamma'_p \leq \gamma''_1$, this will be sufficient (as in this case, $l \geq 2$). If it is not the case, then $l = 1$, and a common vertex for all the generating simplices is required. Note that if $p = 1$, then $\gamma'_p = \gamma'_1 \leq \gamma''_1$. Thus, if $l = 1$, then $p \geq 2$.

Now consider the vertex $\begin{pmatrix} \gamma'_1 \\ \gamma''_1 + 1 \end{pmatrix}$. This vertex is clearly common to all the $2n-1$-simplices on which $\overline{f}$ is already defined. It is also common to all the $2n-2$-simplices which derive from $\delta_k NerSd[n-1]$ since these are only relevant when $k \neq \gamma'_j, \gamma''_j$ for $1 \leq j \leq p$. Lastly, it is common to both x_0 and x_1, since $1 \leq \gamma'_1 \leq n-1$ and $1 \leq \gamma''_1 + 1 \leq \gamma''_2 \leq n-1$.

This concludes the proof of the theorem. ∎

Chapter 6

A Van Kampen Type Theorem

The aim of this chapter is describe how the ideas of subdivision and extension outlined in the last two chapters may be used to obtain Van Kampen type results in the category of simplicially enriched groupoids.

It should be stressed at this point that the classical Van Kampen theorem, and similar results in the literature all deal with topological data, rather than with topological spaces: the classical Van Kampen Theorem deals with pointed spaces, the Van Kampen for the fundamental groupoid deals with spaces with a set of base points (see [9]), the Van Kampen Theorem for crossed complexes deals with filtered topological spaces (see [11]), and the Van Kampen Theorem for cat^n-groups deals with n-cubes of spaces (see [13]). It is also noticeable that strong connectivity conditions are needed in all cases. However, these results do make the actual calculation of homotopy invariants for certain types of data easier.

The main result of this chapter is simply about the preservation of certain pushouts in the category of topological spaces by a functor which is a quotient of the loop groupoid functor. It does not give an explicit way of calculating homotopy types of particular spaces; it is best thought of as a staging post between the loop groupoid functor (which does not satisfy such a theorem) and the fundamental constructions mentioned in the last paragraph (which do). Subtler questions, for example the particular place of CW-complexes and free objects in relation to this work, will not be discussed, since the theory is not yet in an advanced enough state to be able to handle such question.

Before starting, recall the cosimplicial topological morphism $\tau_* : \triangle^* \longrightarrow \triangle^* * \triangle^*$ defined by

$$\tau_n(t_0, t_1, \cdots, t_n) = (\frac{t_0}{2}, \frac{t_1}{2}, \cdots, \frac{t_n}{2}, \frac{t_0}{2}, \cdots, \frac{t_n}{2})$$

which was introduced in section 4.2.

It induced a cosimplicial simplicial morphism $\tau_*^* : SdSing\mathcal{W} \longrightarrow Sing\mathcal{W}$, which took a subdivided simplex in $SdSing\mathcal{W}$ to what was geometrically expected of a subdivision.

It will also be necessary to work with three different adjunctions: $|-| \dashv Sing$ (introduced in section 2.6), $\Pi \dashv Ner$ (introduced in section 2.2) and $G \dashv \overline{W}$ (introduced in section 2.3). Although the unit and counit of the $|-| \dashv Sing$ adjunction will not be explicitly needed, the units of the other two adjunctions will be. To avoid confusion, the following convention has been adopted: η and ϵ shall denote the unit and counit of the $G \dashv \overline{W}$ adjunction respectively; the unit of the $\Pi \dashv Ner$ adjunction will be denoted by λ. The counit is always an identity, and will be ignored.

Further, given a topological space $\mathcal{W}$ and a simplicial set X, then $\sigma \in SS(X, Sing\mathcal{W})$ will also be used to denote the bijective image in $\mathcal{T}op(|X|, \mathcal{W})$, as well as the subspace $\sigma(|X|) \subset \mathcal{W}$.

6.1 Preliminaries

Let $\mathcal{W} = \mathcal{U} \cup \mathcal{V}$, where $\mathcal{U}$ and $\mathcal{V}$ are open path-connected topological subspaces of $\mathcal{W}$ such that $\mathcal{U} \cap \mathcal{V}$ is open. This means that the commuting diagram

$$\begin{array}{ccc} \mathcal{U} \cap \mathcal{V} & \longrightarrow & \mathcal{U} \\ \downarrow & \ulcorner & \downarrow \\ \mathcal{V} & \longrightarrow & \mathcal{W} \end{array}$$

is a push out in the category of topological spaces.

For the rest of this chapter, a "Van Kampen type theorem" is a theorem about the preservation of this pushout in topological spaces by some functor from topological spaces to an algebraic category. As noted at the beginning of this chapter, this is a very specific use of the term. It is justified on the grounds that this chapter is a move towards connecting the general properties of the loop groupoid functor with the special (and powerful) properties of fundamental constructions.

The loop groupoid functor is a left adjoint, and so preserves all colimits in simplicial sets. However, the singular complex functor from topological spaces to simplicial sets is a right adjoint, and it does not preserve pushouts.

Specifically, the pushout of the diagram:

$$\begin{array}{ccc} Sing(\mathcal{U}\cap\mathcal{V}) & \longrightarrow & Sing\mathcal{U} \\ \downarrow & & \\ Sing\mathcal{V} & & \end{array}$$

is

$$\begin{array}{ccc} Sing(\mathcal{U}\cap\mathcal{V}) & \longrightarrow & Sing\mathcal{U} \\ \downarrow & \ulcorner & \downarrow \\ Sing\mathcal{V} & \longrightarrow & Sing\mathcal{U}\cup Sing\mathcal{V} \end{array}$$

and in general , $Sing\mathcal{U}\cup Sing\mathcal{V} \neq Sing\mathcal{W}$.

Further, $Sing\mathcal{W}$ is a Kan complex, for any $\mathcal{W}$. This is because there is a retraction in $\mathcal{T}op$, $r : \triangle^n \longrightarrow |\wedge^k[n]|$, and so for any arrow $f : |\wedge^k[n]| \longrightarrow \mathcal{W}$, there is a "filler", that is the arrow fr. However, in general $Sing\mathcal{U}\cup Sing\mathcal{V}$ is not Kan.

The problem of obtaining a Van Kampen Type theorem for a functor which factors through the singular complex functor, is essentially the problem of inverting the unique morphism, (defined by the pushout) which embeds $Sing\mathcal{U}\cup Sing\mathcal{V}$ into $Sing\mathcal{W}$.

It has been noted already that $|NerSd[n]|$, $|Sd\triangle[n]|$ and $|\triangle[n]|$ are all contractible. It turns out that there are retractions from $|NerSd[n]|$ onto each of the other two spaces.

6.2 Retractions

Recall that $|\triangle[n]| \cong \triangle^n := \{(t_0, \cdots, t_n) \,|\, \sum_{i=0}^{n} t_i = 1\,, t_i \geq 0\}$.

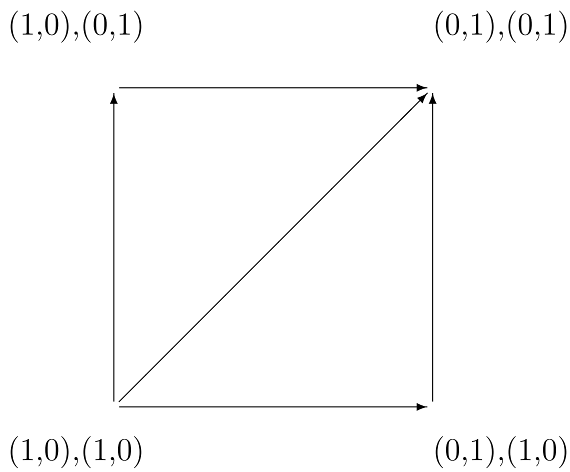
Now $|Sd\triangle[n]|$ comprises (as was noted earlier) 2^n copies of $\triangle^n$ glued together into a "larger" copy of $\triangle^n$. However, it will be easier to think of

$Sd\triangle[n]$ as it was first defined: $Sd\triangle[n] \cong \int^{p,q} \Delta([p]or[q],[n]) \cdot (\triangle[p] \times \triangle[q])$. In this way, $|Sd\triangle[n]| \cong \int^{p,q} \Delta([p]or[q],[n]) \cdot (\triangle^p \times \triangle^q)$.

Similarly, recall (section 5.2) $NerSd[n] \cong Subdiag(\triangle[n] \times \triangle[n])$ (the subdiagonal of $\triangle[n] \times \triangle[n]$), and so $|NerSd[n]| \cong Subdiag(\triangle^n \times \triangle^n)$. This is the topological space $\{(\underline{x}, \underline{y}) \,|\, \underline{x}, \underline{y} \in \triangle^n \text{ and } \underline{x} \leq \underline{y}\,\}$.

Note that for two n-simplices $\underline{x} \leq \underline{y}$ if and only if $\sum_{i=0}^{a} x_i \leq \sum_{i=0}^{a} y_i$ for all $0 \leq a \leq n$.

As an example, consider the square $\triangle[1] \times \triangle[1]$.



The subdiagonal is the lower triangle, and this is defined by

$$\{((x, 1-x), (y, 1-y)) \,|\, x \leq y\,\}.$$

So , for $\underline{x} = (x, 1-x)$ and $\underline{y} = (y, 1-y)$, $\underline{x} \leq \underline{y}$ means $x \leq y$.

Proposition 6.2.1

Recall that ∂ denotes the diagonal embedding, and so $\partial : \triangle[n] \longrightarrow NerSd[n]$

is the map which sends the vertex i to the vertex $\begin{pmatrix} i \\ i \end{pmatrix}$. Then, let $|\partial| : \triangle^n \longrightarrow |NerSd[n]|$ be the map induced by $\partial : \triangle[n] \longrightarrow NerSd[n]$.

Then there is a morphism $Av_n : |NerSd[n]| \longrightarrow \triangle^n$, such that Av_n is a retraction, and $Av_n|\partial| = id_{\triangle^n}$.

Proof

The map $|\partial| : \triangle^n \longrightarrow |NerSd[n]|$ is given by $|\partial| : \underline{x} \mapsto (\underline{x}, \underline{x})$. Then define $Av : |NerSd[n]| \longrightarrow \triangle^n$ by $Av : (\underline{x}, \underline{y}) \mapsto \dfrac{\underline{x} + \underline{y}}{2}$.

It is clear that $Av_n|\partial| = id_{\triangle^n}$. ∎

Before proving a similar proposition for $|NerSd[n]|$ and $|Sd\triangle[n]|$, it will be necessary to describe the map $|\lambda_n| : |Sd\triangle[n]| \mapsto |NerSd[n]|$, which is induced by $\lambda_{Sd\triangle[n]}$. (Recall that λ_* denotes the unit of the adjunction $\Pi \dashv Ner$.)

The definition of $|Sd\triangle[n]|$ implies that it consists of a set of (p, q) prisms, one for each distinct pair (p, q) with $p + q = n$. Recall the picture of "paths" in the step diagram, from section 4.1. The n-simplices of $Sd\triangle[n]$ all lie in the $(p \times q)$ -rectangles, for $p + q = n$, and the rectangles clearly represent the prisms.

The vertex $\begin{pmatrix} i_0 \\ j_0 \end{pmatrix}$ in $NerSd[n]$ represents the vertex

$$((\underbrace{0, \cdots, 0}_{i_0}, 1, \underbrace{0, \cdots, 0}_{n-i_0}), (\underbrace{0, \cdots, 0}_{j_0}, 1, \underbrace{0, \cdots, 0}_{n-j_0}))$$

in $|NerSd[n]|$. Since both $NerSd[n]$ and $Sd\triangle[n]$ are simplicial complexes, the embedding of the (p, q) prism, $(\triangle[p] \times \triangle[q])$, into $NerSd[n]$ may be described by specifying to which vertices of $NerSd[n]$ the vertices of the prisms are sent.

So consider the vertices of the (p,q)-prism. If $p = 0$, the prism is an n-simplex, and its vertices correspond to the vertices $\begin{pmatrix} 0 \\ j \end{pmatrix}$ for $0 \leq j \leq n$. If $p = 1$, the vertices of the prism are $\begin{pmatrix} i \\ j \end{pmatrix}$ where $0 \leq i \leq 1$ and $1 \leq j \leq n$. In general, the vertices of the (p,q) prism are $\begin{pmatrix} i \\ j \end{pmatrix}$ where $0 \leq i \leq p \leq j \leq n$.

In order to denote how the prisms match up, denote the geometric (p,q)-prism as $((\underbrace{0,\cdots,0}_{q}, x_0, \cdots, x_p), (y_0, \cdots, y_q, \underbrace{0,\cdots,0}_{p}))$, where $\sum_{i=0}^{p} x_i = 1$ and $\sum_{j=0}^{q} y_j = 1$.

Since the point described in this way is clearly an element of $\triangle^n \times \triangle^n$, it is clear that $|\lambda_n|$ merely considers the point of the prism as an element of $|NerSd[n]|$.

Corollary 6.2.2

$$Av_n|\lambda_n| : ((\underbrace{0,\cdots,0}_{q}, x_0, \cdots, x_p), (y_0, \cdots, y_q, \underbrace{0,\cdots,0}_{p}))$$
$$\mapsto (\frac{y_0}{2}, \cdots, \frac{y_q + x_0}{2}, \frac{x_1}{2}, \cdots, \frac{x_p}{2}).$$

Proof

This is clear.

This means that for each prism of $|Sd\triangle[n]|$, only one point of the image of $Av|\lambda_n|$ is a vertex in $\triangle^n$, that is the vertex where x_0 and y_q “meet”. The first diagram of the geometric subdivision in section 4.1 shows this phenomenon clearly, (for the case $n = 3$).

Thus, the composite map $Av|\lambda_n|$ embeds the prisms of the subdivided simplex $|Sd\triangle[n]|$ into $\triangle^n$ in precisely the way described by the picture in section 4.1.

It follows that the composite $|\tau^*(Sd\sigma(Sd\triangle[n]))|$ (that is the image in $\mathcal{W}$) must be the same as $\sigma Av|\lambda_n|(|Sd\triangle[n]|)$ as it too takes $|Sd\triangle[n]|$ to the geometric subdivision of $\sigma(\triangle^n)$ in $\mathcal{W}$.

Corollary 6.2.3 *Let $\sigma : \triangle^n \longrightarrow \mathcal{W}$. Then $\sigma Av_n\lambda_n : |Sd\triangle[n]| \longrightarrow \mathcal{W}$ is a subdivision of σ. Thus, given an n-simplex in $Sing\mathcal{W}$ there is a subdivision in $Sing\mathcal{W}$.*

Proposition 6.2.4

There is a morphism $r_n : |NerSd[n]| \longrightarrow |Sd\triangle[n]|$, such that r_n is a retraction, (that is $r_n|\lambda_n| = id_{|Sd\triangle[n]|}$), $Av_n|\lambda_n|r_n = Av_n$, and $r_n|\partial|Av_n = r_n$.

Proof

Define a function $r_n|NerSd[n]| \longrightarrow |Sd\triangle[n]|$ as follows:

$r_n((x_0, \cdots, x_n), (y_0, \cdots, y_n)) :\mapsto$

$$((\underbrace{0}_{q}, x'_q, x_{q+1} + y_{q+1}, \cdots, x_n + y_n), (x_0 + y_0, \cdots, x_{q-1} + y_{q-1}, y'_q, \underbrace{0}_{p}))$$

where

$$\sum_{i=0}^{q-1}(x_i+y_i) < 1, \ \sum_{i=0}^{q}(x_i+y_i) \geq 1, \ x'_q = 1 - \sum_{i=q+1}^{n}(x_i+y_i) \ \& y'_q = 1 - \sum_{i=0}^{q-1}(x_i+y_i)$$

As $\sum_{1=0}^{q}(x_i + y_i)$ tends to 1 from below for some q, the value of y'_{q+1} tends to 0, and x'_{q+1} tends to $x_{q+1}+y_{q+1}$, while if $\sum_{i=0}^{q}(x_i + y_i)$ tends to 1 from above, the value of y'_q tends to $x_q + y_q$, while the value of x'_q tends to 0: at the point $\sum_{i=0}^{q}(x_i + y_i) = 1$, $y'_q = x_q + y_q$, and $x'_q = 0$. Thus, r_n is continuous.

First,

$r_n|\lambda_n|((\underbrace{0, \cdots, 0}_{q}, x_0, \cdots, x_p), (y_0, \cdots, y_q, \underbrace{0, \cdots, 0}_{p}))$
$= r_n((\underbrace{0, \cdots, 0}_{q}, x_0, \cdots, x_p), (y_0, \cdots, y_q, \underbrace{0, \cdots, 0}_{p}))$
$= ((\underbrace{0, \cdots, 0}_{q}, x'_0, \cdots, x_p), (y_0, \cdots, y'_q, \underbrace{0, \cdots, 0}_{p}))$
where $x'_0 = 1 - \sum_{i=1}^{p} x_i = x_0$ and $y'_q = 1 - \sum_{i=0}^{q-1} y_i = y_q$
and so $r_n|\lambda|$ is the identity map.

If $((x_0, \cdots, x_n), (y_0, \cdots, y_n)) \in |\partial|(\triangle^n)|$, then $x_i = y_i$ for all i, and the formula becomes

$r_n((x_0, \cdot\cdot, x_n), (x_0, \cdot\cdot, x_n)) \mapsto$

$$((\underbrace{0, \cdots, 0}_{q}, x'_q, 2x_{q+1}, \cdot\cdot, 2x_n), (2x_0, \cdot\cdot, 2x_{q-1}, 2x_q - x'_q, \underbrace{0, \cdots, 0}_{p}))$$

where $\sum_{i=0}^{q-1} x_i < \frac{1}{2}$, $\sum_{i=0}^{q} x_i \geq \frac{1}{2}$, & $x'_q = 1 - 2\sum_{i=q+1}^{n} x_i$

Then

$Av_n|\lambda_n|r_n((x_0, \cdots, x_n), (y_0, \cdots, y_n))$

$= Av_n|\lambda|((\underbrace{0}_{p}, x'_q, x_{q+1}+y_{q+1}, \cdot\cdot, x_n+y_n), (x_0+y_0, \cdot\cdot, x_{q-1}+y_{q-1}, y'_q, \underbrace{0}_{q}))$
$= (\frac{x_0+y_0}{2}, \cdots, \frac{x_{q-1}+y_{q-1}}{2}, \frac{x'_q+y'_q}{2}, \frac{x_{q+1}+y_{q+1}}{2}, \cdots, \frac{x_n+y_n}{2})$
$= Av_n((x_0, \cdots, x_n), (y_0, \cdots, y_n)).$

That is, $Av_n|\lambda_n|r_n = Av_n$.

Then

$r_n|\partial|Av_n((x_0, \cdots, x_n), (y_0, \cdots, y_n))$
$= r_n(\frac{x_0+y_0}{2}, \cdots, \frac{x_n+y_n}{2}), (\frac{x_0+y_0}{2}, \cdots, \frac{x_n+y_n}{2})$
$= ((\underbrace{0}_{q}, x'_q, x_{q+1}+y_{q+1}, \cdots, x_n+y_n), (x_0+y_0, \cdots, x_{q-1}+y_{q-1}, y'_q, \underbrace{0}_{p}))$
$= r_n((x_0, \cdots, x_n), (y_0, \cdots, y_n))$
for $\sum_{i=0}^{q-1}(x_i+y_i) < 1 \,\&\, \sum_{i=0}^{q}(x_i+y_i) \geq 1$.
That is $r_n|\partial|Av_n = r_n$. ■

Corollary 6.2.5

$Av_n|\lambda|r_n|\partial_n|$ is the identity on $\triangle^n$.

$r_n|\partial|Av_n|\lambda_n|$ is the identity map on $|Sd\triangle[n]|$.

Proof

$Av_n|\lambda|r_n|\partial_n| = Av_n|\partial| = id_{\triangle^n}$

$r_n|\partial|Av_n|\lambda_n| = r_n|\lambda_n| = id_{|Sd\triangle[n]|}$.

As an example, consider the case $n = 1$.

$$r_1((x, 1-x), (y, 1-y)) \mapsto \begin{cases} ((x+y-1, 2-x-y), 1)) & \text{for } 1 \leq x+y \leq 2 \\ ((1, (x+y, 1-x-y)) & \text{for } 0 \leq x+y \leq 1 \end{cases}$$

These clearly match up when $x + y = 1$.

The corollary implies that $\triangle^n \cong |Sd\triangle[n]|$.

Thus, if $\sigma : \triangle^n \longrightarrow \mathcal{W}$, there is a subdivision $\sigma Av|\lambda_n| : |Sd\triangle[n]| \longrightarrow \mathcal{W}$, which composes back to σ. That is, for any simplex in $Sing\mathcal{W}$ there is a specific subdivided simplex (arising from the bijection of the $|-| \dashv Sing$ adjunction), having the property that by composing it with $r_n|\partial|$, the original simplex is recovered.

Recall the following theorem.

The Lebesgue Covering Theorem 6.2.1

Let $\mathcal{W}$ be a compact metric space, and let $\{U_\alpha : \alpha \in A\}$ be an open cover of $\mathcal{W}$, then $\exists\, \beta > 0$ s.t. if $\mathcal{V} \subset \mathcal{W}$ and diam $\mathcal{V} < \beta$, then $\mathcal{V} \subset U_\alpha$ for some $\alpha \in A$.

Note: The supremum of all such β is called the Lebesgue Number of the Cover.

Proof

This is a standard topological result, and the proof may be found in [9].

Corollary 6.2.2

Let $\mathcal{U}$ and $\mathcal{V}$ be open path-connected topological subspaces of $\mathcal{W}$, where $\mathcal{U} \cup \mathcal{V} = \mathcal{W}$. Then for any $\sigma : \triangle^n \longrightarrow \mathcal{W}$, there is an s such that all the affine n-simplices of $|Sd^{2^s}\triangle[n]|$ have diameter less than β, and so each is contained entirely in either $\mathcal{U}$ or $\mathcal{V}$.

Note that the last result is equally true when the topological maps $\sigma Av_n|\lambda|$ and $(r_n|\partial|)^*$ are used in place of $Sd(\sigma)$ and τ.

Thus given a simplex in $Sing\mathcal{W}$, there is a subspace of $Sing\mathcal{U} \cup Sing\mathcal{V}$, namely $(\tau^*)^s Sd^s\sigma(Sd^s\triangle[n])$.

Consider for the moment, that $s = 1$ is sufficient for this purpose, that is that one subdivision will split the simplex σ so that the constituent simplices lie entirely in either $Sing\mathcal{U}$ or $Sing\mathcal{V}$. Now, there is a subspace of $Sing\mathcal{W}$, namely $\sigma Av_n(NerSd[n])$ which contains both $\sigma Av_n\lambda_n(Sd\triangle[n])$ and $\sigma(\triangle[n])$, and it is contractible.

Now, consider $G(Sing\mathcal{U} \cup Sing\mathcal{V})$. It should be possible to construct an algebraic image of $G(\triangle[n]) \subset G(Sing\mathcal{U} \cup Sing\mathcal{V})$ which uses the elements of the image of $G(\sigma Av_n\lambda_n)$.

6.3 Working with G

Theorem 6.3.1

There is a cosimplicial simplicial groupoid morphism

$$\theta : G(NerSd[-]) \longrightarrow G(Sd\triangle[-])$$

with the property $\theta^* G(\lambda_{Sd\triangle[-]}) = id_{GSd\triangle[-]}$.

Proof

Consider, the morphism $\eta_{Sd\triangle[n]} : Sd\triangle[n] \longrightarrow \overline{W}G(Sd\triangle[n])$. Since for any simplicially enriched groupoid, H, $\overline{W}(H)$ is a Kan complex, then the cosimplicial simplicial set $\overline{W}G(Sd\triangle[-])$ has the property that each simplicial set $\overline{W}G(Sd\triangle[n])$ is weak Kan. Thus, using theorem 5.4.1 (the main result of chapter 5), there is an extension

$$\overline{\eta_{Sd\triangle[-]}} : NerSd[-] \longrightarrow \overline{W}G(Sd\triangle[-])$$

such that $$\overline{\eta_{Sd\triangle[-]}}\lambda_{Sd\triangle[-]} = \eta_{Sd\triangle[-]}.$$

Using the $G \dashv \overline{W}$ adjunction, the morphism

$$\epsilon_{G(Sd\triangle[-])}G(\overline{\eta_{Sd\triangle[-]}}) : GNerSd[-] \longrightarrow GSd\triangle[-]$$

has the property that $\epsilon_{G(Sd\triangle[-])}G(\overline{\eta_{Sd\triangle[-]}})G(\lambda_{Sd\triangle[-]}) = id_{G(Sd\triangle[-])}$. Thus $\epsilon_{G(Sd\triangle[-])}G(\overline{\eta_{Sd\triangle[-]}})$ is a retraction in cosimplicial simplicially enriched groupoids. That is, $\epsilon_{G(Sd\triangle[-])}G(\overline{\eta_{Sd\triangle[-]}})$ is left inverse to $G(\lambda_{Sd\triangle[-]})$.

Define $\theta^* := \epsilon_{G(Sd\triangle[-])}G(\overline{\eta_{Sd\triangle[-]}})$. ■

Note that this implies that $GNerSd[-] \cong Ker\theta^* \rtimes G(\lambda_* Sd\triangle[-])$.

Recall that $\partial_n : \triangle[n] \longrightarrow NerSd[n]$ is the diagonal embedding of $\triangle[n]$ into $NerSd[n]$: (this is most clearly defined by considering $NerSd[n]$ as the subdiagonal of $\triangle[n] \times \triangle[n]$).

Recall that the subdivision of an n-simplex consists of 2^n n-simplices which "fit together"; explicit pictures for this are given in chapter 4. Each such simplex is a morphism $\triangle[n] \longrightarrow Sing\mathcal{W}$, and so they define a morphism $\rho : Sd\triangle[n] \longrightarrow Sing\mathcal{W}$.

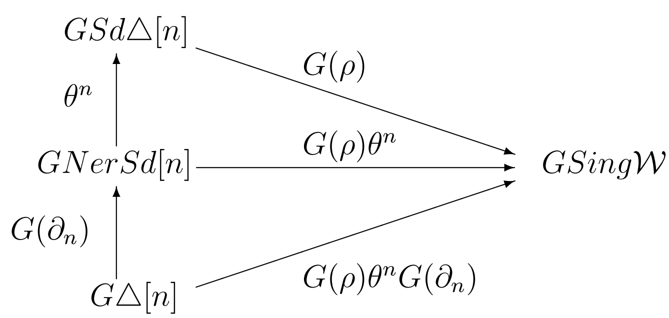
Definition 6.3 (i)

Consider a pair , (X, ρ), where X is a set of 2^n generators of $GSing\mathcal{W}_{n-1}$, which (considered as n-simplices of $Sing\mathcal{W}$) can be collectively described as $\rho : Sd\triangle[n] \longrightarrow Sing\mathcal{W}$. Note that this means that ρ describes the way in which the simplices "fit together", and so forms a pasting scheme for the set.

Then, the *algebraic composite* of (X, ρ) is given by

$$G(\rho)\theta^n G(\partial_n) : G(\triangle[n]) \longrightarrow GSing\mathcal{W}.$$

The following picture describes the various morphisms:



Note that if $\sigma : \triangle^n \longrightarrow \mathcal{W}$, then $\tau^n(Sd\sigma)$ is a "pasting scheme" for the 2^n simplices which make up the subdivision of σ.

It will clarify the idea to consider an example of this definition. So, consider the case $n = 1$. That is, construct the map $\overline{\eta_{Sd\triangle[1]}}$. Note that $Sd\triangle[1] \cong \wedge^1[2]$ and $NerSd[1] \cong \triangle[2]$.

Example 6.3 (ii)

The generators of $Sd\triangle[1]$ are the 1-simplices $s_1 s_0 i_1$ and $s_2 s_1 i_1$ where i_1 is the generator of $\triangle[1]$. The morphism $\eta_{Sd\triangle[1]}$ acts as the identity on 1-simplices.

The filler for this pair in $\overline{W}G(Sd\triangle[1])$ is the 2-simplex $(s_4s_2s_1s_0i_1, s_2s_1i_1)$. This filler is, in fact, uniquely defined. Thus, the 2-simplex $(s_1s_0i_1, s_2s_1i_1)$ in $NerSd[1]$ is mapped to $(s_4s_2s_1s_0i_1, s_2s_1i_1)$ by $\overline{\eta_{Sd\triangle[1]}}$.

The image of $\triangle[1]$ in $NerSd[1]$ is generated by the 1-simplex $(s_1s_0i_1)(s_2s_1i_1)$, the composite of the two generators of $Sd\triangle[1]$.

Using the adjunction, θ^1 is defined on the nondegenerate generator of $(GNerSd[1])_1$ by $(s_1s_0i_1, s_2s_1i_1) \mapsto (s_4s_2s_1s_0i_1) = \sigma_0(s_1s_0i_1) \in (GSd\triangle[1])_1$. Note that in $(NerSd[1])_2$, $\lambda(s_4s_2s_1s_0i_1) = (s_1s_0i_1, s_2s_0i_1)$ and s_2s_0 is the degeneracy from $(Sd\triangle[1])_0$ to $(Sd\triangle[1])_1$. Thus θ^1 identifies the nondegenerate 2-simplex of $NerSd[1]$ with the degeneracy of its 2-face.

Therefore, the map $\epsilon G(\overline{\eta})$ takes the generators of the $(GNerSd[1])_0$ to $(GSd\triangle[1])_0$ as follows:
$s_1s_0i_1 \mapsto s_1s_0i_1$, $s_2s_1i_1 \mapsto s_2s_1i_1$ and $(s_1s_0i_1)(s_2s_1i_1) \mapsto (s_1s_0i_1) \cdot (s_2s_1i_1)$. This is deceptive - the element $(s_1s_0i_1)(s_2s_1i_1)$ in $(GNerSd[1])_0$ is a generator, and is distinct from the composite element

$$(s_1s_0i_1) \cdot (s_2s_1i_1) \in (GNerSd[1])_0.$$

Thus, the algebraic composite of the elements which form a copy of the simplicial set $GSd\triangle[1]$ in $GSing\mathcal{W}$ is the standard composite in the groupoid $GSing\mathcal{W}_0$. Of course, for general n this "composition" will not be as neat as it is for $n = 1$.

It is possible to construct a splitting function (<u>not</u> a morphism) for the embedding $G(Sing\mathcal{V} \cup Sing\mathcal{V}) \longrightarrow GSing\mathcal{W}$. For an element

$$(\sigma) : (\triangle[n]) \longrightarrow Sing\mathcal{W},$$

there is some (iterated) subdivision which has the property that its n-simplices are in $Sing\mathcal{U} \cup Sing\mathcal{V}$. This set of simplices forms the image of the (iterated) subdivision of $\triangle[n]$ for which there is an algebraic composite. (If necessary, the algebraic composite of a set of algebraic composites must be taken). This choice of an element in $G(Sing\mathcal{U} \cup Sing\mathcal{V})$ forms the splitting function.

The aim now is to construct a quotient of $GSing\mathcal{W}$ under which both the embedding $G(Sing\mathcal{U} \cup Sing\mathcal{V}) \longrightarrow GSing\mathcal{W}$ and the splitting function become identity morphisms.

If the quotient identifies the two morphism $G(\sigma)$ and $G(\sigma Av_n\lambda_n)\theta^n G(\partial)$, then it will identify all algebraic composites of subdivisions with the original $G(\sigma)$, and since the Lebesgue Covering Theorem states that there is some finite subdivision of σ which sits in $Sing\mathcal{U} \cup Sing\mathcal{V}$, then the quotient will identify $G(Sing\mathcal{U} \cup Sing\mathcal{V})$ with $GSing\mathcal{W}$.

Lemma 6.3.2

For $g \in GSing\mathcal{W}$, $((G\partial_n)(g))^{-1}((G\lambda_n)\theta^n(G\partial)(g)) \in Ker\theta^n$.

Proof

$\theta^n G(\lambda_n) = id$.

Let $\sigma : \triangle^n \longrightarrow \mathcal{W}$, and consider $\sigma Av_n : |NerSd[n]| \longrightarrow \mathcal{W}$. The bijective image of this map under the $|-| \dashv Sing$ is also written σAv_n. Now

$\sigma A v_n \partial_n = \sigma$, and so,

$$\left(\int^n Sing\mathcal{W}_n \cdot G(\sigma A v_n)\right) G(\partial_n) = G(\sigma) : G(\triangle[n]) \longrightarrow GSing\mathcal{W}$$

where $\int^n Sing\mathcal{W}_n \cdot G(\sigma A v_n) : GNerSd[n] \longrightarrow GSing\mathcal{W}$.

Write $\Psi := \int^n Sing\mathcal{W} \cdot G(\sigma A v_n)$.

Note $\int^n Sing\mathcal{W}_n \cdot GSd\triangle[n] \cong G\left(\int^n Sing\mathcal{W}_n \cdot Sd\triangle[n]\right) \cong GSdSing\mathcal{W}$. Then, the morphism

$$\int^n Sing\mathcal{W}_n \cdot \theta^n : \int^n Sing\mathcal{W}_n \cdot GNerSd[n] \longrightarrow \int^n Sing\mathcal{W}_n \cdot GSd\triangle[n]$$

takes $GNerSd[n]_\sigma$ to $\theta^n GNerSd[n]$ in $GSdSing\mathcal{W}$. Note that $\int^n Sing\mathcal{W}_n \cdot GSd\triangle[n] = GSdSing\mathcal{W}$.

Consider the following pushout diagram:

$$\begin{array}{ccc}
\int^n Sing\mathcal{W}_n \cdot GNerSd[n] & \xrightarrow{\quad \Psi \quad} & GSing\mathcal{W} \\
\Big\downarrow {\scriptstyle \int^n Sing\mathcal{W}_n \cdot \theta} & & \Big\downarrow \\
\int^n Sing\mathcal{W}_n \cdot GSd\triangle[n] & \xrightarrow{\qquad\quad} & G'\mathcal{W}
\end{array}$$

This defines a functor $G' : \mathcal{T}op \longrightarrow \underline{SGpds_*}$: if $f : \mathcal{W} \longrightarrow \mathcal{W}'$, then $G'(f)$ is defined to be the unique arrow arising from the pushout in the obvious commuting cube. It is clear from the definition that $G'\mathcal{W}$ is a quotient of $GSing\mathcal{W}$.

This next result is the central aim of the thesis.

Theorem 6.3.3

The functor G' preserves the pushout

$$\begin{array}{ccc} \mathcal{U}\cap\mathcal{V} & \longrightarrow & \mathcal{U} \\ \downarrow & \ulcorner & \downarrow \\ \mathcal{V} & \longrightarrow & \mathcal{W} \end{array}$$

Proof

First note that $\int^n (Sing\mathcal{U} \cup Sing\mathcal{V})_n \cdot GNerSd[n]$ is the pushout of the diagram

$$\begin{array}{ccc} \int^n (Sing\mathcal{U} \cap Sing\mathcal{V})_n \cdot GNerSd[n] & \longrightarrow & \int^n Sing\mathcal{V}_n \cdot GNerSd[n] \\ \downarrow & & \\ \int^n Sing\mathcal{U}_n \cdot GNerSd[n] & & \end{array}$$

Similarly, since both G and Sd are left adjoints, $G(Sing\mathcal{U} \cup Sing\mathcal{V})$ is the pushout of the diagram

$$\begin{array}{ccc} G(Sing\mathcal{U} \cap Sing\mathcal{V}) & \longrightarrow & GSing\mathcal{V} \\ \downarrow & \ulcorner & \downarrow \\ GSing\mathcal{U} & \longrightarrow & G(Sing\mathcal{U} \cup Sing\mathcal{V}) \end{array}$$

and $GSd(Sing\mathcal{U} \cup Sing\mathcal{V})$ is the pushout of the diagram:

$$\begin{array}{ccc} GSd(Sing\mathcal{U} \cap Sing\mathcal{V}) & \longrightarrow & GSdSing\mathcal{V} \\ \downarrow & \ulcorner & \downarrow \\ GSdSing\mathcal{U} & \longrightarrow & GSd(Sing\mathcal{U} \cup Sing\mathcal{V}) \end{array}$$

Write $G'(\mathcal{U} \cup \mathcal{V})$ for the pushout of the diagram

$$\begin{array}{ccc} \int^n (Sing\mathcal{U} \cup Sing\mathcal{V})_n \cdot GNerSd[n] & \xrightarrow{\Psi} & G(Sing\mathcal{U} \cup Sing\mathcal{V}) \\ \Big\downarrow {\scriptstyle \int^n (Sing\mathcal{U} \cup Sing\mathcal{V})_n \cdot \theta} & & \Big\downarrow \\ \int^n (Sing\mathcal{U} \cup Sing\mathcal{V})_n \cdot GSd\triangle[n] & \longrightarrow & G'(\mathcal{U} \cup \mathcal{V}) \end{array}$$

It follows that $G'(\mathcal{U} \cup \mathcal{V})$ is also the pushout of the diagram

$$\begin{array}{ccc} G'(\mathcal{U} \cap \mathcal{V}) & \longrightarrow & G'\mathcal{V} \\ \downarrow & \ulcorner & \downarrow \\ G'\mathcal{U} & \longrightarrow & G'(\mathcal{U} \cup \mathcal{V}) \end{array}$$

Now to prove the theorem, it is necessary to prove that the morphism $G'(\mathcal{U} \cup \mathcal{V}) \longrightarrow G'\mathcal{W}$ induced by $G(Sing\mathcal{U} \cup Sing\mathcal{V}) \longrightarrow GSing\mathcal{W}$, and also by the pushout property of $G'(\mathcal{U} \cup \mathcal{V})$ is an isomorphism.

Since $G'\mathcal{W}$ is a quotient of $GSing\mathcal{W}$, each generator of $G'\mathcal{W}$ has a preimage that is a generator of $GSing\mathcal{W}$. So, for $\sigma \in Sing\mathcal{W}_n$, there is a generator $[G(\sigma)] \in G'\mathcal{W}$ which is the equivalence class of $G(\sigma)$ under the quotient.

Then, $G(\sigma Av_n \lambda_n)\theta^n G(\partial_n)$ is the algebraic composite of the subdivision of σ, (which is assumed to be in $G(Sing\mathcal{U} \cup Sing\mathcal{V})$, without loss of generality: see the paragraphs following the proof of theorem 6.3.1). Note that $G(\sigma Av_n)G(\partial_n) = G(\sigma)$, since $Av_n\partial_n = id$. Therefore, there is a splitting function for $G'(\mathcal{U} \cup \mathcal{V}) \longrightarrow G'\mathcal{W}$, which takes $[G(\sigma)] = [G(\sigma Av_n \partial_n)]$ to $[G(\sigma Av_n \lambda_n)\theta^n G(\partial_n)]$.

So consider $(G(\sigma Av_n\partial_n))^{-1} (G(\sigma Av_n \lambda_n)\theta^n G(\partial_n))$. This is in the kernel of

θ^n, by lemma 6.3.2, and so in the kernel of $\int^n Sing\mathcal{W}_n \cdot \theta^n$. Thus, the element is the identity in $GSd(Sing\mathcal{W})$, and hence in $G'(\mathcal{W})$. This is precisely the required result, that $[G(\sigma Av_n\lambda_n)\theta^n G(\partial_n)] = [G(\sigma Av_n\partial_n)] = [G(\sigma)]$ in $G'\mathcal{W}$. Thus the splitting is the identity function, so the embedding is the identity morphism, and $G'\mathcal{W} \cong G'(\mathcal{U} \cup \mathcal{V})$. ∎

This shows that G' satisfies a Van Kampen Type Theorem.

There is one remaining problem, which needs a "constructive" proof: ideally $G'\mathcal{W}$ and $GSing\mathcal{W}$ should have the same homotopy type.

Recall that $GNerSd[n] \xrightarrow{\theta^n} GSd\triangle[n]$ is split by $G(\lambda_n)$. It follows that the morphism

$$\int^n Sing\mathcal{W}_n \cdot \theta^n : \int^n Sing\mathcal{W}_n \cdot GNerSd[n] \longrightarrow \int^n Sing\mathcal{W}_n \cdot GSd\triangle[n]$$

is split by the morphism

$$\int^n Sing\mathcal{W}_n \cdot G\lambda_n : \int^n Sing\mathcal{W}_n \cdot GSd\triangle[n] \longrightarrow \int^n Sing\mathcal{W}_n \cdot GNerSd[n]$$

and the kernel of

$$\int^n Sing\mathcal{W}_n \cdot \theta^n : \int^n Sing\mathcal{W}_n \cdot GNerSd[n] \longrightarrow \int^n Sing\mathcal{W}_n \cdot GSd\triangle[n]$$

is $\int^n Sing\mathcal{W}_n \cdot Ker\theta^n$.

Therefore the quotient map, induced by the pushout, which takes $GSing\mathcal{W}$ to $G'\mathcal{W}$ is split (since the pushout preserves splittings), and the kernel of the quotient map is $\Psi(\int^n Sing\mathcal{W}_n \cdot Ker\theta^n)$.

Thus to show that $G'\mathcal{W}$ and $GSing\mathcal{W}$ have the same homotopy type it is necessary to prove that $\Psi(\int^n Sing\mathcal{W}_n \cdot Ker\theta^n)$ is contractible. Although it is not unreasonable to believe this to be true, time constraints have prevented further investigation, and a proof will not be provided in this thesis. However, the next proposition shows that G' is not trivial.

Proposition 6.3.4

The fundamental groupoid $\pi_1(Sing\mathcal{W}, (Sing\mathcal{W})_0)$ is a quotient of $G'(\mathcal{W})_0$. Further, theorem 6.3.3 implies the Van Kampen Theorem for the fundamental groupoid (where homotopy is rel the base points).

Proof

To avoid confusion, write d_i for the face operators of $Sing\mathcal{W}$, and δ_i for the face operators of $GSing\mathcal{W}$.

$G'\mathcal{W}$ is a quotient of $GSing\mathcal{W}$, so

$$(G'\mathcal{W})_0 \cong \frac{(GSing\mathcal{W})_0}{Q}$$

for some normal subgroupoid $Q \subset (GSing\mathcal{W})_0$.

Also, the fundamental groupoid $\pi_1(Sing\mathcal{W}, (Sing\mathcal{W})_0)$ is $\pi_0(GSing\mathcal{W})$, ([20] et al) that is,

$$\pi_1(Sing\mathcal{W}, (Sing\mathcal{W})_0) \cong \frac{(GSing\mathcal{W})_0}{\delta_0^1(Ker\delta_1^1)}$$

The aim is to show that the $Q \subset \delta_0^1(Ker\delta_1^1)$.

Recall that $\delta_0^1(Ker\delta_1^1)$ is generated by elements of the form $(\delta_1 x)(\delta_0 x)^{-1}$ for $x \in Sing\mathcal{W}_2$ (see [20]).

Now, theorem 6.3.3 shows that the equivalence relation represented by Q is generated by identifying each 1-simplex, σ with the"algebraic composite" of the two 1-simplices, σ_1 and σ_2, which comprise its subdivision (and so $d_0\sigma_1 = d_1\sigma_2$). Further, example 6.3 (ii) shows that the "algebraic composite" of two such elements is the normal groupoid composite in $GSing\mathcal{W}$.

Define a 2-simplex of $Sing\mathcal{W}$ as follows:

$$z : (t_0, t_1, t_2) \mapsto \sigma(\frac{2t_0 + t_1}{2}, \frac{t_1 + 2t_2}{2})$$

This element has σ_1, σ and σ_2 as 2, 1 and 0 faces, respectively. Thus, $(\delta_1 z)(\delta_0 z)^{-1}$ is an an element of $\delta_0^1(Ker\delta_1^1)$ which identifies σ with $(\sigma_1)(\sigma_2)$, which is precisely the generating relation for Q.

Thus $Q \subset \delta_0^1(Ker\delta_1^1)$, and $\pi_1(Sing\mathcal{W})$ is a quotient of $G'(\mathcal{W})_0$.

Now consider $\pi_0(G'\mathcal{W})$. Clearly there is an epimorphism from $\pi_0(GSing\mathcal{W})$ onto $\pi_0(G'\mathcal{W})$. Let $x, y \in (GSing\mathcal{W})_0$, so $[x], [y] \in (G'\mathcal{W})_0$. If $[x] \sim [y]$ in $\pi_0(G'\mathcal{W})$, then there is $[z] \in (G'\mathcal{W})_1$ with $\delta_0[z] = [x]$ and $\delta_1[z] = [y]$. Thus, $[\delta_0 z] = [x]$ and $[\delta_1 z] = [y]$. Thus $Q\delta_0 z = Qx$ and $Q\delta_1 z = Qy$, where Qx denotes (as usual) the coset of x under the quotient Q.

However, the elements which generate Q are trivial under π_0, as the first part of this proof demonstrates, and so elements in the same Q-coset are equivalent in $\pi_0(GSing\mathcal{W})$.

Therefore $x \sim \delta_0 z \sim \delta_1 z \sim y$, and so $[x] \sim [y] \in \pi_0(G'\mathcal{W})$ implies that $x \sim y \in \pi_0(GSing\mathcal{W})$. Thus the epimorphism is also a monomorphism, and hence an isomorphism.

Thus theorem 6.3.3 implies the Van Kampen Theorem for the fundamental groupoid.

■

In Conclusion

Given a simplicial set, X, it is possible to build a filtered topological space $|X|_n$, by taking the realisation of the n-skeleton for each n. The fundamental crossed complex of this construction is:-

$$\cdots \to \{\pi_n(X_n, X_{n-1}, p)\}_{p \in X_0} \to \cdots \to \{\pi_2(X_2, X_1, p)\}_{p \in X_0} \to \pi_1(X_1, X_0)$$

It is also possible to construct a crossed complex from a simplicial set by passing to GX (the loop groupoid), taking the Moore complex, NGX, and factoring $(NGX)_n$ by $((NGX)_n \cap D_n)d_0((NGX)_{n+1} \cap D_{n+1})$, where D_n is the subgroupoid of $(GX)_n$ generated by the degenerate elements. There are two methods of proof, one a combinatorial proof by Porter, the other a proof which shows that

$$\frac{(NGX)_n}{((NGX)_n \cap D_n)d_0((NGX)_{n+1} \cap D_{n+1})} \cong \{\pi_n(X_n, X_{n-1}, p)\}_{p \in X_0},$$

by Carrasco and Cegarra. Thus, fundamental crossed complex of a simplicial set may be obtained by passing through simplicially enriched groupoids.

Now, recall that the construction of θ was as $\epsilon_{GSd\triangle[n]}G(\overline{\eta_{Sd\triangle[n]}})$ and $(Sd\triangle[n])_m$ consists entirely of degenerate elements for $m > n$. Thus, the elements of $G(Sd\triangle[n])_m$ are all generated by degenerate elements for $m \geq n$; in particular for $n = m$. Therefore, the image of θ^n is a degenerate element.

Thus, when the Moore complex is divided out by degenerate elements, the identification collapses the filling constructed between a simplex and its subdivision, and so the fundamental crossed complex may be seen to satisfy a Van Kampen Type Theorem for any skeletally filtrated topological space.

This is not a proof, of course, however time constraints have prevented further investigation on these lines.

However, it suggests that when algebraic models can be thought of as arising as quotients of the loop groupoid, then they should satisfy a Van Kampen Theorem, so long as the quotient identifies those elements which the pushout diagram for G' identifies.

Glossary of Notation

The page number given is the first occurrence of the notation.

Categories

Δ	Finite Ordinals and monotonic maps	4
Δ^+	Non-empty Finite Ordinals	5
$\triangle^n$	The affine n-simplex	5
$\underline{Cat}$	Small categories	6
$\mathbf{N}$	Finite Totally Ordered Sets	6
$\mathbf{N}^+$	Non-empty Finite Totally Ordered Sets	6
Δ_0	Non-empty Finite Ordinals and monotonic maps which fix 0	7
$\underline{Sets}$	Sets and functions	15
SS	Simplicial Sets	15
CSS	Contractible Simplicial Sets	15
ASS	Augmented Simplicial Sets	15
$BiSS$	BiSimplicial Sets	15
$\triangle[n]$	The Standard n-simplex	15
	Simplicial Complex	16
	Kan complex	16,72
	weak Kan complex	16,73
$\underline{SGpds}$	Simplicial groupoids	28
$\underline{SGpds_*}$	Simplicial groupoids	28
$BAiSS$	BiSimplicial Sets	35
$\mathcal{T}op$	Topological Spaces	35
$\underline{Sets}^{\Delta^+}$	Cosimplicial Sets	47

Operations and Constructions

or	Ordinal Sum in Δ	10
$\vee$	Join in $\mathbf{N}$	10
$*$	Topological Join	11, 51
sk_n	the n-skeleton construction	16
ι_n	the unique nondegenerate n-simplex of $\triangle[n]$	16,72
	maximal simplex	16
$\wedge^k[n]$	the generic k-horn	20
$ConjX$	Conjugation	31
$\pi_0 X$	connected components of X	32
	the canonical augmentation	33
	the trivial augmentation	33
$X \cdot Y$	The X indexed copower of Y	36
CX	The cone over the codomain of the augmentation	42

Functors

Bibliography

[1] J. F. Adams
ALGEBRAIC TOPOLOGY - A STUDENT'S GUIDE
C. U. P., L.M.S. Lecture Note Series 4, 1972 1

[2] M. Artin & B. Mazur *On The Van Kampen Theorem*
Topology, Vol. 5 pp.179-189, 1966

[3] M. Artin & B. Mazur *ETALE HOMOTOPY*
Springer Lecture Notes in Maths, 100, 1969

[4] N. Ashley *Simplicial-T-complexes and Crossed Complexes*
Dissertationes Math. 265, 1978

[5] M. A. Batanin *The Coherent Categories with respect to the Monad and Coherent ProHomotopy Theory*
Preprint, Universitetskiy Prospect, 4 Novosibirsk-90, c.1990

[6] A. Blakers *Some Relations between Homology and Homotopy Groups*
Annals of Maths, 49(2) pp.428-461 1948

[7] A. K. Bousfield & D. M. Kan *Homotopy Limits, Completions and Localizations*
Springer Lecture Notes in Maths, 304, 1972 63, 70

[8] W. Breckenridge, H. Gastineau-Hills & A. Nelson *Lattice paths and Catalan Numbers*
Sidney Preprint pp.1-15 1990 88

[9] R. Brown *TOPOLOGY*
Ellis Horwood, 1988 2, 12, 58, 59, 62, 112, 122

[10] R. Brown & P. J. Higgins *On the Algebra of Cubes*
J. P. A. A., 21, pp.233-260, 1981

[11] R. Brown & P. J. Higgins *Colimit Theorems for Relative Homotopy Groups*
J. P. A. A., 22, pp.11-41, 1981 2, 112

[12] R. Brown & P. J. Higgins *The Classifying Space of a Crossed Complex*
U. C. N. W. preprint, 1990

[13] R. Brown & J.-L. Loday *Van Kampen Theorems for Diagrams of Spaces*
Topology, Vol. 26 no. 3 pp.311-335, 1987 112

[14] P. Carrasco *Complejos Hipercruzados: Comologia y Extensiones*
Cuadernos de Algebra, Granada no.6, 1987

[15] D. Conduché *Modules Croissés Géneralisés de longeur 2*
J. P. A. A., 22, pp.11-41 , 1984

[16] E. B. Curtis *Simplicial Homotopy Theory*
Advances in Mathematics, Vol. 6, no.2, pp.107ff 1971 41

[17] K. Dakin *Kan Complexes and Multiple Groupoid Structures*
U. C. N. W. PhD Thesis , 1978

[18] J.Duskin & D.Van Osdol *Unpublished Notes* 13, 31, 42, 43, 46, 47

[19] W. Dwyer & D. Kan *Homotopy Theory and Simplicial Groupoids*
Kon. Ned. Acad. v. Wet., 87(4) pp.379-385, 1984 31

[20] P.J.Ehlers *Simplicial Groupoids as Models for Homotopy Type*
MSc.Thesis, U. C. N. W., 1991 13, 31, 42, 44, 47, 132, 133

[21] G. Ellis *Crossed Modules and their Higher Dimensional Analogues*
U. C. N. W. preprint, 1984

[22] G. Ellis & R. Steiner *Higher Dimensional Crossed Modules and the Homotopy Groups of (n+1)-ads* J.P.A.A., 46, pp.117-136, 1987

[23] R. Fritsch & D. M. Latch *Homotopy Inverses for Nerve*
Bulletin A. M. S., 1(1) 1979 30

[24] R. Fritsch & R. A. Piccinnini
CELLULAR STRUCTURES IN TOPOLOGY
C. U. P. Cambridge Studies in Advanced Mathematics 19 1990 5

[25] P. Gabriel & M. Zisman
CALCULUS OF FRACTIONS AND HOMOTOPY THEORY
Springer-Verlag, 1967 83, 96

[26] N. Gilbert *The Fundamental Cat^n-Group of an n-Cube of Spaces*
Springer Lecture Notes in Maths J.,31 pp.17-29 1989
(Proc.Conf.Algebraic Topology, Barcelona 1986)

[27] P. Hilton & J. Pedersen *Catalan Numbers, their Generalization and their Uses*
Mathematical Intelligencer 13 (2), pp.64-75 1991 88

[28] S.-T. Hu *The Homotopy Addition Lemma*
Annals of Maths. 58, pp.108-122 1953

[29] L. Illusie
COMPLEXE COTANGENT ET DEFORMATIONS II
Springer Lecture Notes in Maths, 283, 1972 44

[30] D. M. Kan *A Combinatorial Definition of Homotopy*
Annals of Maths, 67(2) pp.282-312, 1957 1, 5

[31] D. M. Kan *On Homotopy Theory and C.S.S. Groups*
Annals of Maths, 68(1) pp.38-53, 1958 1, 31

[32] D. M. Latch *The Uniqueness of Homology for the Category of Small Categories*
J.P.A.A., 9, pp.221-237, 1977 30

[33] D. M. Latch, R. W. Thomason & W. S. Wilson *Simplicial Sets from Categories*
Preprint, (to appear) c.1979

[34] Loday *Spaces with Finitely many Non -Trivial Homotopy Groups*
J. P. A. A., 24, pp.179-202, 1982

[35] S.Maclane
CATEGORIES FOR THE WORKING MATHEMATICIAN
Springer-Verlag, 1971 4, 5, 7, 9, 11, 41, 46, 76

[36] J. P. May
SIMPLICIAL OBJECTS IN ALGEBRAIC TOPOLOGY
Van Nostrand, Math. Studies 11, 1967 5

[37] J.C.Moore *Semi-Simplicial Complexes and Postnikov Systems*
Symposium Internaçional de Topologia Algebraica, 1958 2

[38] T. Porter *n-Types of Simplicial Groups and Crossed n-cubes*
U. C. N. W. Preprint, 1991

[39] D. Quillen *HOMOTOPICAL ALGEBRA*
Springer Lecture Notes in Maths, 43, 1967 2, 85

[40] D. Quillen
HIGHER ALGEBRAIC K-THEORY, I, HIGHER K-THEORIES
Springer Lecture Notes in Maths, 341, pp.85-147 1973 30

[41] G. Segal, *Classifying Spaces and Spectral Sequences* Inst. Hautes Etudes Sci. Publ. Math., 34, pp.105-112, 1968 30

[42] R. W. Thomason *Homotpy Colimits in the Category of Small Categories* Math. Proc. Camb. Phil. Soc.,85,91 pp.91-109, 1979 30

[43] J. H. C. Whitehead *Combinatorial Homotopy I* Bulletin A. M. S., 55, pp.213-245, 1959 2

[44] J. H. C. Whitehead *Combinatorial Homotopy II* Bulletin A. M. S., 55, pp.496-543, 1959 2