HOMOTOPY AND TYPE THEORY
(PROJECT DESCRIPTION)

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Abstract. This proposal pursues a new connection between Geometry, Algebra, and Logic, in the form of an interpretation of constructive Martin-Löf type theory into homotopy theory, resulting in new examples of certain algebraic structures which are important in topology.

Martin-Löf type theory is a framework for constructive mathematics at least as strong as second-order logic, which has been used to formalize large parts of constructive mathematics and for the development of high-level programming languages. It is prized for its combination of expressive strength and desirable proof-theoretic properties. One aspect of the type theory that has led to special difficulties in providing semantics, however, is the intensionality in the treatment of equality.

The current research constructs a bridge between type theory and contemporary algebraic topology, exploiting both axiomatic homotopy theory in the form of Quillen model structures, and related algebraic methods involving (weak) higher-dimensional groupoids. In doing so, it provides *two* new domains of interpretation for type theory, permiting logical methods to be combined with the traditional algebraic and topological approaches to homotopy. It also opens up a new field of possible applications of type theory in the study of homotopy and higher-dimensional algebra. Existing machine implementations of the type theory also present the promise of computational applications in homotopy theory.

A doctoral student is partially supported under this research project. The student will be trained in the relevant areas of logic, topology, and algebra, and will conduct joint research with the PI.

Keywords: type theory, constructive mathematics, homotopy theory, topology, higher-dimensional algebra, higher categories.

1. Introduction

This research proposal pursues a new and surprising connection between Geometry, Algebra, and Logic, which has recently come to light in the form of an interpretation of the constructive type theory of Martin-Löf into homotopy theory, resulting in new examples of certain algebraic structures which are important in topology. This connection was recently discovered by the PI and his students, and various aspects of it are now also under active investigation by several other researchers worldwide. (See [AW09, AHW09, War08, BG09, GG08, Garar, GvdB08, Lum09].)

1.1. **Type theory.** Martin-Löf type theory is a formal system originally intended to provide a rigorous framework for constructive mathematics [ML75, ML98, ML84]. It is at base an extension of the typed lambda-calculus admitting dependent types and terms. Under the Curry-Howard correspondence [How80], one identifies types with propositions, and terms with proofs; viewed thus, the system is at least as strong as second-order logic, and it is known to interpret constructive set theory [Acz74]. Indeed, Martin-Löf type theory has been used successfully to formalize large parts of constructive mathematics, such as the theory of generalized recursive definitions [NPS90, ML79]. Moreover, it is also employed extensively as a framework for the development of high-level programming languages, in virtue of its combination of expressive strength and desirable proof-theoretic properties [NPS90, Str91].

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The type theory has two variants: an intensional, and an extensional version. The difference between them lies mainly in the treatment of equality of terms. In the intensional version (with which we are mainly concerned in the present work), one has two different kinds of equality: the first kind is called **definitional equality**, and behaves much like equality between terms in the simply-typed lambda-calculus, or any other conventional equational theory. The second kind is a more subtle relation, called **propositional equality**, which, under the Curry-Howard correspondence, represents the equality formulas of first-order logic. Specifically, given two terms a, b of the same type A, one may form a new type $\mathtt{Id}_A(a, b)$, which we think of as the proposition that a and b are equal; a term of this type thus represents a proof of the proposition that a equals b (hence the name "propositional equality").

When a and b are definitially equal, then (since they can be freely substituted for each other) they are also propositionally equal, in the sense that the identity type $\mathtt{Id}_A(a, b)$ is inhabited by a term; but the converse is generally not true in the intensional version of the theory (the rules for intensional identity types are given in the appendix at the end of this document). In the extensional version, by contrast, the two notions of equality are forced by an additional rule to coincide. As a consequence, the extensional version of the theory is essentially a dependent type theory with a standard, extensional equality relation. As is well-known, however, the price one pays for this simplification is a loss of desirable proof-theoretic properties, such as strong normalization and decidable type checking and equality of terms [Str93, Str91, Hof95a].

The intensional theory thus endows the identity types $\mathtt{Id}_A(a, b)$ with a non-trivial structure. Indeed, these satisfy certain conditions which were observed by Hofmann and Streicher in [HS98] to be analogous to the familiar laws for groupoids.[1] Specifically, the posited reflexivity of propositional equality produces identity proofs $\mathtt{r}(a) : \mathtt{Id}_A(a, a)$ for any term $a : A$, playing the role of a unit arrow for a; and when $f : \mathtt{Id}_A(a, b)$ is an identity proof, then (corresponding to the symmetry of identity) there also exists a proof $f^{-1} : \mathtt{Id}_A(b, a)$, to be thought of as the inverse of f; finally, when $f : \mathtt{Id}_A(a, b)$ and $g : \mathtt{Id}_A(b, c)$ are identity proofs, then (corresponding to transitivity) there is a new proof $g \circ f : \mathtt{Id}_A(a, c)$, thought of as the composite of f and g. Moreover, this structure on each type A can be shown to satisfy the usual groupoid laws, but significantly, only **up to propositional equality**. We shall return to this point below.

The constructive character, computational tractability, and proof-theoretic clarity of the type theory are owed in part to this rather subtle treatment of equality between terms, which itself is expressible within the theory using the identity types $\mathtt{Id}_A(a, b)$. Unlike extensional equality, which is computationally intractable, the expressibility of intensional equality within the theory leads to a system that is both powerful and expressive while retaining its important computational character. The cost of intensionality, however, has long been the resulting difficulty of providing a conventional semantic interpretation. Even sophisticated topological and categorical approaches have failed to fully capture the subtle structure of intensional type theory [Hof97, Car86, Hof95b, Dyb96].

The current research constructs a bridge from intensional type theory to contemporary algebraic topology, exploiting both the axiomatic approach to homotopy of Quillen model categories, as well as the related algebraic methods involving (weak) higher-dimensional groupoids. This at once provides **two** new domains of interpretation for type theory. In doing so, it also permits logical methods to be combined with the traditional algebraic and topological approaches to homotopy theory, opening up a range of possible new applications of type theory in the study of homotopy and higher-dimensional algebra. It also allows the importation into homotopy theory of computational tools based on the type theory, such as the computer proof assistants Coq and Agda (cf. [TLG06]).

[1] A **groupoid** is a like a group, but with a partially-defined composition operation. Precisely, a groupoid can be defined as a category in which every arrow has an inverse. A group is thus a groupoid with only one object. Groupoids arise in topology as generalized fundamental groups, not tied to a choice of basepoint (see below).

1.2. **Homotopy theory.** In homotopy theory one is concerned with spaces and continuous mappings up to homotopy; a **homotopy** between continuous maps $f, g : X \to Y$ is a continuous map $\vartheta : X \times [0,1] \to Y$ satisfying $\vartheta(x,0) = f(x)$ and $\vartheta(x,1) = g(x)$. Such a homotopy ϑ can be thought of as a "continuous deformation" of f into g, determining a higher-dimensional arrow $\vartheta : f \Rightarrow g$. Two spaces are said to be homotopy-equivalent if there are continuous maps going back and forth, the composites of which are homotopical to the respective identity mappings. Such spaces may be thought of as differing only by a continuous deformation. Algebraic invariants, such as homology or the fundamental group, are homotopy-invariant, in that any spaces that are homotopy-equivalent must have the same invariants.

It is natural to also consider homotopies between homotopies, referred to as **higher homotopies**. When we consider a space X, a distinguished point $p \in X$, and the paths in X beginning and ending at p, and identify such paths up to homotopy, the result is the **fundamental group** $\pi(X,p)$ of the space at the point. Modern homotopy theory generalizes this classical construction in several directions: first, we remove the dependence on the base-point p by considering the **fundamental groupoid** $\pi(X)$, consisting of all points and all paths up to homotopy. Next, rather than identifying homotopic paths, we can consider the homotopies between paths as distinct, new objects of a higher dimension (just as the paths themselves are homotopies between points). Continuing in this way, we obtain a structure consisting of the points of X, the paths in X, the homotopies between paths, the higher homotopies between homotopies, and so on for even higher homotopies. The resulting structure $\pi_\infty(X)$ is called the **fundamental weak ∞-groupoid of X**. Such higher-dimensional algebraic structures play a central role in homotopy theory (see e.g. [KV91]); they capture much more of the homotopical information of a space than does the fundamental group $\pi(X,p)$, or the groupoid $\pi(X) = \pi_1(X)$, which is a quotient of $\pi_\infty(X)$ by identifying the higher homotopies. As discussed in subsection 2.4 below, it has recently been shown that such higher-dimensional groupoids also arise naturally in intensional type theory.

Another central concept in modern homotopy theory is that of a **Quillen model structure**, which captures axiomatically some of the essential features of homotopy of topological spaces, enabling one to "do homotopy" in different mathematical settings, and to express the fact that two settings carry the same homotopical information. Quillen [Qui67] introduced model categories as an abstract framework for homotopy theory which would apply to a wide range of mathematical settings. Such a structure consists of the specification of three classes of maps (the fibrations, weak equivalences, and cofibrations) satisfying certain conditions typical of the leading topological examples. The resulting framework of axiomatic homotopy theory allows the development of the main lines of classical homotopy theory (fundamental groups, homotopies of maps, strong and weak equivalence, homotopy limits, etc.) independently of any one specific setting. Thus, for instance, it is also applicable not only in spaces and simplicial sets, but also in new settings, as in the work of Voevodsky on the homotopy theory of schemes [MV99], or that of Joyal [Joy02, Joy] and Lurie [Lur09] on quasicategories. In the research discussed below (subsection 2.3), it is shown that Martin-Löf type theory can be interpreted in any model category. This allows the use of type theory as a calculus to reason systematically about the objects and maps of homotopy theory.

2. Background and Preliminary Results

We review the background to the homotopical interpretation of constructive type theory and then briefly survey the recent research on that and related topics by the PI, his students, and others.

2.1. **Background.** Among the most thorough, recent treatments of the **extensional** type theory are the two papers [MP00, MP02] by Moerdijk and Palmgren from 2000 and 2002. The authors also announced a projected third paper devoted to the intensional theory, which never appeared. Their intention (known through private conversations) was to make use of higher categories and, perhaps, Quillen model structures. No preliminary results were ever announced, however (but see [Pal03]).

In 2006, Vladimir Voevodsky gave the Distinguished Lectures at Stanford University's Mathematics Department, consisting of three lectures entitled "Homotopy lambda-calculus". At that time, the PI and his student Michael Warren had already established the first interpretation of intensional type theory using Quillen model structures. Thus, these roughly simultaneous developments were apparently entirely independent. Later, private distribution of notes from Voevodsky's lectures ([Voe06]) confirmed some overlap in the main ideas, particularly the use of homotopy theory to model the intensionality in Martin-Löf's constructive type theory. Voevodsky's remarks are presented as conjectures and suggestions, however, rather than results, and to our knowledge he has not pursued this subject further.

Recently, several young logicians and topologists have begun investigating different aspects of the homotopical interpretation of type theory, partially in informal collaborations with the PI. These include Richard Garner (a recent PhD of Martin Hyland at Cambridge University), Benno van den Berg (a PhD of Ieke Moerdijk, now at Darmstadt, Germany), Nicola Gambino (a PhD of Peter Aczel, now at Palermo), and Pieter Hofstra (a PhD of Jaap van Oosten, now at Ottawa). In addition, Michael Warren has since finished his Carnegie Mellon PhD with the PI, and is currently a Fields Institute Postdoctoral Fellow at the Mathematics Department in Ottawa. In 2008, many of these individuals, as well as the PI and several senior researchers in type theory and homotopy theory, participated in a workshop devoted to the topic of "Categorical and homotopical structures in proof theory", as part of a larger meeting on *Simplicial Methods and Higher Categories in Homotopy Theory* at the Center for Mathematical Research in Barcelona.

In a new book on *Higher Topos Theory* by Jacob Lurie [Lur09], and a forthcoming one on *Quasi-Categories* by Andre Joyal [Joy], an explicit connection between homotopy theory, higher algebra, and topos theory is developed. In addition to its geometrical roots and applications, topos theory of course also has a distinctly logical aspect. Such a logical aspect is not present in these recent investigations of "higher topos theory", but may indeed be regarded as the main thrust of the current research proposal.

2.2. **Groupoid semantics.** A model of type theory is *extensional* if the following reflection rule is satisfied:

$$\frac{p : \mathtt{Id}_A(a,b)}{a = b : A}\ \text{Id-reflection}$$

I.e., the identity type $\mathtt{Id}_A(a,b)$ in extensional models captures no more information than whether or not the terms a and b are definitionally equal. Although type checking is decidable in the intensional theory, it fails to be so in the extensional theory obtained by adding Id-reflection as a rule governing identity types. This fact is the principal motivation for studying intensional rather than extensional type theories (cf. [Str91] for a discussion of the difference between the intensional and extensional forms of the theory). A good notion of a model for the extensional theory is due to Seely [See84], who showed that one can interpret type dependency in locally cartesian closed categories in a very natural way. (There are certain coherence issues, prompting a later refinement by Hofmann [Hof97], but this need not concern us here.) Of course, intensional type theory can also be interpreted this way, but then the interpretation of the identity types necessarily becomes trivial in the above sense

The first non-trivial semantics for intensional type theory were developed by Hoffmann and Streicher [HS98] using **groupoids**, which are categories in which every arrow is an iso. The category of groupoids is not locally cartesian closed [Pal03], and the model employs certain fibrations (equivalently, groupoid-valued functors) to model type dependency. Intuitively, the identity type over a groupoid G is interpreted as the groupoid $G^{\rightarrow}$ of arrows in G, so that an identity proof $f : \mathtt{Id}_A(a,b)$

becomes an arrow $f : a \to b$ in G. The interpretation no longer validates extensionality, since there can be different elements a, b related by non-identity arrows $f : a \to b$. Indeed, there may be many different such arrows $f, g : a \rightrightarrows b$; however—unlike in the type theory—these cannot in turn be further related by identity terms of higher type $\vartheta : \mathtt{Id}_{\mathtt{Id}_A}(f, g)$, since a (conventional) groupoid generally has no such higher-dimensional structure. Thus the groupoid semantics validates a certain truncation principle, stating that all higher identity types are trivial—a form of extensionality one dimension up. In particular, the groupoid laws for the identity types are strictly satisfied in these models, rather than holding only up to propositional equality.

This situation suggests the use of the higher-dimensional analogues of groupoids, as formulated in homotopy theory, in order to provide models admitting non-trivial higher identity types. Such higher groupoids occur naturally as the (higher) fundamental groupoids of spaces (as discussed above). A step in this direction was made by Garner [Garar], who uses a 2-dimensional notion of fibration to model intensional type theory in 2-groupoids, and shows that when various truncation axioms are added, the resulting theory is sound and complete with respect to this semantics. In his dissertation [War08] (supervised by the PI), Warren showed that infinite-dimensional groupoids also give rise to models which validate no such additional truncation axioms. It seems clear that one will ultimately need *weak* infinite dimensional groupoids in order to faithfully model the full intensional type theory; this is one important aspect of the research proposed here (see subsection 2.4 below).

2.3. **Homotopical models of type theory.** Groupoids and their homomorphisms arise in homotopy theory as a model of topological spaces with homotopy classes of continuous maps. There are other models as well, such as simplicial sets. The idea of a Quillen model structure (cf. [Qui67, Bou77]) is to axiomatize the common features of these different models of homotopy, allowing one to develop the theory in an abstract general setting, and to compare different particular settings.

This axiomatic framework also provides a convenient way of specifying a general semantics for intensional type theory, not tied to a particular choice of groupoids, 2-groupoids, ∞-groupoids, etc., or even spaces themselves. The basic result in this connection is due to the PI and his student Warren in [AW09] (see also Warren's thesis [War08]), stating that it is possible to model the type theory in any Quillen model category (a weak factorization system suffices). In this interpretation, one uses so-called "path objects" to model identity types in a non-trivial way, recovering the original groupoid model as a special case.

Subsequently, in [GG08] it was shown that the type theory itself carries a natural such homotopy structure (i.e. a weak factorization system), so that the type theory is not only sound, but also complete with respect to such abstract homotopical semantics. Together, these results clearly establish not only the viability of the homotopical interpretation as semantics for type theory, but also the possibility of using type theory to reason in Quillen model structures.

In order to describe the interpretation in somewhat more detail, we first recall a few standard definitions. In any category $\mathcal{C}$, given maps $f : A \to B$ and $g : C \to D$, we write $f \pitchfork g$ to indicate that f has the *left-lifting property* (LLP) with respect to g: for any commutative square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "i"'] \arrow[ur, dotted, "j"] & D \end{tikzcd}$$

there exists a map $j : B \to C$ such that $j \circ f = h$ and $g \circ j = i$. If M is any collection of maps, we denote by ${}^{\pitchfork}\mathsf{M}$ the collection of maps in $\mathcal{C}$ having the LLP with respect to all maps in M. The collection of maps $\mathsf{M}^{\pitchfork}$ is defined similarly. A *weak factorization system* (L, R) in a category $\mathcal{C}$ consists of two collections L (the "left-class") and R (the "right-class") of maps in $\mathcal{C}$ such that:

(1) Every map $f : A \to B$ has a factorization as $f = p \circ i$, where $i \in \mathsf{L}$ and $p \in \mathsf{R}$.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "f"'] & C \arrow[d, "p"] \\ & B, \end{tikzcd}$$

(2) $\mathsf{L} = {}^{\pitchfork}\mathsf{R}$ and $\mathsf{L}^{\pitchfork} = \mathsf{R}$.

A *(closed) model category* [Qui67] is a bicomplete category $\mathcal{C}$ equipped with subcategories F (fibrations), C (cofibrations) and W (weak equivalences), satisfying the following two conditions: (1) Given any maps $g \circ f = h$, if any two of f, g, h are weak equivalences, then so is the third; (2) both $(\mathsf{C}, \mathsf{F} \cap \mathsf{W})$ and $(\mathsf{C} \cap \mathsf{W}, \mathsf{F})$ are weak factorization systems. A map f in a model category is a *trivial cofibration* if it is both a cofibration and a weak equivalence. Dually, a *trivial fibration* is a map which is both a fibration and a weak equivalence. An object A is said to be *fibrant* if the canonical map $A \to 1$ is a fibration. Dually, A is *cofibrant* if $0 \to A$ is a cofibration.

Examples of model categories include the following:

(1) The category Top of topological spaces, with fibrations the Serre fibrations, weak equivalences the weak homotopy equivalences, and cofibrations those maps which have the LLP with respect to trivial fibrations. The cofibrant objects in this model structure are retracts of spaces constructed, like CW-complexes, by attaching cells.
(2) The category SSet of simplicial sets, with cofibrations the monomorphisms, fibrations the Kan fibrations, and weak equivalences the weak homotopy equivalences. The fibrant objects for this model structure are the Kan complexes.
(3) The category Gpd of (small) groupoids, with cofibrations the homomorphisms that are on objects, fibrations the Grothendieck fibrations, and weak equivalences the categorical equivalences. Here all objects are both fibrant and cofibrant.

See e.g. [DS95, Hov99] for further examples and details.

Finally, recall that in any model category $\mathcal{C}$, a *(very good) path object* A^I for an object A consists of a factorization

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[dr, "\Delta"'] & A^I \arrow[d, "p"] \\ & A \times A, \end{tikzcd} \tag{1}$$

of the diagonal map $\Delta : A \to A \times A$ as a trivial cofibration r followed by a fibration p (see [Hov99]). Paradigm examples of path objects are given by exponentiation by a suitable "unit interval" I in either Gpd or, when the object A is a Kan complex, in SSet. In e.g. the former case, G^I is just the "arrow groupoid" $G^{\to}$, consisting of all arrows in the groupoid G. Path objects always exist, but are not uniquely determined. In many examples, however, they can be chosen functorially.

We can now describe the homotopy interpretation of type theory. Whereas the idea of the Curry-Howard correspondence is often summarized by the slogan "Propositions as Types", the idea underlying the homotopy interpretation is instead "Fibrations as Types". In classical topology, and in most model categories, a fibration $p : E \to X$ can be thought of as a family of objects E_x varying continuously in a parameter $x \in X$. (The path-lifting property of a topological fibration describes how to get from one fiber $E_x = p^{-1}(x)$ to another E_y along a path $f : x \rightsquigarrow y$). This notion gives the interpretation of type dependency. Specifically, assume that $\mathcal{C}$ is a finitely complete category with (at least) a weak factorization system (L, R). Because most interesting examples arise from model categories, we refer to maps in L as trivial cofibrations and those in R as fibrations. A judgement $\vdash A : \mathtt{type}$ is then interpreted as a fibrant object A of $\mathcal{C}$. Similarly, a dependent type $x : A \vdash B(x) : \mathtt{type}$ is interpreted as a fibration $p : B \to A$. Terms $x : A \vdash b(x) : B$ in context are interpreted as sections $b : A \to B$ of $p : B \to A$, i.e. $p \circ b = 1_A$. Thinking of fibrant objects as types and fibrations as dependent types, the natural interpretation of the identity type $\mathtt{Id}_A(a, b)$ should then be as the *fibration of paths* in A from a to b, so that the type $x, y : A \vdash \mathtt{Id}_A(x, y)$ should be the "fibration of all paths in A". That is, it should be a path object for A.

Theorem 2.1 ([AW09]). *Let $\mathcal{C}$ be a finitely complete category with a weak factorization system and a functorial choice of stable path objects. I.e., given any fibration $B \to A$ and any map $\sigma : A' \to A$, the evident comparison map is an isomorphism,*

$$\sigma^*(B^I) \cong \sigma^*(B)^I.$$

Then $\mathcal{C}$ is a model of Martin-Löf type theory with identity types.

The proof exhibits the close connection between type theory and axiomatic reasoning in this setting: We verify the rules for the identity types (see the Appendix). Given a fibrant object A, the judgement $x, y : A \vdash \mathtt{Id}_A(x, y)$ is interpreted as the path object fibration $p : A^I \to A \times A$ (see (1)). Because p is then a fibration, the formation rule

$$x, y : A \vdash \mathtt{Id}_A(x, y) : \mathtt{type}$$

is satisfied. Similarly, the introduction rule

$$x : A \vdash \mathtt{r}(x) : \mathtt{Id}_A(x, x)$$

is valid because the interpretation $r : A \to A^I$ is a section of p over $\Delta : A \to A \times A$. For the elimination and conversion rules, assume that the following premisses are given

$$\begin{aligned} x : A, y : A, z : \mathtt{Id}_A(x, y) &\vdash D(x, y, z) : \mathtt{type}, \\ x : A &\vdash d(x) : D(x, x, \mathtt{r}(x)). \end{aligned}$$

We have, therefore, a fibration $q : D \to A^I$ together with a map $d : A \to D$ such that $q \circ d = r$. This data yields the following (outer) commutative square:

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[d, "r"'] & D \arrow[d, "q"] \\ A^I \arrow[r, "1"'] \arrow[ur, dotted, "j"] & A^I \end{tikzcd}$$

Because q is a fibration and r is, by definition, a trivial cofibration, there exists a diagonal filler j, which we choose as the interpretation of the term:

$$x, y : A, z : \mathtt{Id}_A(x, y) \vdash \mathtt{J}(d, x, y, z) : D(x, y, z).$$

Commutativity of the bottom triangle is precisely this conclusion of the elimination rule, and commutativity of the top triangle is the required conversion rule:

$$x : A \vdash \mathtt{J}(d, x, x, \mathtt{r}(x)) = d(x) : D(x, x, \mathtt{r}(x)).$$

Examples of categories satisfying the hypotheses of this theorem include groupoids, simplicial sets, and many simplicial model categories [Qui67] (including, e.g., simplicial sheaves and presheaves). There is a question of selecting the diagonal fillers j as interpretations of the $\mathtt{J}$-terms in a "coherent way", i.e. respecting substitutions of terms for variables. Some solutions to this problem are discussed in [AW09, War08, Gar07]. A more thorough study of coherence is one of the proposed research directions of this proposal.

A converse of the theorem just described, showing the completeness of type theory with respect to the homotopy interpretation, has recently been established by Gambino and Garner [GG08]. Their method involves showing that any syntactic system of type theory admits the structure of a weak factorization system, thus serving as a sort of "term model". A close analysis of this term model should serve as a starting point for the investigation of the coherence problem just mentioned.

2.4. **Higher algebraic structures.** Given the soundness and completeness of type theory with respect to the abstract homotopical interpretation in Quillen model structures, we are justified in thinking of types in the intensional theory as **spaces**. From this point of view, the terms of the type A are the points of the "space" A, the identity type $\mathtt{Id}_A(a,b)$ represents the collection of paths from a to b, and the higher identities are homotopies between paths, homotopies between homotopies of paths, etc. The topological fact that paths and homotopies do not form a groupoid, but only a groupoid up to homotopy, is of course precisely the same observation as the logical fact that the identity types only satisfy the groupoid laws up to propositional equality. This parallel between homotopy theory and type theory has now been made precise by the recognition that both cases are instances of one and the same abstract axiomatic theory.

2.4.1. *Weak ω-groupoids.* Pursuing this connection further, it has recently been shown by the PI and his current PhD student Peter Lumsdaine [Lum09] (and independently by van den Berg & Garner [BG09, vdB]) that the tower of identity types over any fixed base type A in the type theory indeed gives rise to a certain infinite dimensional algebraic structure called a weak ω-groupoid. Such structures arose first in homotopy theory, and are still the subject of active research ([KV91, Lei02, Che07, Bro87]).

In somewhat more detail, in the globular approach to higher groupoids [Lei04, Bat98], a weak ω-groupoid has objects ("0-cells"), arrows ("1-cells") between objects, 2-cells between 1-cells, and so on, with various composition operations and laws depending on the kind of groupoid in question (strict or weak, n- or ω-, etc.). One paradigm for interpreting type theory is that types (or contexts) are interpreted as objects $[\![A]\!]$, terms $x : A \vdash \tau : B$ as arrows $[\![\tau]\!] : [\![A]\!] \to [\![B]\!]$, terms of identity type $\rho : \mathrm{Id}_A(\tau, \tau')$ as 2-cells $[\![\rho]\!] : [\![\tau]\!] \Rightarrow [\![\tau']\!]$, terms $\chi : \mathrm{Id}(\rho, \rho')$ as 3-cells, and so on.[2] Now, it can be shown that the terms of any type X, together with those of its higher identity types $\mathtt{Id}_X, \mathtt{Id}_{\mathtt{Id}_X}, \ldots$, already carry the structure of a weak ω-groupoid, and that the interpretation $[\![-]\!]$ just mentioned is then simply a homomorphism.

To describe this weak ω-groupoid more closely, we first require the notion of a globular set, which may be thought of as an "infinite-dimensional" graph. Specifically, a *globular set* ([Bat98, Str00]) is a presheaf on the category $\mathbb{G}$ generated by arrows

$$0 \underset{t_0}{\overset{s_0}{\rightrightarrows}} 1 \underset{t_1}{\overset{s_1}{\rightrightarrows}} 2 \rightrightarrows \cdots$$

subject to the equations $ss = ts$, $st = tt$. More concretely, a globular set $A_\bullet$ has a set A_n of "n-cells" for each $n \in \mathbb{N}$, and each $(n+1)$-cell x has parallel source and target n-cells $s(x)$, $t(x)$. (Cells x, y of dimension > 0 are *parallel* if $s(x) = s(y)$ and $t(x) = t(y)$; all 0-cells are considered parallel.)

Example 2.2. For a type A in a type theory $\mathbb{T}$, the terms of types

$$A, \mathtt{Id}_A, \mathtt{Id}_{\mathtt{Id}_A}, \ldots,$$

together with the evident indexing projections, e.g. $s(p) = a$ and $t(p) = b$ for $p : \mathtt{Id}_A(a,b)$, form a globular set $\hat{A}$.

[2]Actually, this interpretation presumes that the algebraic structure is not an ω-groupoid, but something more general called an $(\infty,1)$-category, in which the cells of dimension 1 need not be invertible. Such categories arise in homotopy [Berar], but have an obvious importance in type-theory as well.

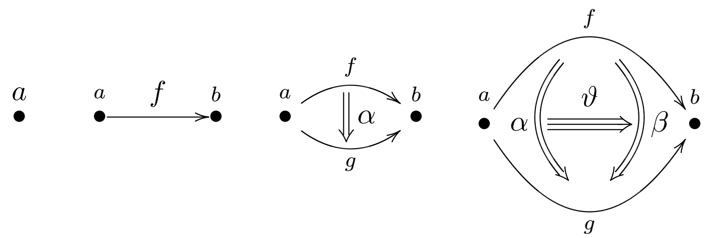

FIGURE 1. Some cells in an ω-groupoid

A strict ω-groupoid is a higher-dimensional groupoid satisfying, in all dimensions, associativity, unit, and inverse laws given by equations between certain cells. Such a groupoid has an underlying globular set consisting of cells of each dimension, and any globular set $A_\bullet$ generates a free strict ω-groupoid $F(A_\bullet)$—just as any set generates a free group, and any graph, a free groupoid. The cells of $F(A_\bullet)$ are free (strictly associative) pastings-together of cells from $A_\bullet$ and their formal duals, including degenerate pastings from the identity cells of $F(A_\bullet)$. In a *strict* ω-groupoid, cells can be composed along a common boundary in any lower dimension, and the composition satisfies various associativity, unit, and interchange laws, captured by the generalized associativity law: each labelled pasting diagram has a unique composite (see Figure 1).

In a *weak* ω-groupoid, by contrast, we do not expect strict associativity, and so we may have multiple composition maps for each shape of pasting diagram; but we do demand that these composites agree *up to cells of the next dimension*, and that these associativity cells satisfy coherence laws of their own, and so on.

Now, this is exactly the situation we find in intensional type theory. For instance, even in constructing a term witnessing the transitivity of identity, one finds that there is no single canonical candidate. Specifically, as a composition for the pasting diagram

$$\cdot \longrightarrow \cdot \longrightarrow \cdot$$

or more concretely, a term c such that

$$x, y, z : X, p : \mathrm{Id}(x, y), q : \mathrm{Id}(y, z) \vdash c(q, p) : \mathrm{Id}(x, z),$$

there are the two equally natural terms c_l, c_r obtained by applying (Id-ELIM) to p and q respectively. These are not definitionally equal, but are propositionally equal, i.e. equal up to a 2-cell, for there is a term e with

$$x, y, z : X, p : \mathrm{Id}(x, y), q : \mathrm{Id}(y, z) \vdash e(q, p) : \mathrm{Id}(c_l(q, p), c_r(q, p)).$$

Indeed, we have the following:

Theorem 2.3 ([Lum09]). *Let A be any type in a system $\mathbb{T}$ of intensional Martin-Löf type theory. Then the globular set $\hat{A}$ of terms of type $A, \mathtt{Id}_A, \mathtt{Id}_{\mathtt{Id}_A}, \ldots$ carries a natural weak ω-groupoid structure.*

It is now quite natural to ask what special properties this particular ω-groupoid has in virtue of its type-theoretic construction. In light of related syntactic constructions of other types of free algebras, a sensible conjecture is that it is the **free weak ω-groupoid**, up to a suitable notion of equivalence. This is one of the proposed research directions of this proposal.

The approach of Garner and van den Berg [GvdB08] differs from that just sketched in that uses an axiomatic description, in the style of Quillen model categories, of the homotopy structure of the type theory to construct the ω-groupoid, rather than doing so explicitly in the type theory. This highlights the common source of such groupoids in type theory and homotopy theory, and suggests pursuing the axiomatic development further. The latter development is another of the proposed research directions of this proposal.

2.4.2. *Weak n-groupoids.* A further step in exploring the connection between type theory and homotopy is to investigate the relationship between type theoretic "truncation" (i.e. higher-dimensional extentionality principles) and topological "truncation" of the higher fundamental groups. Spaces for which the homotopy type is already completely determined by the fundamental groupoid are called **homotopy 1-types**, or simply 1-types [Bau95]. More generally, one has n-types, which are thought of as spaces which have no homotopical information above dimension n. One of the goals of homotopy theory is to obtain good models of homotopy n-types. For example, the category of groupoids is Quillen equivalent to the category of 1-types [KV91]; in this precise sense, groupoids are said to model homotopy 1-types. Recent work [AHW09] by the PI, his former student Warren, and Pieter Hofstra has shown that the 1-truncation of the intensional theory, arrived at by adding the analogue of the Id-reflection rule for all terms of identity type, generates a Quillen model structure on the category of graphs that is Quillen equivalent to that of groupoids. In a precise sense, the truncated system of 1-dimensional type theory thus captures the homotopy 1-types.

In a bit more detail, for every globular set $A_\bullet$ one can define a system of type theory $\mathbb{T}(A_\bullet)$, the basic terms of which are the elements of the various A_n, typed as terms of the corresponding identity type determined by the globular structure: e.g. $a \in A_n$ is a basic term of type $\mathtt{Id}_A(s(a), t(a))$, where $s, t : A_n \rightrightarrows A_{n-1}$ are the source and target maps, at dimension n, of $A_\bullet$. Since we know from the result of Lumsdaine [Lum09], just reviewed, that for any type X, the underlying globular set of terms of the various identity types $X, \mathtt{Id}_X, \mathtt{Id}_{\mathtt{Id}_X}, \ldots$ gives rise to a weak ω-groupoid, we can infer that in particular the globular set of terms over the ground type A_0 in the theory $\mathbb{T}(A_\bullet)$ form such a groupoid, **generated type-theoretically** from the arbitrary globular set $A_\bullet$. Let us call this weak ω-groupoid $G_\omega(A_\bullet)$, the **type-theoretically free** weak ω-groupoid generated by $A_\bullet$. This construction is investigated in depth in [AHW09], where certain groupoids of this kind are called a **Martin-Löf complexes** (technically, these are the algebras for the globular monad just described).

It is clearly of interest to investigate the relationship between this type-theoretic construction of higher groupoids and both the algebraically free higher groupoids, on the one hand, and the higher groupoids arising from spaces as fundamental groupoids, on the other. As a first step, one can consider the 1-dimensional truncation of the above construction, and the resulting (1-) groupoid $G_1(A_\bullet)$. For that case, the following result relating $G_1(A_\bullet)$ to the usual, algebraically free groupoid is established in the work cited:

Theorem 2.4 ([AHW09])**.** *The type-theoretically free groupoid is equivalent to the algebraically free groupoid.*

The proof uses a variation of the proof-theoretic technique of "logical predicates" due to Tait [Tai67]. Furthermore, it is shown that the 1-truncated Martin-Löf complexes admit a Quillen model structure equivalent to that of (1-) groupoids. The following then results from known facts from homotopy theory:

Theorem 2.5 ([AHW09])**.** *The 1-truncated Martin-Löf complexes classify homotopy 1-types.*

Obviously, one should now proceed to higher groupoids and the corresponding type theories truncated at higher dimensions. This constitutes another one of the proposed research directions of this proposal.

3. Proposed Research Directions

The foregoing survey of the present state of knowledge in the field also served to indicate several proposed directions for future research, in the context of the PI's past research and work in progress by the PI and others. We now list the specific research topics mentioned above, and briefly elaborate their objectives and expected significance.

3.1. **Coherence.** As explained above (in 2.3), the interpretation in Quillen model structures (and weak factorization systems generally) need not be coherent, in the sense that substitution into elimination terms is not respected (this is sometimes known as the "Beck condition"). This results quite naturally from the fact that the diagonal fillers in lifting problems are in general not unique, nor are there always canonical choices for such fillers. Even when such choices can be made in a natural way, they need not respect reindexing or change-of-base (pullback). This situation is not a defect, but a virtue of the axiomatic approach to homotopy: it provides flexibility and freedom from having to make choices that have no special significance. Nonetheless, for some purposes it is convenient to have canonical, coherent liftings, and recent work [Gar07] has shown that it is possible to replace many familiar model structures (i.e. those that are cofibrantly generated) by equivalent ones equipped with a natural choice of diagonal liftings. This presents one possible solution to the coherence problem which can be further pursued.

In the case of the particular diagonal fillers involved in the interpretation of type theory, however, it can also be shown [AW09] that all choices are already homotopic, and so a weakened "Beck condition up to homotopy" necessarily holds. Thus one should consider a type theory with explicit substitutions in the style of [ACCL89]; such a theory would be interpreted in a far larger range of homotopy theories, allowing a more direct use of the type theory for homotopical calculations.

One approach to this issue is suggested by the completeness of type theory with respect to the homotopical interpretation ([AW09] and [GG08], see above2.3). Investigating the term model should provide useful insight into the precise coherence laws that need to obtain in all models. It should then be possible to sharpen the soundness and completeness results with an axiomatic description of the required coherence conditions. Such a description could be given within a few months of dedicated study.

Another approach proposed by the PI and pursued in Warren's thesis [War08] avoids coherence issues altogether by deriving the entire homotopy structure from an "interval object", an internal cogroupoid generating the system of fibrations and weak equivalences by lifting properties (as in classical homotopy with respect to the unit interval $[0, 1]$). Preliminary results in [War08, War09] indicate the promise of this approach.

3.2. **Free weak ω-groupoid.** The type-theoretic construction of a weak ω-groupoid outlined in subsection 2.4 above is as plausible a candidate for the free such groupoid as is any known construction (indeed more so, given the known syntactic constructions of other free algebras). The chief difficulty in verifying freeness is in the present lack of a suitable notion of homomorphism for weak ω-groupoids and categories. There are several competing such notions [Lei02, Gar08], all quite elaborate, and one needs to determine whether any of them, or some variant, will serve.

A similar difficulty applies to the truncations of both the full type theory and the notions of weak ω-groupoid. Thus the "2-dimensional type theory" of [Garar] should generate the free weak 2-groupoid, etc. But already here, the existing notion of bi-homomorphism of 2-groupoid is not obviously the correct one for this purpose (although, to be sure, it is the first thing to check). It is quite likely that the type-theoretic construction itself could provide useful insight into the correct notion of homomorphism for weak, higher-dimensional algebras. One could then proceed to compare the higher truncations of type theory with the homotopy types [MS93, LW50].

The logical construction of a free weak ω-groupoid would be a significant contribution to homotopy theory, from an unexpected source. This result is not far away, and could likely be accomplished within the period of this proposal.

A further step in this direction which could then follow is the investigation of the **simplicial aspect of type theory**. The globular structure pursued to date results from focussing on a single type, but the entire theory actually has a structure closer to that of an $(\infty, 1)$-category [Berar] or quasicategory [Joy, Lur09]. It should thus also be the case that any system of type theory generates a "free quasicategory". Work on this conjecture is already underway in collaboration with a PhD student.

3.3. **Axiomatics.** The completeness of the type theory with respect to the homotopy interpretation ([AW09] and [GG08], see subsection 2.3 above) means that logical methods will suffice in principal for many homotopical constructions, such as fundamental groupoids (as is in fact exploited in [GvdB08]). This means that intensional type theory can usefully serve as a logical calculus for reasoning in model categories. To make this an effective tool in practice, one should first systematize the interpretation, in the style of the Kripke-Joyal semantics for categories of sheaves [MM92]. As a next step, one can then investigate the interpretation of sum and product type constructions. It is to be expected that these will relate to the topologists' **homotopy limits and colimits**. This would enable the use of the full type theory as a systematic tool for homotopical calculations and constructions.

In higher-dimensional algebra, too, the benefits of such a rigorous system for reasoning and calculating could be enormous. Current methods of reasoning and even notation begin to fail after just a few dimensions, under the sheer combinatorial complexity of the objects of investigation. The use of type theory as an auxiliary axiomatic framework for higher-dimensional algebra would also bring with it the possibility of computational assistance in the form of computer verification systems like Agda and Coq [TLG06], designed specifically for calculating in type theory. Of course, the same is true with respect to homotopy theory: the use of type theory to formalize homotopical constructions and calculations immediately admits the possibility of computational assistance.

A practically attainable example of such an application would be the verification in higher dimensions of the Breen-Baez-Dolan stabilisation hypothesis.[3] This is a generalization to higher dimensions of the well-known Eckmann-Hilton argument from topology, which shows that the higher homotopy groups are commutative, with applications to the higher homotopy of spheres, cobordism, and topological quantum field theory. A very few low dimensional cases have been successfully verified by hand, but higher dimensions prove combinatorially intractable using current methods [CG08, BS09]. A type theoretic formalization, combined with one of the standard computational implementations, could make a real contribution to advancing this question.

Investigating such applications will be a component of this research project, and it is likely that computational applications will be found quite readily, given the already well-developed implementations of type theory, combined with the soundness and completeness of the theory with respect to the homotopical interpretation.

3.4. **Long-term goals.** The results and directions just surveyed represent the beginnings of a novel research program at the intersection of topology, algebra and logic. The results already achieved and research currently underway strongly indicate the fruitfulness of this program and its potential to make contributions to each of these fields in ways not attainable by current methods.

The application of logic in geometry and topology via categorical algebra has a precedent in the development of topos theory. Invented by Grothendieck as an abstract framework for sheaf cohomology, the notion of a topos was soon discovered to have a logical interpretation, admitting the use of logical methods into topology (see e.g. [JT84] for just one of many examples). Equally important was the resulting flow of geometric and topological ideas and methods into logic, e.g. sheaf-theoretic independence proofs, topological semantics for many non-classical systems, and an abstract treatment of realizability (see the encyclopedic work [Joh03]).

[3]It states that every k-tuply monoidal n-category for $k > n+2$ is equivalent to an $(n+2)$-tuply monoidal n-category.

An important and lively research program in current homotopy theory is the pursuit (again following Grothendieck [Gro83]) of a general concept of "stack," subsuming sheaves of homotopy types, higher groupoids, quasi-categories, and the like (see e.g. the recent Fields Institute Thematic Program on Geometric Applications of Homotopy Theory devoted to this topic). Two important works in this area have either just been published (Lurie, *Higher Topos Theory* [Lur09]), or are available in preprint form (Joyal, *Theory of Quasi-Categories* [Joy]). It may be said, somewhat roughly, that the notion of a "higher-dimensional topos" is to homotopy what that of a topos is to topology (as in [JT91]). This concept also has a clear categorical-algebraic component via Grothendieck's "homotopy hypothesis", which states that n-groupoids are combinatorial models for homotopy n-types, and thus ω-groupoids are models for topological spaces. Missing from the recent development of higher-dimensional toposes, however, is a logical aspect analogous to that of (1-dimensional) topos theory. The current research program by the PI and his students and collaborators indicates that such a component is to be found in intensional type theory. The homotopy interpretation of Martin-Löf type theory into Quillen model categories, and the related results on type-theoretic constructions of higher groupoids, are analogous to basic results interpreting *extensional* type theory and higher-order logic in (1-) toposes, and clearly indicate that the logic of higher toposes, and therewith of higher homotopy theory, is a form of intensional type theory.

Thus the general motivation for the present project can be summarized as investigating, developing, and promoting the conception of homotopy as a model of intensional type theory, which itself is, of course, a formalization of constructive mathematics and logic. The powerful tools of higher-dimensional algebra that have recently been developed for the study of homotopy provide the mathematical framework for this undertaking. A concrete step toward advancing this general program would be the formulation of axioms for an elementary "higher topos", analogous to those for an elementary topos given by Lawvere and Tierney [Tie76]. Such a concept should capture not only the categories of stacks and sheaves of homotopy types currently investigated in homotopy theory (analogous to Grothendieck toposes), but (like elementary toposes) should also highlight the logical aspect, by admitting a direct interpretation of intensional type theory. The first steps toward such a concept are being taken in the research summarized and proposed here.

4. Plan of Work

4.1. **General plan of work.** Work will proceed in collaboration with a doctoral student throughout the academic year and during the summer months. The customary research seminars and weekly meetings will be supplemented by research visits from others working in the field, research visits by the PI and student to other locations where related research is being conducted, and attendance at research meetings and workshops.

In addition, the PI will have a sabbatical leave during the period of the grant, part of which will be devoted to an extended stay at a location where related research is conducted. Possible locations include Cambridge, England; Utrecht, Holland; and Montreal or Ottawa, Canada.

The PI has a successful record of research and publication, as well as doctoral supervision. The methods and practices leading to this success in the past will be continued, and augmented by the support provided by this grant. Specifically, support for summer research, travel support for research visits, and graduate student support will permit a more dedicated and focused effort.

4.2. **Timetable.** The specific topics discussed in section 3 above will be pursued as follows:

1. Free weak ω-groupoid: Work toward this result is already underway. Along with its generalization to quasi-categories, this will likely form part of the PhD student's thesis.
2. Axiomatics: The development of the axiomatic approach and its associated computational implementation will begin with the start of the grant, as an independent research project, in collaboration with a faculty-member and student from Carnegie Mellon's Department

of Computer Science. It is projected that first results could be attained within one year, and a practically useful, working implementation could be in place by the end of the grant period.

(3) Coherence: Work on this topic will proceed parallel to the others. As a mainly theoretical issue, it serves as a guide to other long-range developments. The immediate, more ad hoc solutions, can be worked out quite quickly, probably within the first year, but the fundamental issue, and the knowledge gained from its further investigation, will guide the long-range development of the theory. The implementation of the axiomatic and computational tools will also be a benefit in the later investigation of coherence.

(4) Higher topos: An axiomatic description of higher-dimensional toposes will be developed and refined as work on the other, specific problems sheds sufficient light on the essential features of the concept.

4.3. **Dissemination of results.** The results of the research supported by this project will be presented, documented, and distributed in three distinct scientific communities: type theory and constructive mathematics and logic, algebraic topology and homotopy theory, and higher-dimensional algebra and category theory. Each of these communities has its own workshops, conferences, electronic preprint distribution, and scientific journals. The PI and his student will strive to participate, to the extent practical, in each of these communities. This participation serves not only the purpose of dissemination of results, but perhaps even more importantly, allows the PI and student to interact with researchers in these different fields, in order to learn more of the relevant theory, keep up with new developments, and gauge the success of current project research.

The PI is already an active member of each of the three research communities mentioned, through attendance and presentation of results at conferences and workshops, publication in scientific journals, research collaborations, and supervision of doctoral research. Specific examples include:

- Invited speaker at the workshop on "Proof-theoretic methods in homotopy theory" as part of a special course on Simplicial Methods and Higher Categories in Homotopy Theory at the Center for Mathematical Research in Barcelona, 2008.
- Recent publications in both the *Mathematical Proceedings of the Cambridge Philosophical Society* (2008) and the *Journal of Symbolic Logic* (2009).
- Refereed contributions to Typed Lambda-Calculus and its Application, 2009 (two papers in collaboration with students).
- Recent PhD supervisee (2008) is now a Fields Institute post-doctoral fellow in mathematics at the University of Ottowa.

Such involvement and interaction will continue and be strengthened by the support provided by this grant.

5. Broader Impacts of the Proposed Research

As discussed in the foregoing, the broader impact of the proposed research is to be found in both graduate education and in applications in science and technology.

5.1. **Doctoral education.** The PI is actively involved in the supervision of doctoral students in logic and mathematics in Carnegie Mellon's program in Pure and Applied Logic. A doctoral student in that program is supported under this research project. The student will be trained in the relevant areas of topology, algebra, and logic, and will conduct joint research with the PI, leading to the degree of PhD. Moreover, the student will visit other locations where related research is being conducted and attend research conferences and workshops to learn of recent developments and present results of this project. He/she will also be trained in the essential scientific activities of conducting independent research, lecturing, writing scientific papers for publication, and refereeing of research papers. The PI has a successful track record of graduate supervision, with two current PhD students, and two recent successful ones currently holding post-doctoral fellowships in mathematics.

The PI also collaborates regularly with faculty and students from Carnegie Mellon's distinguished Department of Computer Science. Such collaboration strengthens interdisciplinary ties in both education and research, enhancing the educational infrastructure and building lasting research partnerships.

5.2. **Potential applications.** As already explained in the narrative of the proposal, this research has a strong potential for direct applications in computer science. Constructive type theories of the kind under investigation are used extensively in programing language design and implementation. Homotopical and algebraic interpretations can be of use both in securing the soundness of various applied systems, and as a new theoretical model of the computational paradigms implemented by such systems. Conversely, computational applications in the related areas of mathematics, i.e. topology and higher-dimensional algebra, are made likely by the well-developed computational implementations of constructive type theory.

Appendix A. Rules of type theory

This appendix recalls the rules for identity types in intensional Martin-Löf type theory. See [ML84, NPS90, Jac99] for detailed presentations.

Judgement forms. There are four basic forms of judgement:

$$A : \mathtt{type}, \qquad a : A, \qquad a = b : A, \qquad A = B : \mathtt{type}.$$

Each form can occur also with free variables: e.g. if A is a type, then

$$x : A \vdash B(x) : \mathtt{type}$$

is called a *dependent type*, regarded as an A-indexed family of types.

Rules for identity types.

$$\frac{A : \mathtt{type}}{x : A, y : A \ \vdash \ \mathtt{Id}_A(x,y) : \mathtt{type}} \ \mathtt{Id}\text{ formation}$$

$$\frac{a : A}{\mathtt{r}(a) : \mathtt{Id}_A(a,a)} \ \mathtt{Id}\text{ introduction}$$

$$\frac{c : \mathtt{Id}_A(a,b) \qquad \begin{array}{c} x : A, y : A, z : \mathtt{Id}_A(x,y) \ \vdash \ B(x,y,z) : \mathtt{type} \\ x : A \ \vdash \ d(x) : B\big(x,x,\mathtt{r}(x)\big) \end{array}}{\mathtt{J}(d,a,b,c) : B(a,b,c)} \ \mathtt{Id}\text{ elimination}$$

$$\frac{a : A}{\mathtt{J}\big(d,a,a,\mathtt{r}(a)\big) \ = \ d(a) : B\big(a,a,\mathtt{r}(a)\big)} \ \mathtt{Id}\text{ conversion}$$

The introduction rule provides a witness $\mathtt{r}(a)$ that a is identical to itself, called the *reflexivity term*. The distinctive elimination rule can be recognized as a form of Leibniz's law. The variable $x : A$ is bound in the the notation $\mathtt{J}(d,a,b,c)$.

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