$\mathcal{N} = 2$ Dualities and $2d$ TQFT

Abhijit Gadde

with L. Rastelli, S. Razamat and W. Yan

arXiv:0910.2225
arXiv:1003.4244
arXiv:1104.3850
arXiv:1110:3740
...

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A new paradigm for 4d $\mathcal{N} = 2$ SCFTs [Gaiotto, ...]

Compactify of the $6d$ $(2,0)$ A_{N-1} theory on a 2d surface Σ, with punctures.
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  - Free hypermultiplets
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6=4+2: beautiful and unexpected 4d/2d connections. For ex.,

- Correlators of Liouville/Toda on Σ compute the 4d partition functions (on S^4)

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Superconformal Index
$=$ twisted partition function on $S^3 \times S^1$
$=$ Witten index in radial quantization

- Independent of the gauge theory coupling and invariant under S-duality.
- Independent of coupling $=$ Independent of complex structure on $\Sigma \Longrightarrow$ 2d Topological QFT
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Our aim: To compute the superconformal index of these $\mathcal{N} = 2$ theories (even the strongly coupled ones) by exploiting TQFT structure.

Outline

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The Superconformal Index [Romelsberger; Kinney, Maldacena,Minwalla, Raju 2005]

- The SC index is the Witten index

$$\mathcal{I} = Tr(-1)^F \, e^{-\beta H + M}$$

  Here M is a generic combination of charges (weighted by chemical potentials) which commutes with S and Q.
- States with $H > 0$ come in pairs, boson + fermion, and cancel out,so $\mathcal{I}$ is β-independent.

The SC Index counts (with signs) the (semi)short multiplets, up to equivalence relations that sets to zero $\oplus_i$ Short$_i$ =Long.

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Consider a 4d SCFT. On $S^3 \times \mathbb{R}$ (radial quantization), $Q^\dagger = S$.

- The superconformal algebra implies (taking $Q = \bar{Q}_{1\dot{-}}$)

$$2\,\{S,\, Q\} = \Delta - 2\,j_2 - 2\,R + r \equiv H \geq 0\,.$$

where Δ is the conformal dimension, (j_1, j_2) the $SU(2)_1 \otimes SU(2)_2$ Lorentz spins, and $(R\,, r)$ the quantum numbers under the $SU(2)_R \otimes U(1)_r$ R-symmetry.

$$\mathcal{I}\,(p,\, q\,, t\,, \dots) = \mathrm{Tr}(-1)^F p^{j_1+j_2-r} q^{-j_1+j_2-r} t^{R+r} \dots\,.$$

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$$\mathcal{I}\,(p, q, t; x_1, \dots, x_i) = \mathrm{Tr}(-1)^F p^{j_1+j_2-r} q^{-j_1+j_2-r} t^{R+r} x_1^{F_1} \dots x_i^{F_i}$$

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The Index as a Matrix Integral

If the theory has Lagrangian description there is a simple recipe to compute the index.

- One defines a single-letter partition function as the index evaluated on the set of the basic objects (letters) in the theory with $H = 0$ and in a definite representation of the gauge and flavor groups:

$$f^{\mathcal{R}_j}(p, q, t),$$

where $\mathcal{R}_j$ labels the representation.

- Then the index is computed by enumerating the gauge-invariant words,

$$\boxed{\mathcal{I}(p, q, t, \mathbf{V}) = \int [d\mathbf{U}] \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_j f^{\mathcal{R}_j}(p^n, q^n, t^n) \cdot \chi_{\mathcal{R}_j}(\mathbf{U}^n, \mathbf{V}^n)\right)},$$

Here $\mathbf{U}$ is the matrix of the gauge group, $\mathbf{V}$ the matrix of the flavor group and $\mathcal{R}_j$ label representations of the fields under the flavor and gauge groups.

- $\chi_{\mathcal{R}_j}(\mathbf{U})$ is the character of the group element in representation $\mathcal{R}_j$.
- The measure of integration $[d\,\mathbf{U}]$ is the invariant Haar measure.

$$\int [d\mathbf{U}] \prod_{j=1}^{n} \chi_{\mathcal{R}_j}(\mathbf{U}) = \#\text{of singlets in } \mathcal{R}_1 \otimes \cdots \otimes \mathcal{R}_n \,.$$

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Any punctured Riemann surface can be obtained by gluing pair of pants in more than one way.

- Building blocks: 3-punctured sphere $\Leftrightarrow$ 4d SCFT T_N with $SU(N)^3$ flavor symmetry
  - T_2: Free hypermultiplet
  - T_3: Strongly coupled Minahan-Nemeschensky theory with E_6 symmetry
  - T_N: Strongly coupled theories
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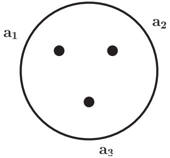
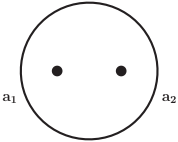


: $I(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$



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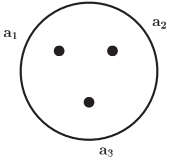

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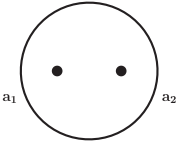

$: \eta(\mathbf{a_1}, \mathbf{a_2}) = \Delta(\mathbf{a_1}) I^V(\mathbf{a_1}) \delta(\mathbf{a_1}, \mathbf{a_2})$

- 4 punctured sphere:

$$I(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}) = \quad \oint [d\mathbf{a}][d\mathbf{b}] \quad I(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a}) \quad \eta(\mathbf{a}, \mathbf{b}) \quad I(\mathbf{b}, \mathbf{a_3}, \mathbf{a_4})$$

- S duality $\Rightarrow I(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4})$ is invariant under permutations of $\mathbf{a_i}$.

Discrete basis

- 3 pt function:

$$I(\mathbf{a},\mathbf{b},\mathbf{c}) \quad = \quad \sum_{\alpha,\beta,\gamma} \quad C_{\alpha\beta\gamma} \quad f^{\alpha}(\mathbf{a})f^{\beta}(\mathbf{b})f^{\gamma}(\mathbf{c})$$

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- $C_{\alpha\beta\gamma} \equiv [N_{\alpha}]^{\gamma}_{\beta}$ $\qquad$ Associativity $\Rightarrow [N_{\alpha}, N_{\beta}] = 0.$

  Simultaneously diagonalize N_{α}!

Discrete basis

Conclusion: Choose $f^{\alpha}(\mathbf{a})$ s.t.

$$\begin{aligned} \eta^{\alpha\beta} &= \delta^{\alpha\beta} \\ C_{\alpha\beta\gamma} &= C_{\alpha\alpha\alpha} \quad \delta_{\alpha\beta}\delta_{\beta\gamma}! \end{aligned}$$

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$$f^{\alpha}(a)_{SU(2)} \quad \longrightarrow \quad f^{\alpha}(\mathbf{a})_{SU(N)}$$

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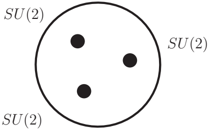
$$\begin{aligned} f^{\alpha}(a)_{SU(2)} &\longrightarrow f^{\alpha}(\mathbf{a})_{SU(N)} \\ \sum_{\alpha,\beta,\gamma\in SU(2) reps.} &\longrightarrow \sum_{\alpha,\beta,\gamma\in SU(N) reps.} \end{aligned}$$

Outline

Example: Hall-Littlewood polynomial $p=0, q=0$



: Free hypermultiplet

$$
\begin{aligned}
I(a_1,a_2,a_3)_{SU(2)} &= \exp\Big(\sum_{n=1}^{\infty}\frac{1}{n}t^n(a_1^n+\frac{1}{a_1^n})(a_2^n+\frac{1}{a_2^n})(a_3^n+\frac{1}{a_3^n})\Big)\\
&\sim \sum_{\lambda=0}^{\infty}\frac{1}{P_\lambda^{HL}(t^{\frac{1}{2}},t^{-\frac{1}{2}}|t)}\prod_{i=1}^{3}P_\lambda^{HL}(a_i,a_i^{-1}|t)\\
&\quad \sum_{\lambda=0}^{\infty}{\color{red}C_{\lambda\lambda\lambda}} \qquad\qquad \prod_{i=1}^{3}{\color{blue}f^\lambda(a_i)}
\end{aligned}
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where, $\qquad P_\lambda^{HL}(a)\sim\chi_\lambda(a)-t\chi_{\lambda-2}(a)$

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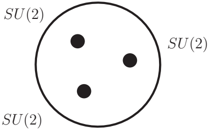

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- Immediate nontrivial result: $SU(2) \longrightarrow SU(3)$

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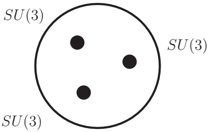

: Minahan-Nemeschansky Theory

$SU(3)^3 \to E_6$

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$$P_\lambda^{HL}(x_1, \ldots, x_k)_{U(k)} \sim \sum_{\sigma \in S_k} \left(x_1^{\lambda_1} \ldots x_k^{\lambda_k} \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j} \right)$$

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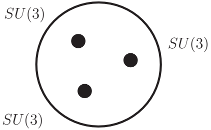

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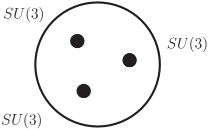

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- $SU(3)^3$ flavor symmetry is enhanced: $SU(3)^3 \to E_6$!
- This expression agrees with the index of E_6 theory obtained from Argyres-Seiberg duality [AG,Rastelli,Razamat,Yan]

Outline

1. Short review of superconformal index
2. 2d TQFT and orthogonal polynomials
   - TQFT structure
   - Example: Hall-Littlewood polynomials
3. Results and Applications
   - Large N limit
   - Instanton partition function

Summary of results

$I(p,q,t)$

$I(0,q,t)$

$f^{\alpha}(a)$: Macdonald polynomial

$I(0,0,t)$

$f^{\alpha}(a)$: Hall-Littlewood polynomial

$I(0,t,t)$

$f^{\alpha}(a)$: Schur polynomial

Outline

Large N limit

$\mathcal{N} = 4$ SYM:

- 1/16 BPS states in $\mathcal{N} = 4$ SYM $\Leftrightarrow$ gravitons, giant gravitons (D branes), black holes [Gutowski,Reall] in $AdS_5 \times S^5$
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Large class of $\mathcal{N} = 2$ theories:

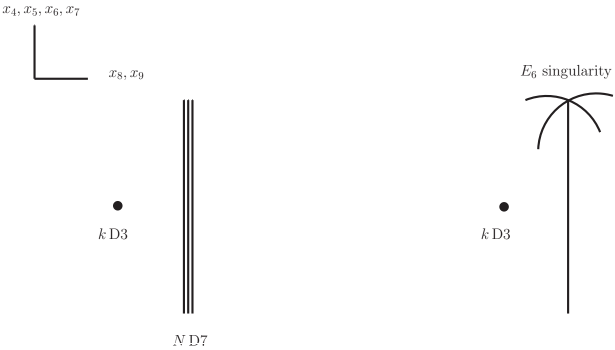
- Large N limit of the index of the 4d theory corresponding to the genus g surface:

$$\mathcal{I}_g^{N\to\infty} = \prod_{j=2}^{\infty} (1 - t^j)^{g-1}$$

- Index of all the $\mathcal{N} = 2$ theories is also independent of N in the large N limit
- Puzzle: what is the reason for this general mysterious cancellation?

Outline

Instanton partition function



- Higgs branch of k D3 $=$ Instanton moduli space of k instantons
- Index of rank k E_6 theory $=$ k E_6 instanton partition function
- Index of T_3 theory $=$ partition function of single E_6 instanton

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- I expect one can play similar games in other dimensions and with other exact observables

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Thank You!!