CHAPTER 2

Generalized Cohomology

In the 1950s, several examples of generalized (co)homology theories were discovered. Each of them has its own geometric origin but it turns out that they can be expressed as homotopy sets by using the notion of spectrum. Before we list the axioms for generalized homology and cohomology, let us take a look at classical examples.

Let A be an Abelian group and consider the ordinary cohomology theory with coefficients in A, $H^n(X;A)$. It is well-known that $H^n(X;A)$ is represented by the Eilenberg-Mac Lane space $K(A,n)$ as a homotopy set

$$\widetilde{H}^n(X;A) \cong [X, K(A,n)]_0.$$

By the definition of the Eilenberg-Mac Lane space (cf. the last paragraph of §1.6), we have $\Omega K(A,n+1) \simeq K(A,n)$. This homotopy equivalence induces the suspension isomorphism of cohomology under the above isomorphism

$$\begin{tikzcd}
{[X, K(A,n)]_0} \arrow[rr, "\cong"] \arrow[d, "\cong"'] & & \widetilde{H}^n(X;A) \arrow[d, "\cong"] \\
{[X, \Omega K(A,n+1)]_0} \arrow[dr, "\cong"] & & \widetilde{H}^{n+1}(\Sigma X;A). \\
& {[\Sigma X, K(A,n+1)]_0} \arrow[ur, "\cong"] &
\end{tikzcd}$$

Thus ordinary cohomology theory with coefficients in A is represented by the sequence of spaces $\{K(A,n)\}_{n\geq 0}$ satisfying $\Omega K(A,n+1) \simeq K(A,n)$.

A sequence of pointed spaces $\{X_n\}$ together with homotopy equivalences $\varepsilon_n : X_n \xrightarrow{\simeq} \Omega X_{n+1}$ is called an Ω-spectrum (and the initial space X_0 is called an infinite loop space).

K-theory is also represented by an Ω-spectrum.

Consider the set of isomorphism classes of n-dimensional complex vector bundles over a paracompact space X, $\mathrm{Vect}_n(X)$. We have a natural bijection between $\mathrm{Vect}_n(X)$ and the homotopy set $[X, BU(n)]$ by taking the associated principal $U(n)$-bundle and by using the Steenrod classification theorem. While adding the trivial 1-dimensional bundle gives us a map $\mathrm{Vect}_n(X) \longrightarrow \mathrm{Vect}_{n+1}(X)$, we have a map $BU(n) \longrightarrow BU(n+1)$ induced by the injective homomorphism $U(n) \longrightarrow U(n+1)$ defined by $A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Thus we have the following natural map:

$$[X, BU(n)] \longrightarrow [X, BU(n+1)]$$

making the following diagram commutative:

$$\begin{tikzcd} \mathrm{Vect}_n(X) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Vect}_{n+1}(X) \arrow[d, "\cong"] \\ {[X, BU(n)]} \arrow[r] & {[X, BU(n+1)].} \end{tikzcd}$$

Let us take the limit of this diagram as $n \to \infty$.

Roughly speaking, the limit of $\mathrm{Vect}_n(X)$ is the (reduced) complex K-group of X, $\widetilde{K}(X)$, which can be identified with the homotopy set $[X, BU]$ where $BU = \mathrm{colim}_n\, BU(n)$. This space BU can be shown to be an infinite loop space as follows. R. Bott defines a map

$$\varphi_n : U(2n)/U(n) \times U(n) \longrightarrow \Omega SU(2n)$$

by

$$\varphi_n(A)(t) = AR_t(n)A^{-1}R_t(n)^{-1},$$

where $0 \leq t \leq 1$ and

$$R_t(n) = \begin{pmatrix} e^{\pi t\sqrt{-1}}E_n & 0 \\ 0 & e^{-\pi t\sqrt{-1}}E_n \end{pmatrix}.$$

Since $BU \simeq \mathrm{colim}_n\, U(2n)/U(n) \times U(n)$, we have a map

$$\varphi = \operatorname*{colim}_n \varphi_n : BU \longrightarrow \Omega SU.$$

Bott proves the famous Bott periodicity theorem, which states that φ is a homotopy equivalence. This theorem, together with the fact $SU \simeq \Omega BSU$, gives us a homotopy equivalence

$$BU \simeq \Omega SU \simeq \Omega^2 BSU.$$

By modifying φ, we have the following homotopy equivalence:

$$BU \times \mathbb{Z} \simeq \Omega U(n) \simeq \Omega^2(BU \times \mathbb{Z}) \cdots$$

and we obtain an Ω-spectrum representing (unreduced) K-theory.

The following construction is a little bit different.

Consider a compact C^∞-manifold M of dimension k and its $(k-n)$-dimensional compact submanifold V. A framing is an embedding $\phi : V \times \mathbb{R}^n \longrightarrow M$ with $\phi(p,0) = p$. The pair (V,ϕ) is called a framed submanifold of M. We say framed submanifolds (V_0,ϕ_0) and (V_1,ϕ_1) of M are framed cobordant if there exists a framed submanifold (W,ψ) of $M \times I$ with $W \cap M \times \{0\} = V_0$, $W \cap M \times \{1\} = V_1$ and φ_0, φ_1 are restrictions of ψ.

A framed submanifold (V,ϕ) of M defines a map

$$h = h_{(V,\phi)} : M \longrightarrow S^n = D^n/\partial D^n$$

as follows:

$$h(x) = \begin{cases} u & \text{if } x = \phi(p,u)(u \in \mathrm{Int}D^n) \\ \infty & \text{if } x \notin \phi(\mathrm{Int}D^n). \end{cases}$$

When (V_0,ϕ_0) and (V_1,ϕ_1) are framed cobordant, it is proved that

$$h_{(V_0,\phi_0)} \simeq h_{(V_1,\phi_1)}$$

and we obtain a map from the framed cobordism classes of submanifolds of codimension n in M to the homotopy set $[M,S^n]$.

This map can be proved to be a bijection by the following argument: Take a representative $f : M \longrightarrow S^n$ of a homotopy class in $[M,S^n]$. We can assume that f is C^∞ in the neighborhood of $f^{-1}(0)$ and has 0 as a regular value. Then $f^{-1}(0)$ is a $(k-n)$-dimensional submanifold of M with a framing.

The above construction of making a homotopy class in $[M,S^n]$ out of a framed cobordism class is called the Pontrjagin-Thom construction.

More generally, let V be a $(k-n)$-dimensional compact submanifold of a compact C^∞-manifold M. The tubular neighborhood of V in M is homeomorphic to the normal bundle $\nu(V)$, which is an n-dimensional real vector bundle. Letting $f : V \longrightarrow BO(n)$ be the classifying map, we have a bundle map

$$\tilde{f} : \nu(V) \longrightarrow \zeta_n,$$

where ζ_n is the universal bundle over $BO(n)$. Let us denote the D^n- and S^{n-1}-bundle associated to ζ_n by $E(\zeta_n)$ and $E_0(\zeta_n)$, respectively, and define $MO(n) = E(\zeta_n)/E_0(\zeta_n)$. This is called the Thom complex of ζ_n. We also have $E(\nu(V))$ and $E_0(\nu(V))$ for $\nu(V)$ and $\tilde{f}$ induces a map between associated bundles. Now define

$$h : M \longrightarrow MO(n)$$

by

$$h(x) = \begin{cases} \tilde{f}(x) & \text{if } x \in E(\nu(\mathrm{Int}D^n)) \\ \infty & \text{if } x \notin E(\nu(\mathrm{Int}D^n)), \end{cases}$$

where ∞ is the point represented by $E_0(\zeta_n)$.

The Thom complex of $\zeta_n \oplus 1$ can be identified with $\Sigma MO(n)$, and the classifying map for $\zeta_n \oplus 1$ induces

$$\varepsilon_n : \Sigma MO(n) \longrightarrow MO(n+1).$$

Let us denote the set of cobordism classes of closed ℓ-dimensional C^∞-manifolds by MO_ℓ. Recall that any compact C^∞-manifold can be embedded in a Euclidean space of a large dimension and a representative V of a cobordism class of MO_ℓ can be regarded as $V \subset \mathbb{R}^{\ell+n} \subset \mathbb{R}^{\ell+n} \cup \{\infty\} = S^{\ell+n}$. The Pontrjagin-Thom construction for this embedding gives us

$$\{[V] \in MO_\ell \mid V \subset S^{\ell+n}\} \longrightarrow [S^{\ell+n}, MO(n)]_0$$

and

$$MO_\ell \longrightarrow \operatorname*{colim}_n [S^{\ell+n}, MO(n)]_0,$$

by taking the limit. MO_ℓ can be made into an Abelian group by taking disjoint union. Thom proves that the above map is an isomorphism of Abelian groups by approximating $BO(n)$ by Grassmannian manifolds and by showing that an element in $[S^{\ell+n}, MO(n)]$ can be represented by a map which is of class C^∞ on the neighborhood of the zero section and transversal to the zero section. Thom also studies the case of orientable manifolds.

Similarly we can define a cobordism relation among the set of continuous maps from closed ℓ-dimensional C^∞-manifolds to a fixed space X. The set of cobordism classes of such maps is denoted by $MO_\ell(X)$. Again, the Pontrjagin-Thom construction gives us an isomorphism

$$MO_\ell(X) \xrightarrow{\cong} \operatorname*{colim}_n [S^{\ell+n}, MO(n) \wedge X_+]_0.$$

This $MO_*(X)$ is an example of generalized homology theory.

As is indicated by this example, a sequence of pointed spaces $\{E_n\}$ together with a sequence of maps $\{\varepsilon_n : \Sigma E_n \longrightarrow E_{n+1}\}$ allows us to define Abelian groups $E_\ell(X) = \operatorname{colim}_n[S^{\ell+n}, E_n \wedge X_+]_0$. Such a sequence $E = \{E_n, \varepsilon_n\}$ is called a spectrum. In general, it can be shown that $E_*(X)$ satisfies the axioms of generalized homology theory.

It should be noted that the homotopy equivalences $\varepsilon_n : E_n \longrightarrow \Omega E_{n+1}$ of an Ω-spectrum give us maps $\varepsilon' : \Sigma E_n \longrightarrow E_{n+1}$, which make $\{E_n, \varepsilon'\}$ into a spectrum.

This correspondence between (co)homology theories and spectra has been a driving force for the efforts to import algebraic structures appearing in cohomology theories into the world of spectra.

For example, we have a map

$$MO_\ell \times MO_{\ell'} \longrightarrow MO_{\ell+\ell'},$$

by taking a product of manifolds, which makes $MO_* = \oplus MO_\ell$ into a graded ring. On the other hand, the classifying map for the product bundle $\zeta_n \times \zeta_m$ induces

$$\lambda : MO(n) \wedge MO(m) \longrightarrow MO(n+m).$$

The ring structure on MO_* corresponds to λ in the obvious way. This ring structure has motivated the definition of ring spectrum. Nowadays, the concept of ring spectrum is replaced by S-algebra and more highly structured ring spectra. Unfortunately we cannot take full advantage of these new objects in this book in order to make the preliminary knowledge as little as possible. Instead, we explain some of the ideas in Appendix C.

1. Axioms for Generalized Cohomology

In this section, we state a set of axioms for generalized (co)homology theory on the category of CW-pairs.

A pair (X, A) of a CW-complex X and its subcomplex A is called a CW-pair. Any CW-complex X can be regarded as a CW-pair $(X, \emptyset)$ with $A = \emptyset$.

The category of CW-pairs is denoted by **CW-pairs**. The full subcategory of pairs of a finite CW-complex and its subcomplex is denoted by **CW-pairs**^{f}.

Define a covariant functor

$$\rho : \mathbf{CW\text{-}pairs} \longrightarrow \mathbf{CW\text{-}pairs}$$

by

$$\begin{aligned} \rho(X, A) &= (A, \emptyset) = A \\ \rho(f) &= f|_A. \end{aligned}$$

Suppose we are given a sequence of contravariant functors

$$h^n : \mathbf{CW\text{-}pairs} \longrightarrow \mathbf{Abels}$$

together with natural transformations

$$\delta^n : h^n \circ \rho \longrightarrow h^{n+1}$$

for $n \in \mathbb{Z}$, where **Abels** is the category of Abelian groups.

For a morphism $f : (X, A) \longrightarrow (Y, B)$ in **CW-pairs**, we denote

$$f^* = h^n(f) : h^n(Y, B) \longrightarrow h^n(X, A)$$

for simplicity and call it the induced homomorphism.

When Eilenberg and Steenrod first axiomatized (ordinary) homology theory in [**ES52**], the language of category theory was not popular and they wrote down the conditions for being a contravariant functor and a natural transformation as follows:

(I) For morphisms $f : (X, A) \longrightarrow (Y, B)$ and $g : (Y, B) \longrightarrow (Z, C)$ in **CW-pairs**, we have

$$h^n(g \circ f) = h^n(f) \circ h^n(g),$$

or $(g \circ f)^* = f^* \circ g^*$.

(II) $h^n(1_{(X,A)}) = 1_{h^n(X,A)}$, or $1^*_{(X,A)} = 1_{h^n(X,A)}$.

(III) For a morphism $f : (X, A) \longrightarrow (Y, B)$ in **CW-pairs**, we have

$$\delta^n(Y, B) \circ h^n(f|_A) = h^{n+1}(f) \circ \delta^n(X, A),$$

or $\delta^n \circ (f|_A)^* = f^* \circ \delta^n$.

If these data $h^* = \{h^n, \delta^n\}_{n\in\mathbb{Z}}$ satisfy the following conditions, we say h^* is a (generalized) cohomology theory:

(IV) (Exactness Axiom) For any object (X, A) in **CW-pairs**, the following sequence is exact

$$\begin{aligned} \ldots \longrightarrow h^{n-1}(A) \xrightarrow{\delta^{n-1}} h^n(X, A) \xrightarrow{j^*} h^n(X) \\ \xrightarrow{i^*} h^n(A) \xrightarrow{\delta^n} h^{n+1}(X, A) \longrightarrow \cdots . \end{aligned}$$

(V) (Homotopy Axiom) If two morphisms f and g in CW are homotopic, i.e., if there exists a morphism

$$F : (X \times I, A \times I) \longrightarrow (Y, B)$$

with $f(x) = F(x, 0)$ and $g(x) = F(x, 1)$, then we have

$$f^* = g^* : h^n(Y, B) \longrightarrow h^n(X, A)$$

for all n.

(VI) (Excision Axiom) For objects (X, A) and (X, B) in **CW-pairs**, the inclusion map $k : (A, A \cap B) \longrightarrow (A \cup B, B)$ induces an isomorphism

$$k^* : h^n(A \cup B, B) \longrightarrow h^n(A, A \cap B)$$

for all n. Note that $(A, A \cap B)$ and $(A \cup B, B)$ are also objects in **CW-pairs**.

When (X, A) and (X, B) are objects in **CW-pairs**, $(X; A, B)$ is called a triad in **CW**. When (X, A) and (A, B) are objects in **CW-pairs**, (X, A, B) is called a triple in **CW**.

THEOREM 2.1 (Exact sequence for triple). *For a triple* (X, A, B) *in* **CW***, we have the following exact sequence:*

$$\cdots \longrightarrow h^{n-1}(A, B) \xrightarrow{\delta^{n-1}} h^n(X, A) \xrightarrow{j^*} h^n(X, B)$$
$$\xrightarrow{i^*} h^n(A, B) \xrightarrow{\delta^n} h^{n+1}(X, A) \longrightarrow \cdots$$

where i *and* j *are inclusion maps.*

THEOREM 2.2 (Exact sequence for triad). *For a triad* $(X; A, B)$ *in* **CW***, we have the following exact sequence:*

$$\cdots \longrightarrow h^{n-1}(A, A \cap B) \xrightarrow{\Delta} h^n(X, A \cup B) \xrightarrow{j^*} h^n(X, B)$$
$$\xrightarrow{i^*} h^n(A, A \cap B) \xrightarrow{\Delta} h^{n+1}(X, A \cup B) \longrightarrow \cdots$$

where i *and* j *are inclusion maps and* Δ *is defined by the following composition:*

$$h^{n-1}(A, A \cap B) \cong h^{n-1}(A \cup B, B) \xrightarrow{\delta^{n-1}} h^n(X, A \cup B).$$

THEOREM 2.3 (Mayer-Vietoris exact sequence). *For a triad* $(X; A, B)$ *in* **CW***, we have the following exact sequence:*

$$\cdots \longrightarrow h^{n-1}(A \cap B) \xrightarrow{\Delta} h^n(A \cup B) \xrightarrow{\alpha} h^n(A) \oplus h^n(B)$$
$$\xrightarrow{\beta} h^n(A \cap B) \xrightarrow{\Delta} h^{n+1}(A \cup B) \longrightarrow \cdots$$

where Δ is defined by the following composition:

$$h^{n-1}(A \cap B) \longrightarrow h^n(B, A \cap B) \cong h^n(A \cup B, A) \longrightarrow h^n(A \cup B)$$

and α, β are defined by $\alpha(x) = (j_A^*(x), j_B^*(x))$, $\beta(y, z) = i_A^*(y) - i_B^*(z)$, respectively. The maps i_A, i_B, j_A, j_B are inclusion maps described in the following diagram:

$$\begin{tikzcd} & A \arrow[dr, "j_A"] & \\ A \cap B \arrow[ur, "i_A"] \arrow[dr, "i_B"'] & & A \cup B \\ & B \arrow[ur, "j_B"'] & \end{tikzcd}$$

Eilenberg and Steenrod require one more axiom.

(VII) (Dimension Axiom) For $n \neq 0$, $h^n(pt) = 0$.

When the axioms (I) through (VII) are all satisfied, h^* is called an ordinary cohomology theory. $h^0(pt)$ is called the coefficient group. For a generalized cohomology theory $h^*(-)$, the graded Abelian group $h^*(pt) = \bigoplus_n h^n(pt)$ is called the coefficient group.

While a cohomology theory on **CW-pairs**^{f} is uniquely characterized by its coefficient group, this is not necessarily the case for cohomology theories on **CW-pairs** (see §3).

(Generalized) homology theory is similarly defined:

Suppose we are given a sequence of covariant functors

$$h_n : \textbf{CW-pairs} \longrightarrow \textbf{Abels}$$

together with natural transformations

$$\partial_n : h_n \longrightarrow h_{n-1} \circ \rho$$

for $n \in \mathbb{Z}$. The system $h_* = \{h_n, \partial_n\}_{n \in \mathbb{Z}}$ is called a (generalized) homology theory if it satisfies axioms corresponding to (IV), (V) and (VI). We also have exact sequences for triple and triad for generalized homology theory. The Mayer-Vietoris sequence also exists.

2. Reduced Cohomology

A based CW-complex is an object (X, x_0) in **CW-pairs** where x_0 is a point in $X^{(0)}$. The full subcategory of based CW-complexes is denoted by $\mathbf{CW}_0$. Note that morphisms and homotopies in $\mathbf{CW}_0$

are those that preserve base points. The full subcategory of based finite CW-complexes is denoted by $\mathbf{CW}_0^f$. We sometimes omit the base points when they are not necessary.

For based spaces (X, x_0) and (Y, y_0), recall that the wedge sum is defined by

$$X \vee Y = \{(x, y) \in X \times Y \mid x = x_0 \text{ or } y = y_0\}.$$

Its base point is $* = (x_0, y_0)$. More generally, for a family of objects $\{(X_\lambda, x_\lambda) \mid \lambda \in \Lambda\}$ in $\mathbf{CW}_0$, we define the wedge sum

$$\bigvee_{\lambda \in \Lambda} X_\lambda$$

as a subspace of $\prod_{\lambda \in \Lambda} X_\lambda$. Note that we need to modify the product topology of $\prod_{\lambda \in \Lambda} X_\lambda$ in order to make it into a CW-complex.

For an object (X, A) in **CW-pairs**, $(X/A, *)$ with $* = A/A$ is an object of $\mathbf{CW}_0$. Thus for (X, x_0) and (Y, y_0) in $\mathbf{CW}_0$, the smash product

$$(X \wedge Y, *) = (X \times Y/X \vee Y, X \vee Y/X \vee Y)$$

is also an object of $\mathbf{CW}_0$. Note that the smash product of $(X_1, x_0^{(1)})$, $\cdots$, $(X_n, x_0^{(n)})$ is defined to be

$$(X_1 \wedge \cdots \wedge X_n, *) = ((X_1 \times \cdots \times X_n)/W(X_1, \cdots, X_n), *),$$

where

$$W(X_1, \cdots, X_n) = \{(x_1, \cdots, x_n) \in X_1 \times \cdots \times X_n \mid x_i = x_0^{(i)} \text{ for some } i\}$$

is the fat wedge.

For an object X in $\mathbf{CW}_0$, $\Sigma X = X \wedge S^1$ is called the reduced suspension of X. Σ defines a functor

$$\Sigma : \mathbf{CW}_0 \longrightarrow \mathbf{CW}_0$$

by $\Sigma f = f \wedge 1_{S^1} : \Sigma X \longrightarrow \Sigma Y$ for a morphism $f : X \longrightarrow Y$. The functor Σ also restricts to give a functor $\Sigma : \mathbf{CW}_0^f \longrightarrow \mathbf{CW}_0^f$.

A reduced cohomology (theory) on $\mathbf{CW}_0$ is a sequence of contravariant functors

$$\tilde{h}^n : \mathbf{CW}_0 \longrightarrow \mathbf{Abels}$$

together with natural transformations

$$\sigma^n : \tilde{h}^n \longrightarrow \tilde{h}^{n+1} \circ \Sigma$$

satisfying the following three axioms:

(IV′) (Exactness Axiom) For objects $A \subset X$ in $\mathbf{CW}_0$, the following sequence is exact:

$$\tilde{h}^n(X/A) \xrightarrow{p^*} \tilde{h}^n(X) \xrightarrow{i^*} \tilde{h}^n(A).$$

(V′) (Homotopy Axiom) If two morphisms f and g in CW_0 are homotopic, then we have

$$f^* = g^* : \tilde{h}^n(Y) \longrightarrow \tilde{h}^n(X)$$

for all n.

(VI′) (Suspension Axiom) For an object X in $\mathbf{CW}_0$,

$$\sigma^n(X) : \tilde{h}^n(X) \longrightarrow \tilde{h}^{n+1}(\Sigma X)$$

is an isomorphism for all n.

Remark 2.4. The definition of reduced homology theory is analogous.

In order to relate reduced and unreduced theories, consider the functor

$$\pi : \textbf{CW-pairs} \longrightarrow \mathbf{CW}_0$$

given by $\pi(X, A) = (X/A, *)$ for objects. It is easy to verify that, for any reduced cohomology theory $\tilde{h}^*(-)$, $\tilde{h}^* \circ \pi$ is an unreduced cohomology theory. On the other hand, the natural transformation defined by the projection $p : (X, A) \longrightarrow \pi(X, A)$ factors as follows:

$$p : (X, A) \xrightarrow{k} (X \cup CA, CA) \xrightarrow{q} (X/A, *),$$

where k is the inclusion into the bottom of the cone and q is defined by collapsing the cone. Theorem 1.5 together with the fact that CA is contractible implies that q is a homotopy equivalence.

For an unreduced cohomology theory $h^*(-)$, k^* is an isomorphism by the Excision Axiom and q^* is also an isomorphism by the Homotopy Axiom. Thus we have the following.

Theorem 2.5. *For any reduced cohomology theory $\tilde{h}^*$ on $\mathbf{CW}_0$, $h^* = \tilde{h}^* \circ \pi$ is a cohomology theory on* **CW-pairs**. *Conversely any cohomology theory on* **CW-pairs** *can be written as $\tilde{h}^* \circ \pi$ with a reduced cohomology $\tilde{h}^*$ on $\mathbf{CW}_0$. The same is true for unreduced and reduced cohomology theories on* $\textbf{CW-pairs}^f$ *and* $\mathbf{CW}_0^f$.

We also have an analogous relation between reduced and unreduced homology theories. The details are omitted.

We close this section with the following relation between induced homomorphisms in ordinary homology theory and generalized cohomology theory.

THEOREM 2.6. *For a morphism $f : X \longrightarrow Y$ in $\mathbf{CW}_0$, if*

$$f_* : H_*(X) \longrightarrow H_*(Y)$$

is an isomorphism, then

$$f^* : \tilde{h}^*(Y) \longrightarrow \tilde{h}^*(X)$$

is also an isomorphism for any reduced cohomology theory $\tilde{h}^*$.

When f is a morphism in $\mathbf{CW}_0^{\text{finite}}$, we can replace the assumption with the condition

$$f^* : H^*(Y) \longrightarrow H^*(X)$$

is an isomorphism.

PROOF. Consider the double suspension of f:

$$\Sigma^2(f) : \Sigma^2 X \longrightarrow \Sigma^2 Y.$$

Suppose

$$f_* : H_*(X) \longrightarrow H_*(Y)$$

is an isomorphism. Then

$$\Sigma^2(f)_* : H_*(\Sigma^2 X) \longrightarrow H_*(\Sigma^2 Y)$$

is also an isomorphism. Since $\Sigma^2 X$ and $\Sigma^2 Y$ are 1-connected, Corollary 1.8 and Corollary 1.22 imply that $\Sigma^2(f)$ is a homotopy equivalence. Now the following commutative diagram completes the proof.

$$\begin{tikzcd}
\tilde{h}^{n+2}(\Sigma^2 X) & \tilde{h}^{n+2}(\Sigma^2 Y) \arrow[l, "\Sigma^2(f)^*"', "\cong"] \\
\tilde{h}^n(X) \arrow[u, "\cong", "\sigma^{n+1}\circ\sigma^n"'] & \tilde{h}^n(Y) \arrow[l, "f^*"] \arrow[u, "\cong", "\sigma^{n+1}\circ\sigma^n"']
\end{tikzcd}$$

$\square$

3. Uniqueness and Milnor's Additivity Axiom

In order to compare two cohomology theories h^* and k^* on the category **CW-pairs**, we need the following definition. A sequence of natural transformations $\Psi^n : h^n \longrightarrow k^{n+s}$ that commutes with connecting homomorphisms ($\delta^{n+s} \circ \Psi^n = \Psi^{n+1} \circ \delta^n$) is called a natural transformation (between cohomology theories) of degree s and written as

$$\Psi : h^* \longrightarrow k^{*+s}.$$

When $s = 0$, it is simply called a natural transformation (of cohomology theories). When $h^* = k^*$, it is called a stable cohomology operation of degree s. Ψ determines a natural transformation between corresponding reduced cohomology theories

$$\widetilde{\Psi} : \tilde{h}^* \longrightarrow \tilde{k}^{*+s}$$

by Theorem 2.5, which satisfies $\sigma^{n+s} \circ \widetilde{\Psi}^n = \widetilde{\Psi}^{n+1} \circ \sigma^n$. The converse also holds. The same is true for cohomology theories on **CW-pairs**^{f}.

The following theorem gives us a criterion to compare two cohomology theories on **CW-pairs**^{f} and can be easily proved by using the long exact sequences induced by a cellular decomposition.

THEOREM 2.7. *Let h^* and k^* be cohomology theories on* **CW-pairs**^{f} *and $\Psi : h^* \longrightarrow k^{*+s}$ be a natural transformation of degree s.*

If $\Psi : h^*(pt) \longrightarrow k^{*+s}(pt)$ is an isomorphism, then

$$\Psi(X, A) : h^*(X, A) \longrightarrow k^{*+s}(X, A)$$

is an isomorphism for all (X, A) in **CW-pairs**^{f}.

In order to prove an analogous theorem for cohomology theories on **CW-pairs**, we need to add the following to our list of axioms for cohomology theory.

(VIII′) (Additivity Axiom) For a family of objects $\{X_\lambda \mid \lambda \in \Lambda\}$ in $\mathbf{CW}_0$, let $X = \vee X_\lambda$ and $i_\lambda : X_\lambda \hookrightarrow X$ be the inclusion map. Then

$$\prod i_\lambda^* : \tilde{h}^*(X) \longrightarrow \prod_{\lambda \in \Lambda} \tilde{h}^*(X_\lambda)$$

is an isomorphism.

The Additivity Axiom (VIII) for unreduced cohomology theory is analogously defined by replacing $\vee$ by $\coprod$. When a reduced cohomology theory $\tilde{h}^*$ satisfies the Additivity Axiom, $h^* = \tilde{h} \circ \pi$ also satisfies the Additivity Axiom.

Note that the axioms (I) through (VI) do not imply the isomorphism in the Additivity Axiom for a wedge of an infinite number of summands.

THEOREM 2.8. *Let $\tilde{h}^*$ be a reduced cohomology theory on* $\mathbf{CW}_0$ *satisfying the Additivity Axiom. Then for an object X in* $\mathbf{CW}_0$ *and an increasing sequence of subcomplexes*

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X$$

with $X = \bigcup_n X_n$, we have the following short exact sequence:

$$0 \longrightarrow \lim_q{}^1 \tilde{h}^{n-1}(X_q) \longrightarrow \tilde{h}^n(X) \longrightarrow \lim_q \tilde{h}^n(X_q) \longrightarrow 0$$

for all n, where $\lim^1$ *is the derived functor of* $\lim$ *discussed in Appendix B.1.*

The above short exact sequence is called the Milnor exact sequence.

For an object X in $\mathbf{CW}_0$ we have a filtration defined by its skeletons

$$X^{(0)} \subset X^{(1)} \subset X^{(2)} \subset \cdots \subset \bigcup_n X^{(n)} = X.$$

Milnor [**Mil62**] proves the following uniqueness theorem for cohomology theories on $\mathbf{CW}$ by applying the Additivity Axiom to the quotient $X^{(p)}/X^{(p-1)} = \bigvee S^p_\alpha$ and then using the Milnor exact sequence.

THEOREM 2.9. *Let h^* and k^* be cohomology theories on the category* **CW-pairs** *and $\Psi : h^* \longrightarrow k^{*+s}$ be a natural transformation of degree s.*

If $\Psi : h^*(pt) \longrightarrow k^{*+s}(pt)$ is an isomorphism, then

$$\Psi(X, A) : h^*(X, A) \longrightarrow k^{*+s}(X, A)$$

is an isomorphism for all (X, A) in **CW-pairs**.

The following is a sketch of a proof of this theorem: once $\Psi(X^{(p)})$ is proved to be an isomorphism for all p, the result follows from the Milnor exact sequence. In order to prove that $\Psi(X^{(p)})$ is an isomorphism, we use the following identification:

$$h^{p+q}(X^{(p)}, X^{(p-1)}) \cong \tilde{h}^{p+q}\left(\bigvee S^p_\alpha\right) \cong \mathrm{Hom}(C_p(X), h^q(pt)), \tag{2}$$

where $C_p(X)$ is the free Abelian group generated by p-cells of X.

Since $\Psi(pt)$ is an isomorphism by the assumption, it follows from the naturality of the above isomorphism and the induction on p that $\Psi(X^{(p)})$ is an isomorphism.

The isomorphism (2) appearing in the above argument is used to identify the E^2-term of the Atiyah-Hirzebruch spectral sequence in section 2 of Chapter 5.

Given a generalized cohomology theory h^* defined on **CW-pairs** and a set of primes $\mathbb{P}$, since the $\mathbb{P}$-localization of Abelian groups is an exact functor, the functors $(X, A) \longmapsto h^n(X, A)_{\mathbb{P}}$ form a generalized cohomology theory $h^*(-)_{\mathbb{P}}$ together with a natural transformation

$$L_{\mathbb{P}} : h^*(-) \longrightarrow h^*(-)_{\mathbb{P}}.$$

If $h^*(-)$ satisfies the Additivity Axiom, so does $h^*(-)_{\mathbb{P}}$.

THEOREM 2.10. *Let h^* be a generalized cohomology theory defined on* **CW** *satisfying the Additivity Axiom and $\mathbb{P}$ be a set of primes. If $h^*(pt)$ is $\mathbb{P}$-local, so is $h^*(X, A)$ for any object (X, A) in* **CW-pairs** *and $L_{\mathbb{P}}$ is an isomorphism.*

The same is true for cohomology theories on $\textbf{CW-pairs}^f$.

PROOF. By assumption, $L_{\mathbb{P}}(pt) : h^*(pt) \longrightarrow h^*(pt)_{\mathbb{P}}$ is an isomorphism. □

THEOREM 2.11. *For a morphism $f : X \longrightarrow Y$ in $\mathbf{CW}_0$, if*

$$f_* : H_*(X; \mathbb{Z}_{\mathbb{P}}) \longrightarrow H_*(Y; \mathbb{Z}_{\mathbb{P}})$$

is an isomorphism, then

$$f^* : \tilde{h}^*(Y) \otimes \mathbb{Z}_{\mathbb{P}} \longrightarrow \tilde{h}^*(X) \otimes \mathbb{Z}_{\mathbb{P}}$$

is also an isomorphism for any reduced cohomology theory $\tilde{h}^*$ satisfying the Additivity Axiom.

When f is a morphism in $\mathbf{CW}_0^f$, we can replace the assumption with the condition on cohomology.

PROOF. By assumption,

$$f_{\mathbb{P}} : (\Sigma^2 X)_{\mathbb{P}} \longrightarrow (\Sigma^2 Y)_{\mathbb{P}}$$

is a homotopy equivalence. Letting $\tilde{k}^*(-) = \tilde{h}^*(-)_{\mathbb{P}}$, the result follows from the fact that $k^*(\Sigma^2 X) \cong k^*((\Sigma^2 X)_{\mathbb{P}})$. □

4. Brown Functor and Representability Theorem

The full subcategory of $\mathbf{CW}_0$ (or $\mathbf{CW}_0^f$) consisting of connected CW-complexes is denoted by $\mathbf{CW}_*$ (or $\mathbf{CW}_*^f$). Ed Brown studies contravariant functors from $\mathbf{CW}_*$ to the category of sets satisfying the "homotopy axiom" [**Bro62, Bro65**].

A contravariant functor

$$H : \mathbf{CW}_* \longrightarrow \mathbf{Sets}$$

is called a homotopy functor if $H(f) = H(g)$ holds whenever $f \simeq g$. An object Y in $\mathbf{CW}_*$ defines a typical example of a homotopy functor F_Y by

$$\begin{aligned} F_Y(X) &= [X, Y]_0 \\ F_Y(f) &= f^* : [X', Y]_0 \longrightarrow [X, Y]_0 \end{aligned}$$

for $f : X \longrightarrow X'$ in $\mathbf{CW}_*$.

A homotopy functor H is said to be representable if there exists an object Y in $\mathbf{CW}_*$ with a natural isomorphism

$$F_Y \cong H,$$

in which case we say H is represented by Y.

When H is representable, the following properties are consequences of elementary properties of homotopy sets.

Wedge Axiom: When $X = \vee_\alpha X_\alpha$, the map $H(i_\alpha) : H(X) \longrightarrow H(X_\alpha)$ induced by the inclusion map $i_\alpha : X_\alpha \hookrightarrow X$ gives rise to a bijection

$$\prod_\alpha : H(X) \longrightarrow \prod_\alpha H(X_\alpha).$$

Mayer-Vietoris Axiom: For the inclusion $A \hookrightarrow X$ of a subcomplex and $u \in H(X)$, denote $H(i)(u)$ by $u|_A$.

For subcomplexes $A, B \subset X$, $a \in H(A)$ and $b \in H(B)$ with $a|_{A\cap B} = b|_{A\cap B}$, there exists $c \in H(A \cup B)$ with $c|_A = a$ and $c|_B = b$.

Brown proves that a homotopy functor satisfying these "Axioms" is always representable.

Definition 2.12. A homotopy functor satisfying the above two axioms is called a Brown functor.

Since $pt \vee pt = pt$, we have $H(pt) \times H(pt) \cong H(pt)$ by the Wedge Axiom. Therefore if $H(pt) \neq \emptyset$, $H(pt) = \{*\}$. We assume $H(pt) \neq \emptyset$ in the rest of this section.

THEOREM 2.13. *A Brown functor H has its range in the category of pointed sets. $H(\Sigma X)$ is a group and $H(\Sigma^k X)$ is an Abelian group for $k \geq 2$.*

For a Brown functor $H : \mathbf{CW}_* \longrightarrow \mathbf{Sets}$, $Y \in \mathbf{CW}_*$ and an element $u \in H(Y)$, define a natural transformation

$$T_u : F_Y \longrightarrow H$$

by

$$T_u(X)([f]) = H(f)(u)$$

for $[f] \in [X, Y]_0 = F_Y(X)$. Note that $T_u(\Sigma^k X)$ is a homomorphism of groups for $k \geq 1$. When $T_u(S^n)$ is an isomorphism for $n \geq 1$, u is called a universal element and Y is called a classifying space of H. Y is unique in the following sense: suppose we have two universal elements $u \in H(Y)$ and $u' \in H(Y')$. Since $T_{u'}$ is an isomorphism, there exists $f : Y \longrightarrow Y'$ with $H(f)(u') = u$. By the definition of universal element, f is a weak homotopy equivalence, hence a homotopy equivalence by Corollary 1.8.

Brown proves that every Brown functor has a classifying space.

THEOREM 2.14. *Any Brown functor $H : \mathbf{CW}_* \longrightarrow \mathbf{Sets}$ is representable.*

When the domain is $\mathbf{CW}_*^f$, we need a few assumptions.

THEOREM 2.15. *A Brown functor $H : \mathbf{CW}_*^f \longrightarrow \mathbf{Sets}$ is representable if one of the following conditions is satisfied:*

(1) *$H(S^n)$ is countable for all $n \geq 1$;*
(2) *H has its values in the category of groups.*

REMARK 2.16. When a Brown functor $H : \mathbf{CW}_* \longrightarrow \mathbf{Sets}$ takes its values in the category of groups, its classifying space has a structure of a homotopy associative Hopf space.

5. Generalized Cohomology as a Representable Functor

Suppose h^* is a generalized cohomology theory defined on the category **CW-pairs**. Then it follows from the Additivity Axiom and Theorem 2.3 that $\tilde{h}^n$ is a functor defined on $\mathbf{CW}_*$ satisfying the Wedge Axiom and the Mayer-Vietoris Axiom of a Brown functor.

When h^* is a generalized cohomology theory defined on $\mathbf{CW\text{-}pairs}^f$, $\tilde{h}^n$ satisfies the condition for Theorem 2.15, since it takes values in the category of Abelian groups. Hence $\tilde{h}^n$ satisfies the conditions either for Theorem 2.14 or Theorem 2.15 and is a representable functor.

Let us consider the case when h^* is defined on **CW-pairs**. There exists $E'_n \in \mathbf{CW}_0$ and $u_n \in \tilde{h}^n(E'_n)$ for which

$$T_{u_n} : F_{E'_n} \longrightarrow \tilde{h}^n|_{\mathbf{CW}_*}$$

is a natural isomorphism of functors. The suspension isomorphism $\sigma : \tilde{h}^n(X) \cong \tilde{h}^{n+1}(\Sigma X)$ implies an isomorphism of corresponding homotopy sets

$$[X, E'_n]_0 \cong [\Sigma X, E'_{n+1}]_0 \cong [X, \Omega E'_{n+1}]_0.$$

The uniqueness of a universal element for Brown functors gives us a homotopy equivalence (of Hopf spaces)

$$f_n : E'_n \longrightarrow \Omega_0 E'_{n+1}$$

where $\Omega_0 E'_{n+1}$ denotes the connected component of $\Omega E'_{n+1}$ containing the constant loop.

Definition 2.17. A sequence $E = \{E_k, \varepsilon_k\}_{k\in\mathbb{Z}}$ of objects E_k in $\mathbf{CW}_0$ and morphisms $\varepsilon_k : \Sigma E_k \longrightarrow E_{k+1}$ is called a spectrum. ε_k is called a structure map. If the adjoint map of ε_k

$$\varepsilon'_k : E_k \longrightarrow \Omega E_{k+1}$$

is a homotopy equivalence for all $k \in \mathbb{Z}$, it is called an Ω-spectrum.

Let us return to the case of spaces E'_n representing a cohomology theory $\tilde{h}^n$. Define $E_n = \Omega E'_{n+1}$ and

$$\varepsilon_n = \Omega f_{n+1} : E_n = \Omega E'_{n+1} \xrightarrow{\simeq} \Omega\Omega_0 E'_{n+2} = \Omega^2 E'_{n+2} = \Omega E_{n+1}.$$

Then E_n together with the adjoint map ε_n to ε'_n form an Ω-spectrum $\{E_n, \varepsilon_n\}$.

Theorem 2.18. *A generalized cohomology theory h^* defined on* **CW** *satisfying the Additivity Axiom can be represented by an Ω-spectrum. The Ω-spectrum representing h^* is unique up to homotopy equivalence. A generalized cohomology theory defined on* $\mathbf{CW\text{-}pairs}^f$ *can also be represented by an Ω-spectrum.*

On the other hand, we can define a cohomology theory by a spectrum as follows. Recall that the homotopy set $[\Sigma X, Y]_0$ has a structure of a group under the multiplication which assigns $f, g : \Sigma X \longrightarrow Y$ to the composition

$$\Sigma X \xrightarrow{\xi} \Sigma X \vee \Sigma X \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\nabla} Y,$$

where ξ is the map that collapses $X \times \{\frac{1}{2}\}$ to the base point. Furthermore $[\Sigma^2 X, Y]_0$ becomes an Abelian group.

Given a spectrum $E = \{E_n, \varepsilon_n\}$, define

$$[X, E]^n = \operatorname*{colim}_{k \geq 1} [\Sigma^k X, E_{n+k}]_0$$

via the map

$$[\Sigma^k X, E_{n+k}]_0 \xrightarrow{E} [\Sigma^{k+1} X, \Sigma E_{n+k}]_0 \xrightarrow{\varepsilon_{n+k}} [\Sigma^{k+1} X, E_{n+k+1}]_0.$$

Note that the Freudenthal suspension map

$$E : [\Sigma X, Y]_0 \longrightarrow [\Sigma^2 X, \Sigma Y]_0$$

is a homomorphism of groups and

$$[X, E]^n = \operatorname*{colim}_{k \geq 1} [\Sigma^k X, E_{n+k}]_0 = \operatorname*{colim}_{k \geq 2} [\Sigma^k X, E_{n+k}]_0$$

is an Abelian group for all n.

The Freudenthal suspension map also gives us a suspension isomorphism

$$\sigma : [X, E]^n \longrightarrow [X, E]^{n+1}$$

by $[f] \longmapsto [E(f)]$.

THEOREM 2.19. *For a spectrum E, define*

$$\widetilde{E}^n(X) = [X, E]^n$$

for $X \in \mathbf{CW}_0$. Then $\widetilde{E}^*(-)$ is a generalized cohomology theory.

Note that when E is an Ω-spectrum, $\widetilde{E}^n(X)$ can be described as a homotopy set

$$\widetilde{E}^n(X) \cong [X, E_n]_0.$$

From this description it is clear that $\widetilde{E}^*(-)$ satisfies the Additivity Axiom.

Recall that, for an Abelian group A, $K(A,n)$ denotes the Eilenberg-Mac Lane space of type (A,n), defined at the end of section 6 of Chapter 1. It is a CW-complex with

$$\pi_j(K(A,n)) = \begin{cases} 0 & j \neq n \\ A & j = n. \end{cases}$$

By the construction, $\Omega K(A,n+1) \simeq K(A,n)$ and we have an Ω-spectrum with $E_k = K(A,k)$. This is called the Eilenberg-Mac Lane spectrum and denoted by HA.

The Dimension Axiom for ordinary cohomology theory $H^*(-;A)$ implies that the Ω-spectrum representing the reduced cohomology theory $\widetilde{H}^*(-;A)$ is the Eilenberg-Mac Lane spectrum.

An object X in $\mathbf{CW}_*$ defines a spectrum by $E_k = \Sigma^k X$ in an obvious way. This is called the suspension spectrum of X and denoted by $\Sigma^\infty X$. When $X = S^0$, it is called the sphere spectrum.

Recall from section 5 of Chapter 1 that a connected pointed space X can be regarded as a subspace of $\Omega\Sigma X$ under the Freudenthal suspension map

$$E : X \longrightarrow \Omega\Sigma X.$$

For an Ω-spectrum $E = \{E_k, \varepsilon_k\}$, we have a map going the other way

$$\xi_1 : \Omega\Sigma E_k \xrightarrow{\Omega\varepsilon_k} \Omega E_{k+1} \xrightarrow{\varepsilon_k'^{-1}} E_k,$$

where ε_k' is the adjoint to ε_k.

More generally, define

$$\begin{aligned} \xi_k : \Omega^k\Sigma^k E_n &\xrightarrow{\Omega^k\Sigma^{k-1}\varepsilon_n} \Omega^k\Sigma^{k-1}E_{n+1} \longrightarrow \cdots \longrightarrow \Omega^k E_{n+k} \\ &\xrightarrow{\Omega^{k-1}\varepsilon_{n+k-1}'^{-1}} \Omega^{k-1}E_{n+k-1} \longrightarrow \cdots \longrightarrow E_n. \end{aligned}$$

Then we have the following commutative diagram:

$$\begin{tikzcd} \Omega^{k+1}\Sigma^{k+1}E_n \arrow[r, "\xi_{k+1}"] & E_n \\ \Omega^k\Sigma^k E_n \arrow[u, hook] \arrow[ur, "\xi_k"'] & \end{tikzcd}$$

Recall that the colimit of the left column is denoted by QE_n and the commutativity of the above diagrams induces a well-defined map

$$\xi : QE_n \longrightarrow E_n.$$

On the other hand, for a pointed space X, we have a homotopy equivalence

$$Q(X) \simeq \Omega Q(\Sigma X)$$

and the sequence $E_k = Q(\Sigma^k X)$ forms an Ω-spectrum. This is the Ω-spectrum associated with the suspension spectrum of X. In general, we can construct an Ω-spectrum from a spectrum in a natural way.

This functor Q can be used to obtain a homology theory as follows. Thanks to the commutativity of colimit and homotopy group, for a pointed space X, we have

$$\operatorname*{colim}_k \pi_{n+k}(\Sigma^k X) \cong \pi_n(QX).$$

This group is usually denoted by $\pi_n^S(X)$ and is called the n-th stable homotopy group of X. The fact that the functor $\pi_*^S(-)$ satisfies the axioms for homology theory is mainly due to the homotopy excision theorem of Blakers-Massey (Theorem 1.31).

PROPOSITION 2.20. *The functor $\pi_*^S(-)$ is a (reduced) generalized homology theory.*

As we have seen at the beginning of this chapter, (unoriented) cobordism theory has a similar description. In general, we have the following theorem.

THEOREM 2.21. *For a spectrum E, the sequence of functors defined by*

$$\widetilde{E}_n(X) = \operatorname*{colim}_k \pi_{n+k}(E_k \wedge X)$$

gives rise to a generalized homology theory.

Another important object related to the Ω-spectrum is infinite loop space. If there exists an Ω-spectrum $E = \{E_k, \varepsilon_k\}$ with $X = E_0$, we say X is an infinite loop space. E_k is called the k-fold delooping of X.

6. Multiplicative Structure

Let h_1^*, h_2^* and h_3^* be generalized cohomology theories on the category of CW-pairs. Suppose, for each $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ and (X,A), (Y,B) in **CW-pairs**, we have a homomorphism

$$\mu_{m,n} : h_1^m(X,A) \otimes h_2^n(Y,B) \longrightarrow h_3^{m+n}(X \times Y, A \times Y \cup X \times B)$$

which is

(1) a natural transformation from **CW-pairs** $\times$ **CW-pairs** to **CW-pairs**
(2) and the following diagrams are commutative:

$$\begin{tikzcd}
h_1^m(A) \otimes h_2^n(Y,B) \arrow[r, "\delta_1^m \otimes 1"] \arrow[d, "\mu_{m,n}"'] & h_1^{m+1}(X,A) \otimes h_2^n(Y,B) \arrow[dd, "\mu_{m+1,n}"] \\
h_3^{m+n}(A \times Y, A \times B) & \\
h_3^{m+n}(A \times Y \cup X \times B, X \times B) \arrow[u, "e"] \arrow[r, "\delta_3^{m+n}"] & h_3^{m+n+1}((X,A) \times (Y,B))
\end{tikzcd}$$

$$\begin{tikzcd}
h_1^m(X,A) \otimes h_2^n(B) \arrow[r, "(-1)^m \otimes \delta_2^n"] \arrow[d, "\mu_{m,n}"'] & h_1^m(X,A) \otimes h_2^{n+1}(Y,B) \arrow[dd, "\mu_{m,n+1}"] \\
h_3^{m+n}(X \times B, A \times B) & \\
h_3^{m+n}(A \times Y \cup X \times B, A \times Y) \arrow[u, "e"] \arrow[r, "\delta_3^{m+n}"] & h_3^{m+n+1}((X,A) \times (Y,B))
\end{tikzcd}$$

where e is the excision isomorphism.

If the above two conditions are satisfied, the family of natural transformations $\mu = \{\mu_{m,n}\}$ is called a pairing of cohomology theories from h_1^* and h_2^* to h_3^* and denoted by

$$\mu : h_1^* \otimes h_2^* \longrightarrow h_3^*.$$

Furthermore, suppose $h_1^* = h_2^* = h_3^*$ $(= h^*)$. We say h^* has a multiplication μ if the following conditions are satisfied:

(3) (Associativity) $\mu \circ (1 \otimes \mu) = \mu \circ (\mu \otimes 1)$
(4) (Existence of 1) There exists an element $1 \in h^0(pt)$ which satisfies

$$\mu(1 \otimes x) = \mu(x \otimes 1) = x$$

for any $x \in h^n(X,A)$.

Define the switching map

$$t : h^m(X,A) \otimes h^n(Y,B) \longrightarrow h^n(Y,B) \otimes h^m(X,A)$$

by $t(u \otimes v) = (-1)^{mn} v \otimes u$ and

$$T : (X,A) \times (Y,B) \longrightarrow (Y,B) \times (X,A)$$

by $T(x,y) = (y,x)$. If the following condition

(5) (Commutativity) $\mu = T^* \circ \mu \circ t$

is satisfied μ is said to be commutative. A cohomology theory equipped with a commutative multiplication μ is called a multiplicative cohomology theory.

When h^* has a multiplication μ, the composition with the map Δ^* induced by the diagonal map $\Delta : X \longrightarrow X \times X$ defines a natural homomorphism

$$\varphi : h^m(X, A) \otimes h^n(X, B) \longrightarrow h^{m+n}(X, A \cup B).$$

This is called the internal multiplication or the cup product. The map μ is called the external multiplication and, for $x \in h^*(X, A)$ and $y \in h^*(Y, B)$, $\mu(x \otimes y)$ is denoted by $x \times y$ and is called the cross product.

Conversely, the external multiplication can be written as

$$\mu = \varphi \circ (p_1^* \otimes p_2^*),$$

where p_1 and p_2 are projections:

$$\begin{aligned} p_1 &: (X \times Y, A \times Y) \longrightarrow (X, A), \\ p_2 &: (X \times Y, X \times B) \longrightarrow (Y, B). \end{aligned}$$

When $A = B = \emptyset$, the direct sum

$$h^*(X) = \bigoplus_n h^n(X)$$

is a graded ring under the internal multiplication and $1 = \pi^*(1) \in h^0(X)$, where $\pi : X \longrightarrow pt$. If μ is commutative, $h^*(X)$ is a graded commutative ring. When $A = \emptyset$,

$$h^*(X, B) = \bigoplus_n h^n(X, B)$$

is a graded module over this graded ring $h^*(X)$.

We also have multiplications on the corresponding reduced cohomology theory. We can also define notions of pairing and multiplication on spectra which correspond to the pairing and the multiplication on cohomology theories. See Appendix C for details.

Before we end this section, let us briefly take a look at the relationship between cohomology theory and homology theory.

As we have seen in §5, a generalized cohomology theory is always represented by a spectrum $E = \{E_n\}$, and the spectrum E gives rise to a generalized homology theory as described in Theorem 2.21. In other words, given a cohomology theory $h^*(-)$, we always have a corresponding homology theory $h_*(-)$. Suppose that the cohomology theory is multiplicative and that the multiplication is induced by maps

$$E_k \wedge E_\ell \longrightarrow E_{k+\ell}.$$

Recall that

$$\begin{aligned}\tilde{h}^p(X) &= \operatorname*{colim}_k [\Sigma^k, E_{p+k}]_0,\\ \tilde{h}_q(X) &= \operatorname*{colim}_\ell [S^{q+\ell}, E_\ell \wedge X]_0.\end{aligned}$$

The composition

$$\begin{aligned}
& \tilde{h}^p(X) \otimes_{h_*(pt)} \tilde{h}_{p+q}(X)\\
= \quad & \operatorname*{colim}_k [\Sigma^k X, E_{p+k}]_0 \otimes_{h_*(pt)} \operatorname*{colim}_\ell [S^{p+q+\ell}, E_\ell \wedge X]\\
\xrightarrow{1\otimes\Delta_*} \quad & \operatorname*{colim}_k [\Sigma^k X, E_{p+k}]_0 \otimes_{h_*(pt)} \operatorname*{colim}_\ell [S^{p+q+\ell}, E_\ell \wedge X \wedge X]_0\\
\longrightarrow \quad & \operatorname*{colim}_k [\Sigma^k X, E_{p+k}]_0 \otimes_{h_*(pt)} \operatorname*{colim}_\ell [S^{p+q+k+\ell}, E_\ell \wedge (\Sigma^k X) \wedge X]_0\\
\longrightarrow \quad & \operatorname*{colim}_{k,\ell} [S^{p+q+k+\ell}, E_\ell \wedge E_{p+k} \wedge X]_0\\
\longrightarrow \quad & \operatorname*{colim}_{k,\ell} [S^{p+q+k+\ell}, E_{p+k+\ell} \wedge X]_0\\
\cong \quad & \operatorname*{colim}_k [S^{q+k}, E_k \wedge X]_0\\
= \quad & \tilde{h}_q(X)
\end{aligned}$$

is bilinear and called the cap product. Following the notation of ordinary (co)homology theory, it is denoted by

$$\cap : \tilde{h}^p(X) \otimes_{h_*(pt)} \tilde{h}_{p+q}(X) \longrightarrow \tilde{h}_q(X).$$