Universal property of Kasparov bivariant K-theory

Hvedri Inassaridze*
(Correspondent member of the Georgian Academy of Sciences)
A.Razmadze Mathematical Institute
Georgian Academy of Sciences

Abstract

It will be proved that Kasparov's bivariant K-theory is the theory of satellites of the Grothendieck functor of homotopy classes of homomorphisms with respect to pre(co)sheaves of semi-split extensions of separable C^*-algebras.

To this end the theory of satellites of arbitrary functors with respect to (co)presheaves of categories (constructed in [2]) will be used.

In what follows we will work in the category $\underline{A}^*_{\mathbf{C}}$ of separable C^*-algebras. So all considered C^*-algebras will be separable.The basic notion we shall need is the notion of semi-split extension of C^*-algebras.

Recall some definitions and results concerning extensions of C^*-algebras ([1,3,4]) needed to expose the main theorem.

Let

$$0 \longrightarrow B \xrightarrow{\varphi} X \xrightarrow{\psi} A \longrightarrow 0 \tag{1}$$

be an extension of A by B , i.e. the sequence (1) is an exact sequence of C^*-algebras.It will be said that (1) is a split extension if there is a commutative diagram of C^*-algebras

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$$\begin{array}{ccccccc} 0 \longrightarrow & B & \xrightarrow{\varphi} & X & \xrightarrow{\psi} & A & \longrightarrow 0 \\ & \downarrow 0 & & \downarrow \alpha & & \| & \\ 0 \longrightarrow & B & \xrightarrow{\varphi} & X & \xrightarrow{\psi} & A & \longrightarrow 0 \end{array}$$

where $0 : B \longrightarrow B$ is the trivial map.We will investigate only extensions of the form $E : 0 \longrightarrow K \otimes B \xrightarrow{\varphi} X \xrightarrow{\psi} A \longrightarrow 0$ where K is the C^*-algebra of compact operators on the infinite dimensional Hilbert space and $K \otimes B$ is the spatial tensor product of K and B.

Two extensions E and E' of A by $K \otimes B$ will be called isomorphic if there is a commutative diagram

$$\begin{array}{ccccccc} E : 0 \longrightarrow & K \otimes B & \xrightarrow{\varphi} & X & \xrightarrow{\psi} & A & \longrightarrow 0 \\ & \| & & \downarrow & & \| & \\ E' : 0 \longrightarrow & K \otimes B & \xrightarrow{\varphi'} & X' & \xrightarrow{\psi'} & A & \longrightarrow 0 \end{array}.$$

Let $E^1(A, B)$ be the set of equivalence classes of isomorphic extensions of A by $K \otimes B$. If $f : A' \longrightarrow A$ is a homomorphism of C^*-algebras the map

$$E^1(f, B) : E^1(A, B) \longrightarrow E^1(A', B)$$

is defined in the usual way. Namely for $E : 0 \longrightarrow K \otimes B \xrightarrow{\varphi} X \xrightarrow{\psi} A \longrightarrow 0$ take the fiber product X' of $X \xrightarrow{\psi} A \xleftarrow{f} A'$. Then $E^1(f, B)([E]) = [E']$ where $E' : 0 \longrightarrow K \otimes B \xrightarrow{\varphi'} X' \xrightarrow{\psi'} A' \longrightarrow 0$ with φ' and ψ' natural maps. $E^1(-, B)$ becomes a contravariant functor from $\underline{A^*_C}$ to the category $\underline{Sets}$.

For any extension (1) of C^*-algebras there is a uniquely defined commutative diagram

$$\begin{array}{ccccccc} E : 0 \longrightarrow & B & \xrightarrow{\varphi} & X & \xrightarrow{\psi} & A & \longrightarrow 0 \\ & \| & & \downarrow & & \downarrow \tau_E & \\ E : 0 \longrightarrow & B & \xrightarrow{\sigma} & M(B) & \xrightarrow{\eta} & O(B) & \longrightarrow 0 \end{array}$$

where $M(B)$ is the multiplier algebra of B, σ is the natural injection and $\eta : M(B) \longrightarrow O(B) = M(B)/\sigma(B)$ is the canonical surjection. The homomorphism τ_E is called the Busby invariant associated to the given extension E of A by B.

$E^1(A,B)$ can be defined also as a covariant functor in the second variable. In effect let $g : B \longrightarrow B'$ be a homomorphism of C^*-algebras. Take by Lemma 1.2 [4] the homomorphism

$$(\widetilde{K \otimes g})_{\neq} : M(K \otimes B) \longrightarrow M(K \otimes B').$$

For $E : 0 \longrightarrow K \otimes B \xrightarrow{\varphi} X \xrightarrow{\psi} A \longrightarrow 0$ one gets a commutative diagram

$$\begin{array}{cccccccc}
E : 0 \rightarrow & K \otimes B & \xrightarrow{\varphi} & X & \xrightarrow{\psi} & A & \rightarrow 0 \\
& \| & & \downarrow & & \downarrow \tau_E & \\
0 \rightarrow & K \otimes B & \rightarrow & M(K \otimes B) & \rightarrow & O(K \otimes B) & \rightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow \lambda_g & \\
0 \rightarrow & K \otimes B' & \rightarrow & M(K \otimes B') & \rightarrow & O(K \otimes B') & \rightarrow 0
\end{array}$$

with $(K \otimes g)_{\neq} : K \otimes B \longrightarrow K \otimes B'$ and let E' be the extension of A by $K \otimes B'$ whose Busby invariant is $\lambda_g \tau_E$. Then define

$$E^1(A,g) : E^1(A,B) \longrightarrow E^1(A,B')$$

by $[E] \longmapsto [E']$ .So $E^1(A,-)$ becomes a covariant functor from $\underline{A_C^*}$ to $\underline{Sets}$.

A sum $\oplus$ is defined on the set $E^1(A,B)$ as follows.Let $\tau_{E_1}$and τ_{E_2} be the Busby invariant of E_1 and E_2 respectively where $[E_1]$, $[E_2] \in E^1(A,B)$. Consider the homomorphism $\tau : A \longrightarrow O(K \otimes B)$ given by

$$\tau(a) = \begin{pmatrix} \tau_{E_1}(a) & 0 \\ 0 & \tau_{E_2}(a) \end{pmatrix} \in M_2 \otimes O(K \otimes B) \approx O(K \otimes B)$$

and take the extension E denoted by $E_1 \oplus E_2$ with Busby invariant τ . Define

$$[E_1] \oplus [E_2] = [E] \ .$$

We arrive to the definition of a semi-split extension of A by $K \otimes B$. Let A and B be C^*-algebras. An extension E of A by $K \otimes B$ is called a semi-split extension if there is an extension E_- of A by $K \otimes B$ such that $E \oplus E_-$ is a split extension.

It will be said that two semi-split extensions E_1 and E_2 of A by $K \otimes B$ are unitary equivalent up to splitting if there exists split extensions F_1 , F_2 of A by $K \otimes B$ and a unitary element $u \in M(K \otimes B)$ such that there is a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & K \otimes B & \xrightarrow{\varphi_1} & E_1 \oplus F_1 & \xrightarrow{\psi_1} & A & \longrightarrow 0 \\ & \downarrow ad\ u & & \downarrow \alpha & & \| & \\ 0 \longrightarrow & K \otimes B & \xrightarrow{\varphi_2} & E_2 \oplus F_2 & \xrightarrow{\psi_2} & A & \longrightarrow 0 \end{array}$$

where $ad\ u$ is a derivation given by $x \longmapsto \sigma^{-1}(u\ \sigma(x)\ u^*)$ with $x \in K \otimes B$ and $\sigma : K \otimes B \longrightarrow M(K \otimes B)$.

Let $ext^1(A, B)$ be the set of semi-split extensions of A by $K \otimes B$. Then $ext^1(-,-)$ is a subbifunctor of $E^1(-,-)$. Moreover $ext^1(A, B)$ is a commutative monoid under the sum $\oplus$ and its quotient set $Ext^1(A, B)$ by the unitary equivalence up to splitting becomes an abelian group with sum induced by $\oplus$ and it is a bifunctor from $\underline{A}_C^*$ to the category $\underline{\mathbf{Ab}}$ of abelian groups. It was proved by Kasparov [3] that in fact $Ext^1(A, B)$ is isomorphic to $KK^1(A, B)$ and so is a homotopy functor under both variables .

Define a presheaf $\mathbf{G}$ of categories over the category $\underline{A}_C^*$ of separable C^*-algebras as follows.For any $A \in Ob\ \underline{A}_C^*$ the objects of the category $\mathbf{G}(A)$ are semi-split extensions E of the C^*-algebra A

$$E : 0 \longrightarrow K \otimes X \xrightarrow{\varphi} Y \xrightarrow{\psi} A \longrightarrow 0\ .$$

A morphism of $\mathbf{G}(A)$ is a triple $(\alpha, \beta, 1_A) : E \longrightarrow E^{/}$ such that the diagram

$$\begin{array}{ccccccc} E : 0 \longrightarrow & K \otimes X & \xrightarrow{\varphi} & Y & \xrightarrow{\psi} & A & \longrightarrow 0 \\ & \downarrow \alpha & & \downarrow \beta & & \| & \\ E' : 0 \longrightarrow & K \otimes X' & \xrightarrow{\varphi'} & Y' & \xrightarrow{\psi'} & A & \longrightarrow 0 \end{array}$$

is commutative. If $f : A' \longrightarrow A$ is a homomorphism of C^*-algebras then the covariant functor $\mathbf{G}(f) : \mathbf{G}(A) \longrightarrow \mathbf{G}(A')$ is given by

$$\mathbf{G}(f)(E) = ext^1(f, K \otimes X)(E)$$

for $E : 0 \longrightarrow K \otimes X \xrightarrow{\varphi} Y \xrightarrow{\psi} A \longrightarrow 0 \in Ob\ \mathbf{G}(A)$ and for a morphism $E \longrightarrow E'$ of $\mathbf{G}(A)$ the morphism $\mathbf{G}(f)(E) \longrightarrow \mathbf{G}(f)(E')$ is defined in a natural way. The trace (S, s) in the category $\underline{A}_C^*$ of the presheaf $\mathbf{G}$ is given by $S_A(E) = K \otimes X$ for $E : 0 \longrightarrow K \otimes X \xrightarrow{\varphi} Y \xrightarrow{\psi} A \longrightarrow 0$ and for any C^*-algebra A , and $S_A(\alpha, \beta, 1_A) = \alpha$ for $(\alpha, \beta, 1_A) : E \longrightarrow E'$. If $f : A' \longrightarrow A$ is a homomorphism of C^*-algebras then for $E : 0 \longrightarrow K \otimes X \xrightarrow{\varphi} Y \xrightarrow{\psi} A \longrightarrow 0$

and $A \in Ob\ \underline{A}_C^*$ the homomorphism $s_E(f) : S_A(\mathbf{G}(f)(E)) \longrightarrow S_A(E)$ is the identity map $1_{K\otimes X} : K \otimes X \longrightarrow K \otimes X$.

We see that the presheaf $\mathbf{G}(S, s)$ of semi-split extensions over $\underline{A}_C^*$ is completely analogous to the presheaf of short exact sequences of modules with its trace over the category of modules [2].

Let A and B be two C^*-algebras and let $\hom(A, K \otimes B)$ be the set of all C^*-homomorphisms from A into $K \otimes B$. Let $\hom^*(A, K \otimes B)$ be the set of equivalence classes of homotopic C^*-homomorphisms from A into $K \otimes B$. Then one can define on $\hom(A, K \otimes B)$ a sum $\oplus$ by $f \oplus g = h$ where

$$h(a) = \begin{pmatrix} f(a) & 0 \\ 0 & g(a) \end{pmatrix} \in M_2 \otimes (K \otimes B) \approx K \otimes B$$

for $a \in A$ and $f, g \in \hom(A, K \otimes B)$. The sum $\oplus$ induces on $\hom^*(A, K \otimes B)$ a structure of commutative monoid and let $K\hom^*(A, K \otimes B)$ be its Grothendieck group. One gets a bifunctor $K\hom^*(-,-)$ from $\underline{A}_C^*$ to $\underline{\mathbf{Ab}}$.

Definition 1. It will be said that a connected pair (T^0, ϑ, T^1) of contravariant functors from $\underline{A}_C^*$ to $\underline{\mathbf{Ab}}$ with respect to the presheaf $\mathbf{G}(S, s)$ of semi-split extensions satisfies condition (i) if for any unitary element $u \in M(K \otimes B)$ the equality

$$\delta_E T^0(ad\ u) = \delta_E$$

holds for any $E : 0 \longrightarrow K \otimes B \xrightarrow{\varphi} X \xrightarrow{\psi} A \longrightarrow 0 \in Ob\ \mathbf{G}(A)$, $A \in Ob\ \underline{A}_C^*$.

Denote by $\mathbf{L}$ be the class of all connected pairs of functors satisfying condition (i). Let $E : 0 \longrightarrow K \otimes X \xrightarrow{\varphi} Y \xrightarrow{\psi} A \longrightarrow 0 \in \mathbf{G}(A)$. Define a homomorphism

$$\widetilde{\vartheta}_E : \hom^*(K \otimes X, K \otimes B) \longrightarrow Ext^1(A, B)$$

by $\widetilde{\vartheta}_E([g]) = ext^1(A, g)(E)$ for $g : K \otimes X \longrightarrow K \otimes B$ and extend $\widetilde{\vartheta}_E$ to a homomorphism

$$\vartheta_E : K\hom^*(K \otimes X, K \otimes B) \longrightarrow Ext^1(A, B).$$

Theorem 2. The pair $(K\hom^*(-, K \otimes B)\ ,\ \vartheta\ ,\ Ext^1(-, B))$ is a right universal pair of contravariant functors with respect to the class $\mathbf{L}$.

Let $\mathbf{H}$ be the copresheaf of categories of semi-split extensions over the category $S\underline{A}_C^*$ of stable separable C^*-algebras with its (dually defined) natural trace (S, s) in the category $\underline{A}_C^*$.

Definition 3. It will be said that a connected pair (T_0, κ, T_1) of functors $T_0 : \underline{A}_C^* \longrightarrow \underline{\mathbf{Ab}}$, $T_1 : S\underline{A}_C^* \longrightarrow \underline{\mathbf{Ab}}$ with respect to $\mathbf{H}(S, s)$ satifies condition (j) if for any unitary element $u \in M(K \otimes B)$ the equality

$$T_1(ad\ u)\kappa_E = \kappa_E$$

holds for any $E : K \otimes B \longrightarrow X \longrightarrow Y \in Ob\, \mathbf{H}(K \otimes B)$, $K \otimes B \in Ob\ S\underline{A}_C^*$.

For any $E : K \otimes B \longrightarrow X \longrightarrow Y$ define a connecting homomorphism

$$\eta_E : K \hom{}^*(A, Y) \longrightarrow Ext^1(A, B)$$

given by $[g] \longmapsto [ext^1(g, K \otimes B)]$ where A is a separable C^*-algebra.

Theorem 4. The pair $(K \hom{}^*(A, -), \eta, Ext^1(A, -))$ is a universal pair of functors with respect to the class of all connected pairs of functors satisfying condition (j).

Note that Theorems 2 and 4 can be extended in a natural way to the functors Ext^n.

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