K-THEORY AND MORITA THEORY

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The Quillen Q-construction allosws us to define the K-theory of an exact 1-category $\mathcal{C}$. The K-theory functor $\mathbf{K} : \mathbf{Exact} \to \mathbf{Top}$, taking the category of exact 1-categories to the category of topological spaces, is defined by $\mathbf{K} : \mathcal{C} \mapsto \Omega\,|\mathrm{N}\mathbf{Q}(\mathcal{C})|$. We may generalize and define an analog of the Quillen Q-construction for a colored operad $\mathcal{M}$ satisfying certain conditions, where the K-theory $\mathbf{K} : \mathbf{A} \to \mathbf{Sp}$ takes a subcategory $\mathbf{A}$ of the category of colored operads $\mathbf{Op}$ to the category of spectra $\mathbf{Sp}$. It is natural to generalize in a different way, from exact 1-categories to exact ∞-categories, to define the Quillen Q-construction. Barwick has done so in [Bar], which we now outline.

Let X be a simplicial set; we can define another simplicial set by $(\mathscr{O}(X))_n := \mathrm{Map}(\Delta^n \star \Delta^n, X)$, where $\star$ is the concatenation operator on $\mathbf{\Delta}$. The Quillen Q-construction uses $\mathscr{O}(\Delta^n)$, but in order to define the Quillen Q-construction, we have to define ambigressive pullbacks and ambigressive functors.

Let $\mathcal{C}_\infty$ be an exact ∞-category, and let $\mathcal{C}^!_\infty$ and $\mathcal{C}^\bullet_\infty$ be full subcategories of $\mathcal{C}_\infty$ containing all the equivalences. Given a pullback square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \end{array}$$

we call it ambigressive if $X' \to Y'$ and $Y \to Y'$ are morphisms in $\mathcal{C}^!_\infty$ and $\mathcal{C}^\bullet_\infty$, respectively. We call a functor $\mathscr{O}(\Delta^n) \to \mathcal{C}_\infty$ ambigressive if for all integers $0 \leq i \leq k \leq l \leq j \leq n$, the pullback square

$$\begin{array}{ccc} X_{ij} & \longrightarrow & X_{kj} \\ \downarrow & & \downarrow \\ X_{il} & \longrightarrow & X_{kl} \end{array}$$

is ambigressive. We may now finally proceed to the Quillen Q-construction: define a simplicial set $\mathbf{Q}(\mathcal{C}_\infty)$, whose n-simplices are the ambigressive functors $\mathscr{O}(\Delta^n)^{op} \to \mathcal{C}_\infty$. The K-theory is then simply $\Omega\mathbf{Q}(\mathcal{C}_\infty)$, and this defines a functor from the ∞-category $\mathbf{Exact}_\infty$ of exact ∞-categories and exact functors between them to the ∞-category $\mathcal{C}\mathrm{at}_\infty$ of ∞-categories.

Let $\mathcal{C}_\infty$ be an exact ∞-category. If we equip it with a map $\mathcal{C}_\infty \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ satisfying certain conditions that make it an ∞-operad, we call $\mathcal{C}_\infty$ a unital ∞-operad. To signify that it is equipped

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with a map $\mathcal{C}_\infty$, we will write it as $\mathcal{C}_\infty^\otimes$. Let $\mathbf{Exact}_\infty^\otimes$ be the subcategory of $\mathbf{Exact}_\infty$ spanned by the unital ∞-category. The K-theory construction for exact ∞-categories passes over to unital ∞-categories, so we can ask what additional structure/properties $\mathbf{K}(\mathcal{C}_\infty^\otimes)$ have? This can be answered by looking at a pattern in the codomain of the K-theory functors; a simple analysis shows that for ∞-operads, the K-theory takes $\mathbf{K} : \mathbf{Exact}_\infty^\otimes \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$, where $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$ is the ∞-category of stable ∞-categories and exact ∞-functors between them. Note that what we call exact functors are exact functors between exact ∞-categories, in the sense of [Bar], and what we call exact ∞-functors are exact ∞-functors between stable ∞-categories.

Consider the homotopy category $\mathrm{h}\mathbf{K}(\mathcal{C}_\infty^\otimes)$, which, because $\mathbf{K}(\mathcal{C}_\infty^\otimes)$ is a stable ∞-category, is a triangulated category. We would like to develop some sort of derived Morita theory, and so we'd like to consider the homotopy category of some ∞-operad of module objects.

Lurie has defined such objects in [Lura]; more specifically, he has defined an ∞-operad $\mathrm{Mod}^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes$ of $\mathcal{O}$-module objects over $\mathcal{C}_\infty^\otimes$, and an ∞-category $\mathcal{A}\mathrm{lg}_{/\mathcal{O}}(\mathcal{C}_\infty^\otimes)$ of $\mathcal{O}$-algebra objects over $\mathcal{C}_\infty^\otimes$, where $\mathcal{O}^\otimes$ is an ∞-category. We can define the ∞-operad $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes$ of $\mathcal{O}$-module objects over an $\mathcal{O}$-algebra object A over $\mathcal{C}_\infty^\otimes$ as the pushout $\mathrm{Mod}^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes \prod_{\mathcal{A}\mathrm{lg}_{/\mathcal{O}}(\mathcal{C}_\infty^\otimes)}\{A\}$. Since we'd like to provide a derived category structure on $\mathrm{h}\mathbf{K}(\mathcal{C}_\infty^\otimes)$ through $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes$, we will study $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes$ first. We provide two interesting properties that it satisfies, one of which will help us define the derived category of an algebra over an ∞-operad.

The first follows from induction using [Lura, Corollary 3.4.1.9]:

Theorem 0.1. *Let* $(\mathrm{Mod}_A^{\mathcal{O}})^n(\mathcal{C}_\infty^\otimes)$ *denote* $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes$ *iterated* n *times. Then* $(\mathrm{Mod}_A^{\mathcal{O}})^n(\mathcal{C}_\infty^\otimes)$ *is equivalent to* $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes$ *for any* $n \geq 1$.

Consider the identity morphism $\mathrm{id}_{\mathbf{K}(\mathcal{C}_\infty^\otimes)} : \mathbf{K}(\mathcal{C}_\infty^\otimes) \to \mathbf{K}(\mathcal{C}_\infty^\otimes)$, which is an equivalence of categories. Since $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes \simeq \mathcal{C}_\infty^\otimes$ when $\mathcal{O}^\otimes = \mathbb{E}_0^\otimes$, we expect one of the following three statements to hold true:

(1) There is a fully faithful non-essentially surjective functor $\mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathcal{C}_\infty^\otimes)) \to \mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes)$.

(2) There is a fully faithful non-essentially surjective functor $\mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes) \to \mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathcal{C}_\infty^\otimes))$.

(3) There is a fully faithful essentially surjective functor $\mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes) \to \mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathcal{C}_\infty^\otimes))$.

We will proceed to inspect each of these points separately:

(1') *There is a fully faithful non-essentially surjective functor* $\mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathcal{C}_\infty^\otimes)) \to \mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes)$. This induces a map $\mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathcal{C}_\infty^\otimes)) \to \mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes))$, which implies the existence of a forgetful functor $\mathbf{K}(\mathcal{C}_\infty^\otimes) \to \mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes))$, and this is obviously false, meaning that there is no fully faithful non-essentially surjective functor $\mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathcal{C}_\infty^\otimes)) \to \mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes)$. When $\mathcal{O}^\otimes = \mathbb{E}_0^\otimes$, this means that there is no fully faithful non-essentially surjective functor $\mathbf{K}(\mathcal{C}_\infty^\otimes) \to \mathbf{K}(\mathcal{C}_\infty^\otimes)$.

(2') *There is a fully faithful non-essentially surjective functor* $\mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes) \to \mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathcal{C}_\infty^\otimes))$. This reduces to the statement that there is a fully faithful non-essentially surjective functor $\mathbf{K}(\mathcal{C}_\infty^\otimes) \to \mathbf{K}(\mathcal{C}_\infty^\otimes)$, and we just showed this to be false. This implies that the only left option must hold true:

(3') *There is a fully faithful essentially surjective functor* $\mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes) \to \mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathcal{C}_\infty^\otimes))$.

We will state this as a theorem to emphasize that this is a very important result:

Theorem 0.2. *There is a fully faithful essentially surjective functor* $\mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^{\otimes})^{\otimes}) \to \mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathcal{C}_\infty^{\otimes}))$.

Returning to derived categories, we see that we can define the derived category $\mathcal{D}(A)$ to be the homotopy category $\mathrm{h}\mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^{\otimes})^{\otimes})$, because of two reasons:

(i) $\mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^{\otimes})^{\otimes})$ is a stable ∞-category, and so its homotopy category must have the structure of a triangulated category.

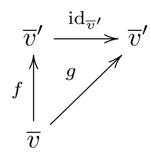
(ii) $\mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^{\otimes})^{\otimes})$ has the structure of an ∞-operad of modules by Theorem 0.2, so its homotopy category must be similar to the derived category of an algebra.

Derived Morita theory is concerned with the following question:

Question 0.3. When are the derived categories $\mathcal{D}(A)$ and $\mathcal{D}(A')$ equivalent as triangulated categories?

In order to answer this question, we'll introduce a model structure on the homotopy category $\mathrm{h}\mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^{\otimes})^{\otimes})$.

The category $\mathrm{h}\mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^{\otimes})^{\otimes})$ admits finite limits and colimits, which allows us to define the model structure on it. We will define the cofibrations and fibrations as the isomorphisms. Let $f : \overline{v} \to \overline{v}'$ be a morphism in $\mathrm{Mod}^{\mathcal{O}}(\mathcal{C}_\infty^{\otimes})^{\otimes}$. We call f a weak equivalence if for any map $g : \overline{v} \to \overline{v}'$, there is a 2-simplex:



We are now ready to state our theorem regarding the derived Morita theory of algebras over ∞-operads:

Theorem 0.4. *Let* $F : \mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathcal{C}_\infty^{\otimes})) \to \mathrm{Mod}_{A'}^{\mathcal{O}'}(\mathbf{K}(\mathcal{C}_\infty^{\otimes}))$ *be a functor that induces a map between homotopy categories* $\mathbf{L}F : \mathrm{hMod}_A^{\mathcal{O}}(\mathbf{K}(\mathcal{C}_\infty^{\otimes})) \to \mathrm{hMod}_{A'}^{\mathcal{O}'}(\mathbf{K}(\mathcal{C}_\infty^{\otimes}))$*, and hence a map between the derived categories* $\mathbf{L}F : \mathcal{D}(A) \to \mathcal{D}(A')$*. The following statements are equivalent:*

(1) *F is an equivalence of ∞-categories.*

(2) *$\mathbf{L}F$ is a Quillen equivalence.*

(3) *$\mathbf{L}F$ is a triangulated equivalence of derived categories.*

We would now like to briefly discuss one more application of our equivalence theorem (Theorem 0.2) to a question asked by Gunnar Carlsson. Carlsson asked the following question (see [EM]): what structure on a permutative category $\mathcal{C}$ would give $\mathbf{K}(\mathcal{C})$ a module structure over $\mathbf{K}(\mathcal{D})$, for $\mathcal{D}$ a bipermutative category? We have already given an answer in the ∞-operadical context, at least when $\mathcal{C}$ is bipermuatative. I am not sure whether our result can be used when $\mathcal{C}$ is not necessarily bipermutative; perhaps one must use the theory of ∞-*pre*operads...

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