The following is a pretty trivial and half-baked note on the question of how to relate (left) Kan extensions of a functor $\nabla : X \to F$ along a functor $p : X \to \Sigma$ to the σ-model pull-push operation on sections of the transgression of ∇ associated to a cospan

$$\begin{tikzcd} & \Sigma & \\ * \arrow[ur, "\sigma"] & & * \arrow[ul, "\sigma'"'] \end{tikzcd}$$

as discussed in [1].

Warning. The word "section of a functor" appears in two different ways throughout the following. The functor $p : X \to \Sigma$ we regard as a kind of fibration and call a section of it a way to split this fibration weakly. But our "background field" $\nabla : X \to F$ we consider as a placeholder for the fibration $P \to X$ that it classifies (as described in [1]). So a section of ∇ is actually a natural transformation from the functor constant on the point $\mathrm{pt} : * \to F$ of F to ∇. This overlap of terms might appear unfortunate, but actually it reflects accurately the situtation that we want to describe in physics: the physical fields are supposed to be sections of some bundle over parameter space Σ and then the quantum states are supposed to be sections of the background field transgressed to the space of physical fields.

The point here is that if we we modify the prescription of [1] slightly in that we take the space of fields over Σ to be not the full functor category $[\Sigma, X]$, but the category of *weak left sections* $\Gamma^p(\Sigma, X)$ of p, then we arrive naturally at a pull-push prescription where over a point $\sigma : * \to \Sigma$ we don't compute the flat sections of the functor $[*, \nabla] : [*, \Sigma] \to [*, F]$ i.e. of ∇ itself, as in [1], but instead its pullback to the comma category $(p/\mathrm{const}_\sigma)$: the comma category here appears as the restriction of $\Gamma^p(\Sigma, X)$ to the point.

So with this slight modification, which, as we have seen, has good motivation from physical models, it turns out that the pull-push quantization prescription coincides *on points* with the left Kan extension of ∇ along p. (At least if we don't pass to generalized sections of ∇ as in [1]).

Unfortunately, I am not sure yet how both constructions relate on morphisms. But I thought I'd make the following remark nevertheless.

1 Weak sections and comma categories

Definition 1 (weak sections of a functor) *Given a functor $p : X \to \Sigma$ a* weak left section *of p is a functor $\phi : \Sigma \to X$ and a transformation*

$$\begin{tikzcd} & X \arrow[dr, "p"] \arrow[d, phantom, "\Downarrow \eta_\phi"] & \\ \Sigma \arrow[ur, "\phi"] \arrow[rr, "\mathrm{Id}"'] & {} & \Sigma \end{tikzcd}$$

. A morphism $h : \phi \to \phi'$ of left sections of p is a transformation

$$\begin{tikzcd} X \\ \Sigma \arrow[u, bend left=60, "\phi"] \arrow[u, bend right=60, "\phi'"'] \arrow[u, phantom, "\overset{h}{\Rightarrow}"] \end{tikzcd}$$

such that

$$\begin{tikzcd} & X \arrow[dr, "p"] & \\ \Sigma \arrow[ur, bend left, "\phi"] \arrow[ur, "\phi'"'] \arrow[rr, "\mathrm{Id}"'] & & \Sigma \end{tikzcd} \;=\; \begin{tikzcd} & X \arrow[dr, "p"] \arrow[d, phantom, "\Downarrow \eta_{\phi'}"] & \\ \Sigma \arrow[ur, "\phi'"] \arrow[rr, "\mathrm{Id}"'] & {} & \Sigma \end{tikzcd}$$

. This defines a category of left sections $\Gamma^p(\Sigma, X)$.

Remark. Right sections would be defined similarly, with the direction of η_ϕ reversed. The category of right sections would be $\Gamma_p(\Sigma, X)$. Maybe "left"/"right" is not so good terminology here, but it will correspond to left/right Kan extension below.

Definition 2 (functors out of the category of sections) *For $p : X \to \Sigma$ a functor we have the following canonical functors out of $\Gamma^p(\Sigma, X)$.*

- *Write $[\Sigma, X]$ for category of functors from Σ to X. There is the obvious forgetful functor*

$$\Gamma^p(\Sigma, X) \to [\Sigma, X] \,.$$

- *For every object $\sigma \in \Sigma$ there is a functor from $\Gamma^p(\Sigma, X)$ to the comma category $(p/\mathrm{const}_\sigma)$*

$$(-)|_\sigma : \Gamma(\Sigma, X) \to (p/\mathrm{const}_\sigma)$$

given by precomposition with $\sigma : * \to \Sigma$

$$(-)|_\sigma \;:\; \begin{tikzcd} & X \arrow[dr, "p"] \arrow[d, phantom, "\Downarrow\eta_\sigma"] & \\ \Sigma \arrow[ur, "\phi"] \arrow[rr, "\mathrm{Id}"'] & & \Sigma \end{tikzcd} \;\mapsto\; \begin{tikzcd} & & X \arrow[dr, "p"] \arrow[d, phantom, "\Downarrow\eta_\sigma"] & \\ * \arrow[urr, bend left, "\phi|_\sigma"] \arrow[r, "\sigma"] & \Sigma \arrow[ur, "\phi"] \arrow[rr, "\mathrm{Id}"'] & & \Sigma \end{tikzcd} \;\mapsto\; \left(\phi_\sigma,\ p(\phi_\sigma) \xrightarrow{\eta_\phi(\sigma)} \sigma\right)$$

Proposition 1 *Let $p : X \to \Sigma$ and $\nabla : X \to F$ be functors. Write $j : (p/\mathrm{const}_\sigma) \to X$ for the canonical projection out of the comma category.*

For every cospan $\begin{tikzcd} & \Sigma & \\ * \arrow[ur, "\sigma"] & & * \arrow[ul, "\sigma'"'] \end{tikzcd}$ *the canonical functors from definition 2 make the top part of the following span construction commute (the bottom part commutes obviously)*

$$\begin{tikzcd} & \Gamma(\Sigma, X) \arrow[dl, "(-)|_\sigma"'] \arrow[d] \arrow[dr, "(-)|_{\sigma'}"] & \\ (p/\mathrm{const}_\sigma) \arrow[d, "j"] & {[\Sigma, X]} \arrow[dl, "(-)|_\sigma"'] \arrow[d, "{[\Sigma,g]}"] \arrow[dr, "(-)|_{\sigma'}"] & (p/\mathrm{const}_{\sigma'}) \arrow[d, "j"] \\ {[*, X]} \arrow[d, "{[*,\nabla]}"] & {[\Sigma, F]} \arrow[dl, "(-)|_\sigma"'] \arrow[dr, "(-)|_{\sigma'}"] & {[*, X]} \arrow[d, "{[*,\nabla]}"] \\ {[*, F]} & & {[*, F]} \end{tikzcd}$$

Remarks.

- Here of course $[*, \nabla] = \nabla$, but we keep the notation for emphasis of the more general case where instead of $*$ we have Σ_{in} and Σ_{out}.
- In [1] is considered the lower part of this span construction. Here the suggestion is that it is quite natural to consider the outer part with $[X, \Sigma]$ refined by $\Gamma^p(X, \Sigma)$.

Observation 1 *The pull-push quantization of [1] would assign to the above span construction some morphism between the collections of sections of the left and right vertical transgressed background fields. With just the lower part of the above span construction this would be* $\mathrm{colim}\ X \xrightarrow{\nabla} F$ *over each point σ (if we don't pass to generalized sections, at least). But with the full span construction it would now be*

$$\mathrm{colim}\ (p/\mathrm{const}_\sigma) \longrightarrow X \xrightarrow{\nabla} F$$

which is of course $(\mathrm{Lan}_p \nabla)(\sigma)$.

References

[1] *Nonabelian cocycles and their σ-model QFTs*