Lecture 8: Parallel and Killing spinor fields

In this lecture we will characterise manifolds admitting spinor fields satisfying some natural differential equations. We will first revisit parallel (or covariantly constant) spinor fields, which were already discussed in §6.3, from the point of view of the holonomy representation. We will then introduce the notion of a (real) Killing spinor field, as a special case of a "twistor" spinor field.

8.1 Manifolds admitting parallel spinor fields

Recall that a covariantly constant spinor field ψ — that is, one obeying $d^\nabla\psi = 0$ — is invariant under parallel transport and hence its value at any point m is a spinor which is invariant under (the spin lift of) the holonomy group $\mathrm{Hol}(m)$. We also learnt that in positive-definite signature, a spin manifold admits parallel spinor fields only if it is Ricci-flat. This means that if $\mathrm{Hol}(m)$ leaves invariant a (nonzero) spinor, the manifold must be Ricci-flat. As we discussed in the last lecture, there are four Ricci-flat holonomy representations: $\mathrm{SU}(n) \subset \mathrm{SO}(2n)$, $\mathrm{Sp}(n) \subset \mathrm{SO}(4n)$, $\mathrm{G}_2 \subset \mathrm{SO}(7)$ and $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$. Curiously, as shown by Wang [Wan89], each of these representations preserve a nonzero spinor. His results can be summarised in the following table. The column labelled "Parallel spinors" lists the dimension of the space of parallel spinors. In even dimensions, this is further refined according to chirality, in such a way that (n_+, n_-) means that the space of positive (resp. negative) parallel half-spinors has (real) dimension n_+ (resp. n_-). Of course, changing the orientation of the manifold interchanges n_+ and n_-.

Table 1: Irreducible, simply-connected manifolds admitting parallel spinors

Holonomy representation	Geometry	Parallel spinors
$\mathrm{SU}(2n+1) \subset \mathrm{SO}(4n+2)$	Calabi–Yau	$(1,1)$
$\mathrm{SU}(2n) \subset \mathrm{SO}(4n)$	Calabi–Yau	$(2,0)$
$\mathrm{Sp}(n) \subset \mathrm{SO}(4n)$	hyperkähler	$(k+1,0)$
$\mathrm{G}_2 \subset \mathrm{SO}(7)$	exceptional	1
$\mathrm{Spin}(7) \subset \mathrm{SO}(8)$	exceptional	$(1,0)$

We will concentrate on two examples: $\mathrm{SU}(3) \subset \mathrm{SO}(6)$ and $\mathrm{G}_2 \subset \mathrm{SO}(7)$.

8.1.1 Calabi–Yau 3-folds

We start with the following lemma.

Lemma 8.1. *The spin representation gives an isomorphism* $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$.

Proof. First of all we remark that $\mathrm{Spin}(6) \subset C\ell(6)_0 \cong \mathbb{C}(4)$. Thus we have an injective homomorphism $\iota : \mathrm{Spin}(6) \to \mathrm{GL}(4,\mathbb{C})$, which is the spin representation. Since $\mathrm{Spin}(6)$ is compact, its image in $\mathrm{GL}(4,\mathbb{C})$ must lie inside a maximal compact subgroup of $\mathrm{GL}(4,\mathbb{C})$: namely, a copy of $\mathrm{U}(4)$. Since $\mathrm{Spin}(6)$ is simple, its image must be inside $\mathrm{SU}(4)$. Finally, since $\dim \mathrm{Spin}(6) = \dim \mathrm{SU}(4) = 15$, and since both $\mathrm{Spin}(6)$ and $\mathrm{SU}(4)$ are connected, ι is an isomorphism. □

This means that the spinor representation of $\mathrm{Spin}(6)$ is the defining representation of $\mathrm{SU}(4)$ on $\mathbb{C}^4$. A nonzero spinor is a vector $\psi \in \mathbb{C}^4$. Without loss of generality we can assume that $\psi = (z,0,0,0)$ for some $0 \neq z \in \mathbb{C}$. It is then clear that the subgroup of $\mathrm{SU}(4)$ leaving that vector invariant is an $\mathrm{SU}(3)$ subgroup, which is the image under ι of an $\mathrm{SU}(3)$ subgroup of $\mathrm{Spin}(6)$. Since $-\mathbf{1} \in \mathrm{Spin}(6)$ does not leave ψ invariant, it does not belong to $\mathrm{SU}(3)$ whence its image under $\widetilde{\mathrm{Ad}} : \mathrm{Spin}(6) \to \mathrm{SO}(6)$ is an $\mathrm{SU}(3)$ subgroup of $\mathrm{SO}(6)$. This is precisely the holonomy representation $\mathrm{SU}(3) \subset \mathrm{SO}(6)$ in Berger's table. The complex conjugate spinor $\overline{\psi}$ has the opposite chirality to ψ and is also left invariant by the same $\mathrm{SU}(3)$ subgroup, whence the $(1,1)$ in the corresponding entry in the table.

8.1.2 Manifolds of G_2 holonomy

Let $\mathbb{O}$ denote the real division algebra of octonions, obtained from the quaternions by the Cayley–Dickson doubling construction. It is a normed algebra with a positive-definite inner product $B(x, y) = \operatorname{Re}(x\overline{y})$. The octonions are *not* associative, but they are alternating, which means that the subalgebra generated by any two elements is associative. In particular, if $x, y \in \mathbb{O}$, then $x(xy) = x^2 y$ and $(yx)x = yx^2$. This is equivalent to associator $(xy)z - x(yz)$ being totally skewsymmetric in x, y, z. Consider now the linear maps defined by $\ell : x \mapsto \ell_x$ and $r : x \mapsto r_x$, where ℓ_x and r_x are, respectively, left and right multiplication by $x \in \mathbb{O}$.

Lemma 8.2. *The linear maps* $\ell, r : \operatorname{Im}\mathbb{O} \to \operatorname{End}_{\mathbb{R}}(\mathbb{O})$ *defined above are Clifford.*

Proof. We will prove the lemma for ℓ and leave r as an exercise. First of all, notice that the alternating property of $\mathbb{O}$ says that for all $x, y, z \in \mathbb{O}$,

$$x(yz) - (xy)z = -y(xz) + (yx)z \,, \tag{103}$$

whence

$$\begin{aligned} \ell_x \ell_y z + \ell_y \ell_x z &= x(yz) + y(xz) \\ &= (xy)z + (yx)z \qquad\qquad \text{by equation (103)} \\ &= (xy + yx)z \,. \end{aligned}$$

But notice that since $x, y \in \operatorname{Im}\mathbb{O}$, $xy + yx \in \mathbb{R} \subset \mathbb{O}$ and is indeed equal to $-2B(x, y)$, whence we conclude that

$$\ell_x \ell_y + \ell_y \ell_x = -2B(x, y)\mathbf{1} \,.$$

□

This means that ℓ and r extend to representations of the Clifford algebra $C\ell(\operatorname{Im}\mathbb{O}) \cong C\ell(7)$. Indeed, the isomorphism $C\ell(7) \cong \mathbb{R}(8) \oplus \mathbb{R}(8)$ is the Clifford extension of the Clifford map $x \mapsto (\ell_x, r_x)$. The spinor representation of $\mathrm{Spin}(7)$ is obtained by restricting either of these two Clifford modules to $\mathrm{Spin}(7) \subset C\ell(7)$. This defines a map $\mathrm{Spin}(7) \to \mathrm{GL}(8, \mathbb{R})$ whose image, since $\mathrm{Spin}(7)$ is compact and connected, lies inside $\mathrm{SO}(8)$, for some $\mathrm{SO}(8)$ subgroup of $\mathrm{GL}(8, \mathbb{R})$. Indeed, it is the $\mathrm{SO}(8)$ which preserves the octonionic inner product. This follows form the fact that $\mathbb{O}$ is a normed algebra, whence $B(xy, xy) = B(x, x)B(y, y)$, whence if $B(x, x) = 1$ then both ℓ_x and r_x are isometries.

Let ψ be a nonzero spinor, which we may take to correspond to $1 \in \mathbb{R} \subset \mathbb{O}$. The subgroup of $\mathrm{Spin}(7)$ which fixes ψ is a G_2 subgroup of $\mathrm{Spin}(7)$ which does not contain $-\mathbf{1}$ and hence projects under $\widetilde{\mathrm{Ad}} : \mathrm{Spin}(7) \to \mathrm{SO}(7)$ to a G_2 subgroup of $\mathrm{SO}(7)$, which is precisely the holonomy representation $G_2 \subset \mathrm{SO}(7)$. Any other spinor left invariant by this G_2 subgroup is proportional to ψ.

8.1.3 Some comments about indefinite signature

In physical applications it is often necessary to determine the lorentzian (or even higher index) spin manifolds admitting parallel spinors. There is a classification of lorentzian holonomy groups due to Leistner and Galaev [GL08], as well as earlier results of Bryant [Bry00a] and myself [FO00]. A lorentzian spin n-dimensional manifold admits parallel spinors if its holonomy representation is $G \ltimes \mathbb{R}^{n-2} \subset \mathrm{SO}_0(n-1, 1)$, where $G \subset \mathrm{SO}(n-2)$ is one of the riemannian holonomy representations admitting parallel spinors. The subgroup $G \ltimes \mathbb{R}^{n-2}$ of $\mathrm{SO}(n-1, 1)$ is such that G acts on $\mathbb{R}^{n-2}$ via the holonomy representation $G \subset \mathrm{SO}(n-2)$, and the abelian normal subgroup $\mathbb{R}^{n-2}$ acts as *null rotations* on $\mathbb{R}^{n-1,1}$. The situation for higher index is much less clear and still the subject of investigation.

8.2 Manifolds admitting (real) Killing spinor fields

On a spin manifold one can define natural equations satisfied by spinor fields other than $d^\nabla \psi = 0$. In this section we will discuss the Killing spinor equation which is a special case of the twistor spinor equation, about which we will not say anything beyond its definition.

8.2.1 The Dirac operator

Let $\mathscr{E} = (e_i)$ be a local frame and let (e^i) denote the dual frame, so that $g(e^i, e_j) = \delta^i_j$. The **Dirac operator** is the differential operator D acting on a spinor field ψ as

$$\mathrm{D}\psi = \sum_i e^i \cdot \nabla_{e_i}\psi \,, \tag{104}$$

where the dot $(\cdot)$ stands for Clifford action. More invariantly, it is defined as the composition of the following two maps

$$C^\infty(M, S(M)) \xrightarrow{d^\nabla} C^\infty(M, T^*M \otimes S(M)) \xrightarrow{\mathrm{cl}} C^\infty(M, S(M)) \tag{105}$$

where the first map is the covariant derivative and the second map is the fibrewise Clifford action $T^*M \otimes S(M) \to S(M)$.

Example 8.3. The original Dirac operator was defined on four-dimensional Minkowski spacetime. Relative to flat coordinates x^μ and the associated frame, the Dirac operator takes the form

$$\mathrm{D}\psi = \sum_\mu \Gamma^\mu \cdot \nabla_{\frac{\partial}{\partial x^\mu}} \psi = \sum_\mu \Gamma^\mu \frac{\partial \psi}{\partial x^\mu} \,, \tag{106}$$

where $\psi : \mathbb{R}^{3,1} \to \mathbb{C}^4$ and $\Gamma^\mu = \sum_\nu \eta^{\mu\nu}\Gamma_\nu$ and Γ_μ are the 4×4 gamma matrices representing the Clifford action by the frame vectors $\frac{\partial}{\partial x^\mu}$.

Spinors which are annihilated by the Dirac operator are known as **harmonic spinors**. The origin of the name is due to the fact that squaring the original Dirac operator, one gets the laplacian:

$$\mathrm{D}^2\psi = -\eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\psi = \Box\psi \,. \tag{107}$$

In the general case, squaring the Dirac operator results in a curvature-dependent correction:

$$\mathrm{D}^2\psi = \nabla^*\nabla\psi + \frac{s}{4}\psi \,, \tag{108}$$

where s is the scalar curvature and $\nabla^*\nabla$ is the covariant laplacian.

An immediate corollary of this calculation is the following theorem due to Lichnerowicz.

Theorem 8.4 (Lichnerowicz)**.** *If* (M, g) *is a compact positive-definite riemannian spin manifold with* $s \geq 0$ *and* $s > 0$ *at at least one point, then* (M, g) *admits no nonzero harmonic spinor fields; whereas if* $s \equiv 0$ *then a harmonic spinor field is parallel.*

Proof. Indeed, let $(-,-)$ denote the invariant inner product on the spinor bundle, and consider the integral

$$\int_M \left(\psi, \mathrm{D}^2\psi\right) = \int_M |d^\nabla \psi|^2 + \frac{1}{4}\int_M s|\psi|^2 \,.$$

Let $\mathrm{D}\psi = 0$, so that the LHS vanishes. Then if $s \geq 0$, the RHS is positive-semidefinite and in particular we see that $d^\nabla\psi = 0$. This being the case, ψ is determined uniquely by its value at any point, so that in particular if it vanishes anywhere, it must vanish everywhere. If $s > 0$ at at least one point, then it $s > 0$ is a neighbourhood of that point and hence $\psi = 0$ in a neighbourhood of that point and hence $\psi = 0$ everywhere. □

8.2.2 The Penrose operator and twistor spinor fields

Let $W \subset T^*M \otimes S(M)$ denote the subbundle defined as the kernel of the Clifford action $T * M \otimes S(M) \to S(M)$. Let $\pi : T^*M \otimes S(M) \to W$ denote the projection onto W along S(M). The **Penrose operator** $P : C^\infty(M, S(M)) \to C^\infty(M, W)$ is defined as the composition

$$C^\infty(M, S(M)) \xrightarrow{d^\nabla} C^\infty(M, T^*M \otimes S(M)) \xrightarrow{\pi} C^\infty(M, W) \,. \tag{109}$$

Explicitly, we can write for all spinor fields ψ and all vector fields X,

$$P_X\psi = \nabla_X\psi + \tfrac{1}{n}X \cdot \mathrm{D}\psi \,, \tag{110}$$

where $n = \dim M$. Spinor fields in the kernel of the Penrose operator are known as **twistor fields**.

8.2.3 Killing spinor fields

A **Killing spinor field** is a special type of twistor field ψ which satisfies the stronger equation

$$\nabla_X \psi = \lambda X \cdot \psi \,, \tag{111}$$

for some constant $\lambda \in \mathbb{C}$ called the **Killing constant**. A calculation similar to that in section 6.3 reveals that the integrability condition for the existence of Killing spinor fields is

$$(R(X) - 4\lambda^2 (n-1) X) \cdot \psi = 0 \,,$$

for all vector fields X and where $X \mapsto R(X)$ is the Ricci operator and $n = \dim M$. In positive-definite signature, it says that $R(X) = 4\lambda^2(n-1)X$ for all vector fields X, or equivalently after taking the inner product with a second vector field Y, that $r(X, Y) = 4\lambda^2(n-1) g(X, Y)$, whence (M, g) is Einstein. In indefinite signature this is no longer the case, but we can take the Clifford trace of the above equation to conclude that

$$s\psi = 4\lambda^2 n(n-1)\psi \,,$$

whence if ψ is not identically zero, the scalar curvature is constrained in terms of λ: namely, $s = 4\lambda^2 n(n-1)$. Since the scalar curvature is real, we see that λ^2 is real, whence it is either real or pure imaginary. The nature of the Killing constant gives rise to two different kinds of Killing spinor fields: **real** and **imaginary**, respectively. They each have a very different flavour and in the rest of this lecture we will concentrate on the real case. Furthermore via a homothety (i.e., rescaling the metric by a constant positive number) we can further assume that $\lambda = \pm\frac{1}{2}$. Finally, we will concentrate on positive-definite signature, whence we will be interested in characterising those positive-definite riemannian spin manifolds admitting nonzero spinor fields ψ satisfying

$$\nabla_X \psi = \pm\tfrac{1}{2} X \cdot \psi \,, \tag{112}$$

for all vector fields X.

Bär's cone construction [Bär93] will relate such Killing spinor fields to parallel spinor fields in an auxiliary geometry. To at least demonstrate the plausibility of such a construction, let us first of all notice that a spinor field obeying equation (112) is actually parallel with respect to the connection $\mathscr{D}_X = \nabla_X \mp \frac{1}{2} X \cdot$. The connection one-form associated with $\mathscr{D}$ is given, relative to a local frame $\mathscr{E} = (e_i)$, by

$$\tfrac{1}{4} \sum_{i,j} \omega_{ij} e^i e^j \mp \tfrac{1}{2} \sum_i \theta_i e^i \,, \tag{113}$$

where $\omega_{ij}(X) = g(\nabla_X e_i, e_j)$ and $\theta_i(X) = g(X, e_i)$. But now notice that $\frac{1}{4}[e^i, e^j]$ and $\mp\frac{1}{2} e_i$ in $C\ell(n)$ span an $\mathfrak{so}(n+1)$ subalgebra of $C\ell(n)$, whence the above connection one-form is $\mathfrak{so}(n+1)$-valued, which suggests that it could very well be the spin connection of an $(n+1)$-dimensional manifold. This manifold is the metric cone as we now review.

8.2.4 The cone construction

Let (M, g) be an n-dimensional riemannian manifold and let $\widetilde{M} = \mathbb{R}^+ \times M$. We parametrise $\mathbb{R}^+$ by $r > 0$ and define a metric $\tilde{g}$ on $\widetilde{M}$ by $\tilde{g} = dr^2 + r^2 g$. The riemannian manifold $(\widetilde{M}, \tilde{g})$ thus constructed is the **metric cone** of (M, g). (M, g) embeds isometrically into $(\widetilde{M}, \tilde{g})$ as the submanifold at $r = 1$. Generically the metric on $\widetilde{M}$ cannot be extended smoothly to $r = 0$. The exception occurs when (M, g) is the round n-sphere, in which case the cone is $\mathbb{R}^{n+1} \setminus \{0\}$ with the flat euclidean metric, since in that case the flat metric is clearly regular at the origin and can be extended there.

The cone $\widetilde{M}$ admits a homothetic action by R^+, where $e^t \in \mathbb{R}^+$ acts by rescaling the "radial" coordinate: $(r, x) \mapsto (e^t r, x)$. The conformal Killing vector generating this action is the **Euler vector** $\xi = \frac{1}{r}\frac{\partial}{\partial r}$. A vector field $X \in \mathscr{X}(M)$ admits a unique lift to $\widetilde{M}$, also denoted X with a little abuse of notation, such that it is orthogonal to ξ and such that it maps to X under the natural projection $\widetilde{M} \to M$, sending (r, x) to x. Let $\widetilde{\nabla}$ denote the Levi-Civita connection on $\widetilde{M}$.

Lemma 8.5. *Let* $X, Y \in \mathscr{X}(\widetilde{M})$ *be lifts of vector fields on* M. *Then*

$$\widetilde{\nabla}_\xi \xi = \xi\,, \qquad \widetilde{\nabla}_\xi X = X\,, \qquad \widetilde{\nabla}_X \xi = X \qquad \textit{and} \qquad \widetilde{\nabla}_X Y = \nabla_X Y - g(X,Y)\xi\,.$$

Remark 8.6. In fact, a result of Gibbons and Rychenkova [GR98] states that a riemannian manifold is a metric cone if and only if there exists a vector field ξ such that $\nabla_V \xi = V$ for all vector fields V, where ∇ is the Levi-Civita connection.

Now given a local frame $\mathscr{E} = (e_i)$ for M, we extend it to a local frame $\widetilde{\mathscr{E}} = (\widetilde{e}_0 = \frac{\partial}{\partial r}, \widetilde{e}_i = \frac{1}{r} e_i)$ for $\widetilde{M}$. The connection coefficients of $\widetilde{\nabla}$ relative to $\widetilde{\mathscr{E}}$ are given in terms of the connection coefficients of ∇ relative to $\mathscr{E}$ by the following formulae.

Lemma 8.7. *Let* $\widetilde{\omega}_{ab} = \widetilde{g}(\widetilde{\nabla}\widetilde{e}_a, \widetilde{e}_b)$ *be the connection 1-form in* $\widetilde{M}$ *relative to the local frame* $\widetilde{\mathscr{E}}$. *Then*

$$\widetilde{\omega}_{ab}(\tfrac{\partial}{\partial r}) = 0\,, \qquad \widetilde{\omega}_{0i}(e_j) = \delta_{ij} \qquad \textit{and} \qquad \widetilde{\omega}_{ij}(e_k) = \omega_{ij}(e_j)\,.$$

Since $\mathbb{R}^+$ is contractible, the cone $\widetilde{M}$ is homotopy equivalent to M, whence if M is spin, so is $\widetilde{M}$. Furthermore, if M is spin, the embedding (at $r = 1$) of M into $\widetilde{M}$ sets up a bijective correspondence between the spin structures on M and on $\widetilde{M}$. From now on we assume that both M and $\widetilde{M}$ are spin, with corresponding spin structures. Now let $\widetilde{\psi}$ be a spinor field on $\widetilde{M}$. Its covariant derivative can be computed from equation (89) and the previous lemma and one finds

$$\widetilde{\nabla}_{\frac{\partial}{\partial r}} \widetilde{\psi} = \tfrac{\partial}{\partial r}\widetilde{\psi} \qquad \text{and} \qquad \widetilde{\nabla}_{e_k}\widetilde{\psi} = \nabla_{e_k}\widetilde{\psi} + \tfrac{1}{2}\widetilde{e}_0\widetilde{e}_k\widetilde{\psi}\,.$$

Therefore a parallel spinor field $\widetilde{\psi}$ on $\widetilde{M}$ satisfies $\frac{\partial}{\partial r}\widetilde{\psi} = 0$ and $\nabla_{e_k}\widetilde{\psi} = \frac{1}{2}\widetilde{e}_k\widetilde{e}_0\widetilde{\psi}$. The restriction of $\widetilde{\psi}$ to $r = 1$ is a spinor field on M which satisfies the second of the above equations. To understand this equation intrinsically, we recall the isomorphism $C\ell(n) \cong C\ell(n+1)_0$ given in Proposition 2.8. In fact, there are two possible isomorphisms, distinguished by a sign: $e_i \mapsto \varepsilon\widetilde{e}_i\widetilde{e}_0$, for $\varepsilon^2 = 1$. It is now that we must make a distinction between even- and odd-dimensional M. Consider the volume element $e_1 \cdots e_n \in C\ell(n)$. Its image in $C\ell(n+1)_0$ under the above isomorphism is given by

$$e_1 \cdots e_n \mapsto \begin{cases} -\varepsilon\widetilde{e}_0\widetilde{e}_1\cdots\widetilde{e}_n & n \text{ odd} \\ \widetilde{e}_1\cdots\widetilde{e}_n & n \text{ even.} \end{cases} \tag{114}$$

If $n = \dim M$ is odd, then there are two inequivalent Clifford modules, each determined by the action of the volume element $e_1 \cdots e_n$ in $C\ell(n)$, which goes over to $-\varepsilon$ times the action of the volume element $\widetilde{e}_0\widetilde{e}_1\cdots\widetilde{e}_n$ in $C\ell(n+1)$. This means that ε can be fixed in order to relate Killing spinor fields on M (with respect to one choice of Clifford module) to the chirality of the parallel spinor field on $\widetilde{M}$. Hence the sign of the Killing constant and the chirality of the parallel spinor field are correlated. On the other hand, if n is even, then ε is not fixed and for every parallel spinor field on $\widetilde{M}$ we obtain a Killing spinor field on M with either sign of the Killing constant, simply by making the right choice of ε.

8.2.5 The classification

We have just reduced the problem of which riemannian manifolds admit real Killing spinor fields to which metric cones admit parallel spinors. We will assume that (M, g) is complete and admits real Killing spinor fields. Then since it is Einstein, Myers Theorem [CE75, Theorem 1.26] implies that it is compact. Then a result of Gallot's [Gal79, Proposition 3.1] says that if (M, g) is in addition simply connected, the cone $(\widetilde{M}, \widetilde{g})$ is either irreducible or flat. If the latter, (M, g) is the round sphere; if the former it is one of the geometries in Table 8.1.

Every geometry in Table 8.1 admits parallel forms, constructed via the holonomy principle from the invariants under the holonomy representation and indeed constructed out of the parallel spinors. Since in addition the manifold in question is a cone, and hence we have at our disposal also the Euler vector field ξ, we can construct a number of geometric structures on the manifold M, which are listed in Table 8.2.5, where $N_\pm$ is the dimension of the space of Killing spinor fields with Killing constant $\pm\frac{1}{2}$.

Table 2: Simply-connected, complete riemannian manifolds with real Killing spinor fields

dim	Geometry	Cone	(N_+, N_-)
n	round sphere	flat	$(2^{\lfloor n/2 \rfloor}, 2^{\lfloor n/2 \rfloor})$
$4k-1$	3-Sasaki	hyperkähler	$(k+1, 0)$
$4k-1$	Sasaki–Einstein	Calabi–Yau	$(2, 0)$
$4k+1$	Sasaki–Einstein	Calabi–Yau	$(1, 1)$
6	nearly Kähler	G_2	$(1, 1)$
7	weak G_2	Spin(7)	$(1, 0)$

For example, if the cone is Calabi–Yau, then we have a parallel complex structure J. The vector field $\chi = J\xi$ is orthogonal to ξ and it is the lift of a vector field on M, which we also denote χ. It is easy to show that χ is a Killing vector and has unit norm. Its dual one-form θ is (the restriction to $r = 1$ of) the contraction of the Euler vector into the Kähler form on the cone. The covariant derivative $\nabla\chi$ defines a skewsymmetric endomorphism T of the TM such that $T(X) = \nabla_X \chi$. The fact that J is parallel means that

$$(\nabla_X T)(Y) = \theta(Y)X - g(X,Y)\chi \ . \tag{115}$$

The triple (χ, θ, T) defines a **Sasakian structure** on M, whence M is Sasaki–Einstein.

For another example, consider the case of a G_2-holonomy cone. We have a parallel 3-form ϕ into which we contract the Euler vector field ξ to define a 2-form $\omega \in \Omega^2(M)$: $\omega(X,Y) = \phi(\xi, X, Y)$, evaluated at $r = 1$. This defines an endomorphism J of TM by $g(J(X), Y) = \omega(X,Y)$. One can show that J is an orthogonal almost complex structure. It is not parallel, but it satisfies $(\nabla_X J)(X) = 0$ for all vector fields $X \in \mathscr{X}(M)$. This defines a (non-Kähler) **nearly Kähler** structure on M.

These geometries defined via Killing spinors are presently under very active investigation, largely due to their rôle in the gauge/gravity correspondence (see, e.g., [AFOHS98, MP99]).