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Semi-Free Circle Actions on Spin^c-Manifolds

By

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Introduction

When a compact Lie group acts differentiably on smooth manifolds, various results have been known concerning the characteristic numbers of the manifolds. The most frequently used tool is Atiyah-Singer Lefschetz formula [2]. However, several approaches have been made to obtain similar results by geometric methods. Hattori and Taniguchi [6] investigated the cobordism groups of oriented or weakly almost complex manifolds with S^1-actions and recovered Kosniowski formula [8] and Atiyah-Singer formula [2]. But as for Spin-manifolds, no cobordism theoretic interpretation of Atiyah-Hirzebruch theorem [3] has been known so far.

In this paper we consider Spin^c-manifolds with semi-free S^1-actions. By purely geometric methods, we obtain Todd genus formula which relates the Todd genus of the manifold and the local behaviour of the S^1-action around the fixed point sets. A similar formula has been given by Petrie [9] using Atiyah-Singer Lefschetz formula and the Dirac operator.

As applications of our Todd genus formula, we can prove the results of Kosniowski [8] and Atiyah-Hirzebruch [3] in the semi-free case.

§1. Equivariant Characteristic Classes

Let M^n be an oriented closed smooth manifold of dimension n. We choose a Riemannian metric on the tangent bundle τ_M of M and denote by F_M its associated $SO(n)$-bundle. By a Spin^c-structure on

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M, we mean a $\mathrm{Spin}^c(n)$-bundle P_M on M with an equivalence $F_M \cong P_M \times_{\mathrm{Spin}^c(n)} SO(n)$ of $SO(n)$-bundles. Here $\mathrm{Spin}^c(n)$ acts on $SO(n)$ via the canonical projection $\phi^c\colon \mathrm{Spin}^c(n) \to SO(n)$ (see the Appendix). Usually in cobordism theories, Spin^c-structures are defined on the stable tangent bundle of the manifold. But since stable Spin^c-structures are in one-to-one correspondence with the Spin^c-structures in our sense (see e.g. [10]), there will arise no confusion.

Let G be a compact Lie group acting effectively and differentiably on M from the left. We may assume that the Riemannian metric on M is G-invariant by the usual averaging process and then G induces a bundle map action on F_M. That is, there exists a left action of G on F_M which commutes with the right principal $SO(n)$-action and the G-action is compatible with the G-action on M shown by the commutativity of the diagram below.

$$\begin{array}{ccc} G \times F_M & \longrightarrow & F_M \\ \Big\downarrow{\scriptstyle id \times proj} & & \Big\downarrow{\scriptstyle proj} \\ G \times M & \longrightarrow & M \end{array}$$

If, in addition, M has a Spin^c-structure P_M and G acts on P_M commuting with the right principal $\mathrm{Spin}^c(n)$-action compatibly with the reduction $P_M \to F_M$, we say that G acts on M preserving the Spin^c-structure or that G acts on (M, P_M).

Take $G = S^1$ the circle group and let $\mathscr{F}$ and $\mathscr{F}'$ be families of closed subgroups of S^1 with $\mathscr{F} \supset \mathscr{F}'$. Consider the objects (φ, M^n, P_M) where M^n is an oriented smooth manifold with a Spin^c-structure P_M and φ is an S^1-action on (M^n, P_M) with the additional condition that the isotropy subgroup $(S^1)_x$ belongs to $\mathscr{F}$ if $x \in M$ and $(S^1)_x$ belongs to $\mathscr{F}'$ if $x \in \partial M$. Introducing a usual cobordism relation to these objects, we obtain cobordism groups $\Omega_n^{\mathrm{Spin}^c}(S^1; \mathscr{F}, \mathscr{F}')$ and $\Omega_n^{\mathrm{Spin}^c}(S^1; \mathscr{F}) = \Omega_n^{\mathrm{Spin}^c}(S^1; \mathscr{F}, \phi)$ as in [5].

If $p\colon P \to X$ is a right principal $\mathrm{Spin}^c(n)$-bundle over a space X, it is well known that it determines an element $\omega(P)$ in $H^2(X; \boldsymbol{Z})$ whose reduction modulo 2 is the second Stiefel-Whitney class of P. This class is usually called the "c_1-class" of the $\mathrm{Spin}^c(n)$-bundle, but we shall call it ω-class instead.

Let X be a space with a left action of a compact Lie group G and $EG \to BG$ be a universal right principal G-bundle. We define $X_G = EG \times_G X$ to be the orbit space of $EG \times X$ under the left G-action $g(e, x) = (eg^{-1}, gx)$. The orbit space $G \backslash X$ of a left G-space X is denoted by $\overline{X}$. When $p: P \to X$ is a right principal $\mathrm{Spin}^c(n)$-bundle and G acts on (X, P) compatibly with the projection p and commuting with the right principal $\mathrm{Spin}^c(n)$-action, then we define its G-equivariant ω-class by $\omega^G(P) = \omega(P_G) \in H^2(X_G; \boldsymbol{Z})$. If moreover P is a Spin^c-structure of a manifold X, we write $\omega_X = \omega(P)$ and $\omega_X^G = \omega^G(P)$.

Let $p: P \to X$ be a $\mathrm{Spin}^c(n)$-bundle with an S^1-action and consider maps

$$X \xleftarrow{p_2} ES^1 \times X \xrightarrow{\pi} X_{S^1}.$$

Lemma 1.1.

$$\pi^* \omega^{S^1}(P) = p_2^* \omega(P).$$

Proof. From the diagram of bundle maps

$$\begin{array}{ccccc} P & \xleftarrow{p_2} & ES^1 \times P & \longrightarrow & P_{S^1} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle id \times p} & & \downarrow{\scriptstyle p_{S^1}} \\ X & \xleftarrow[p_2]{} & ES^1 \times X & \xrightarrow[\pi]{} & X_{S^1} \end{array}$$

we see that $\pi^*(P_{S^1}) \cong p_2^*(P)$ and the Lemma follows.

Proposition 1.2. *Let $p: P \to X$ be a* $\mathrm{Spin}^c(n)$*-bundle and S^1 act on P as bundle automorphisms (trivially on X). Then the action determines a homomorphism $r: S^1 \to \mathrm{Spin}^c(n)$ (see Conner and Floyd* [5]*). Then we have*

$$\omega^{S^1}(P) = (\deg r)\alpha \oplus \omega(P).$$

Here $\deg r$ *is the degree of the map*

$$\det{}^c \circ r: S^1 \longrightarrow \mathrm{Spin}^c(n) \longrightarrow SO(2) \qquad (\textit{see the Appendix})$$

and we made identifications $H^2(X_{S^1}; \boldsymbol{Z}) = H^2(BS^1 \times X; \boldsymbol{Z}) \cong H^2(BS^1; \boldsymbol{Z}) \otimes 1 \oplus 1 \otimes H^2(X; \boldsymbol{Z}) \cong H^2(BS^1; \boldsymbol{Z}) \oplus H^2(X; \boldsymbol{Z})$ by the natural homeomorphism $X_{S^1} = BS^1 \times X$ and the Künneth formula. α is the canonical *generator of* $H^2(BS^1; \mathbf{Z})$. *In particular, if* P *is an extension of a* Spin(n)*-bundle* $\tilde{P}$ *and the* S^1*-action on* P *is induced by an* S^1*-action of* $\tilde{P}$, *then* $\omega^{S^1}(P)=0$.

Proof. Let $\omega^{S^1}(P)=m\alpha \oplus u$ where $m \in \mathbf{Z}$ and $u \in H^2(X; \mathbf{Z})$. By Lemma 1.1, we know that $u=\omega(P)$. Since we have only to compute m, we shall restrict ourselves to a fiber over a point $x \in X$. Then $(P_x)_{S^1}$ is a Spin$^c(n)$-bundle over BS^1 induced by the map $Br: BS^1 \to B\,\mathrm{Spin}^c(n)$. ω-class is induced by the map $B(\det^c): B\,\mathrm{Spin}^c(n) \to BSO(2)$ by definition. Hence $\omega^{S^1}(P_x)=(\deg r)\alpha$. If P is an extension of a Spin (n)-bundle, then r factors through Spin(n). Hence $\deg r=0$ and $\omega(P)=0$.

§2. Free S^1-actions on Spin^c-manifolds

Let (M^n, P_M) be a Spin^c-manifold with a free S^1-action. The tangent bundle τ_M of M^n has a subbundle τ' composed of tangent vectors orthogonal to the S^1-orbits of M. The associated $SO(n-1)$-bundle F'_M is a reduction of the tangent oriented orthonormal n-frame bundle F_M of M^n. F'_M has a Spin$^c(n-1)$-reduction P'_M obtained as the fiber product of $P_M \to F_M$ and $F'_M \to F_M$. All these bundles have induced S^1-actions. Let $\pi: M \to \overline{M}$ be the orbit map, then this defines a principal S^1-bundle denoted by ξ. Under these conditions we have the following lemma whose proof is clear from the definitions.

Lemma 2.1. $F_{\overline{M}}=S^1 \backslash F'_M$ *is a tangent frame bundle of* $\overline{M}=S^1 \backslash M$ *and* $P_{\overline{M}}=S^1 \backslash P'_M$ *is a* Spin^c*-structure on* $\overline{M}$. *And we have equivalences of bundles with* S^1*-actions*:

$$\pi^* P_{\overline{M}}=P'_M \quad \text{and} \quad \pi^*\xi=M \times S^1 .$$
$$\text{(the action on } M \times S^1 \text{ is trivial in the fiber } S^1\text{)}$$

Let M^n be as before and consider the $(n+1)$-manifold $W^{n+1}=M \times D^2/\sim$ where $(x, v) \sim (gx, gv)$ for $x \in M$, $v \in D^2$ (unit disk in $\mathbf{C}$) and $g \in S^1$ (unit sphere in $\mathbf{C}$). Define maps $i: M \to W$, $p: W \to \overline{M}$ and $j: \overline{M} \to W$ by $i(x)=[x, 1]$, $p([x, v])=[x]$ and $j([x])=[x, 0]$. Let S^1 act on W by $g[x, v]=[gx, v]$. Then i, j, and p are S^1-equivariant maps. Consider the Spin$^c(n-1) \times U(1)$-bundle $Q=p^*(P_{\overline{M}} \oplus \xi)$ over W. Then Q is a reduction of the tangent bundle of W and the restriction of Q to M gives an S^1-equivariant isomorphism $i^*Q \cong P'_M \oplus (M \times U(1))$ by Lemma 2.1. Therefore we get the lemma below.

Lemma 2.2.

$$i^*\omega(Q) = \omega(P) = \omega_M = \pi^*\omega_{\overline{M}}$$

$$\omega(Q) = p^*(\omega_{\overline{M}} + c)$$

where c is the Euler class of ξ.

Now, with the use of the ω^{S^1}-classes, we can give cobordism-theoretic description of Spin^c-manifolds with free S^1-actions.

Proposition 2.3. *Let $\{1\}$ denote the family of closed subgroups of S^1 consisting of the trivial subgroup only. Then*

$$\Omega_n^{\mathrm{Spin}^c}(S^1; \{1\}) \cong \Omega_{n-1}^{\mathrm{Spin}^c}(BU(1))$$

where the right hand side is the bordism group of $BU(1)$ associated with the Spin^c *spectrum* $M\mathrm{Spin}^c(k)$ *(see [10] for a precise definition).*

Proof. To a Spin^c-manifold (M^n, P_M) with a free S^1-action φ, we assign the manifold $(\overline{M}, P_{\overline{M}})$ and the $U(1)$-bundle ξ defined before. Clearly, this construction defines a homomorphism

$$A\colon\ \Omega_n^{\mathrm{Spin}^c}(S^1; \{1\}) \longrightarrow \Omega_{n-1}^{\mathrm{Spin}^c}(BU(1)).$$

Conversely, for each representative (N^{n-1}, P_N, ζ) of $\Omega_{n-1}^{\mathrm{Spin}^c}(BU(1))$, let M^n be the total space E_ζ of ζ with the S^1-action $gx = xg^{-1}$ $(x \in E_\zeta, g \in S^1)$. We can give a Spin^c-structure P_M on M by the extension of the Spin$^c(n-1)$-bundle $\pi^*P_{\overline{M}}$. This procedure leads to a well-defined homomorphism

$$A'\colon\ \Omega_{n-1}^{\mathrm{Spin}^c}(BU(1)) \longrightarrow \Omega_n^{\mathrm{Spin}^c}(S^1; \{1\}).$$

In view of Lemmas 2.1 and 2.2, it is clear that A and A' are inverses to each other.

Before going over to the next section, we shall compute the ω^{S^1}-classes in the case of free actions. The results will be crucial in the treatment of semi-free S^1-actions.

Let (M, P_M) be a Spin^c-manifold with a free S^1-action as before. Let the bundle $\xi, \pi\colon M\to\overline{M}$, be classified by the map $c\colon \overline{M}\to BS^1$ and $\tilde{c}\colon M\to ES^1$ be a lift of c.

$$\begin{tikzcd} M \arrow[r, "\tilde{c}"] \arrow[d, "\pi"'] & ES^1 \arrow[d] \\ \overline{M} \arrow[r, "c"] & BS^1 \end{tikzcd}$$

Then $(\tilde{c}, id)\colon M\to ES^1\times M$ is a homotopy equivalence of free S^1-spaces. Hence we get a homotopy equivalence

$$\bar{c}\colon \overline{M}\longrightarrow M_{S^1}$$

whose homotopy inverse $\bar{p}_2$ is induced from the second projection of $ES^1\times M$.

Lemma 2.4. *Under these conditions,*

$$\bar{c}^*(\omega_M^{S^1})=\omega_{\overline{M}}$$

holds.

Proof. Since we have seen that P_M is the extension of P'_M, $(P_M)_{S^1}$ is the extension of $(P'_M)_{S^1}$. Therefore, $\omega^{S^1}(P_M)=\omega^{S^1}(P'_M)$.
From Lemma 2.1,

$$\omega^{S^1}(P'_M)=\omega^{S^1}(\pi^*P_{\overline{M}})=(\pi_{S^1})^*\omega^{S^1}(P_{\overline{M}})=\pi_{S^1}^*p_2^*\omega(P_{\overline{M}})$$

where the maps are as follows:

$$M_{S^1}\xrightarrow{\pi_{S^1}}(\overline{M})_{S^1}=BS^1\times\overline{M}\xrightarrow{p_2}\overline{M}\,.$$

Since $\omega^{S^1}(P'_M)=\omega^{S^1}(P_M)$ and $p_2\circ\pi_{S^1}=\bar{p}_2$, we get the assertion.

Let W^{n+1} be as in Lemma 2.2, and $j_{S^1}\colon \overline{M}_{S^1}\to W_{S^1}$ be the map induced by $id\times j\colon ES^1\times\overline{M}\to ES^1\times W$. Since S^1 acts trivially on $\overline{M}$, $\overline{M}_{S^1}$ is homeomorphic to $BS^1\times\overline{M}$ and we make the canonical identification

$$H^2(\overline{M}_{S^1};\boldsymbol{Z})=H^2(BS^1;\boldsymbol{Z})\oplus H^2(\overline{M};\boldsymbol{Z})$$

as in Proposition 1.2.

Lemma 2.5. *Under these conditions*

$$(j_{S^1})^*\omega^{S^1}(Q) = -\alpha\oplus(\omega_{\overline{M}}+c)$$

where α is the canonical generator of $H^2(BS^1; \boldsymbol{Z})$.

Proof. By Proposition 1.2, it is easy to see that

$$\omega^{S^1}(\xi) = -\alpha\oplus c\,.$$

On the other hand, $j^*Q = P_{\overline{M}}\oplus\xi$ holds. And the result is immediate.

§3. Semi-free S^1-actions on Spin^c-manifolds

It is well known that we have an exact sequence of abelian groups (see e.g. [5], [11]).

$$0 \longrightarrow \Omega_n^{\mathrm{Spin}^c}(S^1;\ \{S^1,\ 1\}) \xrightarrow{\beta} \Omega_n^{\mathrm{Spin}^c}(S^1;\{S^1,\ 1\},\ \{1\})$$

$$\xrightarrow{\partial} \Omega_{n-1}^{\mathrm{Spin}^c}(S^1;\ \{1\}) \longrightarrow 0$$

where $\{S^1, 1\}$ is the family of subgroups of S^1 consisting of the whole group S^1 and the trivial group.

First, remark that we have already constructed a right inverse of ∂ implicitly in §2. To be precise, let (N^{n-1}, P_N) be an $(n-1)$-dimensional Spin^c-manifold with a free S^1-action φ. Then, the orbit manifold $\overline{N}$ has a Spin^c-structure by Lemma 2.1, and by taking the associated D^2-bundle W of the principal S^1-bundle ξ given by $N\to\overline{N}$, we know by Lemma 2.2 that W has a natural S^1-action φ' which preserves the natural Spin^c-structure P_W obtained as the extension of the Spin$^c(n-2)\times U(1)$-bundle $p^*(P_{\overline{N}}\oplus\xi)$ where $p\colon W\to\overline{N}$ is the projection of the D^2-bundle. In this way, we define a map

$$\psi\colon\ \Omega_{n-1}^{\mathrm{Spin}^c}(S^1;\ \{1\}) \longrightarrow \Omega_n^{\mathrm{Spin}^c}(S^1;\ \{S^1,\ 1\},\ \{1\})$$

by $\psi([\varphi, N, P_N]) = [\varphi', W, P_W]$.

Lemma 3.1. *ψ is a right inverse of ∂.*

Proof. It is clear from the construction that $\partial W=N$ with the original S^1-action. But by the remark just before Lemma 2.2, $P_W|N=P'_N\oplus$ trivial $U(1)$-bundle. Hence $P_N=P'_N\times_{\mathrm{Spin}^c(n-1)}\mathrm{Spin}^c(n)$ is induced by the Spin^c-structure P_W.

The next step is to clarify the structure of the group $\Omega_n^{\mathrm{Spin}^c}(S^1;\{S^1,1\},\{1\})$. Take a representative (φ, M^n, P_M) of this group and let $\{X_i\}$ be the fixed point set components of the S^1-action on M^n. Choose a small S^1-invariant closed tubular neighborhood V_i for each X_i so that no V_i meets the boundary of M^n. Then each V_i, as an n-manifold, has the Spin^c-structure P_{V_i} induced by P_M. It is easy to see that $[\varphi, M, P_M]=\sum_i[\varphi, V_i, P_{V_i}]$ in $\Omega_n^{\mathrm{Spin}^c}(S^1;\{S^1,1\},\{1\})$ (see e.g. [6]). Let $\mathscr{B}_n$ be a collection of triples (φ, V, X) such that

i) V is an n-dimensional Spin^c-manifold.

ii) V is a linear disk bundle over the manifold X with projection $p\colon V\to X$. The dimension of the fibers may vary over connected components of X.

iii) φ is a semi-free S^1-action on V which preserves the Spin^c-structure P_V of V.

iv) The fixed point set of φ equals exactly X.

v) The S^1-action defines linear bundle automorphisms of V.

$\mathscr{B}_n$ forms an abelian semi-group under disjoint union. We introduce a natural cobordism relation in $\mathscr{B}_n$. Let B_n be the set of equivalence classes of $\mathscr{B}_n$ under this relation. Then B_n becomes an abelian group.

Lemma 3.2. *The group* $\Omega_n^{\mathrm{Spin}^c}(S^1;\{S^1,1\},\{1\})$ *is isomorphic to* B_n.

Proof. Use similar arguments as in [5].

Take a representative (φ, V, X) of B_n. Let $\{X_i\}$ be the connected components of X and $2q_i=\mathrm{codim}(X_i)$ in V. Put $V_i=p^{-1}(X_i)$ and $p_i=p|V_i$. Since $(V_i)_{S^1}$ is homotopy equivalent to $(X_i)_{S^1}=BS^1\times X_i$, we shall identify $H^2((V_i)_{S^1};\boldsymbol{Z})$ with $H^2(BS^1;\boldsymbol{Z})\oplus H^2(X_i;\boldsymbol{Z})$ as in Proposition 1.2. Then the equivariant ω-class of V_i is given by $\omega^{S^1}(P_{V_i})=l_i\alpha\oplus x_i$ where $l_i\in\boldsymbol{Z}$, $x_i\in H^2(X_i;\boldsymbol{Z})$ and α is the canonical generator of $H^2(BS^1;$

Z). Let $s_i\colon X_i \to V_i$ be the zero-section and consider maps

$$X_i \xleftarrow{p_2} ES^1 \times X_i \xrightarrow{\pi} (X_i)_{S^1} = BS^1 \times X_i .$$

Then

$$\pi^* \omega^{S^1}(s_i^* P_{V_i}) = p_2^* \omega(s_i^* P_{V_i}) = p_2^* s_i^* \omega(P_{V_i}) = p_2^* s_i^* \omega_{V_i} .$$

By Lemma 1.1, $x_i = s_i^* \omega_{V_i}$.

Let $W_i = \partial V_i \times D^2 / \sim$ where $(v, a) \sim (gv, ga)$ for $g \in S^1$ and $K_i = V_i \cup (-W_i)$ where we identify ∂V_i with ∂W_i via $v \mapsto [v, 1]$. W_i has a natural S^1-action $g[v, a] = [gv, a]$ and K_i has an S^1-action compatible with those on V_i and W_i. From the arguments just before Lemma 2.2, we see that K_i has a natural Spin^c-structure P_{K_i} and the S^1-action preserves the Spin^c-structure. We have the following diagram of maps.

$$\begin{tikzcd}
\overline{\partial V_i} \arrow[r, shift right, "j_i"'] \arrow[drr, "\bar{\mu}_i"'] & & W_i \arrow[ll, shift right, "\pi_i"'] \arrow[d, "h_i"] & \partial V_i \arrow[l, "e_i"'] \arrow[d, "k_i"] \arrow[dl, dashed, "\mu_i"'] \\
K_i \arrow[rr, "\rho_i"'] & & X_i \arrow[r, shift right, "s_i"'] & V_i \arrow[l, shift right, "p_i"']
\end{tikzcd}$$

where e_i, j_i, k_i and s_i are inclusions and $\pi_i, \mu_i, p_i, \bar{\mu}_i, h_i$ and ρ_i are projections of bundles. We will compute $\omega^{S^1}(P_{K_i})$ using the Mayer-Victoris sequence of the triad $((K_i)_{S^1}; (V_i)_{S^1}, (W_i)_{S^1})$.

$$0 \longrightarrow H^2((K_i)_{S^1}) \xrightarrow{s} H^2((V_i)_{S^1}) \oplus H^2((W_i)_{S^1}) \xrightarrow{t} H^2((\partial V_i)_{S^1}) \longrightarrow 0$$

If we identify $H^2((V_i)_{S^1}) \oplus H^2((W_i)_{S^1})$ with $H^2((BS^1 \times X_i) \oplus H^2(BS^1 \times \overline{\partial V_i})$ by the natural isomorphism induced by the homotopy equivalences, we see that

$$t((n_1 \alpha \otimes 1 + 1 \otimes x) \oplus (n_2 \alpha \otimes 1 + 1 \otimes y)) = (n_1 - n_2) c + \bar{\mu}_i^* x - y .$$

If we put $\omega^{S^1}(P_{V_i}) = l_i \alpha + s_i^* \omega_{V_i}$, then by Lemma 2.5,

$$s(\omega^{S^1}(P_{K_i})) = (l_i \alpha \otimes 1 + 1 \otimes s_i^* \omega_{V_i}) \oplus (-\alpha \otimes 1 + 1 \otimes (\omega_{\overline{\partial V_i}} + c)) .$$

Since $ts(\omega^{S^1}(P_{K_i})) = 0$, we have

$\omega_{\partial V_i} = l_i c + \bar{\mu}_i^* s_i^* \omega_{V_i}$ and

$$s(\omega^{S^1}(P_{K_i}))=(l_i\alpha\otimes 1+1\otimes s_i^*\omega_{V_i})\oplus(-\alpha\otimes 1+1\otimes((l_i+1)c+\bar{\mu}_i^*s_i^*\omega_{V_i})$$

$$=(l_i+1)(\alpha\otimes 1\oplus 1\otimes c)-(\alpha\otimes 1\oplus\alpha\otimes 1)$$

$$+((1\otimes s_i^*\omega_{V_i})\oplus(1\otimes\bar{\mu}_i^*s_i^*\omega_{V_i}))\,.$$

By Lemma 1.1, we can compute $\omega(P_{K_i})$ from the maps

$$K_i \xleftarrow{p_2} ES^1\times K_i \xrightarrow{\pi} (K_i)_{S^1}\,.$$

Then it can be shown that

$$(p_2^*)^{-1}\pi^*s^{-1}(\alpha\otimes 1\oplus 1\otimes c)=\beta_i$$

$$(p_2^*)^{-1}\pi^*s^{-1}(\alpha\otimes 1\oplus\alpha\otimes 1)=0$$

$$(p_2^*)^{-1}\pi^*s^{-1}((1\otimes s_i^*\omega_{V_i})\oplus(1\otimes\bar{\mu}_i^*s_i^*\omega_{V_i}))=\rho_i^*s_i^*\omega_{V_i}.$$

Here β_i is the Euler class of the canonical S^1-bundle over K_i. Hencc we have $\omega_{K_i}=(l_i+1)\beta_i+\rho_i^*s_i^*\omega_{V_i}$. When we consider the second Stiefel-Whitney class of $\rho_i^{-1}(x)$ $(x\in X_i)$, it is seen that $w_2(\rho_i^{-1}(x))=(l_i+1)(\beta_i|\rho_i^{-1}(x))$ modulo 2. On the other hand, since $\rho_i^{-1}(x)$ is diffeomorphic to CP^{q_i} where $2q_i=\mathrm{codim}(X_i)$, we have $l_i\equiv q_i$ modulo 2. Thus we have proven the lemma below.

Lemma 3.3. *Let $(\varphi, V, X)\in\mathscr{B}_n$. For each component X_i of X, let $2q_i=\mathrm{codim}\,(X_i)$ in V, then $\omega^{S^1}(P_{V_i})=l_i\oplus s_i^*\omega_{V_i}$ for some integer l_i satisfying $l_i\equiv q_i\,(\mathrm{mod.}\ 2)$ and K_i has a natural* Spin^c*-structure with $\omega_{K_i}=(l_i+1)\beta_i+\rho_i^*s_i^*\omega_{V_i}$. The natural semi-free S^1-action on K_i preserves this* Spin^c*-structure.*

Henceforth we put $m_i=(l_i-q_i)/2$ in the remainder of this paper.

Using this lemma, we can clarify the structure of Spin^c-manifolds with semi-free S^1-actions.

Theorem 3.4.

$$\Omega_n^{\mathrm{Spin}^c}(S^1;\{S^1, 1\}, \{1\})\cong\sum_{q\geq 1}\Omega_{n-2q}^{\mathrm{Spin}^c}(\boldsymbol{Z}\times BU(q))$$

$$\Omega_n^{\mathrm{Spin}^c}(S^1;\{S^1, 1\})\cong\Omega_{n-2}^{\mathrm{Spin}^c}(\boldsymbol{Z}^*\times BU(1))+\sum_{q>1}\Omega_{n-2q}^{\mathrm{Spin}^c}(\boldsymbol{Z}\times BU(q))$$

where $\boldsymbol{Z}^*=\boldsymbol{Z}-\{0\}$ *and* $\Omega_n^{\mathrm{Spin}^c}(\cdot)$ *is the bordism group associated to the spectrum* $M\mathrm{Spin}^c(k)$.

Proof. Take an element (φ, V, X) of $\mathscr{B}_n$ which can be regarded as a representative of $\Omega_n^{\mathrm{Spin}^c}(S^1; \{S^1, 1\}, \{1\})$ by Lemma 3.2. For each component X_i of X, V_i has a complex structure defined by the S^1-action. V_i with this complex structure is written by V_i^c. Then X_i has a Spin^c-structure and the correspondence $(\varphi, V, X)\to\{(X_i, V_i^c, m_i)\}_i$ defines a well-defined homomorphism

$$\Phi\colon \Omega_n^{\mathrm{Spin}^c}(S^1; \{S^1, 1\}, \{1\}) \longrightarrow \sum_{q\geqq 1} \Omega_{n-2q}^{\mathrm{Spin}^c}(\boldsymbol{Z}\times BU(q)).$$

In order to show that Φ is an isomorphism of abelian groups, we shall construct an inverse Ψ of Φ. Take a representative (X, V, m) of $\Omega_{n-2q}^{\mathrm{Spin}^c}(\boldsymbol{Z}\times BU(q))$ where V is a complex q-dimensional vector bundle over an $(n-2q)$-dimensional connected manifold X with a Spin^c-structure and $m\in\boldsymbol{Z}$. Let $p\colon V\to X$ be the projection. Since $\tau_V=p^*\tau_X\oplus p^*V$, we have a $\mathrm{Spin}^c(n-2q)\times U(q)$-structure $P_1\oplus P_2$ on V. Let the S^1-action on P_2 be given by a homomorphism $f\colon S^1\to U(q)$ in the sense of Conner and Floyd [5]. Then define $f'\colon S^1\to SO(2q)\times SO(2)$ by $f'(z)=(rf(z), z^{\deg(f)+2m})$ where $r\colon U(q)\to SO(2q)$ is the canonical injective homomorphism. It is known ([1]) that f' lifts to a homomorphism $f''\colon S^1\to\mathrm{Spin}^c(2q)$. Letting S^1 act on P_1 trivially and on P_2 by f'', we get an S^1-action on the Spin$^c(n)$-extension P_V of $P_1\oplus P_2$. Then we define a homomorphism

$$\Psi\colon \sum_{q\geqq 1} \Omega_{n-2q}^{\mathrm{Spin}^c}(\boldsymbol{Z}\times BU(q)) \longrightarrow \Omega_n^{\mathrm{Spin}^c}(S^1; \{S^1, 1\}, \{1\})$$

by $\Psi[X, V, m]=[\varphi, V, X]$. Then it is easy to see that $\Phi\Psi=$identity by Proposition 1.2.

Conversely, let $[X, V^c, m]=\Phi([\varphi, V, X])$. From the construction of Ψ and Φ we see that the Spin^c-structures on V are equal in $\Psi([X, V^c, m])$ and $[\varphi, V, X]$. So we have only to show that the S^1-actions on P_V are equal. Let f and f' be the homomorphisms $S^1\to\mathrm{Spin}^c(n)$ corresponding to $[\varphi, V, X]$ and $\Psi([X, V^c, m])$ respectively. Since f and f' induce the same action on the tangent frame bundle F_V of V, $\phi^c\circ f=\phi^c\circ f'$ holds where ϕ^c is the canonical projection $\mathrm{Spin}^c(n)\to\mathrm{SO}(n)$.

But since $\deg(\det^c \circ f)=\deg(\det^c \circ f')=q+2m$, f and f' must be conjugate and therefore homotopic by a homotopy of homomorphisms (see the Appendix). This homotopy gives a cobordism. Hence $\Psi([X, V^c, m])=[\varphi, V, X]$ proving that $\Psi\Phi=$identity. Proposition 2.3 together with Lemma 3.1 shows that we have a splitting (also denoted by ψ):

$$\psi\colon\ \Omega_{n-2}^{\mathrm{Spin}^c}(BU(1)) \longrightarrow \sum_{q\geqq 1} \Omega_{n-2q}^{\mathrm{Spin}^c}(\boldsymbol{Z}\times BU(q)).$$

But by Lemma 2.5, we know that the image of ψ is given by $q=1$, $m=0$. This completes the proof.

§4. Todd Genus Formula for Semi-free S^1-actions on Spin^c-manifolds and its Applications

Take a representative $(\varphi, V, X)=\{(\varphi, V_i, X_i)\}_i$ of $\Omega_n^{\mathrm{Spin}^c}(S^1; \{S^1, 1\}, \{1\})\cong B_n$. In the argument of Lemma 3.3, we have manifolds $\{K_i\}$ with semi-free S^1-actions which preserve the Spin^c-structures $\{P_{K_i}\}$. This defines a homomorphism

$$b\colon\ \Omega_n^{\mathrm{Spin}^c}(S^1; \{S^1, 1\}, \{1\}) \longrightarrow \Omega_n^{\mathrm{Spin}^c}(S^1; \{S^1, 1\})$$

which is clearly a left inverse of β.

Let (φ, M^n, P_M) be a Spin^c-manifold with a semi-free S^1-action φ with fixed point set components $\{X_i\}$ and their closed tubular neighborhoods $\{V_i\}$. Then $[\varphi, M^n, P_M]=\sum_i b[\varphi, V_i, X_i]=\sum_i [\varphi, K_i, P_{K_i}]$ in $\Omega_n^{\mathrm{Spin}^c}(S^1; \{S^1, 1\})$. Recall that the $\hat{\mathfrak{A}}$-class is defined by a multiplicative sequence of polynomials associated to $(\sqrt{z}/2)/(\sinh(\sqrt{z}/2))$. $\hat{\mathfrak{A}}(M; d)=\exp(d)\hat{\mathfrak{A}}(M)$ is defined for $d\in H^2(M; \boldsymbol{Q})$ and is called the generalized Todd class $\tilde{\mathscr{T}}(M)$ when M^n is a Spin^c-manifold and $d=\omega_M/2$.

In our case, $\tilde{T}(M)=\tilde{\mathscr{T}}(M)[M]$ is given by $\tilde{T}(M)=\sum_i \tilde{T}(K_i)$. We shall follow the line of Borel and Hirzebruch [4] §22 to compute each $\tilde{T}(K_i)=\hat{\mathfrak{A}}(K_i; \omega_{K_i}/2)[K_i]$. The normal bundle ν_i of X_i in M^n has a natural complex structure induced by the given S^1-action. Then the bundle $\hat{\rho}_i$ along the fibers of ρ_i has a natural almost complex structure and by [4] §7 and §15, we have an isomorphism of complex vector bundles over K_i:

$$\hat{\rho}_i \oplus 1_{\mathbf{C}} \cong \rho_i^*(\nu_i \oplus 1_{\mathbf{C}}) \otimes \eta_i$$

where η_i is the canonical complex line bundle over K_i with $c_1(\eta_i)=\beta_i$. Hence $c_1(\hat{\rho}_i)=\rho_i^*(c_1(\nu_i))+(q_i+1)\beta_i$ and $\hat{\mathfrak{A}}(\hat{\rho}_i)=\exp(-(\rho_i^*(c_1(\nu_i))+(q_i+1)\beta_i)/2)\mathscr{T}(\hat{\rho}_i)$ where $\mathscr{T}$ is the usual complex Todd class defined by $z/(1-\exp(-z))$.

$$\tilde{T}(K_i)=\exp(\omega_{K_i}/2)\hat{\mathfrak{A}}(K_i)[K_i]$$

$$=\rho_{i\#}(\exp(\omega_{K_i}/2)\hat{\mathfrak{A}}(K_i))[X_i]$$

where $\rho_{i\#}$ is the Gysin homomorphism induced by the projection

$$\rho_i\colon K_i \longrightarrow X_i.$$

Using the fact that $\hat{\mathfrak{A}}(K_i)=\rho_i^*\hat{\mathfrak{A}}(X_i)\hat{\mathfrak{A}}(\hat{\rho}_i)$ and $\omega_{K_i}=(q_i+2m_i+1)\beta_i+\rho_i^*s_i^*\omega_M$ by Lemma 3.3, we have

$$\tilde{T}(K_i)=\rho_{i\#}(\exp(m_i\beta_i)\mathscr{T}(\hat{\rho}_i))\hat{\mathfrak{A}}(X_i;\,(s_i^*\omega_M-c_1(\nu_i))/2)[X_i]$$

where $s_i\colon X_i \to M$ is the inclusion map.
We can calculate $\rho_{i\#}(\exp(m_i\beta_i)\mathscr{T}(\hat{\rho}_i))$ by the methods developed in [4] §22. As a consequence, we get the following results.

$$\tilde{T}(K_i)=\begin{cases}(1+ch\bar{\nu}_i)^{m_i}\hat{\mathfrak{A}}(X_i;\,(s_i^*\omega_M-c_1(\nu_i))/2)[X_i] & \text{if } m_i\geqq 0\\ (1+ch\nu_i)^{-(m_i+q_i+1)}\hat{\mathfrak{A}}(X_i;\,(s_i^*\omega_M-c_1(\bar{\nu}_i))/2)[X_i] & \\ & \text{if } m_i\leqq -(q_i+1)\\ 0 \qquad \text{if } -q_i\leqq m_i\leqq -1. & \end{cases}$$

Here $\bar{\nu}_i$ is the complex conjugate of ν_i. Thus we have obtained the following formula for semi-free S^1-actions on Spin^c-manifolds.

Theorem 4.1. (*Todd genus formula*). *Suppose that S^1 acts semi-freely on a* Spin^c*-manifold M^n preserving its* Spin^c*-structure. Then the generalized Todd genus of M is given by*

$$\tilde{T}(M)=\sum_{m_i\geqq 0}(1+ch\bar{\nu}_i)^{m_i}\hat{\mathfrak{A}}(X_i;\,(s_i^*\omega_M-c_1(\nu_i))/2)[X_i]$$
$$+\sum_{m_i\leqq-(q_i+1)}(1+ch\nu_i)^{-(m_i+q_i+1)}\hat{\mathfrak{A}}(X_i;\,(s_i^*\omega_M-c_1(\bar{\nu}_i))/2)[X_i]$$

where $\{X_i\}$ are fixed point set components of the action.

Now we are in a position to apply the Todd genus formula for manifolds which admit semi-free S^1-actions. We shall begin with Spin-manifolds.

Theorem 4.2. (*Atiyah and Hirzebruch* [3]). *If a connected* Spin-*manifold M^n admits a nontrivial semi-free S^1-action, then $\hat{A}(M)=0$.*

Proof. Suppose that S^1 acts semi-freely on $(M^n, \tilde{P}_M)$ where $\tilde{P}_M$ is the Spin-structure of M^n. Let X_i be the fixed point set components and $\tilde{P}_i=\tilde{P}_M|X_i$. Consider the $\mathrm{Spin}^c(n)$-extensions $P_M=\tilde{P}_M\times_{\mathrm{Spin}(n)}\mathrm{Spin}^c(n)$ and $P_i=\tilde{P}_i\times_{\mathrm{Spin}(n)}\mathrm{Spin}^c(n)$. Then $P_i=P_M|X_i$ and $(P_i)_{S^1}$ is also a $\mathrm{Spin}^c(n)$-extension of the $\mathrm{Spin}(n)$-bundle $(\tilde{P}_i)_{S^1}$, and we have $l_i=0$ for each i by Proposition 1.2. Hence $m_i=(l_i-q_i)/2$ satisfies $-q_i\leqq m_i\leqq -1$. Consequently, $\hat{A}(M)=\tilde{T}(M)=0$ by the Todd genus formula.

Remark: It seems wothwhile noting that in the Spin case each term $\tilde{T}(K_i)$ vanishes if the action is semi-free.

Next we shall consider semi-free S^1-actions on almost complex manifolds. Let M^n be an almost complex manifold and suppose that we are given a semi-free S^1-action on M^n which preserves the almost complex structure U_M whose structure group is $U(p)$ where $2p=n$. Then the normal bundle of each fixed point set component X_i has a decomposition $\nu_i=\nu_i^+\oplus\nu_i^-$ of complex vector bundles where $g\in S^1\in C$ acts on ν_i^+ (resp. ν_i^-) as the multiplication of the complex number g (resp. g^{-1}). Then, if we put $d_i^+=\dim_C\nu_i^+$ and $d_i^-=\dim_C\nu_i^-$, $d_i^++d_i^-=q_i=\mathrm{codim}\,(X_i)/2$.

Theorem 4.3. (*Kosniowski* [8]). *If an almost complex manifold M^n admits an almost complex semi-free S^1-action, then its Todd genus is given by*

$$T(M)=\sum_{d_i^+=q_i}T(X_i)=\sum_{d_i^-=q_i}T(X_i).$$

Proof. Around a fixed point set component X_i, the S^1-action can be expressed ([5]) by the map $f_i\colon S^1\to U(p)$ where

$$f_i(g)=\begin{pmatrix} \overbrace{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}}^{p-q_i} & & \\ & \overbrace{\begin{matrix}g & & \\ & \ddots & \\ & & g\end{matrix}}^{d_i^+} & \\ & & \overbrace{\begin{matrix}g^{-1} & & \\ & \ddots & \\ & & g^{-1}\end{matrix}}^{d_i^-} \end{pmatrix}.$$

Then by Proposition 1.2, $l_i=\deg(\det(f_i))=d_i^+-d_i^-$ hence $m_i=-d_i^-$. The first part of the theorem follows from the Todd genus formula since the non-vanishing terms occur when $m_i=0$ namely $d_i^+=q_i$. For the second equality, consider the reversed S^1-action given by $(g, x)\to g^{-1}x$ $(g\in S^1, x\in M)$ and apply the Todd genus formula with $m_i=-d_i^+$.

Appendix

We shall present necessary elementary facts on the group $\mathrm{Spin}^c(n)$. The main reference here is Husemoller [7].

Let $T(\mathbf{R}^n)$ be the tensor algebra over $\mathbf{R}^n$. The standard orthonormal basis of $\mathbf{R}^n$ is expressed by $(e_1,\dots, e_n)$.

Definition. Clifford algebra C_n is an $\mathbf{R}$-algebra defined by $T(\mathbf{R}^n)/I$ where I is the ideal generated by elements of the form $x\otimes x+<x, x>1$ $(x\in\mathbf{R}^n)$. Here, $<\ ,\ >$ is the standard inner product in $\mathbf{R}^n$.

There is a canonical inclusion $\mathbf{R}^n\to C_n$ and we shall consider $\mathbf{R}^n$ as embedded in C_n. Then C_n is an $\mathbf{R}$-algebra with generators 1, e_1, $e_2,\dots, e_n$ and relations are:

$$e_ie_j+e_je_i=0 \qquad (i\neq j)$$

$$(e_i)^2=-1\,.$$

C_n is a $\boldsymbol{Z}_2$-graded algebra with decomposition:

$$C_n=C_n^0\oplus C_n^1\,.$$

We define a linear involution on C_n by

$$(x_1 \dots x_p)^* = x_p \dots x_1 \qquad (x_i \in \mathbf{R}^n).$$

Definition. $\mathrm{pin}(n)$ is the subgroup of C_n generated by S^{n-1} in the units of C_n and $\mathrm{Spin}(n)$ is $\mathrm{pin}(n) \cap C_n^0$.

Definition. ϕ: $\mathrm{Spin}(n) \to SO(n)$ is defined by $\phi(u)(x) = uxu^*$.

ϕ is the well known double covering of $SO(n)$. Next, consider the map ϕ: $\mathrm{Spin}(n+2) \to SO(n+2)$ and the subgroup $SO(n) \times SO(2)$ of $SO(n+2)$.

Definition. $\mathrm{Spin}^c(n) = \phi^{-1}(SO(n) \times SO(2))$.

$$\phi^c = p_1 \circ \phi \colon \mathrm{Spin}^c(n) \longrightarrow SO(n) \times SO(2) \longrightarrow SO(n)$$

$$\det{}^c = p_2 \circ \phi \colon \mathrm{Spin}^c(n) \longrightarrow SO(n) \times SO(2) \longrightarrow SO(2).$$

It is known that we have a commutative diagram of homomorphisms (see e.g. [1]):

$$\begin{tikzcd} U(n) \arrow[r, "\det"] \arrow[d, "r"'] \arrow[dr, "l"] & SO(2) \\ SO(2n) & \mathrm{Spin}^c(2n) \arrow[l, "\phi^c"] \arrow[u, "\det^c"'] \end{tikzcd}$$

where r is the canonical injection.

Proposition. *Let f, g: $S^1 \to \mathrm{Spin}^c(n)$ be homomorphisms such that there exists an element $\alpha \in SO(n)$ with $\alpha(\phi^c \circ f(z))\alpha^{-1} = \phi^c \circ g(z)$ for all $z \in S^1$. Then there exists an element u in $\mathrm{Spin}^c(n)$ with $u(f(z))u^{-1} = g(z)$ for all $z \in S^1$ if and only if $\det^c \circ f = \det^c \circ g$.*

Proof. Let $\det^c \circ f = \det^c \circ g$ and take $u \in \mathrm{Spin}^c(n)$ so that $\phi^c(u) = \alpha$. Then $h(z) = uf(z)u^{-1}$ is a homomorphism $S^1 \to \mathrm{Spin}^c(n)$ with $\phi \circ h = \phi \circ g$. Since ϕ has discrete kernel, h and g must coincide. The converse is trivial.

Remark. Under the conditions of this proposition, f and g are homotopic by a homotopy of homomorphisms since $\mathrm{Spin}^c(n)$ is path-connected.

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