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K-Theory, Hermitian K-Theory and the Karoubi Tower

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Abstract. The goal of this paper is to present a new view on the link between the well known K-theory of finitely generated projective modules and much less understood K-theory of Hermitian forms on finitely generated projective modules.

For a given Hermitian ring (R, α, ϵ), we obtain the K-theory space KR equipped with an involution and the Hermitian K-theory space $K\mathrm{Herm}(R, \alpha, \epsilon)$. The fixed subspace $KR^{\mathbb{Z}_2}$ is homotopy equivalent to the Hermitian K-theory space $K\mathrm{Herm}(R, \alpha, \epsilon)$. We construct the Karoubi Tower diagram, which is obtained by iterating Karoubi's construction of the homotopy fibers of the forgetful and hyperbolic maps. Using an interesting factorization of these maps, we prove various homotopy properties of the Karoubi Tower. The homotopy inverse limit of the Karoubi Tower is homotopy equivalent to the homotopy fibre of the inclusion of the fixed set $KR^{\mathbb{Z}_2}$ into the homotopy fixed set $KR^{h\mathbb{Z}_2}$. Considering Karoubi's fundamental periodicity theorem, the Karoubi Tower generalizes the low dimensional connections between Hermitian K-theory and K-theory groups. Illustrative examples of the Karoubi Tower are given by the finite field case and the classical Hermitian rings over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. Considering topological K-theory for these cases, the Karoubi Tower comprises the classical Bott periodicity.

Another important application of the Karoubi Tower is an elegant and comprehensive generalization of the classical invariants of quadratic forms.

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0. Introduction

Let (R, α, ϵ) be a Hermitian ring, i.e. R is a ring with unit element 1, α is an anti-involution and ϵ is a central element, such that $\alpha(\epsilon)\epsilon = 1$. Let $\mathcal{P}(R)$ be the category of finitely generated projective left R-modules and let $\mathcal{H}(R, \alpha, \epsilon)$ be the category of Hermitian modules, that is the category of finitely generated projective left R-modules with an (R, α, ϵ)-Hermitian form. Let KR and $K\mathrm{Herm}(R, \alpha, \epsilon)$ be the associated K-theory spaces. KR and $K\mathrm{Herm}(R, \alpha, \epsilon)$ are infinite loop spaces. α induces an involution on KR and the fixed subspace $KR^{\mathbb{Z}_2}$ is homotopy equivalent to $K\mathrm{Herm}(R, \alpha, \epsilon)$. The lower K-theory groups of $\mathcal{P}(R)$, were extensively studied before Quillen's definition of the higher K-theory groups. Similarly, the lower $K\mathrm{Herm}$-theory groups, i.e. (Hermitian) forms were extensively studied by Bass, Bak, Scharlau, Hahn-O'Meara, Wall and others.