Localization of Model Categories

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CHAPTER 1

Local spaces and localization

1.1. Spaces and function spaces

We will be working simultaneously in several different categories of spaces, and a central question will be whether a map of spaces induces a weak equivalence of mapping spaces. In order to make statements that are valid in each of our categories, we will refer uniformly to the *simplicial mapping space* (i.e., the simplicial set of maps) between two spaces no matter what the category of spaces. Notation 1.1.2 describes our categories of spaces, and Definition 1.1.5 describes the simplicial mapping space. Corollary 1.1.9 implies that a map of spaces induces a weak equivalence of these simplicial mapping spaces if and only if it induces a weak equivalence of the usual internal mapping spaces.

1.1.1. Our categories of spaces. We will be working with both topological spaces and simplicial sets, and for each of these we will consider both the category of pointed spaces and the category of unpointed spaces. In order to keep the terminology concise, the word *space* will be used to mean either a topological space or a simplicial set, and we will use the following notation for our categories of spaces.

NOTATION 1.1.2. We will use the following notation for our categories of spaces:

SS : The category of simplicial sets.
SS_* : The category of pointed simplicial sets.
Top : The category of compactly generated Hausdorff topological spaces.
Top_*: The category of pointed compactly generated Hausdorff topological spaces.

Since much of our discussion will apply to more than one of these categories, we will use the following notation:

$\mathsf{SS}_{(*)}$: Either SS or SS_*.
$\mathsf{Top}_{(*)}$: Either Top or Top_*.
Spc : A category of unpointed spaces, i.e., either Top or SS.
Spc_* : A category of pointed spaces, i.e., either Top_* or SS_*.
$\mathsf{Spc}_{(*)}$: Any of the categories SS, SS_*, Top, or Top_*.

DEFINITION 1.1.3. We will use the notions of fibration, cofibration, and weak equivalence appropriate to the standard model category structures on $\mathsf{SS}_{(*)}$ and $\mathsf{Top}_{(*)}$ (see Theorem 10.1.4). Thus,

- A *fibration* of simplicial sets is a Kan fibration (see, e.g., [**43**, page 25]), a *cofibration* of simplicial sets is an inclusion map, and a *weak equivalence* of simplicial sets is a map whose geometric realization is a homotopy equivalence.
- A *fibration* of topological spaces is a Serre fibration, a *cofibration* of topological spaces is a retract of a relative cell complex (see Definition 2.2.1 or

[**46**, Part II Section 3]), and a *weak equivalence* of topological spaces is a map whose total singular complex is a homotopy equivalence.

1.1.4. Function spaces. Given spaces X and Y in $\mathsf{Spc}_{(*)}$, we will need two spaces of maps from X to Y. The first is the *simplicial set of maps* from X to Y, which is the simplicial mapping space used as part of the usual simplicial model category structure on $\mathsf{Spc}_{(*)}$ (see Definition 10.1.2). The second is an *internal mapping space*, i.e., an object of $\mathsf{Spc}_{(*)}$. These two mapping spaces are closely related (see Proposition 1.1.7). In particular, if $\mathsf{Spc}_{(*)} = \mathsf{SS}$, then these two mapping spaces are the same.

DEFINITION 1.1.5 (Simplicial mapping spaces). Let X and Y be spaces in $\mathsf{Spc}_{(*)}$.

- If $\mathsf{Spc}_{(*)} = \mathsf{SS}$, then $\mathrm{Map}(X,Y)$ is the simplicial set with n-simplices the simplicial maps $X \times \Delta[n] \to Y$ and face and degeneracy maps induced by the standard maps between the $\Delta[n]$.
- If $\mathsf{Spc}_{(*)} = \mathsf{SS}_*$, then $\mathrm{Map}(X,Y)$ is the simplicial set with n-simplices the basepoint preserving simplicial maps $X \wedge \Delta[n]^+ \to Y$ and face and degeneracy maps induced by the standard maps between the $\Delta[n]$.
- If $\mathsf{Spc}_{(*)} = \mathsf{Top}$, then $\mathrm{Map}(X,Y)$ is the simplicial set with n-simplices the continuous functions $X \times \big|\Delta[n]\big| \to Y$ and face and degeneracy maps induced by the standard maps between the $\Delta[n]$.
- If $\mathsf{Spc}_{(*)} = \mathsf{Top}_*$, then $\mathrm{Map}(X,Y)$ is the simplicial set with n-simplices the continuous functions $X \wedge \big|\Delta[n]\big|^+ \to Y$ and face and degeneracy maps induced by the standard maps between the $\Delta[n]$.

Note that, in all cases, $\mathrm{Map}(X,Y)$ is an *unpointed* simplicial set.

DEFINITION 1.1.6 (Internal mapping spaces). Let X and Y be spaces in $\mathsf{Spc}_{(*)}$.

- If $\mathsf{Spc}_{(*)} = \mathsf{SS}$, then the internal mapping space Y^X equals the simplicial mapping space $\mathrm{Map}(X,Y)$ (see Definition 1.1.5).
- If $\mathsf{Spc}_{(*)} = \mathsf{SS}_*$, then Y^X is the pointed simplicial set with n-simplices the basepoint preserving simplicial maps $X \wedge \Delta[n]^+ \to Y$, and face and degeneracy maps induced by the standard maps between the $\Delta[n]$. When we need to emphasize the category in which we work, we will use the notation $\mathrm{Map}_*(X,Y)$ for the pointed simplicial set of basepoint preserving maps.
- If $\mathsf{Spc}_{(*)} = \mathsf{Top}$, then Y^X is the topological space of continuous functions from X to Y with the compactly generated compact open topology. When we need to emphasize the category in which we work, we will use the notation $\mathrm{map}(X,Y)$ for the unpointed topological space of continuous functions.
- If $\mathsf{Spc}_{(*)} = \mathsf{Top}_*$, then Y^X is the pointed topological space of basepoint preserving continuous functions from X to Y with the compactly generated compact open topology. When we need to emphasize the category in which we work, we will use the notation $\mathrm{map}_*(X,Y)$ for the pointed topological space of basepoint preserving continuous functions.

PROPOSITION 1.1.7. *The internal mapping spaces Y^X of Definition 1.1.6 are related to the simplicial mapping spaces $\mathrm{Map}(X,Y)$ of Definition 1.1.5 as follows:*

- *If $\mathsf{Spc}_{(*)} = \mathsf{SS}$, then $\mathrm{Map}(X,Y)$ equals Y^X.*
- *If $\mathsf{Spc}_{(*)} = \mathsf{SS}_*$, then $\mathrm{Map}(X,Y)$ is obtained from Y^X by forgetting the basepoint.*

- *If* $\mathsf{Spc}_{(*)} = \mathsf{Top}$*, then the simplicial set* $\mathrm{Map}(X, Y)$ *is the total singular complex of* Y^X*.*
- *If* $\mathsf{Spc}_{(*)} = \mathsf{Top}_*$*, the simplicial set* $\mathrm{Map}(X, Y)$ *is the total singular complex of the unpointed space obtained from* Y^X *by forgetting the basepoint.*

PROOF. This follows from the natural isomorphisms of sets

$$\mathsf{Top}\big(\big|\Delta[n]\big|, Y^X\big) \approx \mathsf{Top}\big(X \times \big|\Delta[n]\big|, Y\big)$$
$$\mathsf{Top}_*\big(\big|\Delta[n]\big|^+, Y^X\big) \approx \mathsf{Top}_*\big(X \wedge \big|\Delta[n]\big|^+, Y\big).$$

□

COROLLARY 1.1.8. *If* $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$*, then* $\mathrm{Map}(X, Y)$ *is always a fibrant simplicial set.*

PROOF. Since the total singular complex of a topological space is always fibrant, this follows from Proposition 1.1.7. □

COROLLARY 1.1.9. *Let* $g \colon W \to X$ *and* $h \colon Y \to Z$ *be maps in* $\mathsf{Spc}_{(*)}$*.*

1. *The map* $h_* \colon Y^X \to Z^X$ *is a weak equivalence in* $\mathsf{Spc}_{(*)}$ *if and only if the map* $h_* \colon \mathrm{Map}(X, Y) \to \mathrm{Map}(X, Z)$ *is a weak equivalence of simplicial sets.*
2. *The map* $g^* \colon Y^X \to Y^W$ *is a weak equivalence in* $\mathsf{Spc}_{(*)}$ *if and only if the map* $g^* \colon \mathrm{Map}(X, Y) \to \mathrm{Map}(W, Y)$ *is a weak equivalence of simplicial sets.*

PROOF. Since a map of pointed spaces is a weak equivalence if and only if it is a weak equivalence of unpointed spaces after forgetting the basepoint, and a map of topological spaces is a weak equivalence if and only if its total singular complex is a weak equivalence of simplicial sets, this follows from Proposition 1.1.7. □

1.1.10. Topological spaces and simplicial sets.

DEFINITION 1.1.11. If X and Y are objects of $\mathsf{Spc}_{(*)}$ and K is a simplicial set, then $X \otimes K$ and Y^K will denote the objects of $\mathsf{Spc}_{(*)}$ characterized by the natural isomorphisms of simplicial sets

$$\mathsf{Spc}_{(*)}(X \otimes K, Y) \approx \mathsf{SS}\big(K, \mathrm{Map}(X, Y)\big) \approx \mathsf{Spc}_{(*)}(X, Y^K)$$

(see Definition 10.1.2). Thus,

$$\begin{array}{lll} \text{If } \mathsf{Spc}_{(*)} = \mathsf{SS}, & X \otimes K = X \times K & \text{and } X^K = \mathrm{Map}(K, X). \\ \text{If } \mathsf{Spc}_{(*)} = \mathsf{SS}_*, & X \otimes K = X \wedge K^+ & \text{and } X^K = \mathrm{Map}_*(K^+, X). \\ \text{If } \mathsf{Spc}_{(*)} = \mathsf{Top}, & X \otimes K = X \times \big|K\big| & \text{and } X^K = \mathrm{map}\big(\big|K\big|, X\big). \\ \text{If } \mathsf{Spc}_{(*)} = \mathsf{Top}_*, & X \otimes K = X \wedge \big|K\big|^+ & \text{and } X^K = \mathrm{map}_*\big(\big|K\big|^+, X\big) \end{array}$$

(see Definition 1.1.6).

LEMMA 1.1.12. *If* X *is a space in* $\mathsf{SS}_{(*)}$ *and* K *is a simplicial set, then there is a natural homeomorphism* $\big|X \otimes K\big| \approx \big|X\big| \otimes K$*.*

PROOF. Since $\mathsf{Top}_{(*)}$ is the category of *compactly generated* Hausdorff spaces, the natural map $\big|X \times K\big| \to \big|X\big| \times \big|K\big|$ is a homeomorphism. □

LEMMA 1.1.13. *If* K *is a simplicial set and* W *is a topological space (either both pointed or both unpointed), then the standard adjunction of the geometric* *realization and total singular complex functors extends to a natural isomorphism of simplicial mapping spaces*

$$\mathrm{Map}(|K|, W) \approx \mathrm{Map}(K, \mathrm{Sing}\, W).$$

PROOF. This follows from the natural homeomorphism $|K \otimes \Delta[n]| \approx |K| \otimes |\Delta[n]|$ (see Definition 1.1.11). □

PROPOSITION 1.1.14. *If A and X are objects of $\mathsf{SS}_{(*)}$ and X is fibrant, then there is a natural weak equivalence of simplicial sets*

$$\mathrm{Map}(A, X) \cong \mathrm{Map}(|A|, |X|).$$

PROOF. Since all simplicial sets are cofibrant, the natural map $X \to \mathrm{Sing}|X|$ induces a weak equivalence $\mathrm{Map}(A, X) \cong \mathrm{Map}(A, \mathrm{Sing}|X|)$ (see Corollary 10.2.2). The proposition now follows from Lemma 1.1.13. □

PROPOSITION 1.1.15. *If A and X are objects of $\mathsf{Top}_{(*)}$ and A is cofibrant, then there is a natural weak equivalence of simplicial sets*

$$\mathrm{Map}(A, X) \cong \mathrm{Map}(\mathrm{Sing}\, A, \mathrm{Sing}\, X).$$

PROOF. Since all topological spaces are fibrant, the natural map $|\mathrm{Sing}\, A| \to A$ induces a weak equivalence $\mathrm{Map}(A, X) \cong \mathrm{Map}(|\mathrm{Sing}\, A|, X)$ (see Corollary 10.2.2). The proposition now follows from Lemma 1.1.13. □

DEFINITION 1.1.16. Each of our categories of spaces has a functor to SS, and this functor has a left adjoint $\mathsf{SS} \to \mathsf{Spc}_{(*)}$, i.e., for an unpointed simplicial set K and an object X of $\mathsf{Spc}_{(*)}$, we have natural isomorphisms

$$\begin{aligned} \mathsf{SS}(K, X) &\approx \mathsf{SS}(K, X) \\ \mathsf{SS}_*(K^+, X) &\approx \mathsf{SS}(K, X^-) \\ \mathsf{Top}(|K|, X) &\approx \mathsf{SS}(K, \mathrm{Sing}\, X) \\ \mathsf{Top}_*(|K|^+, X) &\approx \mathsf{SS}(K, \mathrm{Sing}\, X^-) \end{aligned}$$

where "X^-" means "forget the basepoint of X". If K is an (unpointed) simplicial set, then $\mathsf{Spc}_{(*)}(K)$ will denote the image of K in $\mathsf{Spc}_{(*)}$ under this left adjoint. Thus,

$$\begin{array}{ll} \text{If } \mathsf{Spc}_{(*)} = \mathsf{SS}, & \text{then } \mathsf{Spc}_{(*)}(K) = K. \\ \text{If } \mathsf{Spc}_{(*)} = \mathsf{SS}_*, & \text{then } \mathsf{Spc}_{(*)}(K) = K^+. \\ \text{If } \mathsf{Spc}_{(*)} = \mathsf{Top}, & \text{then } \mathsf{Spc}_{(*)}(K) = |K|. \\ \text{If } \mathsf{Spc}_{(*)} = \mathsf{Top}_*, & \text{then } \mathsf{Spc}_{(*)}(K) = |K|^+. \end{array}$$

EXAMPLE 1.1.17. In the standard model category structure on $\mathsf{Spc}_{(*)}$, a map is defined to be a fibration if it has the right lifting property (see Definition 8.2.1) with respect to the maps $\mathsf{Spc}_{(*)}(\Lambda[n, k]) \to \mathsf{Spc}_{(*)}(\Delta[n])$ for all $n > 0$ and $0 \le k \le n$.

1.2. Local spaces and localization

1.2.1. Definitions.

DEFINITION 1.2.2. Let $f\colon A \to B$ be a map between cofibrant spaces in $\mathsf{Spc}_{(*)}$ (see Notation 1.1.2).

1. A space W is *f-local* if W is fibrant and the induced map of simplicial sets $f^*\colon \operatorname{Map}(B,W) \to \operatorname{Map}(A,W)$ is a weak equivalence. If f is a map $* \to A$, then an f-local space will also be called *A-local* or *A-null*. Bousfield ([**12**]) has used the term *A-periodic* for what we here call A-local.
2. A map $g\colon X \to Y$ is an *f-local equivalence* if there is a cofibrant approximation $\tilde{g}\colon \widetilde{X} \to \widetilde{Y}$ to g (see Definition 9.1.8) such that, for every f-local space W, the induced map of simplicial sets $\tilde{g}^*\colon \operatorname{Map}(\widetilde{Y},W) \to \operatorname{Map}(\widetilde{X},W)$ is a weak equivalence. (Proposition 10.5.2 implies that if this is true for any one cofibrant approximation to g, then it is true for every cofibrant approximation to g.)

If $\mathsf{Spc}_{(*)} = \mathsf{SS}_{(*)}$, then every space is cofibrant, and so a map $g\colon X \to Y$ is an f-local equivalence if and only if, for every f-local space W, the map $g^*\colon \operatorname{Map}(Y,W) \to \operatorname{Map}(X,W)$ is a weak equivalence. If $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$, then all CW-complexes are cofibrant, and so a CW-replacement for a space serves as a cofibrant approximation to that space.

A paraphrase of Definition 1.2.2 is that a fibrant space is f-local if it makes f look like a weak equivalence (see Proposition 10.2.1), and a map is an f-local equivalence if all f-local spaces make it look like a weak equivalence. In Theorem 2.1.2, we show that there is a model category structure on $\mathsf{Spc}_{(*)}$ in which the local spaces are the fibrant objects (see Proposition 2.1.3) and the f-local equivalences are the weak equivalences. For a discussion of the relation of our definition of f-local equivalence to earlier definitions, see Remark 1.2.9.

PROPOSITION 1.2.3. *Let both f and $\tilde{f}$ be maps between cofibrant spaces. If the class of f-local spaces equals the class of $\tilde{f}$-local spaces, then the class of f-local equivalences equals the class of $\tilde{f}$-local equivalences.*

PROOF. This follows directly from the definitions. □

EXAMPLE 1.2.4. Let A be a simplicial set (if $\mathsf{Spc}_{(*)} = \mathsf{SS}_{(*)}$) or a cell complex (if $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$), and let $\mathrm{C}A$ be the cone on A. If $f\colon * \to A$ is the inclusion of a vertex and $\tilde{f}\colon A \to \mathrm{C}A$ is the standard inclusion, then a space is f-local (i.e., A-local; see Definition 1.2.2) if and only if it is $\tilde{f}$-local, and so the class of f-local equivalences equals the class of $\tilde{f}$-local equivalences.

PROPOSITION 1.2.5. *Let $f\colon A \to B$ be a map of cofibrant spaces. If X and Y are fibrant spaces and $g\colon X \to Y$ is a weak equivalence, then X is f-local if and only if Y is f-local.*

PROOF. We have a commutative diagram

$$\begin{tikzcd} \operatorname{Map}(B,X) \arrow[r] \arrow[d, "\cong"'] & \operatorname{Map}(A,X) \arrow[d, "\cong"] \\ \operatorname{Map}(B,Y) \arrow[r] & \operatorname{Map}(A,Y) \end{tikzcd}$$

in which the vertical maps are weak equivalences (see Corollary 10.2.2). Thus, the top map is a weak equivalence if and only if the bottom map is a weak equivalence. □

PROPOSITION 1.2.6. *Let both $f\colon A \to B$ and $\tilde{f}\colon \widetilde{A} \to \widetilde{B}$ be maps between cofibrant spaces. If there are weak equivalences $A \to \widetilde{A}$ and $B \to \widetilde{B}$ such that the square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\cong"'] & B \arrow[d, "\cong"] \\ \widetilde{A} \arrow[r, "\tilde{f}"'] & \widetilde{B} \end{tikzcd}$$

commutes, then

1. *the class of f-local spaces equals the class of $\tilde{f}$-local spaces, and*
2. *the class of f-local equivalences equals the class of $\tilde{f}$-local equivalences.*

PROOF. Proposition 1.2.3 implies that part 1 implies part 2, and so it is sufficient to prove part 1.

If W is a fibrant space, then we have the commutative square

$$\begin{tikzcd} \mathrm{Map}(\widetilde{B}, W) \arrow[r, "\tilde{f}^*"] \arrow[d, "\cong"'] & \mathrm{Map}(\widetilde{A}, W) \arrow[d, "\cong"] \\ \mathrm{Map}(B, W) \arrow[r, "f^*"'] & \mathrm{Map}(A, W) \end{tikzcd}$$

in which the vertical maps are weak equivalences (see Corollary 10.2.2). Thus, f^* is a weak equivalence if and only if $\tilde{f}^*$ is a weak equivalence, and so W is f-local if and only if it is $\tilde{f}$-local. □

REMARK 1.2.7. Proposition 1.2.6 and Proposition 13.2.16 imply that we can always replace our map $f\colon A \to B$ with an inclusion of simplicial sets (if $\mathsf{Spc}_{(*)} = \mathsf{SS}_{(*)}$) or an inclusion of cell complexes (if $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$) without changing the class of f-local spaces or the class of f-local equivalences. We will often assume that we have done this, and we will summarize this assumption by saying that f is an *inclusion of cell complexes.* (This usage is consistent with the definition of *cell complex* in a cofibrantly generated model category (see Definition 13.2.4).)

DEFINITION 1.2.8. Let $f\colon A \to B$ be a map between cofibrant spaces.

1. An *f-localization* of a space X is an f-local space $\widehat{X}$ (see Definition 1.2.2) together with an f-local equivalence $j_X : X \to \widehat{X}$. We will sometimes use the phrase *f-localization* to refer to the space $\widehat{X}$, without explicitly mentioning the f-local equivalence j. A *cofibrant f-localization* of X is an f-localization in which the f-local equivalence is also a cofibration.
2. An *f-localization* of a map $g\colon X \to Y$ is an f-localization $(\widehat{X}, j_X)$ of X, an f-localization $(\widehat{Y}, j_Y)$ of Y, and a map $\hat{g}\colon \widehat{X} \to \widehat{Y}$ such that the square

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "j_X"'] & Y \arrow[d, "j_Y"] \\ \widehat{X} \arrow[r, "\hat{g}"'] & \widehat{Y} \end{tikzcd}$$

commutes. We will sometimes use the term *f-localization* to refer to the map $\hat{g}$, without explicitly mentioning the f-localizations $(\widehat{X}, j_X)$ of X and $(\widehat{Y}, j_Y)$ of Y.

We will show in Corollary 1.4.13 that all spaces and maps have f-localizations.

The reader should note the similarity between the definitions of f-localization and fibrant approximation (see Definition 9.1.1 and Definition 9.1.8). In Theorem 2.1.2, we prove that there is an f-local model category structure on $\mathsf{Spc}_{(*)}$ in which the local spaces are the fibrant objects and the f-local equivalences are the weak equivalences. In the f-local model category, an f-localization of a space or map is exactly a fibrant approximation to that space or map.

REMARK 1.2.9. In most earlier work on localization [**21, 19, 24, 23, 12, 17**], an f-local equivalence was defined to be a map $g\colon X \to Y$ such that, for every f-local space W, the map of function spaces $g^*\colon \operatorname{Map}(Y, W) \to \operatorname{Map}(X, W)$ is a weak equivalence. In fact, this earlier work considered only the subcategory of cofibrant spaces. Since a cofibrant space is a cofibrant approximation to itself, this earlier definition coincides with ours.

1.2.10. f-local equivalences.

PROPOSITION 1.2.11. *If $f\colon A \to B$ is a map between cofibrant spaces, then every weak equivalence is an f-local equivalence.*

PROOF. Since a cofibrant approximation to a weak equivalence must also be a weak equivalence, this follows from Corollary 10.2.2. □

PROPOSITION 1.2.12. *If $f\colon A \to B$ is a map between cofibrant spaces, then the class of f-local equivalences satisfies the "two out of three" axiom, i.e., if g and h are composable maps, and if two of g, h, and hg are f-local equivalences, then so is the third.*

PROOF. Given maps $g\colon X \to Y$ and $h\colon Y \to Z$, we can apply a functorial cofibrant approximation (see Proposition 9.1.2) to g and h to obtain the diagram

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{\tilde{g}} & \widetilde{Y} & \xrightarrow{\tilde{h}} & \widetilde{Z} \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow[g]{} & Y & \xrightarrow[h]{} & Z \end{array}$$

in which $\tilde{g}$, $\tilde{h}$, and $\tilde{h}\tilde{g}$ are cofibrant approximations to g, h, and hg, respectively. If W is a fibrant space, then two of the maps $\tilde{g}^*\colon \operatorname{Map}(\widetilde{Y}, W) \to \operatorname{Map}(\widetilde{X}, W)$, $\tilde{h}^*\colon \operatorname{Map}(\widetilde{Z}, W) \to \operatorname{Map}(\widetilde{Y}, W)$, and $(\tilde{h}\tilde{g})^*\colon \operatorname{Map}(\widetilde{A}, W) \to \operatorname{Map}(\widetilde{X}, W)$ are weak equivalences, and so the third is as well. □

PROPOSITION 1.2.13. *If $f\colon A \to B$ is a map between cofibrant spaces, then a retract (see Definition 8.1.1) of an f-local equivalence is an f-local equivalence.*

PROOF. If $g\colon X \to Y$ is an f-local equivalence and $h\colon V \to W$ is a retract of g, then we apply a functorial cofibrant approximation (see Proposition 9.1.2) to obtain cofibrant approximations $\tilde{g}\colon \widetilde{X} \to \widetilde{Y}$ to g and $\tilde{h}\colon \widetilde{V} \to \widetilde{W}$ such that $\tilde{h}$ is a retract of $\tilde{g}$. If Z is an f-local space, then $\tilde{h}^*\colon \operatorname{Map}(\widetilde{W}, Z) \to \operatorname{Map}(\widetilde{V}, Z)$ is a retract of the weak equivalence $\tilde{g}^*\colon \operatorname{Map}(\widetilde{Y}, Z) \to \operatorname{Map}(\widetilde{X}, Z)$, and so $\tilde{h}^*$ is a weak equivalence. □

PROPOSITION 1.2.14. *Let $f\colon A \to B$ be a map of cofibrant spaces. If $g\colon X \to Y$ is a cofibration of cofibrant spaces, then g is an f-local equivalence if and only if it has the left lifting property (see Definition 8.2.1) with respect to the map $W^{\Delta[n]} \to W^{\partial\Delta[n]}$ for all $n \geq 0$ and all f-local spaces W.*

PROOF. This follows from Proposition 10.3.3 and Lemma 10.3.6. □

PROPOSITION 1.2.15. *Let $f\colon A \to B$ be a map of cofibrant spaces, and let T be a totally ordered set. If $\boldsymbol{W}\colon T \to \mathsf{Spc}_{(*)}$ is a functor such that, if $s, t \in T$ and $s \leq t$, then $\boldsymbol{W}_s \to \boldsymbol{W}_t$ is a cofibration of cofibrant spaces that is an f-local equivalence, then, for every $s \in T$, the map $\boldsymbol{W}_s \to \operatorname{colim}_{t \geq s} \boldsymbol{W}_t$ is an f-local equivalence.*

PROOF. This follows from Proposition 1.2.14, Lemma 12.2.20, and Proposition 12.2.21. □

PROPOSITION 1.2.16. *Let $f\colon A \to B$ be a map of cofibrant spaces and let $g\colon C \to D$ be a cofibration between cofibrant spaces that is also an f-local equivalence. If the square*

$$\begin{tikzcd} C \arrow[r] \arrow[d, "g"'] & X \arrow[d, "h"] \\ D \arrow[r] & Y \end{tikzcd}$$

is a pushout, then h is an f-local equivalence.

PROOF. Factor the map $C \to X$ as $C \xrightarrow{u} P \xrightarrow{v} X$, where u is a cofibration and v is a trivial fibration. If we let Q be the pushout $D \amalg_C P$, then we have the commutative diagram

$$\begin{tikzcd} C \arrow[r, "u"] \arrow[d, "g"'] & P \arrow[r, "v"] \arrow[d, "k"] & X \arrow[d, "h"] \\ D \arrow[r, "s"'] & Q \arrow[r, "t"'] & Y \end{tikzcd}$$

in which u and s are cofibrations, and so P and Q are cofibrant. Since k is a cofibration, we are in a proper model category (see Theorem 11.1.16), and Proposition 8.2.12 implies that Y is the pushout $Q \amalg_P X$, the map t is a weak equivalence. Thus, k is a cofibrant approximation to h (see Definition 9.1.8), and so it is sufficient to show that k induces a weak equivalence of mapping spaces to every f-local space. Since g is a cofibration and an f-local equivalence and k is a cofibration, this follows from Proposition 10.3.3 and Lemma 10.3.7. □

1.2.17. f-local Whitehead theorems.

LEMMA 1.2.18. *If $f\colon A \to B$ is a map between cofibrant spaces, W is an f-local space, and $g\colon X \to Y$ is an f-local equivalence of cofibrant spaces, then g induces an isomorphism of the sets of simplicial homotopy classes of maps $g^*\colon [Y, W] \approx [X, W]$.*

PROOF. This follows from Corollary 10.4.9. □

THEOREM 1.2.19 (Strong f-local Whitehead theorem). *Let $f\colon A \to B$ be a map between cofibrant spaces. If X and Y are cofibrant f-local spaces and $g\colon X \to Y$ is an f-local equivalence, then g is a simplicial homotopy equivalence.*

PROOF. This follows from Lemma 1.2.18 and Proposition 10.4.24. ☐

THEOREM 1.2.20 (Weak f-local Whitehead theorem). *Let $f\colon A \to B$ be a map of cofibrant spaces. If X and Y are f-local spaces and $g\colon X \to Y$ is an f-local equivalence, then g is a weak equivalence.*

PROOF. Choose a cofibrant approximation $\tilde{g}\colon \widetilde{X} \to \widetilde{Y}$ to g such that $j_X : \widetilde{X} \to X$ and $j_Y : \widetilde{Y} \to Y$ are trivial fibrations (see Proposition 9.1.9). Proposition 1.2.5 implies that $\widetilde{X}$ and $\widetilde{Y}$ are f-local spaces, and Proposition 1.2.11 and Proposition 1.2.12 imply that $\tilde{g}$ is an f-local equivalence. Theorem 1.2.19 now implies that $\tilde{g}$ is a weak equivalence, which implies that g is a weak equivalence. ☐

1.2.21. Characterizing f-local spaces and f-local equivalences.

THEOREM 1.2.22. *Let $f\colon A \to B$ be a map between cofibrant spaces. If X is a fibrant space and $j : X \to \widehat{X}$ is an f-localization of X (see Definition 1.2.8), then j is a weak equivalence if and only if X is f-local.*

PROOF. If X is f-local, then Theorem 1.2.20 implies that j is a weak equivalence. Conversely, if j is a weak equivalence, then Proposition 1.2.5 implies that X is f-local. ☐

THEOREM 1.2.23. *Let $f\colon A \to B$ be a map between cofibrant spaces. If $\hat{g}\colon \widehat{X} \to \widehat{Y}$ is an f-localization of $g\colon X \to Y$ (see Definition 1.2.8), then g is an f-local equivalence if and only if $\hat{g}$ is a weak equivalence.*

PROOF. Proposition 1.2.11 and Proposition 1.2.12 imply that g is an f-local equivalence if and only if $\hat{g}$ is an f-local equivalence. Since $\widehat{X}$ and $\widehat{Y}$ are f-local spaces, Theorem 1.2.20 and Proposition 1.2.11 imply that $\hat{g}$ is an f-local equivalence if and only if it is a weak equivalence, and so the proof is complete. ☐

In Definition 1.4.11, we define a functorial f-localization (L_f, j). Theorem 1.2.22 then implies that a fibrant space X is f-local if and only if the localization map $j(X)\colon X \to \mathrm{L}_f X$ is a weak equivalence (see Theorem 1.4.14), and Theorem 1.2.23 implies that a map $g\colon X \to Y$ is an f-local equivalence if and only if $\mathrm{L}_f(g)\colon \mathrm{L}_f X \to \mathrm{L}_f Y$ is a weak equivalence (see Theorem 1.4.15).

1.2.24. Topological spaces and simplicial sets.

PROPOSITION 1.2.25. *Let $f\colon A \to B$ be a map between cofibrant spaces in $\mathsf{Top}_{(*)}$.*

1. *A space is f-local if and only if it is $\left|\operatorname{Sing} f\right|$-local.*
2. *A map $g\colon X \to Y$ is an f-local equivalence if and only if it is a $\left|\operatorname{Sing} f\right|$-local equivalence.*

PROOF. This follows from Proposition 1.2.6 and Proposition 1.2.3. ☐

PROPOSITION 1.2.26. *Let $f\colon A \to B$ be a map in $\mathsf{SS}_{(*)}$.*

1. *A space is f-local if and only if it is $\left(\operatorname{Sing}\left|f\right|\right)$-local.*

2. *A map $g\colon X \to Y$ is an f-local equivalence if and only if it is a $(\operatorname{Sing}|f|)$-local equivalence.*

PROOF. Since every simplicial set is cofibrant, this follows from Proposition 1.2.6 and Proposition 1.2.3. □

PROPOSITION 1.2.27. *If $f\colon A \to B$ is a map in $\mathsf{SS}_{(*)}$, then a topological space W in $\mathsf{Top}_{(*)}$ is $|f|$-local if and only if $\operatorname{Sing} W$ is f-local.*

PROOF. Lemma 1.1.13 gives us the commutative square

$$\begin{tikzcd}
\operatorname{Map}(|B|, W) \arrow[r] \arrow[d, "\approx"'] & \operatorname{Map}(|A|, W) \arrow[d, "\approx"] \\
\operatorname{Map}(B, \operatorname{Sing} W) \arrow[r] & \operatorname{Map}(A, \operatorname{Sing} W)
\end{tikzcd}$$

in which the vertical maps are isomorphisms, from which the proposition follows. □

PROPOSITION 1.2.28. *If $f\colon A \to B$ is a map in $\mathsf{SS}_{(*)}$ and K is a fibrant simplicial set in $\mathsf{SS}_{(*)}$, then K is f-local if and only if $|K|$ is $|f|$-local.*

PROOF. Since K is fibrant the natural map $K \to \operatorname{Sing}|K|$ is a weak equivalence of fibrant spaces, and so we have the commutative square

$$\begin{tikzcd}
\operatorname{Map}(B, K) \arrow[r] \arrow[d, "\approx"'] & \operatorname{Map}(A, K) \arrow[d, "\approx"] \\
\operatorname{Map}(B, \operatorname{Sing}|K|) \arrow[r] & \operatorname{Map}(A, \operatorname{Sing}|K|)
\end{tikzcd}$$

in which the vertical maps are weak equivalences (see Corollary 10.2.2). Thus, K is f-local if and only if $\operatorname{Sing}|K|$ is f-local, and so the proposition follows from Proposition 1.2.27. □

PROPOSITION 1.2.29. *If $f\colon A \to B$ is a map in $\mathsf{SS}_{(*)}$, then the map $g\colon C \to D$ in $\mathsf{SS}_{(*)}$ is an f-local equivalence if and only if the map $|g|\colon |C| \to |D|$ in $\mathsf{Top}_{(*)}$ is a $|f|$-local equivalence.*

PROOF. Since every simplicial set is cofibrant, g is an f-local equivalence if and only if, for every f-local simplicial set K, the map of simplicial sets $g^*\colon \operatorname{Map}(D, K) \to \operatorname{Map}(C, K)$ is a weak equivalence. If K is an f-local simplicial set, then K is fibrant, and so Corollary 10.2.2 implies that g is an f-local equivalence if and only if, for every f-local simplicial set K, the map of simplicial sets $g^*\colon \operatorname{Map}(D, \operatorname{Sing}|K|) \to \operatorname{Map}(C, \operatorname{Sing}|K|)$ is a weak equivalence. Lemma 1.1.13 implies that this is true if and only if $\operatorname{Map}(|D|, |K|) \to \operatorname{Map}(|C|, |K|)$ is a weak equivalence. Proposition 1.2.27 and Proposition 1.2.28 imply that this is true if and only if, for every $|f|$-local topological space W, the map $\operatorname{Map}(|D|, W) \to \operatorname{Map}(|C|, W)$ is a weak equivalence. Since $|C|$ and $|D|$ are cofibrant, this is true if and only if $|g|\colon |C| \to |D|$ is a $|f|$-local equivalence, and the proof is complete. □

PROPOSITION 1.2.30. *If $f\colon A \to B$ is a map in $\mathsf{SS}_{(*)}$, then the map $g\colon X \to Y$ in $\mathsf{Top}_{(*)}$ is a $|f|$-local equivalence if and only if the map $(\operatorname{Sing} g)\colon \operatorname{Sing} X \to \operatorname{Sing} Y$ in $\mathsf{SS}_{(*)}$ is an f-local equivalence.*

PROOF. The map $|\operatorname{Sing} g|: |\operatorname{Sing} X| \to |\operatorname{Sing} Y|$ is a cofibrant approximation to g (see Definition 9.1.8), and so g is a $|f|$-local equivalence if and only if, for every $|f|$-local topological space W, the map of simplicial sets $\operatorname{Map}(|\operatorname{Sing} Y|, W) \to \operatorname{Map}(|\operatorname{Sing} X|, W)$ is a weak equivalence. Lemma 1.1.13 implies that this is true if and only if, for every $|f|$-local topological space W, the map $\operatorname{Map}(\operatorname{Sing} Y, \operatorname{Sing} W) \to \operatorname{Map}(\operatorname{Sing} X, \operatorname{Sing} W)$ is a weak equivalence. If K is an f-local simplicial set, then K is fibrant, and so the natural map $K \to \operatorname{Sing}|K|$ is a weak equivalence of fibrant objects. Thus, Corollary 10.2.2 and Proposition 1.2.28 imply that g is a $|f|$-local equivalence if and only if, for every f-local simplicial set K, the map $\operatorname{Map}(\operatorname{Sing} Y, \operatorname{Sing}|K|) \to \operatorname{Map}(\operatorname{Sing} X, \operatorname{Sing}|K|)$ is a weak equivalence. Since every simplicial set is cofibrant, this completes the proof. $\square$

1.3. Constructing an f-localization functor

If $f: A \to B$ is a map of cofibrant spaces in $\mathsf{Spc}_{(*)}$, we describe in this section how to construct a functorial f-localization on $\mathsf{Spc}_{(*)}$. The construction that we present is essentially the one used by Bousfield in [**10**].

1.3.1. Horns on f. Given a map $f: A \to B$ of cofibrant spaces in $\mathsf{Spc}_{(*)}$, we want to construct a functorial f-localization (see Definition 1.2.8) on $\mathsf{Spc}_{(*)}$. That is, for every space X we want to construct a natural f-local space $\widehat{X}$ together with a natural f-local equivalence $X \to \widehat{X}$. Remark 1.2.7 implies that we can assume that f is an inclusion of cell complexes, and we will assume that f is such an inclusion.

If $\widehat{X}$ is to be an f-local space, then it must first of all be fibrant. Thus, the map $\widehat{X} \to *$ must have the right lifting property with respect to the inclusions $\mathsf{Spc}_{(*)}(\Lambda[n,k]) \to \mathsf{Spc}_{(*)}(\Delta[n])$ (see Definition 1.1.16) for all $n > 0$ and $n \ge k \ge 0$.

If $\widehat{X}$ is a fibrant space, then $f^*: \operatorname{Map}(B, \widehat{X}) \to \operatorname{Map}(A, \widehat{X})$ is already a fibration of simplicial sets (see Proposition 10.1.6). Thus, if $\widehat{X}$ is fibrant, then the assertion that $\widehat{X}$ is f-local is equivalent to the assertion that f^* is a trivial fibration of simplicial sets. Since a map of simplicial sets is a trivial fibration if and only if it has the right lifting property with respect to the inclusions $\partial\Delta[n] \to \Delta[n]$ for $n \ge 0$, this implies that a fibrant space $\widehat{X}$ is f-local if and only if the dotted arrow exists in every solid arrow diagram of the form

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d] & \operatorname{Map}(B, \widehat{X}) \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dotted] & \operatorname{Map}(A, \widehat{X}) \end{tikzcd}$$

and the isomorphisms of Definition 1.1.11 imply that this is true if and only if the dotted arrow exists in every solid arrow diagram of the form

$$\begin{tikzcd} A \otimes \Delta[n] \amalg_{A \otimes \partial\Delta[n]} B \otimes \partial\Delta[n] \arrow[r] \arrow[d] & \widehat{X} \arrow[d] \\ B \otimes \Delta[n] \arrow[r] \arrow[ur, dotted] & * \end{tikzcd}$$

Thus, a space $\widehat{X}$ is f-local if and only if the map $\widehat{X} \to *$ has the right lifting property with respect to the maps $\mathsf{Spc}_{(*)}(\Lambda[n,k]) \to \mathsf{Spc}_{(*)}(\Delta[n])$ for all $n > 0$ and

$n \geq k \geq 0$ and the maps $A \otimes \Delta[n] \amalg_{A \otimes \partial\Delta[n]} B \otimes \partial\Delta[n] \to B \otimes \Delta[n]$ for all $n \geq 0$. This is the motivation for the definition of the set $\overline{\Lambda\{f\}}$ of *augmented f-horns*.

Definition 1.3.2. Let $f\colon A \to B$ be an inclusion of cell complexes (see Remark 1.2.7).

- The set $\Lambda\{f\}$ of *horns on f* is the set of maps
$$\Lambda\{f\} = \{A \otimes \Delta[n] \amalg_{A \otimes \partial\Delta[n]} B \otimes \partial\Delta[n] \to B \otimes \Delta[n] \bigm| n \geq 0\}.$$
If $\mathsf{Spc}_{(*)} = \mathsf{Spc}_*$ and f is the map $f\colon * \to A$, then $\Lambda\{f\}$ is the set of maps
$$\Lambda\{A\} = \{A \otimes \partial\Delta[n] \to A \otimes \Delta[n] \bigm| n \geq 0\},$$
and it will also be called the set of *horns on A*.
- The set $\overline{\Lambda\{f\}}$ of *augmented f-horns* is the set of maps
$$J_f = \Lambda\{f\} \cup \{\mathsf{Spc}_{(*)}(\Lambda[n,k]) \to \mathsf{Spc}_{(*)}(\Delta[n]) \bigm| n > 0, n \geq k \geq 0\}$$
(see Definition 1.1.16).

Proposition 1.3.3. *If $f\colon A \to B$ is an inclusion of cell complexes (see Remark 1.2.7), then a space X is f-local if and only if the map $X \to *$ has the right lifting property (see Definition 8.2.1) with respect to all augmented f-horns (see Definition 1.3.2).*

Proof. This follows from the discussion preceding Definition 1.3.2. □

We will construct the map $X \to \widehat{X}$ as a transfinite composition (see Definition 12.2.2) of inclusions of cell complexes $X = \mathrm{E}^0 \to \mathrm{E}^1 \to \mathrm{E}^2 \to \cdots \to \mathrm{E}^\beta \to \cdots \quad (\beta < \lambda)$, $\widehat{X} = \operatorname{colim}_{\beta<\lambda} \mathrm{E}^\beta$. To ensure that $\widehat{X}$ is f-local, we will construct the E^β so that if the map $C \to D$ is an element of $\overline{\Lambda\{f\}}$, then

1. for every map $h\colon C \to \widehat{X}$ there is an ordinal $\alpha < \lambda$ such that h factors through the map $\mathrm{E}^\alpha \to \widehat{X}$, and
2. for every ordinal $\alpha < \lambda$, the dotted arrow exists in every solid arrow diagram of the form
$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathrm{E}^\alpha \arrow[r] & \mathrm{E}^{\alpha+1} \\ D \arrow[urr, dotted] & & \end{tikzcd}$$

Thus, if the map $C \to D$ is an element of $\overline{\Lambda\{f\}}$, then the dotted arrow will exists in every solid arrow diagram of the form
$$\begin{tikzcd} C \arrow[r] \arrow[d] & \widehat{X} \\ D \arrow[ur, dotted] & \end{tikzcd}$$
and so the map $\widehat{X} \to *$ will have the right lifting property with respect to every element of $\overline{\Lambda\{f\}}$ (see Proposition 1.3.3).

1.3.4. Choice of the ordinal λ. If A and B are finite complexes, then we let λ be the first infinite ordinal. Otherwise, we let λ be the first cardinal greater than that of the set of simplices (or cells) of $A \amalg B$ (in which case λ is a successor cardinal). In either case, λ is a *regular cardinal* (see Proposition 12.1.15).

Suppose we now construct a λ-sequence (see Definition 12.2.1) of inclusions of cell complexes

$$X = \mathrm{E}^0 \to \mathrm{E}^1 \to \mathrm{E}^2 \to \cdots \to \mathrm{E}^\beta \to \cdots \qquad (\beta < \lambda)$$

and let $\widehat{X} = \operatorname{colim}_{\beta<\lambda} \mathrm{E}^\beta$. If $A \otimes \Delta[n] \amalg_{A\otimes\partial\Delta[n]} B \otimes \partial\Delta[n] \to \widehat{X}$ is any map, then for each simplex (or cell) of $A \otimes \Delta[n] \amalg_{A\otimes\partial\Delta[n]} B \otimes \partial\Delta[n]$ there is an ordinal $\beta < \lambda$ such that that simplex (or cell) lands in E^β. (If $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$, then this follows from Corollary 2.2.5.) If we let α be the union of the ordinals β obtained in this way for each simplex (or cell) in $A \otimes \Delta[n] \amalg_{A\otimes\partial\Delta[n]} B \otimes \partial\Delta[n]$, then the regularity of λ ensures that $\alpha < \lambda$. Thus, our map factors through E^α. The same argument applies to maps $\mathsf{Spc}_{(*)}(\Lambda[n,k]) \to \widehat{X}$.

1.3.5. Construction of the sequence. It remains only to show how to construct the E^β. We begin the sequence by letting $\mathrm{E}^0 = X$. If $\beta < \lambda$, and we have constructed the sequence through E^β, we let

$$C_\beta = \coprod_{\substack{(C\to D)\in\overline{\Lambda\{f\}} \\ \mathsf{Spc}_{(*)}(C,\mathrm{E}^\beta)}} C \qquad \text{and} \qquad D_\beta = \coprod_{\substack{(C\to D)\in\overline{\Lambda\{f\}} \\ \mathsf{Spc}_{(*)}(C,\mathrm{E}^\beta)}} D$$

We then have a natural map $C_\beta \to \mathrm{E}^\beta$, and we define $\mathrm{E}^{\beta+1}$ by letting the square

$$\begin{tikzcd} C_\beta \arrow[r] \arrow[d] & \mathrm{E}^\beta \arrow[d, dotted] \\ D_\beta \arrow[r, dotted] & \mathrm{E}^{\beta+1} \end{tikzcd}$$

be a pushout. If γ is a limit ordinal, we let $\mathrm{E}^\gamma = \operatorname{colim}_{\beta<\gamma} \mathrm{E}^\beta$. We let $\widehat{X} = \operatorname{colim}_{\beta<\lambda} \mathrm{E}^\beta$.

It remains only to show that the map $X \to \widehat{X}$ that we have constructed is an f-local equivalence. This will follow from Theorem 1.3.11.

1.3.6. Horns on f and f-local equivalences.

PROPOSITION 1.3.7. *If $f\colon A \to B$ is an inclusion of cell complexes (see Remark 1.2.7), then every horn on f is an f-local equivalence.*

PROOF. Since every horn on f is a cofibration between cofibrant spaces, this follows from Proposition 10.3.3 and Proposition 10.3.10. ☐

DEFINITION 1.3.8. If $f\colon A \to B$ is an inclusion of cell complexes (see Remark 1.2.7), then a *relative* $\overline{\Lambda\{f\}}$*-cell complex* is defined to be a map that can be constructed as a transfinite composition (see Definition 12.2.2) of pushouts (see Definition 8.2.10) of elements of $\overline{\Lambda\{f\}}$ (see Definition 1.3.2). If the map from the initial object to a space X is a relative $\overline{\Lambda\{f\}}$-cell complex, then X will be called an $\overline{\Lambda\{f\}}$ *complex.*

THEOREM 1.3.9. *If $f: A \to B$ is an inclusion of cell complexes (see Remark 1.2.7), then every relative $\overline{\Lambda\{f\}}$-cell complex is both a cofibration and an f-local equivalence.*

PROOF. Since every element of $\overline{\Lambda\{f\}}$ (see Definition 1.3.2) is a cofibration, and cofibrations are closed under both pushouts and transfinite compositions (see Proposition 12.2.19), every relative $\overline{\Lambda\{f\}}$-cell complex is a cofibration. Thus, it remains only to show that a relative $\overline{\Lambda\{f\}}$-cell complex is an f-local equivalence.

If $\mathsf{Spc}_{(*)} = \mathsf{SS}_{(*)}$ (in which every object is cofibrant), then Proposition 10.3.3, Proposition 10.3.10, and Proposition 12.2.18 imply that every relative $\overline{\Lambda\{f\}}$-cell complex is an f-local equivalence.

If $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$, then Proposition 1.3.7 implies that every element of $\overline{\Lambda\{f\}}$ is an f-local equivalence. Since every f-cell has cofibrant domain and codomain, Proposition 1.2.16 now implies that every pushout of an element of $\overline{\Lambda\{f\}}$ is an f-local equivalence.

If λ is an ordinal and

$$X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \qquad (\beta < \lambda)$$

is a λ-sequence of pushouts of elements of $\overline{\Lambda\{f\}}$, then Proposition 11.1.22 implies that we can find a λ-sequence of cofibrations together with a map of λ-sequences

$$\begin{tikzcd} \widetilde{X}_0 \arrow[r] \arrow[d] & \widetilde{X}_1 \arrow[r] \arrow[d] & \widetilde{X}_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & \widetilde{X}_\beta \arrow[r] \arrow[d] & \cdots \\ X_0 \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & \cdots \arrow[r] & X_\beta \arrow[r] & \cdots \end{tikzcd}$$

such that each vertical map $\widetilde{X}_\beta \to X_\beta$ is a cofibrant approximation to X_β and $\operatorname{colim}_{\beta<\lambda} \widetilde{X}_\beta \to \operatorname{colim}_{\beta<\lambda} X_\beta$ is a cofibrant approximation to $\operatorname{colim}_{\beta<\lambda} X_\beta$. If W is an f-local space, then $\operatorname{Map}(\operatorname{colim}_{\beta<\lambda} \widetilde{X}_\beta, W)$ is isomorphic to $\lim_{\beta<\lambda} \operatorname{Map}(\widetilde{X}_\beta, W)$. Since each $X_\beta \to X_{\beta+1}$ is an f-local equivalence and each $\widetilde{X}_\beta \to \widetilde{X}_{\beta+1}$ is a cofibration, each $\operatorname{Map}(\widetilde{X}_{\beta+1}, W) \to \operatorname{Map}(\widetilde{X}_\beta, W)$ is a trivial fibration. Thus,

$$\operatorname{Map}(\widetilde{X}_0, W) \leftarrow \operatorname{Map}(\widetilde{X}_1, W) \leftarrow \operatorname{Map}(\widetilde{X}_2, W) \leftarrow \cdots \leftarrow \operatorname{Map}(\widetilde{X}_\beta, W) \leftarrow \cdots$$

is a tower of trivial fibrations, and so the composition $\operatorname{Map}(\operatorname{colim}_{\beta<\lambda} \widetilde{X}_\beta, W) \to \operatorname{Map}(\widetilde{X}_0, W)$ is a weak equivalence, and so the composition $X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$ is an f-local equivalence. □

PROPOSITION 1.3.10. *If $f: A \to B$ is an inclusion of cell complexes (see Remark 1.2.7), then for every space X, the map $X \to \widehat{X}$ constructed in Section 1.3.5 is a relative $\overline{\Lambda\{f\}}$-cell complex.*

PROOF. The map $X \to \widehat{X}$ is constructed as a transfinite composition of pushouts of coproducts of elements of $\overline{\Lambda\{f\}}$, and to the result follows from Proposition 12.2.12. □

THEOREM 1.3.11. *If $f: A \to B$ is an inclusion of cell complexes (see Remark 1.2.7), then for every space X the map $X \to \widehat{X}$ constructed in Section 1.3.5 is a natural f-localization of X.*

PROOF. This follows from Proposition 1.3.10, Theorem 1.3.9, Proposition 1.3.3, and the discussion following Proposition 1.3.3. $\square$

1.4. Concise description of the f-localization

1.4.1. f-cofibrations and f-injectives.

DEFINITION 1.4.2. Let $f\colon A \to B$ be an inclusion of cell complexes (see Remark 1.2.7).

1. A $\overline{\Lambda\{f\}}$-*injective* is defined to be a map that has the right lifting property (see Definition 8.2.1) with respect to every element of $\overline{\Lambda\{f\}}$ (see Definition 1.3.2). A space X will be called a $\overline{\Lambda\{f\}}$-*injective* if the map $X \to *$ is a $\overline{\Lambda\{f\}}$-injective. If f is a cofibration $f\colon * \to A$, then a $\overline{\Lambda\{f\}}$-injective will also be called a $\overline{\Lambda\{A\}}$-*injective*.
2. A $\overline{\Lambda\{f\}}$-*cofibration* is defined to be a map that has the left lifting property with respect to all $\overline{\Lambda\{f\}}$-injectives. If the map from the initial object to a space X is a $\overline{\Lambda\{f\}}$-cofibration, then X will be called $\overline{\Lambda\{f\}}$-*cofibrant*. If f is a cofibration $f\colon * \to A$, then a $\overline{\Lambda\{f\}}$-cofibration will also be called a $\overline{\Lambda\{A\}}$-*cofibration*, and a $\overline{\Lambda\{f\}}$-cofibrant space will also be called a $\overline{\Lambda\{A\}}$-*cofibrant space*.

REMARK 1.4.3. The term $\overline{\Lambda\{f\}}$-*injective* comes from the theory of injective classes ([**32**]). A space X is a $\overline{\Lambda\{f\}}$-injective if and only if it is injective in the sense of [**32**] relative to the elements of $\overline{\Lambda\{f\}}$, and we will show in Proposition 1.4.5 that a map $p\colon X \to Y$ is a $\overline{\Lambda\{f\}}$-injective if and only if, in the category $(\mathsf{Spc}_{(*)}\downarrow Y)$ of spaces over Y (see Definition 14.4.1), the object p is injective relative to the class of maps whose image under the forgetful functor $(\mathsf{Spc}_{(*)}\downarrow Y) \to \mathsf{Spc}_{(*)}$ is a relative $\overline{\Lambda\{f\}}$-cell complex (see Definition 1.3.8).

PROPOSITION 1.4.4. *If $f\colon A \to B$ is an inclusion of cell complexes (see Remark 1.2.7), then a map $p\colon X \to Y$ is a $\overline{\Lambda\{f\}}$-injective if and only if it is a fibration with the homotopy right lifting property with respect to f.*

PROOF. This follows from Lemma 10.3.6. $\square$

PROPOSITION 1.4.5. *If $f\colon A \to B$ is an inclusion of cell complexes (see Remark 1.2.7), then every relative $\overline{\Lambda\{f\}}$-cell complex (see Definition 1.3.8) is a $\overline{\Lambda\{f\}}$-cofibration.*

PROOF. This follows from Proposition 1.4.4. $\square$

PROPOSITION 1.4.6. *If $f\colon A \to B$ is an inclusion of cell complexes (see Remark 1.2.7), then every trivial cofibration is a $\overline{\Lambda\{f\}}$-cofibration.*

PROOF. This follows from Proposition 8.2.3. $\square$

PROPOSITION 1.4.7. *If $f\colon A \to B$ is an inclusion of cell complexes (see Remark 1.2.7), then a space X is a $\overline{\Lambda\{f\}}$-injective if and only if it is f-local (see Definition 1.2.2).*

PROOF. This follows from Proposition 10.3.3 and Proposition 1.4.4. $\square$

1.4.8. The functorial localization.

PROPOSITION 1.4.9. *Let $f\colon A \to B$ be an inclusion of cell complexes (see Remark 1.2.7). If $j\colon X \to \widehat{X}$ is a relative $\overline{\Lambda\{f\}}$-cell complex and $\widehat{X}$ is a $\overline{\Lambda\{f\}}$-injective, then the pair $(\widehat{X}, j)$ is a cofibrant f-localization of X.*

PROOF. This follows from Proposition 1.4.7 and Theorem 1.3.9. □

THEOREM 1.4.10. *If $f\colon A \to B$ is an inclusion of cell complexes (see Remark 1.2.7), then there is a natural factorization of every map $X \to Y$ as*

$$X \xrightarrow{j} \mathrm{E}_f \xrightarrow{p} Y$$

in which j is a relative $\overline{\Lambda\{f\}}$-cell complex (see Definition 1.3.8) and p is a $\overline{\Lambda\{f\}}$-injective (see Definition 1.4.2).

PROOF. This follows from Proposition 12.4.12. □

DEFINITION 1.4.11. Let $f\colon A \to B$ be an inclusion of cell complexes (see Remark 1.2.7). The *f-localization* of a space X is the space $\mathrm{L}_f X$ obtained by applying the factorization of Theorem 1.4.10 to the map $X \to *$ from X to the terminal object of $\mathsf{Spc}_{(*)}$. This factorization defines a natural transformation $j\colon 1 \to \mathrm{L}_f$ such that $j_X : X \to \mathrm{L}_f X$ is a relative $\overline{\Lambda\{f\}}$-cell complex.

THEOREM 1.4.12. *If $f\colon A \to B$ is an inclusion of cell complexes (see Remark 1.2.7), then for every space X, the f-localization $j_X : X \to \mathrm{L}_f X$ is a cofibrant f-localization of X.*

PROOF. This follows from Proposition 1.4.9. □

COROLLARY 1.4.13. *If $f\colon A \to B$ is an inclusion of cell complexes (see Remark 1.2.7), then every space has an f-localization.*

PROOF. This follows from Theorem 1.4.12. □

THEOREM 1.4.14. *Let $f\colon A \to B$ be an inclusion of cell complexes (see Remark 1.2.7). If X is a fibrant space, then X is f-local if and only if the f-localization map $j_X : X \to \mathrm{L}_f X$ is a weak equivalence.*

PROOF. This follows from Theorem 1.2.22. □

THEOREM 1.4.15. *Let $f\colon A \to B$ be an inclusion of cell complexes (see Remark 1.2.7). The map $g\colon X \to Y$ is an f-local equivalence if and only if its f-localization $\mathrm{L}_f(g)\colon \mathrm{L}_f X \to \mathrm{L}_f Y$ is a weak equivalence.*

PROOF. This follows from Theorem 1.2.23. □

PROPOSITION 1.4.16. *If $f\colon A \to B$ is an inclusion of cell complexes (see Remark 1.2.7), then every $\overline{\Lambda\{f\}}$-cofibration (see Definition 1.4.2) is a retract of a relative $\overline{\Lambda\{f\}}$-cell complex.*

PROOF. This follows form Theorem 1.4.10 and the retract argument (see Proposition 8.2.2). □

COROLLARY 1.4.17. *If $f\colon A \to B$ is an inclusion of cell complexes (see Remark 1.2.7), then every $\overline{\Lambda\{f\}}$-cofibration is an f-local equivalence.*

PROOF. This follows from Proposition 1.4.16, Theorem 1.3.9, and Proposition 1.2.13. □

1.4.18. Properties of the localization functor.

PROPOSITION 1.4.19. *Let $f\colon A \to B$ be an inclusion of cell complexes (see Remark 1.2.7), let $X \to X'$ and $Y \to Y'$ be cofibrations, and let the square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd}$$

be commutative. If we apply the factorization of Theorem 1.4.10 to each of the horizontal maps to obtain the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{E}_f \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & \mathrm{E}'_f \arrow[r] & Y', \end{tikzcd}$$

then the map $\mathrm{E}_f \to \mathrm{E}'_f$ is a cofibration.

PROOF. Using Lemma 8.2.13, one can check inductively that at each stage in the construction of the factorization, we have a cofibration $\mathrm{E}^\beta \to (\mathrm{E}^\beta)'$. □

COROLLARY 1.4.20. *Let $f\colon A \to B$ be an inclusion of cell complexes (see Remark 1.2.7). If $g\colon X \to Y$ is a cofibration, then so is $\mathrm{L}_f(g)\colon \mathrm{L}_f X \to \mathrm{L}_f Y$ (see Definition 1.4.11).*

PROOF. This follows from Proposition 1.4.19. □

1.5. Topological spaces and simplicial sets

Warning: This section is a collection of leftovers in need of reorganization!

The main results of this section (Corollary 1.5.5 and Corollary 1.5.7) imply, roughly speaking, that when using the localization functor of Definition 1.4.11, one can pass freely through the geometric realization and total singular complex functors, at the cost of only a natural weak equivalence.

LEMMA 1.5.1. *Let K and C be simplicial sets and let X be is a topological space.*

1. *A map of topological spaces $|K| \to X$ defines a simplicial map $\mathrm{Map}(C, K) \to \mathrm{Map}(|C|, X)$ that is natural in C and in the map $|K| \to X$.*
2. *A map of simplicial sets $K \to \mathrm{Sing}\, X$ defines a simplicial map $\mathrm{Map}(C, K) \to \mathrm{Map}(|C|, X)$ that is natural in C and in the map $K \to \mathrm{Sing}\, X$.*

PROOF. The map of part 1 is defined as the composition

$$\mathrm{Map}(C, K) \to \mathrm{Map}(|C|, |K|) \to \mathrm{Map}(|C|, X)$$

and the map of part 2 is defined as the composition

$$\mathrm{Map}(C, K) \to \mathrm{Map}(C, \mathrm{Sing}\, X) \to \mathrm{Map}(|C|, X).$$

□

Proposition 1.5.2. *If $K \to L$ is a map of simplicial sets, $X \to Y$ a map of topological spaces, and*

$$\begin{tikzcd} |K| \arrow[r] \arrow[d] & X \arrow[d] \\ |L| \arrow[r] & Y \end{tikzcd}$$

a commutative square, then there is a natural map from the geometric realization of the pushout

$$\begin{tikzcd} C \times \bigl(\mathrm{Map}(C,K) \times_{\mathrm{Map}(C,L)} \mathrm{Map}(D,L)\bigr) \arrow[r] \arrow[d] & K \arrow[r, dotted] \arrow[d] & P \arrow[dl, dotted] \\ D \times \bigl(\mathrm{Map}(C,K) \times_{\mathrm{Map}(C,L)} \mathrm{Map}(D,L)\bigr) \arrow[r] \arrow[urr, dotted] & L & \end{tikzcd}$$

to the pushout

$$\begin{tikzcd} |C| \times \bigl|\mathrm{Map}(|C|,X) \times_{\mathrm{Map}(|C|,Y)} \mathrm{Map}(|D|,Y)\bigr| \arrow[r] \arrow[d] & X \arrow[r, dotted] \arrow[d] & Q \arrow[dl, dotted] \\ |D| \times \bigl|\mathrm{Map}(|C|,X) \times_{\mathrm{Map}(|C|,Y)} \mathrm{Map}(|D|,Y)\bigr| \arrow[r] \arrow[urr, dotted] & Y & \end{tikzcd}$$

that makes the diagram

$$\begin{tikzcd} |K| \arrow[r] \arrow[d] & |P| \arrow[r] \arrow[d] & |L| \arrow[d] \\ X \arrow[r] & Q \arrow[r] & Y \end{tikzcd}$$

commute.

Proof. Since the geometric realization functor commutes with pushouts, this follows from Lemma 1.5.1. □

Proposition 1.5.3. *If $K \to L$ is a map of simplicial sets, $X \to Y$ a map of topological spaces, and*

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \mathrm{Sing}\, X \arrow[d] \\ L \arrow[r] & \mathrm{Sing}\, Y \end{tikzcd}$$

a commutative square, then there is a natural map from the pushout

$$\begin{tikzcd} C \times \bigl(\mathrm{Map}(C,K) \times_{\mathrm{Map}(C,L)} \mathrm{Map}(D,L)\bigr) \arrow[r] \arrow[d] & K \arrow[r, dotted] \arrow[d] & P \arrow[dl, dotted] \\ D \times \bigl(\mathrm{Map}(C,K) \times_{\mathrm{Map}(C,L)} \mathrm{Map}(D,L)\bigr) \arrow[r] \arrow[urr, dotted] & L & \end{tikzcd}$$

to the total singular complex of the pushout

$$\begin{tikzcd}
\left|C\right| \times \left|\mathrm{Map}\left(\left|C\right|, X\right) \times_{\mathrm{Map}(|C|,Y)} \mathrm{Map}\left(\left|D\right|, Y\right)\right| \arrow[r] \arrow[d] & X \arrow[r, dotted] \arrow[d] & Q \arrow[dl, dotted] \\
\left|D\right| \times \left|\mathrm{Map}\left(\left|C\right|, X\right) \times_{\mathrm{Map}(|C|,Y)} \mathrm{Map}\left(\left|D\right|, Y\right)\right| \arrow[r] \arrow[urr, dotted] & Y &
\end{tikzcd}$$

that makes the diagram

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & L \arrow[d] \\
\operatorname{Sing} X \arrow[r] & \operatorname{Sing} Q \arrow[r] & \operatorname{Sing} Y
\end{tikzcd}$$

commute.

PROOF. This follows from Lemma 1.5.1, using the natural map from the pushout of the total singular complexes to the total singular complex of the pushout. □

THEOREM 1.5.4. *Let $f\colon A \to B$ be a cofibration of simplicial sets and let $g\colon X \to Y$ be a map of topological spaces. If $\mathrm{E}_f(\operatorname{Sing} g)$ is the simplicial set obtained by applying the factorization of Theorem* 1.4.10 *to the map $\operatorname{Sing} g\colon \operatorname{Sing} X \to \operatorname{Sing} Y$ and $E_{|f|}g$ is the topological space obtained by applying the factorization of Theorem* 1.4.10 *(with respect to the map $\left|f\right|\colon \left|A\right| \to \left|B\right|$) to the map g, then there is a natural map $\left|\mathrm{E}_f(\operatorname{Sing} g)\right| \to \mathrm{E}_{|f|}\, g$ that makes the diagram*

$$\begin{tikzcd}
\left|\operatorname{Sing} X\right| \arrow[r] \arrow[d] & \left|\mathrm{E}_f(\operatorname{Sing} g)\right| \arrow[r] \arrow[d] & \left|\operatorname{Sing} Y\right| \arrow[d] \\
X \arrow[r] & \mathrm{E}_{|f|}\, g \arrow[r] & Y
\end{tikzcd}$$

commute.

PROOF. Using Proposition 1.5.2, one can construct the map inductively at each stage in the construction of the factorization. □

COROLLARY 1.5.5. *If $f\colon A \to B$ is a cofibration of simplicial sets, then, for every topological space X, there is a natural map $\left|\mathrm{L}_f \operatorname{Sing} X\right| \to \mathrm{L}_{|f|}X$ that makes the square*

$$\begin{tikzcd}
\left|\operatorname{Sing} X\right| \arrow[r] \arrow[d] & X \arrow[d] \\
\left|\mathrm{L}_f \operatorname{Sing} X\right| \arrow[r] & \mathrm{L}_{|f|}X
\end{tikzcd}$$

commute, and this natural map is a weak equivalence.

PROOF. The existence of the natural map follows from Theorem 1.5.4. Proposition 1.2.28 implies that $\left|\mathrm{L}_f \operatorname{Sing} X\right|$ is $\left|f\right|$-local, and so Proposition 1.2.29 implies that our natural map is a $\left|f\right|$-localization of the weak equivalence $\left|\operatorname{Sing} X\right| \to X$ (see Definition 1.2.8). Proposition 1.2.11 and Theorem 1.2.23 now imply that our natural map is a weak equivalence. □

THEOREM 1.5.6. *Let $f\colon A \to B$ be a cofibration of simplicial sets and let $g\colon K \to L$ be a map of simplicial sets. If $\mathrm{E}_f\, g$ is the simplicial set obtained by applying the factorization of Theorem 1.4.10 to the map g and $E_{|f|}|g|$ is the topological space obtained by applying the factorization of Theorem 1.4.10 (with respect to the map $|f|\colon |A| \to |B|$) to the map $|g|\colon |K| \to |L|$, then there is a natural map $\mathrm{E}_f\, g \to \mathrm{E}_{|f|}|g|$ that makes the diagram*

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \mathrm{E}_f\, g \arrow[r] \arrow[d] & L \arrow[d] \\ \mathrm{Sing}|K| \arrow[r] & \mathrm{Sing}\,\mathrm{E}_{|f|}|g| \arrow[r] & \mathrm{Sing}|Y| \end{tikzcd}$$

commute.

PROOF. Using Proposition 1.5.3, one can construct the map inductively at each stage in the construction of the factorization. □

COROLLARY 1.5.7. *If $f\colon A \to B$ is a cofibration of simplicial sets then for every simplicial set K there is a natural map $\mathrm{L}_f K \to \mathrm{Sing}\,\mathrm{L}_{|f|}K$ that makes the square*

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \mathrm{Sing}|K| \arrow[d] \\ \mathrm{L}_f K \arrow[r] & \mathrm{Sing}\,\mathrm{L}_{|f|}|K| \end{tikzcd}$$

commute, and this natural map is a simplicial homotopy equivalence.

PROOF OF COROLLARY 1.5.7. The existence of the natural map follows from Theorem 1.5.6. Proposition 1.2.27 implies that $\mathrm{Sing}\,\mathrm{L}_{|f|}|K|$ is f-local, and Proposition 1.2.12, Proposition 1.2.11, and Proposition 1.2.30 imply that our natural map is an f-local equivalence of cofibrant f-local spaces. The result now follows from Theorem 1.2.19. □

PROPOSITION 1.5.8. *If $f\colon A \to B$ is a cofibration in $\mathsf{SS}_{(*)}$, $(\mathrm{M}_f, j\colon 1 \to \mathrm{M}_f)$ is a functorial cofibrant f-localization on $\mathsf{SS}_{(*)}$, and $(\mathrm{N}_{|f|}, k\colon 1 \to \mathrm{N}_{|f|})$ is a functorial cofibrant $|f|$-localization on $\mathsf{Top}_{(*)}$, then, for every topological space X, there is a map $|\mathrm{M}_f\,\mathrm{Sing}\,X| \to \mathrm{N}_{|f|}X$, unique up to simplicial homotopy, that makes the square*

(1.5.9)
$$\begin{tikzcd} |\mathrm{Sing}\,X| \arrow[r] \arrow[d] & X \arrow[d] \\ |\mathrm{M}_f\,\mathrm{Sing}\,X| \arrow[r, dotted] & \mathrm{N}_{|f|}X \end{tikzcd}$$

commute, and any such map is a weak equivalence. (Since $|\mathrm{M}_f\,\mathrm{Sing}\,X|$ is cofibrant and $\mathrm{N}_{|f|}X$ is fibrant, all notions of homotopy of maps $|\mathrm{M}_f\,\mathrm{Sing}\,X| \to \mathrm{N}_{|f|}X$ coincide and are equivalence relations (see Proposition 10.4.4).) This map is natural *up to homotopy, i.e., if* $g\colon X \to Y$ *is a map of topological spaces, then the square*

$$\begin{tikzcd}
\bigl|\mathrm{M}_f \operatorname{Sing} X\bigr| \arrow[r] \arrow[d] & \mathrm{N}_{|f|}X \arrow[d] \\
\bigl|\mathrm{M}_f \operatorname{Sing} Y\bigr| \arrow[r] & \mathrm{N}_{|f|}Y
\end{tikzcd}$$

commutes up to homotopy.

Proof. Since Proposition 1.2.29 implies that the map $\bigl|\operatorname{Sing} X\bigr| \to \bigl|\mathrm{M}_f \operatorname{Sing} X\bigr|$ is a $\bigl|f\bigr|$-local equivalence, the existence and uniqueness of the map follow from Lemma 1.2.18. Since Proposition 1.2.28 implies that $\bigl|\mathrm{M}_f \operatorname{Sing} X\bigr|$ is $\bigl|f\bigr|$-local, Theorem 1.2.20 implies that the map is a weak equivalence.

For the naturality statement, we note that we have the cube

$$\begin{tikzcd}
\bigl|\operatorname{Sing} X\bigr| \arrow[rr] \arrow[dr] \arrow[dd] & & X \arrow[dr] \arrow[dd] & \\
& \bigl|\operatorname{Sing} Y\bigr| \arrow[rr, crossing over] & & Y \arrow[dd] \\
\bigl|\mathrm{M}_f \operatorname{Sing} X\bigr| \arrow[rr] \arrow[dr] & & \mathrm{N}_{|f|}X \arrow[dr] & \\
& \bigl|\mathrm{M}_f \operatorname{Sing} Y\bigr| \arrow[rr] \arrow[from=uu, crossing over] & & \mathrm{N}_{|f|}Y
\end{tikzcd}$$

in which the top and side squares commute and the front and back squares commute up to simplicial homotopy. This implies that the composition

$$\bigl|\operatorname{Sing} X\bigr| \to \bigl|\mathrm{M}_f \operatorname{Sing} X\bigr| \to \bigl|\mathrm{M}_f \operatorname{Sing} Y\bigr| \to \mathrm{N}_{|f|}Y$$

is simplicially homotopic to the composition

$$\bigl|\operatorname{Sing} X\bigr| \to \bigl|\mathrm{M}_f \operatorname{Sing} X\bigr| \to \mathrm{N}_{|f|}X \to \mathrm{N}_{|f|}Y,$$

and so the result follows from Lemma 1.2.18. □

Proposition 1.5.10. *If* $f\colon A \to B$ *is a cofibration in* $\mathsf{SS}_{(*)}$, $(\mathrm{M}_f, j\colon 1 \to \mathrm{M}_f)$ *is a functorial cofibrant* f*-localization on* $\mathsf{SS}_{(*)}$, *and* $(\mathrm{N}_{|f|}, k\colon 1 \to \mathrm{N}_{|f|})$ *is a functorial cofibrant* $\bigl|f\bigr|$*-localization on* $\mathsf{Top}_{(*)}$, *then, for every simplicial set* K, *there is a map* $\mathrm{M}_f K \to \operatorname{Sing} \mathrm{N}_{|f|}\bigl|K\bigr|$, *unique up to homotopy, that makes the square*

(1.5.11)
$$\begin{tikzcd}
K \arrow[r] \arrow[d] & \operatorname{Sing}\bigl|K\bigr| \arrow[d] \\
\mathrm{M}_f K \arrow[r, dotted] & \operatorname{Sing} \mathrm{N}_{|f|}\bigl|K\bigr|
\end{tikzcd}$$

commute, and any such map is a homotopy equivalence. (Since every simplicial set is cofibrant and $\operatorname{Sing} \mathrm{N}_{|f|}\bigl|K\bigr|$ *is fibrant, all notions of homotopy of maps* $\mathrm{M}_f K \to \operatorname{Sing} \mathrm{N}_{|f|}\bigl|K\bigr|$ *coincide and are equivalence relations (see Proposition 10.4.4).) This map is natural up to homotopy, i.e., if* $g\colon K \to L$ *is a map of simplicial sets, then* *the square*

$$\begin{array}{ccc}
\mathrm{M}_f K & \longrightarrow & \mathrm{Sing}\,\mathrm{N}_{|f|}\big|K\big| \\
\big\downarrow & & \big\downarrow \\
\mathrm{M}_f L & \longrightarrow & \mathrm{Sing}\,\mathrm{N}_{|f|}\big|L\big|
\end{array}$$

commutes up to homotopy.

PROOF. Proposition 1.2.30 implies that the map $\mathrm{Sing}\big|K\big| \to \mathrm{Sing}\,\mathrm{N}_{|f|}\big|K\big|$ is an f-local equivalence and Proposition 1.2.27 implies that $\mathrm{Sing}\,\mathrm{N}_{|f|}\big|K\big|$ is f-local. Since every simplicial set is cofibrant, the existence and uniqueness of the map now follows from Lemma 1.2.18, and Theorem 1.2.19 implies that it is a homotopy equivalence. The naturality statement follows as in the proof of Proposition 1.5.8. □

1.6. A continuous localization functor

In this section, we will define a variant $\mathrm{L}_f^{\mathrm{cont}}$ of the f-localization functor L_f that is "continuous". If we were using topological spaces of functions (instead of simplicial sets of functions; see Section 1.1.4) then we would want to define a function

$$\mathrm{Map}(X,Y) \to \mathrm{Map}(\mathrm{L}_f X, \mathrm{L}_f Y) \tag{1.6.1}$$

that is a continuous function of topological spaces. Since we are considering $\mathsf{Spc}_{(*)}$ as a simplicial model category (see Definition 10.1.2), we want to define $\mathrm{L}_f^{\mathrm{cont}}$ to be a *simplicial functor*, i.e., we want a functor $\mathrm{L}_f^{\mathrm{cont}}$ that defines a map of simplicial sets (1.6.1) (see [**46**, Chapter II, Section 1]). Note that not every functor can be extended to a simplicial functor; for a counterexample, see Example 1.6.11.

1.6.2. Constructing relative $\overline{\Lambda\{f\}}$-cell complexes.

LEMMA 1.6.3. *If $f\colon A \to B$ is an inclusion of cell complexes (see Remark 1.2.7), then a pushout of a relative $\overline{\Lambda\{f\}}$-cell complexes is a relative $\overline{\Lambda\{f\}}$-cell complex.*

PROOF. This follows from Lemma 8.2.11. □

LEMMA 1.6.4. *If $f\colon A \to B$ is an inclusion of cell complexes (see Remark 1.2.7), then a coproduct of relative $\overline{\Lambda\{f\}}$-cell complexes is a relative $\overline{\Lambda\{f\}}$-cell complex.*

PROOF. This follows from Proposition 12.2.5, Lemma 1.6.3, and Lemma 12.2.11. □

PROPOSITION 1.6.5. *Let $f\colon A \to B$ be an inclusion of cell complexes (see Remark 1.2.7). If (K,L) is a pair of simplicial sets, then the map*

$$A \otimes K \amalg_{A\otimes L} B \otimes L \to B \otimes K$$

is a relative $\overline{\Lambda\{f\}}$-cell complexes.

PROOF. The inclusion $L \to K$ can be written as a transfinite composition (see Definition 12.2.2) of inclusions each of which is a pushout of an inclusion $\partial\Delta[n] \to \Delta[n]$ (for various values of n). Thus, $A \otimes K \amalg_{A\otimes L} B \otimes L \to B \otimes K$ is a transfinite composition of pushouts of $A\otimes\Delta[n] \amalg_{A\otimes\partial\Delta[n]} B\otimes\partial\Delta[n] \to B\otimes\Delta[n]$. □

COROLLARY 1.6.6. *If $f\colon A \to B$ is an inclusion of cell complexes (see Remark 1.2.7) and K is a simplicial set, then the maps*

$$\begin{aligned} (A\otimes\Delta[n] \amalg_{A\otimes\partial\Delta[n]} B\otimes\partial\Delta[n])\otimes K &\to (B\otimes\Delta[n])\otimes K \quad &&\text{for } n\geq 0\\ \Lambda[n,k]\otimes K &\to \Delta[n]\otimes K &&\text{for } n>0 \text{ and } 0\leq k\leq n \end{aligned}$$

are relative $\overline{\Lambda\{f\}}$-cell complexes.

PROOF. Lemma 10.2.3 and axiom M6 (see Definition 10.1.2) imply that the map

$$(A\otimes\Delta[n] \amalg_{A\otimes\partial\Delta[n]} B\otimes\partial\Delta[n])\otimes K \to (B\otimes\Delta[n])\otimes K$$

is isomorphic to the map

$$A\otimes(\Delta[n]\times K) \amalg_{A\otimes(\partial\Delta[n]\times K)} B\otimes(\partial\Delta[n]\times K) \to B\otimes(\Delta[n]\times K),$$

and so Proposition 1.6.5 implies that it is a relative $\overline{\Lambda\{f\}}$-cell complex. The map $\Lambda[n,k]\otimes K\to\Delta[n]\otimes K$ can be written as a transfinite composition (see Definition 12.2.2) of inclusions each of which is a pushout of an inclusion $\Lambda[n,k]\to\Delta[n]$ (for various values of n and k), and so the proof is complete. □

1.6.7. Constructing the continuous f-localization. We follow the procedure described in Section 1.3, using the same ordinal λ, except that we use a new construction to define the space $\mathrm{E}^{\beta+1}$ in terms of the space E^{β} (see Section 1.3.5).

1.6.8. Construction of the sequence. As in Section 1.3.5, we begin the sequence by letting $\mathrm{E}^0 = X$. If $\beta<\lambda$, and we have constructed the sequence through E^{β}, we let

$$\begin{aligned} C_\beta^{\mathrm{cont}} &= \coprod_{(C\to D)\in\overline{\Lambda\{f\}}} C\otimes \mathrm{Map}(C,\mathrm{E}^{\beta})\\ D_\beta^{\mathrm{cont}} &= \coprod_{(C\to D)\in\overline{\Lambda\{f\}}} D\otimes \mathrm{Map}(C,\mathrm{E}^{\beta}) \end{aligned}$$

We then have a natural map $C_\beta^{\mathrm{cont}}\to\mathrm{E}^{\beta}$, and we define $\mathrm{E}^{\beta+1}$ by letting the square

$$\begin{tikzcd} C_\beta^{\mathrm{cont}} \arrow[r] \arrow[d] & \mathrm{E}^{\beta} \arrow[d, dotted] \\ D_\beta^{\mathrm{cont}} \arrow[r, dotted] & \mathrm{E}^{\beta+1} \end{tikzcd}$$

be a pushout. If γ is a limit ordinal, we let $\mathrm{E}^{\gamma} = \mathrm{colim}_{\beta<\gamma}\mathrm{E}^{\beta}$. We let $\mathrm{L}_f^{\mathrm{cont}}X = \mathrm{colim}_{\beta<\lambda}\mathrm{E}^{\beta}$.

THEOREM 1.6.9. *Let $f\colon A\to B$ be an inclusion of cell complexes (see Remark 1.2.7). If X is a space, then the map $X\to\mathrm{L}_f^{\mathrm{cont}}X$ constructed in Section 1.6.8 is a cofibrant f-localization of X.*

PROOF. Corollary 1.6.6, Lemma 1.6.4, and Lemma 1.6.3 imply that each $\mathrm{E}^{\beta}\to\mathrm{E}^{\beta+1}$ is a relative $\overline{\Lambda\{f\}}$-cell complex, and so Lemma 12.2.11 implies that $X\to\mathrm{L}_f^{\mathrm{cont}}X$ is a relative $\overline{\Lambda\{f\}}$-cell complex. Theorem 1.3.9 now implies that the map $X\to\mathrm{L}_f^{\mathrm{cont}}X$ is both a cofibration and an f-local equivalence, and so it remains only to show that $L_f^{\mathrm{cont}}X$ is f-local. The 0-skeleton of $\mathrm{Map}(C, \mathrm{E}^\beta)$ is $\mathsf{Spc}_{(*)}(C, \mathrm{E}^\beta)$, and so $C \otimes \mathrm{Map}(C, \mathrm{E}^\beta)$ contains

$$C \otimes \mathsf{Spc}_{(*)}(C, \mathrm{E}^\beta) \approx \coprod_{\mathsf{Spc}_{(*)}(C, \mathrm{E}^\beta)} C$$

as a subcomplex. The discussion in Section 1.3.4 now explains why the space $L_f^{\mathrm{cont}}X$ is a $\overline{\Lambda\{f\}}$-injective, and so the map $X \to L_f^{\mathrm{cont}}X$ is a functorial cofibrant f-localization of X. □

THEOREM 1.6.10. *The functor L_f^{cont} can be extended to a simplicial functor.*

PROOF. If C and X are spaces and K is a simplicial set, then there is a natural map $\mathrm{Map}(C, X) \times K \to \mathrm{Map}(C, X \otimes K)$ that takes the n-simplex $(\alpha\colon C \otimes \Delta[n] \to X, \tau)$ of $\mathrm{Map}(C, X) \times K$ to the n-simplex $\sigma(\alpha, \tau)\colon C \otimes \Delta[n] \to X \otimes K$ of $\mathrm{Map}(C, X \otimes K)$ where the projection of $\sigma(\alpha, \tau)$ on X is α and the projection on K is the composition of the projection $C \otimes \Delta[n] \to \Delta[n]$ with the map that takes the non-degenerate n-simplex of $\Delta[n]$ to τ. This natural map σ has the properties required by Theorem 10.6.4, and so we can use it to inductively define σ for all the spaces used in the construction of the localization (see Section 1.6.8). The theorem now follows from Proposition 10.6.6 and Theorem 10.6.4. □

EXAMPLE 1.6.11 (Counterexample to continuity). If A is any nonempty space in Top, we define a functor $W_A = W\colon \mathsf{Top} \to \mathsf{Top}$ by $W_A X = WX = \coprod_{A \to X} A$, that is, we take the disjoint union of one copy of A for each continuous function $g\colon A \to X$. This defines a functor in which the copy of A corresponding to g as above maps under $W(f)\colon WX \to WY$ by the identity map to the copy corresponding to $f \circ g$, but W cannot be extended to a simplicial functor. To see this, take $X = A$ and $Y = A \times I$. The simplicial set $\mathrm{Map}(X, Y) = \mathrm{Map}(A, A \times I)$ has vertices (i.e., maps $A \to A \times I$) the inclusions i_0 and i_1 (where $i_0(a) = (a, 0)$ and $i_1(a) = (a, 1)$), and these vertices of $\mathrm{Map}(A, A \times I)$ are connected by a 1-simplex $A \times \Delta[1] \to A \times I$ of $\mathrm{Map}(A, A \times I)$. The functions $W(i_0)$ and $W(i_1)$, however, take each point of WA into different components of $W(A \times I)$, and so there can be no 1-simplex of $\mathrm{Map}\bigl(WA, W(A \times I)\bigr)$ connecting these vertices.

EXAMPLE 1.6.12. If we change Example 1.6.11 slightly, we can construct a functor that *is* continuous. Define $W_A^c = W^c$ by $W^cX = X^A \times X$ (where X^A is the compactly generated topological space of continuous functions $A \to X$). We have a natural transformation $W \to W^c$ such that $WX \to W^cX$ is always a continuous bijection, but it is not, in general, a homeomorphism.

1.7. Pointed and unpointed localization

There is a functor from the category of pointed spaces to the category of unpointed spaces that forgets the basepoint. If $f\colon A \to B$ is a cofibration of cofibrant pointed spaces, we can consider the notions of *pointed* f-local spaces and *pointed* f-local equivalences in Spc_*, or we can still consider spaces with basepoint (i.e., spaces in Spc_*) but consider the notions of *unpointed* f-local spaces and *unpointed* f-local equivalences in Spc by forgetting the basepoints.

NOTATION 1.7.1. In this section, if X and Y are objects of Spc_*, then $\mathrm{Map}(X, Y)$ will continue to denote the unpointed simplicial set of maps between the pointed spaces X and Y, and $\mathrm{UMap}(X, Y)$ will denote the unpointed simplicial set of maps between the unpointed spaces obtained from X and Y by forgetting the basepoint.

PROPOSITION 1.7.2. *Let A be a cofibrant object of Spc_* and let X be a fibrant object of Spc_*.*

1. *If $\mathsf{Spc}_* = \mathsf{SS}_*$, then there is a natural fibration of unpointed simplicial sets*
$$\mathrm{Map}(A, X) \to \mathrm{UMap}(A, X) \to X.$$
2. *If $\mathsf{Spc}_* = \mathsf{Top}_*$, then there is a natural fibration of unpointed simplicial sets*
$$\mathrm{Map}(A, X) \to \mathrm{UMap}(A, X) \to \mathrm{Sing}\, X.$$

PROOF. Since $* \to A$ is a cofibration of pointed spaces and X is a fibrant pointed space, $* \to A$ is also a cofibration of unpointed spaces (after forgetting the basepoints) and X is also a fibrant pointed space (after forgetting the basepoint). Thus, Proposition 10.1.6 implies that we have a natural fibration of simplicial sets $\mathrm{UMap}(A, X) \to \mathrm{UMap}(*, X)$. The fiber of this fibration is $\mathrm{Map}(A, X)$. If $\mathsf{Spc}_* = \mathsf{SS}_*$, then $\mathrm{UMap}(*, X)$ is naturally isomorphic to the unpointed simplicial set X. If $\mathsf{Spc}_* = \mathsf{Top}_*$, then $\mathrm{UMap}(*, X)$ is naturally isomorphic to the unpointed simplicial set $\mathrm{Sing}\, X$. □

DEFINITION 1.7.3. If $f \colon A \to B$ is a cofibration of cofibrant pointed spaces and X is a pointed space, then we will say that X is *pointed f-local* if it is an f-local space in Spc_*, and we will say that X is *unpointed f-local* if X is an f-local space in Spc when we forget the basepoints of all the spaces involved. Similarly, a map $f \colon X \to Y$ will be called a *pointed f-local equivalence* if it is an f-local equivalence in Spc_*, and an *unpointed f-local equivalence* if it is an f-local equivalence in Spc after forgetting all basepoints.

PROPOSITION 1.7.4. *Let $A \to B$ be a map of cofibrant pointed spaces and let W be a fibrant pointed space.*

1. *If $\mathrm{UMap}(B, W) \to \mathrm{UMap}(A, W)$ (see Notation 1.7.1) is a weak equivalence, then $\mathrm{Map}(B, W) \to \mathrm{Map}(A, W)$ is a weak equivalence.*
2. *If W is path connected and $\mathrm{Map}(B, W) \to \mathrm{Map}(A, W)$ is a weak equivalence, then $\mathrm{UMap}(B, W) \to \mathrm{UMap}(A, W)$ is a weak equivalence.*

PROOF. This follows from Proposition 1.7.2. □

PROPOSITION 1.7.5. *Let $f \colon A \to B$ be a cofibration of cofibrant pointed spaces and let X be a pointed space.*

1. *If X is an unpointed f-local space, then it is also a pointed f-local space.*
2. *If X is a path connected pointed f-local space, then it is also an unpointed f-local space.*

PROOF. This follows from Proposition 1.7.4. □

COROLLARY 1.7.6. *Let $f \colon A \to B$ be a cofibration of cofibrant pointed spaces. If X is a path connected pointed space, then X is pointed f-local if and only if it is unpointed f-local.*

PROOF. This follows from Proposition 1.7.5. □

LEMMA 1.7.7. *If A is a path connected pointed space, X is a pointed space, and X_b is the path component of X containing the basepoint, then the natural map* $\mathrm{Map}(A, X_b) \to \mathrm{Map}(A, X)$ *is an isomorphism.*

PROOF. Since the image of a path connected space is path connected, the image of a pointed map from A to X is contained in X_b. □

THEOREM 1.7.8. *If $f\colon A \to B$ is a map of path connected cofibrant pointed spaces and X is a pointed space, then the following are equivalent:*

1. *X is pointed f-local.*
2. *Every path component of X is fibrant and the path component of X containing the basepoint is pointed f-local.*
3. *Every path component of X is fibrant and the path component of X containing the basepoint is unpointed f-local.*

PROOF. This follows from Lemma 1.7.7 and Corollary 1.7.6. □

COROLLARY 1.7.9. *If $f\colon A \to B$ is a map of path connected cofibrant pointed spaces and X is a fibrant pointed space, then X is unpointed f-local if and only if every path component of X is pointed f-local when you choose a basepoint for each path component.*

PROOF. If the path components of X are $\{X_s\}_{s\in S}$, then there is an isomorphism $\mathrm{Map}(A, X) \approx \coprod_{s\in S} \mathrm{Map}(A, X_s)$ that is natural in A, and (after you choose a basepoint for each path component) an isomorphism $\mathrm{UMap}(A, X) \approx \coprod_{s\in S} \mathrm{UMap}(A, X_s)$ that is natural in A. The result now follows from Corollary 1.7.6. □

COROLLARY 1.7.10. *If $f\colon A \to B$ is a map of path connected cofibrant pointed spaces, X is a pointed space, and X_b is the path component of X containing the basepoint, then the natural map*

$$(X - X_b) \amalg \mathrm{L}_f X_b \to \mathrm{L}_f X$$

is a weak equivalence (where L_f denotes pointed f-localization).

PROOF. This follows from Theorem 1.7.8, Lemma 1.7.7, and Theorem 1.2.20. □

LEMMA 1.7.11. *If W is an f-local space, then any space consisting of a nonempty union of path components of W is an f-local space.*

PROOF. A nonempty union of path components of a cofibrant space is a retract of that space. □

PROPOSITION 1.7.12. *Let $f\colon A \to B$ be a map of cofibrant pointed spaces. If $X \to Y$ is an unpointed f-local equivalence of path connected pointed spaces, then it is also a pointed f-local equivalence.*

PROOF. If $\widetilde{X} \to \widetilde{Y}$ is a pointed cofibrant approximation (see Definition 9.1.8) to $X \to Y$, then it is also an unpointed cofibrant approximation. If W is a pointed f-local space, let W_b be the path component of W containing the basepoint. Lemma 1.7.11 and Proposition 1.7.5 imply that W_b is an unpointed f-local space, and so the map $\mathrm{UMap}(\widetilde{Y}, W_b) \to \mathrm{UMap}(\widetilde{X}, W_b)$ is a weak equivalence. Proposition 1.7.4 now implies that the map $\mathrm{Map}(\widetilde{Y}, W_b) \to \mathrm{Map}(\widetilde{X}, W_b)$ is a weak equivalence. Since both $\widetilde{X}$ and $\widetilde{Y}$ are path connected, the horizontal maps in the commutative square

$$\begin{tikzcd} \mathrm{Map}(\widetilde{Y}, W_b) \arrow[r] \arrow[d] & \mathrm{Map}(\widetilde{Y}, W) \arrow[d] \\ \mathrm{Map}(\widetilde{X}, W_b) \arrow[r] & \mathrm{Map}(\widetilde{X}, W) \end{tikzcd}$$

are isomorphisms, and so the map $\mathrm{Map}(\widetilde{Y}, W) \to \mathrm{Map}(\widetilde{X}, W)$ is a weak equivalence. ☐

THEOREM 1.7.13. *If $f\colon A \to B$ is a cofibration of cofibrant pointed spaces and X is a cofibrant path connected pointed space, then the pointed f-localization of X is weakly equivalent to the unpointed f-localization of X.*

PROOF. Let $X \to Y$ be the unpointed f-localization of X. Proposition 1.7.5 implies that Y is pointed f-local and Proposition 1.7.12 implies that the map $X \to Y$ is a pointed f-local equivalence, and so the result follows from Proposition 6.1.10. ☐

THEOREM 1.7.14. *If $f\colon A \to B$ is a cofibration of path connected cofibrant pointed spaces and X is a pointed space, then the unpointed f-localization of X is weakly equivalent to the space obtained by choosing a basepoint for each path component of X and taking the pointed f-localization of each path component.*

1.8. Comparing localizations

PROPOSITION 1.8.1. *Let $f\colon A \to B$ and $\tilde{f}\colon \widetilde{A} \to \widetilde{B}$ be maps between cofibrant objects. If f is an $\tilde{f}$-local equivalence, then every f-local equivalence is an $\tilde{f}$-local equivalence.*

PROOF. Since f is an $\tilde{f}$-local equivalence, every $\tilde{f}$-local space is f-local. ☐

COROLLARY 1.8.2. *Let $f\colon A \to B$ and $\tilde{f}\colon \widetilde{A} \to \widetilde{B}$ be maps between cofibrant spaces. If f is an $\tilde{f}$-local equivalence, then for every object X, the f-localization map $X \to \mathrm{L}_f X$ is an $\tilde{f}$-local equivalence.*

PROOF. This follows from Proposition 1.8.1. ☐

PROPOSITION 1.8.3. *If $n > 0$ and f is the inclusion $S^n \subset D^{n+1}$ in Top, then a space X is f-local if and only if $\pi_i X \approx 0$ for $i \geq n$ and every choice of basepoint in X.*

PROOF. If $k \geq 0$, then the inclusion $S^n \otimes \Delta[k] \amalg_{S^n \otimes \partial\Delta[k]} D^{n+1} \otimes \partial\Delta[k] \to D^{n+1} \otimes \Delta[k]$ is a relative CW-complex that attaches a single cell of dimension $n+k+1$. Thus, any map $S^n \otimes \Delta[k] \amalg_{S^n \otimes \partial\Delta[k]} D^{n+1} \otimes \partial\Delta[k] \to X$ can be extended over $D^{n+1} \otimes \Delta[k]$. ☐

PROPOSITION 1.8.4. *If $n > 0$ and f is the inclusion $S^n \subset D^{n+1}$ in Top, then a map $g\colon X \to Y$ is an f-local equivalence if and only if it induces isomorphisms $g_*\colon \pi_i X \approx \pi_i Y$ for $i < n$ and every choice of basepoint in X.*

PROOF. We can choose a cofibrant approximation $\tilde{g}\colon \widetilde{X} \to \widetilde{Y}$ to g such that $\widetilde{Y}$ is a CW-complex and $\tilde{g}$ is the inclusion of a subcomplex that contains the n-skeleton of $\widetilde{Y}$. Thus, if $k \geq 0$, then the map $\widetilde{X} \otimes \Delta[k] \amalg_{\widetilde{X} \otimes \partial\Delta[k]} \widetilde{Y} \otimes \partial\Delta[k] \to \widetilde{Y} \otimes \Delta[k]$ is the inclusion of a subcomplex that contains the $(n+k)$-skeleton. If Z is an f-local space, then Proposition 1.8.3 implies that every map $\widetilde{X} \otimes \Delta[k] \amalg_{\widetilde{X} \otimes \partial\Delta[k]} \widetilde{Y} \otimes \partial\Delta[k] \to Z$ can be extended over $\widetilde{Y} \otimes \Delta[k]$. □

PROPOSITION 1.8.5. *If $n > 0$ and f is the inclusion $S^n \subset D^{n+1}$ in* Top*, then the functor that projects a space onto its $(n-1)$st Postnikov piece is an f-localization map.*

PROOF. This follows from Proposition 1.8.3 and Proposition 1.8.4. □

THEOREM 1.8.6. *If $n > 0$ and $f: A \to B$ is a map in* Spc *that induces isomorphisms $f_*: \pi_i X \approx \pi_i Y$ for $i \leq n$ and every choice of basepoint in A, then, for every space X, the f-localization map $X \to \mathrm{L}_f X$ induces isomorphisms $\pi_k X \approx \pi_k \mathrm{L}_f X$ for $k \leq n$ and every choice of basepoint in X.*

COROLLARY 1.8.7. *If $f\colon A \to B$ is a map between n-connected spaces, then, for every space X, the f-localization map $X \to \mathrm{L}_f X$ induces isomorphisms $\pi_k X \approx \pi_k \mathrm{L}_f X$ for $k \leq n$ and every choice of basepoint in X.*

CHAPTER 2

The localization model category for spaces

2.1. The Bousfield localization model category structure

In this section, we show that for every map $f\colon A \to B$ in $\mathsf{Spc}_{(*)}$ there is a model category structure on $\mathsf{Spc}_{(*)}$ in which the weak equivalences are the f-local equivalences and the fibrant objects are the f-local spaces (see Theorem 2.1.2 and Proposition 2.1.3). This is a generalization of the h_*-local model category structure for a generalized homology theory h_* on the category of simplicial sets defined by A.K. Bousfield in [**9**]. It is also an example of a *left Bousfield localization* (see Definition 3.2.1). This model category structure has also been obtained by Bousfield in [**13**] for the category of simplicial sets, where he deals as well with localizing certain proper classes of maps of simplicial sets.

DEFINITION 2.1.1. Let $f\colon A \to B$ be a map between cofibrant spaces in $\mathsf{Spc}_{(*)}$.

1. An *f-local weak equivalence* is defined to be an f-local equivalence (see Definition 1.2.2).
2. An *f-local cofibration* is defined to be a cofibration.
3. An *f-local fibration* is defined to be a map with the right lifting property (see Definition 8.2.1) with respect to all maps that are both f-local cofibrations and f-local weak equivalences. If the map from a space to a point is an f-local fibration, then we will say that the space is *f-local fibrant.*

THEOREM 2.1.2. *If $f\colon A \to B$ is a map between cofibrant spaces in $\mathsf{Spc}_{(*)}$, then there is a simplicial model category structure on $\mathsf{Spc}_{(*)}$ in which the weak equivalences are the f-local weak equivalences, the cofibrations are the f-local cofibrations, the fibrations are the f-local fibrations, and the simplicial structure is the usual simplicial structure on $\mathsf{Spc}_{(*)}$.*

PROPOSITION 2.1.3. *If $f\colon A \to B$ is an inclusion of cell complexes (see Remark 1.2.7), then a space is f-local if and only if it is fibrant in the f-local model category structure of Theorem 2.1.2.*

The proof of Theorem 2.1.2 will use the following proposition.

PROPOSITION 2.1.4. *If $f\colon A \to B$ is a map of cofibrant spaces in $\mathsf{Spc}_{(*)}$, then there is a set J of inclusions of cell complexes (see Remark 1.2.7) such that*

1. *every map in J is an f-local equivalence, and*
2. *the class of J-cofibrations (see Definition 12.4.1) equals the class of cofibrations that are also f-local equivalences.*

We will present the proof of Proposition 2.1.4 in Section 2.4, after some necessary preparatory work in Section 2.3.

PROOF OF THEOREM 2.1.2. We begin by using Theorem 13.3.1 to show that there is a cofibrantly generated model category structure on $\mathsf{Spc}_{(*)}$ with weak equivalences, cofibrations, and fibrations as described in the statement of Theorem 2.1.2.

Proposition 1.2.12 implies that the class of f-local equivalences satisfies the "two out of three" axiom, and Proposition 1.2.13 implies that it is closed under retracts.

Let I be the set of maps

$$I = \{\mathsf{Spc}_{(*)}(\partial\Delta[n]) \to \mathsf{Spc}_{(*)}(\Delta[n]) \mid n \geq 0\}$$

(see Definition 1.1.16) and let J be the set of maps provided by Proposition 2.1.4. Since every map in either I or J is an inclusion of simplicial sets (if $\mathsf{Spc}_{(*)} = \mathsf{SS}_{(*)}$) or an inclusion of cell complexes (if $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$), Example 12.3.4 and Example 12.3.5 imply that condition 1 of Theorem 13.3.1 is satisfied.

The subcategory of I-cofibrations is the subcategory of cofibrations in the usual model category structure in $\mathsf{Spc}_{(*)}$, and the I-injectives are the usual trivial fibrations. Thus, Proposition 2.1.4 implies that condition 2 of Theorem 13.3.1 is satisfied.

Since the J-cofibrations are a subcategory of the I-cofibrations, every I-injective must be a J-injective. Proposition 1.2.11 implies that every J-injective is an f-local equivalence, and so condition 3 is satisfied.

Proposition 2.1.4 implies that condition 4a of Theorem 13.3.1 is satisfied, and so Theorem 13.3.1 now implies that we have a model category.

To show that our model category is a simplicial model category, we note that, since the simplicial structure is the usual one, axiom M6 of Definition 10.1.2 holds because it does so in the usual simplicial model category structure on $\mathsf{Spc}_{(*)}$. For axiom M7 of Definition 10.1.2, we note that the class of f-local cofibrations equals the usual class of cofibrations and the class of f-local fibrations is contained in the usual class of fibrations. Thus, the first requirement of axiom M7 is clear. In the case that the map p is an f-local equivalence, the rest of axiom M7 follows from the fact that, since the class of f-local cofibrations equals the usual class of cofibrations, the class of f-local trivial fibrations equals the usual class of trivial fibrations (see Proposition 8.2.3).

In the case that the map i is an f-local equivalence, we choose a cofibrant approximation $\tilde{\imath}\colon \widetilde{A} \to \widetilde{B}$ to i such that $\tilde{\imath}$ is a cofibration (see Proposition 9.1.9). Proposition 10.3.3 and Proposition 10.3.10 imply that, for every $n \geq 0$, the map $\widetilde{A} \otimes \Delta[n] \amalg_{\widetilde{A}\otimes\partial\Delta[n]} \widetilde{B} \otimes \partial\Delta[n] \to \widetilde{B} \otimes \Delta[n]$ is also an f-local equivalence, and so it has the left lifting property with respect to the map p. Lemma 10.3.6 now implies that the map $\tilde{\imath}$ has the left lifting property with respect to the map $X^{\Delta[n]} \to Y^{\Delta[n]} \times_{Y^{\partial\Delta[n]}} X^{\partial\Delta[n]}$ for every $n \geq 0$. Since $\mathsf{Spc}_{(*)}$ is a left proper model category (see Theorem 11.1.16), Proposition 11.1.18 implies that the map i has the left lifting property with respect to the map $X^{\Delta[n]} \to Y^{\Delta[n]} \times_{Y^{\partial\Delta[n]}} X^{\partial\Delta[n]}$ for every $n \geq 0$, and so the result follows from Lemma 10.3.6. □

PROOF OF PROPOSITION 2.1.3. If W is fibrant in the f-local model category structure, then the map $W \to *$ has the right lifting property with respect to every cofibration that is an f-local equivalence. Proposition 1.3.7 implies that every horn on f is both a cofibration and an f-local equivalence, and so Proposition 1.3.3 implies that W is f-local.

Conversely, assume that W is f-local. If $i\colon A \to B$ is both a cofibration and an f-local equivalence, then Proposition 9.1.9 implies that there is a cofibrant approximation $\tilde{\imath}\colon \widetilde{A} \to \widetilde{B}$ to i such that $\tilde{\imath}$ is a cofibration, and Proposition 11.1.18 and Proposition 8.2.3 imply that it is sufficient to show that $\tilde{\imath}$ has the left lifting property with respect to the map $W \to *$. Proposition 1.2.11 and Proposition 1.2.12 imply that $\tilde{\imath}$ is an f-local equivalence, and so Proposition 10.3.3 and Proposition 10.3.4 imply that $\tilde{\imath}$ has the left lifting property with respect to the map $W \to *$. $\square$

Corollary 1.4.17 implies that every $\overline{\Lambda\{f\}}$-cofibration is an f-local equivalence. The following example (due to A. K. Bousfield) shows that, among the cofibrations that are f-local equivalences, there are maps that are not $\overline{\Lambda\{f\}}$-cofibrations.

Example 2.1.5. Let $\mathsf{Spc}_{(*)} = \mathsf{Top}_*$, and let $f\colon A \to B$ be the inclusion $S^n \to D^{n+1}$. The path space fibration $p\colon \mathrm{P\,K}(\mathbb{Z}, n) \to \mathrm{K}(\mathbb{Z}, n)$ is a $\overline{\Lambda\{f\}}$-injective (see Definition 1.4.2), and so every $\overline{\Lambda\{f\}}$-cofibration has the homotopy left lifting property with respect to p (see Definition 10.3.2). The cofibration $* \to S^n$ does not have the homotopy left lifting property with respect to p, and so it is not a $\overline{\Lambda\{f\}}$-cofibration (see **Fix this reference!**). However, since both the composition $* \to S^n \to D^{n+1}$ and f itself are f-local equivalences (see Proposition 1.2.11), the "two out of three" property of weak equivalences implies that the inclusion $* \to S^n$ is an f-local equivalence. Thus, $* \to S^n$ is both a cofibration and an f-local equivalence, but it is not a $\overline{\Lambda\{f\}}$-cofibration.

2.2. Cell complexes of topological spaces

A cell complex in $\mathsf{Top}_{(*)}$ is a topological space built by a sequential process of attaching cells. The class of cell complexes includes the class of CW-complexes, but the attaching map of a cell in a cell complex need not be contained in a union of cells of lower dimension. Thus, while a CW-complex can be built by a countable process of attaching unions of cells, a general cell complex may require an arbitrarily long transfinite construction. Cell complexes and their retracts are the cofibrant objects in the standard model category of topological spaces.

Definition 2.2.1.

- A *relative cell complex* in Top is a map that is a transfinite composition (see Definition 12.2.2) of pushouts (see Definition 8.2.10) of maps of the form $\left|\partial\Delta[n]\right| \to \left|\Delta[n]\right|$ for $n \geq 0$. The topological space X in Top is a *cell complex* if the map $\emptyset \to X$ is a relative cell complex, and it is a *finite cell complex* if the map $\emptyset \to X$ is a finite composition of pushouts of maps of the form $\left|\partial\Delta[n]\right| \to \left|\Delta[n]\right|$ for $n \geq 0$.
- A *relative cell complex* in Top_* is a map that is a transfinite composition of pushouts of maps of the form $\left|\partial\Delta[n]\right|^+ \to \left|\Delta[n]\right|^+$ for $n \geq 0$. The topological space X in Top_* is a *cell complex* if the map $* \to X$ is a relative cell complex, and it is a *finite cell complex* if the map $* \to X$ is a finite composition of pushouts of maps of the form $\left|\partial\Delta[n]\right|^+ \to \left|\Delta[n]\right|^+$ for $n \geq 0$.

Example 2.2.2. A CW-complex in $\mathsf{Top}_{(*)}$ is a cell complex.

Remark 2.2.3. Definition 2.2.1 implies that a relative cell complex in $\mathsf{Top}_{(*)}$ is a map that can be constructed as a transfinite composition of pushouts of inclusions of the boundary of a cell into that cell, but there will generally be many different possible such constructions. When dealing with a topological space that is a cell complex or a map that is a relative cell complex, we will often assume that we have chosen some specific such construction. Furthermore, we may choose a construction of the map as a transfinite composition of pushouts of *coproducts* of cells, i.e., we will consider constructions as transfinite compositions in which more than one cell is attached at a time (see Proposition 12.2.5).

PROPOSITION 2.2.4. *If $X \to Y$ is a relative cell complex, then a compact subset of Y can intersect the interiors of only finitely many cells of $Y - X$.*

PROOF. Let C be a subset of Y; we will show that if C intersects the interiors of infinitely many cells of $Y - X$, then C has an infinite subset that has no accumulation point (which implies that C is not compact).

Suppose now that C intersects the interiors of infinitely many cells of $Y - X$. We construct a subset P of C by choosing one point of C from the interior of each cell whose interior intersects C. We will now show that this infinite subset P of C has no accumulation point in C. We will do this by showing that, for every point $c \in C$, there is an open subset U of Y such that $c \in U$ and $U \cap P$ has at most one point.

Let e_c be the unique cell of $Y - X$ that contains c in its interior. Since there is at most one point of P in the interior of any cell of $Y - X$, we can choose an open subset U_c of the interior of e_c that contains no points of P (except for c, if $c \in P$). We will use Zorn's lemma to show that we can enlarge U_c to an open subset of Y that contains no points of P (except for c, if $c \in P$).

Let α be the presentation ordinal (see Definition 12.5.4) of the cell e_c. If the presentation ordinal of the relative cell complex $X \to Y$ is γ, consider the set T of ordered pairs (β, U) where $\alpha \le \beta \le \gamma$ and U is an open subset of Y^β such that $U \cap Y^\alpha = U_c$ and U contains no points of P except possibly c. We define a preorder on T by defining $(\beta_1, U_1) < (\beta_2, U_2)$ if $\beta_1 < \beta_2$ and $U_2 \cap Y^{\beta_1} = U_1$.

If $\{(\beta_s, U_s)\}_{s \in S}$ is a chain in T, then $(\bigcup_{s\in S} \beta_s, \bigcup_{s \in S} U_s)$ (see Section 12.1.1) is an upper bound in T for the chain, and so Zorn's lemma implies that T has a maximal element (β_m, U_m). We will complete the proof by showing that $\beta_m = \gamma$.

If $\beta_m < \gamma$, then consider the cells of presentation ordinal $\beta_m + 1$. Since Y has the weak topology determined by X and the cells of $Y - X$, we need only enlarge U_m so that its intersection with each cell of presentation ordinal $\beta_m + 1$ is open in that cell, and so that it still contains no points of P except possibly c. If $h\colon S^{n-1} \to Y^{\beta_m}$ is the attaching map for a cell of presentation ordinal $\beta_m + 1$, then $h^{-1}U_m$ is open in S^{n-1}, and so we can "thicken" $h^{-1}U_m$ to an open subset of D^n, avoiding the (at most one) point of P that is in the interior of the cell. If we let U' equal the union of U_m with these thickenings in the interiors of the cells of presentation ordinal $\beta_m + 1$, then the pair $(\beta_m + 1, U')$ is an element of T greater than the maximal element (β_m, U_m) of T. This contradiction implies that $\beta_m = \gamma$, and so the proof is complete. □

COROLLARY 2.2.5. *A compact subset of a cell complex can intersect the interiors of only finitely many cells.*

PROOF. This follows from Proposition 2.2.4. □

PROPOSITION 2.2.6. *Every cell of a cell complex is contained in a finite subcomplex of the cell complex.*

PROOF. If we choose a presentation of the cell complex X (see Definition 12.5.2), then the proposition follows from Corollary 2.2.5, using a transfinite induction on the presentation ordinal of the cell. The attaching map of any cell intersects the interiors of only finitely many cells, each of which (by the induction hypothesis) is contained in a finite subcomplex of X. □

COROLLARY 2.2.7. *A compact subset of a cell complex is contained in a finite subcomplex of the cell complex.*

PROOF. This follows from Corollary 2.2.5 and Proposition 2.2.6. □

2.3. Subcomplexes of relative $\overline{\Lambda\{f\}}$-cell complexes

The proof of Proposition 2.1.4 (in Section 2.4) will require a careful analysis of the localization of a space. Since the localization map is a relative $\overline{\Lambda\{f\}}$-cell complex, we need to study subcomplexes of relative $\overline{\Lambda\{f\}}$-cell complexes.

DEFINITION 2.3.1. Let $f: A \to B$ be an inclusion of cell complexes (see Remark 1.2.7).

- If $C \to D$ is an element of $\overline{\Lambda\{f\}}$ (see Definition 1.3.2), then D will also be called a $\overline{\Lambda\{f\}}$*-cell*, C will be called the *boundary* of the $\overline{\Lambda\{f\}}$-cell, and $D - C$ will be called the *interior* of the $\overline{\Lambda\{f\}}$-cell. (The interior of a $\overline{\Lambda\{f\}}$-cell is not, in general, a subcomplex.)
- If $C \to D$ is a map in $\overline{\Lambda\{f\}}$ and

$$\begin{tikzcd} C \arrow[r] \arrow[d] & X \arrow[d] \\ D \arrow[r] & Y \end{tikzcd}$$

is a pushout, then we will refer to the image of D in Y as a $\overline{\Lambda\{f\}}$*-cell*.

2.3.2. Presentations of relative $\overline{\Lambda\{f\}}$-cell complexes. A relative $\overline{\Lambda\{f\}}$-cell complex is a map that can be constructed as a transfinite composition of pushouts of elements of $\overline{\Lambda\{f\}}$ (see Definition 1.3.8). To consider subcomplexes of a relative $\overline{\Lambda\{f\}}$-cell complex, we need to choose a particular such construction.

DEFINITION 2.3.3. If $g: X \to Y$ is a relative $\overline{\Lambda\{f\}}$-cell complex (see Definition 1.3.8), then a *presentation* of g is a pair consisting of a λ-sequence

$$X = X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \qquad (\beta < \lambda)$$

(for some ordinal λ) and a set of ordered triples

$$\{(T^\beta, e^\beta, h^\beta)\}_{\beta<\lambda}$$

such that

1. the composition of the λ-sequence is the map $g: X \to Y$,
2. each T^β is a set,
3. each e^β is a function $e^\beta : T^\beta \to \overline{\Lambda\{f\}}$ (see Definition 1.3.2),
4. for every $\beta < \lambda$, if $i \in T^\beta$ and e^β_i is the $\overline{\Lambda\{f\}}$-cell $C_i \to D_i$, then h^β_i is a map $h^\beta_i : C_i \to X_\beta$, and

5. every $X_{\beta+1}$ is the pushout

$$\begin{tikzcd}
\coprod_{T^\beta} C_i \arrow[r] \arrow[d, "\coprod h_i^\beta"'] & \coprod_{T^\beta} D_i \arrow[d] \\
X_\beta \arrow[r] & X_{\beta+1}.
\end{tikzcd}$$

DEFINITION 2.3.4. Let $g\colon X \to Y$ be a relative $\overline{\Lambda\{f\}}$-cell complex with presentation $\big(X = X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \quad (\beta < \lambda), \{T^\beta, e^\beta, h^\beta\}_{\beta<\lambda}\big)$.

1. If e is a $\overline{\Lambda\{f\}}$-cell of g (see Definition 1.3.2), the *presentation ordinal* of e is defined to be the first ordinal β such that e is in X_β.
2. If $\beta < \lambda$, then the *β-skeleton* of g is defined to be X_β. We will sometimes abuse language and refer to the image of X_β in Y as the β-skeleton of g.

2.3.5. Constructing a subcomplex of a relative $\overline{\Lambda\{f\}}$-cell complex.

DEFINITION 2.3.6. If $g\colon X \to Y$ is a relative $\overline{\Lambda\{f\}}$-cell complex with presentation $\big(X = X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \quad (\beta < \lambda), \{T^\beta, e^\beta, h^\beta\}_{\beta<\lambda}\big)$, then a *subcomplex* of g relative to that presentation consists of a family of sets $\{\widetilde{T}^\beta\}_{\beta<\lambda}$ such that

1. for every $\beta < \lambda$, the set $\widetilde{T}^\beta$ is a subset of T^β,
2. there is a λ-sequence
$$X = \widetilde{X}_0 \to \widetilde{X}_1 \to \widetilde{X}_2 \to \cdots \to \widetilde{X}_\beta \to \cdots \qquad (\beta < \lambda)$$
(called the λ-sequence *associated* with the subcomplex) and a map of λ-sequences
$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & \widetilde{X}_0 \arrow[r] \arrow[d] & \widetilde{X}_1 \arrow[r] \arrow[d] & \widetilde{X}_2 \arrow[r] \arrow[d] & \cdots \\
X \arrow[r, equal] & X_0 \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & \cdots
\end{tikzcd}$$
such that, for every $\beta < \lambda$ and every $i \in \widetilde{T}^\beta$, the map $h_i^\beta : C_i \to X_\beta$ factors through the map $\widetilde{X}_\beta \to X_\beta$, and
3. for every $\beta < \lambda$, the square
$$\begin{tikzcd}
\coprod_{\widetilde{T}^\beta} C_i \arrow[r] \arrow[d] & \coprod_{\widetilde{T}^\beta} D_i \arrow[d] \\
\widetilde{X}_\beta \arrow[r] & \widetilde{X}_{\beta+1}.
\end{tikzcd}$$
is a pushout.

REMARK 2.3.7. Although a subcomplex of a relative $\overline{\Lambda\{f\}}$-cell complex can only be defined relative to some particular presentation of that relative $\overline{\Lambda\{f\}}$-cell complex, we will often discuss subcomplexes of a relative $\overline{\Lambda\{f\}}$-cell complex without explicitly mentioning the presentation relative to which the subcomplexes are defined.

REMARK 2.3.8. Although a subcomplex of a relative $\overline{\Lambda\{f\}}$-cell complex with some particular presentation is defined to be a family of sets $\{\widetilde{T}^\beta\}_{\beta<\lambda}$ (see Definition 2.3.6), we will often abuse language and refer to the λ-sequence associated with the subcomplex, or the composition of that λ-sequence, as a "subcomplex".

REMARK 2.3.9. Note that the definition of a subcomplex implies that the maps $\widetilde{X}_\beta \to X_\beta$ are all relative $\overline{\Lambda\{f\}}$-cell complexes. Since a relative $\overline{\Lambda\{f\}}$-cell complex is a monomorphism, the factorization of each h_i^β through $\widetilde{X}_\beta \to X_\beta$ is unique. Thus, a subcomplex of a relative $\overline{\Lambda\{f\}}$-cell complex is itself a relative $\overline{\Lambda\{f\}}$-cell complex with a natural presentation.

PROPOSITION 2.3.10. *Given a relative $\overline{\Lambda\{f\}}$-cell complex $X \to Y$ with presentation $\big(X = X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \quad (\beta < \lambda), \{T^\beta, e^\beta, h^\beta\}_{\beta<\lambda}\big)$, an arbitrary subcomplex can be constructed by the following inductive procedure.*

1. *Choose an arbitrary subset $\widetilde{T}^0$ of T^0.*
2. *If $\beta < \lambda$ and we have defined $\{\widetilde{T}^\gamma\}_{\gamma<\beta}$, then we have determined the space $\widetilde{X}_\beta$ and the map $\widetilde{X}_\beta \to X_\beta$ (where $\widetilde{X}_\beta$ is the space that appears in the λ-sequence associated to the subcomplex). Consider the set*

$$\{i \in T^\beta \mid h_i^\beta \colon C_i \to X_\beta \text{ factors through } \widetilde{X}_\beta \to X_\beta\}$$

Choose an arbitrary subset $\widetilde{T}^\beta$ of this set. For every $i \in \widetilde{T}^\beta$, there is a unique map $\tilde{h}_i^\beta : C_i \to \widetilde{X}_\beta$ that makes the diagram

$$\begin{tikzcd} C_i \arrow[d, "\tilde{h}_i^\beta"'] \arrow[dr, "h_i^\beta"] & \\ \widetilde{X}_\beta \arrow[r] & X_\beta \end{tikzcd}$$

commute. We let $\widetilde{X}_{\beta+1}$ be the pushout

$$\begin{tikzcd} \coprod_{\widetilde{T}^\beta} C_i \arrow[r] \arrow[d, "\amalg \tilde{h}^\beta"'] & \coprod_{\widetilde{T}^\beta} D_i \arrow[d] \\ \widetilde{X}_\beta \arrow[r] & \widetilde{X}_{\beta+1} \end{tikzcd}$$

PROOF. This follows directly from the definitions. □

PROPOSITION 2.3.11. *Let $g\colon X \to Y$ be a relative $\overline{\Lambda\{f\}}$-cell complex with presentation $\big(X = X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \quad (\beta < \lambda), \{T^\beta, e^\beta, h^\beta\}_{\beta<\lambda}\big)$. If $\big\{\{\widetilde{T}^{u\beta}\}_{\beta<\lambda}\big\}_{u\in U}$ is a set of subcomplexes of g, then the intersection $\{\widetilde{T}^\beta\}_{\beta<\lambda}$ of the set of subcomplexes (where $\widetilde{T}^\beta = \bigcap_{u\in U} \widetilde{T}^{u\beta}$ for every $\beta < \lambda$) is a subcomplex of g.*

PROOF. It is sufficient to show that, if $\beta < \lambda$ and we have constructed the β-skeleton of the associated λ-sequence $X = \widetilde{X}^0 \to \widetilde{X}_1 \to \widetilde{X}_2 \to \cdots \to \widetilde{X}_\beta$, then, for every $i \in \widetilde{T}^\beta$, the map $h_i^\beta \colon C_i \to X_\beta$ factors through $\widetilde{X}_\beta \to X_\beta$. If $i \in \widetilde{T}^\beta$, then $i \in \widetilde{T}^{u\beta}$ for every $u \in U$, and so h_i^β factors uniquely through $\widetilde{X}_\beta^u \to X_\beta$ for every $u \in U$. Since $\widetilde{X}_\beta$ is the limit of the diagram that contains the map $\widetilde{X}_\beta^u \to X_\beta$

for every $u \in U$, the map h_i^β factors uniquely through $\widetilde{X}_\beta \to X_\beta$, and the proof is complete. □

COROLLARY 2.3.12. *Let $g\colon X \to Y$ be a relative $\overline{\Lambda\{f\}}$-cell complex with presentation $\bigl(X = X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \quad (\beta < \lambda), \{T^\beta, e^\beta, h^\beta\}_{\beta<\lambda}\bigr)$. If e is an f-cell of g, then there is a smallest subcomplex of g that contains e, i.e., a subcomplex of g containing e that is a subcomplex of every subcomplex of g that contains e.*

PROOF. Proposition 2.3.11 implies that we can take the intersection of all subcomplexes of g that contain e. □

DEFINITION 2.3.13. If e is a $\overline{\Lambda\{f\}}$-cell of the relative $\overline{\Lambda\{f\}}$-cell complex $g\colon X \to Y$ with some particular presentation, then the smallest subcomplex of g that contains e (whose existence is guaranteed by Corollary 2.3.12) will be called the subcomplex *generated* by e.

2.3.14. Subcomplexes of the localization. If $f\colon A \to B$ is an inclusion of cell complexes (see Remark 1.2.7), then for every space X, the localization $j_X : X \to \mathrm{L}_f X$ has a natural presentation as a relative $\overline{\Lambda\{f\}}$-cell complex. When we discuss subcomplexes of j_X, it will be with respect to that natural presentation.

LEMMA 2.3.15. *Let $f\colon A \to B$ be an inclusion of cell complexes (see Remark 1.2.7), and let X be a simplicial set (or a cell complex). If W is a subcomplex of X, then $\mathrm{L}_f W$ is naturally isomorphic (or homeomorphic) to a subcomplex of $\mathrm{L}_f X$ (where by "naturally" we mean that this isomorphism is a functor on the category of subcomplexes of X).*

PROOF. The construction of $\mathrm{L}_f X$ from X defines an obvious presentation of the relative $\overline{\Lambda\{f\}}$-cell complex $j_X : X \to \mathrm{L}_f X$. Since an inclusion of a subcomplex is a monomorphism, the construction of $\mathrm{L}_f W$ from W defines an obvious natural isomorphism of the relative $\overline{\Lambda\{f\}}$-cell complex $W \to \mathrm{L}_f W$ with a subcomplex of $j(X)$. □

PROPOSITION 2.3.16. *Let $f\colon A \to B$ be an inclusion of cell complexes (see Remark 1.2.7). If X is a simplicial set (or a cell complex) and W is a subcomplex of X, then $\mathrm{L}_f W$ is naturally isomorphic (or homeomorphic) to the subcomplex of $\mathrm{L}_f X$ consisting of those $\overline{\Lambda\{f\}}$-cells of $\mathrm{L}_f X$ for which the zero skeleton of the subcomplex of $\mathrm{L}_f X$ generated by that $\overline{\Lambda\{f\}}$-cell (see Definition 2.3.13) is a subcomplex of W.*

PROOF. We identify $\mathrm{L}_f W$ with a subcomplex of $\mathrm{L}_f X$ as in Lemma 2.3.15, and we will show by transfinite induction on the presentation ordinal (see Definition 2.3.4) of the $\overline{\Lambda\{f\}}$-cell that a $\overline{\Lambda\{f\}}$-cell of $\mathrm{L}_f X$ is in $\mathrm{L}_f W$ if and only if the zero skeleton of the subcomplex of $\mathrm{L}_f X$ generated by that $\overline{\Lambda\{f\}}$-cell (see Definition 2.3.13) is a subcomplex of W.

If e is a $\overline{\Lambda\{f\}}$-cell of presentation ordinal 1, then the subcomplex of $\mathrm{L}_f X$ generated by e consists of the union of e and the subcomplexes of X generated by those simplices (or cells) of X whose interiors intersect the image of the attaching map of e. Thus, the zero skeleton of the subcomplex of $\mathrm{L}_f X$ generated by e is a subcomplex of W if and only if the attaching map of e factors through the inclusion $W \to X$, which is true if and only if e is contained in $\mathrm{L}_f W$.

Since there are no $\overline{\Lambda\{f\}}$-cells whose presentation ordinal is a limit ordinal, we assume that $\beta + 1 < \lambda$ and that the assertion is true for all $\overline{\Lambda\{f\}}$-cells of presentation ordinal less than or equal to β. Let e be a $\overline{\Lambda\{f\}}$-cell of presentation ordinal $\beta + 1$. The subcomplex of $\mathrm{L}_f X$ generated by e consists of the union of e and the subcomplexes of $\mathrm{L}_f X$ generated by those $\overline{\Lambda\{f\}}$-cells and simplices (or cells) of X whose interiors intersect the image of the attaching map of e. Each of those $\overline{\Lambda\{f\}}$-cells is of presentation ordinal at most β, and so it is in $\mathrm{L}_f W$ if and only if the zero skeleton of the subcomplex of $\mathrm{L}_f X$ it generates is contained in W, and the inductive hypothesis implies that this is true if and only if that $\overline{\Lambda\{f\}}$-cell is in $\mathrm{L}_f W$. Thus, the subcomplex of $\mathrm{L}_f X$ generated by e is contained in $\mathrm{L}_f W$ if and only if the attaching map for e factors through $W_\beta \to X_\beta$, i.e., if and only if e is in $\mathrm{L}_f W$. □

PROPOSITION 2.3.17. *Let $f\colon A \to B$ be an inclusion of cell complexes (see Remark 1.2.7). If X is a simplicial set (or a cell complex) and $\{W_s\}_{s\in S}$ is a family of subcomplexes of X, then $\mathrm{L}_f(\bigcap_{s\in S} W_s) = \bigcap_{s\in S} \mathrm{L}_f W_s$.*

PROOF. This follows from Proposition 2.3.16. □

PROPOSITION 2.3.18. *Let $f\colon A \to B$ be an inclusion of cell complexes (see Remark 1.2.7). If X is a simplicial set (or a cell complex) and $W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_\beta \subset \cdots \quad (\beta < \lambda)$ is a λ-sequence of subcomplexes of X (where λ is the ordinal chosen in Section 1.3.4), then the natural map $\mathrm{colim}_{\beta<\lambda} \mathrm{L}_f W_\beta \to \mathrm{L}_f \,\mathrm{colim}_{\beta<\lambda} W_\beta$ is an isomorphism (or a homeomorphism).*

PROOF. Proposition 2.3.16 implies that the map is an isomorphism onto a subcomplex; it remains only to show that every $\overline{\Lambda\{f\}}$-cell of $\mathrm{L}_f \,\mathrm{colim}_{\beta<\lambda} W_\beta$ is contained in some $\mathrm{L}_f W_\beta$. We will do this by a transfinite induction on the presentation ordinal of the $\overline{\Lambda\{f\}}$-cell (see Definition 2.3.4).

If e is a $\overline{\Lambda\{f\}}$-cell of $\mathrm{L}_f \,\mathrm{colim}_{\beta<\lambda} W_\beta$ of presentation ordinal 1, then its attaching map is a map to $\mathrm{colim}_{\beta<\lambda} W_\beta$, and the discussion in Section 1.3.4 explains why there is an ordinal $\beta < \lambda$ such that the image of the attaching map is contained in W_β. Thus, the $\overline{\Lambda\{f\}}$-cell is in $\mathrm{L}_f W_\beta$.

Since there are no $\overline{\Lambda\{f\}}$-cells of presentation ordinal equal to a limit ordinal, we now let γ be an ordinal such that $\gamma + 1 < \lambda$, and we assume that the assertion is true for all $\overline{\Lambda\{f\}}$-cells of presentation ordinal less than or equal to γ. If e is a $\overline{\Lambda\{f\}}$-cell of presentation ordinal $\gamma + 1$, then e has fewer than λ simplices (or cells). Thus, the image of the attaching map of e is contained in the interiors of fewer than λ many $\overline{\Lambda\{f\}}$-cells, each of presentation ordinal less than or equal to γ. (If $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$, then this follows from Corollary 2.2.5.) The induction hypothesis implies that each of these is contained in some $\mathrm{L}_f W_\beta$. Since λ is a regular cardinal, there must exist $\beta < \lambda$ such that the union of these $\overline{\Lambda\{f\}}$-cells is contained in $\mathrm{L}_f W_\beta$, and so e is also contained in $\mathrm{L}_f W_\beta$. □

2.4. The Bousfield-Smith cardinality argument

The proof of Proposition 2.1.4 is at the end of this section. The cardinality argument that we use here was first used by A. K. Bousfield [**9**] to define a model category structure on the category of simplicial sets in which a weak equivalence was a map that induced a homology isomorphism (for some chosen homology theory).

This was extended to more general localizations of cofibrantly generated model categories (see Definition 13.2.1) by J. H. Smith. We are indebted to D. M. Kan for explaining this argument to us.

We will prove Proposition 2.1.4 by showing that there is a set J of cofibrations that are f-local equivalences such that every cofibration that is an f-local equivalence has the left lifting property (see Definition 8.2.1) with respect to every J-injective. Proposition 2.1.4 will then follow from Corollary 12.4.17.

We will find the set J by showing (in Proposition 2.4.8) that there is a cardinal γ such that, if a map has the right lifting property with respect to all inclusions of simplicial sets (or of cell complexes) that are f-local equivalences between complexes of size no larger than γ, then it has the right lifting property with respect to all cofibrations that are f-local equivalences. By the "size" of a simplicial set (or a cell complex) X, we will mean the cardinal of the set of simplices (or cells) of X. We will then let J be a set of representatives of the isomorphism classes of of these "small enough" inclusions of complexes that are f-local equivalences.

We must first deal with an inconvenient aspect of the categories Top and Top_*: Not all spaces are cell complexes. This requires Lemma 2.4.1, which shows that, for a fibration to have the right lifting property (see Definition 8.2.1) with respect to all cofibrations that are f-local equivalences, it is sufficient for it to have the right lifting property with respect to all such cofibrations that are inclusions of cell complexes.

LEMMA 2.4.1. *Let $f\colon A \to B$ be a map of cofibrant spaces in $\mathsf{Top}_{(*)}$. If $p\colon E \to B$ is a fibration with the right lifting property with respect to all inclusions of cell complexes that are f-local equivalences, then it has the right lifting property with respect to all cofibrations that are f-local equivalences.*

PROOF. Let $g\colon X \to Y$ be a cofibration that is an f-local equivalence. Proposition 13.2.16 implies that there is a cofibrant approximation (see Definition 9.1.8) g_c to g such that g_c is an inclusion of cell complexes. Proposition 1.2.11 and Proposition 1.2.12 imply that g_c is an f-local equivalence, and so the lemma now follows from Proposition 11.1.18. □

We can now restrict our attention to inclusions of simplicial sets (if $\mathsf{Spc}_{(*)} = \mathsf{SS}_{(*)}$) or inclusions of cell complexes (if $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$). We need to find a cardinal γ with two properties:

1. The cardinal γ is "large enough" in that, for every complex X, every subcomplex of $\mathrm{L}_f X$ of size no greater than γ is contained in the localization of a subcomplex of X of size no greater than γ.
2. The cardinal γ is "stable" in that, if X is a complex of size no greater than γ, then $\mathrm{L}_f X$ will also have size no greater than γ.

Once we have such a cardinal γ, Proposition 2.4.7 (which uses Lemma 2.4.5) will show that any inclusion of complexes that is an f-local equivalence can be built out of ones of size no greater than γ. This will be used in Proposition 2.4.8 to show that if a map has the right lifting property with respect to all "small" inclusions of complexes that are f-local equivalences, then it has the right lifting property with respect to all inclusions of complexes that are f-local equivalences. We define our cardinal γ in Definition 2.4.4.

DEFINITION 2.4.2. If the set of simplices (or cells) of the complex X has cardinal κ, then we will say that X is of *size* κ.

LEMMA 2.4.3. *Let $f\colon A \to B$ be an inclusion of cell complexes (see Remark 1.2.7), and let λ be the first infinite cardinal greater than that of the simplices (or cells) of $A \amalg B$. For any complex X, we have $\mathrm{L}_f X \approx \operatorname{colim} \mathrm{L}_f X_s$, where X_s varies over the subcomplexes of X of size less than λ.*

PROOF. Proposition 2.3.16 implies that each $\mathrm{L}_f X_s$ is a subcomplex of $\mathrm{L}_f X$, and so we need only show that every $\overline{\Lambda\{f\}}$-cell of $\mathrm{L}_f X$ is contained in $\mathrm{L}_f X_s$ for some small subcomplex X_s of X. We will do this by a transfinite induction on the presentation ordinal of the $\overline{\Lambda\{f\}}$-cell (see Definition 2.3.4). To ease the strain of terminology, for the remainder of this proof, the word "small" will mean "of size less than λ".

The induction is begun by noting that the zero skeleton of $X \to \mathrm{L}_f X$ equals X. Since there are no $\overline{\Lambda\{f\}}$-cells of sequential dimension equal to a limit ordinal, we need only consider the case of successor ordinals.

Now let $\beta + 1 < \lambda$, and assume that each $\overline{\Lambda\{f\}}$-cell of presentation ordinal less than or equal to β is contained in $\mathrm{L}_f X_s$ for some small subcomplex X_s of X. Any $\overline{\Lambda\{f\}}$-cell of presentation ordinal $\beta + 1$ must be attached by a map of its boundary to the β-skeleton of $\mathrm{L}_f X$ (see Definition 2.3.4). Since the boundary of an $\overline{\Lambda\{f\}}$-cell has size less than λ, the image of the attaching map can intersect the interiors of fewer than λ other simplices (or cells), each of which is either in X or in an $\overline{\Lambda\{f\}}$-cell of sequential dimension less than or equal to β. (If $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$, then this uses Corollary 2.2.5.) Thus, our $\overline{\Lambda\{f\}}$-cell is attached to the union of X with some $\overline{\Lambda\{f\}}$-cells, each of which is contained in the localization of a small subcomplex of X. If we let Z be the union of those small subcomplexes of X and the subcomplexes of X generated by the (fewer than λ) simplices (or cells) of X in the image of the attaching map of our $\overline{\Lambda\{f\}}$-cell, then Z is the union a collection of size less than λ of subcomplexes of X, each of which is of size less than λ. Since λ is a regular cardinal, this implies that Z is of size less than λ, and our $\overline{\Lambda\{f\}}$-cell is contained in $\mathrm{L}_f Z$. $\square$

DEFINITION 2.4.4. We let $\mathfrak{c}$ denote the cardinal of the continuum, i.e., $\mathfrak{c}$ is the cardinal of the set of real numbers. We let λ denote the ordinal (which is also a cardinal) selected in Section 1.3.4, i.e., if $f\colon A \to B$, then λ is the first infinite cardinal greater than that of the set of simplices (or cells) of $A \amalg B$. We now define γ as

$$\gamma = \begin{cases} \lambda^{\lambda} & \text{if } \mathsf{Spc}_{(*)} = \mathsf{SS}_{(*)} \\ (\lambda\mathfrak{c})^{\lambda\mathfrak{c}} & \text{if } \mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}. \end{cases}$$

Thus, if $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$, then $\gamma = (\lambda\mathfrak{c})^{\lambda\mathfrak{c}} = \max(\lambda^{\lambda}, \mathfrak{c}^{\mathfrak{c}}) = (\lambda^{\lambda})(\mathfrak{c}^{\mathfrak{c}})$ (since the maximum of two infinite cardinals equals their product (see, e.g., [**25**, Chapter 2])).

LEMMA 2.4.5. *Let $f\colon A \to B$ be an inclusion of cell complexes (see Remark 1.2.7), and let X be a simplicial set (or a cell complex). If Z is a subcomplex of $\mathrm{L}_f X$ of size less than or equal to γ, then there exists a subcomplex W of X, of size less than or equal to γ, such that $Z \subset \mathrm{L}_f W$.*

PROOF. Lemma 2.4.3 implies that each simplex (or cell) of Z is contained in the localization of some subcomplex of X of size less than λ, and so Proposition 2.3.16 implies that Z is contained in the localization of the union of those subcomplexes. Since $\lambda < \gamma$ (see Definition 2.4.4), $\lambda \times \gamma = \gamma$, and so that union of subcomplexes is of size less than or equal to γ. $\square$

LEMMA 2.4.6. *Let $f\colon A \to B$ be an inclusion of cell complexes (see Remark 1.2.7). If X is a simplicial set (or a cell complex) of size less than or equal to γ (see Definition 2.4.4), then $\mathrm{L}_f X$ has size less than or equal to γ.*

PROOF. Let $X = X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \quad (\beta < \lambda)$ be the λ-sequence that is part of the natural presentation of the relative $\overline{\Lambda\{f\}}$-cell complex $X \to \mathrm{L}_f X$ (see Definition 2.3.3). We will prove by transfinite induction that, for every $\beta < \lambda$, the complex X_β has size less than or equal to γ. Since $\mathrm{L}_f X = \operatorname{colim}_{\beta<\lambda} X_\beta$ and $\mathrm{Succ}(\gamma)$ (see Definition 12.1.11) is a regular cardinal (see Definition 12.1.15), this will imply the lemma.

We begin the induction by noting that $X_0 = X$. If we now assume that X_β has size less than or equal to γ, then (since the boundary of a $\overline{\Lambda\{f\}}$-cell is of size less than λ) there are fewer than $\gamma^\lambda = \gamma$ (if $\mathsf{Spc}_{(*)} = \mathsf{SS}_{(*)}$) or $\gamma^{\lambda^{\mathfrak{c}}} = \gamma$ (if $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$) (see Proposition 12.1.16) many maps from the boundary of a $\overline{\Lambda\{f\}}$-cell to X_β. Since there are only countably many $\overline{\Lambda\{f\}}$-cells, there are fewer than γ many $\overline{\Lambda\{f\}}$-cells attached to X_β to form $X_{\beta+1}$. Since each $\overline{\Lambda\{f\}}$-cell has fewer than λ many simplices (or cells), $X_{\beta+1}$ has size less than or equal to γ.

If β is a limit ordinal, then X_β is a colimit of complexes, each of which is of size less than or equal to γ. Since $\beta < \lambda < \gamma$, this implies that X_β has size less than or equal to γ, and the proof is complete. $\square$

The following proposition will be used in Proposition 2.4.8 to extend a map over an arbitrary inclusion of a subcomplex that is an f-local equivalence by extending it over a subcomplex of size no greater than γ.

PROPOSITION 2.4.7. *Let $f\colon A \to B$ be an inclusion of cell complexes (see Remark 1.2.7), and let D be a simplicial set (or a cell complex). If $i\colon C \to D$ is the inclusion of a proper subcomplex and an f-local equivalence, then there is a subcomplex K of D such that*

1. *the subcomplex K is not contained in the subcomplex C,*
2. *the size of K is less than or equal to γ (see Definition 2.4.4), and*
3. *the inclusions $K \cap C \to \mathrm{K}$ and $C \to C \cup K$ are both f-local equivalences.*

PROOF. Since $i\colon C \to D$ is the inclusion of a subcomplex and an f-local equivalence, Lemma 2.3.15, and Theorem 1.4.15 imply that $\mathrm{L}_f(i)\colon \mathrm{L}_f C \to \mathrm{L}_f D$ is a trivial cofibration of fibrant spaces, and so it is the inclusion of a strong deformation retract (see Corollary 10.4.20). We choose a strong deformation retraction $R\colon \mathrm{L}_f D \otimes I \to \mathrm{L}_f D$ (where $I = \Delta[1]$), which will remain fixed throughout this proof.

We will show that there exists a subcomplex K of D of size less than or equal to γ such that

1. K is not contained in C,
2. $R|_{\mathrm{L}_f K \otimes I}$ is a deformation retraction of $\mathrm{L}_f K$ to $\mathrm{L}_f(K \cap C)$, and
3. $R|_{\mathrm{L}_f(C \cup K) \otimes I}$ is a deformation retraction of $\mathrm{L}_f(C \cup K)$ to $\mathrm{L}_f C$.

We will do this by constructing a λ-sequence

$$K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_\beta \subset \cdots \qquad (\beta < \lambda)$$

(where λ is as in Definition 2.4.4) of subcomplexes of D such that, for every $\beta < \lambda$,

1. K_β has size less than or equal to γ,
2. $R(\mathrm{L}_f K_\beta \otimes I) \subset \mathrm{L}_f K_{\beta+1}$,

and such that no K_β is contained in C. If we then let $K = \bigcup_{\beta<\lambda} K_\beta$, then Proposition 2.3.18 will imply that K has the properties that we require.

We begin by choosing a simplex (or cell) of D that isn't contained in C, and letting K_0 equal the subcomplex generated by that simplex (or cell).

For successor ordinals, suppose that $\beta + 1 < \gamma$, and that we've constructed K_β. Lemma 2.4.6 implies that $\mathrm{L}_f K_\beta$ has size less than or equal to γ, and so $R(\mathrm{L}_f K_\beta \otimes I)$ is contained in a subcomplex of $\mathrm{L}_f D$ of size less than or equal to γ. (If $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$, then this uses Corollary 2.2.7.) Lemma 2.4.5 now implies that we can find a subcomplex Z_β of D, of size less than or equal to γ, such that $R(\mathrm{L}_f K_\beta \otimes I) \subset \mathrm{L}_f Z_\beta$. We let $K_{\beta+1} = K_\beta \cup Z_\beta$. It is clear that $K_{\beta+1}$ has the properties required of it, and so the proof is complete. $\square$

PROPOSITION 2.4.8. *Let $f\colon A \to B$ be an inclusion of cell complexes (see Remark 1.2.7). If $p\colon X \to Y$ has the right lifting property with respect to those inclusions of subcomplexes $i\colon C \to D$ that are f-local equivalences and such that the size of D is less than or equal to γ (see Definition 2.4.4), then p has the right lifting property with respect to all inclusions of subcomplexes that are f-local equivalences.*

PROOF. Let $i\colon C \to D$ be an inclusion of a subcomplex that is an f-local equivalence, and let the solid arrow diagram

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ D \arrow[r, "k"'] \arrow[ur, dotted] & Y \end{tikzcd}$$

be commutative; we must show that there exists a dotted arrow making both triangles commute. To do this, we will consider the subcomplexes of D over which our map can be defined, and use Zorn's lemma to show that we can define it over all of D.

Let S be the set of pairs (D_s, g_s) such that

1. D_s is a subcomplex of D containing C, and the inclusion $i_s\colon C \to D_s$ is an f-local equivalence
2. g_s is a function $D_s \to X$ such that $g_s i_s = h$ and $pg_s = k|_{D_s}$.

We define a preorder on S by defining $(D_s, g_s) < (D_t, g_t)$ if $D_s \subset D_t$ and $g_t|_{D_s} = g_s$.

If $S' \subset S$ is a chain (i.e., a totally ordered subset of S), let $D_u = \operatorname{colim}_{(D_s,g_s)\in S'} D_s$, and define $g_u\colon D_u \to X$ by $g_u = \operatorname{colim}_{(D_s,g_s)\in S'} g_s$. The universal mapping property of the colimit implies that $g_u i_u = h$ and $pg_u = k|_{D_u}$, and Proposition 1.2.15 implies that the map $C \to D_u$ is an f-local equivalence. Thus, (D_u, g_u) is an element of S, and so it is an upper bound for S'. Zorn's lemma now implies that S has a maximal element (D_m, g_m). We will complete the proof by showing that $D_m = D$.

If $D_m \neq D$, then Proposition 2.4.7 implies that there is a subcomplex K of D such that K is not contained in D_m, the size of K is less than or equal to γ, and the inclusions $K \cap D_m \to K$ and $D_m \to D_m \cup K$ are both f-local equivalences. Thus, there is a map $g_K\colon K \to X$ such that $pg_K = k|_K$ and $g_K|_{K\cap D_m} = g_m|_{K\cap D_m}$, and so g_m and g_K combine to define a map $g_{mK}\colon K \cup D_m \to X$ such that $pg_{mK} = k|_{K\cup D_m}$ and $g_{mK} i = h$. Thus, $(K \cup D_m, g_{mK})$ is an element of S strictly greater than (D_m, g_m). This contradicts (D_m, g_m) being a maximal element of S, and so our assumption that $D_m \neq D$ must have been false, and the proof is complete. $\square$

Proof of Proposition 2.1.4. Let J be a set of representatives of the isomorphism classes of inclusions of subcomplexes that are f-local equivalences of complexes of size less than or equal to γ. Proposition 2.4.8, Corollary 12.4.17 and Lemma 2.4.1 (if $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$) imply that the J-cofibrations are exactly the cofibrations that are f-local equivalences, and so the proof is complete. $\square$

CHAPTER 3

Localization of model categories

3.1. Introduction

The purpose of a model category is to serve as a presentation of its homotopy theory (where we loosely define the "homotopy theory" of a model category as its homotopy category together with the function complexes between its objects). Thus, a "localization" of a model category should not be a construction that adds inverses for maps in the underlying category, but rather one that adds inverses for maps in the homotopy category. If $\mathcal{M}$ is a model category and $\mathcal{C}$ is a class of maps in $\mathcal{M}$, a localization of $\mathcal{M}$ with respect to $\mathcal{C}$ will be a map of model categories $\mathrm{F}\colon \mathcal{M} \to \mathcal{N}$ such that the images in $\mathrm{Ho}\,\mathcal{M}$ of the elements of $\mathcal{C}$ go to isomorphisms in $\mathrm{Ho}\,\mathcal{N}$ and such that F is initial among such maps of model categories. Since there are two different varieties of maps of model categories, *left Quillen functors* and *right Quillen functors* (see Definition 9.8.1), we will define (in Definition 3.2.1) two different varieties of localizations of model categories, the *left localizations* and the *right localizations*.

If $\mathrm{F}\colon \mathcal{M} \to \mathcal{N}$ is a left Quillen functor, $g\colon X \to Y$ is a map in $\mathcal{M}$, and $[g]\colon X \to Y$ is the image of g in $\mathrm{Ho}\,\mathcal{M}$, then the total left derived functor $\mathbf{L}\mathrm{F}\colon \mathrm{Ho}\,\mathcal{M} \to \mathrm{Ho}\,\mathcal{N}$ of F (see Definition 9.7.9) takes $[g]$ to the image of $\mathrm{F}(\tilde{g})$ in $\mathrm{Ho}\,\mathcal{N}$ for some cofibrant approximation $\tilde{g}$ to g. Thus, if $\mathbf{L}\mathrm{F}[g]$ is to be an isomorphism for every element g of $\mathcal{C}$, then Proposition 9.6.8 and Proposition 9.3.2 imply that F must take every cofibrant approximation to an element of $\mathcal{C}$ into a weak equivalence. Thus, if $\mathcal{C}$ is a class of maps in $\mathcal{M}$, then a left localization of $\mathcal{M}$ with respect to $\mathcal{C}$ will be a left Quillen functor that takes cofibrant approximations to elements of $\mathcal{C}$ into weak equivalences, and is initial among such left Quillen functors. Similarly, a right localization of $\mathcal{M}$ with respect to $\mathcal{C}$ will be a right Quillen functor that takes fibrant approximations to elements of $\mathcal{C}$ into weak equivalences, and is initial among such right Quillen functors.

3.2. Localizations of model categories

Definition 3.2.1. Let $\mathcal{M}$ be a model category and let $\mathcal{C}$ be a class of maps in $\mathcal{M}$.

1. A *left localization of* $\mathcal{M}$ *with respect to* $\mathcal{C}$ is a model category $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ together with a left Quillen functor (see Definition 9.8.1) $j\colon \mathcal{M} \to \mathrm{L}_{\mathcal{C}}\mathcal{M}$ such that
   (a) the total left derived functor $\mathbf{L}j\colon \mathrm{Ho}\,\mathcal{M} \to \mathrm{Ho}\,\mathcal{N}$ (see Definition 9.7.9) of j takes the images in $\mathrm{Ho}\,\mathcal{M}$ of the elements of $\mathcal{C}$ into isomorphisms in $\mathrm{Ho}\,\mathrm{L}_{\mathcal{C}}\mathcal{M}$, and
   (b) j is initial among left Quillen functors satisfying condition 1a.
2. A *right localization of* $\mathcal{M}$ *with respect to* $\mathcal{C}$ is a model category $\mathrm{R}_{\mathcal{C}}\mathcal{M}$ together with a right Quillen functor $j\colon \mathcal{M} \to \mathrm{R}_{\mathcal{C}}\mathcal{M}$ such that

(a) the total right derived functor $\mathbf{R}j\colon \mathrm{Ho}\,\mathcal{M} \to \mathrm{Ho}\,\mathcal{N}$ of j takes the images in $\mathrm{Ho}\,\mathcal{M}$ of the elements of $\mathcal{C}$ into isomorphisms in $\mathrm{Ho}\,\mathrm{R}_{\mathcal{C}}\mathcal{M}$, and
(b) j is initial among right Quillen functors satisfying condition 2a.

PROPOSITION 3.2.2. *Let $\mathcal{M}$ be a model category and let $\mathcal{C}$ be a class of maps in $\mathcal{M}$. If a (left or right) localization of $\mathcal{M}$ with respect to $\mathcal{C}$ exists, it is unique up to a unique isomorphism.*

PROOF. The standard argument applies. □

PROPOSITION 3.2.3. *Let $\mathcal{M}$ be a model category, and let $\mathcal{C}$ be a class of maps in $\mathcal{M}$.*

1. *If $\mathrm{F}\colon \mathcal{M} \to \mathcal{N}$ is a left Quillen functor, then the total left derived functor $\mathbf{L}\mathrm{F}\colon \mathrm{Ho}\,\mathcal{M} \to \mathrm{Ho}\,\mathcal{N}$ (see Definition 9.7.9) of F takes the images in $\mathrm{Ho}\,\mathcal{M}$ of the elements of $\mathcal{C}$ into isomorphisms in $\mathrm{Ho}\,\mathcal{N}$ if and only if F takes every cofibrant approximation to an element of $\mathcal{C}$ (see Definition 9.1.8) into a weak equivalence in $\mathcal{N}$.*
2. *If $\mathrm{F}\colon \mathcal{M} \to \mathcal{N}$ is a right Quillen functor, then the total right derived functor $\mathbf{R}\mathrm{F}\colon \mathrm{Ho}\,\mathcal{M} \to \mathrm{Ho}\,\mathcal{N}$ of F takes the images in $\mathrm{Ho}\,\mathcal{M}$ of the elements of $\mathcal{C}$ into isomorphisms in $\mathrm{Ho}\,\mathcal{N}$ if and only if F takes every fibrant approximation to an element of $\mathcal{C}$ into a weak equivalence in $\mathcal{N}$.*

PROOF. We will prove part 1; the proof of part 2 is dual.

If $g\colon X \to Y$ is a map in $\mathcal{M}$, then the total left derived functor of F takes the image of g in $\mathrm{Ho}\,\mathcal{M}$ to the image in $\mathrm{Ho}\,\mathcal{N}$ of $\mathrm{F}(\tilde{g})$ for some cofibrant approximation $\tilde{g}$ to g (see the proof of Proposition 9.7.6). Since a map in $\mathcal{N}$ is a weak equivalence if and only if its image in $\mathrm{Ho}\,\mathcal{N}$ is an isomorphism (see Proposition 9.6.8), the result now follows from Proposition 9.3.2. □

COROLLARY 3.2.4. *Let $\mathcal{M}$ be a model category and let $\mathcal{C}$ be a class of maps in $\mathcal{M}$.*

1. *A left Quillen functor $j\colon \mathcal{M} \to \mathrm{L}_{\mathcal{C}}\mathcal{M}$ is a left localization of $\mathcal{M}$ with respect to $\mathcal{C}$ if and only if it takes all cofibrant approximations to elements of $\mathcal{C}$ into weak equivalences, and is initial among such left Quillen functors.*
2. *A right Quillen functor $j\colon \mathcal{M} \to \mathrm{R}_{\mathcal{C}}\mathcal{M}$ is a right localization of $\mathcal{M}$ with respect to $\mathcal{C}$ if and only if takes all fibrant approximations to elements of $\mathcal{C}$ into weak equivalences, and is initial mong such right Quillen functors.*

PROOF. This follows from Proposition 3.2.3. □

3.2.5. Bousfield localization.

DEFINITION 3.2.6. Let $\mathcal{M}$ be a model category, and let $\mathcal{C}$ be a class of maps in $\mathcal{M}$.

- A left localization $j\colon \mathcal{M} \to \mathrm{L}_{\mathcal{C}}\mathcal{M}$ of $\mathcal{M}$ with respect to $\mathcal{C}$ (see Definition 3.2.1) will be called a *left Bousfield localization* if
  1. the underlying category of $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ equals that of $\mathcal{M}$ and j is the identity functor,
  2. every weak equivalence of $\mathcal{M}$ is a weak equivalence of $\mathrm{L}_{\mathcal{C}}\mathcal{M}$,
  3. every element of $\mathcal{C}$ is a weak equivalence of $\mathrm{L}_{\mathcal{C}}\mathcal{M}$, and
  4. the class of cofibrations of $\mathcal{M}$ equals the class of cofibrations of $\mathrm{L}_{\mathcal{C}}\mathcal{M}$.

We will often call a left Bousfield localization of $\mathcal{M}$ a *localization* of $\mathcal{M}$.

- A right localization $j\colon \mathcal{M} \to \mathrm{R}_{\mathcal{C}}\mathcal{M}$ of $\mathcal{M}$ with respect to $\mathcal{C}$ will be called a *right Bousfield localization* if
  1. the underlying category of $\mathrm{R}_{\mathcal{C}}\mathcal{M}$ equals that of $\mathcal{M}$ and j is the identity functor,
  2. every weak equivalence of $\mathcal{M}$ is a weak equivalence of $\mathrm{L}_{\mathcal{C}}\mathcal{M}$,
  3. every element of $\mathcal{C}$ is a weak equivalence of $\mathrm{R}_{\mathcal{C}}\mathcal{M}$, and
  4. the class of fibrations of $\mathcal{M}$ equals the class of fibrations of $\mathrm{R}_{\mathcal{C}}\mathcal{M}$.

  We will often call a right Bousfield localization of $\mathcal{M}$ a *colocalization* of $\mathcal{M}$.

PROPOSITION 3.2.7. *Let $\mathcal{M}$ be a model category, and let $\mathcal{C}$ be a class of maps in $\mathcal{M}$.*

1. *If $j\colon \mathcal{M} \to \mathrm{L}_{\mathcal{C}}\mathcal{M}$ is a left Bousfield localization of $\mathcal{M}$ with respect to $\mathcal{C}$, then*
   (a) *the cofibrations and trivial fibrations of $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ equal those of $\mathcal{M}$,*
   (b) *the trivial cofibrations of $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ contain those of $\mathcal{M}$, and*
   (c) *the fibrations of $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ are contained in those of $\mathcal{M}$.*
2. *If $j\colon \mathcal{M} \to \mathrm{R}_{\mathcal{C}}\mathcal{M}$ is a right Bousfield localization of $\mathcal{M}$ with respect to $\mathcal{C}$, then*
   (a) *the fibrations and trivial cofibrations of $\mathrm{R}_{\mathcal{C}}\mathcal{M}$ equal those of $\mathcal{M}$,*
   (b) *the trivial fibrations of $\mathrm{R}_{\mathcal{C}}\mathcal{M}$ contain those of $\mathcal{M}$, and*
   (c) *the cofibrations of $\mathrm{R}_{\mathcal{C}}\mathcal{M}$ are contained in those of $\mathcal{M}$.*

PROOF. This follows from Proposition 8.2.3. □

In Section 3.3, we show that if $\mathcal{M}$ is a left proper cellular model category (see Definition 15.1.1 and Definition 11.1.1) and S is a set of maps in $\mathcal{M}$, then the left Bousfield localization of $\mathcal{M}$ with respect to S exists (see Theorem 3.3.11) and is itself a left proper cellular model category (see Theorem 3.3.13). In Section 3.4, we show that if $\mathcal{M}$ is a right proper cellular model category, then certain right Bousfield localizations of $\mathcal{M}$ (the S-colocalizations; see Theorem 3.4.9) exist.

3.3. Left Bousfield localization

In this section, we show that if $\mathcal{M}$ is a left proper cellular model category and S is a set of maps in $\mathcal{M}$, then a left Bousfield localization of $\mathcal{M}$ with respect to S exists (see Theorem 3.3.11). We begin by showing that if $\mathcal{M}$ is a model category, $\mathcal{C}$ is a class of maps in $\mathcal{M}$, and a left Quillen functor takes cofibrant approximations to elements of $\mathcal{C}$ (see Corollary 3.2.4) into weak equivalences, then it takes all $\mathcal{C}$-local equivalences (see Definition 3.3.2) into weak equivalences (see Proposition 3.3.5).

3.3.1. Structure of a left localization.

DEFINITION 3.3.2. Let $\mathcal{M}$ be a model category, and let $\mathcal{C}$ be a class of maps in $\mathcal{M}$.

1. An object W of $\mathcal{M}$ is *$\mathcal{C}$-local* if W is fibrant and, for every element $f\colon A \to B$ of $\mathcal{C}$, the induced map of homotopy function complexes (see Definition 17.2.4) $f^*\colon \operatorname{map}(B, W) \to \operatorname{map}(A, W)$ is a weak equivalence. (Theorem 17.6.6 implies that if this is true for any one homotopy function complex, then it is true for every homotopy function complex.) If $\mathcal{C}$ consists of the single map $f\colon A \to B$, then a $\mathcal{C}$-local object will also be called *f-local*, and if $\mathcal{C}$ consists of the single map from the initial object of $\mathcal{M}$ to an object A, then an $\mathcal{C}$-local object will also be called *A-local* or *A-null*.

2. A map $g\colon X \to Y$ in $\mathcal{M}$ is a $\mathcal{C}$-*local equivalence* if, for every $\mathcal{C}$-local object W, the induced map of homotopy function complexes (see Definition 17.2.4) $g^*\colon \operatorname{map}(Y, W) \to \operatorname{map}(X, W)$ is a weak equivalence. (Theorem 17.6.6 implies that if this is true for any one homotopy function complex, then it is true for every homotopy function complex.) If $\mathcal{C}$ consists of the single map $f\colon A \to B$, then a $\mathcal{C}$-local equivalence will also be called an f-*local equivalence*, and if $\mathcal{C}$ consists of the single map from the initial object of $\mathcal{M}$ to an object A, then a $\mathcal{C}$-local equivalence will also be called an A-*local equivalence*.

PROPOSITION 3.3.3. *If $\mathcal{M}$ is a model category and $\mathcal{C}$ is a class of maps in $\mathcal{M}$, then every weak equivalence is a $\mathcal{C}$-local equivalence.*

PROOF. This follows from Theorem 17.5.2. □

LEMMA 3.3.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories, let $\mathcal{C}$ be a class of maps in $\mathcal{M}$, and let $\mathrm{F}\colon \mathcal{M} \rightleftarrows \mathcal{N} :\mathrm{U}$ be a Quillen pair. If F takes every cofibrant approximation to an element of $\mathcal{C}$ into a weak equivalence in $\mathcal{N}$, then U takes every fibrant object in $\mathcal{N}$ into a $\mathcal{C}$-local object in $\mathcal{M}$.*

PROOF. If $f\colon A \to B$ is an element of $\mathcal{C}$ and $\tilde{f}\colon \widetilde{\boldsymbol{A}} \to \widetilde{\boldsymbol{B}}$ is a cosimplicial resolution of f in $\mathcal{M}$, then $\mathrm{F}(\tilde{f})\colon \mathrm{F}(\widetilde{\boldsymbol{A}}) \to \mathrm{F}(\widetilde{\boldsymbol{B}})$ is a cosimplicial resolution in $\mathcal{N}$ of $\mathrm{F}(\tilde{f}_0)\colon \mathrm{F}(\widetilde{\boldsymbol{A}}_0) \to \mathrm{F}(\widetilde{\boldsymbol{B}}_0)$ (see Proposition 18.6.2). Since $\tilde{f}_0\colon \widetilde{\boldsymbol{A}}_0 \to \widetilde{\boldsymbol{B}}_0$ is a cofibrant approximation to f, if W is a fibrant object in $\mathcal{N}$, Theorem 17.5.2 implies that the map of simplicial sets $\mathcal{N}\big(\mathrm{F}(\widetilde{\boldsymbol{B}}), W\big) \to \mathcal{N}\big(\mathrm{F}(\widetilde{\boldsymbol{A}}), W\big)$ is a weak equivalence. Thus, the map of simplicial sets $\mathcal{M}\big(\widetilde{\boldsymbol{B}}, \mathrm{U}(W)\big) \to \mathcal{M}\big(\widetilde{\boldsymbol{A}}, \mathrm{U}(W)\big)$ is a weak equivalence, and so $\mathrm{U}(W)$ is $\mathcal{C}$-local. □

PROPOSITION 3.3.5. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories, let $\mathcal{C}$ be a class of maps in $\mathcal{M}$, and let $\mathrm{F}\colon \mathcal{M} \rightleftarrows \mathcal{N} :\mathrm{U}$ be a Quillen pair. If F takes every cofibrant approximation to an element of $\mathcal{C}$ into a weak equivalence in $\mathcal{N}$, then F takes every $\mathcal{C}$-local equivalence between cofibrant objects into a weak equivalence in $\mathcal{N}$.*

PROOF. Let $g\colon A \to B$ be a $\mathcal{C}$-local equivalence between cofibrant objects in $\mathcal{M}$. If W is a fibrant object in $\mathcal{N}$ and $\widehat{\boldsymbol{W}}$ is a simplicial resolution of W in $\mathcal{N}$, then $\mathrm{U}(\widehat{\boldsymbol{W}})$ is a simplicial resolution of $\mathrm{U}(W)$ in $\mathcal{M}$ (see Proposition 18.6.2), and so Lemma 3.3.4 implies that the map of simplicial sets $g^*\colon \mathcal{M}\big(B, \mathrm{U}(\widehat{\boldsymbol{W}})\big) \to \mathcal{M}\big(A, \mathrm{U}(\widehat{\boldsymbol{W}})\big)$ is a weak equivalence. Thus, the map of simplicial sets $\mathrm{F}(g)^*\colon \mathcal{N}\big(\mathrm{F}(B), \widehat{\boldsymbol{W}}\big) \to \mathcal{M}\big(\mathrm{F}(A), \widehat{\boldsymbol{W}}\big)$ is a weak equivalence, and so Theorem 18.1.6 implies that $\mathrm{F}(g)$ is a weak equivalence. □

3.3.6. Existence of left Bousfield localizations.

DEFINITION 3.3.7. Let $\mathcal{M}$ be a left proper cellular model category (see Definition 15.1.1), and let S be a set of maps in $\mathcal{M}$.

1. An S-*local weak equivalence* is defined to be an S-local equivalence (see Definition 3.3.2).
2. An S-*local cofibration* is defined to be a cofibration.
3. An S-*local fibration* is defined to be a map with the right lifting property (see Definition 8.2.1) with respect to all maps that are both S-local cofibrations and S-local weak equivalences. If the map $X \to *$ from an object X to

the terminal object of $\mathcal{M}$ is an S-local fibration, then we will say that X is *S-local fibrant.*

THEOREM 3.3.8. *If $\mathcal{M}$ is a left proper cellular model category (see Definition* 11.1.1 *and Definition* 15.1.1*) and S is a set of maps in $\mathcal{M}$, then there is a model category structure on $\mathcal{M}$ (which we call the S-*local model category structure*) in which*

1. *the weak equivalences are the S-local weak equivalences (see Definition* 3.3.7*),*
2. *the cofibrations are the S-local cofibrations, and*
3. *the fibrations are the S-local fibrations.*

If $\mathcal{M}$ is a simplicial model category, then that simplicial structure gives the S-local model category the structure of a simplicial model category.

The proof of Theorem 3.3.8 is in Section 4.6.

PROPOSITION 3.3.9. *If $\mathcal{M}$ is a left proper cellular model category and S is a set of maps in $\mathcal{M}$, then an object W of $\mathcal{M}$ is S-local if and only if it is a fibrant object in the S-local model category structure on $\mathcal{M}$ (see Theorem* 3.3.8*).*

The proof of Proposition 3.3.9 is in Section 4.6.

PROPOSITION 3.3.10. *If $\mathcal{M}$ is a left proper cellular model category and S is a set of maps in $\mathcal{M}$, then the model category structure of Theorem* 3.3.8 *is a left Bousfield localization of $\mathcal{M}$ with respect to S (see Definition* 3.2.1*).*

The proof of Proposition 3.3.10 is in Section 4.6.

THEOREM 3.3.11. *If $\mathcal{M}$ is a left proper cellular model category (see Definition* 11.1.1 *and Definition* 15.1.1*) and S is a set of maps in $\mathcal{M}$, then a left Bousfield localization of $\mathcal{M}$ with respect to S exists.*

PROOF. This follows from Proposition 3.3.10. □

DEFINITION 3.3.12. If $\mathcal{M}$ is a left proper cellular model category and S is a set of maps in $\mathcal{M}$, then the model category structure of Theorem 3.3.8 will be called *the* left Bousfield localization of $\mathcal{M}$ with respect to S (see Proposition 3.3.10).

THEOREM 3.3.13. *If $\mathcal{M}$ is a left proper cellular model category and S is a set of maps in $\mathcal{M}$, then the left Bousfield localization of $\mathcal{M}$ with respect to S (see Definition* 3.3.12*) is a left proper cellular model category.*

The proof of Theorem 3.3.13 is in Section 4.6.

3.3.14. Examples of left proper cellular model categories.

PROPOSITION 3.3.15. *The categories* SS, Top, SS_*, *and* Top_* *are left proper cellular model categories.*

PROPOSITION 3.3.16. *If $\mathcal{M}$ is a left proper cellular model category and $\mathcal{C}$ is a small category, then the diagram category $\mathcal{M}^{\mathcal{C}}$ is a left proper cellular model category.*

PROPOSITION 3.3.17. *If $\mathcal{M}$ is a left proper cellular model category and Z is an object of $\mathcal{M}$, then the overcategory $(\mathcal{M}\downarrow Z)$ is a left proper cellular model category.*

PROPOSITION 3.3.18. *If $\mathcal{M}$ is a left proper cellular simplicial model category and $\mathcal{C}$ is a small simplicial category, then the category $\mathcal{M}^{\mathcal{C}}$ of simplicial diagrams is a left proper cellular model category.*

PROPOSITION 3.3.19. *If $\mathcal{M}$ is a pointed left proper cellular model category with an action by pointed simplicial sets, then the category of spectra over $\mathcal{M}$ (as in* [**14**]*) is a left proper cellular model category.*

PROPOSITION 3.3.20. *If $\mathcal{M}$ is a pointed left proper cellular model category with an action by pointed simplicial sets, then J. H. Smith's category of symmetric spectra over $\mathcal{M}$* [**52, 36**] *is a left proper cellular model category.*

3.4. Right Bousfield localization

3.4.1. Structure of a right localization.

DEFINITION 3.4.2. Let $\mathcal{M}$ be a model category, and let $\mathcal{C}$ be a class of objects in $\mathcal{M}$.

1. A map $g\colon X \to Y$ is a $\mathcal{C}$*-colocal equivalence* if for every element A of $\mathcal{C}$ the induced map of homotopy function complexes $g_*\colon \operatorname{map}(A, X) \to \operatorname{map}(A, Y)$ is a weak equivalence. (Theorem 17.6.6 implies that if this is true for any one homotopy function complex, then it is true for every homotopy function complex.) If $\mathcal{C}$ consists of the single object A, then a $\mathcal{C}$-colocal equivalence will be called an A*-colocal equivalence*.
2. An object W is $\mathcal{C}$*-colocal* if W is cofibrant and, for every $\mathcal{C}$-colocal equivalence $g\colon X \to Y$, the induced map of homotopy function complexes $g_*\colon \operatorname{map}(W, X) \to \operatorname{map}(W, Y)$ is a weak equivalence. (Theorem 17.6.6 implies that if this is true for any one homotopy function complex, then it is true for every homotopy function complex.) If $\mathcal{C}$ consists of the single object A, then a $\mathcal{C}$-colocal object will be called A*-colocal*.

For a discussion of the relation between our definitions (in the case $\mathcal{M} = \mathsf{Spc}_*$) of $\mathcal{C}$-colocal spaces and $\mathcal{C}$-colocal equivalences and earlier definitions (which used the terms "A-cellular space" and "A-cellular equivalences"), see Remark 5.1.2.

PROPOSITION 3.4.3. *Let $\mathcal{M}$ be a model category. If $\mathcal{C}$ is a class of objects in $\mathcal{M}$, then every weak equivalence is a $\mathcal{C}$-colocal equivalence.*

PROOF. This follows from Theorem 17.5.2. □

LEMMA 3.4.4. *Let $\mathcal{M}$ be a model category, and let $\mathcal{C}$ be a class of objects in $\mathcal{M}$. If $\mathrm{R}_{\mathcal{C}}\mathcal{M}$ exists, then every cofibrant object in $\mathrm{R}_{\mathcal{C}}\mathcal{M}$ is $\mathcal{C}$-colocal.*

PROOF. Let $g\colon X \to Y$ be a $\mathcal{C}$-colocal equivalence, and let $\hat{g}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ be a simplicial resolution of g in the original model category structure on $\mathcal{M}$. Since $j\colon \mathcal{M} \to \mathrm{R}_{\mathcal{C}}\mathcal{M}$ is a right Quillen functor, $\hat{g}$ is also a simplicial resolution of g in $\mathrm{R}_{\mathcal{C}}\mathcal{M}$. Thus, if W is a cofibrant object in $\mathrm{R}_{\mathcal{C}}\mathcal{M}$, then Theorem 17.5.2 implies that the map of simplicial sets $\hat{g}_*\colon \mathcal{M}(W, \widehat{\boldsymbol{X}}) \to \mathcal{M}(W, \widehat{\boldsymbol{Y}})$ is a weak equivalence, and so W is $\mathcal{C}$-colocal. □

PROPOSITION 3.4.5. *Let $\mathcal{M}$ be a model category, and let $\mathcal{C}$ be a class of objects in $\mathcal{M}$. If $\mathrm{R}_{\mathcal{C}}\mathcal{M}$ exists, then every $\mathcal{C}$-colocal equivalence is a weak equivalence in $\mathrm{R}_{\mathcal{C}}\mathcal{M}$.*

PROOF. This follows from Lemma 3.4.4 and Theorem 18.1.6. □

REMARK 3.4.6. If $\mathcal{M} = \mathsf{Spc}$, a category of unpointed spaces (see Notation 1.1.2), and A is a non-empty space, then every space X is a retract of X^A (see Definition 1.1.6), and so an $\mathcal{C}$-colocal equivalence of unpointed spaces must be a weak equivalence (see Corollary 1.1.9). Thus, to consider the notion of $\mathcal{C}$-colocal equivalence of unpointed spaces would be pointless. (According to E. Dror Farjoun, this joke is due to W. G. Dwyer.)

3.4.7. Existence of right Bousfield localizations.

DEFINITION 3.4.8. Let $\mathcal{M}$ be a right proper cellular model category, and let S be a set of objects in $\mathcal{M}$.

1. An *S-colocal weak equivalence* is defined to be an S-colocal equivalence (see Definition 3.4.2).
2. An *S-colocal fibration* is defined to be a fibration.
3. An *S-colocal cofibration* is defined to be a map with the left lifting property with respect to all maps that are both S-colocal weak equivalences and S-colocal fibrations.

THEOREM 3.4.9. *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of objects in $\mathcal{M}$, then there is a model category structure on $\mathcal{M}$ (called the* S-colocal model category*) in which*

1. *the weak equivalences are the S-colocal weak equivalences,*
2. *the fibrations are the S-colocal fibrations, and*
3. *the cofibrations are the S-colocal cofibrations.*

If $\mathcal{M}$ is a simplicial model category, then the given simplicial structure on $\mathcal{M}$ gives the S-colocal model category the structure of a simplicial model category.

The proof of Theorem 3.4.9 is in Chapter 5. Theorem 3.4.9 for the category of pointed topological spaces was first obtained by Nofech [**44**].

PROPOSITION 3.4.10. *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of objects in $\mathcal{M}$, then an object is S-colocal (see Definition 3.4.2) if and only if it is a cofibrant object in the S-colocal model category structure on $\mathcal{M}$.*

PROOF. If W is a cofibrant object in the S-colocal model category structure, then the map $\emptyset \to W$ has the left lifting property with respect to all maps that are both fibrations and S-colocal equivalences (where $\emptyset$ is the initial object of $\mathcal{M}$). If $g\colon X \to Y$ is an S-colocal equivalence, let $\hat{g}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ be a simplicial resolution of g in the original model category structure on $\mathcal{M}$ such that $\hat{g}$ is a Reedy fibration. Proposition 18.3.5, Proposition 18.3.13, and Proposition 17.3.7 imply that the map $\widehat{\boldsymbol{X}}^{\Delta[n]} \to \widehat{\boldsymbol{Y}}^{\Delta[n]} \times_{\widehat{\boldsymbol{Y}}^{\partial\Delta[n]}} \widehat{\boldsymbol{X}}^{\partial\Delta[n]}$ is both a fibration and an S-colocal equivalence. Proposition 18.3.8 now implies that W is S-colocal.

Conversely, assume that W is S-colocal. Proposition 8.2.3 implies that it is sufficient to show that if $p\colon X \to Y$ is both a fibration and an S-colocal equivalence, then the map $\emptyset \to W$ has the left lifting property with respect to p. Proposition 17.1.12 implies that we can choose a simplicial resolution $\hat{p}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ of p such that $\hat{p}$ is a Reedy fibration. Proposition 18.3.5 and Proposition 18.3.9 imply that the map $\emptyset \to W$ has the left lifting property with respect to $\hat{p}_0\colon \widehat{\boldsymbol{X}}_0 \to \widehat{\boldsymbol{Y}}_0$. Since $\mathcal{M}$ is right proper, Proposition 11.1.18 and Proposition 17.1.6 imply that the map $\emptyset \to W$ has the left lifting property with respect to p. □

PROPOSITION 3.4.11. *If $\mathcal{M}$ is a right proper cellular model category and S is a set of objects in $\mathcal{M}$, then the model category structure of Theorem 3.4.9 is a right Bousfield localization of $\mathcal{M}$ with respect to the class of S-colocal equivalences.*

PROOF. This follows from Proposition 3.4.3 and Proposition 3.4.5. $\square$

THEOREM 3.4.12. *If $\mathcal{M}$ is a right proper cellular model category and S is a set of objects in $\mathcal{M}$, then the right Bousfield localization of $\mathcal{M}$ with respect to the class of S-colocal equivalences exists.*

PROOF. This follows from Proposition 3.4.11. $\square$

PROPOSITION 3.4.13. *Let $\mathcal{M}$ be a right proper cellular model category, and let S be a set of objects in $\mathcal{M}$. If $g\colon X \to Y$ is a S-colocal equivalence, $h\colon Z \to Y$ is a map, at least one of g and h is a fibration, and the square*

$$\begin{tikzcd} W \arrow[r] \arrow[d, "k"'] & X \arrow[d, "g"] \\ Z \arrow[r, "h"'] & Y \end{tikzcd}$$

is a pullback, then k is an S-colocal equivalence.

PROOF. This follows from Proposition 18.3.5 and Proposition 18.5.6. $\square$

PROPOSITION 3.4.14. *If $\mathcal{M}$ is a right proper cellular model category and S is a set of objects in $\mathcal{M}$, then the right Bousfield localization of $\mathcal{M}$ with respect to the class of S-colocal equivalences is right proper.*

PROOF. This follows from Proposition 3.4.13. $\square$

CHAPTER 4

Left Bousfield localization

The main purpose of this chapter is to prove Theorem 3.3.8. This is done in Section 4.6.

Section 4.1 discuses S-localizations of objects in $\mathcal{M}$. Section 4.2 has some technical results (motivated by the discussion of Section 1.3) needed for the construction of a functorial cofibrant S-localization in Section 4.3 (see Definition 4.3.2 and Theorem 4.3.3). Section 4.4 contains some technical results needed for the cardinality argument in Section 4.5, and the proof of Theorem 3.3.8 is in Section 4.6.

Theorem 4.2.12 might lead one to hope that the factorization of Theorem 4.3.1 would serve as the required factorization into an S-local trivial cofibration followed by an S-local fibration (see Definition 8.1.2). Unfortunately, Example 4.2.14 shows that not all S-local trivial cofibrations need be $\widetilde{\Lambda S}$-cofibrations, and so there may be $\widetilde{\Lambda S}$-injectives that are not S-local fibrations. Thus, we must establish Proposition 4.5.1, which shows that there is a *set* J_S of generating trivial cofibrations (see Definition 13.2.1) for the S-local model category structure on $\mathcal{M}$.

4.1. Localizing objects and maps

DEFINITION 4.1.1. Let $\mathcal{M}$ be a model category, and let S be a set of maps in $\mathcal{M}$.

1. An *S-localization* of an object X is an S-local object $\widehat{X}$ (see Definition 3.3.2) together with an S-local equivalence $j\colon X \to \widehat{X}$. We will sometimes use the phrase *S-localization* to refer to the object $\widehat{X}$, without explicitly mentioning the S-local equivalence j. A *cofibrant S-localization* of X is an S-localization in which the S-local equivalence j is also a cofibration.
2. An *S-localization* of a map $g\colon X \to Y$ is an S-localization $(\widehat{X}, j_X)$ of X, an S-localization $(\widehat{Y}, j_Y)$ of Y, and a map $\hat{g}\colon \widehat{X} \to \widehat{Y}$ such that the square

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "j_X"'] & Y \arrow[d, "j_Y"] \\ \widehat{X} \arrow[r, "\hat{g}"'] & \widehat{Y} \end{tikzcd}$$

commutes. We will sometimes use the term *S-localization* to refer to the map $\hat{g}$, without explicitly mentioning the S-localizations $(\widehat{X}, j_X)$ of X and $(\widehat{Y}, j_Y)$ of Y.

LEMMA 4.1.2. *Let $\mathcal{M}$ be a model category, and let S be a set of maps in $\mathcal{M}$. If X and Y are fibrant objects and $g\colon X \to Y$ is a weak equivalence, then X is S-local if and only if Y is S-local.*

PROOF. If $f\colon A \to B$ is an element of S, then we have a commutative diagram

$$\begin{tikzcd} \mathrm{Map}(B,X) \arrow[r] \arrow[d, "\approx"'] & \mathrm{Map}(A,X) \arrow[d, "\approx"] \\ \mathrm{Map}(B,Y) \arrow[r] & \mathrm{Map}(A,Y) \end{tikzcd}$$

in which the vertical maps are weak equivalences (see Theorem 17.5.2). Thus, the top map is a weak equivalence if and only if the bottom map is a weak equivalence. $\square$

PROPOSITION 4.1.3. *Let $\mathcal{M}$ be a model category, and let S be a set of maps in $\mathcal{M}$. If X and Y are fibrant objects that are weakly equivalent (see Definition 9.5.2), then X is S-local if and only if Y is S-local.*

PROOF. This follows from Lemma 4.1.2. $\square$

4.1.4. S-local equivalences.

PROPOSITION 4.1.5. *If $\mathcal{M}$ is a model category and S is a set of maps in $\mathcal{M}$, then the class of S-local equivalences satisfies the "two out of three" axiom, i.e., if g and h are composable maps, and if two of g, h, and hg are S-local equivalences, then so is the third.*

PROOF. Given maps $g\colon X \to Y$ and $h\colon Y \to Z$, we can apply a functorial cofibrant approximation (see Proposition 9.1.2) to g and h to obtain the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\tilde{g}"] \arrow[d] & \widetilde{Y} \arrow[r, "\tilde{h}"] \arrow[d] & \widetilde{Z} \arrow[d] \\ X \arrow[r, "g"'] & Y \arrow[r, "h"'] & Z \end{tikzcd}$$

in which $\tilde{g}$, $\tilde{h}$, and $\tilde{h}\tilde{g}$ are cofibrant approximations to g, h, and hg, respectively. If W is an S-local object, $\widehat{\boldsymbol{W}}$ is a simplicial resolution of W, and two of the maps $\tilde{g}^*\colon \mathcal{M}(\widetilde{Y}, \widehat{\boldsymbol{W}}) \to \mathcal{M}(\widetilde{X}, \widehat{\boldsymbol{W}})$, $\tilde{h}^*\colon \mathcal{M}(\widetilde{Z}, \widehat{\boldsymbol{W}}) \to \mathcal{M}(\widetilde{Y}, \widehat{\boldsymbol{W}})$, and $(\tilde{h}\tilde{g})^*\colon \mathcal{M}(\widetilde{Z}, \widehat{\boldsymbol{W}}) \to \mathcal{M}(\widetilde{X}, \widehat{\boldsymbol{W}})$ are weak equivalences, then the third is as well. $\square$

PROPOSITION 4.1.6. *If $\mathcal{M}$ is a model category and S is a set of maps in $\mathcal{M}$, then a retract (see Definition 8.1.1) of an S-local equivalence is an S-local equivalence.*

PROOF. If $g\colon X \to Y$ is an S-local equivalence and $h\colon V \to W$ is a retract of g, then we apply the functorial factorization of the maps from the initial object to each of X, Y, V, and W to obtain cofibrant approximations $\tilde{g}\colon \widetilde{X} \to \widetilde{Y}$ to g and $\tilde{h}\colon \widetilde{V} \to \widetilde{W}$ to h such that $\tilde{h}$ is a retract of $\tilde{g}$. If Z is an S-local object and $\widehat{\boldsymbol{Z}}$ is a simplicial resolution of Z, then $\tilde{h}^*\colon \mathcal{M}(\widetilde{W}, \widehat{\boldsymbol{Z}}) \to \mathcal{M}(\widetilde{V}, \widehat{\boldsymbol{Z}})$ is a retract of the weak equivalence $\tilde{g}^*\colon \mathcal{M}(\widetilde{Y}, \widehat{\boldsymbol{Z}}) \to \mathcal{M}(\widetilde{X}, \widehat{\boldsymbol{Z}})$, and so $\tilde{h}^*$ is a weak equivalence. $\square$

PROPOSITION 4.1.7. *Let $\mathcal{M}$ be a simplicial model category, and let S be a set of maps in $\mathcal{M}$. If $g\colon X \to Y$ is a cofibration of cofibrant objects, then g is an S-local equivalence if and only if it has the left lifting property (see Definition 8.2.1) with respect to the map $\widehat{\boldsymbol{W}}^{\Delta[n]} \to \widehat{\boldsymbol{W}}^{\partial\Delta[n]}$ for every simplicial resolution $\widehat{\boldsymbol{W}}$ of every S-local object W and every $n \geq 0$.*

PROOF. This follows from Proposition 18.3.5 and Proposition 18.3.8. ☐

PROPOSITION 4.1.8. *Let $\mathcal{M}$ be a model category, let S be a set of maps in $\mathcal{M}$, and let T be a totally ordered set. If $\boldsymbol{W}: T \to \mathcal{M}$ is a functor such that, if $s, t \in T$ and $s \le t$, then $\boldsymbol{W}_s \to \boldsymbol{W}_t$ is a cofibration of cofibrant objects that is an S-local equivalence, then, for every $s \in T$, the map $\boldsymbol{W}_s \to \operatorname{colim}_{t \ge s} \boldsymbol{W}_t$ is an S-local equivalence.*

PROOF. This follows from Proposition 4.1.7, Lemma 12.2.20 and Proposition 12.2.21. ☐

4.1.9. S-local Whitehead theorems.

THEOREM 4.1.10 (Weak S-local Whitehead theorem). *Let $\mathcal{M}$ is a model category, and let S be a set of maps in $\mathcal{M}$. If X and Y are S-local objects and $g: X \to Y$ is an S-local equivalence, then g is a weak equivalence.*

PROOF. This follows from Proposition 18.1.5. ☐

THEOREM 4.1.11 (Strong S-local Whitehead theorem). *Let $\mathcal{M}$ be a model category, and let S be a set of maps in $\mathcal{M}$. If X and Y are cofibrant S-local objects and $g: X \to Y$ is an S-local equivalence, then g is a homotopy equivalence.*

PROOF. This follows from Theorem 4.1.10 and Proposition 8.3.26. ☐

4.1.12. Characterizing S-local objects and S-local equivalences.

THEOREM 4.1.13. *Let $\mathcal{M}$ be a model category, and let S be a set of maps in $\mathcal{M}$. If X is a fibrant object and $j: X \to \widehat{X}$ is an S-localization of X (see Definition 4.1.1), then j is a weak equivalence if and only if X is S-local.*

PROOF. If X is S-local, then Theorem 4.1.10 implies that j is a weak equivalence. Conversely, if j is a weak equivalence, then Proposition 4.1.3 implies that X is S-local. ☐

THEOREM 4.1.14. *Let $\mathcal{M}$ be a model category, and let S be a set of maps in $\mathcal{M}$. If $\hat{g}: \widehat{X} \to \widehat{Y}$ is an S-localization of $g: X \to Y$ (see Definition 4.1.1), then g is an S-local equivalence if and only if $\hat{g}$ is a weak equivalence.*

PROOF. Proposition 3.3.3 and Proposition 4.1.5 imply that g is an S-local equivalence if and only if $\hat{g}$ is an S-local equivalence. Since $\widehat{X}$ and $\widehat{Y}$ are S-local, Theorem 4.1.10 and Proposition 3.3.3 imply that $\hat{g}$ is an S-local equivalence if and only if it is a weak equivalence. ☐

If $\mathcal{M}$ is a left proper cellular model category (see Definition 15.1.1) and S is a set of maps in $\mathcal{M}$, then, in Definition 4.3.2, we define a functorial S-localization (L_S, j). Theorem 4.1.13 then implies that a fibrant object X is S-local if and only if the S-localization map $j(X): X \to \mathrm{L}_S X$ is a weak equivalence (see Theorem 4.3.5), and Theorem 4.1.14 implies that a map $g: X \to Y$ is an S-local equivalence if and only if $\mathrm{L}_S(g): \mathrm{L}_S X \to \mathrm{L}_S Y$ is a weak equivalence (see Theorem 4.3.6).

4.2. Horns on S and S-local equivalences

This section contains some technical constructions and results that are needed for our construction of a natural cofibrant S-localization in Section 4.3. For the motivation for the definition of a horn on S, see Section 1.3.

DEFINITION 4.2.1. If $\mathcal{M}$ is a model category and S is a set of maps in $\mathcal{M}$, then a *horn on* S is a map constructed by

1. choosing an element $f\colon A \to B$ of S,
2. choosing a cosimplicial resolution $\tilde{f}\colon \widetilde{A} \to \widetilde{B}$ (see Definition 17.1.10) of f such that $\tilde{f}$ is a Reedy cofibration,
3. choosing an integer $n \geq 0$, and then
4. constructing the map $\widetilde{A} \otimes \Delta[n] \amalg_{\widetilde{A}\otimes\partial\Delta[n]} \widetilde{B} \otimes \partial\Delta[n] \to \widetilde{B} \otimes \Delta[n]$.

PROPOSITION 4.2.2. *If $\mathcal{M}$ is a model category and S is a set of maps in $\mathcal{M}$, then every horn on S is a cofibration.*

PROOF. This follows from Proposition 17.3.13. $\square$

PROPOSITION 4.2.3. *If $\mathcal{M}$ is a model category and S is a set of weak equivalences in $\mathcal{M}$, then every horn on S is a trivial cofibration.*

PROOF. This follows from Proposition 17.1.14 and Proposition 17.3.12. $\square$

DEFINITION 4.2.4. Let $\mathcal{M}$ be a left proper cellular model category with generating cofibrations I and generating trivial cofibrations J, and let S be a set of maps in $\mathcal{M}$.

- A *full set of horns on* S is a set $\Lambda(S)$ of maps obtained by choosing, for every element $f\colon A \to B$ of S, a cosimplicial resolution $\tilde{f}\colon \widetilde{\boldsymbol{A}} \to \widetilde{\boldsymbol{B}}$ of f (see Definition 17.1.10) such that $\tilde{f}$ is a Reedy cofibration (see Proposition 17.1.12) and letting $\Lambda(S)$ be the set

$$\Lambda(S) = \{\widetilde{\boldsymbol{A}} \otimes \Delta[n] \amalg_{\widetilde{\boldsymbol{A}}\otimes\partial\Delta[n]} \widetilde{\boldsymbol{B}} \otimes \partial\Delta[n] \to \widetilde{\boldsymbol{B}} \otimes \Delta[n] \bigm| (A \to B) \in S, n \geq 0\}.$$

  We will use the symbol $\Lambda(S)$ to denote some full set of horns on S, even though it depends on the choices of cosimplicial resolutions of the elements of S.
- A *full set of augmented S-horns* is a set $\overline{\Lambda(S)}$ of maps

$$\overline{\Lambda(S)} = \Lambda(S) \cup J$$

  for some full set of horns $\Lambda(S)$ on S.

PROPOSITION 4.2.5. *Let $\mathcal{M}$ be a left proper cellular model category, and let S be a set of maps in $\mathcal{M}$. An object X of $\mathcal{M}$ is S-local if and only if the map $X \to *$ (where $*$ is the terminal object of $\mathcal{M}$) has the right lifting property with respect to every element of a full set of augmented S-horns (see Definition* 4.2.4*).*

PROOF. This follows from Proposition 13.2.9, Proposition 18.3.5 and Proposition 18.3.8. $\square$

PROPOSITION 4.2.6. *If $\mathcal{M}$ is a left proper cellular model category and S is a set of maps in $\mathcal{M}$, then every element of a full set of horns on S (see Definition* 4.2.4*) is an S-local equivalence.*

PROOF. This follows from Proposition 18.3.5 and Proposition 18.3.13. $\square$

PROPOSITION 4.2.7. *If $\mathcal{M}$ is a left proper cellular model category with generating cofibrations I and S is a set of maps in $\mathcal{M}$, then there is a set $\widetilde{\Lambda S}$ of relative I-cell complexes with cofibrant domains such that*

1. *every element of $\widetilde{\Lambda S}$ is an S-local equivalence, and*

2. *an object X of $\mathcal{M}$ is S-local if and only if the map $X \to *$ (where $*$ is the terminal object of $\mathcal{M}$) is a $\widetilde{\Lambda S}$-injective.*

PROOF. Choose a full set of horns on S (see Definition 4.2.4.) Factor each element $g\colon C \to D$ of $\Lambda(S)$ as $C \xrightarrow{\tilde{g}} \widetilde{D} \xrightarrow{p} D$ where $\tilde{g}$ is a relative I-cell complexes and p is a trivial fibration (see Corollary 13.2.12). The retract argument (see Proposition 8.2.2) implies that g is a retract of $\tilde{g}$. Since p and g are S-local equivalences (see Proposition 3.3.3 and Proposition 4.2.6), Proposition 4.1.5 implies that $\tilde{g}$ is an S-local equivalence.

Proposition 13.2.14 implies that there is a set $\widetilde{J}$ of generating trivial cofibrations for $\mathcal{M}$ such that every element of $\widetilde{J}$ is a relative I-cell complexe with cofibrant domain. We let

$$\widetilde{\Lambda S} = \widetilde{J} \cup \{\tilde{g}\}_{g\in\Lambda(S)}.$$

It remains only to show that condition 2 is satisfied. If the map $X \to *$ is a $\widetilde{\Lambda S}$-injective, then Proposition 4.2.5 and Lemma 8.2.7 imply that X is S-local. Conversely, if X is S-local, then X is fibrant and every element of $\widetilde{\Lambda S}$ is a cofibration between cofibrant objects, and so Proposition 18.3.5, Theorem 17.1.28, Proposition 17.1.27, and Proposition 18.3.8 imply that the map $X \to *$ is a $\widetilde{\Lambda S}$-injective. □

DEFINITION 4.2.8. If $\mathcal{M}$ is a left proper cellular model category and S is a set of maps in $\mathcal{M}$, then a *relative $\widetilde{\Lambda S}$-cell complex* is a map that can be constructed as a transfinite composition (see Definition 12.2.2) of pushouts (see Definition 8.2.10) of elements of $\widetilde{\Lambda S}$ (see Proposition 4.2.7).

PROPOSITION 4.2.9. *Let $\mathcal{M}$ be a left proper cellular model category, and let S be a set of maps in $\mathcal{M}$. An object X of $\mathcal{M}$ is S-local if and only if the map $X \to *$ (where $*$ is the terminal object of $\mathcal{M}$) has the right lifting property with respect to all relative $\widetilde{\Lambda S}$-cell complexes.*

PROOF. This follows from Proposition 4.2.7, Lemma 8.2.5, and Lemma 12.2.16. □

4.2.10. Regular $\widetilde{\Lambda S}$-cofibrations and S-local equivalences. The main result of this section is Theorem 4.2.12, which asserts that if $\mathcal{M}$ is a left proper cellular model category and S is a set of maps in $\mathcal{M}$, then every relative $\widetilde{\Lambda S}$-cell complexe is an S-local equivalence.

PROPOSITION 4.2.11. *Let $\mathcal{M}$ be a left proper cellular model category, and let S be a set of maps in $\mathcal{M}$. If $g\colon C \to D$ is a cofibration that is also an S-local equivalence, then any pushout of g is also an S-local equivalence.*

PROOF. This follows from Proposition 4.6.4. □

THEOREM 4.2.12. *If $\mathcal{M}$ is a left proper cellular model category and S is a set of maps in $\mathcal{M}$, then every relative $\widetilde{\Lambda S}$-cell complex (see Definition 4.2.8) is both a cofibration and an S-local equivalence.*

PROOF. Since every element of $\widetilde{\Lambda S}$ (see Proposition 4.2.7) is a cofibration and cofibrations are closed under both pushouts and transfinite compositions (see Proposition 12.2.19), every relative $\widetilde{\Lambda S}$-cell complex is a cofibration. Thus, it remains only to show that a relative $\widetilde{\Lambda S}$-cell complex is an S-local equivalence.

Proposition 4.2.7 implies that every element of $\widetilde{\Lambda S}$ is an S-local equivalence, and so Proposition 4.2.11 implies that every pushout of an element of $\widetilde{\Lambda S}$ is an S-local equivalence. Thus, it remains only to show that a transfinite composition of pushouts of elements of $\widetilde{\Lambda S}$ is an S-local equivalence.

If λ is an ordinal and

$$X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \qquad (\beta < \lambda)$$

is a λ-sequence of pushouts of elements of $\widetilde{\Lambda S}$, then Proposition 18.5.3 implies that we can find a λ-sequence of cofibrations together with a map of λ-sequences

$$\begin{tikzcd}
\widetilde{X}_0 \arrow[r] \arrow[d] & \widetilde{X}_1 \arrow[r] \arrow[d] & \widetilde{X}_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & \widetilde{X}_\beta \arrow[r] \arrow[d] & \cdots \\
X_0 \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & \cdots \arrow[r] & X_\beta \arrow[r] & \cdots
\end{tikzcd}$$

such that each vertical map $\widetilde{X}_\beta \to X_\beta$ is a cofibrant approximation to X_β and $\operatorname{colim}_{\beta<\lambda} \widetilde{X}_\beta \to \operatorname{colim}_{\beta<\lambda} X_\beta$ is a cofibrant approximation to $\operatorname{colim}_{\beta<\lambda} X_\beta$. If W is an S-local object and $\widehat{\boldsymbol{W}}$ is a simplicial resolution of W, then, since each $X_\beta \to X_{\beta+1}$ is an S-local equivalence and each $\widetilde{X}_\beta \to \widetilde{X}_{\beta+1}$ is a cofibration, each $\mathcal{M}(\widetilde{X}_{\beta+1}, \widehat{\boldsymbol{W}}) \to \mathcal{M}(\widetilde{X}_\beta, \widehat{\boldsymbol{W}})$ is a trivial fibration of simplicial sets (see Theorem 18.3.7). Thus,

$$\mathcal{M}(\widetilde{X}_0, \widehat{\boldsymbol{W}}) \leftarrow \mathcal{M}(\widetilde{X}_1, \widehat{\boldsymbol{W}}) \leftarrow \mathcal{M}(\widetilde{X}_2, \widehat{\boldsymbol{W}}) \leftarrow \cdots \leftarrow \mathcal{M}(\widetilde{X}_\beta, \widehat{\boldsymbol{W}}) \leftarrow \cdots$$

is a tower of trivial fibrations of simplicial sets, and so the projection $\lim_{\beta<\lambda} \mathcal{M}(\widetilde{X}_\beta, \widehat{\boldsymbol{W}}) \to \mathcal{M}(\widetilde{X}_0, \widehat{\boldsymbol{W}})$ is a weak equivalence. Since $\mathcal{M}(\operatorname{colim}_{\beta<\lambda} \widetilde{X}_\beta, \widehat{\boldsymbol{W}})$ is isomorphic to $\lim_{\beta<\lambda} \mathcal{M}(\widetilde{X}_\beta, \widehat{\boldsymbol{W}})$, this implies that the composition $X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$ is an S-local equivalence. □

PROPOSITION 4.2.13. *Let $\mathcal{M}$ be a left proper cellular model category, and let S be a set of maps in $\mathcal{M}$. If $j\colon X \to \widehat{X}$ is a relative $\widetilde{\Lambda S}$-cell complex and $\widehat{X}$ is a $\widetilde{\Lambda S}$-injective, then the pair $(\widehat{X}, j)$ is a cofibrant S-localization of X.*

PROOF. This follows from Theorem 4.2.12 and Proposition 4.2.7. □

Theorem 4.2.12 and Proposition 4.1.6 imply that every $\widetilde{\Lambda S}$-cofibration is an S-local equivalence. The following example (due to A. K. Bousfield) shows that, among the cofibrations that are S-local equivalences, there are maps that are not $\widetilde{\Lambda S}$-cofibrations.

EXAMPLE 4.2.14. Let $\mathcal{M} = \mathsf{Top}_*$, and let $f\colon A \to B$ be the inclusion $S^n \to D^{n+1}$. The path space fibration $p\colon \mathrm{P}\,\mathrm{K}(\mathbb{Z}, n) \to \mathrm{K}(\mathbb{Z}, n)$ is an f-injective (see Definition 1.4.2), and so every f-cofibration has the homotopy left lifting property with respect to p (see Definition 10.3.2). The cofibration $* \to S^n$ does not have the homotopy left lifting property with respect to p, and so it is not an f-cofibration. However, since both the composition $* \to S^n \to D^{n+1}$ and f itself are f-local equivalences (see Proposition 3.3.3), the "two out of three" property of weak equivalences implies that the inclusion $* \to S^n$ is an f-local equivalence. Thus, $* \to S^n$ is both a cofibration and an f-local equivalence, but it is not an f-cofibration.

4.3. A functorial localization

THEOREM 4.3.1. *If $\mathcal{M}$ is a left proper cellular model category and S is a set of maps in $\mathcal{M}$, then there is a natural factorization of every map $X \to Y$ in $\mathcal{M}$ as*

$$X \xrightarrow{j} \mathrm{E}_S \xrightarrow{p} Y$$

in which j is a relative $\widetilde{\Lambda S}$-cell complex (see Definition 4.2.8) and p is a $\widetilde{\Lambda S}$-injective.

PROOF. Proposition 4.2.7 and Theorem 15.4.3 imply that the domains of the elements of $\widetilde{\Lambda S}$ are small relative to the subcategory of relative $\widetilde{\Lambda S}$-cell complexes, and so Lemma 12.3.6 implies that there is a cardinal κ such that the domain of every element of $\widetilde{\Lambda S}$ is κ-small relative to the subcategory of relative $\widetilde{\Lambda S}$-cell complexes. We let $\lambda = \mathrm{Succ}(\kappa)$ (see Definition 12.1.11), so that λ is a regular cardinal (see Proposition 12.1.15). The result now follows from Corollary 12.4.16. □

DEFINITION 4.3.2. Let $\mathcal{M}$ be a left proper cellular model category, and let S be a set of maps in $\mathcal{M}$. The *S-localization* of an object X is the object $\mathrm{L}_S X$ obtained by applying the factorization of Theorem 4.3.1 to the map $X \to *$ (where $*$ is the terminal object of $\mathcal{M}$). This factorization defines a natural transformation $j\colon 1 \to \mathrm{L}_S$ such that $j(X)\colon X \to \mathrm{L}_S X$ is a relative $\widetilde{\Lambda S}$-cell complex for every object X of $\mathcal{M}$.

THEOREM 4.3.3. *If $\mathcal{M}$ is a left proper cellular model category and S is a set of maps in $\mathcal{M}$, then, for every object X, the S-localization $j(X)\colon X \to \mathrm{L}_S X$ (see Definition 4.3.2) is a cofibrant S-localization of X.*

PROOF. This follows from Proposition 4.2.13. □

COROLLARY 4.3.4. *If $\mathcal{M}$ is a left proper cellular model category and S is a set of maps in $\mathcal{M}$, then every object has an S-localization.*

PROOF. This follows from Theorem 4.3.3. □

THEOREM 4.3.5. *Let $\mathcal{M}$ be a left proper cellular model category, and let S be a set of maps in $\mathcal{M}$. If X is a fibrant object, then X is S-local if and only if the S-localization map $j(X)\colon X \to \mathrm{L}_S X$ (see Definition 4.3.2) is a weak equivalence.*

PROOF. This follows from Theorem 4.1.13. □

THEOREM 4.3.6. *Let $\mathcal{M}$ be a left proper cellular model category, and let S be a set of maps in $\mathcal{M}$. The map $g\colon X \to Y$ is an S-local equivalence if and only if its S-localization $\mathrm{L}_S(g)\colon \mathrm{L}_S X \to \mathrm{L}_S Y$ (see Definition 4.3.2) is a weak equivalence.*

PROOF. This follows from Theorem 4.1.14. □

4.4. Localization of subcomplexes

This section contains some technical results on the S-localization (see Definition 4.3.2) needed for the cardinality argument of Section 4.5.

PROPOSITION 4.4.1. *Let $\mathcal{M}$ be a left proper cellular model category, and let S be a set of maps in $\mathcal{M}$. If $g\colon X \to Y$ is the inclusion of a subcomplex, then so is $\mathrm{L}_S(g)\colon \mathrm{L}_S X \to \mathrm{L}_S Y$ (see Definition 4.3.2).*

PROOF. This follows from Proposition 15.4.6. □

PROPOSITION 4.4.2. *Let $\mathcal{M}$ be a left proper cellular model category, and let S be a set of maps in $\mathcal{M}$. If $g\colon X \to Y$ is the inclusion of a subcomplex, then it is an S-local equivalence if and only if $\mathrm{L}_S(g)\colon \mathrm{L}_S X \to \mathrm{L}_S Y$ is the inclusion of a strong deformation retract.*

PROOF. If $\mathrm{L}_S(g)$ is the inclusion of a strong deformation retract, then it is a weak equivalence, and so Theorem 4.3.6 implies that g is an S-local equivalence.

Conversely, if g is an S-local equivalence, then Theorem 4.3.6 and Proposition 4.4.1 imply that $\mathrm{L}_S(g)$ is a trivial cofibration of fibrant objects, and so Corollary 10.4.20 implies that it is the inclusion of a strong deformation retract. □

REMARK 4.4.3. If we take S to be the empty set, then $\mathrm{L}_S X$ is a functorial fibrant approximation to X (see Definition 9.1.1). In this case, Proposition 4.4.1 asserts that if W is a subcomplex of X, then this fibrant approximation to W is a subcomplex of this fibrant approximation to X.

PROPOSITION 4.4.4. *Let $\mathcal{M}$ be a left proper cellular model category, and let S be a set of maps in $\mathcal{M}$. If X is a cell complex and $W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_\beta \subset \cdots$ $(\beta < \lambda)$ is a λ-sequence of subcomplexes (see Remark 12.5.7) of X (where λ is the ordinal chosen in the proof of Theorem 4.3.1), then the natural map $\mathrm{colim}_{\beta<\lambda} \mathrm{L}_S W_\beta \to \mathrm{L}_S \,\mathrm{colim}_{\beta<\lambda} W_\beta$ is an isomorphism.*

PROOF. Proposition 4.4.1 implies that the map is an isomorphism onto a subcomplex; it remains only to show that every $\widetilde{\Lambda S}$-cell of $\mathrm{L}_S \,\mathrm{colim}_{\beta<\lambda} W_\beta$ is contained in some $\mathrm{L}_S W_\beta$. We will do this by a transfinite induction on the presentation ordinal of the $\widetilde{\Lambda S}$-cell.

Since there are no $\widetilde{\Lambda S}$-cells of presentation ordinal equal to a limit ordinal, we let γ be an ordinal such that $\gamma + 1 < \lambda$, and we assume that the assertion is true for all $\widetilde{\Lambda S}$-cells of presentation ordinal at most γ. This assumption implies that the γ-skeleton of $\mathrm{L}_S \,\mathrm{colim}_{\beta<\lambda} W_\beta$ is isomorphic to $\mathrm{colim}_{\beta<\lambda}\big((\mathrm{L}_S W_\beta)^\gamma\big)$. Thus, the γ-skeleta of the $\mathrm{L}_S W_\beta$ form a λ-sequence whose colimit is the γ-skeleton of $\mathrm{L}_S \,\mathrm{colim}_{\beta<\lambda} W_\beta$. If e is a $\widetilde{\Lambda S}$-cell of $\mathrm{L}_S \,\mathrm{colim}_{\beta<\lambda} W_\beta$ of presentation ordinal $\gamma + 1$, then the attaching map of e must factor through $(\mathrm{L}_S W_\beta)^\gamma$ for some $\beta < \lambda$, and so e is contained in $\mathrm{L}_S W_\beta$. □

4.5. The Bousfield-Smith cardinality argument

The purpose of this section is to prove the following proposition, which will be used in Section 4.6 to prove Theorem 3.3.8.

PROPOSITION 4.5.1. *If $\mathcal{M}$ is a left proper cellular model category and S is a set of maps in $\mathcal{M}$, then there is a set J_S of inclusions of cell complexes such that*

1. *every element of J_S is an S-local equivalence, and*
2. *the class of J_S-cofibrations (see Definition 12.4.1) equals the class of cofibrations that are also S-local equivalences.*

The set J_S will serve as our set of generating trivial cofibrations (see Definition 13.2.1) for the S-local model category structure on $\mathcal{M}$ (see Theorem 3.3.8 and Section 4.6).

The proof of Proposition 4.5.1 is at the end of this section (on page 69). We will prove Proposition 4.5.1 by showing that there is a set J_S of cofibrations that are S-local equivalences such that every cofibration that is an S-local equivalence has the left lifting property (see Definition 8.2.1) with respect to every J_S-injective. Proposition 4.5.1 will then follow from Corollary 12.4.17.

We will find the set J_S by showing (in Proposition 4.5.6) that there is a cardinal γ (see Definition 4.5.3) such that, if a map has the right lifting property with respect to all inclusions of cell complexes that are S-local equivalences between complexes of size at most γ, then it has the right lifting property with respect to all cofibrations that are S-local equivalences. We will then let J_S be a set of representatives of the isomorphism classes of of these "small enough" inclusions of cell complexes that are S-local equivalences.

We begin with the following lemma, which implies that it is sufficient to find a set J_S such that the J_S-injectives have the right lifting property with respect to all inclusions of cell complexes that are S-local equivalences.

LEMMA 4.5.2. *Let $\mathcal{M}$ be a left proper cellular model category, and let S be a set of maps in $\mathcal{M}$. If $p\colon E \to B$ is a fibration with the right lifting property with respect to all inclusions of cell complexes that are S-local equivalences, then it has the right lifting property with respect to all cofibrations that are S-local equivalences.*

PROOF. Let $g\colon X \to Y$ be a cofibration that is an S-local equivalence. Proposition 13.2.16 implies that there is a cofibrant approximation (see Definition 9.1.8) $\tilde{g}$ to g such that $\tilde{g}$ is an inclusion of cell complexes. Proposition 3.3.3 and Proposition 4.1.5 imply that $\tilde{g}$ is an S-local equivalence, and so the lemma now follows from Proposition 11.1.18. □

DEFINITION 4.5.3. Let $\mathcal{M}$ be a left proper cellular model category, and let S be a set of maps in $\mathcal{M}$. If σ denotes the cardinal of Definition 15.1.1, η denotes a cardinal such that the domain of every element of I is η-compact (see Proposition 13.4.6), λ denotes the cardinal selected in the proof of Theorem 4.3.1, ρ denotes the cardinal described in Definition 15.5.4, and κ denotes the cardinal described in Proposition 15.5.3 for the set $\widetilde{\Lambda S}$, then we let γ denote the cardinal $\gamma = \sigma\eta\lambda\rho\kappa$.

LEMMA 4.5.4. *Let $\mathcal{M}$ be a left proper cellular model category, and let S be a set of maps in $\mathcal{M}$. If X is a cell complex of size at most γ (see Definition 4.5.3), then $\mathrm{L}_S X$ has size at most γ.*

PROOF. This follows from Proposition 15.5.3. □

The following proposition will be used in Proposition 4.5.6 to extend a map over an arbitrary inclusion of a subcomplex that is an S-local equivalence by extending it over a subcomplex of size at most γ.

PROPOSITION 4.5.5. *Let $\mathcal{M}$ be a left proper cellular model category, let S be a set of maps in $\mathcal{M}$, and let D be a cell complex. If $i\colon C \to D$ is the inclusion of a proper subcomplex and an S-local equivalence, then there is a subcomplex K of D such that*

1. *the subcomplex K is not contained in the subcomplex C,*

2. *the size of K is at most γ (see Definition 4.5.3), and*
3. *the inclusions $K \cap C \to K$ (see Theorem 15.2.6) and $C \to C \cup K$ are both S-local equivalences.*

PROOF. Since $i\colon C \to D$ is an inclusion of a subcomplex and an S-local equivalence, Proposition 4.4.2 implies that $\mathrm{L}_S(i)\colon \mathrm{L}_S C \to \mathrm{L}_S D$ is the inclusion of a deformation retract. Thus, there is a retraction $r\colon \mathrm{L}_S D \to \mathrm{L}_S C$, and Proposition 8.3.16 implies that we can choose a homotopy $R\colon \mathrm{Cyl}^{\mathcal{M}}(\mathrm{L}_S D) \to \mathrm{L}_S D$ (see Definition 15.5.4) from the identity map of $\mathrm{L}_S D$ to $\mathrm{L}_S(i) \circ r$.

We will show that there exists a subcomplex K of D, of size at most γ, such that

1. K is not contained in C,
2. $R|_{\mathrm{Cyl}^{\mathcal{M}}(\mathrm{L}_S K)}$ is a deformation retraction of $\mathrm{L}_S K$ onto $\mathrm{L}_S(K \cap C)$, and
3. $R|_{\mathrm{Cyl}^{\mathcal{M}}(\mathrm{L}_S (C\cup K))}$ is a deformation retraction of $\mathrm{L}_S(C \cup K)$ onto $\mathrm{L}_S C$.

We will do this by constructing a λ-sequence $K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_\beta \subset \cdots$ ($\beta < \lambda$) of subcomplexes of D (where λ is the ordinal selected in the proof of Theorem 4.3.1) such that, for every $\beta < \lambda$,

1. K_β has size at most γ,
2. $R|_{\mathrm{Cyl}^{\mathcal{M}}(\mathrm{L}_S K_\beta)}$ factors through the subcomplex $\mathrm{L}_S K_{\beta+1}$ of $\mathrm{L}_S D$ (see Proposition 4.4.1),

and such that no K_β is contained in C. If we then let $K = \bigcup_{\beta<\lambda} K_\beta$, then Proposition 4.4.4 will imply that $\mathrm{L}_S K \approx \mathrm{colim}_{\beta<\lambda} \mathrm{L}_S K_\beta$. Thus, $R|_{\mathrm{Cyl}^{\mathcal{M}}(\mathrm{L}_S K)}$ factors through $\mathrm{L}_S K$, $r|_{\mathrm{L}_S K}$ factors through $(\mathrm{L}_S K) \cap (\mathrm{L}_S C)$ (see Theorem 15.2.6 and Proposition 15.2.3), and $R|_{\mathrm{Cyl}^{\mathcal{M}}(\mathrm{L}_S K)}$ is a deformation retraction of $\mathrm{L}_S K$ onto $(\mathrm{L}_S K) \cap (\mathrm{L}_S C)$.

We begin by choosing a cell of D that isn't contained in C. Since the domains of the elements of I are γ-compact, we can choose a subcomplex K_0 of D, of size at most γ, through which the inclusion of that cell factors.

For successor ordinals, suppose that $\beta+1 < \gamma$, and that we've constructed K_β. Lemma 4.5.4 and Proposition 15.5.5 imply that $\mathrm{Cyl}^{\mathcal{M}}(\mathrm{L}_S K_\beta)$ has size at most γ, and so Definition 15.1.1 implies that $R|_{\mathrm{Cyl}^{\mathcal{M}}(\mathrm{L}_S K_\beta)}$ factors through a subcomplex of $\mathrm{L}_S D$ of size at most $\sigma\gamma = \gamma$. The zero skeleton of this subcomplex is a subcomplex Z_β of D, of size at most γ, such that $R|_{\mathrm{Cyl}^{\mathcal{M}}(\mathrm{L}_S K_\beta)}$ factors through $\mathrm{L}_S Z_\beta$. We let $K_{\beta+1} = K_\beta \cup Z_\beta$. It is clear that $K_{\beta+1}$ has the properties required of it, and so the proof is complete. $\square$

PROPOSITION 4.5.6. *Let $\mathcal{M}$ be a left proper cellular model category, and let S be a set of maps in $\mathcal{M}$. If $p\colon X \to Y$ has the right lifting property with respect to those inclusions of subcomplexes $i\colon C \to D$ that are S-local equivalences and such that the size of D is at most γ (see Definition 4.5.3), then p has the right lifting property with respect to all inclusions of subcomplexes that are S-local equivalences.*

PROOF. Let $i\colon C \to D$ be an inclusion of a subcomplex that is an S-local equivalence, and let the solid arrow diagram

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ D \arrow[r, "k"'] \arrow[ur, dotted] & Y \end{tikzcd}$$

be commutative; we must show that there exists a dotted arrow making both triangles commute. To do this, we will consider the subcomplexes of D over which our map can be defined, and use Zorn's lemma to show that it can be defined over all of D.

Let T be the set of pairs (D_t, g_t) such that

1. D_t is a subcomplex of D containing C such that the inclusion $i_t\colon C \to D_t$ is an S-local equivalence, and
2. g_t is a function $D_t \to X$ such that $g_t i_t = h$ and $pg_t = k|_{D_t}$.

We define a preorder on T by defining $(D_t, g_t) < (D_u, g_u)$ if $D_t \subset D_u$ and $g_u|_{D_t} = g_t$. If $T' \subset T$ is a chain (i.e., a totally ordered subset of T), let $D_u = \operatorname{colim}_{(D_t,g_t)\in T'} D_t$, and define $g_u\colon D_u \to X$ by $g_u = \operatorname{colim}_{(D_t,g_t)\in T'} g_t$. The universal mapping property of the colimit implies that $g_u i_u = h$ and $pg_u = k|_{D_u}$, and Proposition 4.1.8 implies that the map $C \to D_u$ is an S-local equivalence. Thus, (D_u, g_u) is an element of T, and so it is an upper bound for T'. Zorn's lemma now implies that T has a maximal element (D_m, g_m). We will complete the proof by showing that $D_m = D$.

If $D_m \neq D$, then Proposition 4.5.5 implies that there is a subcomplex K of D such that K is not contained in D_m, the size of K is at most γ, and the inclusions $K \cap D_m \to K$ and $D_m \to D_m \cup K$ are both S-local equivalences. Thus, there is a map $g_K : K \to X$ such that $pg_K = k|_K$ and $g_K|_{K\cap D_m} = g_m|_{K\cap D_m}$, and so g_m and g_K combine to define a map $g_{mK} : K \cup D_m \to X$ such that $pg_{mK} = k|_{K\cup D_m}$ and $g_{mK} i = h$. Thus, $(K \cup D_m, g_{mK})$ is an element of T strictly greater than (D_m, g_m). This contradicts (D_m, g_m) being a maximal element of T, and so our assumption that $D_m \neq D$ must have been false, and the proof is complete. □

Proof of Proposition 4.5.1. Let J_S be a set of representatives of the isomorphism classes of inclusions of subcomplexes that are S-local equivalences of complexes of size at most γ (see Definition 4.5.3). Proposition 4.5.6, Lemma 4.5.2, and Corollary 12.4.17 imply that the J_S-cofibrations are exactly the cofibrations that are S-local equivalences, and so the proof is complete. □

4.6. Completion of the proofs

Proof of Theorem 3.3.8. We begin by using Theorem 13.3.1 to show that there is a cofibrantly generated model category structure on $\mathcal{M}$ with weak equivalences, cofibrations, and fibrations as described in the statement of Theorem 3.3.8.

Proposition 4.1.5 implies that the class of S-local equivalences satisfies the "two out of three" axiom, and Proposition 4.1.6 implies that it is closed under retracts.

Let J_S be the set of maps provided by Proposition 4.5.1, and let I be the set of generating cofibrations of the original cofibrantly generated model category structure on $\mathcal{M}$. Condition 1 of Theorem 13.3.1 is thus satisfied for I and, since every element of J_S has a cofibrant domain, Theorem 15.4.3 implies that condition 1 of Theorem 13.3.1 is satisfied for J.

The subcategory of I-cofibrations is the subcategory of cofibrations in the given model category structure in $\mathcal{M}$, and the I-injectives are the trivial fibrations in that model category. Thus, Proposition 4.5.1 implies that condition 2 of Theorem 13.3.1 is satisfied.

Since the J_S-cofibrations are a subcategory of the I-cofibrations, every I-injective must be a J_S-injective. Proposition 3.3.3 implies that every J_S-injective is an S-local equivalence, and so condition 3 is satisfied.

Proposition 4.5.1 implies that condition 4a of Theorem 13.3.1 is satisfied, and so Theorem 13.3.1 now implies that we have a model category.

If $\mathcal{M}$ is a simplicial model category, we note that, since the simplicial structure is the given one, axiom M6 of Definition 10.1.2 holds because it does so in the given simplicial model category structure on $\mathcal{M}$. For axiom M7 of Definition 10.1.2, we note that the class of S-local cofibrations equals the given class of cofibrations and the class of S-local fibrations is contained in the given class of fibrations. Thus, the first requirement of axiom M7 is clear. In the case that the map p is an S-local equivalence, the rest of axiom M7 follows from the fact that, since the class of S-local cofibrations equals the given class of cofibrations, the class of S-local trivial fibrations equals the given class of trivial fibrations (see Proposition 8.2.3).

In the case that the map i is an S-local equivalence, we can choose a cofibrant approximation $\tilde{\imath}\colon \tilde{\boldsymbol{A}} \to \tilde{\boldsymbol{B}}$ to i such that $\tilde{\imath}$ is a cofibration, and Proposition 18.3.4 and Proposition 18.3.6 imply that $(\tilde{\imath}, p)$ is a homotopy orthogonal pair. Example 17.1.30, Proposition 18.3.13, and Proposition 18.3.9 imply that the map $\widetilde{A} \otimes \Delta[n] \amalg_{\widetilde{A}\otimes\partial\Delta[n]} \widetilde{B} \otimes \partial\Delta[n] \to \widetilde{B} \otimes \Delta[n]$ has the left lifting property with respect to p for every $n \geq 0$. Thus, $\tilde{\imath}\colon \widetilde{A} \to \widetilde{B}$ has the left lifting property with respect to the fibration $X^{\Delta[n]} \to Y^{\Delta[n]} \times_{Y^{\partial\Delta[n]}} X^{\partial\Delta[n]}$ for every $n \geq 0$ (see Lemma 10.3.6). Since $\mathcal{M}$ is left proper, $i\colon A \to B$ also has the left lifting property with respect to the map $X^{\Delta[n]} \to Y^{\Delta[n]} \times_{Y^{\partial\Delta[n]}} X^{\partial\Delta[n]}$ for every $n \geq 0$, and so our result follows from Lemma 10.3.6. □

LEMMA 4.6.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories and let $\mathrm{F}\colon \mathcal{M} \rightleftarrows \mathcal{N} :\mathrm{U}$ be a Quillen pair. If $g\colon A \to B$ is a map of cofibrant objects in $\mathcal{M}$ and $h\colon C \to D$ is a horn on g (see Definition 4.2.1), then $\mathrm{F}(h)$ is a horn on $\mathrm{F}(g)$.*

PROOF. Since the left adjoint F commutes with colimits, this follows from Corollary 18.6.3. □

PROPOSITION 4.6.2. *Let $\mathcal{M}$ be a left proper cellular model category and let S be a set of maps in $\mathcal{M}$. If $\mathcal{N}$ is a model category and $\mathrm{F}\colon \mathcal{M} \to \mathcal{N}$ is a left Quillen functor that takes every cofibrant approximation to an element of S into a weak equivalence in $\mathcal{N}$, then F is a left Quillen functor when considered as a functor $\mathrm{L}_S\mathcal{M} \to \mathcal{N}$.*

PROOF. Since the underlying category of $\mathrm{L}_S\mathcal{M}$ equals that of $\mathcal{M}$, F has a right adjoint whether we consider it to be a functor $\mathrm{F}\colon \mathcal{M} \to \mathcal{N}$ or a functor $\mathrm{F}\colon \mathrm{L}_S\mathcal{M} \to \mathcal{N}$. Thus, it remains only to show that $\mathrm{F}\colon \mathrm{L}_S\mathcal{M} \to \mathcal{N}$ preserves both cofibrations and trivial cofibrations. Since the class of cofibrations of $\mathrm{L}_S\mathcal{M}$ equals that of $\mathcal{M}$, we need only consider the trivial cofibrations of $\mathrm{L}_S\mathcal{M}$.

If $g\colon A \to B$ is a trivial cofibration of $\mathrm{L}_S\mathcal{M}$, then g is a cofibration in $\mathcal{M}$ such that $\mathrm{L}_S(g)\colon \mathrm{L}_S A \to \mathrm{L}_S B$ is a weak equivalence in $\mathcal{M}$ (see Theorem 4.3.6). Since g is a cofibration, so is $\mathrm{F}(g)$, and so it remains only to show that $\mathrm{F}(g)$ is a weak equivalence. Since the natural S-localization $\mathrm{L}_S\colon \mathcal{M} \to \mathcal{M}$ preserves cofibrations, $\mathrm{L}_S(g)$ is actually a trivial cofibration in $\mathcal{M}$, and so $\mathrm{FL}_S(g)$ is a trivial cofibration in $\mathcal{N}$. Thus, we have the diagram in $\mathcal{N}$

$$\begin{tikzcd} \mathrm{F}A \arrow[r, "\mathrm{F}j(A)"] \arrow[d, "\mathrm{F}(g)"'] & \mathrm{FL}_S A \arrow[d, "\mathrm{FL}_S(g)"] \\ \mathrm{F}B \arrow[r, "\mathrm{F}j(B)"'] & \mathrm{FL}_S B \end{tikzcd}$$

in which $\mathrm{FL}_S(g)$ is a weak equivalence. Since $j(A)\colon A \to \mathrm{L}_S A$ is a transfinite composition of pushouts of horns on S, and the left adjoint F commutes with transfinite compositions and pushouts, $\mathrm{F}j(A)$ is a transfinite composition of pushouts of horns on S. Since F takes every cofibrant approximation to an element of S into a weak equivalence in $\mathcal{N}$, $\mathrm{F}j(A)$ is a transfinite composition of pushouts of horns on weak equivalences in $\mathcal{N}$ (see Lemma 4.6.1 and Proposition 4.2.3), and so $\mathrm{F}j(A)$ is a weak equivalence in $\mathcal{N}$. Similarly, $\mathrm{F}j(B)$ is a weak equivalence in $\mathcal{N}$, and so $\mathrm{F}(g)$ is a weak equivalence in $\mathcal{N}$. $\square$

THEOREM 4.6.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be left proper cellular model categories and let* $\mathrm{F}\colon \mathcal{M} \rightleftarrows \mathcal{N} :\mathrm{U}$ *be a Quillen pair.*

1. *If S is a set of maps in $\mathcal{M}$, then (F,U) is also a Quillen pair when considered as functors $\mathrm{F}\colon \mathrm{L}_S\mathcal{M} \rightleftarrows \mathrm{L}_{\mathrm{F}S}\mathcal{N} :\mathrm{U}$ between the localizations of $\mathcal{M}$ and $\mathcal{N}$.*
2. *If (F,U) is a pair of Quillen equivalences, then (F,U) is also a pair of Quillen equivalences when considered as functors $\mathrm{F}\colon \mathrm{L}_S\mathcal{M} \rightleftarrows \mathrm{L}_{\mathrm{F}S}\mathcal{N} :\mathrm{U}$ between the localizations of $\mathcal{M}$ and $\mathcal{N}$.*

PROOF. Proposition 4.6.2 implies that the composition $\mathcal{M} \xrightarrow{\mathrm{F}} \mathcal{N} \xrightarrow{1_{\mathcal{N}}} \mathrm{L}_{\mathrm{F}S}\mathcal{N}$ is a left Quillen functor when considered as a functor $\mathrm{L}_S\mathcal{M} \to \mathrm{L}_{\mathrm{F}S}\mathcal{N}$, which proves part 1.

For part 2, we must show that if X is cofibrant in $\mathrm{L}_S\mathcal{M}$ and Y is fibrant in $\mathrm{L}_{\mathrm{F}S}\mathcal{N}$, then a map $g\colon X \to \mathrm{U}Y$ in $\mathrm{L}_S\mathcal{M}$ is an S-local equivalence if and only if the corresponding map $g^\sharp : \mathrm{F}X \to Y$ in $\mathrm{L}_{\mathrm{F}S}\mathcal{N}$ is an $\mathrm{F}S$-local equivalence. Given such a map g, we factor it in $\mathcal{M}$ as $X \xrightarrow{h} \widetilde{Y} \xrightarrow{k} \mathrm{U}Y$ where h is a cofibration in $\mathcal{M}$ and k is a trivial fibration in $\mathcal{M}$. Both X and $\widetilde{Y}$ are cofibrant, and since k is a weak equivalence in $\mathcal{M}$, g is an S-local equivalence if and only if h is an S-local equivalence. The corresponding factorization of $g^\sharp$ in $\mathcal{N}$ is $\mathrm{F}X \xrightarrow{\mathrm{F}h} \mathrm{F}\widetilde{Y} \xrightarrow{k^\sharp} Y$, and since (F,U) is a pair of Quillen equivalences between $\mathcal{M}$ and $\mathcal{N}$, the map $k^\sharp$ is a weak equivalence in $\mathcal{N}$. Thus, both $\mathrm{F}X$ and $\mathrm{F}\widetilde{Y}$ are cofibrant, and $g^\sharp$ is an $\mathrm{F}S$-local equivalence if and only if $\mathrm{F}h$ is an $\mathrm{F}S$-local equivalence. It remains only to show that h is an S-local equivalence if and only if $\mathrm{F}h$ is an $\mathrm{F}S$-local equivalence.

The map $\mathrm{F}h$ is an $\mathrm{F}S$-local equivalence if and only if for every $\mathrm{F}S$-local object W in $\mathcal{N}$ and every simplicial resolution $\widehat{\boldsymbol{W}}$ of W, the map of simplicial sets $\mathcal{N}(\mathrm{F}\widetilde{Y}, \widehat{\boldsymbol{W}}) \to \mathcal{N}(\mathrm{F}X, \widehat{\boldsymbol{W}})$ is a weak equivalence. This map is isomorphic to the map $\mathcal{M}(\widetilde{Y}, \mathrm{U}\widehat{\boldsymbol{W}}) \to \mathcal{M}(X, \mathrm{U}\widehat{\boldsymbol{W}})$, and so it is now sufficient to show that every S-local object Z of $\mathcal{M}$ is weakly equivalent to an object of the form $\mathrm{U}W$ for some $\mathrm{F}S$-local object W of $\mathcal{N}$. Since a fibrant object W in $\mathcal{N}$ is $\mathrm{F}S$-local if and only if $\mathrm{U}W$ is S-local, it is sufficient to show that every S-local object Z of $\mathcal{M}$ is weakly equivalent to an object of the form $\mathrm{U}W$ for some fibrant object W in $\mathcal{N}$. Given such an object Z, we can choose a trivial fibration $Z' \to Z$ in $\mathcal{M}$ with Z' cofibrant, and then choose a trivial cofibration in $\mathcal{N}$ $\mathrm{F}Z' \to W$ with W fibrant. Since Z' is cofibrant and W

is fibrant, the corresponding map $Z' \to \mathrm{U}W$ is a weak equivalence, and so we have the diagram of weak equivalences $Z \leftarrow Z' \to \mathrm{U}W$. $\square$

Proof of Proposition 3.3.9. If W is S-local fibrant (see Definition 3.3.7), then the map $W \to *$ (where $*$ is the terminal object of $\mathcal{M}$) has the right lifting property with respect to all maps that are both cofibrations and S-local equivalences. If $f\colon A \to B$ is an element of S and $\tilde{f}\colon \widetilde{\boldsymbol{A}} \to \widetilde{\boldsymbol{B}}$ is a cosimplicial resolution of f (in the original model category structure on $\mathcal{M}$) such that $\tilde{f}$ is a Reedy cofibration, then for every $n \ge 0$ the map $\widetilde{\boldsymbol{A}} \otimes \Delta[n] \amalg_{\widetilde{\boldsymbol{A}} \otimes \partial\Delta[n]} \widetilde{\boldsymbol{B}} \otimes \partial\Delta[n] \to \widetilde{\boldsymbol{B}} \otimes \Delta[n]$ is an S-local trivial cofibration (see Proposition 18.3.5, Proposition 18.3.13, and Proposition 17.3.7), and so Proposition 18.3.8 implies that W is S-local.

Conversely, assume that W is S-local. Proposition 8.2.3 implies that it is sufficient to show that if $i\colon A \to B$ is both a cofibration and an S-local equivalence, then the map $W \to *$ has the right lifting property with respect to i. Proposition 17.1.12 implies that we can choose a cosimplicial resolution $\tilde{\imath}\colon \widetilde{\boldsymbol{A}} \to \widetilde{\boldsymbol{B}}$ of i such that $\tilde{\imath}$ is a Reedy cofibration. Proposition 18.3.5 and Proposition 18.3.9 imply that the map $W \to *$ has the right lifting property with respect to $\tilde{\imath}^0\colon \widetilde{\boldsymbol{A}}^0 \to \widetilde{\boldsymbol{B}}^0$. Since $\mathcal{M}$ is left proper, Proposition 11.1.18 and Proposition 17.1.6 now imply that the map $W \to *$ has the right lifting property with respect to i. $\square$

Proof of Proposition 3.3.10. **Fill this in!!** $\square$

Proposition 4.6.4. *Let $\mathcal{M}$ be a left proper cellular model category, and let S be a set of maps in $\mathcal{M}$. If $g\colon A \to B$ is an S-local equivalence, $h\colon A \to X$ is a map, at least one of g and h is a cofibration, and the square*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "g"'] & X \arrow[d, "k"] \\ B \arrow[r] & Y \end{tikzcd}$$

is a pushout, then k is an S-local equivalence.

Proof. This follows from Proposition 18.3.5 and Proposition 18.5.5. $\square$

Corollary 4.6.5. *If $\mathcal{M}$ is a left proper cellular model category and S is a set of maps in $\mathcal{M}$, then the left Bousfield localization of $\mathcal{M}$ with respect to S (see Definition 3.3.12) is left proper.*

Proof. This follows from Proposition 4.6.4. $\square$

Proof of Theorem 3.3.13. The proof of Theorem 3.3.8 constructed the localization as a cofibrantly generated model category with the same set of generating cofibrations as in $\mathcal{M}$ and a set of inclusions of cell complexes as generating trivial cofibrations, and so most of the conditions are clear. Finally, Corollary 4.6.5 implies that the localization is left proper. $\square$

CHAPTER 5

Right Bousfield localization

Warning: This chapter is in the midst of serious revision.

The main purpose of this chapter is to prove Theorem 3.4.9. This is done in Section 5.5. We begin by discussing S-colocalizations of objects and maps in $\mathcal{M}$.

5.1. Colocalizing objects and maps

DEFINITION 5.1.1. Let $\mathcal{M}$ be a right proper cellular model category, and let S be a set of objects in $\mathcal{M}$.

1. A *S-colocalization* of an object X is an S-colocal object $\widehat{X}$ together with an S-colocal equivalence $j\colon \widehat{X} \to X$. We will sometimes use the phrase *S-colocalization* to refer to the object $\widehat{X}$, without explicitly mentioning the S-colocal equivalence j. A *fibrant S-colocalization* of X is an S-colocalization in which the S-colocal equivalence is also a fibration.
2. A *S-colocalization* of a map $g\colon X \to Y$ is a S-colocalization $(\widetilde{X}, j_X)$ of X, an S-colocalization $(\widetilde{Y}, j_Y)$ of Y, and a map $\tilde{g}\colon \widetilde{X} \to \widetilde{Y}$ such that the square

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\tilde{g}"] \arrow[d, "j_X"'] & \widetilde{Y} \arrow[d, "j_Y"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

commutes. We will sometimes use the term *S-colocalization* to refer to the map $\tilde{g}$, without explicitly mentioning the S-colocalizations $(\widetilde{X}, j_X)$ of X and $(\widetilde{Y}, j_Y)$ of Y.

We construct S-colocalizations in Definition 5.5.3.

REMARK 5.1.2. Earlier work on colocalization was exclusively in a category of pointed spaces ([**19, 20, 23**]), and was called *cellularization*. Given a pointed space A, an A-cellular equivalence of pointed spaces was defined to be a map $g\colon X \to Y$ for which the induced map $g_*\colon \mathrm{Map}(A, X) \to \mathrm{Map}(A, Y)$ is a weak equivalence, and the class of A-cellular spaces was defined to be the smallest class of cofibrant spaces containing A and closed under homotopy colimits and weak equivalences. Since this earlier work considered only the subcategory of fibrant objects (or worked entirely in the category of topological spaces, in which every object is fibrant), this earlier definition of an A-cellular equivalence coincides with our definition of an A-colocal equivalence (see Example 17.2.3). We will show in Theorem 6.6.4 that this earlier definition of an A-cellular space also coincides with our definition of an A-colocal space.

5.2. S-colocal equivalences

PROPOSITION 5.2.1. *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of objects in $\mathcal{M}$, then the class of S-colocal equivalences is closed under retracts, i.e., if the map g is a retract of an S-colocal equivalence h, then g is an S-colocal equivalence.*

PROOF. If g is a retract of h, then we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "g"'] & W \arrow[r, "p_X"] \arrow[d, "h"] & X \arrow[d, "g"] \\ Y \arrow[r, "i_Y"'] & Z \arrow[r, "p_Y"'] & Y \end{tikzcd} \tag{5.2.2}$$

in which $p_X i_X = 1_X$ and $p_Y i_Y = 1_Y$. If we apply a functorial fibrant approximation to this diagram (see Proposition 9.1.2), we obtain the diagram

$$\begin{tikzcd} \widehat{X} \arrow[r, "\hat{\imath}_X"] \arrow[d, "\hat{g}"'] & \widehat{W} \arrow[r, "\hat{p}_X"] \arrow[d, "\hat{h}"] & \widehat{X} \arrow[d, "\hat{g}"] \\ \widehat{Y} \arrow[r, "\hat{\imath}_Y"'] & \widehat{Z} \arrow[r, "\hat{p}_Y"'] & \widehat{Y} \end{tikzcd}$$

in which $\hat{p}_X \hat{\imath}_X = 1_{\widehat{X}}$ and $\hat{p}_Y \hat{\imath}_Y = 1_{\widehat{Y}}$, and the objects and maps are fibrant approximations to those in Diagram 5.2.2. If $\widetilde{\boldsymbol{A}}$ is a cosimplicial resolution of an element of S, then the map $\mathcal{M}(\widetilde{\boldsymbol{A}}, \widehat{X}) \to \mathcal{M}(\widetilde{\boldsymbol{A}}, \widehat{Y})$ is a retract of the map $\mathcal{M}(\widetilde{\boldsymbol{A}}, \widehat{W}) \to \mathcal{M}(\widetilde{\boldsymbol{A}}, \widehat{Z})$, and so the proposition follows. □

PROPOSITION 5.2.3. *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of objects in $\mathcal{M}$, then the class of S-colocal equivalences satisfies the "two out of three" axiom, i.e., if g and h are composable maps, and if two of g, h, and hg are S-colocal equivalences, then so is the third.*

PROOF. Given maps $g\colon X \to Y$ and $h\colon Y \to Z$, we can apply a functorial fibrant approximation (see Proposition 9.1.2) to obtain the diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & Y \arrow[r, "h"] \arrow[d] & Z \arrow[d] \\ \widehat{X} \arrow[r, "\hat{g}"'] & \widehat{Y} \arrow[r, "\hat{h}"'] & \widehat{Z} \end{tikzcd}$$

in which $\hat{g}$, $\hat{h}$, and $\hat{h}\hat{g}$ are fibrant approximations to g, h, and hg, respectively. If $\widetilde{\boldsymbol{A}}$ is a cosimplicial resolution of an element of S, then two of the maps

$$\hat{g}_*\colon \mathcal{M}(\widetilde{\boldsymbol{A}}, \widehat{X}) \to \mathcal{M}(\widetilde{\boldsymbol{A}}, \widehat{Y})$$
$$\hat{h}_*\colon \mathcal{M}(\widetilde{\boldsymbol{A}}, \widehat{Y}) \to \mathcal{M}(\widetilde{\boldsymbol{A}}, \widehat{Z})$$
$$(\hat{h}\hat{g})_*\colon \mathcal{M}(\widetilde{\boldsymbol{A}}, \widehat{X}) \to \mathcal{M}(\widetilde{\boldsymbol{A}}, \widehat{Z})$$

are weak equivalences, and so the third is as well. □

5.2.4. S-colocal Whitehead theorems.

THEOREM 5.2.5 (S-colocal weak Whitehead theorem). *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of objects in $\mathcal{M}$ and $g\colon X \to Y$ is an S-colocal equivalence between S-colocal objects, then g is a weak equivalence.*

PROOF. This follows from Proposition 18.1.5. □

THEOREM 5.2.6 (S-colocal strong Whitehead theorem). *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of objects in $\mathcal{M}$ and $g\colon X \to Y$ is an S-colocal equivalence between fibrant S-colocal objects, then g is a simplicial homotopy equivalence.*

PROOF. This follows from Theorem 5.2.5 and Proposition 8.3.26. □

5.2.7. Characterizing S-colocal objects and S-colocal equivalences.

PROPOSITION 5.2.8. *Let $\mathcal{M}$ be a right proper cellular model category, and let S be a set of objects in $\mathcal{M}$. If C and D are cofibrant objects and $h\colon C \to D$ is a weak equivalence, then C is S-colocal if and only if D is S-colocal.*

PROOF. If $g\colon X \to Y$ is an S-colocal equivalence, then we have the commutative diagram

$$\begin{array}{ccc} \operatorname{map}(D,X) & \longrightarrow & \operatorname{map}(D,Y) \\ {\scriptstyle\approx}\downarrow & & \downarrow{\scriptstyle\approx} \\ \operatorname{map}(C,X) & \longrightarrow & \operatorname{map}(C,Y) \end{array}$$

in which the vertical maps are weak equivalences (see Theorem 17.5.2). Thus, the top map is a weak equivalence if and only if the bottom map is a weak equivalence. □

THEOREM 5.2.9. *Let $\mathcal{M}$ be a right proper cellular model category, and let S be a set of objects in $\mathcal{M}$. If X is cofibrant and $j\colon \widetilde{X} \to X$ is an S-colocalization of X (see Definition 5.1.1), then j is a weak equivalence if and only if X is S-colocal.*

PROOF. If X is S-colocal, then Theorem 5.2.5 implies that j is a weak equivalence. Conversely, if j is a weak equivalence, then Proposition 5.2.8 implies that X is S-colocal. □

THEOREM 5.2.10. *Let $\mathcal{M}$ be a right proper cellular model category, and let S be a set of objects in $\mathcal{M}$. If $\tilde{g}\colon \widetilde{X} \to \widetilde{Y}$ is an S-colocalization of $g\colon X \to Y$ (see Definition 5.1.1), then g is an S-colocal equivalence if and only if $\tilde{g}$ is a weak equivalence.*

PROOF. Proposition 5.2.3 implies that g is an S-colocal equivalence if and only if $\tilde{g}$ is an S-colocal equivalence. Proposition 3.4.3 and Theorem 5.2.5 imply that $\tilde{g}$ is an S-colocal equivalence if and only if it is a weak equivalence. □

5.3. Regular $\overline{\Lambda(S)}$-cofibrations and $\overline{\Lambda(S)}$-injectives

DEFINITION 5.3.1. Let $\mathcal{M}$ be a right proper cellular model category with generating cofibrations I and generating trivial cofibrations J, and let S be a set of objects in $\mathcal{M}$.

- A *full set of horns on S* is a set $\Lambda(S)$ of maps obtained by choosing a cosimplicial resolution $\widetilde{\boldsymbol{A}}$ of every element A of S and letting
$$\Lambda(S) = \{\widetilde{\boldsymbol{A}} \otimes \partial\Delta[n] \to \widetilde{\boldsymbol{A}} \otimes \Delta[n] \mid A \in S, n \geq 0\}.$$
(This is exactly a full set of horns on the maps from the initial object of $\mathcal{M}$ to the elements of S; see Definition 4.2.4.) If S consists of the single object A, then $\Lambda(S)$ is the set of maps
$$\Lambda\{A\} = \{\widetilde{\boldsymbol{A}} \otimes \partial\Delta[n] \to \widetilde{\boldsymbol{A}} \otimes \Delta[n] \mid n \geq 0\},$$
and it will also be called a *full set of horns on A*.
- A *full set of augmented S-horns* is a set $\overline{\Lambda(S)}$ of maps
$$\overline{\Lambda(S)} = \Lambda(S) \cup J$$
for some full set of horns $\Lambda(S)$ on S. If S consists of the single object A, then $\overline{\Lambda(S)}$ will also be denoted $\overline{\Lambda\{A\}}$, and will be called *a full set of augmented A-horns*.

Definition 5.3.2. Let $\mathcal{M}$ be a right proper cellular model category, and let S be a set of objects in $\mathcal{M}$.

- A $\overline{\Lambda(S)}$*-injective* is a map with the right lifting property with respect to every element of $\overline{\Lambda(S)}$.
- A $\overline{\Lambda(S)}$*-cofibration* is a map with the left lifting property with respect to every $\overline{\Lambda(S)}$-injective.
- A *relative* $\overline{\Lambda(S)}$*-cell complex* is a transfinite composition of pushouts of elements of $\overline{\Lambda(S)}$.
- An object of $\mathcal{M}$ is a $\overline{\Lambda(S)}$*-cell complex* if the map to it from the initial object of $\mathcal{M}$ is a relative $\overline{\Lambda(S)}$-cell complex.

Proposition 5.3.3. *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of objects in $\mathcal{M}$, then there is a functorial factorization of every map $X \to Y$ as $X \xrightarrow{p} W \xrightarrow{q} Y$ where p is a relative $\overline{\Lambda(S)}$-cell complex and q is a $\overline{\Lambda(S)}$-injective.*

Proof. This follows from Proposition 15.4.5. □

Proposition 5.3.4. *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of cofibrant objects in $\mathcal{M}$, then a map $g\colon X \to Y$ is a $\overline{\Lambda(S)}$-injective if and only if g is a fibration that induces a weak equivalence of homotopy function complexes $g_*\colon \operatorname{map}(A, X) \cong \operatorname{map}(A, Y)$ for every element A of S.*

Corollary 5.3.5. *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of objects in $\mathcal{M}$ and X and Y are fibrant objects, then a map $g\colon X \to Y$ is an $\overline{\Lambda(S)}$-injective if and only if it is both a fibration and an S-colocal equivalence.*

Proof. Since a fibrant object is a fibrant approximation to itself, this follows from Proposition 5.3.4. □

Proposition 5.3.6. *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of cofibrant objects in $\mathcal{M}$, then a relative $\overline{\Lambda(S)}$-cell complex is an S-colocal cofibration.*

Proof. If $g\colon X \to Y$ is both an S-colocal weak equivalence and a S-colocal fibration, then Proposition 9.1.9 implies that we can choose a fibrant approximation

$\tilde{g}$ to g such that $\tilde{g}$ is a fibration. Proposition 5.3.4 implies that $\tilde{g}$ is a $\overline{\Lambda(S)}$-injective, and so Proposition 12.4.8 implies that $\tilde{g}$ has the right lifting property with respect to all relative $\overline{\Lambda(S)}$-cell complexes. Since $\mathcal{M}$ is a right proper model category, Proposition 11.1.18 implies that g has the right lifting property with respect to all relative $\overline{\Lambda(S)}$-cell complexes. □

EXAMPLE 5.3.7. We present here an example of an $\overline{\Lambda(S)}$-injective that is not a S-colocal equivalence. Let $\mathcal{M} = \mathsf{SS}_*$ (the category of pointed simplicial sets), and let $S = \{A\}$, where A is the quotient of $\Delta[1]$ obtained by identifying the two vertices of $\Delta[1]$ (so that the geometric realization of A is homeomorphic to a circle). Let Y be $\partial\Delta[2]$, i.e., let Y consist of three 1-simplices with vertices identified so that its geometric realization is homeomorphic to a circle. Let X be the simplicial set built from six 1-simplices by identifying vertices so that the geometric realization of X is homeomorphic to a circle and there is a map $g\colon X \to Y$ whose geometric realization is the double cover of the circle. The map g is a fibration, since it is a fiber bundle with fiber two discrete points (see [**6**, Section IV.2] or [**43**, Lemma 11.9]).

Since no nondegenerate 1-simplex of X has its vertices equal, the only pointed map from A to X is the constant map to the basepoint. One can now show by induction on n that the only pointed map from $A \wedge \Delta[n]^+$ to X is the constant map to the basepoint. Thus, $\operatorname{Map}(A, X)$ has only one simplex in each dimension. Similarly, $\operatorname{Map}(A, Y)$ has only one simplex in each dimension, and so the map $g_*\colon \operatorname{Map}(A, X) \to \operatorname{Map}(A, Y)$ is an isomorphism. Thus, g is an A-injective (see Corollary 5.3.5).

To see that g is not an A-colocal equivalence, we note that $\operatorname{Sing}|g|\colon \operatorname{Sing}|X| \to \operatorname{Sing}|Y|$ is a fibrant approximation to g, and the map $\operatorname{Map}(A, \operatorname{Sing}|X|) \to \operatorname{Map}(A, \operatorname{Sing}|Y|)$ is isomorphic to the map $\operatorname{Map}(|A|, |X|) \to \operatorname{Map}(|A|, |Y|)$ (see Lemma 1.1.13). Since the map $|g|\colon |X| \to |Y|$ is homeomorphic to the double covering map of the circle, the induced map $\operatorname{Map}(|A|, |X|) \to \operatorname{Map}(|A|, |Y|)$ is not surjective on the set of components, and so g is not an A-colocal equivalence.

REMARK 5.3.8. Example 5.3.7 shows that, if $\mathcal{M} = \mathsf{SS}_*$, then not every $\overline{\Lambda(S)}$-injective need be an S-colocal weak equivalence. Since the $\overline{\Lambda(S)}$-cofibrations are exactly the maps with the left lifting property with respect to all $\overline{\Lambda(S)}$-injectives, this implies that the S-colocal cofibrations must consist of more than just the $\overline{\Lambda(S)}$-cofibrations (see Proposition 5.3.6).

5.4. S-colocal cofibrations

The main results of this section are Proposition 5.4.2 and Proposition 5.4.4, which together provide the factorizations needed for the proof of Theorem 3.4.9.

LEMMA 5.4.1. *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of objects in $\mathcal{M}$, then every S-colocal cofibration is a cofibration.*

PROOF. This follows from Proposition 8.2.3 and Proposition 3.4.3. □

PROPOSITION 5.4.2. *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of objects in $\mathcal{M}$, then a map $g\colon X \to Y$ is both a S-colocal cofibration and an S-colocal weak equivalence if and only if it is a trivial cofibration.*

Proof. If g is a trivial cofibration, then Proposition 3.4.3 implies that it is an S-colocal weak equivalence and Proposition 8.2.3 implies that it is an S-colocal cofibration.

Conversely, let $g\colon X \to Y$ be both an S-colocal cofibration and an S-colocal weak equivalence. Lemma 5.4.1 implies that we need only show that g is a weak equivalence. Proposition 9.1.9 implies that there is a solid arrow diagram

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "g"'] & \widetilde{X} \arrow[d, "\tilde{g}"] \\ Y \arrow[r, "i_Y"'] \arrow[ur, dotted, "h"] & \widetilde{Y} \end{tikzcd}$$

in which $\widetilde{X}$ and $\widetilde{Y}$ are fibrant, i_X and i_Y are trivial cofibrations, and $\tilde{g}$ is a fibration. We will show that $\tilde{g}$ is a simplicial homotopy equivalence. Since i_X and i_Y are trivial cofibrations, this will imply that g is a weak equivalence, and the proof will be complete.

Since g is an S-colocal weak equivalence and $\tilde{g}$ is a fibrant approximation to g, the map $\tilde{g}$ is also a S-colocal weak equivalence. Thus, g has the left lifting property with respect to $\tilde{g}$, and so there exists a map $h\colon Y \to \widetilde{X}$ such that $hg = i_X$ and $\tilde{g}h = i_Y$. Since i_Y is a trivial cofibration and $\widetilde{X}$ is fibrant, there exists a map $\hat{g}\colon \widetilde{Y} \to \widetilde{X}$ such that $\hat{g}i_Y = h$. We will show that $\hat{g}$ is a simplicial homotopy inverse to $\tilde{g}$.

We have $\tilde{g}\hat{g}i_Y = \tilde{g}h = i_Y$. Since i_Y is a trivial cofibration and $\widetilde{Y}$ is fibrant, Corollary 10.4.10 implies that $\tilde{g}\hat{g} \overset{s}{\simeq} 1_{\widetilde{Y}}$. We also have $\hat{g}\tilde{g}i_X = \hat{g}i_Y g = hg = i_X$. Since i_X is a trivial cofibration and $\widetilde{X}$ is fibrant, Corollary 10.4.10 implies that $\hat{g}\tilde{g} \overset{s}{\simeq} 1_{\widetilde{X}}$. Thus, $\hat{g}$ is a simplicial homotopy inverse to $\tilde{g}$. ☐

Lemma 5.4.3. *Let $\mathcal{M}$ be a right proper cellular model category, and let S be a set of objects in $\mathcal{M}$. If $g\colon A \to B$ is a cofibration, $h\colon B \to C$ is a weak equivalence, and the composition $hg\colon A \to C$ is an S-colocal cofibration, then g is an S-colocal cofibration.*

Proof. If $f\colon X \to Y$ is both an S-colocal weak equivalence and a S-colocal fibration, then Proposition 9.1.9 implies that we can choose a fibrant approximation $\tilde{f}\colon \widetilde{X} \to \widetilde{Y}$ to f such that $\tilde{f}$ is a fibration. Since $\mathcal{M}$ is a right proper model category, Proposition 11.1.18 implies that it is sufficient to show that g has the left lifting property with respect to $\tilde{f}$. Proposition 3.4.3 and Proposition 5.2.3 imply that $\tilde{f}$ is an S-colocal weak equivalence.

Suppose that we have the commutative solid arrow diagram

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "g"'] & \widetilde{X} \arrow[d, "\tilde{f}"] \\ B \arrow[r, "t"] \arrow[d, "h"'] & \widetilde{Y} \\ C \arrow[uur, dotted, "k"] \arrow[ur, dotted, "j"'] & \end{tikzcd}$$

In the category $(A\downarrow\mathcal{M})$ of object of $\mathcal{M}$ under A, h is a weak equivalence of cofibrant objects (see Lemma 5.4.1) and $\widetilde{Y}$ is fibrant. Thus, Corollary 8.5.4 implies that there is a map $j\colon C \to \widetilde{Y}$ in $(A\downarrow\mathcal{M})$ such that $jh \simeq t$ in $(A\downarrow\mathcal{M})$. Since hg is an S-colocal cofibration and $\tilde{f}$ is both an S-colocal weak equivalence and an S-colocal fibration, there exists a map $k\colon C \to \widetilde{X}$ such that $khg = s$ and $\tilde{f}k = j$.

Since $\tilde{f}kh = jh \simeq t$ in $(A\downarrow\mathcal{M})$, if we let $u = kh$, then $u\colon B \to \widetilde{X}$, and $\tilde{f}u \simeq t$ in $(A\downarrow\mathcal{M})$. Since B is cofibrant in $(A\downarrow\mathcal{M})$ and $\tilde{f}$ is a fibration, the homotopy lifting property of fibrations (see Proposition 8.3.8) implies that there is a map $v\colon B \to \widetilde{X}$ in $(A\downarrow\mathcal{M})$ such that $v \simeq u$ and $\tilde{f}v = t$. The map v satisfies $vg = s$ and $\tilde{f}v = t$, and so g has the left lifting property with respect to $\tilde{f}$. $\square$

PROPOSITION 5.4.4. *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of objects in $\mathcal{M}$, then there is a functorial factorization of every map $g\colon X \to Y$ in $\mathcal{M}$ as $X \xrightarrow{p} W \xrightarrow{q} Y$ in which p is an S-colocal cofibration and q is both an S-colocal weak equivalence and an S-colocal fibration.*

PROOF. Choose a functorial cofibrant fibrant approximation $j\colon Y \to \widetilde{Y}$ to Y. Proposition 5.3.3 implies that there is a functorial factorization of the composition $jg\colon X \to \widetilde{Y}$ as $X \xrightarrow{r} \widetilde{W} \xrightarrow{s} \widetilde{Y}$, in which r is a relative $\overline{\Lambda(S)}$-cell complex and s is a $\overline{\Lambda(S)}$-injective. If we let Z be the pullback $Y \times_{\widetilde{Y}} \widetilde{W}$, then we can factor the natural map $X \to Z$ in $\mathcal{M}$ as $X \xrightarrow{p} W \xrightarrow{u} Z$ where p is a cofibration and u is a trivial fibration. If we let $q = vu$, then we have the diagram

$$\begin{tikzcd}
X \arrow[dr, "p"] \arrow[rrr, bend left, "r"] \arrow[ddrr, bend right, "g"'] & & & \\
& W \arrow[r, "u"] \arrow[dr, "q"] & Z \arrow[r, "t"] \arrow[d, "v"] & \widetilde{W} \arrow[d, "s"] \\
& & Y \arrow[r, "j"'] & \widetilde{Y}
\end{tikzcd}$$

Since j is a weak equivalence, s is a fibration, and $\mathcal{M}$ is a right proper model category, t is a weak equivalence. Thus, the composition tu is a weak equivalence, and so s is a fibrant approximation to q. Since Corollary 5.3.5 implies that s is an S-colocal equivalence, q (which is the composition of two fibrations) is both an S-colocal weak equivalence and a S-colocal fibration. Since r is an S-colocal cofibration (see Proposition 5.3.6), Lemma 5.4.3 implies that p is an S-colocal cofibration. $\square$

PROPOSITION 5.4.5. *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of objects in $\mathcal{M}$, then every S-colocal cofibration has the homotopy left lifting property with respect to every map that is both an S-colocal weak equivalence and an S-colocal fibration.*

PROOF. If $g\colon X \to Y$ is both an S-colocal weak equivalence and a S-colocal fibration, Proposition 9.1.9 implies that we can choose a fibrant approximation $\tilde{g}\colon \widetilde{X} \to \widetilde{Y}$ to g such that $\tilde{g}$ is a fibration. Since $\mathcal{M}$ is a right proper model category, Corollary 11.1.19 implies that it is sufficient to show that every S-colocal cofibration has the homotopy left lifting property with respect to $\tilde{g}$. Thus, Lemma 10.3.6 implies that it is sufficient to show that every S-colocal cofibration has the left lifting property with respect to the map $\widetilde{X}^{\Delta[n]} \to \widetilde{Y}^{\Delta[n]} \times_{\widetilde{Y}^{\partial\Delta[n]}} \widetilde{X}^{\partial\Delta[n]}$ for every $n \geq 0$.

Proposition 5.3.4 implies that $\tilde{g}$ is a $\overline{\Lambda(S)}$-injective, and so Proposition 10.3.10 implies that $\widetilde{X}^{\Delta[n]} \to \widetilde{Y}^{\Delta[n]} \times_{\widetilde{Y}^{\partial\Delta[n]}} \widetilde{X}^{\partial\Delta[n]}$ is also a $\overline{\Lambda(S)}$-injective for every $n \geq 0$. Corollary 5.3.5 implies that each of these maps is both an S-colocal weak equivalence and an S-colocal fibration, and so the proof is complete. □

5.5. The colocalization model category

This section contains the proof of Theorem 3.4.9.

Proof of Theorem 3.4.9. We must show that axioms M1 through M5 of Definition 8.1.2 and axioms M6 and M7 of Definition 10.1.2 are satisfied.

Axiom M1 is clear, axiom M2 follows from Proposition 5.2.3, and axiom M3 follows from Proposition 5.2.1. Axiom M4 part (1) follows from the definition of S-colocal cofibration, and axiom M4 part (2) follows from Proposition 5.4.2. Axiom M5 part (1) follows from Proposition 5.4.4, and axiom M5 part (2) follows from Proposition 5.4.2.

Axiom M6 follows because the simplicial structure is the given one on $\mathcal{M}$. Axiom M7 follows from Lemma 5.4.1, Proposition 5.4.2, and Proposition 5.4.5. □

Proposition 5.5.1. *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of objects in $\mathcal{M}$, then a map is an S-colocal cofibration if and only if it is a retract of a cofibration $X \to Y$ for which there is a weak equivalence $Y \to Z$ such that the composition $X \to Z$ is a relative $\overline{\Lambda(S)}$-cell complex.*

Proof. This follows from the factorization constructed in the proof of Proposition 5.4.4 and the retract argument (see Proposition 8.2.2). □

Corollary 5.5.2. *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of objects in $\mathcal{M}$, then an object is S-colocal if and only if it is a retract of an object that is both cofibrant and weakly equivalent to a $\overline{\Lambda(S)}$-complex.*

Proof. This follows from Proposition 3.4.10 and Proposition 5.5.1. □

Definition 5.5.3. Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of objects in $\mathcal{M}$, then we can choose a functorial fibrant cofibrant approximation (CW_S, p) for the S-colocal model category structure on $\mathcal{M}$ (see Proposition 9.1.2). We define the *S-colocalization* of an object X to be the object $\mathrm{CW}_S\, X$ together with the S-colocal trivial fibration $p(X)\colon \mathrm{CW}_S\, X \to X$.

Proposition 5.5.4. *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of objects in $\mathcal{M}$, then $\mathrm{CW}_S\, X$ is weakly equivalent to a $\overline{\Lambda(S)}$-complex for every object X.*

Proof. This follows from the definition of $\mathrm{CW}_S\, X$ (see Definition 5.5.3) and the factorization constructed in Proposition 5.4.4. □

Proposition 5.5.5. *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of objects in $\mathcal{M}$, then the S-colocal model category structure on $\mathcal{M}$ is right proper.*

Proof. This follows from Proposition 3.4.13. □

PROPOSITION 5.5.6. *Let $\mathcal{M}$ be a right proper cellular model category in which every object is fibrant. If S is a set of objects in $\mathcal{M}$, then*

1. *every S-colocal cofibration is a $\overline{\Lambda(S)}$-cofibration, and*
2. *every S-colocal cofibrant object is a retract of a $\overline{\Lambda(S)}$-complex.*

5.6. Topological spaces and simplicial sets

Warning: This section is a collection of leftovers.

5.6.1. Topological spaces and simplicial sets.

PROPOSITION 5.6.2. *Let A be a cofibrant pointed topological space.*

1. *A map in Top_* is an A-colocal equivalence if and only if it is a $|\mathrm{Sing}\, A|$-colocal equivalence.*
2. *A cofibrant pointed topological space is A-colocal if and only if it is $|\mathrm{Sing}\, A|$-colocal.*

PROOF. This follows from Proposition 6.4.3 and Proposition 6.4.1. □

PROPOSITION 5.6.3. *Let A be a pointed simplicial set.*

1. *A map in SS_* is an A-colocal equivalence if and only if it is a $\mathrm{Sing}|A|$-colocal equivalence.*
2. *A pointed simplicial set is A-colocal if and only if it is $\mathrm{Sing}|A|$-colocal.*

PROOF. Since every simplicial set is cofibrant, this follows from Proposition 6.4.3 and Proposition 6.4.1. □

PROPOSITION 5.6.4. *If A is a pointed simplicial set, then a map of pointed topological spaces $g\colon X \to Y$ is a $|A|$-colocal equivalence if and only if $(\mathrm{Sing}\, g)\colon \mathrm{Sing}\, X \to \mathrm{Sing}\, Y$ is an A-colocal equivalence.*

PROOF. Lemma 1.1.13 gives us the commutative square

$$\begin{tikzcd}
\mathrm{Map}(|A|, X) \arrow[r] \arrow[d, "\approx"'] & \mathrm{Map}(|A|, Y) \arrow[d, "\approx"] \\
\mathrm{Map}(A, \mathrm{Sing}\, X) \arrow[r] & \mathrm{Map}(A, \mathrm{Sing}\, Y)
\end{tikzcd}$$

in which the vertical maps are isomorphisms. Since all topological spaces are fibrant and the total singular complex of a topological space is fibrant, the proposition follows. □

PROPOSITION 5.6.5. *If A is a pointed simplicial set, then a map $g\colon X \to Y$ of simplicial sets is an A-colocal equivalence if and only if $|g|\colon |X| \to |Y|$ is a $|A|$-colocal equivalence.*

PROOF. The map $(\mathrm{Sing}|g|)\colon \mathrm{Sing}|X| \to \mathrm{Sing}|Y|$ is a fibrant approximation to g, and so g is an A-colocal equivalence if and only if $(\mathrm{Sing}|g|)_*\colon \mathrm{Map}(A, \mathrm{Sing}|X|) \to \mathrm{Map}(A, \mathrm{Sing}|Y|)$ is a weak equivalence. Lemma 1.1.13 implies that this is true if and only if $|g|_*\colon \mathrm{Map}(|A|, |X|) \to \mathrm{Map}(|A|, |Y|)$ is a weak equivalence. Since all topological spaces are fibrant, this is true if and only if $|g|\colon |X| \to |Y|$ is a $|A|$-colocal equivalence. □

PROPOSITION 5.6.6. *If A is a cofibrant pointed topological space, then a pointed simplicial set W is A-colocal if and only if $|W|$ is $|A|$-colocal.*

PROOF. If W is A-colocal, let $g\colon X \to Y$ be a $|A|$-colocal equivalence of topological spaces. Proposition 5.6.4 implies that $(\operatorname{Sing} g)_*\colon \operatorname{Map}(W, \operatorname{Sing} X) \to \operatorname{Map}(W, \operatorname{Sing} Y)$ is a weak equivalence, and so Lemma 1.1.13 implies that $g_*\colon \operatorname{Map}(|W|, X) \to \operatorname{Map}(|W|, Y)$ is a weak equivalence. Since all topological spaces are fibrant, this implies that $|W|$ is $|A|$-colocal.

Conversely, assume that $|W|$ is $|A|$-colocal, and let $g\colon X \to Y$ be an A-colocal equivalence of simplicial sets. Proposition 5.6.5 implies that $|g|\colon |X| \to |Y|$ is a $|A|$-colocal equivalence, and so $|g|_*\colon \operatorname{Map}(|W|, |X|) \to \operatorname{Map}(|W|, |Y|)$ is a weak equivalence. Lemma 1.1.13 now implies that $(\operatorname{Sing}|g|)_*\colon \operatorname{Map}(W, \operatorname{Sing}|X|) \to \operatorname{Map}(W, \operatorname{Sing}|Y|)$ is a weak equivalence. Since $\operatorname{Sing}|g|$ is a fibrant approximation to g, this implies that W is A-colocal. $\square$

PROPOSITION 5.6.7. *If A is a cofibrant pointed topological space, then a cofibrant pointed topological space W is A-colocal if and only if $\operatorname{Sing} W$ is $(\operatorname{Sing} A)$-colocal.*

PROOF. If W is A-colocal, let $g\colon X \to Y$ be a $(\operatorname{Sing} A)$-colocal equivalence of simplicial sets. Since $\operatorname{Sing}|g|$ is a fibrant approximation to g, we must show that $(\operatorname{Sing}|g|)_*\colon \operatorname{Map}(\operatorname{Sing} W, \operatorname{Sing}|X|) \to \operatorname{Map}(\operatorname{Sing} W, \operatorname{Sing}|Y|)$ is a weak equivalence. This is true if and only if $|g|_*\colon \operatorname{Map}(W, |X|) \to \operatorname{Map}(W, |Y|)$ is a weak equivalence. Proposition 5.6.5 and Proposition 5.6.2 imply that $|g|\colon |X| \to |Y|$ is an A-colocal equivalence, and so $\operatorname{Sing} W$ is $(\operatorname{Sing} A)$-colocal.

Conversely, assume that $\operatorname{Sing} W$ is $(\operatorname{Sing} A)$-colocal, and let $g\colon X \to Y$ be an A-colocal equivalence of topological spaces; we must show that $g_*\colon \operatorname{Map}(W, X) \to \operatorname{Map}(W, Y)$ is a weak equivalence. This is true if and only if $(\operatorname{Sing} g)_*\colon \operatorname{Map}(\operatorname{Sing} W, \operatorname{Sing} X) \to \operatorname{Map}(\operatorname{Sing} W, \operatorname{Sing} Y)$ is a weak equivalence (see Proposition 1.1.15). Proposition 5.6.4 and Proposition 5.6.2 imply that $\operatorname{Sing} g$ is a $(\operatorname{Sing} A)$-colocal equivalence, and so the proof is complete. $\square$

5.6.8. Topological spaces and simplicial sets.

COROLLARY 5.6.9. *If A is a pointed simplicial set then, for every topological space X, there is a natural map $|\mathrm{CW}_A \operatorname{Sing} X| \to \mathrm{CW}_{|A|} X$ that makes the square*

$$\begin{tikzcd}
|\mathrm{CW}_A \operatorname{Sing} X| \arrow[r] \arrow[d] & \mathrm{CW}_{|A|} X \arrow[d] \\
|\operatorname{Sing} X| \arrow[r] & X
\end{tikzcd}$$

commute, and this natural map is a simplicial homotopy equivalence.

PROOF. The existence of the natural map follows from Theorem 1.5.4. Proposition 5.6.6 implies that $|\mathrm{CW}_A \operatorname{Sing} X|$ is $|A|$-colocal, and Proposition 5.2.3, Proposition 3.4.3, and Proposition 5.6.5 imply that our natural map is a $|A|$-colocal equivalence of fibrant $|A|$-colocal spaces. The result now follows from Theorem 5.2.6. $\square$

COROLLARY 5.6.10. *If A is a pointed simplicial set then for every simplicial set K there is a natural map $\mathrm{CW}_A K \to \mathrm{Sing}\,\mathrm{CW}_{|A|}|K|$ that makes the square*

$$\begin{array}{ccc} \mathrm{CW}_A X & \longrightarrow & \mathrm{Sing}\,\mathrm{CW}_{|A|}|K| \\ \big\downarrow & & \big\downarrow \\ K & \longrightarrow & \mathrm{Sing}|K| \end{array}$$

commute, and this natural map is a weak equivalence.

PROOF. The existence of the natural map follows from Theorem 1.5.6. Proposition 5.6.7 and Proposition 5.6.2 imply that $\mathrm{Sing}\,\mathrm{CW}_{|A|}|K|$ is A-colocal, and Proposition 5.2.3, Proposition 3.4.3, and Proposition 5.6.4 imply that our natural map is an A-colocal equivalence of A-colocal spaces. The result now follows from Theorem 5.2.5. □

5.6.11. Topological spaces and simplicial sets.

PROPOSITION 5.6.12. *If A is a pointed simplicial set, then the map $g\colon X \to Y$ of pointed topological spaces is a $|A|$-colocal equivalence if and only if the map $\mathrm{Sing}\,g\colon \mathrm{Sing}\,X \to \mathrm{Sing}\,Y$ of pointed simplicial sets is an A-colocal equivalence.*

PROOF. Since every topological space is fibrant, g is a $|A|$-colocal equivalence if and only if $g_*\colon \mathrm{Map}(|A|, X) \to \mathrm{Map}(|A|, Y)$ is a weak equivalence of simplicial sets. Lemma 1.1.13 implies that this is true if and only if the map of simplicial sets $(\mathrm{Sing}\,g)_*\colon \mathrm{Map}(A, \mathrm{Sing}\,X) \to \mathrm{Map}(A, \mathrm{Sing}\,Y)$ is a weak equivalence. Since $\mathrm{Sing}\,X$ and $\mathrm{Sing}\,Y$ are fibrant, this is true if and only if $(\mathrm{Sing}\,g)$ is an A-colocal equivalence. □

PROPOSITION 5.6.13. *If A is a pointed simplicial set, then the map $g\colon C \to D$ of pointed simplicial sets is an A-colocal equivalence if and only if the map $|g|\colon |C| \to |D|$ of pointed topological spaces is a $|A|$-colocal equivalence.*

PROOF. The map $\mathrm{Sing}|g|\colon \mathrm{Sing}|C| \to \mathrm{Sing}|D|$ is a fibrant approximation to g (see Definition 9.1.8), and so g is an A-colocal equivalence if and only if the map of simplicial sets $\big(\mathrm{Sing}|g|\big)_*\colon \mathrm{Map}\big(A, \mathrm{Sing}|C|\big) \to \mathrm{Map}\big(A, \mathrm{Sing}|D|\big)$ is a weak equivalence. Lemma 1.1.13 implies that this is true if and only if the map $|g|_*\colon \mathrm{Map}\big(|A|, |C|\big) \to \mathrm{Map}\big(|A|, |D|\big)$ is a weak equivalence. Since $|C|$ and $|D|$ are fibrant, the result follows. □

PROPOSITION 5.6.14. *If A is a pointed simplicial set, then the simplicial set K is A-colocal if and only if $|K|$ is $|A|$-colocal.*

PROOF. If $|K|$ is $|A|$-colocal and $g\colon C \to D$ is a fibration of fibrant simplicial sets that is an A-colocal equivalence, then $|g|\colon |C| \to |D|$ is a fibration and Proposition 5.6.13 implies that it is a $|A|$-colocal equivalence. Proposition 6.6.1 implies that the map $\mathrm{Map}\big(|K|, |C|\big) \to \mathrm{Map}\big(|K|, |D|\big)$ is a weak equivalence. Lemma 1.1.13 now implies that $\mathrm{Map}\big(K, \mathrm{Sing}|C|\big) \to \mathrm{Map}\big(K, \mathrm{Sing}|D|\big)$ is a weak equivalence. Since $C \to \mathrm{Sing}|C|$ and $D \to \mathrm{Sing}|D|$ are weak equivalences of fibrant simplicial sets and every simplicial set is cofibrant, Corollary 10.2.2 implies that $\mathrm{Map}(K, C) \to \mathrm{Map}(K, D)$ is a weak equivalence, and so Proposition 6.6.1 implies that K is A-colocal.

Conversely, if K is A-colocal and $g\colon X \to Y$ is a fibration of topological spaces that is a $|A|$-colocal equivalence, then $\operatorname{Sing} g\colon \operatorname{Sing} X \to \operatorname{Sing} Y$ is a fibration and Proposition 5.6.12 implies that it is an A-colocal equivalence. Lemma 1.1.13 and Proposition 6.6.1 imply that we have a commutative square

$$\begin{tikzcd}
\operatorname{Map}(|K|, X) \arrow[r] \arrow[d, "\approx"'] & \operatorname{Map}(|K|, Y) \arrow[d, "\approx"] \\
\operatorname{Map}(K, \operatorname{Sing} X) \arrow[r] & \operatorname{Map}(K, \operatorname{Sing} Y)
\end{tikzcd}$$

in which all the maps except the top one are weak equivalences, and so this last one must also be a weak equivalence. Proposition 6.6.1 now implies that $|K|$ is $|A|$-colocal, and the proof is complete. $\square$

Proposition 5.6.15. *If A is a pointed simplicial set, then a cofibrant topological space X is $|A|$-colocal if and only if $\operatorname{Sing} X$ is A-colocal.*

Proof. If X is $|A|$-colocal and $g\colon C \to D$ is a fibration of fibrant simplicial sets that is an A-colocal equivalence, then we have the commutative diagram

$$\begin{tikzcd}
\operatorname{Map}(\operatorname{Sing} X, C) \arrow[r] \arrow[d, "\cong"'] & \operatorname{Map}(\operatorname{Sing} X, D) \arrow[d, "\cong"] \\
\operatorname{Map}(\operatorname{Sing} X, \operatorname{Sing}|C|) \arrow[r] \arrow[d, "\approx"'] & \operatorname{Map}(\operatorname{Sing} X, \operatorname{Sing}|D|) \arrow[d, "\approx"] \\
\operatorname{Map}(|\operatorname{Sing} X|, |C|) \arrow[r] & \operatorname{Map}(|\operatorname{Sing} X|, |D|) \\
\operatorname{Map}(X, |C|) \arrow[r] \arrow[u, "\cong"] & \operatorname{Map}(X, |D|) \arrow[u, "\cong"']
\end{tikzcd}$$

in which all the vertical maps are weak equivalences (see Corollary 10.2.2 and Lemma 1.1.13). Since $|C| \to |D|$ is a fibration and a $|A|$-colocal equivalence (see Proposition 5.6.13), the bottom map is a weak equivalence, and so the top map is also a weak equivalence, and so Proposition 6.6.1 implies that $\operatorname{Sing} X$ is A-colocal.

Conversely, if $\operatorname{Sing} X$ is A-colocal and $g\colon Y \to Z$ is a fibration of topological spaces that is a $|A|$-colocal equivalence, then we have the commutative diagram

$$\begin{tikzcd}
\operatorname{Map}(X, Y) \arrow[r] \arrow[d, "\cong"'] & \operatorname{Map}(X, Z) \arrow[d, "\cong"] \\
\operatorname{Map}(|\operatorname{Sing} X|, Y) \arrow[r] \arrow[d, "\approx"'] & \operatorname{Map}(|\operatorname{Sing} X|, Z) \arrow[d, "\approx"] \\
\operatorname{Map}(\operatorname{Sing} X, \operatorname{Sing} Y) \arrow[r] & \operatorname{Map}(\operatorname{Sing} X, \operatorname{Sing} Z)
\end{tikzcd}$$

in which all the vertical maps are weak equivalences. Since $\operatorname{Sing} Y \to \operatorname{Sing} Z$ is a fibration of fibrant simplicial sets and an A-colocal equivalence (see Proposition 5.6.12), the bottom map is a weak equivalence, and so the top map is a weak equivalence, and so the result follows from Proposition 6.6.1. $\square$

Proposition 5.6.16. *If A is a pointed simplicial set, (M, p) is a functorial fibrant A-colocalization on SS_*, and (N, q) is a functorial fibrant $|A|$-colocalization on Top_*, then, for every topological space X, there is a map $|\mathrm{M}\,\mathrm{Sing}\,X| \to \mathrm{N}X$, unique up to simplicial homotopy, that makes the square*

$$\begin{tikzcd} {|\mathrm{M}\,\mathrm{Sing}\,X|} \arrow[r, dotted] \arrow[d] & \mathrm{N}X \arrow[d] \\ {|\mathrm{Sing}\,X|} \arrow[r] & X \end{tikzcd} \tag{5.6.17}$$

commute, and any such map is a homotopy equivalence. This map is natural up to simplicial homotopy, i.e., if $g\colon X \to Y$ is a map of topological spaces then the square

$$\begin{tikzcd} {|\mathrm{M}\,\mathrm{Sing}\,X|} \arrow[r] \arrow[d] & \mathrm{N}X \arrow[d] \\ {|\mathrm{M}\,\mathrm{Sing}\,Y|} \arrow[r] & \mathrm{N}Y \end{tikzcd}$$

commutes up to simplicial homotopy.

Proof. This is similar to the proof of Proposition 1.5.8. □

Theorem 5.6.18. *If A is a pointed simplicial set, then, for every topological space X, there is a natural homotopy equivalence (i.e., a natural map that is a homotopy equivalence, not just a homotopy class of maps) $|\mathrm{CW}_A\,\mathrm{Sing}\,X| \to \mathrm{CW}_{|A|}\,X$ (see Definition 5.5.3).*

Proof. Corollary 5.6.9 implies that there is a natural map $|\mathrm{CW}_A\,\mathrm{Sing}\,X| \to \mathrm{CW}_{|A|}\,X$ that makes the square (5.6.17) commute, and so the theorem follows from Proposition 5.6.16. □

Proposition 5.6.19. *If A is a pointed simplicial set, (M, p) is a functorial fibrant A-colocalization on SS_*, and (N, q) is a functorial fibrant $|A|$-colocalization on Top_*, then, for every simplicial set K, there is a map $\mathrm{M}K \to \mathrm{Sing}\,\mathrm{N}|K|$, unique up to homotopy, that makes the square*

$$\begin{tikzcd} \mathrm{M}K \arrow[r, dotted] \arrow[d] & \mathrm{Sing}\,\mathrm{N}|K| \arrow[d] \\ K \arrow[r] & \mathrm{Sing}|K| \end{tikzcd} \tag{5.6.20}$$

commute, and any such map is a weak equivalence. This map is natural up to simplicial homotopy, i.e., if $g\colon K \to L$ is a map of simplicial sets then the square

$$\begin{tikzcd} \mathrm{M}K \arrow[r] \arrow[d] & \mathrm{Sing}\,\mathrm{N}|K| \arrow[d] \\ \mathrm{M}L \arrow[r] & \mathrm{Sing}\,\mathrm{N}|L| \end{tikzcd}$$

commutes up to simplicial homotopy.

PROOF. This is similar to the proof of Proposition 5.6.16. □

THEOREM 5.6.21. *If A is a pointed simplicial set then for every simplicial set K there is a natural weak equivalence (i.e., a natural map that is a weak equivalence, not just a homotopy class of maps)* $\mathrm{CW}_A K \to \mathrm{Sing}\,\mathrm{CW}_{|A|}\bigl|K\bigr|$ *(see Definition* 5.5.3*).*

PROOF. Corollary 5.6.10 implies that there is a natural map $\mathrm{CW}_A K \to \mathrm{Sing}\,\mathrm{CW}_{|A|}\bigl|K\bigr|$ that makes the square (5.6.20) commute, and so the theorem follows from Proposition 5.6.19. □

CHAPTER 6

Localization functors and colocalization functors

Warning: This chapter is in need of revision.

6.1. Characterizing localization functors

6.1.1. Functorial localizations. If $\mathcal{M}$ is a left proper cellular model category and S is a set of maps in $\mathcal{M}$, then the definition of the S-local model category structure (see Definition 3.3.7) implies that an S-localization of an object is exactly an S-local fibrant approximation to that object (see Definition 9.1.1), and that a cofibrant S-localization of an object is exactly an S-local cofibrant fibrant approximation to that object.

DEFINITION 6.1.2. If $\mathcal{M}$ is a model category, a *coaugmented functor* on $\mathcal{M}$ is a pair (F, j) where F is a functor $\mathrm{F}\colon \mathcal{M} \to \mathcal{M}$ and j is a natural transformation $j\colon 1 \to \mathrm{F}$.

DEFINITION 6.1.3. If $\mathcal{M}$ is a model category, then a coaugmented functor F on $\mathcal{M}$ will be called *homotopy idempotent* if, for every object X in $\mathcal{M}$, the natural maps $j(\mathrm{F}X), \mathrm{F}\bigl(j(X)\bigr) : \mathrm{F}X \to \mathrm{FF}X$ are homotopic under X (see Definition 8.4.3) and are homotopy equivalences under X.

REMARK 6.1.4. Definition 6.1.3 is the lifting to $\mathcal{M}$ of J. F. Adams' notion of an idempotent functor on the homotopy category of $\mathcal{M}$ (see [**1**]).

DEFINITION 6.1.5. If $\mathcal{M}$ is a model category and S is a set of maps in $\mathcal{M}$, then a *functorial S-localization* is a coaugmented functor (F, j) on $\mathcal{M}$ such that, for every object X, the coaugmentation $j(X)\colon X \to \mathrm{F}X$ is an S-localization of X (see Definition 4.1.1). A *functorial cofibrant S-localization* is a functorial S-localization for which the coaugmentation $j(X)$ is a cofibrant S-localization for every object X.

PROPOSITION 6.1.6. *If $\mathcal{M}$ is a model category and S is a set of maps in $\mathcal{M}$, then a functorial cofibrant S-localization (M_S, j) is homotopy idempotent, i.e., for every object X, the maps $j(\mathrm{M}_S X)$ and $\mathrm{M}_S\bigl(j(X)\bigr)$ are homotopic under X (see Definition 8.4.3), and are homotopy equivalences $\mathrm{M}_S X \cong \mathrm{M}_S\mathrm{M}_S X$ under X.*

PROOF. Since j is a natural transformation, for every object X we have a commutative square

$$\begin{tikzcd}
X \arrow[r, "j(X)"] \arrow[d, "j(X)"'] & \mathrm{M}_S X \arrow[d, "j(\mathrm{M}_S X)"] \\
\mathrm{M}_S X \arrow[r, "\mathrm{M}_S(j(X))"'] & \mathrm{M}_S\mathrm{M}_S X
\end{tikzcd}$$

Since $j(X)$ is an S-local trivial cofibration (see Definition 3.3.7) and $\mathrm{M}_S X$ is S-local fibrant, Proposition 8.3.21 implies that $j(\mathrm{M}_S X) \overset{r}{\simeq} \mathrm{M}_S\bigl(j(X)\bigr)$ in $(X \downarrow \mathcal{M})$. Since $\mathrm{M}_S X$ and $\mathrm{M}_S \mathrm{M}_S X$ are both cofibrant and fibrant in $(X \downarrow \mathcal{M})$, Proposition 8.3.18 implies that $j(\mathrm{M}_S X) \simeq \mathrm{M}_S\bigl(j(X)\bigr)$ in $(X \downarrow \mathcal{M})$. Since $\mathrm{M}_S X$ and $\mathrm{M}_S \mathrm{M}_S X$ are both cofibrant and fibrant in $(X \downarrow \mathcal{M})$ and Theorem 4.1.13 implies that $j(\mathrm{M}_S X)$ is a weak equivalence, Proposition 8.3.26 implies that $j(\mathrm{M}_S X)$ (and so $\mathrm{M}_S\bigl(j(X)\bigr)$ as well) is a homotopy equivalence. □

PROPOSITION 6.1.7. *If $\mathcal{M}$ is a left proper cellular model category and S is a set of maps in $\mathcal{M}$, then the pair (L_S, j) of Definition* 4.3.2 *is a functorial S-localization (see Definition* 6.1.5*).*

PROOF. This follows from Theorem 4.3.3. □

PROPOSITION 6.1.8. *If $\mathcal{M}$ is a left proper cellular model category and S is a set of maps in $\mathcal{M}$, (M_S, j) is a functorial cofibrant S-localization on $\mathcal{M}$, and X is a fibrant object in $\mathcal{M}$, then the following are equivalent:*

1. *The object X is S-local.*
2. *The S-localization map $j(X)\colon X \to \mathrm{M}_S X$ is a weak equivalence.*
3. *The S-localization map $j(X)\colon X \to \mathrm{M}_S X$ is a homotopy equivalence under X (see Definition* 8.4.3*).*
4. *The S-localization map $j(X)\colon X \to \mathrm{M}_S X$ is the inclusion of a strong deformation retract (see Definition* 8.4.6*).*

PROOF. A cofibrant S-localization of an object is an S-local trivial cofibration to an S-local fibrant object (see Definition 3.3.7). Thus, Proposition 8.4.7 implies that condition 1 implies condition 4. It is obvious that condition 4 implies condition 3 and that condition 3 implies condition 2, and Proposition 4.1.3 implies that condition 2 implies condition 1. □

6.1.9. Uniqueness of localizations.

PROPOSITION 6.1.10. *Let $\mathcal{M}$ be a left proper cellular model category and let S be a set of maps in $\mathcal{M}$. If $g\colon X \to Y$ is an S-localization of X, then there is a map $\phi\colon \mathrm{L}_S X \to Y$ (see Definition* 1.4.11*), unique up to homotopy under X (see Definition* 8.4.3*), such that $\phi j(X) = g$, and any such map ϕ is a weak equivalence.*

PROOF. This follows from Theorem 3.3.8 and Proposition 9.1.6. □

PROPOSITION 6.1.11. *Let $\mathcal{M}$ be a left proper cellular model category and let S be a set of maps in $\mathcal{M}$. If $(\mathrm{M}_S, j\colon 1 \to \mathrm{M}_S)$ and $(\mathrm{N}_S, k\colon 1 \to \mathrm{N}_S)$ are functorial cofibrant S-localizations, then, for every object X, there is a map $\phi_X\colon \mathrm{M}_S X \to \mathrm{N}_S X$, unique up to homotopy under X (see Definition* 8.4.3*), that makes the triangle*

$$\begin{tikzcd} & X \arrow[dl, "j(X)"'] \arrow[dr, "k(X)"] & \\ \mathrm{M}_f X \arrow[rr, "\phi_X"'] & & \mathrm{N}_f X \end{tikzcd}$$

commute, and any such map is a homotopy equivalence under X.

PROOF. This follows from Theorem 3.3.8 and Corollary 9.1.7. □

PROPOSITION 6.1.12. *Let $\mathcal{M}$ be a left proper cellular model category and let S be a set of maps in $\mathcal{M}$. If $(\mathrm{M}_S, j\colon 1 \to \mathrm{M}_S)$ is a functorial cofibrant S-localization, then, for every object X, the homotopy class of the S-localization $j(X)\colon X \to \mathrm{M}_S X$ in $(X \downarrow \mathcal{M})$ is initial in the category of homotopy classes of maps from X to S-local objects, i.e., if W is S-local and $k\colon X \to W$ is a map, then there is a map $\phi\colon \mathrm{M}_S X \to W$, unique up to homotopy in $(X \downarrow \mathcal{M})$, such that $\phi j(X) \simeq k$.*

PROOF. Theorem 3.3.8 implies that $j(X)$ is a trivial cofibration of cofibrant objects in $(X \downarrow \mathcal{M})$, and $X \to W$ is a fibrant object in $(X \downarrow \mathcal{M})$. Thus, the result follows from Proposition 8.3.21 and Proposition 8.3.18. □

PROPOSITION 6.1.13. *Let $\mathcal{M}$ be a left proper cellular model category and let S be a set of maps in $\mathcal{M}$. If $(\mathrm{M}_S, j\colon 1 \to \mathrm{M}_S)$ is a functorial cofibrant S-localization, then for every object X, the homotopy class of the S-localization $j(X)\colon X \to \mathrm{M}_S X$ in $(X \downarrow \mathcal{M})$ is terminal in the category of homotopy classes of cofibrations that are S-local equivalences, i.e., if $k\colon X \to W$ is a cofibration and an S-local equivalence, then there is a map $\phi\colon W \to \mathrm{M}_f X$, unique up to homotopy in $(X \downarrow \mathcal{M})$, such that $\phi k \simeq j(X)$.*

PROOF. Since k is an S-local trivial cofibration (see Theorem 3.3.8) and $\mathrm{M}_S X$ is an S-local fibrant object, this follows from Proposition 8.3.21 and Proposition 8.3.18. □

PROPOSITION 6.1.14. *Let $\mathcal{M}$ be a left proper cellular model category and let S be a set of maps in $\mathcal{M}$. If $(\mathrm{M}_S, j\colon 1 \to \mathrm{M}_S)$ is a functorial cofibrant S-localization, $g\colon X \to Y$ is an S-local equivalence, and Y is an S-local object, there is a map $\phi\colon \mathrm{M}_f X \to Y$, unique up to homotopy under X (see Definition 8.4.3), such that the triangle*

$$\begin{tikzcd} & X \arrow[dl, "j(X)"'] \arrow[dr, "g"] & \\ \mathrm{M}_S X \arrow[rr, "\phi"'] & & Y \end{tikzcd}$$

commutes, and any such map ϕ is a weak equivalence.

PROOF. This follows from Proposition 6.1.12 and Theorem 4.1.10. □

6.2. Comparing localizations

PROPOSITION 6.2.1. *Let $\mathcal{M}$ be a model category, and let f and f' be maps in $\mathcal{M}$. If the class of f-local objects equals the class of f'-local objects, then a map $g\colon X \to Y$ is an f-local equivalence if and only if it is an f'-local equivalence.*

PROOF. This follows directly from the definitions. □

PROPOSITION 6.2.2. *Let $\mathcal{M}$ be a model category, and let both $f\colon A \to B$ and $f'\colon A' \to B'$ be maps in $\mathcal{M}$. If there are weak equivalences $A \to A'$ and $B \to B'$ such that the square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A' \arrow[r, "f'"] & B' \end{tikzcd}$$

commutes, then the class of f-local objects equals the class of f'-local objects.

PROOF. This follows from Proposition 18.3.5 and Proposition 18.3.6. □

PROPOSITION 6.2.3. *Let $\mathcal{M}$ be a model category, and let S and S' be classes of maps in $\mathcal{M}$.*

1. *If for every element f of S there is an element f' of S' such that every f'-local object is f-local, then every S'-local object is S-local and every S-local equivalence is an S'-local equivalence.*
2. *If condition 1 holds and, in addition, for every element f' of S' there is an element f of S such that every f-local object is f'-local, then the class of S-local objects equals the class of S'-local objects and the class of S-local equivalences equals the class of S'-local equivalences.*

PROOF. This follows directly from the definitions. □

6.3. Simplicial localization functors

In this section, we show that if $\mathcal{M}$ is a left proper cellular model category that is a *simplicial* model category and if S is a set of maps in $\mathcal{M}$, then we can define a cofibrant S-localization on $\mathcal{M}$ that is a *simplicial functor* (see Section 10.6).

THEOREM 6.3.1. *If $\mathcal{M}$ is a left proper simplicial cellular model category and S is a set of maps in $\mathcal{M}$, then there is a cofibrant S-localization functor on $\mathcal{M}$ that is a simplicial functor.*

PROOF. **Fill this in!** □

6.4. Comparing colocalizations

PROPOSITION 6.4.1. *Let $\mathcal{M}$ be a right proper cellular model category. If S and S' are sets of objects in $\mathcal{M}$ such that the class of S-colocal equivalences equals the class of S'-colocal equivalences, then the class of S-colocal cofibrations equals the class of S'-colocal cofibrations.*

PROOF. This follows directly from the definitions. □

PROPOSITION 6.4.2. *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of objects in $\mathcal{M}$, then a map is an S-colocal equivalence if and only if it is an A-colocal equivalence for every element A of S.*

PROOF. This follows directly from the definitions. □

PROPOSITION 6.4.3. *Let $\mathcal{M}$ be a right proper cellular model category, and let A and B be objects in $\mathcal{M}$. If there is a weak equivalence $A \to B$, then*

1. *the class of A-colocal equivalences equals the class of B-colocal equivalences, and*
2. *the class of A-colocal objects equals the class of B-colocal objects.*

PROOF. Part 1 follows from Theorem 17.5.2. Part 2 now follows from Proposition 6.4.1. □

PROPOSITION 6.4.4. *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of objects in $\mathcal{M}$, then there is a set S' of cofibrant objects such that*

1. *the class of S-colocal equivalences equals the class of S'-colocal equivalences, and*
2. *the class of S-colocal cofibrations equals the class of S'-colocal cofibrations.*

PROOF. Let S' be a set of cofibrant approximations to the elements of S. The result now follows from Proposition 6.4.2, Proposition 6.4.3 and Proposition 6.4.1. □

6.5. Functorial colocalizations

DEFINITION 6.5.1. If $\mathcal{M}$ is a model category, an *augmented functor* on $\mathcal{M}$ is a pair (F, j) where F is a functor $\mathrm{F}\colon \mathcal{M} \to \mathcal{M}$ and j is a natural transformation $j\colon \mathrm{F} \to 1$.

DEFINITION 6.5.2. If $\mathcal{M}$ is a model category, then an augmented functor (F, j) on $\mathcal{M}$ will be called *homotopy idempotent* if, for every object X in $\mathcal{M}$, the natural maps $j(\mathrm{F}X), \mathrm{F}j(X)\colon \mathrm{FF}X \rightrightarrows \mathrm{F}X$ are homotopic over X (see Definition 8.4.3) and are homotopy equivalences over X.

PROPOSITION 6.5.3. *If $\mathcal{M}$ is a model category and S is a set of objects in $\mathcal{M}$, then a functorial fibrant S-colocal approximation (F, p) (see Definition 5.1.1) is homotopy idempotent.*

PROOF. Since p is a natural transformation, for each object X we have a commutative square

$$\begin{tikzcd} \mathrm{FF}X \arrow[r, "p(\mathrm{F}X)"] \arrow[d, "\mathrm{F}(p(X))"'] & \mathrm{F}X \arrow[d, "p(X)"] \\ \mathrm{F}X \arrow[r, "p(X)"'] & X \end{tikzcd}$$

Since $\mathrm{FF}X$ is S-colocal cofibrant and $p(X)$ is an S-colocal trivial fibration (see Theorem 3.4.9), Proposition 10.2.1 implies that $p(X)$ induces a weak equivalence of homotopy function complexes $\big(p(X)\big)_*\colon \mathrm{map}(\mathrm{FF}X, \mathrm{F}X) \cong \mathrm{map}(\mathrm{FF}X, X)$. **Fix this!!** Corollary 10.4.9 now implies that $\mathrm{F}\big(p(X)\big) \overset{s}{\simeq} p(\mathrm{F}X)$. Theorem 5.2.10 implies that $\mathrm{F}\big(p(X)\big)$ is a weak equivalence, and so $p(\mathrm{F}X)$ must be a weak equivalence as well. □

PROPOSITION 6.5.4. *If (F, p) is a functorial fibrant A-colocalization and X is a cofibrant object, then the following are equivalent:*

1. *The object X is A-colocal.*
2. *The A-colocalization map $p(X)\colon \mathrm{F}X \to X$ is a weak equivalence.*
3. *The A-colocalization map $p(X)\colon \mathrm{F}X \to X$ is a simplicial homotopy equivalence.*
4. *The A-colocalization map $p(X)\colon \mathrm{F}X \to X$ is a simplicial homotopy equivalence that has a right inverse that is a simplicial homotopy inverse,*

PROOF. Proposition 10.4.19 implies that condition 1 implies condition 4. It is obvious that condition 4 implies condition 3 and that condition 3 implies condition 2, and Proposition 5.2.8 implies that condition 2 implies condition 1. □

6.6. Closed classes of objects

In this section, we show that if $\mathcal{M}$ is a right proper cellular model category and S is a set of cofibrant objects in $\mathcal{M}$, then the class of S-colocal objects is the smallest class of cofibrant objects that contains S and is closed under homotopy colimits and weak equivalences (see Theorem 6.6.4).

PROPOSITION 6.6.1. *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of objects in $\mathcal{M}$, then a cofibrant object W is S-colocal if and only if, for every S-colocal equivalence that is a fibration of fibrant objects $g\colon X \to Y$, the induced map of simplicial sets $g_*\colon \operatorname{Map}(W,X) \to \operatorname{Map}(W,Y)$ is a weak equivalence.*

PROOF. Since a fibrant object is a fibrant approximation to itself, the condition is clearly necessary.

Conversely, if $g\colon X \to Y$ is an S-colocal equivalence, then Proposition 9.1.9 implies that we can choose a fibrant approximation $\tilde{g}\colon \widetilde{X} \to \widetilde{Y}$ to g such that $\tilde{g}$ is a fibration. Since $\tilde{g}$ is itself an S-colocal equivalence, $\tilde{g}$ induces a weak equivalence $\tilde{g}_*\colon \operatorname{Map}(W,\widetilde{X}) \approx \operatorname{Map}(W,\widetilde{Y})$, and so W is S-colocal. ☐

LEMMA 6.6.2. *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of objects in $\mathcal{M}$, then the homotopy colimit of a diagram of S-colocal objects is an S-colocal object.*

PROOF. Let $\mathcal{C}$ be a small category, and let $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ be a diagram of S-colocal objects. Corollary 20.6.8 implies that $\operatorname{hocolim}\boldsymbol{X}$ is cofibrant, and so Proposition 6.6.1 implies that it is sufficient to show that $\operatorname{hocolim}\boldsymbol{X}$ has the homotopy left lifting property with respect to all $\overline{\Lambda(S)}$-injectives between fibrant objects. Thus, it is sufficient to show that if Y and Z are fibrant and $g\colon Y \to Z$ is a $\overline{\Lambda(S)}$-injective, then the map

$$g_*\colon \operatorname{Map}(\operatorname{hocolim}\boldsymbol{X}, Y) \to \operatorname{Map}(\operatorname{hocolim}\boldsymbol{X}, Z)$$

is a trivial fibration. This map is isomorphic to the map

$$\operatorname{holim}\operatorname{Map}(\boldsymbol{X}, Y) \to \operatorname{holim}\operatorname{Map}(\boldsymbol{X}, Z)$$

(see Corollary 20.3.19). Since $\operatorname{Map}(\boldsymbol{X}_\alpha, Y) \to \operatorname{Map}(\boldsymbol{X}_\alpha, Z)$ is a trivial fibration for every object α in $\mathcal{C}$ and the homotopy limit of such a diagram of maps is a trivial fibration (see Theorem 20.6.9), the proof is complete. ☐

LEMMA 6.6.3. *Let $\mathcal{M}$ be a right proper cellular model category, and let S be a set of objects in $\mathcal{M}$. If W is an S-colocal object and K is a simplicial set, then the object $W \otimes K$ is S-colocal.*

PROOF. This follows from Proposition 3.4.10 and Lemma 17.3.11. ☐

THEOREM 6.6.4. *Let $\mathcal{M}$ be a right proper cellular model category. If S is a set of objects in $\mathcal{M}$, then the class of S-colocal objects is the smallest class of cofibrant objects that contains S and is closed under homotopy colimits and weak equivalences.*

PROOF. Let $\mathcal{C}$ be a class of cofibrant objects that contains S and is closed under homotopy colimits and weak equivalences. Lemma 6.6.3 implies that $\mathcal{C}$ contains $A \otimes \partial\Delta[n]$ for every element A of S and every $n \geq 1$, and so Proposition 20.3.8 implies that, if X is an object in $\mathcal{C}$, then $\mathcal{C}$ contains the pushout of the diagram

$X \leftarrow A \otimes \partial\Delta[n] \to A \otimes \Delta[n]$. Together with Proposition 20.9.9, this implies that $\mathcal{C}$ contains all $\overline{\Lambda(S)}$-complexes (see Definition 5.3.2), and so Corollary 5.5.2 and Proposition 19.5.14 imply that $\mathcal{C}$ contains $\mathrm{CW}_S\, X$ for every object X. Since every S-colocal object X is weakly equivalent to $\mathrm{CW}_S\, X$ (see Proposition 6.5.4), $\mathcal{C}$ must contain all S-colocal objects. Lemma 6.6.2 shows that the class of S-colocal objects is closed under homotopy colimits, and Proposition 6.4.1 shows that it is closed under weak equivalences, and so the proof is complete. □

CHAPTER 7

Fiberwise localization

7.1. Fiberwise localization

If S is a set of maps in $\mathsf{Spc}_{(*)}$ and $p\colon Y \to Z$ is a fibration in $\mathsf{Spc}_{(*)}$, then a fiberwise S-localization of p should be a map from p to another fibration q over Z

$$\begin{tikzcd} Y \arrow[rr, "i"] \arrow[dr, "p"'] & & \widehat{Y} \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

that "localizes the fibers of p", i.e., for every point z in Z the map $p^{-1}(z) \to q^{-1}(z)$ should be an S-localization of $p^{-1}(z)$. The actual definition is a generalization of this that deals with maps p that may not be fibrations.

Definition 7.1.1. Let S be a set of maps in $\mathsf{Spc}_{(*)}$ (see Notation 1.1.2). If $p\colon Y \to Z$ is a map in $\mathsf{Spc}_{(*)}$, then a *fiberwise S-localization* of p is a factorization $Y \xrightarrow{i} \widehat{Y} \xrightarrow{q} Z$ of p such that

1. q is a fibration, and
2. for every point z of Z, the induced map of homotopy fibers (see Definition 11.2.19) $\mathrm{HFib}_z(p) \to \mathrm{HFib}_z(q)$ is an S-localization of $\mathrm{HFib}_z(p)$ (see Definition 4.1.1).

Proposition 7.1.2. *If S is a set of maps in $\mathsf{Spc}_{(*)}$, $p\colon Y \to Z$ is a fibration in $\mathsf{Spc}_{(*)}$, and $Y \xrightarrow{i} \widehat{Y} \xrightarrow{q} Z$ is a factorization of p, then this factorization is a fiberwise S-localization of p if and only if*

1. *q is a fibration, and*
2. *for every point z of Z, the induced map of fibers $p^{-1}(z) \to q^{-1}(z)$ is an S-localization of $p^{-1}(z)$.*

Proof. This follows from Proposition 11.2.22. $\square$

In this chapter, we show that for every set S of maps in Spc (a category of *unpointed* spaces; see Notation 1.1.2), every map $p\colon Y \to Z$ has a natural fiberwise S-localization $Y \to \widehat{Y} \to Z$. We also show that if $p\colon Y \to Z$ is a map in Spc and $Y \to \widehat{Y}' \to Z$ is some other fiberwise S-localization of p, then there is a map $\widehat{Y} \to \widehat{Y}'$ under Y and over Z, unique up to simplicial homotopy in $(Y \downarrow \mathsf{Spc} \downarrow Z)$, and any such map is a weak equivalence.

We construct our fiberwise localization for the categories of unpointed spaces Top and SS. Since the pointed localization of a connected space is weakly equivalent to its unpointed localization (see Theorem 1.7.13), our construction will also serve as a fiberwise pointed localization for fibrations with connected fibers. This is the strongest possible result; in Proposition 7.1.3, we show that it is not possible to construct a fiberwise pointed localization for fibrations with fibers that are not connected.

PROPOSITION 7.1.3. *Let $f\colon A \to B$ be the inclusion $\mathrm{S}^2 \to \mathrm{D}^3$ in Top_*. If $X = \mathrm{S}^2 \times \mathbb{R}$, $Z = \mathrm{S}^1$, and $p\colon X \to Z$ is the composition of the projection $\mathrm{S}^2 \times \mathbb{R} \to \mathbb{R}$ with the universal covering map $\mathbb{R} \to \mathrm{S}^1$, then there is no fiberwise f-localization of p in the category Top_* of pointed spaces.*

PROOF. The fiber F of p is a countable disjoint union of copies of S^2, and so if there were a fiberwise pointed localization of p, its fiber would have countably many path components: one contractible, and the others weakly equivalent to S^2 (see Corollary 1.7.10).

To see that this is not possible, note that $\pi_1 Z$ acts transitively on $\pi_0 F$, and so $\pi_1 Z$ would act transitively on the path components of the fiber of any fiberwise localization of p. Since $\pi_1 Z$ acts on the fiber through (unpointed) weak equivalences, this is impossible, and so there does not exist a fiberwise pointed localization of p. $\square$

The following theorem summarizes the main results of this chapter.

THEOREM 7.1.4. *If S is a set of maps in Spc (see Notation 1.1.2), then there is a natural factorization of every map $p\colon X \to Z$ as $X \xrightarrow{i} \widetilde{\mathrm{L}}_S X \xrightarrow{q} Z$ such that*

1. *q is a fibration with S-local fibers,*
2. *for every point z in Z the induced map of homotopy fibers $\mathrm{HFib}_z(p) \to \mathrm{HFib}_z(q)$ (see Definition 11.2.19) is an S-localization of $\mathrm{HFib}_z(p)$,*
3. *i is both a cofibration and an S-local equivalence,*
4. *if we have a solid arrow diagram*

$$\begin{tikzcd} X \arrow[r, "i"'] \arrow[rr, bend left, "j"] \arrow[dr, "p"'] & \widetilde{\mathrm{L}}_S X \arrow[r, dotted, "k"'] \arrow[d, "q"] & W \arrow[dl, "r"] \\ & Z & \end{tikzcd}$$

   in which r is a fibration with S-local fibers, then there is a map $k\colon \widetilde{\mathrm{L}}_S X \to W$, unique up to simplicial homotopy in $(X \downarrow \mathsf{Spc} \downarrow Z)$, such that $ki = j$, and
5. *if we have a diagram as in the previous part such that for every point z in Z the map $\mathrm{HFib}_z(p) \to \mathrm{HFib}_z(r)$ of homotopy fibers over z induced by j is an S-local equivalence (i.e., if j is another fiberwise S-localization of p), then the map k is a weak equivalence.*

7.2. The fiberwise local model category structure

DEFINITION 7.2.1. Let S be a set of maps in Spc. If Z is a space in Spc, then we define $\mathrm{Fib}_Z(S)$ (which we call the *set of elements of S fiberwise over Z*) to be the set of maps in $(\mathsf{Spc}\downarrow Z)$

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \arrow[dl] \\ & Z & \end{tikzcd}$$

where $f\colon A \to B$ is an element of S and the images of the maps $A \to Z$ and $B \to Z$ are a single point of Z.

PROPOSITION 7.2.2. *If Z is a space in Spc, then the category $(\mathsf{Spc}\downarrow Z)$ of objects of Spc over Z is a left proper cellular model category.*

PROOF. This follows from Proposition 3.3.17. $\square$

THEOREM 7.2.3. *Let Z be a space in Spc, and let S be a set of maps in Spc. If we define*

1. a fiberwise over Z S-local equivalence *to be a $\mathrm{Fib}_Z(S)$-local equivalence in $(\mathsf{Spc}\downarrow Z)$ (see Definition 3.3.2),*
2. a fiberwise over Z S-local cofibration *to be a $\mathrm{Fib}_Z(S)$-local cofibration, and*
3. a fiberwise over Z S-local fibration *to be a $\mathrm{Fib}_Z(S)$-local fibration,*

then there is a simplicial model category structure on $(\mathsf{Spc}\downarrow Z)$ in which the weak equivalences are the fiberwise over Z S-local equivalences, the cofibrations are the fiberwise over Z S-local cofibrations, and the fibrations are the fiberwise over Z S-local fibrations.

PROOF. This follows from Theorem 3.3.8 and Proposition 7.2.2. $\square$

PROPOSITION 7.2.4. *If S is a set of maps in Spc and Z is a space in Spc, then an object of $(\mathsf{Spc}\downarrow Z)$ is fibrant in the fiberwise over Z S-local model category structure if and only if it is a fibration and the fiber over every point of Z is an S-local space.*

PROOF. This follows from Proposition 3.3.9. $\square$

7.3. Localizing the fiber

The purpose of this section is to prove the following theorem.

THEOREM 7.3.1. *If S is a set of maps in Spc, Z is a space in Spc, and*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

is a $\overline{\Lambda(\mathrm{Fib}_Z(S))}$-cofibration (see Definition 4.2.4), then for every point z of Z the induced map of homotopy fibers $\mathrm{HFib}_z(p) \to \mathrm{HFib}_z(q)$ is an S-local equivalence.

The proof of Theorem 7.3.1 is at the end of this section.

LEMMA 7.3.2. *Let S be a set of maps in Spc, let $\mathcal{C}$ be a small category, and let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be diagrams in Spc indexed by $\mathcal{C}$. If $g\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a map of diagrams such that the map $g_\alpha\colon \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ is an S-local equivalence between cofibrant spaces* *for each* $\alpha \in \mathrm{Ob}(\mathcal{C})$*, then the induced map of homotopy colimits* $\operatorname{hocolim} \boldsymbol{X} \to \operatorname{hocolim} \boldsymbol{Y}$ *is an* S*-local equivalence.*

PROOF. Let W be an S-local space. Since $\mathrm{Map}(\operatorname{hocolim} \boldsymbol{X}_\alpha, W)$ is naturally isomorphic to $\operatorname{holim} \mathrm{Map}(\boldsymbol{X}_\alpha, W)$ (see Corollary 20.3.19), our map is isomorphic to $\operatorname{holim} \mathrm{Map}(\boldsymbol{Y}_\alpha, W) \to \operatorname{holim} \mathrm{Map}(\boldsymbol{X}_\alpha, W)$. Since for each α we have a weak equivalence of fibrant simplicial sets $\mathrm{Map}(\boldsymbol{Y}_\alpha, W) \to \mathrm{Map}(\boldsymbol{X}_\alpha, W)$, and a homotopy limit of such maps is also a weak equivalence (see Theorem 20.6.10), the proposition follows. $\square$

PROPOSITION 7.3.3. *If* $q\colon X \to Z$ *is a map of simplicial sets and* $z \in Z$*, then there is a contractible simplicial set* C *(which depends naturally on the pair* (Z, z)*) and a natural* (ΔC)*-diagram (see Definition* 16.1.11*) of simplicial sets* $\boldsymbol{F}\colon (\Delta C) \to \mathsf{SS}$ *such that*

1. *for every simplex* σ *of* C *there is a simplex* τ *of* Z *such that* $\boldsymbol{F}(\sigma) = \tilde{q}(\tau)$ *(see Example* 20.10.1*), and*
2. *there is a natural weak equivalence* $\operatorname{hocolim} \boldsymbol{F} \cong \mathrm{HFib}_z(q)$ *(where* $\mathrm{HFib}_z(q)$ *is the homotopy fiber of* q *over* z*).*

By "natural" we mean that the simplicial set C *is a functor of the pair* (Z, z) *and, for a fixed pair* (Z, z)*, the diagram* $\boldsymbol{F}$ *is a functor of the object* $q\colon X \to Z$ *of* $(\mathsf{SS} \downarrow Z)$.

PROOF. If $* \to Z$ is the map to the point z in Z, let $* \to C \xrightarrow{p} Z$ be a natural factorization of it into a trivial cofibration followed by a fibration. The homotopy fiber of q over z is then naturally weakly equivalent to the pullback of the diagram $C \xrightarrow{p} Z \xleftarrow{q} X$ (see Proposition 11.2.7). If we let F be that pullback and $r\colon F \to C$ its projection onto C, then the construction of Example 20.10.1 applied to r yields a diagram $\boldsymbol{F}\colon (\Delta C) \to \mathsf{SS}$ that satisfies condition 1. Proposition 20.11.11 implies that $\boldsymbol{F}$ is Reedy cofibrant, and so condition 2 follows from Theorem 20.11.10 and the natural isomorphism $\operatorname{colim} \boldsymbol{F} \approx F$. $\square$

PROPOSITION 7.3.4 (E. Dror Farjoun, [**22**]). *Let* S *be a set of maps in* SS*, let* Z *be a simplicial set, let* $p\colon X \to Z$ *and* $q\colon Y \to Z$ *be objects of* $(\mathsf{SS} \downarrow Z)$*, and let*

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

be a map in $(\mathsf{SS} \downarrow Z)$*. If for every simplex* σ *of* Z *the induced map* $\tilde{p}(\sigma) \to \tilde{q}(\sigma)$ *(see Example* 20.10.1*) is an* S*-local equivalence, then for every point* z *in* Z *the induced map of homotopy fibers* $\mathrm{HFib}_z(p) \to \mathrm{HFib}_z(q)$ *is an* S*-local equivalence.*

PROOF. This follows from Proposition 7.3.3 and Lemma 7.3.2. $\square$

LEMMA 7.3.5. *If* $f\colon A \to B$ *is a cofibration in* SS*,* Z *is a space in* Top*,*

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

is a map in $(\mathsf{Top}\downarrow Z)$*, and* z *is a point in* Z*, then the induced map of homotopy fibers* $\mathrm{HFib}_z(p) \to \mathrm{HFib}_z(q)$ *is a* $|f|$*-local equivalence if and only if the induced map of the corresponding homotopy fibers of* $(\mathrm{Sing}\, p)\colon \mathrm{Sing}\, X \to \mathrm{Sing}\, Z$ *and* $(\mathrm{Sing}\, q)\colon \mathrm{Sing}\, Y \to \mathrm{Sing}\, Z$ *is an* f*-local equivalence.*

PROOF. Proposition 11.2.26 implies that the "homotopy fiber" and "total singular complex" functors commute up to a natural weak equivalence, and so the result follows from Proposition 1.2.30. $\square$

PROPOSITION 7.3.6. *Let* $f\colon A \to B$ *be an inclusion of cell complexes in* Spc*, and let* Z *be a space in* Spc*. If the map*

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

in $(\mathsf{Spc}\downarrow Z)$ *is a pushout of an element of* $\overline{\Lambda\big(\mathrm{Fib}_Z\{f\}\big)}$ *(see Definition 4.2.4), then* g *is both a cofibration and an* f*-local equivalence in* Spc*, and for every point* z *in* Z *the induced map of homotopy fibers* $\mathrm{HFib}_z(p) \to \mathrm{HFib}_z(q)$ *is an* f*-local equivalence.*

PROOF. There are two types of maps in the set $\overline{\Lambda\big(\mathrm{Fib}_Z\{f\}\big)}$. The first type is an element of $\Lambda\big(\mathrm{Fib}_Z\{f\}\big)$ (see Definition 4.2.4); a map of this type is an S-local equivalence in Spc, and its domain and codomain lie over a single point z of Z. The second type is a generating trivial cofibration of Spc. If Y is obtained from X by pushing out a map of the second type, then the map g is a weak equivalence, and so the induced map of homotopy fibers is a weak equivalence. Thus, we need only consider the case in which Y is obtained from X by pushing out an element of $\Lambda\big(\mathrm{Fib}_Z\{f\}\big)$.

If $\mathsf{Spc} = \mathsf{SS}$, then for each simplex σ of Z, the map $\tilde{p}(\sigma) \to \tilde{q}(\sigma)$ (see Example 20.10.1) is obtained by pushing out one copy of our element of $\Lambda\big(\mathrm{Fib}_Z\{f\}\big)$ for each vertex of σ that equals z. Thus, $\tilde{p}(\sigma) \to \tilde{q}(\sigma)$ is an S-local equivalence, and so the lemma follows from Proposition 7.3.4. Thus, we need only consider the case $\mathsf{Spc} = \mathsf{Top}$.

If $\mathsf{Spc} = \mathsf{Top}$, then Proposition 1.2.30 and Proposition 1.2.6 imply that it is sufficient to show that $\mathrm{Sing}\big(\mathrm{HFib}_z(p)\big) \to \mathrm{Sing}\big(\mathrm{HFib}_z(q)\big)$ is a $(\mathrm{Sing}\, f)$-local equivalence, and Proposition 11.2.26 implies that this is equivalent to showing that $\mathrm{HFib}_z(\mathrm{Sing}\, p) \to \mathrm{HFib}_z(\mathrm{Sing}\, q)$ is a $(\mathrm{Sing}\, f)$-local equivalence (where we also use the symbol z to denote the vertex of $\mathrm{Sing}\, Z$ corresponding to the point z of Z).

Let $A\times\big|\Delta[n]\big|\amalg_{A\times|\partial\Delta[n]|} B\times\big|\partial\Delta[n]\big| \to B\times\big|\Delta[n]\big|$ be the element of $\Lambda\big(\mathrm{Fib}_Z\{f\}\big)$ in the pushout that transforms X into Y. We have a pushout square

$$\begin{tikzcd} A \times \big|\Delta[n]\big| \amalg_{A\times|\partial\Delta[n]|} B \times \big|\partial\Delta[n]\big| \arrow[r] \arrow[d] & X \arrow[d] \\ B \times \big|\Delta[n]\big| \arrow[r] & Y \end{tikzcd}$$

and Proposition 11.3.3 implies that $\operatorname{Sing} Y$ is weakly equivalent to the pushout

$$\begin{tikzcd}
\operatorname{Sing}\bigl(A \times |\Delta[n]| \amalg_{A \times |\partial\Delta[n]|} B \times |\partial\Delta[n]|\bigr) \arrow[r] \arrow[d] & \operatorname{Sing} X \arrow[d] \\
\operatorname{Sing}\bigl(B \times |\Delta[n]|\bigr) \arrow[r] & Y'
\end{tikzcd}$$

If we let $q' \colon Y' \to Z$ be the structure map of Y' in $\bigl(\mathsf{SS}{\downarrow}(\operatorname{Sing} Z)\bigr)$, then for every simplex $\sigma \in \operatorname{Sing} Z$ the map $\widetilde{(\operatorname{Sing} p)}(\sigma) \to \tilde{q}'(\sigma)$ (see Example 20.10.1) is obtained by pushing out one copy of $\operatorname{Sing}\bigl(A \times |\Delta[n]| \amalg_{A \times |\partial\Delta[n]|} B \times |\partial\Delta[n]|\bigr) \to \operatorname{Sing}\bigl(B \times |\Delta[n]|\bigr)$ for each vertex of σ that equals the image of $\operatorname{Sing}\bigl(B \times |\Delta[n]|\bigr)$ in $\operatorname{Sing} Z$. Proposition 1.2.30 implies that this is a $(\operatorname{Sing} f)$-local equivalence, and so Proposition 7.3.4 implies that $\operatorname{HFib}_z(\operatorname{Sing} p) \to \operatorname{HFib}_z(q')$ is a $(\operatorname{Sing} f)$-local equivalence. This implies that $\operatorname{HFib}_z(\operatorname{Sing} p) \to \operatorname{HFib}_z(\operatorname{Sing} q)$ is a $(\operatorname{Sing} f)$-local equivalence, and the proof is complete. ☐

Proof of Theorem 7.3.1. Every $\operatorname{Fib}_Z(S)$-cofibration is a retract of a transfinite composition of pushouts of elements of $\overline{\Lambda\bigl(\operatorname{Fib}_Z(S)\bigr)}$ (see Corollary 12.4.18). Since S-local equivalences are closed under retracts, Proposition 11.2.25 implies that a retract of a map in $(\mathsf{Spc}{\downarrow}Z)$ inducing an S-local equivalence of homotopy fibers over z must also induce an S-local equivalence of homotopy fibers over z. Thus, it is sufficient to show that if

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[drr, "p_0"'] & X_1 \arrow[r] \arrow[dr, "p_1"] & X_2 \arrow[r] \arrow[d, "p_2"] & \cdots \arrow[r] & X_\beta \arrow[r] \arrow[dll, "p_\beta"] & \cdots & (\beta < \lambda) \\
& & Z & & & &
\end{tikzcd}$$

is a transfinite composition of pushouts of elements of $\overline{\Lambda\bigl(\operatorname{Fib}_Z(S)\bigr)}$, then the induced map of homotopy fibers $\operatorname{HFib}_z(p_0) \to \operatorname{HFib}_z(\operatorname{colim}_{\beta<\lambda} p_\beta)$ is an S-local equivalence.

If $\mathsf{Spc} = \mathsf{Top}$, then we choose a factorization $* \xrightarrow{s} C \xrightarrow{t} Z$ of the map $* \to Z$ whose image is z such that s is a trivial cofibration and t is a fibration, and Proposition 11.2.25 implies that each $\operatorname{HFib}_z(X_\beta)$ is naturally weakly equivalent to $C \times_Z X_\beta$. Each map $C \times_Z X_\beta \to C \times_Z X_{\beta+1}$ is an inclusion (and, thus, a cofibration), and Proposition 7.3.6 implies that it is an S-local equivalence. Thus, it is a trivial cofibration in the S-local model category structure on SS (see Theorem 3.3.8). Proposition 12.2.19 now implies that the transfinite composition $C \times_Z X_0 \to \operatorname{colim}_{\beta<\lambda}(C \times_Z X_\beta) \approx C \times_Z (\operatorname{colim}_{\beta<\lambda} X_\beta)$ is an S-local equivalence, and Proposition 11.2.25 implies that this is weakly equivalent to the map $\operatorname{HFib}_z(p_0) \to \operatorname{HFib}_z(\operatorname{colim}_{\beta<\lambda} p_\beta)$.

If $\mathsf{Spc} = \mathsf{Top}$, then Proposition 11.2.26 and Proposition 1.2.30 imply that it is sufficient to show that the induced map of homotopy fibers of total singular complexes $\operatorname{HFib}_z(p_0) \to \operatorname{HFib}_z(\operatorname{colim}_{\beta<\lambda} \operatorname{Sing} p_\beta) \approx \operatorname{HFib}_z(\operatorname{Sing} \operatorname{colim}_{\beta<\lambda} p_\beta)$ is a $(\operatorname{Sing} S)$-local equivalence (where $(\operatorname{Sing} S) = \{\operatorname{Sing} f \mid f \in S\}$ and we use the symbol z to also denote the vertex of $\operatorname{Sing} Z$ corresponding to z). We choose a factorization $* \xrightarrow{s} C \xrightarrow{t} \operatorname{Sing} Z$ in SS of the map $* \to \operatorname{Sing} Z$ whose image is z such that s is a trivial cofibration and t is a fibration, and the argument proceeds as in the case $\mathsf{Spc} = \mathsf{SS}$. ☐

7.4. Uniqueness of the fiberwise localization

THEOREM 7.4.1. *Let S be a set of maps in $\mathsf{Spc}_{(*)}$. If*

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend left, "g"] \arrow[dr, "q"'] & \mathrm{E}_S \arrow[r] \arrow[d, "r"] & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

is the factorization in $(\mathsf{Spc}_{(*)} \downarrow Z)$ of g into a $\overline{\Lambda(S)}$-cofibration followed by a $\overline{\Lambda(S)}$-injective, then for every $z \in Z$ the induced map of homotopy fibers $\mathrm{HFib}_z(q) \to \mathrm{HFib}_z(r)$ is an S-local equivalence.

PROPOSITION 7.4.2. *If S is a set of maps in $\mathsf{Spc}_{(*)}$, $p\colon X \to Z$ is an object of $(\mathsf{Spc}\downarrow Z)$, $q\colon Y \to Z$ is a fibration with S-local fibers, $g\colon X \to Y$ is a map in $(\mathsf{Spc}\downarrow Z)$ and $X \to \widetilde{\mathrm{L}}_S X$ is the fiberwise S-localization of X over Z, then the dotted arrow exists in the diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend left, "g"] \arrow[dr, "p"'] & \widetilde{\mathrm{L}}_S X \arrow[r, dotted] \arrow[d] & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

and it is unique up to simplicial homotopy in $(\mathsf{Spc}\downarrow Z)$.

PROOF. Since $q\colon Y \to Z$ is a $(\mathrm{Fib}_Z S)$-injective, this follows from Proposition 10.4.16. □

THEOREM 7.4.3 (Uniqueness of fiberwise localization). *Let S be a set of maps in $\mathsf{Spc}_{(*)}$. If $q\colon Y \to Z$ is a fibration in Spc and*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

is a map in $(\mathsf{Spc}\downarrow Z)$ such that for every point z of Z the induced map of homotopy fibers $\mathrm{HFib}_z(p) \to \mathrm{HFib}_z(q)$ is an S-local equivalence, then the map $\widetilde{\mathrm{L}}_S X \to Y$ of Proposition 7.4.2 is a weak equivalence.

PROOF. Since for every point $z \in Z$ the induced map from the homotopy fiber of $\widetilde{\mathrm{L}}_S X \to Z$ over z to the homotopy fiber of q over z is an S-local equivalence between S-local spaces, Theorem 4.1.10 implies that it is a weak equivalence. The theorem now follows from the exact homotopy sequence of a fibration applied over each path component of Z. □

7.5. Other constructions of the fiberwise localization

7.5.1. Decompose the total space. Decompose the total space as a diagram indexed by the category of simplices of the base, and localize each space in the diagram.

7.5.2. Using the classification of fibrations. Use the classification of fibrations of simplicial sets. The continuity of the localization functor gives us a simplicial map $\operatorname{aut} F \to \operatorname{aut} \mathrm{L}_f F$. Either take classifying spaces, or use to alter the twisted cartesian product directly (as in [**11**] or [**2**]).

PROPOSITION 7.5.3. *If W is f-local and X is cofibrant, then W^X is f-local.*

PROOF. This follows from Corollary 1.1.9 and the natural isomorphisms

$$\left(W^X\right)^B \approx W^{B \otimes X} \approx \left(W^B\right)^X .$$

□

PROPOSITION 7.5.4. *If W is a cofibrant f-local space, then* $\operatorname{aut} W$ *(the monoid of self-homotopy equivalences of W) is f-local.*

PROOF. Since $\operatorname{aut} W$ is a nonempty union of a set of path components of W^W, this follows from Proposition 7.5.3 and Lemma 1.7.11. □

Part 2

Model categories

CHAPTER 8

Model categories

8.1. Model categories

We adopt the definition of a model category used in [**26**]. This is a strengthening of Quillen's axioms for a *closed model category* (see [**48**, page 233]) in that it requires the category to contain all small limits and colimits (rather than just the finite ones), and it requires the factorizations described in the fifth axiom to be functorial.

DEFINITION 8.1.1. If we have a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "1_A"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "1_B"'] & D \arrow[r] & B \end{tikzcd}$$

then we will say that the map f is a *retract* of the map g.

DEFINITION 8.1.2. A *model category* is a category $\mathcal{M}$ together with three classes of maps (cofibrations, fibrations, and weak equivalences) satisfying the following five axioms:

M1: (Limit axiom) The category $\mathcal{M}$ is closed under small limits and colimits.

M2: (Two out of three axiom) If g and h are maps in $\mathcal{M}$ such that hg is defined and two of g, h, and hg are weak equivalences, then so is the third.

M3: (Retract axiom) If g and h are maps in $\mathcal{M}$ such that g is a retract of h (in the category of maps of $\mathcal{M}$) and h is a weak equivalence, a fibration, or a cofibration, then so is g.

M4: (Lifting axiom) Given the commutative solid arrow diagram in $\mathcal{M}$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

the dotted arrow exists in each of the following two cases:

1. i is a cofibration and p is a trivial fibration (i.e., a fibration that is also a weak equivalence).
2. p is a fibration and i is a trivial cofibration (i.e., a cofibration that is also a weak equivalence).

M5: (Factorization axiom) Every map $g \in \mathcal{M}$ has two functorial factorizations:

1. $g = hi$, where i is a cofibration and h is a trivial fibration.

2. $g = pj$, where p is a fibration and j is a trivial cofibration.

TERMINOLOGY. We will follow Quillen [**46, 48**] in using the term *trivial fibration* for a fibration that is also a weak equivalence, and the term *trivial cofibration* for a cofibration that is also a weak equivalence.

REMARK 8.1.3. The lifting axiom implies both the homotopy extension property of cofibrations (see Proposition 8.3.7) and the homotopy lifting property of fibrations (see Proposition 8.3.8).

REMARK 8.1.4. The retract axiom implies that any two of the three classes of maps cofibrations, fibrations, and weak equivalences determine the third (see Proposition 8.2.3), and was the reason for the use of the name "closed model category" for what we call simply a "model category".

8.1.5. Duality in model categories. The axioms for a model category are self dual.

PROPOSITION 8.1.6. *If $\mathcal{M}$ is a model category, then its opposite category $\mathcal{M}^{\mathrm{op}}$ is a model category such that*

- *the cofibrations in $\mathcal{M}^{\mathrm{op}}$ are the opposites of the fibrations in $\mathcal{M}$,*
- *the fibrations in $\mathcal{M}^{\mathrm{op}}$ are the opposites of the cofibrations in $\mathcal{M}$, and*
- *the weak equivalences in $\mathcal{M}^{\mathrm{op}}$ are the opposites of the weak equivalences in $\mathcal{M}$.*

PROOF. This follows directly from the definitions. □

REMARK 8.1.7. Thus, any statement that is proved true for all model categories implies a dual statement in which cofibrations are replaced by fibrations, fibrations are replaced by cofibrations, colimits are replace by limits, and limits are replace by colimits.

8.2. Lifting and the retract argument

DEFINITION 8.2.1. If $i\colon A \to B$ and $p\colon X \to Y$ are maps for which the dotted arrow exists in every solid arrow diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

then (i, p) is called a *lifting-extension pair*, i is said to have the *left lifting property* with respect to p, and p is said to have the *right lifting property* with respect to i.

PROPOSITION 8.2.2 (The retract argument). *Let $\mathcal{M}$ be a model category, and let $g\colon X \to Y$ be a map in $\mathcal{M}$.*

1. *If g can be factored as $g = pi$ where p has the right lifting property with respect to g, then g is a retract of i.*
2. *If g can be factored as $g = pi$ where i has the left lifting property with respect to g, then g is a retract of p.*

PROOF. We will prove part 1; the proof of part 2 is similar.

We have the solid arrow diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "g"'] & Z \arrow[d, "p"] \\ Y \arrow[r, equal] \arrow[ur, dotted, "q"] & Y \end{tikzcd}$$

Since p has the right lifting property with respect to g, the dotted arrow q exists. This yields the commutative diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "g"'] & X \arrow[r, equal] \arrow[d, "i"] & X \arrow[d, "g"] \\ Y \arrow[r, "q"'] \arrow[rr, bend right, "1_Y"'] & Z \arrow[r, "p"'] & Y \end{tikzcd}$$

and so g is a retract of i. □

PROPOSITION 8.2.3. *Let $\mathcal{M}$ be a model category.*

1. *The map $i\colon A \to B$ is a cofibration if and only if it has the left lifting property with respect to all trivial fibrations.*
2. *The map $i\colon A \to B$ is a trivial cofibration if and only if it has the left lifting property with respect to all fibrations.*
3. *The map $p\colon X \to Y$ is a fibration if and only if it has the right lifting property with respect to all trivial cofibrations.*
4. *The map $p\colon X \to Y$ is a trivial fibration if and only if it has the right lifting property with respect to all cofibrations.*

PROOF. This follows from the retract argument (Proposition 8.2.2), using axioms M3, M4, and M5 (see Definition 8.1.2). □

PROPOSITION 8.2.4. *If $\mathcal{M}$ is a model category, then the classes of cofibrations and of fibrations are closed under compositions.*

PROOF. This follows from Proposition 8.2.3. □

LEMMA 8.2.5. *Let $\mathcal{M}$ be a model category, and let $p\colon X \to Y$ be a map in $\mathcal{M}$.*

1. *The class of maps with the left lifting property with respect to p is closed under pushouts.*
2. *The class of maps with the right lifting property with respect to p is closed under pullbacks.*

PROOF. This follows directly from the definitions. □

PROPOSITION 8.2.6. *Let $\mathcal{M}$ be a model category.*

1. *The class of cofibrations is closed under pushouts.*
2. *The class of trivial cofibrations is closed under pushouts.*
3. *The class of fibrations is closed under pullbacks.*
4. *The class of trivial fibrations is closed under pullbacks.*

PROOF. This follows from Proposition 8.2.3 and Lemma 8.2.5. □

LEMMA 8.2.7. *Let $\mathcal{M}$ be a model category, and let $p\colon X \to Y$ is a map in $\mathcal{M}$.*

1. *The class of maps with the left lifting property with respect to p is closed under retracts.*
2. *The class of maps with the right lifting property with respect to p is closed under retracts.*

PROOF. This follows directly from the definitions. $\square$

PROPOSITION 8.2.8. *Let $\mathcal{M}$ and $\mathcal{N}$ be categories, and let $\mathrm{F}\colon \mathcal{M} \rightleftarrows \mathcal{N} :\mathrm{U}$ be adjoint functors. If $i\colon A \to B$ is a morphism in $\mathcal{M}$ and $p\colon X \to Y$ is a morphism in $\mathcal{N}$, then $(\mathrm{F}i, p)$ is a lifting-extension pair (see Definition 8.2.1) if and only if $(i, \mathrm{U}p)$ is a lifting-extension pair.*

PROOF. The adjointness implies that there is a one to one correspondence between solid arrow diagrams of the form

$$\begin{tikzcd} \mathrm{F}A \arrow[r] \arrow[d, "\mathrm{F}i"'] & X \arrow[d, "p"] \\ \mathrm{F}B \arrow[r] \arrow[ur, dotted, "h"] & Y \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & \mathrm{U}X \arrow[d, "\mathrm{U}p"] \\ B \arrow[r] \arrow[ur, dotted, "\tilde{h}"] & \mathrm{U}Y. \end{tikzcd}$$

The adjointness also implies that, under this correspondence, the dotted arrow h exists if and only if the dotted arrow $\tilde{h}$ exists. $\square$

8.2.9. Pushouts and pullbacks.

DEFINITION 8.2.10. If the square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] & D \end{tikzcd}$$

is a pushout, then the map g will be called the *pushout* of f along h. If the square is a pullback, then the map f will be called the *pullback* of g along k.

LEMMA 8.2.11. *If $h\colon E \to F$ is a pushout (see Definition 8.2.10) of $g\colon C \to D$ and $k\colon G \to H$ is a pushout of h, then k is a pushout of g.*

PROOF. In the commutative diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "g"'] & E \arrow[r] \arrow[d, "h"] & G \arrow[d, "k"] \\ D \arrow[r] & F \arrow[r] & H \end{tikzcd}$$

if the two squares are pushouts, then the rectangle is a pushout. $\square$

PROPOSITION 8.2.12. *Consider the commutative diagram*

$$\begin{tikzcd} C \arrow[r, "s"] \arrow[d, "f"'] & E \arrow[r, "t"] \arrow[d, "g"] & G \arrow[d, "h"'] \\ D \arrow[r, "u"] & F \arrow[r, "v"] & H \end{tikzcd}$$

1. *If H is the pushout $D \amalg_C G$ and F is the pushout $D \amalg_C E$, then H is the pushout $F \amalg_E G$.*

2. *If C is the pullback $D \times_H G$ and E is the pullback $F \times_H G$, then C is the pullback $D \times_F E$.*

PROOF. We will prove part 1; the proof of part 2 is similar.

If W is an object and $j\colon F \to W$ and $k\colon G \to W$ are maps such that $jg = kt$, then $kts = jgs = juf$. Since H is the pushout $D \amalg_C G$, there exists a unique map $l\colon H \to W$ such that $lvu = ju$ and $lh = k$. Since F is the pushout $D \amalg_C E$ and the maps j and lv satisfy both $(lv)u = (j)u$ and $(j)g = kt = lht = (lv)g$, we have $j = lv$. Thus, the map l satisfies $lh = k$ and $lv = j$. To see that l is the unique such map, note that if $\tilde{l}$ were another map satisfying $\tilde{l}h = k$ and $\tilde{l}v = j$, then $\tilde{l}vu = ju$, and so the universal property of $D \amalg_C G$ implies that $\tilde{l} = l$. □

LEMMA 8.2.13 (C. L. Reedy, [**50**]). *Let $\mathcal{M}$ be a model category. If we have a commutative diagram in $\mathcal{M}$*

$$\begin{tikzcd}
A \arrow[rr] \arrow[dd] \arrow[dr] & & B \arrow[dd] \arrow[dr, "f_B"] & \\
& A' \arrow[rr] \arrow[dd] & & B' \arrow[dd] \\
C \arrow[rr] \arrow[dr] & & D \arrow[dr, "f_D"] & \\
& C' \arrow[rr] & & D'
\end{tikzcd}$$

in which the front and back squares are pushouts and both f_B and $C \amalg_A A' \to C'$ are cofibrations, then f_D is a cofibration.

PROOF. It is sufficient to show that f_D has the left lifting property with respect to all trivial fibrations (see Proposition 8.2.3). If we have a commutative diagram

$$\begin{tikzcd}
D \arrow[r] \arrow[d, "f_D"'] & X \arrow[d, "p"] \\
D' \arrow[r] & Y
\end{tikzcd}$$

in which p is a trivial fibration, then we also have a similar diagram with f_B in place of f_D. Since f_B is a cofibration, there is a map $h_B : B' \to X$ making both triangles commute. Composing h_B with $A' \to B'$ yields a map $h_A\colon A' \to X$ that also makes both triangles commute. This induces a map $C \amalg_A A' \to X$. Since $C \amalg_A A' \to C'$ is a cofibration, there is a map $C' \to X$ making everything commute, and so there is an induced map $D' = C' \amalg_{A'} B' \to X$ making both triangles commute, and the proof is complete. □

8.3. Homotopy

8.3.1. Left homotopy, right homotopy, and homotopy.

DEFINITION 8.3.2. Let $\mathcal{M}$ be a model category, and let $f, g\colon X \to Y$ be maps in $\mathcal{M}$.

1. A *cylinder object* for X is a factorization

$$X \amalg X \xrightarrow{i_0 \amalg i_1} \mathrm{Cyl}(X) \xrightarrow{p} X$$

of the fold map $1_X \amalg 1_X : X \amalg X \to X$ such that $i_0 \amalg i_1$ is a cofibration and p is a weak equivalence.

2. A *left homotopy* from f to g consists of a cylinder object $X \amalg X \xrightarrow{i_0 \amalg i_1} \mathrm{Cyl}(X) \xrightarrow{p} X$ for X and a map $H\colon \mathrm{Cyl}(X) \to Y$ such that $Hi_0 = f$ and $Hi_1 = g$. If there exists a left homotopy from f to g, then we say that f is *left homotopic* to g (written $f \overset{l}{\simeq} g$).
3. A *path object* for Y is a factorization
$$Y \xrightarrow{s} \mathrm{Path}(Y) \xrightarrow{p_0 \times p_1} Y \times Y$$
of the diagonal map $1_Y \times 1_Y : Y \to Y \times Y$ such that s is a weak equivalence and $p_0 \times p_1$ is a fibration.
4. A *right homotopy* from f to g consists of a path object $Y \xrightarrow{s} \mathrm{Path}(Y) \xrightarrow{p_0 \times p_1} Y \times Y$ for Y and a map $H\colon X \to \mathrm{Path}(Y)$ such that $p_0 H = f$ and $p_1 H = g$. If there exists a right homotopy from f to g, then we say that f is right homotopic to g (written $f \overset{r}{\simeq} g$).
5. If f is both left homotopic and right homotopic to g, then we say that f is *homotopic* to g (written $f \simeq g$).

LEMMA 8.3.3. *Let $\mathcal{M}$ be a model category.*

1. *Every object X of $\mathcal{M}$ has a cylinder object $X \amalg X \xrightarrow{i_0 \amalg i_1} \mathrm{Cyl}(X) \xrightarrow{p} X$ in which p is a trivial fibration.*
2. *Every object X of $\mathcal{M}$ has a path object $X \xrightarrow{s} \mathrm{Path}(X) \xrightarrow{p_0 \times p_1} X \times X$ in which s is a trivial cofibration.*

PROOF. Factor the map $1_X \amalg 1_X : X \amalg X \to X$ into a cofibration followed by a trivial fibration, and factor the map $1_X \times 1_X : X \to X \times X$ into a trivial cofibration followed by a fibration. □

PROPOSITION 8.3.4. *Let $\mathcal{M}$ be a model category, and let $f, g\colon X \to Y$ be maps in $\mathcal{M}$.*

1. *The maps f and g are left homotopic if and only if there is a factorization $X \amalg X \xrightarrow{i_0 \amalg i_1} C \xrightarrow{p} X$ of the fold map $1_X \amalg 1_X : X \amalg X \to X$ such that p is a weak equivalence and a map $H\colon C \to Y$ such that $Hi_0 = f$ and $Hi_1 = g$.*
2. *The maps f and g are right homotopic if and only if there is a factorization $Y \xrightarrow{s} P \xrightarrow{p_0 \times p_1} Y \times Y$ of the diagonal map $1_Y \times 1_Y : Y \to Y \times Y$ such that s is a weak equivalence and a map $H\colon X \to P$ such that $p_0 H = f$ and $p_1 H = g$.*

PROOF. We will prove part 1; the proof of part 2 is dual.

The necessity of the condition follows directly from the definition. Conversely, assume the condition is satisfied. If we factor $i_0 \amalg i_1$ as $X \amalg X \xrightarrow{i'_0 \amalg i'_1} C' \xrightarrow{q} C$ where $i'_0 \amalg i'_1$ is a cofibration and q is a trivial fibration, then $X \amalg X \xrightarrow{i'_0 \amalg i'_1} C' \xrightarrow{pq} X$ is a cylinder object for X and $Hq : C' \to Y$ is a left homotopy from f to g. □

LEMMA 8.3.5. *Let $\mathcal{M}$ be a model category, and let X be an object of $\mathcal{M}$.*

1. *If X is cofibrant, then the injections $i_0, i_1 : X \to X \amalg X$ are cofibrations.*
2. *If X is fibrant, then the projections $p_0, p_1 : X \times X \to X$ are fibrations.*

PROOF. We will prove part 1; the proof of part 2 is similar.

Since the diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "i_1"] \\ X \arrow[r, "i_0"'] & X \amalg X \end{tikzcd}$$

(where $\emptyset$ is the initial object of $\mathcal{M}$) is a pushout, the lemma follows from Proposition 8.2.6. $\square$

LEMMA 8.3.6. *Let $\mathcal{M}$ be a model category.*

1. *If $X \amalg X \xrightarrow{i_0 \amalg i_1} \mathrm{Cyl}(X) \xrightarrow{p} X$ is a cylinder object for X, then the injections $i_0, i_1 \colon X \to \mathrm{Cyl}(X)$ are weak equivalences. If X is cofibrant, then they are trivial cofibrations.*
2. *If $Y \xrightarrow{s} \mathrm{Path}(Y) \xrightarrow{p_0 \times p_1} Y \times Y$ is a path object for Y, then the projections $p_0, p_1 \colon \mathrm{Path}(Y) \to Y$ are weak equivalences. If Y is fibrant, then they are trivial fibrations.*

PROOF. This follows from the "two out of three" axiom for weak equivalences (see Definition 8.1.2) and Lemma 8.3.5. $\square$

PROPOSITION 8.3.7 (Homotopy extension property of cofibrations). *Let $\mathcal{M}$ be a model category, let X be fibrant, and let $k \colon A \to B$ be a cofibration. If $f \colon A \to X$ is a map, $\tilde{f} \colon B \to X$ is an extension of f, $X \xrightarrow{s} \mathrm{Path}(X) \xrightarrow{p_0 \times p_1} X \amalg X$ is a path object for X, and $H \colon A \to \mathrm{Path}(X)$ is a right homotopy of f (i.e., a map H such that $p_0 H = f$), then there is a map $\widetilde{H} \colon B \to \mathrm{Path}(X)$ such that $p_0 \widetilde{H} = \tilde{f}$ and $\widetilde{H} k = H$.*

PROOF. We have the solid arrow diagram

$$\begin{tikzcd} A \arrow[r, "H"] \arrow[d, "k"'] & \mathrm{Path}(X) \arrow[d, "p_0"] \\ B \arrow[r, "\tilde{f}"'] \arrow[ur, dotted, "\widetilde{H}"] & X \end{tikzcd}$$

and Lemma 8.3.6 implies that p_0 is a trivial fibration. $\square$

PROPOSITION 8.3.8 (Homotopy lifting property of fibrations). *Let $\mathcal{M}$ be a model category, let A be cofibrant, and let $k \colon X \to Y$ be a fibration. If $f \colon A \to Y$ is a map, $\tilde{f} \colon A \to X$ is a lift of f, $A \amalg A \xrightarrow{i_0 \amalg i_1} \mathrm{Cyl}(A) \xrightarrow{p} A$ is a cylinder object for A, and $H \colon \mathrm{Cyl}(A) \to Y$ is a left homotopy of f (i.e., a map H such that $H i_0 = f$), then there is a map $\widetilde{H} \colon \mathrm{Cyl}(A) \to X$ such that $\widetilde{H} i_0 = \tilde{f}$ and $k \widetilde{H} = H$.*

PROOF. We have the solid arrow diagram

$$\begin{tikzcd} A \arrow[r, "\tilde{f}"] \arrow[d, "i_0"'] & X \arrow[d, "k"] \\ \mathrm{Cyl}(A) \arrow[r, "H"'] \arrow[ur, dotted, "\widetilde{H}"] & Y \end{tikzcd}$$

and Lemma 8.3.6 implies that i_0 is a trivial cofibration. $\square$

8.3.9. Homotopy as an equivalence relation.

DEFINITION 8.3.10. Let $\mathcal{M}$ be a model category and let X and Y be objects in $\mathcal{M}$.

1. If $X \amalg X \xrightarrow{i_0 \amalg i_1} \mathrm{Cyl}(X) \xrightarrow{p} X$ is a cylinder object for X and $H\colon \mathrm{Cyl}(X) \to Y$ is a left homotopy from $f\colon X \to Y$ to $g\colon X \to Y$, then the *inverse* of H is the left homotopy $H^{-1}\colon \mathrm{Cyl}(X)^{-1} \to Y$ from g to f where $X \amalg X \xrightarrow{i_0^{-1} \amalg i_1^{-1}} \mathrm{Cyl}(X)^{-1} \xrightarrow{p^{-1}} X$ is the cylinder object for X defined by $\mathrm{Cyl}(X)^{-1} = \mathrm{Cyl}(X)$, $i_0^{-1} = i_1$, $i_1^{-1} = i_0$, and $p^{-1} = p$, and the map H^{-1} equals the map H.
2. If $Y \xrightarrow{s} \mathrm{Path}(Y) \xrightarrow{p_0 \times p_1} Y \times Y$ is a path object for Y and $H\colon X \to \mathrm{Path}(Y)$ is a right homotopy from $f\colon X \to Y$ to $g\colon X \to Y$, then the *inverse* of H is the right homotopy $H^{-1}\colon X \to \mathrm{Path}(Y)^{-1}$ from g to f where $Y \xrightarrow{s^{-1}} \mathrm{Path}(Y)^{-1} \xrightarrow{p_0^{-1} \times p_1^{-1}} Y \times Y$ is the path object for Y defined by $\mathrm{Path}(Y)^{-1} = \mathrm{Path}(Y)$, $p_0^{-1} = p_1$, $p_1^{-1} = p_0$, and $s^{-1} = s$, and the map H^{-1} equals the map H.

LEMMA 8.3.11. *Let $\mathcal{M}$ be a model category and let X and Y be objects in $\mathcal{M}$.*

1. *If X is cofibrant and $X \amalg X \xrightarrow{i_0 \amalg i_1} \mathrm{Cyl}(X) \xrightarrow{p} X$ and $X \amalg X \xrightarrow{i'_0 \amalg i'_1} \mathrm{Cyl}(X)' \xrightarrow{p'} X$ are cylinder objects for X, then there is a cylinder object $X \amalg X \xrightarrow{i''_0 \amalg i''_1} \mathrm{Cyl}(X)'' \xrightarrow{p''} X$ for X in which*
   (a) *$\mathrm{Cyl}(X)''$ is the pushout of the diagram $\mathrm{Cyl}(X) \xleftarrow{i_1} X \xrightarrow{i'_0} \mathrm{Cyl}(X)'$,*
   (b) *$i''_0\colon X \to \mathrm{Cyl}(X)''$ is the composition $X \xrightarrow{i_0} \mathrm{Cyl}(X) \to \mathrm{Cyl}(X)''$, and*
   (c) *$i''_1\colon X \to \mathrm{Cyl}(X)''$ is the composition $X \xrightarrow{i'_1} \mathrm{Cyl}(X)' \to \mathrm{Cyl}(X)''$.*
2. *If Y is fibrant and $Y \xrightarrow{s} \mathrm{Path}(Y) \xrightarrow{p_0 \times p_1} Y \times Y$ and $Y \xrightarrow{s'} \mathrm{Path}(Y)' \xrightarrow{p'_0 \times p'_1} Y \times Y$ are path objects for Y, then there is a path object $Y \xrightarrow{s''} \mathrm{Path}(Y)'' \xrightarrow{p''_0 \times p''_1} Y \times Y$ for Y in which*
   (a) *$\mathrm{Path}(Y)''$ is the pullback of the diagram $\mathrm{Path}(Y) \xrightarrow{p_1} Y \xleftarrow{p'_0} \mathrm{Path}(Y)'$,*
   (b) *$p''_0\colon \mathrm{Path}(Y)'' \to Y$ is the composition $\mathrm{Path}(Y)'' \to \mathrm{Path}(Y) \xrightarrow{p_0} Y$, and*
   (c) *$p''_1\colon \mathrm{Path}(Y)'' \to Y$ is the composition $\mathrm{Path}(Y)'' \to \mathrm{Path}(Y)' \xrightarrow{p'_1} Y$.*

PROOF. We will prove part 1; the proof of part 2 is dual.

We have the commutative diagram

$$\begin{tikzcd}
& & X \arrow[d, "i_0"] & \\
& X \arrow[r, "i_1"] \arrow[d, "i_0'"'] & \mathrm{Cyl}(X) \arrow[d] \arrow[ddr, bend left, "p"] & \\
X \arrow[r, "i_1'"'] & \mathrm{Cyl}(X)' \arrow[r] \arrow[drr, bend right, "p'"'] & \mathrm{Cyl}(X)'' \arrow[dr, "p''"] & \\
& & & X
\end{tikzcd}$$

Lemma 8.3.6 and Proposition 8.2.6 imply that i_0'' and i_1'' are trivial cofibrations. Together with the "two out of three" property of weak equivalences (see Definition 8.1.2), this implies that p'' is a weak equivalence.

It remains only to show that the map $X \amalg X \xrightarrow{i_0'' \amalg i_1''} \mathrm{Cyl}(X)''$ is a cofibration. This map equals the composition

$$X \amalg X \xrightarrow{i_0 \amalg 1_X} \mathrm{Cyl}(X) \amalg X \xrightarrow{j_0 \amalg j_1 i_1'} \mathrm{Cyl}(X)''.$$

The first of these is the pushout of $i_0 \colon X \to \mathrm{Cyl}(X)$ along the first inclusion $X \to X \amalg X$, and so Lemma 8.3.6 and Proposition 8.2.6 imply that it is a trivial cofibration. The second is the pushout of $i_0' \amalg i_1'$ along $i_1 \amalg 1_X : X \amalg X \to \mathrm{Cyl}(X) \amalg X$, and so Proposition 8.2.6 implies that it is a cofibration. Proposition 8.2.4 now implies that $i_0'' \amalg i_1''$ is a cofibration. $\square$

Definition 8.3.12. Let $\mathcal{M}$ be a model category and let X and Y be objects in $\mathcal{M}$.

1. If X is cofibrant, $X \amalg X \xrightarrow{i_0 \amalg i_1} \mathrm{Cyl}(X) \xrightarrow{p} X$ and $X \amalg X \xrightarrow{i_0' \amalg i_1'} \mathrm{Cyl}(X)' \xrightarrow{p'} X$ are cylinder objects for X, $H \colon \mathrm{Cyl}(X) \to Y$ is a left homotopy from $f \colon X \to Y$ to $g \colon X \to Y$, and $H' \colon \mathrm{Cyl}(X)' \to Y$ is a left homotopy from g to $h \colon X \to Y$, then the *composition* of the left homotopies H and H' is the left homotopy $H \cdot H' \colon \mathrm{Cyl}(X)'' \to Y$ from f to h (where $\mathrm{Cyl}(X)''$ is as in Lemma 8.3.11) defined by H and H'.
2. If Y is fibrant, $Y \xrightarrow{s} \mathrm{Path}(Y) \xrightarrow{p_0 \times p_1} Y \times Y$ and $Y \xrightarrow{s'} \mathrm{Path}(Y)' \xrightarrow{p_0' \times p_1'} Y \times Y$ are path objects for Y, $H \colon X \to \mathrm{Path}(Y)$ is a right homotopy from $f \colon X \to Y$ to $g \colon X \to Y$, and $H' \colon X \to \mathrm{Path}(Y)'$ is a right homotopy from g to $h \colon X \to Y$, then the *composition* of the right homotopies H and H' is the right homotopy $H \cdot H' \colon X \to \mathrm{Path}(Y)''$ from f to h (where $\mathrm{Path}(Y)''$ is as in Lemma 8.3.11) defined by H and H'.

Proposition 8.3.13. *Let $\mathcal{M}$ be a model category, and let X and Y be objects in $\mathcal{M}$.*

1. *If X is cofibrant, then left homotopy is an equivalence relation on the set of maps from X to Y.*
2. *If Y is fibrant, then right homotopy is an equivalence relation on the set of maps from X to Y.*

Proof. We will prove part 1; the proof of part 2 is dual.

Since there is a cylinder object for X in which $\mathrm{Cyl}(X) = X$, left homotopy is reflexive. The inverse of a left homotopy (see Definition 8.3.10) implies that left homotopy is symmetric. Finally, the composition of left homotopies (see Definition 8.3.12) implies that left homotopy is transitive. $\square$

8.3.14. Homotopy classes of maps.

Notation 8.3.15. Let $\mathcal{M}$ be a model category, and let X and Y be objects of $\mathcal{M}$.

1. If X is cofibrant, we let $\pi^l(X, Y)$ denote the set of left homotopy classes of maps from X to Y.
2. If Y is fibrant, we let $\pi^r(X, Y)$ denote the set of right homotopy classes of maps from X to Y.
3. If X is cofibrant and Y is fibrant, we let $\pi(X, Y)$ denote the set of homotopy classes of maps from X to Y.

Proposition 8.3.16. *Let $\mathcal{M}$ be a model category, and let $f, g\colon X \to Y$ be maps in $\mathcal{M}$.*

1. *If X is cofibrant, f is left homotopic to g, and $Y \xrightarrow{s} \mathrm{Path}(Y) \xrightarrow{p_0 \times p_1} Y \times Y$ is a path object for Y, then there is a right homotopy $H\colon X \to \mathrm{Path}(Y)$ from f to g.*
2. *If Y is fibrant, f is right homotopic to g, and $X \amalg X \xrightarrow{i_0 \amalg i_1} \mathrm{Cyl}(X) \xrightarrow{p} X$ is a cylinder object for X, then there is a left homotopy $H\colon \mathrm{Cyl}(X) \to Y$ from f to g.*

Proof. We will prove part 1; the proof of part 2 is dual.

Since f is left homotopic to g, there is a cylinder object $X \amalg X \xrightarrow{i_0 \amalg i_1} \mathrm{Cyl}(X) \xrightarrow{p} X$ for X and a left homotopy $G\colon \mathrm{Cyl}(X) \to Y$ from f to g. Thus, we have the solid arrow diagram

$$\begin{tikzcd} X \arrow[r, "sf"] \arrow[d, "i_0"'] & \mathrm{Path}(Y) \arrow[d, "(p_0, p_1)"] \\ \mathrm{Cyl}(X) \arrow[r, "(fp, G)"'] \arrow[ur, dotted, "h"] & Y \times Y \end{tikzcd}$$

in which (p_0, p_1) is a fibration. Since X is cofibrant, Lemma 8.3.6 implies that i_0 is a trivial cofibration, and so the dotted arrow h exists. If we let $H = hi_1$, then H is the right homotopy we require. $\square$

Proposition 8.3.17. *Let $\mathcal{M}$ be a model category, and let $f, g\colon X \to Y$ be maps in $\mathcal{M}$.*

1. *If X is cofibrant and $f \overset{l}{\simeq} g$, then $f \overset{r}{\simeq} g$.*
2. *If Y is fibrant and $f \overset{r}{\simeq} g$, then $f \overset{l}{\simeq} g$.*

Proof. This follows from Lemma 8.3.3 and Proposition 8.3.16. $\square$

Proposition 8.3.18. *Let $\mathcal{M}$ be a model category. If X is cofibrant and Y is fibrant, then the left homotopy, right homotopy, and homotopy relations coincide and are equivalence relations on the set of maps from X to Y.*

PROOF. This follows from Proposition 8.3.17 and Proposition 8.3.13. □

LEMMA 8.3.19. *Let $\mathcal{M}$ be a model category, and let $f, g\colon X \to Y$ be maps in $\mathcal{M}$.*

1. *If $f \overset{l}{\simeq} g$ and $h\colon Y \to Z$ is a map, them $hf \overset{l}{\simeq} hg$.*
2. *If $f \overset{r}{\simeq} g$ and $k\colon W \to X$, then $fk \overset{r}{\simeq} gk$.*

PROOF. If $X \amalg X \to \mathrm{Cyl}(X) \to X$ is a cylinder object for X and $F\colon \mathrm{Cyl}(X) \to Y$ is a left homotopy from f to g, then hF is a left homotopy from hf to hg. The proof of part 2 is dual. □

PROPOSITION 8.3.20. *Let $\mathcal{M}$ be a model category.*

1. *If $f, g\colon X \to Y$ are left homotopic and Y is fibrant, then there is a cylinder object $X \amalg X \to \mathrm{Cyl}(X) \xrightarrow{p} X$ in which p is a trivial fibration and a left homotopy $H\colon \mathrm{Cyl}(X) \to Y$ from f to g.*
2. *If $f, g\colon X \to Y$ are right homotopic and X is cofibrant, then there is a path object $Y \xrightarrow{s} \mathrm{Path}(Y) \to Y \times Y$ in which s is a trivial cofibration and aright homotopy $H\colon X \to \mathrm{Path}(Y)$ from f to g.*

PROOF. We will prove part 1; the proof of part 2 is dual.

If $X \amalg X \to \mathrm{Cyl}(X)' \xrightarrow{p'} X$ is a cylinder object for X such that there is a left homotopy $H'\colon \mathrm{Cyl}(X)' \to Y$ from f to g, then we factor p as $\mathrm{Cyl}(X)' \xrightarrow{j} \mathrm{Cyl}(X) \xrightarrow{p} X$ where j is a cofibration and p is a trivial fibration. The "two out of three" axiom for weak equivalences (see Definition 8.1.2) implies that j is a trivial cofibration, and so the dotted arrow exists in the diagram

$$\begin{tikzcd}
\mathrm{Cyl}(X)' \arrow[r, "H'"] \arrow[d, "j"'] & Y \arrow[d] \\
\mathrm{Cyl}(X) \arrow[r] \arrow[ur, dotted, "H"] & *
\end{tikzcd}$$

which constructs our left homotopy H. □

PROPOSITION 8.3.21. *Let $\mathcal{M}$ be a model category.*

1. *If A is cofibrant and $p\colon X \to Y$ is a trivial fibration, then p induces an isomorphism of the sets of left homotopy classes of maps $p_*\colon \pi^l(A, X) \to \pi^l(A, Y)$.*
2. *If X is fibrant and $i\colon A \to B$ is a trivial cofibration, then i induces an isomorphism of the sets of right homotopy classes of maps $i_*\colon \pi^r(B, X) \to \pi^r(A, X)$.*

PROOF. We will prove part 1; the proof of part 2 is dual.

Lemma 8.3.19 implies that p_* is well defined. If $g\colon A \to Y$ is a map and $\emptyset$ is the initial object of $\mathcal{M}$, then axiom M4 (see Definition 8.1.2) implies that the dotted arrow exists in the diagram

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
A \arrow[r, "g"'] \arrow[ur, dotted, "f"] & Y,
\end{tikzcd}$$

and so p_* is surjective. To see that p_* is injective, let $f, g\colon A \to X$ be maps such that $pf \overset{l}{\simeq} pg$. There is then a cylinder object $A \amalg A \to \mathrm{Cyl}(A) \to A$ for A and a left homotopy $F\colon \mathrm{Cyl}(A) \to Y$ from pf to pg, and so we have the solid arrow diagram

$$\begin{tikzcd} A \amalg A \arrow[r, "f \amalg g"] \arrow[d] & X \arrow[d, "p"] \\ \mathrm{Cyl}(A) \arrow[r, "F"'] \arrow[ur, dotted, "G"] & Y. \end{tikzcd}$$

Axiom M4 implies that the dotted arrow G exists, and G is a left homotopy from f to g. □

PROPOSITION 8.3.22. *Let $\mathcal{M}$ be a model category, let X, Y, and Z be cofibrant-fibrant objects of $\mathcal{M}$, and let $f, g\colon X \to Y$ and $h, k\colon Y \to Z$ be maps. If $f \simeq g$ and $h \simeq k$, them $hf \simeq kg$, and so composition is well defined on homotopy classes of maps between cofibrant-fibrant objects.*

PROOF. This follows from Lemma 8.3.19. □

8.3.23. The classical homotopy category.

PROPOSITION 8.3.24. *If $\mathcal{M}$ is a model category, then there is a category whose objects are the cofibrant-fibrant objects in $\mathcal{M}$, whose maps are homotopy classes of maps in $\mathcal{M}$, and whose composition of maps is induced by composition of maps in $\mathcal{M}$.*

PROOF. This follows from Proposition 8.3.22. □

DEFINITION 8.3.25. If $\mathcal{M}$ is a model category, then we follow D. M. Kan and define the *classical homotopy category* $\pi\mathcal{M}_{\mathrm{cf}}$ of $\mathcal{M}$ to be the category with objects the cofibrant-fibrant objects of $\mathcal{M}$, and with morphisms from X to Y the homotopy classes of maps from X to Y (see Proposition 8.3.24).

PROPOSITION 8.3.26. *Let $\mathcal{M}$ be a model category. If $f\colon X \to Y$ is a weak equivalence between cofibrant-fibrant objects, then it is a homotopy equivalence.*

PROOF. If we factor f into a cofibration followed by a trivial fibration to obtain $X \xrightarrow{p} W \xrightarrow{q} Y$, then W is also cofibrant-fibrant, and the "two out of three" axiom (see Definition 8.1.2) implies that p is also a weak equivalence. Since a composition of homotopy equivalences between cofibrant-fibrant objects is a homotopy equivalence (see Proposition 8.3.22), it is sufficient to show that a trivial cofibration or trivial fibration between cofibrant-fibrant objects is a homotopy equivalence. We will show this for the trivial cofibration p; the proof for the trivial fibration q is dual.

We have the solid arrow diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "p"'] & X \arrow[d] \\ W \arrow[r] \arrow[ur, dotted, "r"] & * \end{tikzcd}$$

(in which $*$ denotes the terminal object), and so there exists a dotted arrow r such that $rp = 1_X$. Proposition 8.3.21 implies that p induces an isomorphism $p^*\colon \pi^r(W, W) \approx \pi^r(X, W)$, and, since $p^*[pr] = [prp] = [p][rp] = [p][1_X] = [p] =$

$p^*[1_W]$, this implies that $pr \overset{r}{\simeq} 1_W$. Thus, r is a homotopy inverse for p (see Proposition 8.3.18), and so p is a homotopy equivalence. $\square$

PROPOSITION 8.3.27. *Let $\mathcal{M}$ be a model category, let W, X, Y, and Z be cofibrant-fibrant objects, and let $f, g\colon X \to Y$ be a pair of maps.*

1. *If there is a weak equivalence $h\colon Y \to Z$ such that $hf \simeq hg$, then $f \simeq g$.*
2. *If there is a weak equivalence $k\colon W \to X$ such that $fk \simeq gk$, then $f \simeq g$.*

PROOF. We will prove part 1; the proof of part 2 is similar.

Proposition 8.3.26 implies that there is a map $\tilde{h}\colon Z \to Y$ such that $\tilde{h}h \simeq 1_Y$. Thus, $f \simeq 1_Y f \simeq \tilde{h}hf \simeq \tilde{h}hg \simeq 1_Y g \simeq g$. $\square$

PROPOSITION 8.3.28. *Let $\mathcal{M}$ be a model category. If X and Y are cofibrant-fibrant objects in $\mathcal{M}$, then a map $g\colon X \to Y$ is a homotopy equivalence if either of the following two conditions is satisfied:*

1. *The map g induces isomorphisms of the sets of homotopy classes of maps $g_*\colon \pi(X,X) \approx \pi(X,Y)$ and $g_*\colon \pi(Y,X) \approx \pi(Y,Y)$.*
2. *The map g induces isomorphisms of the sets of homotopy classes of maps $g^*\colon \pi(Y,X) \approx \pi(X,X)$ and $g^*\colon \pi(Y,Y) \approx \pi(X,Y)$.*

PROOF. We will prove this using condition 1; the proof using condition 2 is similar.

The isomorphism $g_*\colon \pi(Y,X) \approx \pi(Y,Y)$ implies that there is a map $h\colon Y \to X$ such that $gh \simeq 1_Y$. Proposition 8.3.22 and the isomorphism $g_*\colon \pi(X,X) \approx \pi(X,Y)$ now imply that h induces an isomorphism $h_*\colon \pi(X,Y) \approx \pi(X,X)$, and so there is a map $k\colon X \to Y$ such that $hk \simeq 1_X$. Thus, h is a homotopy equivalence and g is its inverse, and so g is a homotopy equivalence as well. $\square$

8.4. Relative homotopy and fiberwise homotopy

THEOREM 8.4.1. *Let $\mathcal{M}$ be a model category.*

1. *If W is an object in $\mathcal{M}$, then the category $(W\downarrow\mathcal{M})$ of objects of $\mathcal{M}$ under W is a model category in which a map is a weak equivalence, fibration, or cofibration if it is one in $\mathcal{M}$.*
2. *If W is an object in $\mathcal{M}$, then the category $(\mathcal{M}\downarrow W)$ of objects of $\mathcal{M}$ over W is a model category in which a map is a weak equivalence, fibration, or cofibration if it is one in $\mathcal{M}$.*

PROOF. This follows directly from the definitions. $\square$

LEMMA 8.4.2. *If $\mathcal{C}$ is a category and $g\colon X \to Y$ is a map in $\mathcal{C}$, then the functor $g_*\colon (X\downarrow\mathcal{C}) \to (Y\downarrow\mathcal{C})$ that takes the element $X \to Z$ of $(X\downarrow\mathcal{C})$ to its pushout along g is left adjoint to the functor $g^*\colon (Y\downarrow\mathcal{C}) \to (X\downarrow\mathcal{M})$ that takes the element $Y \to W$ of $(Y\downarrow\mathcal{C})$ to its composition with g.*

PROOF. This follows directly from the universal mapping property of the pushout. $\square$

DEFINITION 8.4.3. Let $\mathcal{M}$ be a model category, and let A be an object in $\mathcal{M}$.

1. If $A \to X$ and $A \to Y$ are objects of the category $(A\downarrow\mathcal{M})$ of objects of $\mathcal{M}$ under A, then maps $f, g\colon X \to Y$ in $(A\downarrow\mathcal{M})$ will be called *left homotopic under A*, *right homotopic under A*, or *homotopic under A* if they are, respectively, left homotopic, right homotopic, or homotopic as maps in $(A\downarrow\mathcal{M})$.

A map will be called a *homotopy equivalence under* A if it is a homotopy equivalence in the category $(A \downarrow \mathcal{M})$.

2. If $X \to A$ and $Y \to A$ are objects of the category $(\mathcal{M} \downarrow A)$ of objects of $\mathcal{M}$ over A, then maps $f, g\colon X \to Y$ will be called *left homotopic over* A, *right homotopic over* A, or *homotopic over* A if they are, respectively, left homotopic, right homotopic, or homotopic as maps in $(\mathcal{M} \downarrow A)$. A map will be called a *homotopy equivalence over* A if it is a homotopy equivalence in the category $(\mathcal{M} \downarrow A)$.

PROPOSITION 8.4.4. *Let $\mathcal{M}$ be a model category, and let A be an object in $\mathcal{M}$.*

1. *If maps are left homotopic, right homotopic, or homotopic under A, then they are, respectively, left homotopic, right homotopic, or homotopic.*
2. *If maps are left homotopic, right homotopic, or homotopic over A, then they are, respectively, left homotopic, right homotopic, or homotopic.*

PROOF. This follows from Proposition 8.3.4. □

COROLLARY 8.4.5. *Let $\mathcal{M}$ be a model category, and let A be an object in $\mathcal{M}$. If a map is a homotopy equivalence under A or a homotopy equivalence over A, then it is a homotopy equivalence in $\mathcal{M}$.*

PROOF. This follows from Proposition 8.4.4. □

DEFINITION 8.4.6. If $\mathcal{M}$ is a model category, then a map $i\colon A \to B$ will be called *the inclusion of a deformation retract* (and A will be called a *deformation retract* of B) if there is a map $r\colon B \to A$ such that $ri = 1_A$ and $ir \simeq 1_B$. A deformation retract will be called a *strong deformation retract* if $ir \simeq 1_B$ under A.

PROPOSITION 8.4.7. *Let $\mathcal{M}$ be a model category.*

1. *If $i\colon A \to B$ is a trivial cofibration of fibrant objects, then A is a strong deformation retract of B (see Definition 8.4.6), i.e., there is a map $r\colon B \to A$ such that $ri = 1_A$ and $ir \simeq 1_B$ under A.*
2. *If $p\colon X \to Y$ s a trivial fibration of cofibrant objects, then there is a map $s\colon Y \to X$ such that $ps = 1_Y$ and $sp \simeq 1_X$ over Y.*

PROOF. We will prove part 1; the proof of part 2 is dual.

We have the solid arrow diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "i"'] & A \arrow[d] \\ B \arrow[r] \arrow[ur, dotted, "r"] & * \end{tikzcd}$$

in $(A \downarrow \mathcal{M})$ (see Theorem 8.4.1) in which i is a trivial cofibration and the map on the right is a fibration. Thus, there exists a map $r\colon B \to A$ in $(A \downarrow \mathcal{M})$ such that $ri = 1_A$. Since $i^*(1_B) = i = iri = i^*(ri)$, Proposition 8.3.21 implies that $1_B \overset{r}{\simeq} ri$ in $(A \downarrow \mathcal{M})$. Since both A and B are both cofibrant-fibrant in $(A \downarrow \mathcal{M})$, Proposition 8.3.18 implies that $1_B \simeq ri$ in $(A \downarrow \mathcal{M})$. □

8.4.8. Homotopy uniqueness of lifts.

PROPOSITION 8.4.9. *Let $\mathcal{M}$ be a model category, and let the solid arrow diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted, "h_1", shift left] \arrow[ur, dotted, "h_2"', shift right] & Y \end{tikzcd}$$

be such that either

1. *i is a cofibration and p is a trivial fibration, or*
2. *i is a trivial cofibration and p is a fibration.*

If h_1 and h_2 are maps each of which makes both triangles commute, then $h_1 \simeq h_2$ as maps in $(A\downarrow\mathcal{M}\downarrow Y)$, the category of objects of $\mathcal{M}$ under A and over Y.

PROOF. We will assume that condition 1 holds; the proof in the case that condition 2 holds is similar.

Factor the map $B \amalg_A B \to B$ as $B \amalg_A B \xrightarrow{j} C \xrightarrow{r} B$ where j is a cofibration and r is a trivial fibration. We now have the solid arrow diagram

$$\begin{tikzcd} B \amalg_A B \arrow[rr, "h_1 \amalg h_2"] \arrow[d, "j"'] & & X \arrow[d, "p"] \\ C \arrow[r] \arrow[urr, dotted, "H"] & B \arrow[r] & Y \end{tikzcd}$$

in which j is a cofibration and p is a trivial fibration, and so there exists a dotted arrow H making both triangles commute. In the category $(A\downarrow\mathcal{M}\downarrow Y)$ of objects of $\mathcal{M}$ under A and over Y (see Theorem 8.4.1), $B \amalg_A B \to C \to B$ is a cylinder object for B (see Definition 8.3.2) and H is a left homotopy from h_1 to h_2. Since B is cofibrant and X is fibrant in $(A\downarrow\mathcal{M}\downarrow Y)$, Proposition 8.3.17 implies that h_1 is also right homotopic to h_2, and so h_1 is homotopic to h_2 in $(A\downarrow\mathcal{M}\downarrow Y)$. □

PROPOSITION 8.4.10. *Let $\mathcal{M}$ be a model category. If the solid arrow diagram*

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "q"'] \arrow[ur, dotted, "h"] & Y \end{tikzcd}$$

is such that either

1. *i and j are cofibrations and p and q are trivial fibrations, or*
2. *i and j are trivial cofibrations and p and q are fibrations,*

then there exists a map $h\colon B \to X$ making both triangles commute, unique up to homotopy in $(A\downarrow\mathcal{M}\downarrow Y)$, and any such map is a homotopy equivalence in $(A\downarrow\mathcal{M}\downarrow Y)$.

PROOF. This follows from Proposition 8.4.9. □

8.5. Weak equivalences

LEMMA 8.5.1 (K. S. Brown, [**16**]). *Let $\mathcal{M}$ be a model category.*

1. *If $g\colon X \to Y$ is a weak equivalence between cofibrant objects in $\mathcal{M}$, then g can be factored as $g = ji$ where i is a trivial cofibration and j is a trivial fibration that has a right inverse that is a trivial cofibration.*

2. *If $g\colon X \to Y$ is a weak equivalence between fibrant objects in $\mathcal{M}$, then g can be factored as $g = ji$ where i is a trivial cofibration that has a left inverse that is a trivial fibration and j is a trivial fibration.*

PROOF. We will prove part 1; the proof of part 2 is similar.

Since X and Y are cofibrant, both of the injections $X \to X \amalg Y$ and $Y \to X \amalg Y$ are cofibrations. If we factor the map $g \amalg 1_Y : X \amalg Y \to Y$ as

$$X \amalg Y \xrightarrow{k} Z \xrightarrow{j} Y$$

where k is a cofibration and j is a trivial fibration, then both compositions $X \to X \amalg Y \to Z$ and $Y \to X \amalg Y \to Z$ are cofibrations. Since g and j are weak equivalences, axiom M2 (see Definition 8.1.2) implies that the cofibration $X \to Z$ is a weak equivalence, and the composition of cofibrations $Y \to X \amalg Y \to Z$ is a right inverse to the trivial fibration j. □

COROLLARY 8.5.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories, and let $\mathrm{F}\colon \mathcal{M} \to \mathcal{N}$ be a functor.*

1. *If F takes trivial cofibrations between cofibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$, then F takes all weak equivalences between cofibrant objects to weak equivalences in $\mathcal{N}$.*
2. *If F takes trivial fibrations between fibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$, then F takes all weak equivalences between fibrant objects to weak equivalences in $\mathcal{N}$.*

PROOF. This follows from Lemma 8.5.1. □

COROLLARY 8.5.3. *Let $\mathcal{M}$ be a model category, let $\mathcal{C}$ be a category, and let $\mathrm{F}\colon \mathcal{M} \to \mathcal{C}$ be a functor.*

1. *If F takes trivial cofibrations between cofibrant objects in $\mathcal{M}$ to isomorphisms in $\mathcal{C}$, then F takes all weak equivalences between cofibrant objects to isomorphisms.*
2. *If F takes trivial fibrations between fibrant objects in $\mathcal{M}$ to isomorphisms in $\mathcal{C}$, then F takes all weak equivalences between fibrant objects to isomorphisms.*

PROOF. This follows from Lemma 8.5.1. □

COROLLARY 8.5.4. *Let $\mathcal{M}$ be a model category.*

1. *If $g\colon C \to D$ is a weak equivalence between cofibrant objects in $\mathcal{M}$ and X is a fibrant object of $\mathcal{M}$, then g induces an isomorphism of the sets of homotopy classes of maps $g^*\colon \pi(D, X) \approx \pi(C, X)$.*
2. *If $g\colon X \to Y$ is a weak equivalence between fibrant objects in $\mathcal{M}$ and C is a cofibrant object of $\mathcal{M}$, then g induces an isomorphism of the sets of homotopy classes of maps $g_*\colon \pi(C, X) \approx \pi(C, Y)$.*

PROOF. This follows from Lemma 8.5.1, Proposition 8.3.21, and Proposition 8.3.18. □

COROLLARY 8.5.5. *Let $\mathcal{M}$ be a model category.*

1. *If $g\colon C \to D$ is a weak equivalence between cofibrant objects in $\mathcal{M}$ and X is a fibrant object of $\mathcal{M}$, then there is a map $C \to X$ in $\mathcal{M}$ if and only if there is a map $D \to X$ in $\mathcal{M}$.*

2. *If $g\colon X \to Y$ is a weak equivalence between fibrant objects in $\mathcal{M}$ and C is a cofibrant object of $\mathcal{M}$, then there is a map $C \to X$ in $\mathcal{M}$ if and only if there is a map $C \to Y$ in $\mathcal{M}$.*

PROOF. This follows from Corollary 8.5.4. □

PROPOSITION 8.5.6. *Let $\mathcal{M}$ be a model category, and let $f, g\colon X \to Y$ be maps. If $f \overset{l}{\simeq} g$ or $f \overset{r}{\simeq} g$, then f is a weak equivalence if and only if g is a weak equivalence.*

PROOF. We will consider the case $f \overset{l}{\simeq} g$; the case $f \overset{r}{\simeq} g$ is dual.

Since $f \overset{l}{\simeq} g$, there is a cylinder object $X \amalg X \xrightarrow{i_0 \amalg i_1} \mathrm{Cyl}(X) \xrightarrow{p} X$ for X and a map $H\colon \mathrm{Cyl}(X) \to Y$ such that $hi_0 = f$ and $hi_1 = g$. Lemma 8.3.6 and axiom M2 (see Definition 8.1.2) imply that f is a weak equivalence if and only if H is a weak equivalence, and that this is true if and only if g is a weak equivalence. □

8.6. Homotopy equivalences

LEMMA 8.6.1. *Let $\mathcal{M}$ be a model category and let X and Y be cofibrant-fibrant objects in $\mathcal{M}$.*

1. *Let $X \amalg X \xrightarrow{i_0 \amalg i_1} \mathrm{Cyl}(X) \xrightarrow{p} X$ be a cylinder object for X and let $H : \mathrm{Cyl}(X) \to Y$ be a left homotopy from the map $f : X \to Y$ to the map $g\colon X \to Y$. If H'' is the composition (see Definition 8.3.12) of H and H^{-1} (see Definition 8.3.10), then H'' is homotopic in $\big((X \amalg X)\downarrow \mathcal{M}\big)$ to the constant left homotopy (i.e., the composition $\mathrm{Cyl}(X)'' \xrightarrow{p''} X \xrightarrow{f} Y$).*
2. *Let $Y \xrightarrow{s} \mathrm{Path}(Y) \xrightarrow{p_0 \times p_1} Y \times Y$ be a path object for Y and let $H\colon X \to \mathrm{Path}(Y)$ be a right homotopy from the map $f : X \to Y$ to the map $g\colon X \to Y$. If H'' is the composition (see Definition 8.3.12) of H and H^{-1} (see Definition 8.3.10), then H'' is homotopic in $\big(\mathcal{M}\downarrow(Y \amalg Y)\big)$ to the constant right homotopy (i.e., the composition $X \xrightarrow{f} Y \xrightarrow{s''} \mathrm{Cyl}(Y)''$).*

PROOF. We will prove part 1; the proof of part 2 is dual.

Let $Y \xrightarrow{s} \mathrm{Path}(Y) \xrightarrow{p_0 \times p_1} Y \times Y$ be a path object for Y (see Lemma 8.3.3). We have the solid arrow diagram

$$\begin{tikzcd}
X \arrow[r, "sf"] \arrow[d, "i_0"'] & \mathrm{Path}(Y) \arrow[d, "(p_0, p_1)"] \\
\mathrm{Cyl}(X) \arrow[r, "(fp, H)"'] \arrow[ur, dotted, "K"] & Y \times Y
\end{tikzcd}$$

in which i_0 is a trivial cofibration (see Lemma 8.3.6) and (p_0, p_1) is a fibration, and so the dotted arrow K exists. If we let the map $K'\colon \mathrm{Cyl}(X)' \to \mathrm{Path}(()Y)$ equal the map K, then K and K' combine to define a map $K''\colon \mathrm{Cyl}(X)'' \to \mathrm{Path}(Y)$ that makes the diagram

$$\begin{tikzcd}
X \amalg X \arrow[r, "sf \amalg sf"] \arrow[d, "i_0'' \amalg i_1''"'] & \mathrm{Path}(Y) \arrow[d, "(p_0, p_1)"] \\
\mathrm{Cyl}(X)'' \arrow[r, "(fp'', H'')"'] \arrow[ur, "K''"] & Y \times Y
\end{tikzcd}$$

commutes. Thus, K'' is a right homotopy (see Definition 8.3.2) from the map $fp''\colon \mathrm{Cyl}(X)'' \to Y$ to the map $H''\colon \mathrm{Cyl}(X)'' \to Y$ in the category $\bigl((X \amalg X)\downarrow \mathcal{M}\bigr)$ of objects of $\mathcal{M}$ under $X \amalg X$. Since $\mathrm{Cyl}(X)''$ is cofibrant in $\bigl(\mathcal{M}\downarrow (X \amalg X)\bigr)$ and Y is fibrant in $\bigl(\mathcal{M}\downarrow (X \amalg X)\bigr)$, Proposition 8.3.18 implies that fp'' is also left homotopic to H'' in $\bigl(\mathcal{M}\downarrow (X \amalg X)\bigr)$, and so fp'' is homotopic to H'' in $\bigl(\mathcal{M}\downarrow (X \amalg X)\bigr)$. □

LEMMA 8.6.2. *Let $\mathcal{M}$ be a model category and let $f\colon X \to Y$ be a map between cofibrant-fibrant objects.*

1. *If f is both a cofibration and a homotopy equivalence, then f is the inclusion of a strong deformation retract, i.e., there is a map $g\colon Y \to X$ such that $gf = 1_X$ and $fg \simeq 1_Y$ in $(X\downarrow \mathcal{M})$.*
2. *If f is both a fibration and a homotopy equivalence, then f is the dual of a strong deformation retract, i.e., there is a map $g\colon Y \to X$ such that $fg = 1_Y$ and $gf \simeq 1_X$ in $(\mathcal{M}\downarrow Y)$.*

PROOF. We will prove part 1; the proof of part 2 is dual.

Since f is a homotopy equivalence, there is a map $h\colon Y \to X$ such that $fh \simeq 1_Y$ and $hf \simeq 1_X$. The homotopy extension property of cofibrations (see Proposition 8.3.7) implies that h is homotopic to a map $g\colon Y \to X$ such that $gf = 1_X$ and $fg \simeq 1_Y$ (see Lemma 8.3.19). Let $Y \xrightarrow{s} \mathrm{Path}(Y) \xrightarrow{p_0 \times p_1} Y \times Y$ be a path object for Y and let $H\colon Y \to \mathrm{Path}(Y)$ be a right homotopy from fg to 1_Y. The composition $Hf\colon X \to \mathrm{Path}(Y)$ is then a right homotopy from $fgf = f$ to $1_Y f = f$. The composite homotopy $(Hfg)\cdot H^{-1}\colon Y \to \mathrm{Path}(Y)''$ (see Definition 8.3.12) composed with f is the composite homotopy $(Hf)\cdot(Hf)^{-1}\colon X \to \mathrm{Path}(Y)''$, and Lemma 8.6.1 implies that $(Hf)\cdot(Hf)^{-1}$ is homotopic in $\bigl(\mathcal{M}\downarrow (Y \times Y)\bigr)$ to the constant homotopy $s''f\colon X \to \mathrm{Path}(Y)''$. The homotopy extension property of cofibrations now implies that $(Hfk)\cdot H^{-1}$ is homotopic in $\bigl(\mathcal{M}\downarrow (Y \times Y)\bigr)$ to a right homotopy $K\colon Y \to \mathrm{Path}(Y)''$ such that $Kf\colon X \to \mathrm{Path}(Y)''$ equals $s''f$, i.e., K is a homotopy from gf to 1_Y in $(X\downarrow \mathcal{M})$. □

PROPOSITION 8.6.3. *Let $\mathcal{M}$ be a model category and let X and Y be cofibrant-fibrant objects in $\mathcal{M}$.*

1. *If $g\colon X \to Y$ is both a cofibration and a homotopy equivalence, then g is a weak equivalence.*
2. *If $g\colon X \to Y$ is both a fibration and a homotopy equivalence, then g is a weak equivalence.*

PROOF. We will prove part 1; the proof of part 2 is dual.

If we factor g as $X \xrightarrow{i} W \xrightarrow{p} Y$ where i is a trivial cofibration and p is a fibration, then the retract axiom (see Definition 8.1.2) implies that it is sufficient to show that g is a retract of i. If we can show that the dotted arrow q exists in the diagram

(8.6.4)
$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "g"'] & W \arrow[d, "p"] \\ Y \arrow[r, equal] \arrow[ur, dotted, "q"] & Y \end{tikzcd}$$

then we would have the diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "g"'] & X \arrow[r, equal] \arrow[d, "i"] & X \arrow[d, "g"] \\ Y \arrow[r, "q"'] \arrow[rr, bend right, "1_Y"'] & W \arrow[r, "p"'] & Y \end{tikzcd}$$

which would show that g is a retract of i. thus, it is sufficient to find the dotted arrow q in Diagram 8.6.4. Lemma 8.6.2 implies that there is a map $h\colon Y \to X$ such that $hg = 1_X$ and $gf \simeq 1_Y$ in $(X \downarrow \mathcal{M})$. If we let $k\colon Y \to W$ be defined by $k = ih$, then $kg = i$, and $pk = pih = gh \simeq 1_Y$ in $(X \downarrow \mathcal{M})$. The homotopy lifting property (see Proposition 8.3.8) of the fibration p in the category $(X \downarrow \mathcal{M})$ now implies that k is homotopic in $(X \downarrow \mathcal{M})$ to a map $q\colon Y \to W$ such that $pq = 1_Y$. □

THEOREM 8.6.5. *Let $\mathcal{M}$ be a model category. If X and Y are cofibrant-fibrant objects in $\mathcal{M}$ and $g\colon X \to Y$ is a homotopy equivalence, then g is a weak equivalence.*

PROOF. If we factor g as $X \xrightarrow{h} W \xrightarrow{k} Y$ where h is a cofibration and k is a trivial fibration, then the "two out of three" property of weak equivalences implies that it is sufficient to show that h is a weak equivalence. Since W is also cofibrant-fibrant, Proposition 8.6.3 implies that it is sufficient to show that h is a homotopy equivalence.

If $g^{-1}\colon Y \to X$ is a homotopy inverse for g, then let $r\colon W \to X$ be defined by $r = g^{-1}k$. Since $rh = g^{-1}kh = g^{-1}g \simeq 1_X$, it is sufficient to show that $hr \simeq 1_W$. Proposition 8.3.21 implies that k induced an isomorphism of sets $k_*\colon \pi(X, W) \approx \pi(X, Y)$. Since $khr = gr = gg^{-1}k \simeq k$, this implies that $hr \simeq 1_W$. □

CHAPTER 9

Fibrant and cofibrant approximations

Fibrant and cofibrant approximations are among the most fundamental tools when doing homotopy theory in a model category. When working with topological spaces, a CW-approximation to a space X (i.e., a CW-complex weakly equivalent to X) is the most common cofibrant approximation to X. When working with simplicial sets, a Kan complex weakly equivalent to X (e.g., the total singular complex of the geometric realization of X, or Kan's functor Ex^{∞} (see [**39**])) is a fibrant approximation to X. Cofibrant and fibrant approximations are used to construct the homotopy category of a model category (see Theorem 9.6.4). When doing homological algebra, a resolution of an object is a cofibrant or fibrant approximation in a model category of cosimplicial or simplicial objects (see, e.g., [**45**] or [**46**, Chapter II, Section 4]). When constructing function complexes in a model category (see Chapter 17), a resolution of an object is a cofibrant or fibrant approximation in yet a different model category of cosimplicial or simplicial objects (see Definition 17.1.2).

9.1. Fibrant and cofibrant approximations

DEFINITION 9.1.1. Let $\mathcal{M}$ be a model category.

1. A *cofibrant approximation* to an object X is a pair $(\widetilde{X}, i)$ where $\widetilde{X}$ is a cofibrant object and $i\colon \widetilde{X} \to X$ is a weak equivalence. A *fibrant cofibrant approximation* to X is a cofibrant approximation $(\widetilde{X}, i)$ such that the weak equivalence i is a trivial fibration. We will sometimes use the term *cofibrant approximation* to refer to the object $\widetilde{X}$ without explicitly mentioning the weak equivalence i.
2. A *fibrant approximation* to an object X is a pair $(\widehat{X}, j)$ where $\widehat{X}$ is a fibrant object and $j\colon X \to \widehat{X}$ is a weak equivalence. A *cofibrant fibrant approximation* to X is a fibrant approximation $(\widehat{X}, j)$ such that the weak equivalence j is a trivial cofibration. We will sometimes use the term *fibrant approximation* to refer to the object $\widehat{X}$ without explicitly mentioning the weak equivalence j.

PROPOSITION 9.1.2. *If $\mathcal{M}$ is a model category, then every object has both a functorial fibrant cofibrant approximation and a functorial cofibrant fibrant approximation.*

PROOF. This follows from applying part 1 of the factorization axiom (see Definition 8.1.2) to the map from the initial object and part 2 of the factorization axiom to the map to the terminal object. □

DEFINITION 9.1.3. Let $\mathcal{M}$ be a model category.

1. If $(\widetilde{X}, i)$ and $(\widetilde{X}', i')$ are cofibrant approximations to X, a *map of cofibrant approximations* from $(\widetilde{X}, i)$ to $(\widetilde{X}', i')$ is a map $g\colon \widetilde{X} \to \widetilde{X}'$ such that $i'g = i$.

2. If $(\widehat{X}, j)$ and $(\widehat{X}', j')$ are fibrant approximations to X, a *map of fibrant approximations* from $(\widehat{X}, j)$ to $(\widehat{X}', j')$ is a map $g\colon \widehat{X} \to \widehat{X}'$ such that $gj = j'$.

LEMMA 9.1.4. *Let $\mathcal{M}$ be a model category.*

1. *If $(\widetilde{X}, i)$ and $(\widetilde{X}', i')$ are cofibrant approximations to X and $g\colon \widetilde{X} \to \widetilde{X}'$ is a map of cofibrant approximations, then g is a weak equivalence.*
2. *If $(\widehat{X}, j)$ and $(\widehat{X}', j')$ are fibrant approximations to X and $g\colon \widehat{X} \to \widehat{X}'$ is a map of fibrant approximations, then g is a weak equivalence.*

PROOF. This follows from the "two out of three" axiom for weak equivalences (see Definition 8.1.2). □

PROPOSITION 9.1.5. *Let $\mathcal{M}$ be a model category.*

1. *If $(\widetilde{X}, i)$ is a fibrant cofibrant approximation to X (see Definition 9.1.1) and $g\colon W \to X$ is a map from a cofibrant object W, then there is a map $\phi\colon W \to \widetilde{X}$, unique up to homotopy over X (see Definition 8.4.3), such that $i\phi = g$.*
2. *If $(\widehat{X}, j)$ is a cofibrant fibrant approximation to X and $g\colon X \to Y$ is a map to a fibrant object Y, then there is a map $\phi\colon \widehat{X} \to Y$, unique up to homotopy under X, such that $\phi j = g$.*

PROOF. This follows from Proposition 8.4.9. □

PROPOSITION 9.1.6. *Let $\mathcal{M}$ be a model category.*

1. *If $(\widetilde{X}, i)$ is a cofibrant approximation to X and $(\widetilde{X}', i')$ is a fibrant cofibrant approximation to X, then there is a map of cofibrant approximations $g\colon \widetilde{X} \to \widetilde{X}'$, unique up to homotopy over X (see Definition 8.4.3), and any such map g is a weak equivalence.*
2. *If $(\widehat{X}, j)$ is a cofibrant fibrant approximation to X and $(\widehat{X}', j')$ is a fibrant approximation to X, then there is a map of fibrant approximations $g\colon \widehat{X} \to \widehat{X}'$, unique up to homotopy under X, and any such map g is a weak equivalence.*

PROOF. This follows from Proposition 9.1.5 and Lemma 9.1.4. □

COROLLARY 9.1.7. *Let $\mathcal{M}$ be a model category.*

1. *If $(\widetilde{X}, i)$ and $(\widetilde{X}', i')$ are fibrant cofibrant approximations to X, then there is a map of cofibrant approximations $g\colon \widetilde{X} \to \widetilde{X}'$, unique up to homotopy over X (see Definition 8.4.3), and any such map g is a homotopy equivalence over X.*
2. *If $(\widehat{X}, j)$ and $(\widehat{X}', j')$ are cofibrant fibrant approximations to X, then there is a map of fibrant approximations $g\colon \widehat{X} \to \widehat{X}'$, unique up to homotopy under X, and any such map g is a homotopy equivalence under X.*

PROOF. This follows from Proposition 9.1.6. □

DEFINITION 9.1.8. Let $\mathcal{M}$ be a model category.

1. A *cofibrant approximation* to a map $g\colon X \to Y$ consists of a cofibrant approximation $(\widetilde{X}, i_X)$ to X (see Definition 9.1.1), a cofibrant approximation $(\widetilde{Y}, i_Y)$ to Y, and a map $\tilde{g}\colon \widetilde{X} \to \widetilde{Y}$ such that $i_Y\tilde{g} = gi_X$. We will sometimes use the term *cofibrant approximation* to refer to the map $\tilde{g}$ without explicitly mentioning the cofibrant approximations $(\widetilde{X}, i_X)$ and $(\widetilde{Y}, i_Y)$ to X and Y

(see Definition 9.1.1). The cofibrant approximation $\tilde{g}$ will be called a *fibrant cofibrant approximation* if the cofibrant approximations $(\widetilde{X}, i_X)$ and $(\widetilde{Y}, i_Y)$ are fibrant cofibrant approximations.

2. A *fibrant approximation* to a map $g\colon X \to Y$ consists of a fibrant approximation $(\widehat{X}, j_X)$ to X (see Definition 9.1.1), a fibrant approximation $(\widehat{Y}, j_Y)$ to Y, and a map $\hat{g}\colon \widehat{X} \to \widehat{Y}$ such that $\hat{g}j_X = j_Y g$. We will sometimes use the term *fibrant approximation* to refer to the map $\hat{g}$ without explicitly mentioning the fibrant approximations $(\widehat{X}, j_X)$ and $(\widehat{Y}, j_Y)$ to X and Y. The fibrant approximation $\hat{g}$ will be called a *cofibrant fibrant approximation* if the fibrant approximations $(\widehat{X}, j_X)$ and $(\widehat{Y}, j_Y)$ are cofibrant fibrant approximations.

PROPOSITION 9.1.9. *Let $\mathcal{M}$ be a model category.*

1. *Every map $g\colon X \to Y$ has a natural fibrant cofibrant approximation $\tilde{g}\colon \widetilde{X} \to \widetilde{Y}$ such that $\tilde{g}$ is a cofibration.*
2. *Every map $g\colon X \to Y$ has a natural cofibrant fibrant approximation $\hat{g}\colon \widehat{X} \to \widehat{Y}$ such that $\hat{g}$ is a fibration.*

PROOF. We will prove part 1; the proof of part 2 is similar.

Choose a natural fibrant cofibrant approximation $(\widetilde{X}, i_X)$ to X, and then choose a natural factorization of the composition $gi_X\colon \widetilde{X} \to Y$ as $\widetilde{X} \xrightarrow{\tilde{g}} \widetilde{Y} \xrightarrow{i_Y} Y$ where $\tilde{g}$ is a cofibration and i_Y is a trivial fibration. □

PROPOSITION 9.1.10. *Let $\mathcal{M}$ be a model category.*

1. *If $g\colon X \to Y$ is a map in $\mathcal{M}$, $\widetilde{X} \to X$ is a cofibrant approximation to X, and $\widetilde{Y} \to Y$ is a fibrant cofibrant approximation to Y, then there exists a cofibrant approximation $\tilde{g}\colon \widetilde{X} \to \widetilde{Y}$ to g, and $\tilde{g}$ is unique up to homotopy over Y.*
2. *If $g\colon X \to Y$ is a map in $\mathcal{M}$, $X \to \widehat{X}$ is a cofibrant fibrant approximation to X, and $Y \to \widehat{Y}$ is a fibrant approximation to Y, then there exists a fibrant approximation $\hat{g}\colon \widehat{X} \to \widehat{Y}$ to g, and $\hat{g}$ is unique up to homotopy under X.*

PROOF. This follows from Proposition 8.4.9. □

PROPOSITION 9.1.11. *Let $\mathcal{M}$ be a model category.*

1. *If $i_1(X)\colon \widetilde{\mathrm{C}}_1(X) \to X$ and $i_2(X)\colon \widetilde{\mathrm{C}}_2(X) \to X$ are natural cofibrant approximations defined on some subcategory of $\mathcal{M}$, then $\widetilde{\mathrm{C}}_1(-)$ and $\widetilde{\mathrm{C}}_2(-)$ are naturally weakly equivalent (see Definition 9.5.2) on their domain of definition.*
2. *If $j_1(X)\colon X \to \widehat{\mathrm{F}}_1(X)$ and $j_2(X)\colon X \to \widehat{\mathrm{F}}_2(X)$ are natural fibrant approximations defined on some subcategory of $\mathcal{M}$, then $\widehat{\mathrm{F}}_1(-)$ and $\widehat{\mathrm{F}}_2(-)$ are naturally weakly equivalent on their domain of definition.*

PROOF. We will prove part 1; the proof of part 2 is dual.

If we choose a natural fibrant cofibrant approximation $i(X)\colon \widetilde{\mathrm{C}}(X) \to X$ for every object X in the domain of definition of $\widetilde{\mathrm{C}}_1(-)$ and $\widetilde{\mathrm{C}}_2(-)$ (see Proposition 9.1.2), then it is sufficient to show that each of $\widetilde{\mathrm{C}}_1(-)$ and $\widetilde{\mathrm{C}}_2(-)$ is naturally weakly equivalent to $\widetilde{\mathrm{C}}(-)$. We will do this for $\widetilde{\mathrm{C}}_1(-)$ (the proof for $\widetilde{\mathrm{C}}_2(-)$ is then identical to that).

For every object X in the domain of $\widetilde{\mathrm{C}}_1(-)$, we construct the pullback square

$$\begin{tikzcd} \mathrm{P}_1(X) \arrow[r, "j(X)"] \arrow[d, "j_1(X)"'] & \widetilde{\mathrm{C}}_1(X) \arrow[d, "i_1(X)"] \\ \widetilde{\mathrm{C}}(X) \arrow[r, "i(X)"'] & X \end{tikzcd}$$

and then we choose a functorial cofibrant approximation $k(X)\colon \widetilde{\mathrm{P}}_1(X) \to \mathrm{P}_1(X)$ to $\mathrm{P}_1(X)$. Since $i(X)$ is a trivial fibration, so is $j(X)$, and so the "two out of three" axiom (see Definition 8.1.2) implies that $j_1(X)$ is also a weak equivalence. Thus, $\widetilde{\mathrm{C}}_1(X) \xleftarrow{j(X)k(X)} \widetilde{\mathrm{P}}_1(X) \xrightarrow{j_2(X)k(X)} \widetilde{\mathrm{C}}(X)$ is a natural zig-zag of weak equivalences of cofibrant approximations to X. □

9.2. Approximations and homotopic maps

LEMMA 9.2.1. *Let $\mathcal{M}$ be a model category, let $X \amalg X \to \mathrm{Cyl}(X) \to X$ be a cylinder object for X, and let $X \to \mathrm{Path}(X) \to X \times X$ be a path object for X.*

1. *If $i\colon \widetilde{X} \to X$ is a fibrant cofibrant approximation to X, then*
   (a) *there is a cylinder object $\widetilde{X} \amalg \widetilde{X} \to \mathrm{Cyl}(\widetilde{X}) \to \widetilde{X}$ for $\widetilde{X}$ and a diagram*

$$\begin{tikzcd} \widetilde{X} \amalg \widetilde{X} \arrow[r] \arrow[d, "i \amalg i"'] & \mathrm{Cyl}(\widetilde{X}) \arrow[r] \arrow[d, "\mathrm{Cyl}(i)"] & \widetilde{X} \arrow[d, "i"] \\ X \amalg X \arrow[r] & \mathrm{Cyl}(X) \arrow[r] & X \end{tikzcd}$$

   such that $\mathrm{Cyl}(i)\colon \mathrm{Cyl}(\widetilde{X}) \to \mathrm{Cyl}(X)$ is a fibrant cofibrant approximation to $\mathrm{Cyl}(X)$, and
   (b) *there is a path object $\widetilde{X} \to \mathrm{Path}(\widetilde{X}) \to \widetilde{X} \times \widetilde{X}$ for $\widetilde{X}$ and a diagram*

(9.2.2)
$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d, "i"'] & \mathrm{Path}(\widetilde{X}) \arrow[r] \arrow[d, "\mathrm{Path}(i)"] & \widetilde{X} \times \widetilde{X} \arrow[d, "i \times i"] \\ X \arrow[r] & \mathrm{Path}(X) \arrow[r] & X \times X \end{tikzcd}$$

   such that $\mathrm{Path}(i)\colon \mathrm{Path}(\widetilde{X}) \to \mathrm{Path}(X)$ is a fibrant cofibrant approximation to $\mathrm{Path}(X)$ and the right hand square of Diagram 9.2.2 is a pullback.
2. *If $j\colon X \to \widehat{X}$ is a cofibrant fibrant approximation to X, then*
   (a) *there is a cylinder object $\widehat{X} \amalg \widehat{X} \to \mathrm{Cyl}(\widehat{X}) \to \widehat{X}$ for $\widehat{X}$ and a diagram*

(9.2.3)
$$\begin{tikzcd} X \amalg X \arrow[r] \arrow[d, "j \amalg j"'] & \mathrm{Cyl}(X) \arrow[r] \arrow[d, "\mathrm{Cyl}(j)"] & X \arrow[d, "j"] \\ \widehat{X} \amalg \widehat{X} \arrow[r] & \mathrm{Cyl}(\widehat{X}) \arrow[r] & \widehat{X} \end{tikzcd}$$

   such that $\mathrm{Cyl}(j)\colon \mathrm{Cyl}(X) \to \mathrm{Cyl}(\widehat{X})$ is a cofibrant fibrant approximation to $\mathrm{Cyl}(X)$ and the left hand square of Diagram 9.2.3 is a pushout, and

(b) *there is a path object* $\widehat{X} \to \mathrm{Path}(\widehat{X}) \to \widehat{X} \times \widehat{X}$ *for* $\widehat{X}$ *and a diagram*

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "j"'] & \mathrm{Path}(X) \arrow[r] \arrow[d, "\mathrm{Path}(j)"] & X \times X \arrow[d, "j \times j"] \\
\widehat{X} \arrow[r] & \mathrm{Path}(\widehat{X}) \arrow[r] & \widehat{X} \times \widehat{X}
\end{tikzcd}$$

such that $\mathrm{Path}(j) \colon \mathrm{Path}(X) \to \mathrm{Path}(\widehat{X})$ *is a cofibrant fibrant approximation to* $\mathrm{Path}(X)$.

PROOF. We will prove part 1; the proof of part 2 is dual.

Factor the composition $\widetilde{X} \amalg \widetilde{X} \to X \amalg X \to \mathrm{Cyl}(X)$ as $\widetilde{X} \amalg \widetilde{X} \xrightarrow{k} \mathrm{Cyl}(\widetilde{X}) \xrightarrow{\mathrm{Cyl}(i)} \mathrm{Cyl}(X)$ where k is a cofibration and $\mathrm{Cyl}(i)$ is a trivial fibration. Since i is a trivial fibration, the dotted arrow q exists in the solid arrow diagram

$$\begin{tikzcd}
\widetilde{X} \amalg \widetilde{X} \arrow[rr, "1_{\widetilde{X}} \amalg 1_{\widetilde{X}}"] \arrow[d, "k"'] & & \widetilde{X} \arrow[d, "i"] \\
\mathrm{Cyl}(\widetilde{X}) \arrow[r] \arrow[urr, dotted, "q"] & \mathrm{Cyl}(X) \arrow[r] & X
\end{tikzcd}$$

and the "two out of three" axiom for weak equivalences (see Definition 8.1.2) implies that q is a weak equivalence.

If we let $\mathrm{Path}(\widetilde{X})$ be the pullback $\mathrm{Path}(X) \times_{(X \times X)} (\widetilde{X} \times \widetilde{X})$, then we have the solid arrow diagram

$$\begin{tikzcd}
\widetilde{X} \arrow[r, dotted, "r"] \arrow[rr, bend left, "1_{\widetilde{X}} \times 1_{\widetilde{X}}"] \arrow[d, "i"'] & \mathrm{Path}(\widetilde{X}) \arrow[r] \arrow[d, "\mathrm{Path}(i)"] & \widetilde{X} \times \widetilde{X} \arrow[d, "i \times i"] \\
X \arrow[r] & \mathrm{Path}(X) \arrow[r] & X \times X
\end{tikzcd}$$

and the universal mapping property of the pullback implies that the dotted arrow r exists. Since i is a trivial fibration, so is $i \times i$, and so $\mathrm{Path}(i)$ (which is a pullback of $i \times i$) is a trivial fibration. The "two out of three" axiom for weak equivalences (see Definition 8.1.2) now implies that r is a weak equivalence. $\square$

PROPOSITION 9.2.4. *Let* $\mathcal{M}$ *be a model category, and let* $f, g \colon X \to Y$ *be maps.*

1. *If* $\tilde{f}, \tilde{g} \colon \widetilde{X} \to \widetilde{Y}$ *are fibrant cofibrant approximations to, respectively,* f *and* g*, and if* f *and* g *are left homotopic, right homotopic, or homotopic, then* $\tilde{f}$ *and* $\tilde{g}$ *are, respectively, left homotopic, right homotopic, or homotopic.*
2. *If* $\hat{f}, \hat{g} \colon \widehat{X} \to \widehat{Y}$ *are cofibrant fibrant approximations to, respectively,* f *and* g*, and if* f *and* g *are left homotopic, right homotopic, or homotopic, then* $\hat{f}$ *and* $\hat{g}$ *are, respectively, left homotopic, right homotopic, or homotopic.*

PROOF. We will prove part 1; the proof of part 2 is dual.

If f and g are left homotopic, let $X \amalg X \to \mathrm{Cyl}(X) \to X$ be a cylinder object for X such that there is a left homotopy $H \colon \mathrm{Cyl}(X) \to Y$ from f to g. If $\widetilde{X} \amalg \widetilde{X} \to \mathrm{Cyl}(\widetilde{X}) \to \widetilde{X}$ is the cylinder object of Lemma 9.2.1, then we have the solid arrow diagram

$$\begin{tikzcd} \widetilde{X} \amalg \widetilde{X} \arrow[rr, "\tilde{f} \amalg \tilde{g}"] \arrow[d] & & \widetilde{Y} \arrow[d] \\ \mathrm{Cyl}(\widetilde{X}) \arrow[r] \arrow[urr, dotted, "\widetilde{H}"] & \mathrm{Cyl}(X) \arrow[r, "H"'] & Y \end{tikzcd}$$

Since $\widetilde{Y} \to Y$ is a trivial fibration, the dotted arrow $\widetilde{H}$ exists, and is a left homotopy from $\tilde{f}$ to $\tilde{g}$.

If f and g are right homotopic, let $Y \to \mathrm{Path}(Y) \to Y \times Y$ be a path object for Y such that there is a right homotopy $K : X \to \mathrm{Path}(Y)$ from f to g. If $\widetilde{Y} \to \mathrm{Path}(\widetilde{Y}) \to \widetilde{Y} \times \widetilde{Y}$ is the path object of Lemma 9.2.1, then we have the solid arrow diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, dotted, "\widetilde{K}"'] \arrow[rr, bend left, "\tilde{f} \times \tilde{g}"] \arrow[d] & \mathrm{Path}(\widetilde{Y}) \arrow[r] \arrow[d] & \widetilde{Y} \times \widetilde{Y} \arrow[d] \\ X \arrow[r, "K"'] & \mathrm{Path}(Y) \arrow[r] & Y \times Y \end{tikzcd}$$

Since the right hand square is a pullback, the dotted arrow $\widetilde{K}$ exists and is a right homotopy from $\tilde{f}$ to $\tilde{g}$. $\square$

9.3. Approximations and weak equivalences

LEMMA 9.3.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories, let $g_0, g_1 : X \to Y$ be maps in $\mathcal{M}$, and let $\mathrm{F} : \mathcal{M} \to \mathcal{N}$ be a functor.*

1. *If F takes trivial cofibrations between cofibrant objects in $\mathcal{M}$ into weak equivalences in $\mathcal{N}$, the object X is cofibrant, and g_0 is left homotopic to g_1, then $\mathrm{F}(g_0)$ is a weak equivalence if and only if $\mathrm{F}(g_1)$ is a weak equivalence.*
2. *If F takes trivial fibrations between fibrant objects in $\mathcal{M}$ into weak equivalences in $\mathcal{N}$, the object Y is fibrant, and g_0 is right homotopic to g_1 (see Definition 8.3.2), then $\mathrm{F}(g_0)$ is a weak equivalence if and only if $\mathrm{F}(g_1)$ is a weak equivalence.*

PROOF. This follows from Lemma 9.7.4 and Proposition 8.5.6. $\square$

PROPOSITION 9.3.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories, let $g : X \to Y$ be a map in $\mathcal{M}$, and let $\mathrm{F} : \mathcal{M} \to \mathcal{N}$ be a functor.*

1. *If F takes trivial cofibrations between cofibrant objects in $\mathcal{M}$ into weak equivalences in $\mathcal{N}$ and there is a cofibrant approximation $\tilde{g} : \widetilde{X} \to \widetilde{Y}$ to g (see Definition 9.1.8) such that $\mathrm{F}(\tilde{g})$ is a weak equivalence, then F takes every cofibrant approximation to g into a weak equivalence.*
2. *If F takes trivial fibrations between fibrant objects in $\mathcal{M}$ into weak equivalences in $\mathcal{N}$ and there is a fibrant approximation $\hat{g} : \widehat{X} \to \widehat{Y}$ to g (see Definition 9.1.8) such that $\mathrm{F}(\hat{g})$ is a weak equivalence, then F takes every fibrant approximation to g into a weak equivalence.*

PROOF. We will prove part 1; the proof of part 2 is similar.

Proposition 9.1.9 implies that we can choose a cofibrant approximation $\tilde{g}' : \widetilde{X}' \to \widetilde{Y}'$ to g such that the weak equivalences $i'_X : \widetilde{X}' \to X$ and $i'_Y : \widetilde{Y}' \to Y$ are trivial fibrations. It is sufficient to show that $\mathrm{F}(\tilde{g}')$ is a weak equivalence if and only if F takes every other cofibrant approximation to g into a weak equivalence.

If $\tilde{g} \colon \widetilde{X} \to \widetilde{Y}$ is some other cofibrant approximation to g, then we have the solid arrow diagram

$$\begin{tikzcd}
\widetilde{X}' \arrow[rrr, "\tilde{g}'"] \arrow[dr, "i'_X"] & & & \widetilde{Y}' \arrow[dl, "i'_Y"'] \\
& X \arrow[r, "g"] & Y & \\
\widetilde{X} \arrow[uu, dotted, "h_X"] \arrow[ur, "i_X"'] \arrow[rrr, "\tilde{g}"'] & & & \widetilde{Y} \arrow[uu, dotted, "h_Y"'] \arrow[ul, "i_Y"]
\end{tikzcd}$$

in which i'_X and i'_Y are trivial fibrations and i_X and i_Y are weak equivalences. Proposition 9.1.6 implies that there are weak equivalences $h_X : \widetilde{X} \to \widetilde{X}'$ and $h_Y : \widetilde{Y} \to \widetilde{Y}'$ such that $i'_X h_X = i_X$ and $i'_Y h_Y = i_Y$. Thus, $i'_Y \tilde{g}' h_X = g i'_X h_X = g i_X = i_Y \tilde{g} = i'_Y h_Y \tilde{g}$. Since i'_Y is a trivial fibration and $\widetilde{X}$ is cofibrant, Proposition 8.3.21 implies that $\tilde{g}' h_X$ is left homotopic to $h_Y \tilde{g}$, and so Lemma 9.3.1 implies that $\mathrm{F}(\tilde{g}' h_X)$ is a weak equivalence if and only if $\mathrm{F}(h_Y \tilde{g})$ is a weak equivalence. Since Corollary 8.5.2 implies that $\mathrm{F}(h_X)$ and $\mathrm{F}(h_Y)$ are weak equivalences, the "two out of three" axiom for weak equivalences (see Definition 8.1.2) implies that $\mathrm{F}(\tilde{g}')$ is a weak equivalence if and only if $\mathrm{F}(\tilde{g})$ is a weak equivalence. □

9.4. The classifying space of a small category

DEFINITION 9.4.1. If $\mathcal{C}$ is a small category, then the *classifying space* of $\mathcal{C}$ (also called the *nerve* of $\mathcal{C}$) is the simplicial set $\mathrm{B}\mathcal{C}$ in which an n-simplex σ is a diagram in $\mathcal{C}$ of the form

$$\alpha_0 \xrightarrow{\sigma_0} \alpha_1 \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_{n-1}} \alpha_n$$

and the face and degeneracy maps are defined by

(9.4.2)
$$d_j \sigma = \begin{cases} \alpha_1 \xrightarrow{\sigma_1} \alpha_2 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_{n-1}} \alpha_n & \text{if } j = 0 \\ \alpha_0 \xrightarrow{\sigma_0} \cdots \xrightarrow{\sigma_{j-2}} \alpha_{j-1} \xrightarrow{\sigma_j \sigma_{j-1}} \alpha_{j+1} \xrightarrow{\sigma_{j+1}} \cdots \xrightarrow{\sigma_{n-1}} \alpha_n & \text{if } 0 < j < n \\ \alpha_0 \xrightarrow{\sigma_0} \alpha_1 \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_{n-2}} \alpha_{n-1} & \text{if } j = n \end{cases}$$

$$s_j \sigma = \alpha_0 \xrightarrow{\sigma_0} \cdots \xrightarrow{\sigma_{j-1}} \alpha_j \xrightarrow{1_{\alpha_j}} \alpha_j \xrightarrow{\sigma_j} \alpha_{j+1} \xrightarrow{\sigma_{j+1}} \cdots \xrightarrow{\sigma_{n-1}} \alpha_n$$

If $\mathrm{F} \colon \mathcal{C} \to \mathcal{D}$ is a functor between small categories, then F induces a map of simplicial sets $\mathrm{BF} \colon \mathrm{B}\mathcal{C} \to \mathrm{B}\mathcal{D}$ defined by

$$\mathrm{BF}(\alpha_0 \xrightarrow{\sigma_0} \alpha_1 \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_{n-1}} \alpha_n) = \mathrm{F}\alpha_0 \xrightarrow{\mathrm{F}\sigma_0} \mathrm{F}\alpha_1 \xrightarrow{\mathrm{F}\sigma_1} \cdots \xrightarrow{\mathrm{F}\sigma_{n-1}} \mathrm{F}\alpha_n$$

EXAMPLE 9.4.3. Let G be a discrete group. If we consider G to be a category with one object and with morphisms equal to the group G, then $\mathrm{B}G$ is the standard classifying space of the group G, i.e., $\pi_1 \mathrm{B}G \approx G$ and $\pi_i \mathrm{B}G \approx 0$ for $i \neq 1$.

Proposition 9.4.4. *If the small category $\mathcal{C}$ has either a terminal or an initial object, then the geometric realization of* $\mathrm{B}\mathcal{C}$ *is contractible.*

Proposition 9.4.5. *If $\mathcal{C}$ is a small category, then there is a natural homeomorphism of topological spaces* $|\mathrm{B}\mathcal{C}| \approx |\mathrm{B}\mathcal{C}^{\mathrm{op}}|$.

Definition 9.4.6. Let $\mathcal{M}$ be a model category.

1. If X is an object of $\mathcal{M}$, we let $\mathrm{CofAp}(X)$ denote the category whose objects are cofibrant approximations to X (see Definition 9.1.1) and whose morphisms are maps of cofibrant approximations (see Definition 9.1.3).
2. If X is an object of $\mathcal{M}$, we let $\mathrm{FibAp}(X)$ denote the category whose objects are fibrant approximations to X and whose morphisms are maps of fibrant approximations.

Proposition 9.4.7. *Let $\mathcal{M}$ be a model category.*

1. *If X is an object in $\mathcal{M}$, then* $\mathrm{B}\,\mathrm{CofAp}(X)$ *(see Definition* 9.4.1*) (which may exist only in a higher universe, since* $\mathrm{CofAp}(X)$ *is not, in general, small) (see, e.g.,* [**51**, page 17]*) is contractible. If $\mathcal{C}$ is a small subcategory of* $\mathrm{CofAp}(X)$*, then there exists a small subcategory $\mathcal{D}$ of* $\mathrm{CofAp}(X)$ *such that $\mathcal{C} \subset \mathcal{D}$ and* $\mathrm{B}\mathcal{D}$ *is contractible.*
2. *If X is an object in $\mathcal{M}$, then* $\mathrm{B}\,\mathrm{FibAp}(X)$ *(which may exist only in a higher universe, since* $\mathrm{FibAp}(X)$ *is not, in general, small) is contractible. If $\mathcal{C}$ is a small subcategory of* $\mathrm{FibAp}(X)$*, then there exists a small subcategory $\mathcal{D}$ of* $\mathrm{FibAp}(X)$ *such that $\mathcal{C} \subset \mathcal{D}$ and* $\mathrm{B}\mathcal{D}$ *is contractible.*

Proof. This follows from **Fix This Reference!**, Proposition 9.1.2 and Proposition 9.1.6. □

9.5. Equivalence classes of weak equivalences

Definition 9.5.1. Let $\mathcal{M}$ be a model category, and let $\mathcal{C}$ be a class of maps in $\mathcal{M}$.

1. If X and Y are objects in $\mathcal{M}$ and $n \geq 0$, then a *zig-zag of elements of $\mathcal{C}$ of length n from X to Y* is a diagram of the form
$$X \xrightarrow{f_1} W_1 \xleftarrow{f_2} W_2 \xrightarrow{f_3} \cdots \xleftarrow{f_{n-1}} W_{n-1} \xrightarrow{f_n} Y$$
where
   (a) each f_i is an element of $\mathcal{C}$,
   (b) each f_i can point either to the left or to the right, and
   (c) consecutive f_is can point in either the same direction or in opposite directions.
2. If X, Y, and Z are objects in $\mathcal{M}$ and
$$X \xrightarrow{f_1} W_1 \xleftarrow{f_2} \cdots \xleftarrow{f_{n-1}} W_{n-1} \xrightarrow{f_n} Y \quad \text{and} \quad Y \xrightarrow{g_1} V_1 \xleftarrow{g_2} \cdots \xleftarrow{g_{k-1}} V_{k-1} \xrightarrow{g_k} Z$$
are, respectively, a zig-zag in $\mathcal{C}$ from X to Y and a zig-zag in $\mathcal{C}$ from Y to Z, then the *composition* of those zig-zags is the zig-zag in $\mathcal{C}$ of length $n+k$ from X to Z
$$X \xrightarrow{f_1} W_1 \xleftarrow{f_2} \cdots \xleftarrow{f_{n-1}} W_{n-1} \xrightarrow{f_n} Y \xrightarrow{g_1} V_1 \xleftarrow{g_2} \cdots \xleftarrow{g_{k-1}} V_{k-1} \xrightarrow{g_k} Z$$

Definition 9.5.2. Let $\mathcal{M}$ be a model category.

1. If X and Y are objects in $\mathcal{M}$, then X and Y are *weakly equivalent* if there is a zig-zag of weak equivalences from X to Y (see Definition 9.5.1).
2. If $\mathcal{C}$ is a category and F and G are functors from $\mathcal{C}$ to $\mathcal{M}$, then F and G are *naturally weakly equivalent* if for every object A in $\mathcal{C}$ there is a natural zig-zag of weak equivalences

$$\mathrm{F}(A) \xrightarrow{\cong} W_1(A) \xleftarrow{\cong} W_2(A) \xrightarrow{\cong} W_3(A) \xleftarrow{\cong} \cdots \xrightarrow{\cong} W_n(A) \xleftarrow{\cong} \mathrm{G}(A)$$

from $\mathrm{F}(A)$ to $\mathrm{G}(A)$.

Definition 9.5.3. Let $\mathcal{M}$ be a model category. If X and Y are objects in $\mathcal{M}$, then we define an equivalence relation on the zip-zags of weak equivalences from X to Y (which is a set only in a higher universe) by taking the equivalence relation generated by the relation:

1. If two consecutive maps in a zig-zag point in the same direction, compose them; i.e.,

$$X \xrightarrow{f_1} W_1 \leftarrow \cdots \rightarrow W_{k_1} \xrightarrow{f_k} W_k \xrightarrow{f_{k+1}} W_{k+1} \leftarrow \cdots \rightarrow Y$$

equals

$$X \xrightarrow{f_1} W_1 \leftarrow \cdots \rightarrow W_{k_1} \xrightarrow{f_{k+1}f_k} W_{k+1} \leftarrow \cdots \rightarrow Y$$

2. If a map in a zig-zag is immediately followed by the same map pointing in the opposite direction, remove the pair of maps; i.e.,

$$X \xrightarrow{f_1} W_1 \leftarrow \cdots \rightarrow W_{k-1} \xrightarrow{f_k} W_k \xleftarrow{f_k} W_{k-1} \leftarrow \cdots \rightarrow Y$$

equals

$$X \xrightarrow{f_1} W_1 \leftarrow \cdots \rightarrow W_{k-1} \leftarrow \cdots \rightarrow Y$$

If two zig-zags of weak equivalences are equivalent under the equivalence relation generated by that relation, then they will be called *equivalent zig-zags of weak equivalences*.

Proposition 9.5.4. *Let $\mathcal{M}$ be a model category. If X, Y, and Z are objects in $\mathcal{M}$, then composition of zig-zags of weak equivalences (see Definition 9.5.1) passes to equivalence classes of zig-zags of weak equivalences (see Definition 9.5.3) to define the composition of an equivalence class of zig-zags of weak equivalences from X to Y with an equivalence class of zig-zags of weak equivalences from Y to Z.*

Proof. This follows directory from the definitions. □

Theorem 9.5.5. *Let $\mathcal{M}$ be a category, let $\mathcal{C}$ be a subcategory of $\mathcal{M}$, and let X and Y be objects in $\mathcal{M}$. If every small subcategory of $\mathcal{C}$ is contained in a small subcategory of $\mathcal{C}$ whose classifying space is simply connected, then any two zig-zags in $\mathcal{C}$ from X to Y are equivalent.*

Proof. This follows because the equivalence classes of zig-zags in $\mathcal{C}$ from X to Y are the morphisms in the edge-path groupoid of the classifying space of $\mathcal{C}$ from X to Y. □

9.6. The homotopy category of a model category

DEFINITION 9.6.1. If $\mathcal{M}$ is a category and S is a class of maps in $\mathcal{M}$, then a *localization* of $\mathcal{M}$ with respect to S is a category $\mathrm{L}_S\mathcal{M}$ and a functor $\gamma\colon \mathcal{M} \to \mathrm{L}_S\mathcal{M}$ such that

1. if $s \in S$, then $\gamma(s)$ is an isomorphism, and
2. if $\mathcal{N}$ is a category and $\mathrm{F}\colon \mathcal{M} \to \mathcal{N}$ is a functor such that $\mathrm{F}(s)$ is an isomorphism for every s in S, then there is a unique functor $\delta\colon \mathrm{L}_S\mathcal{M} \to \mathcal{N}$ such that $\delta\gamma = \mathrm{F}$.

The usual argument shows that if the localization of $\mathcal{M}$ with respect to S exists, then it is unique up to a unique isomorphism. Thus, we will speak of *the* localization of $\mathcal{M}$ with respect to S.

DEFINITION 9.6.2. If $\mathcal{M}$ is a model category, then the *Quillen homotopy category* of $\mathcal{M}$ is the localization of $\mathcal{M}$ with respect to the class of weak equivalences, which we denote by $\gamma\colon \mathcal{M} \to \operatorname{Ho}\mathcal{M}$.

We will show that the Quillen homotopy category of a model category $\mathcal{M}$ exists (see Theorem 9.6.4), and that it is equivalent to the classical homotopy category of $\mathcal{M}$ (see Definition 8.3.25 and Theorem 9.6.7).

LEMMA 9.6.3. *Let $\mathcal{M}$ be a model category, let $\mathcal{N}$ be a category, and let $\mathrm{F}\colon \mathcal{M} \to \mathcal{N}$ be a functor that takes weak equivalences in $\mathcal{M}$ to isomorphisms in $\mathcal{N}$. If $f, g\colon X \to Y$ are maps in $\mathcal{M}$ such that either $f \overset{l}{\simeq} g$ or $f \overset{r}{\simeq} g$ (see Definition 8.3.2), then $\mathrm{F}(f) = \mathrm{F}(g)$.*

PROOF. We will consider the case $f \overset{l}{\simeq} g$; the case $f \overset{r}{\simeq} g$ is similar.

If $f \overset{l}{\simeq} g$, then there is a cylinder object (see Definition 8.3.2) $X \amalg X \xrightarrow{i_0 \amalg i_1} \mathrm{Cyl}(X) \xrightarrow{p} X$ for X and a map $H\colon \mathrm{Cyl}(X) \to Y$ such that $Hi_0 = f$ and $Hi_1 = g$. Since p is a weak equivalence, $\mathrm{F}(p)$ is an isomorphism. Since $pi_0 = pi_1$, this implies that $\mathrm{F}(i_0) = \mathrm{F}(i_1)$. Thus, $\mathrm{F}(f) = \mathrm{F}(H)\mathrm{F}(i_0) = \mathrm{F}(H)\mathrm{F}(i_1) = \mathrm{F}(g)$. □

Lemma 9.6.3 implies that a functor $\mathrm{F}\colon \mathcal{M} \to \mathcal{N}$ that takes weak equivalences to isomorphisms must identify homotopic maps. Thus, when searching for the Quillen homotopy category of $\mathcal{M}$ (see Definition 9.6.2), a natural object to consider is the classical homotopy category of $\mathcal{M}$ (see Definition 8.3.25). Proposition 8.3.26 implies that if we restrict ourselves to the full subcategory of $\mathcal{M}$ spanned by the cofibrant-fibrant objects, then identifying homotopic maps turns weak equivalences into isomorphisms, and so the classical homotopy category serves as the Quillen homotopy category of this subcategory.

To deal with objects that are not cofibrant-fibrant, we note that, if $\widetilde{X}$ is weakly equivalent to X and $\widetilde{Y}$ is weakly equivalent to Y, then, in any category in which weak equivalences have become isomorphisms, the set of maps from X to Y will be isomorphic to the set of maps from $\widetilde{X}$ to $\widetilde{Y}$. This suggests that we should choose $\widetilde{X}$ and $\widetilde{Y}$ to be cofibrant-fibrant objects weakly equivalent to, respectively, X and Y, and define $\operatorname{Ho}\mathcal{M}(X,Y)$ to be the set of homotopy classes of maps from $\widetilde{X}$ to $\widetilde{Y}$ in $\mathcal{M}$. This is what we shall do to define $\operatorname{Ho}\mathcal{M}$.

THEOREM 9.6.4. *If $\mathcal{M}$ is a model category, then the Quillen homotopy category of $\mathcal{M}$ (see Definition 9.6.2) exists.*

PROOF. For every cofibrant object X, let $\widetilde{\mathrm{C}}X = X$ and let $i_X\colon \widetilde{\mathrm{C}}X \to X$ be the identity map. For every non-cofibrant object X, factor the map from the initial object to X into a cofibration followed by a trivial fibration to obtain a cofibrant object $\widetilde{\mathrm{C}}X$ and a trivial fibration $i_X\colon \widetilde{\mathrm{C}}X \to X$. (In the terminology of Definition 9.1.1, we have chosen a fibrant cofibrant approximation to X.)

For every fibrant object X, let $\widehat{\mathrm{F}}X = X$ and let $j_X\colon X \to \widehat{\mathrm{F}}X$ be the identity map. For every non-fibrant object X, factor the map from X to the terminal object into a trivial cofibration followed by a fibration to obtain a fibrant object $\widehat{\mathrm{F}}X$ and a trivial cofibration $j_X : X \to \widehat{\mathrm{F}}X$. (In the terminology of Definition 9.1.1, we have chosen a cofibrant fibrant approximation to X.)

We define the category $\mathrm{Ho}\,\mathcal{M}$ as follows:

1. The objects of $\mathrm{Ho}\,\mathcal{M}$ are the objects of $\mathcal{M}$.
2. If X and Y are objects of $\mathcal{M}$, then $\mathrm{Ho}\,\mathcal{M}(X,Y) = \pi(\widehat{\mathrm{F}}\widetilde{\mathrm{C}}X, \widehat{\mathrm{F}}\widetilde{\mathrm{C}}Y)$ (see Notation 8.3.15).
3. If X, Y, and Z are objects of $\mathcal{M}$, then the composition
$$\mathrm{Ho}\,\mathcal{M}(Y,Z) \times \mathrm{Ho}\,\mathcal{M}(X,Y) \to \mathrm{Ho}\,\mathcal{M}(X,Z)$$
is the composition of homotopy classes of maps between cofibrant-fibrant objects in $\mathcal{M}$
$$\pi(\widehat{\mathrm{F}}\widetilde{\mathrm{C}}Y, \widehat{\mathrm{F}}\widetilde{\mathrm{C}}Z) \times \pi(\widehat{\mathrm{F}}\widetilde{\mathrm{C}}X, \widehat{\mathrm{F}}\widetilde{\mathrm{C}}Y) \to \pi(\widehat{\mathrm{F}}\widetilde{\mathrm{C}}X, \widehat{\mathrm{F}}\widetilde{\mathrm{C}}Z).$$

We now define the functor $\gamma\colon \mathcal{M} \to \mathrm{Ho}\,\mathcal{M}$. We let γ be the identity on the class of objects. For every map $f\colon X \to Y$ in $\mathcal{M}$, we have the solid arrow diagram

$$\begin{tikzcd}
\emptyset \arrow[rr] \arrow[d] & & \widetilde{\mathrm{C}}Y \arrow[d, "i_Y"] \\
\widetilde{\mathrm{C}}X \arrow[r, "i_X"'] \arrow[urr, dotted, "\widetilde{\mathrm{C}}(f)"] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

(where $\emptyset$ denotes the initial object of $\mathcal{M}$), and we can choose a dotted arrow $\widetilde{\mathrm{C}}(f)$ that makes the diagram commute. (In the terminology of Definition 9.1.8, $\widetilde{\mathrm{C}}(f)$ is a cofibrant approximation to f.) Proposition 8.3.21 implies that $\widetilde{\mathrm{C}}(f)$ is well defined up to left homotopy, and so Proposition 8.3.17 implies that it is well defined up to right homotopy. We now have the solid arrow diagram

$$\begin{tikzcd}
\widetilde{\mathrm{C}}X \arrow[r, "\widetilde{\mathrm{C}}(f)"] \arrow[d, "j_{\widetilde{\mathrm{C}}X}"'] & \widetilde{\mathrm{C}}Y \arrow[r, "j_{\widetilde{\mathrm{C}}Y}"] & \widehat{\mathrm{F}}\widetilde{\mathrm{C}}Y \arrow[d] \\
\widehat{\mathrm{F}}\widetilde{\mathrm{C}}X \arrow[rr] \arrow[urr, dotted, "\widehat{\mathrm{F}}\widetilde{\mathrm{C}}(f)"] & & *
\end{tikzcd}$$

(where $*$ denotes the terminal object of $\mathcal{M}$), and we can choose a dotted arrow $\widehat{\mathrm{F}}\widetilde{\mathrm{C}}(f)$ that makes the diagram commute. Proposition 8.3.21 implies that $\widehat{\mathrm{F}}\widetilde{\mathrm{C}}(f)$ is well defined up to homotopy, and we define $\gamma(f)$ to the the element of $\pi(\widehat{\mathrm{F}}\widetilde{\mathrm{C}}X, \widehat{\mathrm{F}}\widetilde{\mathrm{C}}Y)$ represented by $\widehat{\mathrm{F}}\widetilde{\mathrm{C}}(f)$ (see Proposition 8.3.18).

To see that γ is a functor, we note that, for every object X of $\mathcal{M}$, Proposition 8.3.21 implies that $\widetilde{\mathrm{C}}(1_X) \overset{l}{\simeq} 1_{\widetilde{\mathrm{C}}X}$, and so $\widetilde{\mathrm{C}}(1_X) \overset{r}{\simeq} 1_{\widetilde{\mathrm{C}}X}$, and so $\widehat{\mathrm{F}}\widetilde{\mathrm{C}}(1_X) \simeq$

$1_{\widehat{\mathrm{F}}\widetilde{\mathrm{C}}X}$. Similarly, if $f\colon X \to Y$ and $g\colon Y \to Z$ are maps in $\mathcal{M}$, then Proposition 8.3.21 implies that $\widetilde{\mathrm{C}}(g)\widetilde{\mathrm{C}}(f) \overset{l}{\simeq} \widetilde{\mathrm{C}}(gf)$, and so $\widehat{\mathrm{F}}\widetilde{\mathrm{C}}(g)\widehat{\mathrm{F}}\widetilde{\mathrm{C}}(g) \simeq \widehat{\mathrm{F}}\widetilde{\mathrm{C}}(gf)$. Thus, we have defined the category $\operatorname{Ho}\mathcal{M}$ and the functor $\gamma\colon \mathcal{M} \to \operatorname{Ho}\mathcal{M}$.

We will now show that γ takes weak equivalences in $\mathcal{M}$ to isomorphisms in $\operatorname{Ho}\mathcal{M}$. If $f\colon X \to Y$ is a weak equivalence, then the "two out of three" property of weak equivalences (see Definition 8.1.2) implies that $\widetilde{\mathrm{C}}(f)$ and $\widehat{\mathrm{F}}\widetilde{\mathrm{C}}(f)$ are weak equivalences, and so Proposition 8.3.26 implies that $\widehat{\mathrm{F}}\widetilde{\mathrm{C}}(f)$ is a homotopy equivalence. Thus, the homotopy class of $\widehat{\mathrm{F}}\widetilde{\mathrm{C}}(f)$ is an isomorphism, i.e., $\gamma(f)$ is an isomorphism in $\operatorname{Ho}\mathcal{M}$.

It remains only to show that if $\mathcal{N}$ is a category and $\mathrm{F}\colon \mathcal{M} \to \mathcal{N}$ is a functor that takes weak equivalences in $\mathcal{M}$ to isomorphisms in $\mathcal{N}$, then there is a unique functor $\delta\colon \operatorname{Ho}\mathcal{M} \to \mathcal{N}$ such that $\delta\gamma = \mathrm{F}$. Let $\mathrm{F}\colon \mathcal{M} \to \mathcal{N}$ be such a functor. For every object X of $\operatorname{Ho}\mathcal{M}$, we let $\delta(X) = \mathrm{F}(X)$. If $g\colon X \to Y$ is a map in $\operatorname{Ho}\mathcal{M}$, then g is a homotopy class of maps $\widehat{\mathrm{F}}\widetilde{\mathrm{C}}X \to \widehat{\mathrm{F}}\widetilde{\mathrm{C}}Y$ in $\mathcal{M}$. Lemma 9.6.3 implies that F takes all elements of that homotopy class to the same map of $\mathcal{N}$, and so we can let

$$\delta(g) = \mathrm{F}(i_Y)\big(\mathrm{F}(j_{\widetilde{\mathrm{C}}Y})\big)^{-1}\mathrm{F}(g)\mathrm{F}(j_{\widetilde{\mathrm{C}}X})\big(\mathrm{F}(i_X)\big)^{-1}$$

(where by $\mathrm{F}(g)$ we mean F applied to some map in the homotopy class g). To see that δ is a functor, we note that an identity map in $\operatorname{Ho}\mathcal{M}$ is a homotopy class of maps in $\mathcal{M}$ containing an identity map, and composition of maps between cofibrant-fibrant objects of $\mathcal{M}$ is well defined on homotopy classes (see Proposition 8.3.22). Thus, δ is a functor.

To see that $\delta\gamma = \mathrm{F}$, we note that γ is the identity on objects, and δ was defined to agree with F on objects. If $f\colon X \to Y$ is a map in $\mathcal{M}$, then we have the commutative diagram

$$\begin{tikzcd}
\widehat{\mathrm{F}}\widetilde{\mathrm{C}}X \arrow[r, "\widehat{\mathrm{F}}\widetilde{\mathrm{C}}(f)"] & \widehat{\mathrm{F}}\widetilde{\mathrm{C}}Y \\
\widetilde{\mathrm{C}}X \arrow[u, "j_{\widetilde{\mathrm{C}}X}"] \arrow[r, "\widetilde{\mathrm{C}}(f)"] \arrow[d, "i_X"'] & \widetilde{\mathrm{C}}Y \arrow[u, "j_{\widetilde{\mathrm{C}}Y}"'] \arrow[d, "i_Y"] \\
X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

Since F takes weak equivalences to isomorphisms in $\mathcal{N}$, we have

$$\mathrm{F}(f) = \mathrm{F}(i_Y)\big(\mathrm{F}(j_{\widetilde{\mathrm{C}}Y})\big)^{-1}\mathrm{F}\big(\widehat{\mathrm{F}}\widetilde{\mathrm{C}}(f)\big)\mathrm{F}(j_{\widetilde{\mathrm{C}}X})\big(\mathrm{F}(i_X)\big)^{-1}.$$

Since $\gamma(f)$ is the homotopy class of $\widehat{\mathrm{F}}\widetilde{\mathrm{C}}(f)$, this implies that $\delta\gamma(f) = \mathrm{F}(f)$.

Finally, to see that δ is the unique functor satisfying $\delta\gamma = \mathrm{F}$, we note that every map of $\operatorname{Ho}\mathcal{M}$ is a composition of maps in the image of γ and inverses of the image under γ of a weak equivalence of $\mathcal{M}$. □

Remark 9.6.5. The proof of Theorem 9.6.4 did not use a functorial cofibrant approximation $\widetilde{\mathrm{C}}$, but instead let $\widetilde{\mathrm{C}}X$ equal X when X is cofibrant (and a similar remark applies to the fibrant approximation $\widehat{\mathrm{F}}$). This was done so that if X and Y are cofibrant-fibrant objects of $\mathcal{M}$, then $\operatorname{Ho}\mathcal{M}(X, Y)$ is the set of homotopy classes of maps in $\mathcal{M}$ from X to Y.

THEOREM 9.6.6. *If $\mathcal{M}$ is a model category, then the classical homotopy category of $\mathcal{M}$ (see Definition 8.3.25) is naturally isomorphic to the full subcategory of the Quillen homotopy category of $\mathcal{M}$ spanned by the cofibrant-fibrant objects.*

PROOF. If X and Y are cofibrant-fibrant objects of $\mathcal{M}$, then the proof of Theorem 9.6.4 sets $\widehat{\mathrm{F}}\widetilde{\mathrm{C}}X$ equal to X, $\widehat{\mathrm{F}}\widetilde{\mathrm{C}}Y$ equal to Y, and $\mathrm{Ho}\,\mathcal{M}(X,Y)$ equal to $\pi(X,Y)$ (see Remark 9.6.5). $\square$

THEOREM 9.6.7. *If $\mathcal{M}$ is a model category, then the embedding of the classical homotopy category into the Quillen homotopy category (see Theorem 9.6.6) is an equivalence of categories.*

PROOF. Let ν denote the embedding $\pi\mathcal{M}_{\mathrm{cf}} \to \mathrm{Ho}\,\mathcal{M}$ described in Theorem 9.6.6. To define $\eta\colon \mathrm{Ho}\,\mathcal{M} \to \pi\mathcal{M}_{\mathrm{cf}}$, let $\widetilde{\mathrm{C}}$ and $\widehat{\mathrm{F}}$ be as in the proof of Theorem 9.6.4. If X is an object of $\mathrm{Ho}\,\mathcal{M}$, let $\eta(X) = \widehat{\mathrm{F}}\widetilde{\mathrm{C}}X$. If X and Y are objects of $\mathrm{Ho}\,\mathcal{M}$, then $\mathrm{Ho}\,\mathcal{M}(X,Y) = \pi(\widehat{\mathrm{F}}\widetilde{\mathrm{C}}X, \widehat{\mathrm{F}}\widetilde{\mathrm{C}}Y)$, and we let η be the "identity map" from $\mathrm{Ho}\,\mathcal{M}(X,Y)$ to $\pi\mathcal{M}_{\mathrm{cf}}(X,Y)$.

Since $\eta\nu$ is the identity functor of $\pi\mathcal{M}_{\mathrm{cf}}$, it remains only to define a natural equivalence θ from the identity functor of $\mathrm{Ho}\,\mathcal{M}$ to $\nu\eta$. If X is an object of $\mathrm{Ho}\,\mathcal{M}$, then $\nu\eta(X) = \widehat{\mathrm{F}}\widetilde{\mathrm{C}}X$, and so $\mathrm{Ho}\,\mathcal{M}\big(X, \nu\eta(X)\big) = \mathrm{Ho}\,\mathcal{M}(X, \widehat{\mathrm{F}}\widetilde{\mathrm{C}}X) = \pi(\widehat{\mathrm{F}}\widetilde{\mathrm{C}}X, \widehat{\mathrm{F}}\widetilde{\mathrm{C}}\widehat{\mathrm{F}}\widetilde{\mathrm{C}}X) = \pi(\widehat{\mathrm{F}}\widetilde{\mathrm{C}}X, \widehat{\mathrm{F}}\widetilde{\mathrm{C}}X)$ (see Remark 9.6.5); we let $\theta(X)\colon X \to \nu\eta X$ be the homotopy class of the identity map of $\widehat{\mathrm{F}}\widetilde{\mathrm{C}}X$ in $\mathcal{M}$. $\square$

PROPOSITION 9.6.8. *Let $\mathcal{M}$ be a model category. If $g\colon X \to Y$ is a map in $\mathcal{M}$, then g is a weak equivalence if and only if $\gamma(g)$ is an isomorphism in* $\mathrm{Ho}\,\mathcal{M}$.

PROOF. If g is a weak equivalence, then Theorem 9.6.4 implies that $\gamma(g)$ is an isomorphism. Conversely, if $\gamma(g)$ is an isomorphism, then $\widehat{\mathrm{F}}\widetilde{\mathrm{C}}(g)$ (see the proof of Theorem 9.6.4) is a homotopy equivalence, and so Theorem 8.6.5 and the "two out of three" property of weak equivalences implies that g is a weak equivalence. $\square$

9.7. Derived functors

DEFINITION 9.7.1. Let $\mathcal{M}$ be a model category, let $\mathcal{C}$ be a category, and let $\mathrm{F}\colon \mathcal{M} \to \mathcal{C}$ be a functor.

1. A *left derived functor* of F is a functor $\mathrm{LF}\colon \mathrm{Ho}\,\mathcal{M} \to \mathcal{C}$ together with a natural transformation $\epsilon\colon \mathrm{L}\circ\gamma \to \mathrm{F}$ such that, if $\mathrm{G}\colon \mathrm{Ho}\,\mathcal{M} \to \mathcal{C}$ is a functor and $\zeta\colon \mathrm{G}\circ\gamma \to \mathrm{F}$ is a natural transformation, then there is a unique natural transformation $\theta\colon \mathrm{G} \to \mathrm{LF}$ such that $\zeta = \epsilon(\theta\circ\gamma)$.
2. A *right derived functor* of F is a functor $\mathrm{RF}\colon \mathrm{Ho}\,\mathcal{M} \to \mathcal{C}$ together with a natural transformation $\epsilon\colon \mathrm{F} \to \mathrm{RF}\circ\gamma$ such that, if $\mathrm{G}\colon \mathrm{Ho}\,\mathcal{M} \to \mathcal{C}$ is a functor and $\zeta\colon \mathrm{F} \to \mathrm{G}\circ\gamma$ is a natural transformation, then there is a unique natural transformation $\theta\colon \mathrm{RF} \to \mathrm{G}$ such that $\zeta = (\theta\circ\gamma)\epsilon$.

REMARK 9.7.2. The usual argument shows that if a left derived functor of F exists, then it is unique up to a unique natural equivalence. Thus, we will speak of *the* left derived functor of F. A similar remark applies to right derived functors.

REMARK 9.7.3. The left derived functor of $\mathrm{F}\colon \mathcal{M} \to \mathcal{C}$ is also known as the *right Kan extension of* F *along* $\gamma\colon \mathcal{M} \to \mathrm{Ho}\,\mathcal{M}$ (see [**41**, page 232]). Similarly, the right derived functor of $\mathrm{F}\colon \mathcal{M} \to \mathcal{C}$ is also known as the *left Kan extension of* F *along* $\gamma\colon \mathcal{M} \to \mathrm{Ho}\,\mathcal{M}$.

LEMMA 9.7.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories, and let* $\mathrm{F}\colon \mathcal{M} \to \mathcal{N}$ *be a functor.*

1. *If* F *takes trivial cofibrations between cofibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$, $f, g\colon X \to Y$ are left homotopic maps in $\mathcal{M}$, and X is cofibrant, then* $\mathrm{F}(f)$ *is left homotopic to* $\mathrm{F}(g)$.
2. *If* F *takes trivial fibrations between fibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$, $f, g\colon X \to Y$ are right homotopic maps in $\mathcal{M}$, and Y is fibrant, then* $\mathrm{F}(f)$ *is right homotopic to* $\mathrm{F}(g)$.

PROOF. We will prove part 1; the proof of part 2 is dual.

Since f and g are left homotopic, there is a cylinder object $X \amalg X \xrightarrow{i_0 \amalg i_1} \mathrm{Cyl}(X) \xrightarrow{p} X$ for X and a map $H\colon \mathrm{Cyl}(X) \to Y$ such that $Hi_0 = f$ and $Hi_1 = g$. Since $pi_0 = 1_X$, we have $\mathrm{F}(p)\mathrm{F}(i_0) = 1_{\mathrm{F}(X)}$, and, since i_0 is a trivial cofibration (see Lemma 8.3.6), the "two out of three" property of weak equivalences (see Definition 8.1.2) implies that $\mathrm{F}(p)$ is a weak equivalence. The result now follows from Proposition 8.3.4. □

LEMMA 9.7.5. *Let $\mathcal{M}$ be a model category, let $\mathcal{C}$ be a category, and let* $\mathrm{F}\colon \mathcal{M} \to \mathcal{C}$ *be a functor.*

1. *If* F *takes trivial cofibrations between cofibrant objects in $\mathcal{M}$ to isomorphisms in $\mathcal{C}$, $f, g\colon X \to Y$ are left homotopic maps in $\mathcal{M}$, and X is cofibrant, then* $\mathrm{F}(f) = \mathrm{F}(g)$.
2. *If* F *takes trivial fibrations between fibrant objects in $\mathcal{M}$ to isomorphisms in $\mathcal{C}$, $f, g\colon X \to Y$ are right homotopic maps in $\mathcal{M}$, and Y is fibrant, then* $\mathrm{F}(f) = \mathrm{F}(g)$.

PROOF. We will prove part 1; the proof of part 2 is dual.

Since f and g are left homotopic, there is a cylinder object $X \amalg X \xrightarrow{i_0 \amalg i_1} \mathrm{Cyl}(X) \xrightarrow{p} X$ for X and a map $H\colon \mathrm{Cyl}(X) \to Y$ such that $Hi_0 = f$ and $Hi_1 = g$. Since $pi_0 = 1_X$, we have $\mathrm{F}(p)\mathrm{F}(i_0) = 1_{\mathrm{F}(X)}$, and, since i_0 is a trivial cofibration (see Lemma 8.3.6), $\mathrm{F}(i_0)$ is an isomorphism, and so $\mathrm{F}(p)$ is an isomorphism. Since $pi_0 = 1_X = pi_1$, $\mathrm{F}(i_0) = \big(\mathrm{F}(p)\big)^{-1} = \mathrm{F}(i_1)$. Thus, $\mathrm{F}(f) = \mathrm{F}(H)\mathrm{F}(i_0) = \mathrm{F}(H)\mathrm{F}(i_1) = \mathrm{F}(g)$. □

PROPOSITION 9.7.6. *Let $\mathcal{M}$ be a model category, let $\mathcal{C}$ be a category, and let* $\mathrm{F}\colon \mathcal{M} \to \mathcal{C}$ *be a functor.*

1. *If* F *takes trivial cofibrations between cofibrant objects to isomorphisms in $\mathcal{C}$, then the left derived functor of* F *exists.*
2. *If* F *takes trivial fibrations between fibrant objects to isomorphisms in $\mathcal{C}$, then the right derived functor of* F *exists.*

PROOF. We will prove part 1; the proof of part 2 is dual.

Let $\widetilde{\mathrm{C}}$ be as in the proof of Theorem 9.6.4. We define a functor $\mathrm{D}\colon \mathcal{M} \to \mathcal{C}$ as follows: If X is an object of $\mathcal{M}$, we let $\mathrm{D}(X) = \mathrm{F}(\widetilde{\mathrm{C}}X)$. If $f\colon X \to Y$ is a map in $\mathcal{M}$, then $\widetilde{\mathrm{C}}(f)\colon \widetilde{\mathrm{C}}X \to \widetilde{\mathrm{C}}Y$ is well defined up to left homotopy, and so Lemma 9.7.5 implies that $\mathrm{F}\big(\widetilde{\mathrm{C}}(f)\big)$ is well defined; we let $\mathrm{D}(f) = \mathrm{F}\big(\widetilde{\mathrm{C}}(f)\big)$. To see that D is a functor, we note that $\widetilde{\mathrm{C}}(1_X) \overset{l}{\simeq} 1_{\widetilde{\mathrm{C}}X}$ and so $\mathrm{D}(1_X) = 1_{\mathrm{D}X}$, and if $f\colon X \to Y$ and $g\colon Y \to Z$ are maps in $\mathcal{M}$, then $\widetilde{\mathrm{C}}(g)\widetilde{\mathrm{C}}(f) \overset{l}{\simeq} \widetilde{\mathrm{C}}(gf)$, and so $\mathrm{D}(g)\mathrm{D}(f) = \mathrm{D}(gf)$.

If $f\colon X \to Y$ is a weak equivalence in $\mathcal{M}$, then $\widetilde{\mathrm{C}}(f)$ is a weak equivalence between cofibrant objects, and so Corollary 8.5.3 implies that $\mathrm{D}(f)$ is an isomorphism. Thus, the universal property of $\mathrm{Ho}\,\mathcal{M}$ (see Definition 9.6.2 and Definition 9.6.1) implies that there is a unique functor $\mathrm{LF}\colon \mathrm{Ho}\,\mathcal{M} \to \mathcal{C}$ such that $\mathrm{LF} \circ \gamma = \mathrm{D}$. We define a natural transformation $\epsilon\colon \mathrm{LF} \circ \gamma \to \mathrm{F}$ by letting $\epsilon(X) = \mathrm{F}(i_X)\colon \mathrm{LF} \circ \gamma(X) = \mathrm{D}(X) = \mathrm{F}(\widetilde{\mathrm{C}}X) \to \mathrm{F}(X)$. We will show that the pair (LF, ϵ) is the left derived functor of F.

If $\mathrm{G}\colon \mathrm{Ho}\,\mathcal{M} \to \mathcal{C}$ is a functor and $\zeta\colon \mathrm{G} \circ \gamma \to \mathrm{F}$ is a natural transformation, then we have the solid arrow diagram

$$\begin{tikzcd}
\mathrm{G} \circ \gamma(\widetilde{\mathrm{C}}X) \arrow[r, "\zeta(\widetilde{\mathrm{C}}X)"] \arrow[d, "(\mathrm{G}\circ\gamma)(i_X)"'] & \mathrm{F}(\widetilde{\mathrm{C}}X) = (\mathrm{LF} \circ \gamma)(X) \arrow[d, "\mathrm{F}(i_X)=\epsilon(X)"] \\
\mathrm{G} \circ \gamma(X) \arrow[r, "\zeta(X)"'] \arrow[ur, dotted, "\theta(X)"] & \mathrm{F}(X)
\end{tikzcd} \tag{9.7.7}$$

and so we define a natural transformation $\theta\colon \mathrm{G} \to \mathrm{LF}$ by letting $\theta(X) = \big(\zeta(\widetilde{\mathrm{C}}X)\big) \circ \big((\mathrm{G} \circ \gamma)(i_X)\big)^{-1}$. If X is cofibrant, then $\mathrm{F}(i_X)$ is an isomorphism, and so $\theta(X)$ is the only possible map that makes Diagram 9.7.7 commute. Since $\widetilde{\mathrm{C}}X \approx X$ for every object X in $\mathrm{Ho}\,\mathcal{M}$, this implies the uniqueness of θ in general. ☐

9.7.8. Total derived functors.

DEFINITION 9.7.9. Let $\mathcal{M}$ and $\mathcal{N}$ be model categories and let $\mathrm{F}\colon \mathcal{M} \to \mathcal{N}$ be a functor.

1. A *total left derived functor of* F is a left derived functor (see Definition 9.7.1) of the composition $\mathcal{M} \xrightarrow{\mathrm{F}} \mathcal{N} \xrightarrow{\nu_{\mathcal{N}}} \mathrm{Ho}\,\mathcal{N}$. Thus, a total left derived functor of F is a functor $\mathbf{L}\mathrm{F}\colon \mathrm{Ho}\,\mathcal{M} \to \mathrm{Ho}\,\mathcal{N}$ together with a natural transformation $\epsilon\colon \mathbf{L}\mathrm{F} \circ \nu_{\mathcal{M}} \to \nu_{\mathcal{N}} \circ \mathrm{F}$ such that the pair $(\mathbf{L}\mathrm{F}, \epsilon)$ is "closest to $\nu_{\mathcal{N}} \circ \mathrm{F}$ from the left" (see Definition 9.7.1). We will often refer to $\mathbf{L}\mathrm{F}\colon \mathrm{Ho}\,\mathcal{M} \to \mathrm{Ho}\,\mathcal{N}$ as the *total left derived functor of* F, without explicitly mentioning the natural transformation ϵ.
2. A *total right derived functor of* F is a right derived functor of the composition $\mathcal{M} \xrightarrow{\mathrm{F}} \mathcal{N} \xrightarrow{\nu_{\mathcal{N}}} \mathrm{Ho}\,\mathcal{N}$. Thus, a total right derived functor of F is a functor $\mathbf{R}\mathrm{F}\colon \mathrm{Ho}\,\mathcal{M} \to \mathrm{Ho}\,\mathcal{N}$ together with a natural transformation $\epsilon\colon \nu_{\mathcal{N}} \circ \mathrm{F} \to \mathbf{R}\mathrm{F} \circ \nu_{\mathcal{M}}$ such that the pair $(\mathbf{R}\mathrm{F}, \epsilon)$ is "closest to $\nu_{\mathcal{N}} \circ \mathrm{F}$ from the right" (see Definition 9.7.1). We will often refer to $\mathbf{R}\mathrm{F}\colon \mathrm{Ho}\,\mathcal{M} \to \mathrm{Ho}\,\mathcal{N}$ as the *total right derived functor of* F, without explicitly mentioning the natural transformation ϵ.

PROPOSITION 9.7.10. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories, and let $\mathrm{F}\colon \mathcal{M} \to \mathcal{N}$ be a functor.*

1. *If* F *takes trivial cofibrations between cofibrant objects in $\mathcal{M}$ into weak equivalences in $\mathcal{N}$, then the total left derived functor $\mathbf{L}\mathrm{F}\colon \mathrm{Ho}\,\mathcal{M} \to \mathrm{Ho}\,\mathcal{N}$ exists.*
2. *If* F *takes trivial fibrations between fibrant objects in $\mathcal{M}$ into weak equivalences in $\mathcal{N}$, then the total right derived functor $\mathbf{R}\mathrm{F}\colon \mathrm{Ho}\,\mathcal{M} \to \mathrm{Ho}\,\mathcal{N}$ exists.*

PROOF. This follows from Proposition 9.7.6. ☐

9.8. Quillen functors

DEFINITION 9.8.1. Let $\mathcal{M}$ and $\mathcal{N}$ be model categories, and let $\mathrm{F}\colon \mathcal{M} \rightleftarrows \mathcal{N} :\mathrm{U}$ be a pair of adjoint functors. We will say that

1. (F, U) *is a Quillen pair*,
2. F *is a left Quillen functor*, and
3. U *is a right Quillen functor*

if

1. the left adjoint F preserves both cofibrations and trivial cofibrations, and
2. the right adjoint U preserves both fibrations and trivial fibrations.

PROPOSITION 9.8.2. *If $\mathcal{M}$ and $\mathcal{N}$ are model categories and $\mathrm{F}\colon \mathcal{M} \rightleftarrows \mathcal{N} :\mathrm{U}$ is a pair of adjoint functors, then the following are equivalent:*

1. *The pair (F, U) is a Quillen pair.*
2. *The left adjoint* F *preserves both cofibrations and trivial cofibrations.*
3. *The right adjoint* U *preserves both fibrations and trivial fibrations.*
4. *The left adjoint* F *preserves cofibrations and the right adjoint* U *preserves fibrations.*
5. *The left adjoint* F *preserves trivial cofibrations and the right adjoint* U *preserves trivial fibrations.*

PROOF. This follows from Proposition 8.2.3 and Proposition 8.2.8. □

DEFINITION 9.8.3. Let $\mathcal{M}$ and $\mathcal{N}$ be model categories, and let $\mathrm{F}\colon \mathcal{M} \rightleftarrows \mathcal{N} :\mathrm{U}$ be a Quillen pair (see Definition 9.8.1). We will say that

1. (F, U) *is a pair of Quillen equivalences*,
2. F *is a left Quillen equivalence*, and
3. U *is a right Quillen equivalence*

if for every cofibrant object B in $\mathcal{M}$, every fibrant object X in $\mathcal{N}$, and every map $f\colon B \to \mathrm{U}X$ in $\mathcal{M}$, the map f is a weak equivalence in $\mathcal{M}$ if and only if the corresponding map $f^\sharp\colon \mathrm{F}B \to X$ is a weak equivalence in $\mathcal{N}$.

LEMMA 9.8.4. *Let $\mathcal{M}$ be a model category and let $i_X : \widetilde{\mathrm{C}}X \to X$ and $j_X : X \to \widehat{\mathrm{F}}X$ be the constructions used in the proof of Theorem 9.6.4.*

1. *If W is cofibrant and X is fibrant, then i_X induces an isomorphism of the sets of homotopy classes of maps $(i_X)_*\colon \pi(W, \widetilde{\mathrm{C}}X) \to \pi(W, X)$ that is natural in both W and X.*
2. *If X is cofibrant and Z is fibrant, then j_X induces an isomorphism of the set of homotopy classes of maps $(j_X)^*\colon \pi(\widehat{\mathrm{F}}X, Z) \to \pi(X, Z)$ that is natural in both X and Z.*

PROOF. This follows from Proposition 8.3.21. □

LEMMA 9.8.5. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories, and let $\mathrm{F}\colon \mathcal{M} \rightleftarrows \mathcal{N} :\mathrm{U}$ be a Quillen pair (see Definition 9.8.1).*

1. *If B is a cofibrant object in $\mathcal{M}$ and $B \vee B \to \mathrm{Cyl}(B) \to B$ is a cylinder object for B, then $\mathrm{F}B \vee \mathrm{F}B \to \mathrm{F}\mathrm{Cyl}(B) \to \mathrm{F}B$ is a cylinder object for $\mathrm{F}B$.*
2. *If X is a fibrant object in $\mathcal{N}$ and $X \to \mathrm{Path}(X) \to X \times X$ is a path object for X, then $\mathrm{U}X \to \mathrm{U}\mathrm{Path}(X) \to \mathrm{U}X \times \mathrm{U}X$ is a path object for $\mathrm{U}X$.*

PROOF. We will prove part 1; the proof of part 2 is dual.

Since B is cofibrant, Lemma 8.3.6 and the "two out of three" property of weak equivalences (see Definition 8.1.2) imply that the map $\mathrm{FCyl}(B) \to \mathrm{F}B$ is a weak equivalence. Since F is a left adjoint, $\mathrm{F}(B \vee B) \approx \mathrm{F}B \vee \mathrm{F}B$, and so $\mathrm{F}B \vee \mathrm{F}B \to \mathrm{FCyl}(B) \to \mathrm{F}B$ is a cylinder object for $\mathrm{F}B$. □

LEMMA 9.8.6. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories and let $\mathrm{F}\colon \mathcal{M} \rightleftarrows \mathcal{N} :\mathrm{U}$ be a Quillen pair (see Definition 9.8.1).*

1. *If $f, g\colon A \to B$ are left homotopic maps in $\mathcal{M}$ and A is cofibrant, then $\mathrm{F}(f)$ is left homotopic to $\mathrm{F}(g)$.*
2. *If $f, g\colon X \to Y$ are right homotopic maps in $\mathcal{N}$ and Y is fibrant, then $\mathrm{U}(f)$ is right homotopic to $\mathrm{U}(g)$.*

PROOF. This follows from Lemma 9.8.5. □

PROPOSITION 9.8.7. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories, and let $\mathrm{F}\colon \mathcal{M} \rightleftarrows \mathcal{N} :\mathrm{U}$ be a Quillen pair (see Definition 9.8.1). If X is a cofibrant object in $\mathcal{M}$ and Y is a fibrant object in $\mathcal{N}$, then the adjointness isomorphism between F and U induces a natural isomorphism of the sets of homotopy classes of maps $\pi(\mathrm{F}X, Y) \approx \pi(X, \mathrm{U}Y)$.*

PROOF. The adjointness of F and U gives us a natural isomorphism of sets of maps $\mathcal{N}(\mathrm{F}X, Y) \approx \mathcal{M}(X, \mathrm{U}Y)$; we must show that this passes to homotopy classes. Lemma 9.8.5 implies that if two maps $X \to \mathrm{U}Y$ in $\mathcal{M}$ are left homotopic, then the corresponding maps $\mathrm{F}X \to Y$ are left homotopic, and that if two maps $\mathrm{F}X \to Y$ in $\mathcal{N}$ are right homotopic, then the corresponding maps $X \to \mathrm{U}Y$ are right homotopic. □

LEMMA 9.8.8. *Let $\mathcal{M}$ be a model category, let A be cofibrant, and let W, X, Y, and Z be fibrant. If $f\colon V \to W$ is a weak equivalence and the diagram*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "f"'] & W \arrow[r] & Y \arrow[d] \\ W \arrow[r] & X \arrow[r] & Z \end{tikzcd}$$

commutes, then the diagram

(9.8.9)
$$\begin{tikzcd} \pi(A, W) \arrow[r] \arrow[d] & \pi(A, Y) \arrow[d] \\ \pi(A, X) \arrow[r] & \pi(A, Z) \end{tikzcd}$$

also commutes.

PROOF. If we choose a functorial fibrant approximation on $\mathcal{M}$ (see Proposition 9.1.2), then we have the diagram

$$\begin{tikzcd}
V \arrow[rr, "f"] \arrow[dr, "j_V"] \arrow[dd, "f"'] & & W \arrow[rr] \arrow[dr, "j_W"] & & Y \arrow[dr, "j_Y"] \arrow[dd] & \\
& \hat{V} \arrow[rr, "\hat{f}"] \arrow[dd, "\hat{f}"'] & & \widehat{W} \arrow[rr] & & \hat{Y} \arrow[dd] \\
W \arrow[rr] \arrow[dr, "j_W"'] & & X \arrow[rr] \arrow[dr, "j_X"] & & Z \arrow[dr, "j_Z"] & \\
& \widehat{W} \arrow[rr] & & \hat{X} \arrow[rr] & & \hat{Z}
\end{tikzcd}$$

in which $\hat{f}$, j_W, j_X, j_Y, and j_Z are weak equivalences between fibrant objects. Since the front rectangle commutes and $\hat{f}_*\colon \pi(A,\hat{V}) \to \pi(A,\widehat{W})$ is an isomorphism (see Corollary 8.5.4), the square

$$\begin{tikzcd}
\pi(A,\widehat{W}) \arrow[r] \arrow[d] & \pi(A,\hat{Y}) \arrow[d] \\
\pi(A,\hat{X}) \arrow[r] & \pi(A,\hat{Z})
\end{tikzcd}$$

commutes, and Corollary 8.5.4 implies that this square is isomorphic to the square of Diagram 9.8.9. $\square$

LEMMA 9.8.10. *Let $\mathcal{M}$ be a model category and let $i_Y : \widetilde{\mathrm{C}}Y \to Y$ and $j_Y : Y \to \widehat{\mathrm{F}}Y$ be the constructions used in the proof of Theorem 9.6.4. If W is cofibrant, then the map $\widehat{\mathrm{F}}(i_Y)\colon \widehat{\mathrm{F}}\widetilde{\mathrm{C}}Y \to \widehat{\mathrm{F}}Y$ induces an isomorphism of the sets of homotopy classes of maps $\widehat{\mathrm{F}}(i_Y)\colon \pi(W,\widehat{\mathrm{F}}\widetilde{\mathrm{C}}Y) \to \pi(W,\widehat{\mathrm{F}}Y)$ that is natural in Y.*

PROOF. The "two out of three" property of weak equivalences (see Definition 8.1.2) implies that $\widehat{\mathrm{F}}(i_Y)$ is a weak equivalence of fibrant objects, and so Corollary 8.5.4 implies that $\widehat{\mathrm{F}}(i_Y)_*\colon \pi(W,\widehat{\mathrm{F}}\widetilde{\mathrm{C}}Y) \to \pi(W,\widehat{\mathrm{F}}Y)$ is an isomorphism. It remains only to show that this is natural in Y.

If $f\colon Y \to Z$ is a map in $\mathcal{M}$, then we have the diagram

$$\begin{tikzcd}
\widetilde{\mathrm{C}}Y \arrow[rr, "\widetilde{\mathrm{C}}f"] \arrow[dr, "j_{\widetilde{\mathrm{C}}Y}"] \arrow[dd, "i_Y"'] & & \widetilde{\mathrm{C}}Z \arrow[dr, "j_{\widetilde{\mathrm{C}}Z}"] \arrow[dd, "i_Z"'] & \\
& \widehat{\mathrm{F}}\widetilde{\mathrm{C}}Y \arrow[rr, "\widehat{\mathrm{F}}\widetilde{\mathrm{C}}(f)"] \arrow[dd, "\widehat{\mathrm{F}}(i_Y)"'] & & \widehat{\mathrm{F}}\widetilde{\mathrm{C}}Z \arrow[dd, "\widehat{\mathrm{F}}(i_Z)"] \\
Y \arrow[rr, "f"] \arrow[dr, "j_Y"'] & & Z \arrow[dr, "j_Z"] & \\
& \widehat{\mathrm{F}}Y \arrow[rr, "\widehat{\mathrm{F}}f"'] & & \widehat{\mathrm{F}}Z
\end{tikzcd}$$

in which all the squares except possibly the front one commute. A diagram chase shows that $\widehat{\mathrm{F}}(i_Z) \circ \widehat{\mathrm{F}}\widetilde{\mathrm{C}}(f) \circ j_{\widetilde{\mathrm{C}}Y} = \widehat{\mathrm{F}}(f) \circ \widehat{\mathrm{F}}(i_Y) \circ j_{\widetilde{\mathrm{C}}Y}$, and so Lemma 9.8.8 now implies that the square

$$\begin{tikzcd} \pi W, \widehat{\mathrm{F}}\widetilde{\mathrm{C}}Y \arrow[r] \arrow[d] & \pi Q, \widehat{\mathrm{F}}\widetilde{\mathrm{C}}Z \arrow[d] \\ \pi(W, \widehat{\mathrm{F}}Y) \arrow[r] & \pi(W, \widehat{\mathrm{F}}Z) \end{tikzcd}$$

commutes. □

THEOREM 9.8.11. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be model categories. If* $\mathrm{F}\colon \mathcal{M} \rightleftarrows \mathcal{N} :\mathrm{U}$ *is a Quillen pair (see Definition* 9.8.1*), then*

1. *the total left derived functor* $\mathbf{L}\mathrm{F}\colon \operatorname{Ho}\mathcal{M} \to \operatorname{Ho}\mathcal{N}$ *of* F *exists,*
2. *the total right derived functor* $\mathbf{R}\mathrm{U}\colon \operatorname{Ho}\mathcal{N} \to \operatorname{Ho}\mathcal{M}$ *of* U *exists, and*
3. *the functors* $\mathbf{L}\mathrm{F}$ *and* $\mathbf{R}\mathrm{U}$ *are an adjoint pair.*

PROOF. The existence of the functors $\mathbf{L}\mathrm{F}$ and $\mathbf{R}\mathrm{U}$ follows from Proposition 9.7.10. To see that $\mathbf{L}\mathrm{F}$ and $\mathbf{R}\mathrm{U}$ are adjoint, let X be an object in $\mathcal{M}$, let Y be an object in $\mathcal{N}$, let $\widetilde{\mathrm{C}}$ and $\widehat{\mathrm{F}}$ be the constructions in $\mathcal{M}$ as in the proof of Theorem 9.6.4, and let $\widetilde{\mathrm{C}}'$ and $\widehat{\mathrm{F}}'$ be the analogous constructions in $\mathcal{N}$; then we have natural isomorphisms

$$\begin{aligned}
\operatorname{Ho}\mathcal{N}(\mathbf{L}\mathrm{F}X, Y) &= \operatorname{Ho}\mathcal{N}\big(\mathrm{F}(\widetilde{\mathrm{C}}X), Y\big) \\
&= \pi\big(\widehat{\mathrm{F}}'\widetilde{\mathrm{C}}'\mathrm{F}(\widetilde{\mathrm{C}}X), \widehat{\mathrm{F}}'\widetilde{\mathrm{C}}'Y\big) \\
&= \pi\big(\widehat{\mathrm{F}}'\mathrm{F}(\widetilde{\mathrm{C}}X), \widehat{\mathrm{F}}'\widetilde{\mathrm{C}}'Y\big) && \text{because } \mathrm{F}(\widetilde{\mathrm{C}}X) \text{ is cofibrant} \\
&\approx \pi\big(\mathrm{F}(\widetilde{\mathrm{C}}X), \widehat{\mathrm{F}}'\widetilde{\mathrm{C}}'Y\big) && \text{see Lemma 9.8.4} \\
&\approx \pi\big(\mathrm{F}(\widetilde{\mathrm{C}}X), \widehat{\mathrm{F}}'Y\big) && \text{see Lemma 9.8.10 and Lemma 9.8.6} \\
&\approx \pi\big(\widetilde{\mathrm{C}}X, \mathrm{U}(\widehat{\mathrm{F}}'Y)\big) && \text{see Proposition 9.8.7} \\
&\approx \pi\big(\widehat{\mathrm{F}}\widetilde{\mathrm{C}}X, \mathrm{U}(\widehat{\mathrm{F}}'Y)\big) && \text{see Lemma 9.8.4} \\
&\approx \pi\big(\widehat{\mathrm{F}}\widetilde{\mathrm{C}}X, \widehat{\mathrm{F}}\widetilde{\mathrm{C}}\mathrm{U}(\widehat{\mathrm{F}}'Y)\big) && \text{because } \widetilde{\mathrm{C}}\mathrm{U}(\widehat{\mathrm{F}}'Y) \text{ is fibrant} \\
&= \operatorname{Ho}\mathcal{M}\big(X, \mathrm{U}(\widehat{\mathrm{F}}'Y)\big) \\
&= \operatorname{Ho}\mathcal{M}(X, \mathbf{R}\mathrm{U}Y).
\end{aligned}$$

□

THEOREM 9.8.12. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be model categories and let* $\mathrm{F}\colon \mathcal{M} \rightleftarrows \mathcal{N} :\mathrm{U}$ *be a Quillen pair. If* (F, U) *is a pair of Quillen equivalences (see Definition* 9.8.3*), then the total derived functors* $\mathbf{L}\mathrm{F}\colon \operatorname{Ho}\mathcal{M} \rightleftarrows \operatorname{Ho}\mathcal{N} :\mathbf{R}\mathrm{U}$ *(see Theorem* 9.8.11*) are equivalences of the homotopy categories* $\operatorname{Ho}\mathcal{M}$ *and* $\operatorname{Ho}\mathcal{N}$.

PROOF. **Fill this in!!** □

CHAPTER 10

Simplicial model categories

10.1. Simplicial model categories

We adopt the definition of a simplicial model category used in [**26**].

NOTATION 10.1.1. If $f\colon A \to B$ and $p\colon X \to Y$ are maps in $\mathsf{Spc}_{(*)}$, then

$$\mathrm{Map}(A,X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B,Y)$$

will denote the pullback of the diagram of simplicial sets

$$\mathrm{Map}(A,X) \to \mathrm{Map}(A,Y) \leftarrow \mathrm{Map}(B,Y).$$

DEFINITION 10.1.2. A *simplicial model category* is a model category $\mathcal{M}$ (see Definition 8.1.2) together with functors

$$\begin{aligned} \mathrm{Map}(X,Y) &\in \mathsf{SS} \quad \text{for } X, Y \in \mathcal{M} \\ X \otimes K &\in \mathcal{M} \quad \text{for } X \in \mathcal{M} \text{ and } K \in \mathsf{SS} \\ X^K &\in \mathcal{M} \quad \text{for } X \in \mathcal{M} \text{ and } K \in \mathsf{SS} \end{aligned}$$

such that the following two axioms hold:

M6: The above functors are a *closed cartesian action* of SS on $\mathcal{M}$, i.e., for $X, Y \in \mathcal{M}$ and $K, L \in \mathsf{SS}$ there are natural isomorphisms

$$\begin{gathered} X \otimes (K \times L) \approx (X \otimes K) \otimes L \\ X \otimes \Delta[0] \approx X \\ \mathrm{Map}(X,Y)_0 \approx \mathcal{M}(X,Y) \\ \mathrm{Map}(X \otimes K, Y) \approx \mathrm{Map}\bigl(K, \mathrm{Map}(X,Y)\bigr) \approx \mathrm{Map}(X, Y^K) \end{gathered}$$

such that the following three diagrams commute:

$$\begin{tikzcd} X \otimes \bigl(K \times (L \times M)\bigr) \arrow[r] \arrow[d] & (X \otimes K) \otimes (L \times M) \arrow[dd] \\ X \otimes \bigl((K \times L) \times M\bigr) \arrow[d] & \\ \bigl(X \otimes (K \times L)\bigr) \otimes M \arrow[r] & \bigl((X \otimes K) \otimes L\bigr) \otimes M \end{tikzcd}$$

$$\begin{tikzcd} X \otimes (K \times \Delta[0]) \arrow[rr] \arrow[dr] & & (X \otimes K) \otimes \Delta[0] \arrow[dl] \\ & X \otimes K & \end{tikzcd}$$

$$\begin{tikzcd} X \otimes (\Delta[0] \times K) \arrow[rr] \arrow[dr] & & (X \otimes \Delta[0]) \otimes K \arrow[dl] \\ & X \otimes K & \end{tikzcd}$$

M7: If $i\colon A \to B$ is a cofibration and $p\colon X \to Y$ is a fibration, then the map of simplicial sets

$$\mathrm{Map}(B, X) \to \mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, Y)$$

is a fibration that is a trivial fibration if either i or p is a weak equivalence.

Remark 10.1.3. Axiom M7 is the *homotopy lifting extension theorem*, which was originally a theorem of D. M. Kan for categories of simplicial objects (see [**38**]).

Theorem 10.1.4. *The definitions of Definition* 1.1.3 *and Definition* 1.1.5 *give each of our categories* SS, SS_*, Top, *and* Top_* *the structure of a simplicial model category.*

Proof. See [**34**] or [**46**, Chapter II, Section 3]. □

Lemma 10.1.5. *Let* $\mathcal{M}$ *be a simplicial model category. If* X *and* Y *are objects of* $\mathcal{M}$, *then, for every* $n \geq 0$, *the set of* n*-simplices of* $\mathrm{Map}(X, Y)$ *is naturally isomorphic to the set of map* $\mathcal{M}(X \otimes \Delta[n], Y)$.

Proof. Since the set of n-simplices of a simplicial set K is naturally isomorphic to the set of maps $\mathsf{SS}(\Delta[n], K)$, axiom M6 of Definition 10.1.2 yields natural isomorphisms

$$\begin{aligned} \mathrm{Map}(X, Y)_n &\approx \mathsf{SS}\big(\Delta[n], \mathrm{Map}(X, Y)\big) \\ &\approx \mathrm{Map}\big(\Delta[n], \mathrm{Map}(X, Y)\big)_0 \\ &\approx \mathrm{Map}(X \otimes \Delta[n], Y)_0 \\ &\approx \mathcal{M}(X \otimes \Delta[n], Y). \end{aligned}$$

□

Proposition 10.1.6. *Let* $\mathcal{M}$ *be a simplicial model category.*

1. *If* $i\colon A \to B$ *is a cofibration and* X *is fibrant, then the map of simplicial sets* $i^*\colon \mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$ *is a fibration.*
2. *If* A *is cofibrant and* $p\colon X \to Y$ *is a fibration, then the map of simplicial sets* $p_*\colon \mathrm{Map}(A, X) \to \mathrm{Map}(A, Y)$ *is a fibration.*

Proof. This follows from axiom M7 (see Definition 10.1.2). □

Lemma 10.1.7. *Let* $\mathcal{M}$ *be a category with a closed cartesian action of* SS *(see Definition* 10.1.2*). If* $A \to B$ *and* $X \to Y$ *are maps in* $\mathcal{M}$ *and* (K, L) *is a pair of simplicial sets, then the following are equivalent:*

1. *The dotted arrow exists in every solid arrow diagram of the form*

$$\begin{tikzcd} L \arrow[r] \arrow[d] & \mathrm{Map}(B, X) \arrow[d] \\ K \arrow[r] \arrow[ur, dotted] & \mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, Y) \end{tikzcd}$$

2. *The dotted arrow exists in every solid arrow diagram of the form*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X^K \arrow[d] \\ B \arrow[r] \arrow[ur, dotted] & X^L \times_{Y^L} Y^K \end{tikzcd}$$

3. *The dotted arrow exists in every solid arrow diagram of the form*

$$\begin{tikzcd} A \otimes K \amalg_{A \otimes L} B \otimes L \arrow[r] \arrow[d] & X \arrow[d] \\ B \otimes K \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

Proof. This follows from the adjointness properties in axiom M6 (see Definition 10.1.2). □

Proposition 10.1.8. *If $\mathcal{M}$ is a model category with a closed cartesian action of* SS *(see Definition* 10.1.2*), then axiom M7 (see Definition* 10.1.2*) is equivalent to each of the following:*

1. *If $i\colon A \to B$ is a cofibration in $\mathcal{M}$ and $j\colon L \to K$ is a cofibration in* SS, *then the induced map*
$$A \otimes K \amalg_{A \otimes L} B \otimes L \to B \otimes K$$
is a cofibration in $\mathcal{M}$ which is also a weak equivalence if either i or j is a weak equivalence.
2. *If $j\colon L \to K$ is a cofibration in* SS *and $p\colon X \to Y$ is a fibration in $\mathcal{M}$, then the induced map*
$$X^K \to X^L \times_{Y^L} Y^K$$
is a fibration in $\mathcal{M}$ which is also a weak equivalence if either j or p is a weak equivalence.

Proof. This follows from Lemma 10.1.7 and Proposition 8.2.3. □

10.2. Weak equivalences of mapping spaces

Proposition 10.2.1. *Let $\mathcal{M}$ be a simplicial model category.*

1. *If X is cofibrant and $g\colon Y \to Z$ is a trivial fibration, then g induces a trivial fibration of simplicial sets $g_*\colon \mathrm{Map}(X, Y) \to \mathrm{Map}(X, Z)$.*
2. *If Z is fibrant and $h\colon X \to Y$ is a trivial cofibration, then h induces a trivial fibration of simplicial sets $h^*\colon \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$.*

Proof. This follows directly from axiom M7 (see Definition 10.1.2). □

Corollary 10.2.2. *Let $\mathcal{M}$ be a simplicial model category.*

1. *If X is cofibrant and $g\colon Y \to Z$ is a weak equivalence of fibrant objects, then g induces a weak equivalence of simplicial sets $g_*\colon \mathrm{Map}(X, Y) \to \mathrm{Map}(X, Z)$.*
2. *If Z is fibrant and $h\colon X \to Y$ is a weak equivalence of cofibrant objects, then h induces a weak equivalence of simplicial sets $h\colon \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$.*

Proof. This follows from Proposition 10.2.1 and Corollary 8.5.2. □

LEMMA 10.2.3. *Let $\mathcal{C}$ be a small category and let $\mathcal{M}$ be a simplicial model category.*

1. *If $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a diagram in $\mathcal{M}$ and K a simplicial set, then there is a natural isomorphism*
$$(\operatorname{colim} \boldsymbol{X}) \otimes K \approx \operatorname{colim}(\boldsymbol{X} \otimes K).$$
2. *If X is an object of $\mathcal{M}$ and $\boldsymbol{K}\colon \mathcal{C} \to \mathsf{SS}$ is a diagram of simplicial sets, then there is a natural isomorphism*
$$X \otimes (\operatorname{colim} \boldsymbol{K}) \approx \operatorname{colim}(X \otimes \boldsymbol{K}).$$

PROOF. We will prove part 1; the proof of part 2 is similar.

If Y is an object of $\mathcal{M}$, then we have natural isomorphisms
$$\begin{aligned}\mathcal{M}\big((\operatorname{colim} \boldsymbol{X}) \otimes K, Y\big) &\approx \mathcal{M}(\operatorname{colim} \boldsymbol{X}, Y^K)\\ &\approx \lim \mathcal{M}(\boldsymbol{X}, Y^K)\\ &\approx \lim \mathcal{M}(\boldsymbol{X} \otimes K, Y)\\ &\approx \mathcal{M}\big(\operatorname{colim}(\boldsymbol{X} \otimes K), Y\big)\end{aligned}$$
(see axiom M6 of Definition 10.1.2), and the composition of these must be induced by a natural isomorphism $(\operatorname{colim} \boldsymbol{X}) \otimes K \approx \operatorname{colim}(\boldsymbol{X} \otimes K)$. $\square$

PROPOSITION 10.2.4. *If $\mathcal{M}$ is a simplicial model category, $\mathcal{C}$ is a small category, $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a diagram in $\mathcal{M}$, and Y is an object of $\mathcal{M}$, then there are natural isomorphisms of simplicial sets*
$$\begin{aligned}\operatorname{Map}(\operatorname{colim} \boldsymbol{X}, Y) &\approx \lim \operatorname{Map}(\boldsymbol{X}, Y)\\ \operatorname{Map}(Y, \lim \boldsymbol{X}) &\approx \lim \operatorname{Map}(Y, \boldsymbol{X}).\end{aligned}$$

PROOF. We will prove that the first isomorphism exists; the proof of the second is similar.

For every $n \geq 0$, Lemma 10.1.5, Lemma 10.2.3, and axiom M6 of Definition 10.1.2 yield natural isomorphisms
$$\begin{aligned}\operatorname{Map}(\operatorname{colim} \boldsymbol{X}, Y)_n &\approx \mathsf{SS}\big(\Delta[n], \operatorname{Map}(\operatorname{colim} \boldsymbol{X}, Y)\big)\\ &\approx \mathcal{M}\big((\operatorname{colim} \boldsymbol{X}) \otimes \Delta[n], Y\big)\\ &\approx \mathcal{M}\big(\operatorname{colim}(\boldsymbol{X} \otimes \Delta[n]), Y\big)\\ &\approx \lim \mathcal{M}(\boldsymbol{X} \otimes \Delta[n], Y)\\ &\approx \lim \mathsf{SS}\big(\Delta[n], \operatorname{Map}(\boldsymbol{X}, Y)\big)\\ &\approx \lim \operatorname{Map}(\boldsymbol{X}, Y)_n.\end{aligned}$$
$\square$

COROLLARY 10.2.5. *Let $\mathcal{M}$ be a simplicial model category, and let Y be an object of $\mathcal{M}$. If S is a set and X_s is an object of $\mathcal{M}$ for every $s \in S$, then there is a natural isomorphism of simplicial sets*
$$\operatorname{Map}\Big(\coprod_{s\in S} X_s, Y\Big) \approx \prod_{s\in S} \operatorname{Map}(X_s, Y).$$

PROOF. This follows from Lemma 10.1.5 and Proposition 10.2.4. $\square$

10.3. Homotopy lifting

If i and p are maps in a model category $\mathcal{M}$, axiom M4 (see Definition 8.1.2) implies that (i, p) is a lifting-extension pair (see Definition 8.2.1) if i is a cofibration, p is a fibration, and at least one of i and p is a weak equivalence. In a simplicial model category, a stronger statement is possible. Axiom M7 (see Definition 10.1.2) says that if $i: A \to B$ is a cofibration and $p: X \to Y$ is a fibration, then the map of simplicial sets

$$\mathrm{Map}(B, X) \longrightarrow \mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, Y) \tag{10.3.1}$$

is a fibration, and it is a trivial fibration if at least one of i and p is a weak equivalence.

Definition 10.3.2. Let $\mathcal{M}$ be a simplicial model category. If $i: A \to B$ and $p: X \to Y$ are maps for which the map of simplicial sets (10.3.1) is a trivial fibration, then (i, p) is called a *homotopy lifting extension pair*, and i is said to have the *homotopy left lifting property* with respect to p and p is said to have the *homotopy right lifting property* with respect to i.

Proposition 10.3.3. *Let $\mathcal{M}$ be a simplicial model category.*

1. *If B is cofibrant and $p: X \to Y$ is a fibration, then p has the homotopy right lifting property with respect to the map from the initial object to B if and only if p induces a weak equivalence $p_*: \mathrm{Map}(B, X) \cong \mathrm{Map}(B, Y)$.*
2. *If X is fibrant and $i: A \to B$ is a cofibration, then i has the homotopy left lifting property with respect to the map from X to the terminal object if and only if i induces a weak equivalence $i^*: \mathrm{Map}(B, X) \cong \mathrm{Map}(A, X)$.*

Proof. This follows from Proposition 10.1.6. □

Proposition 10.3.4. *Let $\mathcal{M}$ be a simplicial model category. If $i: A \to B$ and $p: X \to Y$ are maps such that (i, p) is a homotopy lifting-extension pair (see Definition 10.3.2), then (i, p) is a lifting-extension pair (see Definition 8.2.1).*

Proof. This follows because a trivial fibration of simplicial sets is surjective on the set of vertices. □

Proposition 10.3.5. *Let $\mathcal{M}$ be a simplicial model category.*

1. *A map is a cofibration if and only if it has the homotopy left lifting property with respect to all trivial fibrations.*
2. *A map is a trivial cofibration if and only if it has the homotopy left lifting property with respect to all fibrations.*
3. *A map is a fibration if and only if it has the homotopy right lifting property with respect to all trivial cofibrations.*
4. *A map is a trivial fibration if and only if it has the homotopy right lifting property with respect to all cofibrations.*

Proof. This follows from axiom M7 (see Definition 10.1.2), Proposition 10.3.4, and Proposition 8.2.3. □

The following lemma describes the homotopy lifting property in terms of the lifting property.

Lemma 10.3.6. *If $i: A \to B$ and $p: X \to Y$ are maps in a simplicial model category, then the following are equivalent:*

1. *The pair (i, p) is a homotopy lifting-extension pair (see Definition 10.3.2).*
2. *For every pair of simplicial sets (K, L), the map p has the right lifting property with respect to the map*

$$A \otimes K \amalg_{A\otimes L} B \otimes L \to B \otimes K.$$

3. *For every $n \geq 0$, the map p has the right lifting property with respect to the map*

$$A \otimes \Delta[n] \amalg_{A\otimes \partial\Delta[n]} B \otimes \partial\Delta[n] \to B \otimes \Delta[n].$$

4. *For every pair of simplicial sets (K, L), the map i has the left lifting property with respect to the map*

$$X^K \to Y^K \times_{Y^L} X^L.$$

5. *For every $n \geq 0$, the map i has the left lifting property with respect to the map*

$$X^{\Delta[n]} \to Y^{\Delta[n]} \times_{Y^{\partial\Delta[n]}} X^{\partial\Delta[n]}.$$

Proof. Since a map of simplicial sets is a cofibration if and only if it is an inclusion and a trivial fibration if and only if it has the right lifting property with respect to the maps $\partial\Delta[n] \to \Delta[n]$ for $n \geq 0$, this follows from Lemma 10.1.7 and Proposition 8.2.3. □

Lemma 10.3.7. *Let $\mathcal{M}$ be a simplicial model category, and let p be a map in $\mathcal{M}$.*

1. *The class of maps with the homotopy left lifting property with respect to p is closed under pushouts.*
2. *The class of maps with the homotopy right lifting property with respect to p is closed under pullbacks.*

Proof. This follows from Lemma 10.3.6 and Lemma 8.2.5. □

Lemma 10.3.8. *Let $\mathcal{M}$ be a simplicial model category, and let p be a map in $\mathcal{M}$.*

1. *The class of maps with the homotopy left lifting property with respect to p is closed under retracts.*
2. *The class of maps with the homotopy right lifting property with respect to p is closed under retracts.*

Proof. This follows from Lemma 10.3.6 and Lemma 8.2.7. □

Proposition 10.3.9. *Let $\mathcal{M}$ be a simplicial model category, and let $\mathcal{C}$ be a class of maps in $\mathcal{M}$.*

1. *If every map $g\colon X \to Y$ in $\mathcal{M}$ can be factored as $X \xrightarrow{j} W \xrightarrow{p} Y$ where p is in $\mathcal{C}$ and j has the homotopy left lifting property with respect to every map in $\mathcal{C}$, then a map has the left lifting property with respect to every map in $\mathcal{C}$ if and only if it has the homotopy left lifting property with respect to every map in $\mathcal{C}$.*
2. *If every map $g\colon X \to Y$ in $\mathcal{M}$ can be factored as $X \xrightarrow{j} W \xrightarrow{p} Y$ where j is in $\mathcal{C}$ and p has the homotopy right lifting property with respect to every map in $\mathcal{C}$, then a map has the right lifting property with respect to every map in*

$\mathcal{C}$ if and only if it has the homotopy right lifting property with respect to every map in $\mathcal{C}$.

Proof. We will prove part 1; the proof of part 2 is similar.

Proposition 10.3.4 implies that if a map has the homotopy left lifting property with respect to every map in $\mathcal{C}$, then it has the left lifting property with respect to every map in $\mathcal{C}$.

Conversely, if the map $g\colon X \to Y$ has the left lifting property with respect to every map in $\mathcal{C}$, factor g as $X \xrightarrow{j} W \xrightarrow{p} Y$ where p is in $\mathcal{C}$ and j has the homotopy left lifting property with respect to every map in $\mathcal{C}$. The retract argument (see Proposition 8.2.2) implies that g is a retract of j, and so the result follows from Lemma 10.3.8. $\square$

Proposition 10.3.10. *Let $\mathcal{M}$ be a simplicial model category.*

1. *If $i\colon A \to B$ has the homotopy left lifting property with respect to $p\colon X \to Y$ and (K, L) is a pair of simplicial sets, then $A \otimes K \amalg_{A\otimes L} B \otimes L \to B \otimes K$ has the homotopy left lifting property with respect to p.*
2. *If $p\colon X \to Y$ has the homotopy right lifting property with respect to $i\colon A \to B$ and (K, L) is a pair of simplicial sets, then $X^K \to Y^K \times_{Y^L} X^L$ has the homotopy right lifting property with respect to i.*

Proof. We will prove part 2; the proof of part 1 is similar.

Lemma 10.3.6 implies that it is sufficient to show that the map

$$X^K \to Y^K \times_{Y^L} X^L$$

has the right lifting property with respect to the map

$$A \otimes \Delta[n] \amalg_{A\otimes\partial\Delta[n]} B \otimes \partial\Delta[n] \to B \otimes \Delta[n].$$

Lemma 10.1.7 implies that this is equivalent to showing that the map $X \to Y$ has the right lifting property with respect to the map

$$\begin{aligned}(A \otimes \Delta[n] \amalg_{A\otimes\partial\Delta[n]} B \otimes \partial\Delta[n]) \otimes K \amalg_{(A\otimes\Delta[n]\amalg_{A\otimes\partial\Delta[n]} B\otimes\partial\Delta[n])\otimes L} (B \otimes \Delta[n]) \otimes L \\ \to (B \otimes \Delta[n]) \otimes K.\end{aligned}$$

Lemma 10.2.3 and the isomorphisms of axiom M6 (see Definition 10.1.2) imply that this map is isomorphic to the map

$$\begin{aligned}B \otimes (\partial\Delta[n] \times K \amalg_{\partial\Delta[n]\times L} \Delta[n] \times L) \amalg_{A\otimes(\partial\Delta[n]\times K \amalg_{\partial\Delta[n]\times L}\Delta[n]\times L)} A \otimes (\Delta[n] \times K) \\ \to B \otimes (\Delta[n] \times K).\end{aligned}$$

Lemma 10.3.6 implies that this map has the left lifting property with respect to $X \to Y$, and so the proof is complete. $\square$

Corollary 10.3.11. *Let $\mathcal{M}$ be a simplicial model category, and let (K, L) be a pair of simplicial sets.*

1. *If $A \to B$ is a trivial cofibration in $\mathcal{M}$, then $A \otimes K \amalg_{A\otimes L} B \otimes L \to B \otimes K$ is also a trivial cofibration.*
2. *If $X \to Y$ is a trivial fibration in $\mathcal{M}$, then $X^K \to Y^K \times_{Y^L} X^L$ is also a trivial fibration.*

Proof. This follows from Proposition 10.3.5 and Proposition 10.3.10. $\square$

10.4. Simplicial homotopy

10.4.1. Definitions. If X is cofibrant and Y is fibrant, then all notions of homotopy for maps from X to Y coincide and are equivalence relations (see Proposition 10.4.4). Since not all of our spaces are cofibrant and fibrant, we need to consider the version of homotopy most naturally associated with weak equivalences of function spaces: simplicial homotopy.

DEFINITION 10.4.2. Let X and Y be objects of a simplicial model category, and let g and h be maps $X \to Y$ (i.e., vertices of $\mathrm{Map}(X, Y)$ (see Definition 10.1.2)). We will follow Quillen [**46**, Chapter II, Section 1, Definition 4] and say that g is *strictly simplicially homotopic* to h ($g \overset{ss}{\simeq} h$) if there is a one simplex of $\mathrm{Map}(X, Y)$ whose initial vertex is g and whose final vertex is h, and that g and h are *simplicially homotopic* ($g \overset{s}{\simeq} h$) if they are equivalent under the equivalence relation generated by the relation of strict simplicial homotopy.

DEFINITION 10.4.3. The map $g\colon X \to Y$ is a *simplicial homotopy equivalence* if there is a map $h\colon Y \to X$ such that $gh \overset{s}{\simeq} 1_Y$ and $hg \overset{s}{\simeq} 1_X$.

In general, strict simplicial homotopy need not be an equivalence relation, since $\mathrm{Map}(X, Y)$ need not be a fibrant simplicial set. In $\mathsf{Top}_{(*)}$, however, $\mathrm{Map}(X, Y)$ is isomorphic to the total singular complex of the space (in the category of compactly generated Hausdorff spaces) of continuous maps $X \to Y$, and so it is always a fibrant simplicial set (see Corollary 1.1.8). (Strict simplicial homotopy in $\mathsf{Top}_{(*)}$ is exactly the classical definition of homotopy which is, of course, always an equivalence relation.) In $\mathsf{SS}_{(*)}$ every space is cofibrant, and so $\mathrm{Map}(X, Y)$ will be a fibrant simplicial set if Y is a fibrant space (see Theorem 10.1.4).

PROPOSITION 10.4.4 (Quillen). *If g and h are simplicially homotopic, then they are both left homotopic and right homotopic. If X is cofibrant and Y is fibrant, then the strict simplicial, simplicial, left, and right homotopy relations on the set of maps $X \to Y$ coincide and are equivalence relations.*

PROOF. This is [**46**, Chapter II, Section 2, Proposition 5]. □

This immediately implies the following corollaries.

COROLLARY 10.4.5. *If g and h are simplicially homotopic, then they represent the same morphism in the homotopy category* $\mathrm{Ho}\,\mathcal{M}$.

COROLLARY 10.4.6. *A simplicial homotopy equivalence is a weak equivalence.*

10.4.7. Simplicially homotopic maps.

PROPOSITION 10.4.8. *If X and Y are objects of a simplicial model category and g and h are maps from X to Y, then $g \overset{s}{\simeq} h$ if and only if g and h are in the same component of the simplicial set* $\mathrm{Map}(X, Y)$.

PROOF. This follows directly from the definitions. □

COROLLARY 10.4.9. *Let X, Y, and W be objects of a simplicial model category.*

1. *If the map $g\colon X \to Y$ induces a weak equivalence $g_*\colon \mathrm{Map}(W, X) \cong \mathrm{Map}(W, Y)$, then g induces an isomorphism of the sets of simplicial homotopy classes of maps $g_*\colon [W, X] \approx [W, Y]$.*

2. *If the map $g\colon X \to Y$ induces a weak equivalence $g^*\colon \operatorname{Map}(Y,W) \cong \operatorname{Map}(X,W)$, then g induces an isomorphism of the sets of simplicial homotopy classes of maps $g^*\colon [Y,W] \approx [X,W]$.*

PROOF. This follows from Proposition 10.4.8. □

COROLLARY 10.4.10. *Let X, Y, and W be objects of a simplicial model category.*

1. *If W is cofibrant and $g\colon X \to Y$ is a trivial fibration, then g induces an isomorphism of the sets of simplicial homotopy classes of maps $g_*\colon [W,X] \approx [W,Y]$.*
2. *If W is fibrant and $g\colon X \to Y$ is a trivial cofibration, then g induces an isomorphism of the sets of simplicial homotopy classes of maps $g^*\colon [Y,W] \approx [X,W]$.*

PROOF. This follows from Proposition 10.2.1 and Corollary 10.4.9. □

COROLLARY 10.4.11. *Let X, Y, and W be objects of a simplicial model category.*

1. *If W is cofibrant and $g\colon X \to Y$ is a weak equivalence of fibrant objects, then g induces an isomorphism of the sets of simplicial homotopy classes of maps $g_*\colon [W,X] \approx [W,Y]$.*
2. *If W is fibrant and $g\colon X \to Y$ is a weak equivalence of cofibrant objects, then g induces an isomorphism of the sets of simplicial homotopy classes of maps $g^*\colon [Y,W] \approx [X,W]$.*

PROOF. This follows from Corollary 10.2.2 and Corollary 10.4.9. □

COROLLARY 10.4.12. *If $g,h\colon X \to Y$ are simplicially homotopic, $j\colon W \to X$ and $k\colon Y \to Z$, then $kg \overset{s}{\simeq} kh$ and $gj \overset{s}{\simeq} hj$.*

PROOF. This follows from Proposition 10.4.8. □

DEFINITION 10.4.13. A *generalized interval* is a simplicial set that is a union of finitely many one simplices with vertices identified so that its geometric realization is homeomorphic to a unit interval.

PROPOSITION 10.4.14. *If $g,h\colon X \to Y$, then g and h are simplicially homotopic if and only if there is a generalized interval J and a map of simplicial sets $J \to \operatorname{Map}(X,Y)$ taking the ends of J to g and h.*

PROOF. This follows from Proposition 10.4.8. □

REMARK 10.4.15. A map $J \to \operatorname{Map}(X,Y)$ as in Proposition 10.4.14 will be called a *simplicial homotopy* from g to h. The maps $X \otimes J \to Y$ and $X \to Y^J$ that correspond under the isomorphisms of Definition 10.1.2 will also be called simplicial homotopies from g to h.

PROPOSITION 10.4.16. *If $i\colon A \to B$ has the homotopy left lifting property with respect to $p\colon X \to Y$ (see Definition 10.3.2), then for every commutative solid arrow* *diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted, "h"] & Y \end{tikzcd}$$

there exists a map $h\colon B \to X$ making both triangles commute, and the map h is unique up to simplicial homotopy.

PROOF. This follows from Definition 10.3.2 and Proposition 10.4.8. □

COROLLARY 10.4.17. *If we have the solid arrow diagram*

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & C \arrow[d, "p"] \\ B \arrow[r, "q"'] \arrow[ur, dotted, "h"] & D \end{tikzcd}$$

in a simplicial model category such that both i and j have the homotopy left lifting property with respect to each of p and q, then there exists a map $h\colon B \to C$, unique up to simplicial homotopy, such that $hi = j$ and $ph = q$, and any such map is a simplicial homotopy equivalence.

PROOF. This follows from Proposition 10.4.16. □

LEMMA 10.4.18. *An isomorphism in a simplicial model category has both the homotopy left lifting property and the homotopy right lifting property with respect to every map in the category.*

PROOF. This follows from the fact that an isomorphism induces an isomorphism of the simplicial set of maps from (or to) any fixed object. □

PROPOSITION 10.4.19. *Let $g\colon X \to Y$ be a map in a simplicial model category.*

1. *If g has the homotopy left lifting property with respect to the maps from each of X and Y to the terminal object of the category, then g is the inclusion of a strong deformation retract, i.e., there is a map $r\colon Y \to X$ such that $rg = 1_X$ and $gr \overset{s}{\simeq} 1_Y$, where the simplicial homotopy (see Remark 10.4.15) is constant on X.*
2. *If g has the homotopy right lifting property with respect to the maps from the initial object of the category to each of X and Y, then there is a map $s\colon Y \to X$ such that $gs = 1_Y$ and $sg \overset{s}{\simeq} 1_X$, where the simplicial homotopy (see Remark 10.4.15) lies over the identity map of Y.*

PROOF. We will prove part 1; the proof of part 2 is similar.

We have the solid arrow diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "g"'] & X \arrow[d] \\ Y \arrow[r] \arrow[ur, dotted, "r"] & * \end{tikzcd}$$

(in which "$*$" represents the terminal object of the category), and so Corollary 10.4.17 and Lemma 10.4.18 imply that there exists a map $r\colon Y \to X$ such that $rg = 1_X$.

Thus, we can construct the solid arrow diagram

$$\begin{tikzcd} X \otimes \Delta[1] \amalg_{X\otimes \partial\Delta[1]} Y \otimes \partial\Delta[1] \arrow[r] \arrow[d] & Y \arrow[d] \\ Y \otimes \Delta[1] \arrow[r] \arrow[ur, dotted] & * \end{tikzcd}$$

in which the top map is $g \circ \mathrm{pr}_X$ on $X \otimes \Delta[1]$ and $gr \amalg 1_Y$ on $Y \otimes \partial\Delta[1]$. Proposition 10.3.10 implies that the vertical map on the left has the homotopy left lifting property with respect to the vertical map on the right, and so Proposition 10.3.4 implies that the dotted arrow exists, and the proof is complete. □

COROLLARY 10.4.20. *Let $g\colon X \to Y$ be a map in a simplicial model category.*

1. *If both X and Y are fibrant and g is a trivial cofibration, then g is a simplicial homotopy equivalence. In particular, g is the inclusion of a strong deformation retract.*
2. *If both X and Y are cofibrant and g is a trivial fibration, then g is a simplicial homotopy equivalence. In particular, g has a right inverse that is a simplicial homotopy inverse.*

PROOF. This follows from Proposition 10.4.19. □

10.4.21. Weak equivalences of function spaces.

PROPOSITION 10.4.22. *If $g, h\colon X \to Y$ are simplicially homotopic and W is any object, then $g_* \overset{s}{\simeq} h_*\colon \mathrm{Map}(W, X) \to \mathrm{Map}(W, Y)$ and $g^* \overset{s}{\simeq} h^*\colon \mathrm{Map}(Y, W) \to \mathrm{Map}(X, W)$.*

PROOF. Let $X \to Y^J$ be a simplicial homotopy from g to h (where J is a generalized interval). We then have the map of simplicial sets $\mathrm{Map}(W, X) \to \mathrm{Map}(W, Y^J)$, which corresponds to a map $\mathrm{Map}(W, X) \to \mathrm{Map}(W \otimes J, Y)$, which corresponds to a map $\mathrm{Map}(W, X) \to \mathrm{Map}\big(J, \mathrm{Map}(W, Y)\big)$, which corresponds to a map $\mathrm{Map}(W, X) \otimes J \to \mathrm{Map}(W, Y)$, which is a simplicial homotopy from g_* to h_*.

The second assertion is proved similarly, starting with a simplicial homotopy $X \otimes J \to Y$. □

COROLLARY 10.4.23. *If $g\colon X \to Y$ is a simplicial homotopy equivalence, then, for any object W, the maps $g_*\colon \mathrm{Map}(W, X) \to \mathrm{Map}(W, Y)$ and $g^*\colon \mathrm{Map}(Y, W) \to \mathrm{Map}(X, W)$ are simplicial homotopy equivalences of simplicial sets.*

PROOF. This follows from Proposition 10.4.22. □

Note that these corollaries made no assumptions about whether any of the objects were fibrant or cofibrant.

PROPOSITION 10.4.24. *If $g\colon X \to Y$ is a map in a simplicial model category, then g is a simplicial homotopy equivalence if either of the following two conditions is satisfied:*

1. *The map g induces isomorphisms of the sets of simplicial homotopy classes of maps $g_*\colon [X, X] \approx [X, Y]$ and $g_*\colon [Y, X] \approx [Y, Y]$.*
2. *The map g induces isomorphisms of the sets of simplicial homotopy classes of maps $g^*\colon [Y, X] \approx [X, X]$ and $g^*\colon [Y, Y] \approx [X, Y]$.*

PROOF. We will prove this using condition 1; the proof using condition 2 is similar.

The isomorphism $g_*\colon [Y,X] \approx [Y,Y]$ implies that there is a map $h\colon Y \to X$ such that $gh \overset{s}{\simeq} 1_Y$. Corollary 10.4.12 and the isomorphism $g_*\colon [X,X] \approx [X,Y]$ now imply that h induces an isomorphism $h_*\colon [X,Y] \approx [X,X]$, and so there is a map $k\colon X \to Y$ such that $hk \overset{s}{\simeq} 1_X$. Thus, h is a simplicial homotopy equivalence, and so g is its inverse and is thus a simplicial homotopy equivalence as well. $\square$

PROPOSITION 10.4.25. *If $g\colon X \to Y$ is a map in a simplicial model category, then g is a simplicial homotopy equivalence if either of the following two conditions is satisfied:*

1. *The map g induces weak equivalences of simplicial sets $g_*\colon \mathrm{Map}(X,X) \cong \mathrm{Map}(X,Y)$ and $g_*\colon \mathrm{Map}(Y,X) \cong \mathrm{Map}(Y,Y)$.*
2. *The map g induces weak equivalences of simplicial sets $g^*\colon \mathrm{Map}(Y,X) \cong \mathrm{Map}(X,X)$ and $g^*\colon \mathrm{Map}(Y,Y) \cong \mathrm{Map}(X,Y)$.*

PROOF. This follows from Proposition 10.4.24 and Corollary 10.4.9. $\square$

10.5. Detecting weak equivalences

PROPOSITION 10.5.1. *Let $\mathcal{M}$ be a simplicial model category. If $g\colon X \to Y$ is a map in $\mathcal{M}$, then g is a weak equivalence if either of the following two conditions is satisfied:*

1. *For every fibrant object Z, the map of function spaces $g^*\colon \mathrm{Map}(Y,Z) \to \mathrm{Map}(X,Z)$ is a weak equivalence of simplicial sets.*
2. *For every cofibrant object W, the map of function spaces $g_*\colon \mathrm{Map}(W,X) \to \mathrm{Map}(W,Y)$ is a weak equivalence of simplicial sets.*

PROOF. We will prove part 1; the proof of part 2 is dual.

Choose cofibrant fibrant approximations (see Definition 9.1.1) $i_X\colon X \to \widetilde{X}$ and $i_Y\colon Y \to \widetilde{Y}$ and a fibrant approximation $\tilde{g}\colon \widetilde{X} \to \widetilde{Y}$ to g (see Definition 9.1.8). If Z is a fibrant object, then we have the commutative square

$$\begin{tikzcd}
\mathrm{Map}(X,Z) & \mathrm{Map}(Y,Z) \arrow[l, "g^*"'] \\
\mathrm{Map}(\widetilde{X},Z) \arrow[u, "(i_X)^*"] & \mathrm{Map}(\widetilde{Y},Z) \arrow[l, "\tilde{g}^*"'] \arrow[u, "(i_Y)^*"']
\end{tikzcd}$$

in which all the maps except $\tilde{g}^*$ are weak equivalences of simplicial sets (see Proposition 10.2.1). This implies that $\tilde{g}^*$ is also a weak equivalence, and so Proposition 10.4.25 implies implies that $\tilde{g}$ is a simplicial homotopy equivalence. Thus, $\tilde{g}$ is a weak equivalence, and so g is a weak equivalence and the proof is complete. $\square$

PROPOSITION 10.5.2. *Let $\mathcal{M}$ be a simplicial model category, let $g\colon X \to Y$ be a map in $\mathcal{M}$, and let W be an object in $\mathcal{M}$.*

1. *If W is cofibrant and if $\tilde{g}\colon \widetilde{X} \to \widetilde{Y}$ is a fibrant approximation to g (see Definition 9.1.8) such that the induced map of simplicial sets $\tilde{g}_*\colon \mathrm{Map}(W,\widetilde{X}) \to \mathrm{Map}(W,\widetilde{Y})$ is a weak equivalence, then for any other fibrant approximation $\hat{g}\colon \widehat{X} \to \widehat{Y}$ to g, the induced map of simplicial sets $\hat{g}_*\colon \mathrm{Map}(W,\widehat{X}) \to \mathrm{Map}(W,\widehat{Y})$ is a weak equivalence.*

2. *If W is fibrant and if $\tilde{g}\colon \widetilde{X} \to \widetilde{Y}$ is a cofibrant approximation to g (see Definition 9.1.8) such that the induced map of simplicial sets $\tilde{g}^*\colon \operatorname{Map}(\widetilde{Y}, W) \to \operatorname{Map}(\widetilde{X}, W)$ is a weak equivalence, then for any other cofibrant approximation $\hat{g}\colon \widehat{X} \to \widehat{Y}$ to g, the induced map of simplicial sets $\hat{g}^*\colon \operatorname{Map}(\widehat{Y}, W) \to \operatorname{Map}(\widehat{Y}, W)$ is a weak equivalence.*

Proof. This follows from Proposition 9.3.2 and Proposition 10.2.1. ☐

Proposition 10.5.3. *Let $\mathcal{M}$ be a simplicial model category, let $f\colon X \to Y$ be a map in $\mathcal{M}$, and let W be an object in $\mathcal{M}$.*

1. *If X and Y are fibrant and $\widetilde{W} \to W$ is a cofibrant approximation to W such that the induced map of simplicial sets $f_*\colon \operatorname{Map}(\widetilde{W}, X) \to \operatorname{Map}(\widetilde{W}, Y)$ is a weak equivalence, then for any other cofibrant approximation $\widehat{W} \to W$ to W, the induced map of simplicial sets $f_*\colon \operatorname{Map}(\widehat{W}, X) \to \operatorname{Map}(\widehat{W}, Y)$ is a weak equivalence.*
2. *If X and Y are cofibrant and $W \to \widetilde{W}$ is a fibrant approximation to W such that the induced map of simplicial sets $f^*\colon \operatorname{Map}(Y, \widetilde{W}) \to \operatorname{Map}(X, \widetilde{W})$ is a weak equivalence, then for any other fibrant approximation $W \to \widehat{W}$ to W, the induced map of simplicial sets $f^*\colon \operatorname{Map}(Y, \widehat{W}) \to \operatorname{Map}(X, \widehat{W})$ is a weak equivalence.*

Proof. We will prove part 1; the proof of part 2 is dual.

Choose a fibrant cofibrant approximation $\overline{W} \to W$ to W (see Proposition 9.1.2). There are maps of cofibrant approximations (see Definition 9.1.3) $\widetilde{W} \to \overline{W}$ and $\widehat{W} \to \overline{W}$, both of which are weak equivalences (see Proposition 9.1.6). Thus, we have the diagram

$$\begin{tikzcd}
\operatorname{Map}(\widehat{W}, X) \arrow[d] & \operatorname{Map}(\overline{W}, X) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d] & \operatorname{Map}(\widetilde{W}, X) \arrow[d] \\
\operatorname{Map}(\widehat{W}, Y) & \operatorname{Map}(\overline{W}, Y) \arrow[l, "\cong"] \arrow[r, "\cong"'] & \operatorname{Map}(\widetilde{W}, Y)
\end{tikzcd}$$

and Corollary 10.2.2 implies that all the horizontal maps are weak equivalences. ☐

Theorem 10.5.4. *If $g\colon X \to Y$ is a map in a simplicial model category, then the following are equivalent:*

1. *The map g is a weak equivalence.*
2. *For some fibrant approximation $\tilde{g}\colon \widetilde{X} \to \widetilde{Y}$ to g (see Definition 9.1.8) and every cofibrant object W, the map of simplicial sets $\tilde{g}_*\colon \operatorname{Map}(W, \widetilde{X}) \to \operatorname{Map}(W, \widetilde{Y})$ is a weak equivalence.*
3. *For every fibrant approximation $\tilde{g}\colon \widetilde{X} \to \widetilde{Y}$ to g and every cofibrant object W, the map of simplicial sets $\tilde{g}_*\colon \operatorname{Map}(W, \widetilde{X}) \to \operatorname{Map}(W, \widetilde{Y})$ is a weak equivalence.*
4. *For some cofibrant approximation $\tilde{g}\colon \widetilde{X} \to \widetilde{Y}$ to g (see Definition 9.1.8) and every fibrant object Z, the map of simplicial sets $\tilde{g}^*\colon \operatorname{Map}(\widetilde{Y}, Z) \to \operatorname{Map}(\widetilde{X}, Z)$ is a weak equivalence.*

5. *For every cofibrant approximation* $\tilde{g}\colon \widetilde{X} \to \widetilde{Y}$ *to* g *and every fibrant object* Z*, the map of simplicial sets* $\tilde{g}^*\colon \mathrm{Map}(\widetilde{Y}, Z) \to \mathrm{Map}(\widetilde{X}, Z)$ *is a weak equivalence.*

PROOF. Proposition 10.5.2 implies that 2 is equivalent to 3 and that 4 is equivalent to 5.

Proposition 10.5.1 implies that any of 2, 3, 4 or 5 implies 1.

Corollary 10.2.2 implies that 1 implies both 2 and 4, and so the proof is complete. □

COROLLARY 10.5.5. *Let* $\mathcal{M}$ *be a simplicial model category, and let* $g\colon X \to Y$ *be a map in* $\mathcal{M}$.

1. *If* X *and* Y *are fibrant, then* g *is a weak equivalence if and only if, for every cofibrant object* W *in* $\mathcal{M}$*, the map* $g_*\colon \mathrm{Map}(W, X) \to \mathrm{Map}(W, Y)$ *is a weak equivalence of simplicial sets.*
2. *If* X *and* Y *are cofibrant, then* g *is a weak equivalence if and only if, for every fibrant object* Z *in* $\mathcal{M}$*, the map* $g^*\colon \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ *is a weak equivalence of simplicial sets.*

PROOF. This follows from Theorem 10.5.4. □

PROPOSITION 10.5.6. *If* $\mathcal{M}$ *is a simplicial model category,* λ *is an ordinal, and*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "g_0"'] & X_1 \arrow[r] \arrow[d, "g_1"'] & X_2 \arrow[r] \arrow[d, "g_2"'] & \cdots \\ Y_0 \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r] & \cdots \end{tikzcd}$$

is a map of λ*-sequences in* $\mathcal{M}$ *such that*

1. *each of the maps* $g_\alpha\colon X_\alpha \to Y_\alpha$ *(for* $\alpha < \lambda$*) is a weak equivalence of cofibrant objects and*
2. *each of the maps* $X_\alpha \to X_{\alpha+1}$ *and* $Y_\alpha \to Y_{\alpha+1}$ *(for* $\alpha < \lambda$*) is a cofibration,*

then the induced map of colimits $(\operatorname{colim} g_\alpha)\colon \operatorname{colim} X_\alpha \to \operatorname{colim} Y_\alpha$ *is a weak equivalence.*

PROOF. Let Z be a fibrant object of $\mathcal{M}$. Theorem 10.5.4 implies that it is sufficient to show that the map $\mathrm{Map}(\operatorname{colim} Y_\alpha, Z) \to \mathrm{Map}(\operatorname{colim} X_\alpha, Z)$ is a weak equivalence of simplicial sets.

Corollary 10.2.2 implies that the map $g^*\colon \mathrm{Map}(Y_\alpha, Z) \to \mathrm{Map}(X_\alpha, Z)$ is a weak equivalence of fibrant simplicial sets for every $\alpha < \lambda$, and so the diagram

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{Map}(Y_2, Z) \arrow[r] \arrow[d] & \mathrm{Map}(Y_1, Z) \arrow[r] \arrow[d] & \mathrm{Map}(Y_0, Z) \arrow[d] \\ \cdots \arrow[r] & \mathrm{Map}(X_2, Z) \arrow[r] & \mathrm{Map}(X_1, Z) \arrow[r] & \mathrm{Map}(X_0, Z) \end{tikzcd}$$

is a weak equivalence of towers of fibrations of fibrant simplicial sets. Thus, the induced map $\lim \mathrm{Map}(Y_\alpha, Z) \to \lim \mathrm{Map}(X_\alpha, Z)$ is a weak equivalence, and so the result follows from Proposition 10.2.4. □

10.6. Simplicial functors

If $\mathcal{M}$ and $\mathcal{N}$ are simplicial model categories and $\mathrm{F}\colon \mathcal{M} \to \mathcal{N}$ is a functor, then we often want to consider whether F can be extended to a *simplicial functor*, i.e., whether the definition of F can be extended to define a natural map of simplicial sets

$$\mathrm{Map}(X, Y) \to \mathrm{Map}(\mathrm{F}X, \mathrm{F}Y) \tag{10.6.1}$$

that is compatible with composition and with the isomorphisms $\mathrm{Map}(X, Y)_0 \approx \mathcal{M}(X, Y)$ and $\mathrm{Map}(\mathrm{F}X, \mathrm{F}Y)_0 \approx \mathcal{N}(\mathrm{F}X, \mathrm{F}Y)$.

If F is to be a simplicial functor, then given an n-simplex in $\mathrm{Map}(X, Y)$, i.e., a map $\alpha\colon X \otimes \Delta[n] \to Y$ (see Lemma 10.1.5), we must assign to it an n-simplex of $\mathrm{Map}(\mathrm{F}X, \mathrm{F}Y)$, i.e., a map $\mathrm{F}X \otimes \Delta[n] \to \mathrm{F}Y$. We can attempt to use $\mathrm{F}(\alpha)\colon \mathrm{F}(X \otimes \Delta[n]) \to \mathrm{F}Y$, but then we need a map

$$\sigma\colon \mathrm{F}X \otimes \Delta[n] \to \mathrm{F}(X \otimes \Delta[n])$$

to compose with $\mathrm{F}(\alpha)$. If we ensure that σ yields a natural isomorphism $\sigma\colon \mathrm{F}X \otimes \Delta[0] \approx \mathrm{F}(X \otimes \Delta[0])$ that commutes with the natural isomorphisms $X \otimes \Delta[0] \approx X$, then the map (10.6.1) would be an extension of F on $\mathrm{Map}(X, Y)_0 \approx \mathcal{M}(X, Y)$. This would allow us to define the map (10.6.1) for each pair of objects X and Y, but even if we require that σ be natural in both X and $\Delta[n]$, we still could not be sure that the function (10.6.1) commutes with composition of functions, i.e., that the diagram

$$\begin{tikzcd}
\mathrm{Map}(X, Y) \times \mathrm{Map}(Y, Z) \arrow[r] \arrow[d] & \mathrm{Map}(X, Z) \arrow[d] \\
\mathrm{Map}(\mathrm{F}X, \mathrm{F}Y) \times \mathrm{Map}(\mathrm{F}Y, \mathrm{F}Z) \arrow[r] & \mathrm{Map}(\mathrm{F}X, \mathrm{F}Z)
\end{tikzcd}$$

commutes. For this, σ must have an additional property.

Given n-simplices $\alpha \in \mathrm{Map}(X, Y)_n$ and $\beta \in \mathrm{Map}(Y, Z)_n$, i.e., functions $\alpha\colon X \otimes \Delta[n] \to Y$ and $\beta\colon Y \otimes \Delta[n] \to Z$, their composition in $\mathrm{Map}(X, Z)_n$ is the composition

$$X \otimes \Delta[n] \xrightarrow{1 \otimes D} X \otimes (\Delta[n] \times \Delta[n]) \approx (X \otimes \Delta[n]) \otimes \Delta[n] \xrightarrow{\alpha \otimes 1} Y \otimes \Delta[n] \xrightarrow{\beta} Z$$

(where $D\colon \Delta[n] \to \Delta[n] \times \Delta[n]$ is the diagonal map). If we apply F and compose with the natural transformation σ, then we get the n-simplex

$$\begin{aligned}
\mathrm{F}X \otimes \Delta[n] &\xrightarrow{\sigma} \mathrm{F}(X \otimes \Delta[n]) \\
&\xrightarrow{\mathrm{F}(1 \otimes D)} \mathrm{F}\big(X \otimes (\Delta[n] \times \Delta[n])\big) \approx \mathrm{F}\big((X \otimes \Delta[n]) \otimes \Delta[n]\big) \\
&\qquad\xrightarrow{\mathrm{F}(\alpha \otimes 1)} \mathrm{F}(Y \otimes \Delta[n]) \xrightarrow{\mathrm{F}(\beta)} \mathrm{F}(Z)
\end{aligned}$$

of $\mathrm{Map}(\mathrm{F}X, \mathrm{F}Z)$. Since σ is natural, this can also be written as the composition

$$\begin{aligned}
\mathrm{F}X \otimes \Delta[n] &\xrightarrow{1 \otimes D} \mathrm{F}X \otimes (\Delta[n] \times \Delta[n]) \\
&\xrightarrow{\sigma} \mathrm{F}\big(X \otimes (\Delta[n] \times \Delta[n])\big) \approx \mathrm{F}\big((X \otimes \Delta[n]) \otimes \Delta[n]\big) \\
&\qquad\xrightarrow{\mathrm{F}(\alpha \otimes 1)} \mathrm{F}(Y \otimes \Delta[n]) \xrightarrow{\mathrm{F}(\beta)} \mathrm{F}(Z)
\end{aligned} \tag{10.6.2}$$

If we start with the same n-simplices α and β, apply F to each, and compose each with the natural transformation σ, then we get the pair of simplices

$$\mathrm{F}X \otimes \Delta[n] \xrightarrow{\sigma} \mathrm{F}(X \otimes \Delta[n]) \xrightarrow{\mathrm{F}(\alpha)} \mathrm{F}Y$$
$$\mathrm{F}Y \otimes \Delta[n] \xrightarrow{\sigma} \mathrm{F}(Y \otimes \Delta[n]) \xrightarrow{\mathrm{F}(\beta)} \mathrm{F}Z$$

in $\mathrm{Map}(\mathrm{F}X, \mathrm{F}Y)_n \times \mathrm{Map}(\mathrm{F}Y, \mathrm{F}Z)_n$. If we compose these, then we get the element

$$\begin{aligned}\mathrm{F}X \otimes \Delta[n] &\xrightarrow{1\otimes D} \mathrm{F}X \otimes (\Delta[n] \times \Delta[n])\\ &\approx (\mathrm{F}X \otimes \Delta[n]) \otimes \Delta[n] \xrightarrow{\sigma\otimes 1} \mathrm{F}(X \otimes \Delta[n]) \otimes \Delta[n]\\ &\xrightarrow{\mathrm{F}(\alpha)\otimes 1} \mathrm{F}Y \otimes \Delta[n] \xrightarrow{\sigma} \mathrm{F}(Y \otimes \Delta[n]) \xrightarrow{\mathrm{F}(\beta)} \mathrm{F}Z\end{aligned}$$

of $\mathrm{Map}(\mathrm{F}X, \mathrm{F}Z)_n$. Since σ is natural, this can also be written as the composition

$$\begin{aligned}\mathrm{F}X \otimes \Delta[n] &\xrightarrow{1\otimes D} \mathrm{F}X \otimes (\Delta[n] \times \Delta[n])\\ &\approx (\mathrm{F}X \otimes \Delta[n]) \otimes \Delta[n] \xrightarrow{\sigma\otimes 1} \mathrm{F}(X \otimes \Delta[n]) \otimes \Delta[n]\\ &\xrightarrow{\sigma} \mathrm{F}\big((X \otimes \Delta[n]) \otimes \Delta[n]\big) \xrightarrow{\mathrm{F}(\alpha\otimes 1)} \mathrm{F}(Y \otimes \Delta[n]) \xrightarrow{\mathrm{F}(\beta)} \mathrm{F}Z\end{aligned} \tag{10.6.3}$$

Since we want the composition (10.6.2) to equal the composition (10.6.3), we must require that the diagram

$$\begin{tikzcd}
\mathrm{F}X \otimes (\Delta[n] \times \Delta[n]) \arrow[r, "\approx"] \arrow[dd, "\sigma"'] & (\mathrm{F}X \otimes \Delta[n]) \otimes \Delta[n] \arrow[d, "\sigma\otimes 1"] \\
& \mathrm{F}(X \otimes \Delta[n]) \otimes \Delta[n] \arrow[d, "\sigma"] \\
\mathrm{F}\big(X \otimes (\Delta[n] \times \Delta[n])\big) \arrow[r, "\approx"] & \mathrm{F}\big((X \otimes \Delta[n]) \otimes \Delta[n]\big)
\end{tikzcd}$$

commute.

This leads us to the following theorem.

THEOREM 10.6.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be simplicial model categories. A functor $\mathrm{F}\colon \mathcal{M} \to \mathcal{N}$ can be extended to a simplicial functor if and only if, for every finite simplicial set K and object X of $\mathcal{M}$, there is a map $\sigma\colon \mathrm{F}X \otimes K \to \mathrm{F}(X \otimes K)$, natural in both X and K, such that*

1. *for every object X of $\mathcal{M}$, σ defines an isomorphism $\sigma\colon (\mathrm{F}X) \otimes \Delta[0] \approx \mathrm{F}(X \otimes \Delta[0])$ such that the triangle*

$$\begin{tikzcd}
(\mathrm{F}X) \otimes \Delta[0] \arrow[rr, "\approx"] \arrow[dr, "\approx"'] & & \mathrm{F}(X \otimes \Delta[0]) \arrow[dl, "\approx"] \\
& \mathrm{F}X &
\end{tikzcd}$$

commutes, and

2. *for every object X of $\mathcal{M}$ and finite simplicial sets K and L, the diagram*

$$\begin{tikzcd} \mathrm{F}X \otimes (K \times L) \arrow[r, "\approx"] \arrow[dd, "\sigma"] & (\mathrm{F}X \otimes K) \otimes L \arrow[d, "\sigma \otimes 1"] \\ & \mathrm{F}(X \otimes K) \otimes L \arrow[d, "\sigma"] \\ \mathrm{F}\big(X \otimes (K \times L)\big) \arrow[r, "\approx"] & \mathrm{F}\big((X \otimes K) \otimes L\big) \end{tikzcd}$$

commutes.

Proof. We have isomorphisms

$$\mathrm{Map}(X,Y)_n \approx \mathsf{SS}\big(\Delta[n], \mathrm{Map}(X,Y)\big) \approx \mathcal{M}(X \otimes \Delta[n], Y)$$

that are natural in X, Y, and $\Delta[n]$, and so we can define $\mathrm{F}\colon \mathrm{Map}(X,Y)_n \to \mathrm{Map}(\mathrm{F}X, \mathrm{F}Y)_n$ as the composition

$$\mathcal{M}(X \otimes \Delta[n], Y) \xrightarrow{\mathrm{F}(-,-)} \mathcal{N}\big(\mathrm{F}(X \otimes \Delta[n]), \mathrm{F}Y\big) \xrightarrow{\sigma^*} \mathcal{N}(\mathrm{F}X \otimes \Delta[n], \mathrm{F}Y).$$

The discussion preceding the statement of the theorem explains why this yields a simplicial functor.

Conversely, if $\mathrm{F}\colon \mathcal{M} \to \mathcal{N}$ is simplicial, then we can define σ as the map corresponding to the composition

$$K \to \mathrm{Map}(X, X \otimes K) \xrightarrow{\mathrm{F}(-,-)} \mathrm{Map}\big(\mathrm{F}X, \mathrm{F}(X \otimes K)\big)$$

(where the first map above is adjoint to the identity map of $X \otimes K$) under the isomorphism

$$\mathsf{SS}\big(K, \mathrm{Map}\big(\mathrm{F}X, \mathrm{F}(X \otimes K)\big)\big) \approx \mathcal{N}\big(\mathrm{F}X \otimes K, \mathrm{F}(X \otimes K)\big).$$

□

Example 10.6.5. Let $\mathcal{M}$ be a simplicial model category. If W is an object in $\mathcal{M}$, then the functor $\mathrm{Map}(W,-)\colon \mathcal{M} \to \mathsf{SS}$ is simplicial. In this case, for $(f,k) \in \big(\mathrm{Map}(W,X) \otimes K\big)_n$ we have $\sigma(f,k) = f \times \bar{k}$, where $\bar{k}$ is the composition of the projection $W \otimes \Delta[n] \to \Delta[n]$ with the map $\Delta[n] \to K$ that takes the nondegenerate n-simplex of $\Delta[n]$ to k.

Proposition 10.6.6. *Let $\mathcal{M}$ and $\mathcal{N}$ be simplicial model categories, let $\mathcal{C}$ be a small category, and let $\boldsymbol{X}$ be a $\mathcal{C}$-diagram of functors $\mathcal{M} \to \mathcal{N}$ and natural transformations between them. If for each $\alpha \in \mathcal{C}$ there is a map σ_α as in Theorem 10.6.4 that is natural in α and that extends $\boldsymbol{X}_\alpha$ to a simplicial functor, then there is a map σ that extends $\mathrm{colim}_{\alpha \in \mathcal{C}} \boldsymbol{X}_\alpha$ to a simplicial functor.*

Proof. Let $\sigma = \mathrm{colim}_{\alpha \in \mathcal{C}} \sigma_\alpha$. □

CHAPTER 11

Proper model categories

11.1. Properness

DEFINITION 11.1.1. Let $\mathcal{M}$ be a model category (see Definition 8.1.2), and let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ C \arrow[r, "g"'] & D \end{tikzcd} \tag{11.1.2}$$

be a commutative square in $\mathcal{M}$.

1. The model category $\mathcal{M}$ will be called *left proper* if, whenever f is a weak equivalence, i is a cofibration, and the square (11.1.2) is a pushout, the map g is also a weak equivalence.
2. The model category $\mathcal{M}$ will be called *right proper* if, whenever g is a weak equivalence, j is a fibration, and the square (11.1.2) is a pullback, the map f is also a weak equivalence.
3. The model category $\mathcal{M}$ will be called *proper* if it is both left proper and right proper.

PROPOSITION 11.1.3 (C. L. Reedy, [**50**]). *Let $\mathcal{M}$ be a model category and let*

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "g"'] & X \arrow[d, "p"] \\ D \arrow[r, "k"] & Y \end{tikzcd} \tag{11.1.4}$$

be a commutative square in $\mathcal{M}$.

1. *If p is a fibration, k is a weak equivalence, D and Y are fibrant objects, and the square (11.1.4) a pullback, then h is a weak equivalence.*
2. *If g is a cofibration, h is a weak equivalence, C and X are cofibrant objects, and the square (11.1.4) a pushout, then k is a weak equivalence.*

PROOF. **Fill this in!** $\square$

COROLLARY 11.1.5. *Let $\mathcal{M}$ be a model category.*

1. *If every object of $\mathcal{M}$ is cofibrant, then $\mathcal{M}$ is left proper.*
2. *If every object of $\mathcal{M}$ is fibrant, then $\mathcal{M}$ is right proper.*
3. *If every object of $\mathcal{M}$ is both cofibrant and fibrant, then $\mathcal{M}$ is proper.*

PROOF. This follows from Proposition 11.1.3. $\square$

COROLLARY 11.1.6. *The categories SS and SS_* are both left proper.*

PROOF. This follows from Corollary 11.1.5. $\square$

COROLLARY 11.1.7. *The categories* Top *and* Top_* *are both right proper.*

PROOF. This follows from Corollary 11.1.5. $\square$

11.1.8. Topological spaces.

LEMMA 11.1.9. *Let* $f\colon X \to Y$ *be a map of path connected topological spaces. If* f *induces an isomorphism of fundamental groups* $f_*\colon \pi_1(X, x_0) \approx \pi_1\bigl(Y, f(x_0)\bigr)$ *for some point* $x_0 \in X$ *and an isomorphism of homology* $f_*\colon \mathrm{H}_*(X; f^*A) \approx \mathrm{H}_*(Y; A)$ *for every local coefficient system* A *on* Y*, then* f *is a weak equivalence.*

PROOF. It is sufficient to show that the induced map of total singular complexes is a weak equivalence. Since this is a map of connected simplicial sets inducing an isomorphism of fundamental groups, it is sufficient to show that it induces isomorphisms of all higher homotopy groups, and for this it is sufficient to show that the induced map of universal covers $\widetilde{\operatorname{Sing} f}\colon \widetilde{\operatorname{Sing} X} \to \widetilde{\operatorname{Sing} Y}$ induces an isomorphism of all homology groups. Since the homology groups $\mathrm{H}_*(\widetilde{\operatorname{Sing} X})$ are naturally isomorphic to the local coefficient homology groups $\mathrm{H}_*\bigl(\operatorname{Sing} X; \mathbb{Z}[\pi_1 X]\bigr)$, the proof is complete. $\square$

THEOREM 11.1.10. *A map of topological spaces* $f\colon X \to Y$ *is a weak equivalence if and only if it induces an isomorphism of the sets of path components* $f_*\colon \pi_0 X \approx \pi_0 Y$ *and, for each path component of* X *and the corresponding component of* Y*, isomorphisms of fundamental groups and of homology with all local coefficient systems.*

PROOF. The conditions are clearly necessary, and the converse follows from Lemma 11.1.9. $\square$

PROPOSITION 11.1.11. *Let* $f\colon X \to Y$ *be a weak equivalence of topological spaces. If* $n \ge 0$ *and* $\alpha\colon \mathrm{S}^n \to X$ *is a map, then the induced map* $\hat{f}\colon X \cup_\alpha \mathrm{D}^{n+1} \to Y \cup_{f\alpha} \mathrm{D}^{n+1}$ *is a weak equivalence.*

PROOF. We will use Theorem 11.1.10. It follows immediately that $\hat{f}$ induces an isomorphism on the set of path components.

If $n = 0$ or $n = 1$, then the van Kampen theorem implies that $\hat{f}$ induces an isomorphism on the fundamental group of each path component. If $n > 1$, then the fundamental groups of the components of X and Y were unchanged when the cells were attached.

To see that $\hat{f}$ induces an isomorphism of homology with arbitrary local coefficients, we let

$$\begin{aligned}
\mathrm{T}^{n+1} &= \{x \in \mathbb{R}^{n+1} \mid 0 < |x| \le 1\} \\
\widetilde{X} &= X \cup_f \mathrm{T}^{n+1} \\
\widehat{X} &= X \cup_f \mathrm{D}^{n+1}
\end{aligned}$$

and let $\widetilde{Y}$ and $\widehat{Y}$ be the corresponding constructions for Y. Since X is a deformation retract of $\widetilde{X}$ and Y is a deformation retract of $\widetilde{Y}$, the induced map $\tilde{f}\colon \widetilde{X} \to \widetilde{Y}$ is a weak equivalence. If B^{n+1} is the interior of D^{n+1}, then the subsets $\widetilde{X}$ and B^{n+1} of

$\widehat{X}$ are an excisive pair, and so a Mayer-Vietoris argument shows that $\hat{f}$ induces an isomorphism of homology with arbitrary local coefficients. □

THEOREM 11.1.12. *If $f\colon X \to Y$ is a weak equivalence of topological spaces, $s\colon X \to W$ is a cofibration, and the square*

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"'] & W \arrow[d, "g"] \\ Y \arrow[r] & Z \end{tikzcd}$$

is a pushout, then g is a weak equivalence.

PROOF. The cofibration s must be a retract of a relative cell complex $t\colon X \to U$. If

$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d, "f"'] & U \arrow[d, "h"] \\ Y \arrow[r] & V \end{tikzcd}$$

is a pushout, then g is a retract of h, and so it is sufficient to show that h is a weak equivalence. If we write t as a transfinite composition of maps, each of which attaches a single cell, then a transfinite induction using Proposition 11.1.11 and Proposition 2.2.4 implies that h is a weak equivalence. □

THEOREM 11.1.13. *The categories Top and Top_* are proper model categories.*

PROOF. This follows from Theorem 11.1.12 and Corollary 11.1.7. □

PROPOSITION 11.1.14. *The geometric realization functor commutes with finite limits.*

PROOF. See [**33**, page 49]. □

THEOREM 11.1.15. *The categories $\mathcal{SS}$ and $\mathcal{SS}_*$ are proper model categories.*

PROOF. The geometric realization functor commutes with pullbacks (see Proposition 11.1.14). Since the geometric realization of a fibration of simplicial sets is a fibration (see [**47**]), right properness follows from the right properness of Top and Top_* (see Theorem 11.1.13), and left properness follows from Corollary 11.1.6. □

THEOREM 11.1.16. *The categories Top, Top_*, $\mathcal{SS}$, and $\mathcal{SS}_*$ are proper model categories.*

PROOF. This follows from Theorem 11.1.13 and Theorem 11.1.15. □

11.1.17. Properness and lifting. We are indebted to D. M. Kan for the following proposition.

PROPOSITION 11.1.18. *Let $\mathcal{M}$ be a model category.*

1. *Let $\mathcal{M}$ be left proper, let $g\colon A \to B$ be a cofibration, let $p\colon X \to Y$ be a fibration, and let $\tilde{g}\colon \widetilde{A} \to \widetilde{B}$ be a cofibrant approximation (see Definition 9.1.8) to g such that $\tilde{g}$ is a cofibration. If p has the right lifting property with respect to $\tilde{g}$, then p has the right lifting property with respect to g.*

2. *Let $\mathcal{M}$ be right proper, let $g\colon A \to B$ be a cofibration, let $p\colon X \to Y$ be a fibration, and let $\hat{p}\colon \widehat{X} \to \widehat{Y}$ is a fibrant approximation (see Definition 9.1.8) to p such that $\hat{p}$ is a fibration. If g has the left lifting property with respect to $\hat{p}$, then g has the left lifting property with respect to p.*

PROOF. We will prove part 2; the proof of part 1 is dual.

We have the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & X \arrow[r, "i_X"] \arrow[d, "p"] & \widehat{X} \arrow[d, "\hat{p}"] \\ B \arrow[r] & Y \arrow[r, "i_Y"'] & \widehat{Y} \end{tikzcd}$$

in which both i_X and i_Y are weak equivalences. If we let P be the pullback $Y \times_{\widehat{Y}} \widehat{X}$, then we have the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dd, "g"'] & X \arrow[rr, "i_X"] \arrow[dd, "p"'] \arrow[dr, "k"] & & \widehat{X} \arrow[dd, "\hat{p}"] \\ & & P \arrow[ur, "h"'] \arrow[dl, "j"] & \\ B \arrow[r] & Y \arrow[rr, "i_Y"'] & & \widehat{Y} \end{tikzcd}$$

and, since g has the left lifting property with respect to $\hat{p}$, it also has the left lifting property with respect to j (see Lemma 8.2.5).

If we now consider the category $(A\downarrow\mathcal{M}\downarrow Y)$ of objects of $\mathcal{M}$ under A and over Y, then B, X, and P are objects in this category. Since g has the left lifting property with respect to j, we know that there is a map in this category from B to P, and we must show that there is a map in this category from B to X.

The category of objects under A and over Y is a model category in which a map is a cofibration, fibration, or weak equivalence if and only if it is one in $\mathcal{M}$ (see Theorem 8.4.1). Since j is a pullback of the fibration $\hat{p}$, it is also a fibration, and so X and P are fibrant objects in our category, and B is a cofibrant object. If we knew that k was a weak equivalence, then the result would follow from Corollary 8.5.5.

Since i_Y is a weak equivalence, $\hat{p}$ is a fibration, and $\mathcal{M}$ is a *right proper* model category, the map h is also a weak equivalence. Since $i_X = hk$ and both i_X and h are weak equivalences, k is also a weak equivalence, and the proof is complete.

□

COROLLARY 11.1.19. *Let $\mathcal{M}$ be a simplicial model category.*

1. *Let $\mathcal{M}$ be left proper, let $g\colon A \to B$ be a cofibration, let $p\colon X \to Y$ be a fibration, and let $\tilde{g}\colon \widetilde{A} \to \widetilde{B}$ be a cofibrant approximation (see Definition 9.1.8) to g such that $\tilde{g}$ is a cofibration. If p has the homotopy right lifting property with respect to $\tilde{g}$ (see Definition 10.3.2), then p has the homotopy right lifting property with respect to g.*
2. *Let $\mathcal{M}$ be right proper, let $g\colon A \to B$ be a cofibration, let $p\colon X \to Y$ be a fibration, and let $\hat{p}\colon \widehat{X} \to \widehat{Y}$ be a fibrant approximation (see Definition 9.1.8) to p such that $\hat{p}$ is a fibration. If g has the homotopy left lifting property with respect to $\hat{p}$ (see Definition 10.3.2), then g has the homotopy left lifting property with respect to p.*

PROOF. This follows from Proposition 11.1.18 and Lemma 10.3.6. □

11.1.20. Properness and sequential colimits. We are indebted to D. M. Kan for the following proposition.

PROPOSITION 11.1.21. *Let $\mathcal{M}$ be a left proper simplicial model category (see Definition 11.1.1). If λ is an ordinal and*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "g_0"'] & X_1 \arrow[r] \arrow[d, "g_1"'] & X_2 \arrow[r] \arrow[d, "g_2"'] & \cdots \\ Y_0 \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r] & \cdots \end{tikzcd}$$

is a map of λ-sequences in $\mathcal{M}$ such that

1. *each of the maps $X_\alpha \to X_{\alpha+1}$ and $Y_\alpha \to Y_{\alpha+1}$ (for $\alpha < \lambda$) is a cofibration;*
2. *each of the maps $g_\alpha \colon X_\alpha \to Y_\alpha$ (for $\alpha < \lambda$) is a weak equivalence;*

then the induced map $(\operatorname{colim} g_\alpha) \colon \operatorname{colim} X_\alpha \to \operatorname{colim} Y_\alpha$ is a weak equivalence.

PROOF. We construct a λ-sequence $Z_0 \to Z_1 \to Z_2 \to \cdots$ intermediate between the given ones by letting Z_α be the pushout $Y_0 \amalg_{X_0} X_\alpha$ for every $\alpha < \beta$. Proposition 8.2.12 implies that $Z_\alpha \to Z_{\alpha+1}$ is a cofibration for every $\alpha < \lambda$, and we have maps of λ-sequences

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "h_0"'] & X_1 \arrow[r] \arrow[d, "h_1"'] & X_2 \arrow[r] \arrow[d, "h_2"'] & \cdots \\ Z_0 \arrow[r] \arrow[d, "k_0"'] & Z_1 \arrow[r] \arrow[d, "k_1"'] & Z_2 \arrow[r] \arrow[d, "k_2"'] & \cdots \\ Y_0 \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r] & \cdots \end{tikzcd}$$

such that

1. each of the maps $Z_0 \to Z_\alpha$ (for $\alpha < \lambda$) is a cofibration,
2. the map $k_0 \colon Z_0 \to Y_0$ is an isomorphism, and
3. (since $\mathcal{M}$ is left proper) each of the maps $k_\alpha \colon Z_\alpha \to Y_\alpha$ (for $\alpha < \lambda$) is a weak equivalence.

Since left adjoints commute with colimits, $\operatorname{colim} Z_\alpha$ is isomorphic to the pushout $Y_0 \amalg_{X_0} (\operatorname{colim} X_\alpha)$ (see Lemma 8.4.2); thus, the map $\operatorname{colim} X_\alpha \to \operatorname{colim} Z_\alpha$ is a weak equivalence. Thus, it is sufficient to show that $\operatorname{colim} Z_\alpha \to \operatorname{colim} Y_\alpha$ is a weak equivalence. Since $k_0 \colon Z_0 \to Y_0$ is an isomorphism, each of the maps $k_\alpha \colon Z_\alpha \to Y_\alpha$ (for $\alpha < \lambda$) is a weak equivalence of cofibrant objects in the category $(Z_0 \downarrow \mathcal{M})$ of objects under Z_0 (see Theorem 8.4.1). Thus, Proposition 10.5.6 implies that the map $\operatorname{colim} Z_\alpha \to \operatorname{colim} Y_\alpha$ is a weak equivalence, and the proof is complete. □

PROPOSITION 11.1.22. *Let $\mathcal{M}$ be a left proper simplicial model category. If λ is an ordinal and*

$$X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \qquad (\beta < \lambda)$$

is a λ-sequence in $\mathcal{M}$ such that $X_\beta \to X_{\beta+1}$ is a cofibration for every $\beta < \lambda$, then there is a λ-sequence

$$\widetilde{X}_0 \to \widetilde{X}_1 \to \widetilde{X}_2 \to \cdots \to \widetilde{X}_\beta \to \cdots \qquad (\beta < \lambda)$$

and a map of λ-sequences

$$\begin{tikzcd} \widetilde{X}_0 \arrow[r] \arrow[d, "g_0"'] & \widetilde{X}_1 \arrow[r] \arrow[d, "g_1"'] & \widetilde{X}_2 \arrow[r] \arrow[d, "g_2"'] & \cdots \arrow[r] & \widetilde{X}_\beta \arrow[r] \arrow[d, "g_\beta"'] & \cdots & (\beta < \lambda) \\ X_0 \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & \cdots \arrow[r] & X_\beta \arrow[r] & \cdots & (\beta < \lambda) \end{tikzcd}$$

such that

1. *every $\widetilde{X}_\beta$ is cofibrant,*
2. *every $g_\beta : \widetilde{X}_\beta \to X_\beta$ is a weak equivalence,*
3. *every $\widetilde{X}_\beta \to \widetilde{X}_{\beta+1}$ is a cofibration, and*
4. *the map $\operatorname{colim}_{\beta<\lambda} \widetilde{X}_\beta \to \operatorname{colim}_{\beta<\lambda} X_\beta$ is a weak equivalence.*

PROOF. We will define the $\widetilde{X}_\beta$ inductively. We begin by choosing a cofibrant approximation $g_0 \colon \widetilde{X}_0 \to X_0$ to X_0 (see Proposition 9.1.2). If $\beta + 1 < \lambda$ and we have defined $g_\beta : \widetilde{X}_\beta \to X_\beta$, then we factor the composition $\widetilde{X}_\beta \to X_\beta \to X_{\beta+1}$ into a cofibration followed by a trivial fibration, to obtain $\widetilde{X}_\beta \to \widetilde{X}_{\beta+1} \xrightarrow{g_{\beta+1}} X_{\beta+1}$. If $\beta < \lambda$ and β is a limit ordinal, then Proposition 11.1.21 implies that $\operatorname{colim}_{\alpha<\beta} \widetilde{X}_\alpha \to \operatorname{colim}_{\alpha<\beta} X_\alpha$ is a weak equivalence, and so we can construct the $\widetilde{X}_\beta$ as required. Proposition 11.1.21 implies that the map $\operatorname{colim}_{\beta<\lambda} \widetilde{X}_\beta \to \operatorname{colim}_{\beta<\lambda} X_\beta$ is a weak equivalence, and so the proof is complete. □

11.2. Homotopy pullbacks and homotopy fibers

If all objects in a model category $\mathcal{M}$ were fibrant, then we would define homotopy pullbacks and homotopy fibers in terms of the homotopy limit functor (see Definition 19.1.10). Unfortunately, homotopy limits are homotopy invariants only for diagrams of fibrant objects (see Theorem 20.6.10). However, in a *right proper* model category (see Definition 11.1.1), we can define a homotopy pullback functor (see Definition 11.2.2) that is always homotopy invariant (see Proposition 11.2.4) and that is naturally weakly equivalent to the homotopy limit when all the objects in the diagram are fibrant (see Proposition 19.4.3). The *homotopy fiber* of the map $X \to Y$ over a point (see Definition 11.2.17) of Y will be defined so that it is a fibrant object weakly equivalent to the homotopy pullback of the diagram $X \to Y \leftarrow *$ (where "$*$" denotes the terminal object of $\mathcal{M}$) (see Definition 11.2.19 and Remark 11.2.21).

11.2.1. Homotopy pullbacks. If $\mathcal{M}$ is a right proper model category (see Definition 11.1.1), then the homotopy pullback of the diagram $X \xrightarrow{g} Z \xleftarrow{h} Y$ is constructed by replacing g and h by fibrations and then taking a pullback (see Definition 11.2.2). In order to have a well defined functor, we need to choose a fixed functor to convert our maps into fibrations. We will show, however, that any other factorization into a weak equivalence followed by a fibration yields an object naturally weakly equivalent to the homotopy pullback and that, in fact, only one of the maps must be converted to a fibration (see Proposition 11.2.7). Thus, if either of the maps is already a fibration, then the pullback is naturally weakly equivalent to the homotopy pullback (see Corollary 11.2.8).

DEFINITION 11.2.2. Let $\mathcal{M}$ be a right proper model category (see Definition 11.1.1), and let E be an arbitrary but fixed functorial factorization of every map $g \colon X \to Y$

into $X \xrightarrow{i_g} \mathrm{E}(g) \xrightarrow{p_g} Y$, where i_g is a trivial cofibration and p_g is a fibration. The *homotopy pullback* of the diagram $X \xrightarrow{g} Z \xleftarrow{h} Y$ is defined to be the pullback of the diagram $\mathrm{E}(g) \xrightarrow{p_g} Z \xleftarrow{p_h} \mathrm{E}(h)$.

LEMMA 11.2.3. *Let $\mathcal{M}$ be a right proper model category. If $g\colon X \to Y$ is a weak equivalence and $h\colon W \to Z$ is a fibration, then, for any map $k\colon Y \to Z$, the natural map from the pullback of the diagram $X \xrightarrow{kg} Z \xleftarrow{h} W$ to the pullback of the diagram $Y \xrightarrow{k} Z \xleftarrow{h} W$ is a weak equivalence.*

PROOF. We have the commutative diagram

$$\begin{tikzcd}
X \times_Z W \arrow[r] \arrow[d] & Y \times_Z W \arrow[r] \arrow[d] & W \arrow[d, "h"] \\
X \arrow[r, "g"'] & Y \arrow[r, "k"'] & Z
\end{tikzcd}$$

in which the vertical maps are all fibrations. Since g is a weak equivalence, the result follows from Proposition 8.2.12. $\square$

PROPOSITION 11.2.4 (Homotopy invariance of the homotopy pullback). *Let $\mathcal{M}$ be a right proper model category. If we have the diagram*

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d] & Z \arrow[d] & Y \arrow[l, "h"'] \arrow[d] \\
\widetilde{X} \arrow[r, "\tilde{g}"] & \widetilde{Z} & \widetilde{Y} \arrow[l, "\tilde{h}"']
\end{tikzcd}$$

in which the vertical maps are weak equivalences, then the induced map of homotopy pullbacks

$$\mathrm{E}(g) \times_Z \mathrm{E}(h) \to \mathrm{E}(\tilde{g}) \times_{\widetilde{Z}} \mathrm{E}(\tilde{h})$$

is a weak equivalence.

PROOF. It is sufficient to show that if g, h, $\tilde{g}$, and $\tilde{h}$ are fibrations, then the map of pullbacks $X \times_Z Y \to \widetilde{X} \times_{\widetilde{Z}} \widetilde{Y}$ is a weak equivalence. This map equals the composition

$$X \times_Z Y \to (\widetilde{X} \times_{\widetilde{Z}} Z) \times_Z Y \approx \widetilde{X} \times_{\widetilde{Z}} Y \to \widetilde{X} \times_{\widetilde{Z}} \widetilde{Y}.$$

Since $\mathcal{M}$ is a right proper model category, the map $X \to \widetilde{X} \times_{\widetilde{Z}} Z$ is a weak equivalence, and Lemma 11.2.3 implies that the last map in the composition is a weak equivalence. $\square$

COROLLARY 11.2.5. *Let $\mathcal{M}$ be a right proper model category. If $k\colon W \to X$ is a weak equivalence, then the homotopy pullback of the diagram $X \xrightarrow{g} Z \xleftarrow{h} Y$ is naturally weakly equivalent to the homotopy pullback of the diagram $W \xrightarrow{gk} Z \xleftarrow{h} Y$.*

PROOF. We have the commutative diagram

$$\begin{tikzcd} W \arrow[r, "gk"] \arrow[d, "k"'] & Z \arrow[d, equal] & Y \arrow[l, "h"'] \arrow[d, equal] \\ X \arrow[r, "g"'] & Z & Y \arrow[l, "h"] \end{tikzcd}$$

in which the vertical maps are weak equivalences, and so the result follows from Proposition 11.2.4. □

COROLLARY 11.2.6. *Let $\mathcal{M}$ be a right proper model category. If the maps $r, s\colon X \to Z$ are left homotopic (see Definition 8.3.2), right homotopic, or (if $\mathcal{M}$ is a simplicial model category) simplicially homotopic (see Definition 10.4.2), then the homotopy pullback of the diagram $X \xrightarrow{r} Z \xleftarrow{h} Y$ is weakly equivalent to the homotopy pullback of the diagram $X \xrightarrow{s} Z \xleftarrow{h} Y$.*

PROOF. We will prove this in the case that r and s are left homotopic; the proof in the case that they are right homotopic is similar, and either of these cases implies the corollary in the case that they are simplicially homotopic, since maps that are simplicially homotopic are both left and right homotopic (see Proposition 10.4.4).

If r and s are left homotopic, there is a diagram

$$X \underset{i_1}{\overset{i_0}{\rightrightarrows}} C \xrightarrow{H} Z$$

such that $Hi_0 = r$, $Hi_1 = s$, and both i_0 and i_1 are weak equivalences. The corollary now follows from Corollary 11.2.5. □

PROPOSITION 11.2.7. *Let $\mathcal{M}$ be a right proper model category. If $X \xrightarrow{j_g} W_g \xrightarrow{q_g} Z$ and $Y \xrightarrow{j_h} W_h \xrightarrow{q_h} Z$ are factorizations of, respectively, $g\colon X \to Z$ and $h\colon Y \to Z$, j_g and j_h are weak equivalences, and q_g and q_h are fibrations, then the homotopy pullback of the diagram $X \xrightarrow{g} Z \xleftarrow{h} Y$ is naturally weakly equivalent to each of $W_g \times_Z W_h$, $W_g \times_Z Y$, and $X \times_Z W_h$.*

PROOF. If E is the natural factorization used in Definition 11.2.2, then Lemma 11.2.3 implies that the homotopy pullback $\mathrm{E}(g) \times_Z \mathrm{E}(h)$ is naturally weakly equivalent to both $\mathrm{E}(g) \times_Z Y$ and $X \times_Z \mathrm{E}(h)$. Lemma 11.2.3 implies that these are naturally weakly equivalent to $\mathrm{E}(g) \times_Z W_h$ and $W_g \times_Z \mathrm{E}(h)$ respectively, and that these are naturally weakly equivalent to $X \times_Z W_h$ and $W_g \times_Z Y$, respectively. Lemma 11.2.3 implies that both of these are naturally weakly equivalent to $W_g \times_Z W_h$, and so the proof is complete. □

COROLLARY 11.2.8. *Let $\mathcal{M}$ be a right proper model category. If at least one of the maps $g\colon X \to Z$ and $h\colon Y \to Z$ is a fibration, then the pullback $X \times_Z Y$ is naturally weakly equivalent to the homotopy pullback of the diagram $X \xrightarrow{g} Z \xleftarrow{h} Y$.*

PROOF. This follows from Proposition 11.2.7. □

In Proposition 19.4.3, we show that if $\mathcal{M}$ is a right proper *simplicial* model category and X, Y, and Z are fibrant, then the homotopy pullback of the diagram $X \to Z \leftarrow Y$ is naturally weakly equivalent to the homotopy limit of that diagram (see Definition 19.1.10).

Proposition 11.2.9. *Let $\mathcal{M}$ be a right proper model category. If the vertical maps in the diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z \arrow[d] & Y \arrow[l] \arrow[d] \\ \widetilde{X} \arrow[r] & \widetilde{Z} & \widetilde{Y} \arrow[l] \end{tikzcd}$$

are weak equivalences and at least one map in each row is a fibration, then the map of pullbacks $X \times_Z Y \to \widetilde{X} \times_{\widetilde{Z}} \widetilde{Y}$ is a weak equivalence.

Proof. This follows from Corollary 11.2.8 and Proposition 11.2.4. $\square$

Proposition 11.2.10. *Let $\mathcal{M}$ be a right proper model category. If we have a diagram $X \xrightarrow{g} Z \xleftarrow{h} Y$ in which at least one of g and h is a fibration and if $\hat{h}\colon \widehat{Y} \to \widehat{Z}$ is a fibrant approximation to h, then the pullback of h along g has a fibrant approximation that is a pullback of $\hat{h}$.*

Proof. We have the diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d, "k"'] & Y \arrow[r, "i_Y"] \arrow[d, "h"'] & \widehat{Y} \arrow[d, "\hat{h}"] \\ X \arrow[r, "g"'] & Z \arrow[r, "i_Z"'] & \widehat{Z} \end{tikzcd}$$

in which W is the pullback $X \times_Z Y$ and i_Y and i_Z are weak equivalences, and we must show that there is a pullback of $\hat{h}$ that is a fibrant approximation to k. If we factor the composition $i_Z g\colon X \to \widehat{Z}$ as $X \xrightarrow{i_X} \widehat{X} \xrightarrow{\hat{g}} \widehat{Z}$ where i_X is a trivial cofibration and $\hat{g}$ is a fibration, then we can let $\widehat{W} = \widehat{X} \times_{\widehat{Z}} \widehat{Y}$ and we have the diagram

$$\begin{tikzcd} W \arrow[rr] \arrow[dr, "i_W"] \arrow[dd, "k"'] & & Y \arrow[dr] \arrow[dd, "h"' near start] & \\ & \widehat{W} \arrow[rr] \arrow[dd, "\hat{k}" near start] & & \widehat{Y} \arrow[dd, "\hat{h}"] \\ X \arrow[rr, "g" near start] \arrow[dr] & & Z \arrow[dr, "i_Z"'] & \\ & \widehat{X} \arrow[rr, "\hat{g}"'] & & \widehat{Z} \end{tikzcd}$$

in which the front and back squares are pullbacks. Proposition 11.2.9 now implies that i_W is a weak equivalence, and so the pullback $\hat{k}$ of $\hat{h}$ is a fibrant approximation to k. $\square$

11.2.11. Homotopy fiber squares.

Definition 11.2.12. If $\mathcal{M}$ is a right proper model category, then a square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

will be called a *homotopy fiber square* if the natural map from A to the homotopy pullback (see Definition 11.2.2) of the diagram $B \to D \leftarrow C$ is a weak equivalence.

PROPOSITION 11.2.13. *If $\mathcal{M}$ is a right proper model category and we have the diagram*

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "f_A"] \arrow[dd] & & B \arrow[dr, "f_B"] \arrow[dd] & \\
& A' \arrow[rr] \arrow[dd] & & B' \arrow[dd] \\
C \arrow[rr] \arrow[dr, "f_C"'] & & D \arrow[dr, "f_D"'] & \\
& C' \arrow[rr] & & D'
\end{tikzcd}$$

in which f_A, f_B, f_C, and f_D are weak equivalences, then the front square is a homotopy fiber square if and only if the back square is a homotopy fiber square.

PROOF. If P is the homotopy pullback of the diagram $C \to D \leftarrow B$ and P' is the homotopy pullback of the diagram $C' \to D' \leftarrow B'$, then we have the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f_A"'] & P \arrow[d] \\
A' \arrow[r] & P'
\end{tikzcd}$$

and Proposition 11.2.4 implies that the map on the right is a weak equivalence. Since f_A is a weak equivalence, this implies that the top map is a weak equivalence if and only if the bottom map is a weak equivalence. □

PROPOSITION 11.2.14. *Let $\mathcal{M}$ be a right proper model category. If the front and back squares of the diagram*

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "f_A"] \arrow[dd] & & B \arrow[dr, "f_B"] \arrow[dd] & \\
& A' \arrow[rr] \arrow[dd] & & B' \arrow[dd] \\
C \arrow[rr] \arrow[dr, "f_C"'] & & D \arrow[dr, "f_D"'] & \\
& C' \arrow[rr] & & D'
\end{tikzcd}$$

are homotopy fiber squares and if f_B, f_C, and f_D are weak equivalences, then f_A is a weak equivalence.

PROOF. This follows from Proposition 11.2.4. □

PROPOSITION 11.2.15. *Let $\mathcal{M}$ be a right proper model category. If the right hand square in the diagram*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\
D \arrow[r] & E \arrow[r] & F
\end{tikzcd}$$

is a homotopy fiber square, then the left hand square is a homotopy fibers square if and only if the combined square is a homotopy fiber square.

PROOF. Factor $C \to F$ as $C \xrightarrow{i} G \xrightarrow{p} F$ where i is a trivial cofibration and p is a fibration, and let $P = E \times_F G$ and $P' = D \times_F G$. We now have the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "k"'] & B \arrow[r] \arrow[d, "\cong"', "j"] & C \arrow[d, "\cong"', "i"] \\
P' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, "q"] & G \arrow[d, "p"] \\
D \arrow[r] & E \arrow[r] & F
\end{tikzcd}$$

and Proposition 11.2.7 implies that j is a weak equivalence. Proposition 8.2.12 implies that P' is the pullback $D \times_E P$, and so Proposition 11.2.7 implies that k is a weak equivalence if and only if the (original) left hand square is a homotopy fiber square. Since Proposition 11.2.7 implies that k is a weak equivalence if and only if the (original) combined square is a homotopy fiber square, the proof is complete. □

11.2.16. Homotopy fibers.

DEFINITION 11.2.17. If $\mathcal{M}$ is a model category and Z is an object in $\mathcal{M}$, then by a *point of* Z we will mean a map $* \to Z$ (where "$*$" is the terminal object of $\mathcal{M}$).

DEFINITION 11.2.18. If $\mathcal{M}$ is a model category, $g\colon Y \to Z$ is a map, and $z\colon * \to Z$ is a point of Z (see Definition 11.2.17), then the *fiber* of g over z is the pullback of the diagram $* \xrightarrow{z} Z \xleftarrow{g} Y$.

DEFINITION 11.2.19. Let $\mathcal{M}$ be a right proper model category. If $g\colon Y \to Z$ is a map and $z\colon * \to Z$ is a point of Z, then the *homotopy fiber* $\mathrm{HFib}_z(g)$ of g over z is the pullback of the diagram $* \xrightarrow{z} Z \xleftarrow{p_g} \mathrm{E}(Y)$ (see Definition 11.2.2).

PROPOSITION 11.2.20. *If $\mathcal{M}$ is a right proper model category, $g\colon Y \to Z$ is a map in $\mathcal{M}$, and $z\colon * \to Z$ is a point of Z, then the homotopy fiber of g over Z is a fibrant object in $\mathcal{M}$ that is naturally weakly equivalent to the homotopy pullback of the diagram $* \xrightarrow{z} Z \xleftarrow{g} Y$.*

PROOF. This follows from Proposition 8.2.6 and Proposition 11.2.7. □

REMARK 11.2.21. The homotopy fiber of the map $g\colon Y \to Z$ over a point $z\colon * \to Z$ was not defined to be the homotopy pullback of the diagram $* \xrightarrow{z} Z \xleftarrow{g} Y$ because that homotopy pullback need not be a fibrant object in $\mathcal{M}$.

PROPOSITION 11.2.22. *Let $\mathcal{M}$ be a right proper model category. If $g\colon Y \to Z$ is a fibration and $z\colon * \to Z$ is a point of Z, then the fiber of g over z is naturally weakly equivalent to the homotopy fiber of g over z.*

PROOF. This follows from Proposition 11.2.20 and Corollary 11.2.8. □

PROPOSITION 11.2.23. *Let $\mathcal{M}$ be a right proper model category. If $g\colon Y \to Z$ is a map and $z\colon * \to Z$ and $z'\colon * \to Z$ are points of Z that are (either left or right) homotopic, then the homotopy fiber of g over z is weakly equivalent to the homotopy fiber of g over z'.*

PROOF. This follows from Proposition 11.2.20 and Corollary 11.2.6. □

COROLLARY 11.2.24. *If $h: Y \to Z$ is a map in* Spc *and z and z' are points in the same path component of Z, then the homotopy fiber of h over z is weakly equivalent to the homotopy fiber of h over z'.*

PROOF. This follows from Proposition 11.2.23. □

PROPOSITION 11.2.25. *Let $\mathcal{M}$ be a right proper model category. If Z is an object of $\mathcal{M}$, $z: * \to Z$ is a point of Z, and $* \to P \to Z$ is a factorization of z into a weak equivalence followed by a fibration, then the homotopy fiber of any map $h: Y \to Z$ over z is naturally weakly equivalent to $P \times_Z Y$.*

PROOF. This follows from Proposition 11.2.7. □

PROPOSITION 11.2.26. *If $h: Y \to Z$ is a map in* Top *and z is a point of Z, then the total singular complex of the homotopy fiber of h over z is naturally homotopy equivalent to the corresponding homotopy fiber of $(\operatorname{Sing} h): \operatorname{Sing} Y \to \operatorname{Sing} Z$.*

PROOF. If E is the factorization in Top of Definition 11.2.2 and $i_z: * \to Z$ is the constant map to z, then $\operatorname{Sing}(*) \to \operatorname{Sing} \mathrm{E}(i_z) \to \operatorname{Sing} Z$ is a factorization of $\operatorname{Sing}(*) \to \operatorname{Sing} Z$ into a weak equivalence followed by a fibration. Since the total singular complex functor commutes with pullbacks and all the simplicial sets involved are fibrant, the result now follows from Proposition 11.2.25. □

PROPOSITION 11.2.27. *If $h: Y \to Z$ is a map in* SS *and z is a vertex of Z, then the geometric realization of the homotopy fiber of h over z is naturally weakly equivalent to the corresponding homotopy fiber of $|h|: |Y| \to |Z|$.*

PROOF. Since the geometric realization functor commutes with pullbacks (see [**33**, page 49]), this is similar to the proof of Proposition 11.2.26. □

11.3. Homotopy pushouts and homotopy cofibers

PROPOSITION 11.3.1. *Let $\mathcal{M}$ be a left proper model category. If the vertical maps in the diagram*

$$\begin{array}{ccccc} Z & \longleftarrow & X & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{Z} & \longleftarrow & \widetilde{X} & \longrightarrow & \widetilde{Y} \end{array}$$

are weak equivalences and at least one map in each row is a cofibration, then the induced map of pushouts $Z \amalg_X Y \to \widetilde{Z} \amalg_{\widetilde{X}} \widetilde{Y}$ is a weak equivalence.

PROOF. This follows from Proposition 11.2.9 and Proposition 8.1.6 (see Remark 8.1.7). □

PROPOSITION 11.3.2. *Let $\mathcal{M}$ be a left proper model category. If we have a diagram $Y \xleftarrow{g} X \xrightarrow{h} W$ in which at least one of g and h is a cofibration and if $\tilde{g}: \widetilde{X} \to \widetilde{Y}$ is a cofibrant approximation to g, then the pushout of g along h has a cofibrant approximation that is a pushout of $\tilde{g}$.*

Proof. We have the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "i_X"] \arrow[d, "\tilde{g}"'] & X \arrow[r, "h"] \arrow[d, "g"] & W \arrow[d, "k"] \\ \widetilde{Y} \arrow[r, "i_Y"'] & Y \arrow[r] & Z \end{tikzcd}$$

in which Z is the pushout $Y \amalg_X W$ and i_X and i_Y are weak equivalences, and we must show that there is a pushout of $\tilde{g}$ that is a cofibrant approximation to k. If we factor the composition $hi_x : \widetilde{X} \to W$ as $\widetilde{X} \xrightarrow{\tilde{h}} \widetilde{W} \xrightarrow{i_W} W$ where $\tilde{h}$ is a cofibration and i_W is a trivial fibration, then we can let $\widetilde{Z} = \widetilde{Y} \amalg_{\widetilde{X}} \widetilde{W}$ and we have the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "\tilde{h}"] \arrow[dd, "\tilde{g}"'] \arrow[dr] & & \widetilde{W} \arrow[dd, "\tilde{k}"' near start] \arrow[dr] & \\ & X \arrow[rr] \arrow[dd, "g" near start] & & W \arrow[dd, "k"] \\ \widetilde{Y} \arrow[rr] \arrow[dr] & & \widetilde{Z} \arrow[dr, "i_Z"] & \\ & Y \arrow[rr] & & Z \end{tikzcd}$$

in which the front and back squares are pushouts. Proposition 11.3.1 now implies that i_Z is a weak equivalence, and so the pushout $\tilde{k}$ of $\tilde{g}$ is a cofibrant approximation to □

Proposition 11.3.3. *If the diagram in* $\mathsf{Top}_{(*)}$

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is a pushout and i *is a cofibration, then the natural map of simplicial sets* $\operatorname{Sing} C \amalg_{\operatorname{Sing} A} \operatorname{Sing} B \to \operatorname{Sing} D$ *is a weak equivalence.*

Proof. Since left adjoints commute with pushouts, there is a natural homeomorphism $\left|\operatorname{Sing} C \amalg_{\operatorname{Sing} A} \operatorname{Sing} B\right| \approx \left|\operatorname{Sing} C\right| \amalg_{|\operatorname{Sing} A|} \left|\operatorname{Sing} B\right|$, and so it is sufficient to show that the map $\left|\operatorname{Sing} C\right| \amalg_{|\operatorname{Sing} A|} \left|\operatorname{Sing} B\right| \to \left|\operatorname{Sing} D\right|$ is a weak equivalence. We have the diagram

$$\begin{tikzcd} \left|\operatorname{Sing} C\right| \arrow[d] & \left|\operatorname{Sing} A\right| \arrow[l] \arrow[r] \arrow[d] & \left|\operatorname{Sing} B\right| \arrow[d] \\ C & A \arrow[l] \arrow[r] & B \end{tikzcd}$$

and Proposition 11.3.1 implies that the map $\left|\operatorname{Sing} C\right| \amalg_{|\operatorname{Sing} A|} \left|\operatorname{Sing} B\right| \to D$ is a weak equivalence. Since this map factors through the weak equivalence $\left|\operatorname{Sing} D\right| \to D$, the result follows from the "two out of three" axiom for weak equivalences. □

CHAPTER 12

Ordinals, cardinals, and transfinite composition

12.1. Ordinals and cardinals

For a thorough discussion of the definitions and basic properties of ordinals and cardinals, see [**25**, Chapter II].

12.1.1. Ordinals.

DEFINITION 12.1.2. 1. A *preordered set* is a set with a relation that is reflexive and transitive.
2. A *partially ordered set* is a preordered set in which the relation is also antisymmetric.
3. A *totally ordered set* is a partially ordered set in which every pair of elements is comparable.
4. A *well ordered set* is a totally ordered set in which every nonempty subset has a first element.

We adopt the definition of ordinals that arranges it so that an ordinal is the well ordered set of all lesser ordinals, and every well ordered set is isomorphic to a unique ordinal (see, e.g., [**25**, Chapter II]). Thus, the union of a set of ordinals is an ordinal, and it is the least upper bound of the set.

REMARK 12.1.3. We will often consider a preordered set to be a small category with objects equal to the elements of the set and a single morphism from the object s to the object t if $s \leq t$.

DEFINITION 12.1.4. If S is a totally ordered set and T is a subset of S, then T will be called 0-*right cofinal* (or 0-*terminal*) in S if for every $s \in S$ there exists $t \in T$ such that $s \leq t$.

REMARK 12.1.5. What we here call 0-right cofinal has classically been called *cofinal* (see Definition 14.4.5, Definition 14.4.6, and Remark 14.4.12).

THEOREM 12.1.6. *If $\mathcal{C}$ is a cocomplete category, S is a totally ordered set, T is a* 0*-right cofinal subset of S, and $X\colon S \to \mathcal{C}$ is a functor, then the natural map $\operatorname{colim}_T X \to \operatorname{colim}_S X$ is an isomorphism.*

PROOF. The classical proof works. $\square$

PROPOSITION 12.1.7. *If S is a totally ordered set, then there is a* 0*-right cofinal subset T of S that is well ordered.*

PROOF. We will prove the proposition by considering the set of well ordered subsets of S. We will show that this set has a maximal element, and that a maximal element must be 0-right cofinal in S.

Let U be the set of pairs $(\lambda, f\colon \lambda \to S)$ where λ is an ordinal and f is a one to one order preserving function. We define a preorder on U by defining $(\lambda, f) \le (\kappa, g)$ if $\lambda \le \kappa$ and $f = g|_\lambda$. If $U' \subset U$ is a chain (i.e., a totally ordered subset of U), let $\lambda = \bigcup_{(\lambda_u, f_u)\in U'} \lambda_u$, and define $f\colon \lambda \to S$ to be the colimit of the f_u for $(\lambda_u, f_u) \in U'$. The pair (λ, f) is an element of U, and it is an upper bound for the chain. Thus, Zorn's lemma implies that U has a maximal element, and it remains only to show that a maximal element of U must be 0-right cofinal.

If (λ_m, f_m) is a maximal element of U and the image of $f_m : \lambda_m \to S$ is not 0-right cofinal, then there is an element s of S such that $f_m(\beta) < s$ for all $\beta \in \lambda_m$. Thus, we can define $g\colon (\lambda_m + 1) \to S$ by extending f_m to include s in its image. This would imply that (λ_m, f_m) was not a maximal element of U, and so the image of $f_m : \lambda_m \to S$ must actually be a 0-right cofinal well ordered subset of S. □

12.1.8. Cardinals.

DEFINITION 12.1.9. A *cardinal* is an ordinal that is of greater cardinality than any lesser ordinal.

DEFINITION 12.1.10. If X is a set, then the *cardinal of* X is the unique cardinal whose underlying set has a bijection with X.

DEFINITION 12.1.11. If γ is a cardinal, then by $\mathrm{Succ}(\gamma)$ we will mean the *successor* of γ, i.e., the first cardinal greater then γ.

DEFINITION 12.1.12. A cardinal γ is *regular* if, whenever A is a set whose cardinal is less than γ and for every $a \in A$ there is a set S_a whose cardinal is less than γ, the cardinal of the set $\bigcup_{a\in A} S_a$ is less than γ.

EXAMPLE 12.1.13. The countable cardinal $\aleph_0$ is a regular cardinal. This is just the statement that a finite union of finite sets is finite.

PROPOSITION 12.1.14. *The product of two cardinals, at least one of which is infinite, equals the greater of the two cardinals.*

PROOF. See [**25**, page 53]. □

PROPOSITION 12.1.15. *If γ is infinite and a successor cardinal, then γ is regular.*

PROOF. Let β be the cardinal such that $\gamma = \mathrm{Succ}(\beta)$. If a set has cardinal less than γ, then its cardinal is less than or equal to β. Let B be a set whose cardinal is β and, for every $b \in B$, let S_b be a set whose cardinal is β. Then, if the cardinals of A and every S_a for $a \in A$ are all less than γ, we have $\mathrm{card}(\bigcup_{a\in A} S_a) \le \mathrm{card}(\bigcup_{b\in B} S_b) \le \beta \times \beta = \beta < \gamma$. □

PROPOSITION 12.1.16. *If μ is an infinite cardinal and $\gamma = \mu^\mu$, then $\gamma^\mu = \gamma$.*

PROOF.

$$\begin{aligned} \gamma^\mu &= (\mu^\mu)^\mu \\ &= \mu^{\mu\mu} \\ &= \mu^\mu \\ &= \gamma. \end{aligned}$$

□

LEMMA 12.1.17. *Let S be a set whose cardinal is ν. If $\mu < \nu$, then the collection T of subsets of S whose cardinal is μ has cardinal ν.*

PROOF. The product $\prod_\mu S$ has cardinal $\nu\mu = \nu$, and it has a subset that maps onto T. Thus, the cardinal of T is at most ν. Since the collection of one element subsets of S has cardinal ν, the cardinal of T must be exactly ν. □

LEMMA 12.1.18. *If $\mathcal{M}$ is a category, X, Y, and Z are objects of $\mathcal{M}$, and X is a retract of Y, then the cardinal of $\mathcal{M}(X,Z)$ is less than or equal to the cardinal of $\mathcal{M}(Y,Z)$, and the cardinal of $\mathcal{M}(Z,X)$ is less than or equal to the cardinal of $\mathcal{M}(Z,Y)$.*

PROOF. If $i\colon X \to Y$ and $r\colon Y \to X$ are maps such that $ri = 1_X$, then $(ri)^*\colon \mathcal{M}(X,Z) \to \mathcal{M}(X,Z)$ is the identity map. Thus, $i^*\colon \mathcal{M}(Y,Z) \to \mathcal{M}(X,Z)$ is a surjection. Similarly, $r_*\colon \mathcal{M}(Z,Y) \to \mathcal{M}(Z,X)$ is a surjection. □

12.2. Transfinite composition

DEFINITION 12.2.1. Let $\mathcal{C}$ be a category that is closed under colimits.

1. If λ is an ordinal, then a *λ-sequence* in $\mathcal{C}$ is a functor $X\colon \lambda \to \mathcal{C}$ (see Remark 12.1.3) (i.e., a diagram
$$X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \qquad (\beta < \lambda)$$
in $\mathcal{C}$) such that, for every limit ordinal $\gamma < \lambda$, the induced map $\mathrm{colim}_{\beta<\gamma} X_\beta \to X_\gamma$ is an isomorphism.
2. The *composition* of the λ-sequence is the map $X_0 \to \mathrm{colim}_{\beta<\lambda} X_\beta$.

DEFINITION 12.2.2. If $\mathcal{C}$ is a category and $\mathcal{D}$ is a subcategory of $\mathcal{C}$, then a *transfinite composition* of maps in $\mathcal{D}$ is the composition of some λ-sequence $X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \quad (\beta < \lambda)$ (for some ordinal λ, possibly finite) in $\mathcal{C}$ such that, for every $\beta < \lambda$, the map $X_\beta \to X_{\beta+1}$ is in $\mathcal{D}$. (The significance of the λ-sequence being a λ-sequence in $\mathcal{C}$ is that, for every limit ordinal $\gamma < \lambda$, the colimit $\mathrm{colim}_{\beta<\gamma} X_\beta$ is formed in $\mathcal{C}$.)

LEMMA 12.2.3. *Let $\mathcal{C}$ be a category, let λ be a limit ordinal, and let $X\colon \lambda \to \mathcal{C}$ be a functor. If the functor $Y\colon \lambda \to \mathcal{C}$ is defined by*
$$\begin{aligned} Y_0 &= X_0 \\ Y_{\beta+1} &= X_\beta && \text{if } \beta+1<\lambda \\ Y_\beta &= \operatorname*{colim}_{\gamma<\beta} X_\gamma && \text{if } \beta<\lambda \text{ and } \beta \text{ is a limit ordinal} \end{aligned}$$
then Y is a λ-sequence in $\mathcal{C}$, and $\mathrm{colim}_{\beta<\lambda} X_\beta = \mathrm{colim}_{\beta<\lambda} Y_\beta$.

PROOF. This follows directly from the definitions. □

DEFINITION 12.2.4. If $\mathcal{C}$ is a category, λ is a limit ordinal, and $X\colon \lambda \to \mathcal{C}$ is a functor, then the λ-sequence Y obtained from the functor X as in Lemma 12.2.3 will be called the *reindexing* of X.

PROPOSITION 12.2.5. *If $\mathcal{C}$ is a category, S is a set, and $g_s\colon C_s \to D_s$ is a map in $\mathcal{C}$ for every $s \in S$, then the coproduct $\amalg g_s\colon \amalg C_s \to \amalg D_s$ is a transfinite composition of pushouts of the g_s, one for each element of S.*

PROOF. Choose a well ordering of the set S. There is a unique ordinal λ that is isomorphic to the ordered set S (see, e.g., [**25**, Chapter II]), and we will identify S with λ. We define a λ-sequence (see Definition 12.2.1) by letting

$$X_\beta = \Big(\coprod_{\alpha<\beta} D_\alpha\Big) \amalg \Big(\coprod_{\beta\le\alpha<\lambda} C_\alpha\Big)$$

for all $\beta < \lambda$, with the maps in the sequence being the obvious ones. For each $\beta < \lambda$, we have a pushout diagram

$$\begin{tikzcd} C_\beta \arrow[r, "g_\beta"] \arrow[d] & D_\beta \arrow[d] \\ X_\beta \arrow[r] & X_{\beta+1} \end{tikzcd}$$

and so we have a λ-sequence of pushouts of the g_s whose composition is $\amalg g_s$. $\square$

PROPOSITION 12.2.6. *Let $\mathcal{C}$ be a category. If the map $X \to Y$ is the composition of the λ-sequence*

$$X = X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \qquad (\beta < \lambda) \tag{12.2.7}$$

(for some ordinal λ) in which each map $X_\beta \to X_{\beta+1}$ is the composition of the γ_β-sequence

$$X_\beta = W_0^\beta \to W_1^\beta \to W_2^\beta \to \cdots \to W_\alpha^\beta \to \cdots \qquad (\alpha < \gamma_\beta) \tag{12.2.8}$$

(for some ordinal γ_β), then the set $P = \{(\beta,\alpha) \bigm| \beta < \lambda, \alpha < \gamma_\beta\}$ is well ordered by the dictionary order, i.e.,

$$(\beta_1,\alpha_1) < (\beta_2,\alpha_2) \quad \text{if} \quad \beta_1 < \beta_2 \quad \text{or} \quad \beta_1 = \beta_2 \text{ and } \alpha_1 < \alpha_2.$$

We define a quotient $\widetilde{P}$ of P as follows: For each γ_β that is a successor ordinal (i.e., for each γ_β for which there is an ordinal $\bar\gamma_\beta$ such that $\gamma_\beta = \bar\gamma_\beta + 1$), we identify (β,γ_β) with $(\beta+1,0)$. The well ordering on P induces a well ordering on $\widetilde{P}$, and so there is a unique ordinal κ for which there is an isomorphism of ordered sets $f\colon \kappa \approx \widetilde{P}$, and this isomorphism is also unique. If we define a functor $Y : \kappa \to \mathcal{C}$ by $Y(\gamma) = W\big(f(\gamma)\big)$, then Y is a κ-sequence in $\mathcal{C}$.

PROOF. We need only show that if $\gamma < \kappa$ and γ is a limit ordinal, then $Y(\gamma) = \operatorname{colim}_{\alpha<\gamma} Y(\alpha)$. This follows directly from our hypotheses. $\square$

DEFINITION 12.2.9. The κ-sequence of Proposition 12.2.6 will be said to have been obtained by *interpolating* the sequences of (12.2.8) into the sequence (12.2.7).

PROPOSITION 12.2.10. *The λ-sequence of* (12.2.7) *is* 0*-right cofinal (see Definition* 12.1.4*) in the κ-sequence of Proposition* 12.2.6*.*

PROOF. This follows directly from the definition. $\square$

LEMMA 12.2.11. *Let $\mathcal{C}$ be a category, let $\mathcal{D}$ be a subcategory of $\mathcal{C}$, and let λ be an ordinal. If the map $X \to Y$ is the composition of a λ-sequence $X = X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots$ $(\beta < \lambda)$ in which each map $X_\beta \to X_{\beta+1}$ is a transfinite composition of maps in $\mathcal{D}$, then interpolating (see Definition* 12.2.9*) the sequences for each $X_\beta \to X_{\beta+1}$ into the original λ-sequence gives a κ-sequence (for some ordinal κ) of maps in $\mathcal{D}$ whose composition is the map $X \to Y$.*

PROOF. This follows directly from the definitions. □

PROPOSITION 12.2.12. *Let $\mathcal{C}$ be a category, and let $\mathcal{D}$ be a subcategory of $\mathcal{C}$. If the map $g\colon X \to Y$ is a transfinite composition of pushouts of coproducts of elements of $\mathcal{D}$, then g is a transfinite composition of pushouts of elements of $\mathcal{D}$.*

PROOF. This follows from Proposition 12.2.5 and Lemma 12.2.11. □

PROPOSITION 12.2.13. *Let $\mathcal{C}$ be a category, let I be a set of maps in $\mathcal{C}$, and let λ be a regular cardinal (see Definition 12.1.12). If the map $X \to Y$ is the composition of a λ-sequence*

$$X = X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \qquad (\beta < \lambda) \tag{12.2.14}$$

in which each map $X_\beta \to X_{\beta+1}$ is a transfinite composition, indexed by an ordinal whose cardinal is less than λ, of pushouts of coproducts of elements of I, then interpolating the sequences for the $X_\beta \to X_{\beta+1}$ into the sequence (12.2.14) (see Definition 12.2.9) yields a λ-sequence (indexed by the same ordinal λ) of pushouts of coproducts of elements of I.

PROOF. Lemma 12.2.11 implies that there is an ordinal κ such that the map $X \to Y$ is the composition of a κ-sequence of pushouts of coproducts of elements of I, and so it remains only to show that the ordinal κ constructed in the proof of Lemma 12.2.11 equals λ. Since the cardinal of κ equals that of a union, indexed by λ, of sets of cardinal less than λ, the cardinal of κ equals λ. Since any ordinal less than κ is contained within a subunion, indexed by an ordinal less than λ, of sets of cardinal less than λ, and λ is a regular cardinal, that subunion would have cardinal less than λ, i.e., κ is the first ordinal having its cardinal, and so κ is a cardinal, and so $\kappa = \lambda$. □

12.2.15. Transfinite composition and lifting in model categories.

LEMMA 12.2.16. *If $\mathcal{M}$ is a category and $p\colon X \to Y$ is a map in $\mathcal{M}$, then the class of maps with the left lifting property with respect to p is closed under transfinite composition (see Definition 12.2.1).*

PROOF. Given a λ-sequence of maps with the left lifting property with respect to p and a lifting problem for the composition of the λ-sequence, a lift can be constructed by a transfinite induction. □

PROPOSITION 12.2.17. *If $\mathcal{M}$ is a category and $p\colon X \to Y$ is a map in $\mathcal{M}$, then the class of maps with the left lifting property with respect to p is closed under pushouts, transfinite composition, and retracts.*

PROOF. This follows from Lemma 8.2.5, Lemma 12.2.16, and Lemma 8.2.7. □

PROPOSITION 12.2.18. *If $\mathcal{M}$ is a simplicial model category and $\mathcal{C}$ is a class of maps in $\mathcal{M}$, then the class of maps in $\mathcal{M}$ that have the homotopy left lifting property with respect to every element of $\mathcal{C}$ is closed under pushouts, transfinite compositions, and retracts.*

PROOF. This follows from Lemma 10.3.6 and Proposition 12.2.17. □

PROPOSITION 12.2.19. *If $\mathcal{M}$ is a model category, then the classes of cofibrations and of trivial cofibrations are closed under pushouts, transfinite compositions, and retracts.*

PROOF. This follows from Proposition 8.2.3 and Proposition 12.2.17. □

LEMMA 12.2.20. *Let $\mathcal{M}$ be a model category and let $p\colon X \to Y$ be a map in $\mathcal{M}$. If S is a totally ordered set and $\boldsymbol{W}\colon S \to \mathcal{M}$ is a functor such that if $s, t \in S$ and $s \le t$, then $\boldsymbol{W}_s \to \boldsymbol{W}_t$ has the left lifting property with respect to p, then for every $s \in S$ the map $\boldsymbol{W}_s \to \operatorname{colim}_{t \ge s} \boldsymbol{W}_t$ has the left lifting property with respect to p.*

PROOF. Proposition 12.1.7 implies that we can choose a 0-right cofinal subset T of $\{t \in S \mid t \ge s\}$ such that T is well ordered. There is a unique ordinal λ that is isomorphic to T (see, e.g., [**25**, Chapter II]), and so we have a 0-right cofinal functor $\lambda \to \mathcal{M}$. If we reindex this functor (see Definition 12.2.4), then we have a λ-sequence of maps with the left lifting property with respect to p. The lemma now follows from Lemma 12.2.16 and Theorem 12.1.6. □

PROPOSITION 12.2.21. *Let $\mathcal{M}$ be a model category, and let S be a totally ordered set. If $\boldsymbol{W}\colon S \to \mathcal{M}$ is a functor such that, if $s, t \in S$ and $s \le t$, then $\boldsymbol{W}_s \to \boldsymbol{W}_t$ is a cofibration, then, for every $s \in S$, the map $\boldsymbol{W}_s \to \operatorname{colim}_{t \ge s} \boldsymbol{W}_t$ is a cofibration.*

PROOF. This follows from Proposition 8.2.3 and Proposition 12.2.17. □

12.3. Small objects

DEFINITION 12.3.1. Let $\mathcal{C}$ be a cocomplete category and let $\mathcal{D}$ be a subcategory of $\mathcal{C}$.

1. If κ is a cardinal, then an object W in $\mathcal{C}$ is *κ-small relative to* $\mathcal{D}$ if, for every regular cardinal (see Definition 12.1.12) $\lambda \ge \kappa$ and every λ-sequence (see Definition 12.2.1)
$$X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \qquad (\beta < \lambda)$$
in $\mathcal{C}$ such that the map $X_\beta \to X_{\beta+1}$ is in $\mathcal{D}$ for every ordinal β such that $\beta + 1 < \lambda$, the map of sets $\operatorname{colim}_{\beta<\lambda} \mathcal{C}(W, X_\beta) \to \mathcal{C}(W, \operatorname{colim}_{\beta<\lambda} X_\beta)$ is an isomorphism.
2. An object is *small relative to* $\mathcal{D}$ if it is κ-small relative to $\mathcal{D}$ for some cardinal κ, and it is *small* if it is small relative to $\mathcal{C}$.

EXAMPLE 12.3.2. In the category $\mathsf{SS}_{(*)}$, every simplicial set with finitely many nondegenerate simplices is $\aleph_0$-small relative to the subcategory of inclusions of simplicial sets (where $\aleph_0$ is the first infinite cardinal).

EXAMPLE 12.3.3. Let X be a finite cell complex in $\mathsf{Top}_{(*)}$ (see Definition 2.2.1). Corollary 2.2.7 implies X is $\aleph_0$-small relative to the subcategory of inclusions of cell complexes (where $\aleph_0$ is the first infinite cardinal).

EXAMPLE 12.3.4. Let X be an object of $\mathsf{SS}_{(*)}$. If κ is the first infinite cardinal greater than the cardinal of the set of nondegenerate simplices of X, then X is κ-small relative to the subcategory of inclusions (see Proposition 12.1.15). Thus, every simplicial set is small relative to the subcategory of inclusions.

EXAMPLE 12.3.5. Let X be a cell complex in $\mathsf{Top}_{(*)}$ (see Definition 2.2.1). If κ is the first infinite cardinal greater than the cardinal of the set of cells of X (see Proposition 12.1.15), then Proposition 2.2.4 implies that X is κ-small relative to the subcategory of relative cell complexes. Thus, every cell complex is small relative to the subcategory of relative cell complexes.

LEMMA 12.3.6. *If $\mathcal{C}$ is a cocomplete category, $\mathcal{D}$ is a subcategory of $\mathcal{C}$, and I is a set of objects in $\mathcal{C}$ that are small relative to $\mathcal{D}$, then there is a cardinal κ such that every element of I is κ-small relative to $\mathcal{D}$.*

PROOF. For every object A of I let κ_A be a cardinal such that A is κ_A-small relative to $\mathcal{D}$. If we let κ be the union $\bigcup_{A\in I}\kappa_A$, then every object of I is κ-small relative to $\mathcal{D}$. $\square$

PROPOSITION 12.3.7. *Let $\mathcal{C}$ be a cocomplete category and let $\mathcal{D}$ be a subcategory of $\mathcal{C}$. If κ is a cardinal and X is an object in $\mathcal{C}$ that is κ-small relative to $\mathcal{D}$, then any retract of X is κ-small relative to $\mathcal{D}$.*

PROOF. Let $i\colon W \to X$ and $r\colon X \to W$ be maps such that $ri = 1_W$. If λ is a regular cardinal such that $\lambda \ge \kappa$ and $Z_0 \to Z_1 \to Z_2 \to \cdots \to Z_\beta \to \cdots$ $(\beta < \lambda)$ is a λ-sequence in $\mathcal{D}$, then we have the commutative diagram

$$\begin{tikzcd}
\operatorname*{colim}_{\beta<\lambda} \mathcal{C}(W, Z_\beta) \arrow[r, "\operatorname{colim} r^*"] \arrow[d] \arrow[rr, bend left, "1_{\operatorname{colim} \mathcal{C}(W,Z_\beta)}"] & \operatorname*{colim}_{\beta<\lambda} \mathcal{C}(X, Z_\beta) \arrow[r, "\operatorname{colim} i^*"] \arrow[d] & \operatorname*{colim}_{\beta<\lambda} \mathcal{C}(W, Z_\beta) \arrow[d] \\
\mathcal{C}(W, \operatorname*{colim}_{\beta<\lambda} Z_\beta) \arrow[r, "r^*"] \arrow[rr, bend right, "1_{\mathcal{C}(W,\operatorname{colim} Z_\beta)}"'] & \mathcal{C}(X, \operatorname*{colim}_{\beta<\lambda} Z_\beta) \arrow[r, "i^*"] & \mathcal{C}(W, \operatorname*{colim}_{\beta<\lambda} Z_\beta)
\end{tikzcd}$$

Thus, the map $\operatorname{colim}_{\beta<\lambda} \mathcal{C}(W, Z_\beta) \to \mathcal{C}(W, \operatorname{colim}_{\beta<\lambda} Z_\beta)$ is a retract of the isomorphism $\operatorname{colim}_{\beta<\lambda} \mathcal{C}(X, Z_\beta) \to \mathcal{C}(X, \operatorname{colim}_{\beta<\lambda} Z_\beta)$, and is thus an isomorphism. $\square$

PROPOSITION 12.3.8. *Let $\mathcal{C}$ be a cocomplete category and let $\mathcal{D}$ be a subcategory of $\mathcal{C}$. If $\mathcal{I}$ is a small category and $\boldsymbol{W}\colon \mathcal{I} \to \mathcal{C}$ is a diagram in $\mathcal{C}$ such that $\boldsymbol{W}_i$ is small relative to $\mathcal{D}$ for every object i in $\mathcal{I}$, then $\operatorname{colim}_{i\in\mathcal{I}} \boldsymbol{W}_i$ is small relative to $\mathcal{D}$.*

PROOF. Let γ be a cardinal such that $\boldsymbol{W}_i$ is γ-small relative to $\mathcal{D}$ for every object i in $\mathcal{I}$ (see Lemma 12.3.6), let δ be the cardinal of the set of morphisms in $\mathcal{I}$, and let κ be the first cardinal greater than both γ and δ; we will show that $\operatorname{colim}_{i\in\mathcal{I}} \boldsymbol{W}_i$ is κ-small relative to $\mathcal{D}$.

Let λ be a regular cardinal such that $\lambda \ge \kappa$, and let

$$X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \qquad (\beta < \lambda)$$

be a λ-sequence in $\mathcal{C}$ such that the map $X_\beta \to X_{\beta+1}$ is in $\mathcal{D}$ for all $\beta < \lambda$. If we have a map $f\colon \operatorname{colim}_{i\in\mathcal{I}} \boldsymbol{W}_i \to \operatorname{colim}_{\beta<\lambda} X_\beta$, then for every object j in $\mathcal{I}$ the composition of f with the natural map $\boldsymbol{W}_j \to \operatorname{colim}_{i\in\mathcal{I}} \boldsymbol{W}_i$ defines a map $f_j\colon \boldsymbol{W}_j \to \operatorname{colim}_{\beta<\lambda} X_\beta$. Since $\boldsymbol{W}_j$ is small relative to $\mathcal{D}$ and λ is a large enough regular cardinal, there exists an ordinal $\beta_j < \lambda$ such that f_j factors through X_{β_j}. If we let $\tilde\beta = \bigcup_{j\in\operatorname{Ob}\mathcal{I}} \beta_j$, then (since λ is a regular cardinal) $\tilde\beta < \lambda$, and the dotted arrow $\tilde{g}_j$ exists in the diagram

$$\begin{tikzcd} \boldsymbol{W}_j \arrow[d, dotted, "\tilde{g}_j"'] \arrow[dr, "f_j"] & \\ X_{\tilde{\beta}} \arrow[r] & \operatorname{colim}_{\beta<\lambda} X_\beta \end{tikzcd}$$

for every object j in $\mathcal{J}$.

If $s\colon j \to k$ is a morphism in $\mathcal{J}$, then the composition $\boldsymbol{W}_j \xrightarrow{\boldsymbol{W}_s} \boldsymbol{W}_k \xrightarrow{\tilde{g}_k} X_{\tilde{\beta}}$ need not equal the map $\tilde{g}_j\colon \boldsymbol{W}_j \to X_{\tilde{\beta}}$, but their compositions with the natural map $X_{\tilde{\beta}} \to \operatorname{colim}_{\beta<\lambda} X_\beta$ are equal. Since the natural map of sets $\operatorname{colim}_{\beta<\lambda} \mathcal{C}(\boldsymbol{W}_j, X_\beta) \to \mathcal{C}(\boldsymbol{W}_j, \operatorname{colim}_{\beta<\lambda} X_\beta)$ is an isomorphism, there must exist an ordinal $\hat{\beta}_s < \lambda$ such that their compositions with the map $X_{\tilde{\beta}} \to X_{\hat{\beta}_s}$ are equal. If we let $\hat{\beta} = \bigcup_{(s\colon j\to k)\in\mathcal{J}} \hat{\beta}_s$, then (since λ is a regular cardinal) we have $\hat{\beta} < \lambda$. If, for every object j of $\mathcal{J}$, we let $\hat{g}_j$ equal the composition $\boldsymbol{W}_j \xrightarrow{\tilde{g}_j} X_{\tilde{\beta}} \to X_{\hat{\beta}}$, then for every morphism $s\colon j \to k$ in $\mathcal{J}$ the triangle

$$\begin{tikzcd} \boldsymbol{W}_j \arrow[r, "\boldsymbol{W}_s"] \arrow[dr, "\hat{g}_j"'] & \boldsymbol{W}_k \arrow[d, "\hat{g}_k"] \\ & X_{\hat{\beta}} \end{tikzcd}$$

commutes, and so the $\hat{g}_j$ define a map $g\colon \operatorname{colim}_{i\in\mathcal{J}} \boldsymbol{W}_i \to X_{\hat{\beta}}$ whose composition with the natural map $X_{\hat{\beta}} \to \operatorname{colim}_{\beta<\lambda} X_\beta$ equals f. Thus, the map

$$\operatorname*{colim}_{\beta<\lambda} \mathcal{C}(\operatorname*{colim}_{i\in\mathcal{J}} \boldsymbol{W}_i, X_\beta) \to \mathcal{C}(\operatorname*{colim}_{i\in\mathcal{J}} \boldsymbol{W}_i, \operatorname*{colim}_{\beta<\lambda} X_\beta)$$

is surjective.

To show that that map is also injective, let $g'\colon \operatorname{colim}_{i\in\mathcal{J}} \boldsymbol{W}_i \to X_{\bar{\beta}}$ be a map whose composition with the natural map $X_{\bar{\beta}} \to \operatorname{colim}_{\beta<\lambda} X_\beta$ equals f. For every object j in $\mathcal{J}$ the compositions

$$\boldsymbol{W}_j \to \operatorname*{colim}_{i\in\mathcal{J}} \boldsymbol{W}_i \xrightarrow{g} X_{\hat{\beta}} \to \operatorname*{colim}_{\beta<\lambda} X_\beta$$

and

$$\boldsymbol{W}_j \to \operatorname*{colim}_{i\in\mathcal{J}} \boldsymbol{W}_i \xrightarrow{g'} X_{\bar{\beta}} \to \operatorname*{colim}_{\beta<\lambda} X_\beta$$

are equal, and so there exists an ordinal $\alpha_j < \lambda$ such that the compositions

$$\boldsymbol{W}_j \to \operatorname*{colim}_{i\in\mathcal{J}} \boldsymbol{W}_i \xrightarrow{g} X_{\hat{\beta}} \to X_{\alpha_j}$$

and

$$\boldsymbol{W}_j \to \operatorname*{colim}_{i\in\mathcal{J}} \boldsymbol{W}_i \xrightarrow{g'} X_{\bar{\beta}} \to X_{\alpha_j}$$

are equal. If we let $\alpha = \bigcup_{j\in\mathrm{Ob}(\mathcal{J})} \alpha_j$, then $\alpha < \lambda$, and the compositions $\operatorname{colim}_{i\in\mathcal{J}} \boldsymbol{W}_i \to X_{\hat{\beta}} \to X_\alpha$ and $\operatorname{colim}_{i\in\mathcal{J}} \boldsymbol{W}_i \to X_{\bar{\beta}} \to X_\alpha$ are equal, and so the map

$$\operatorname*{colim}_{\beta<\lambda} \mathcal{C}(\operatorname*{colim}_{i\in\mathcal{J}} \boldsymbol{W}_i, X_\beta) \to \mathcal{C}(\operatorname*{colim}_{i\in\mathcal{J}} \boldsymbol{W}_i, \operatorname*{colim}_{\beta<\lambda} X_\beta)$$

is an isomorphism. □

COROLLARY 12.3.9. *Let $\mathcal{C}$ be a cocomplete category, let $\mathcal{D}$ be a subcategory of $\mathcal{C}$, and let I be a set of maps in $\mathcal{C}$ whose domains and codomains are small relative to $\mathcal{D}$. If X is small relative to $\mathcal{D}$ and the map $X \to Y$ is a transfinite composition of pushouts of elements of I, then Y is small relative to $\mathcal{D}$.*

PROOF. This follows from Proposition 12.3.8. □

12.4. The small object argument

DEFINITION 12.4.1. Let $\mathcal{C}$ be a category, and let I be a set of maps in $\mathcal{C}$.

1. The subcategory of *I-injectives* is the subcategory of maps that have the right lifting property (see Definition 8.2.1) with respect to every element of I.
2. The subcategory of *I-cofibrations* is the subcategory of maps that have the left lifting property (see Definition 8.2.1) with respect to every I-injective. An object is *I-cofibrant* if the map to it from the initial object of $\mathcal{C}$ is an I-cofibration.

REMARK 12.4.2. The term *I-injective* comes from the theory of injective classes ([**32**]). The map $p\colon X \to Y$ is an I-injective if and only if, in the category $(\mathcal{C}\downarrow Y)$ of objects over Y, the object p is injective relative to the class of maps whose image under the forgetful functor $(\mathcal{C}\downarrow Y) \to \mathcal{C}$ is an element of I.

EXAMPLE 12.4.3. If I is the set of inclusions $\partial\Delta[n] \to \Delta[n]$ in SS, then the I-injectives are the trivial fibrations, and the I-cofibrations are the inclusions of simplicial sets (see Proposition 8.2.3).

EXAMPLE 12.4.4. If J is the set of inclusions $\Lambda[n,k] \to \Delta[n]$ in SS, then the J-injectives are the Kan fibrations, and the J-cofibrations are the trivial cofibrations (see Proposition 8.2.3).

PROPOSITION 12.4.5. *Let $\mathcal{C}$ be a category, and let J and K be sets of maps in $\mathcal{C}$. If the subcategory of J-injectives equals the subcategory of K-injectives, then the subcategory of J-cofibrations equals the subcategory of K-cofibrations.*

PROOF. This follows directly from the definitions. □

DEFINITION 12.4.6. If $\mathcal{C}$ is a category that is closed under small colimits and I is a set of maps in $\mathcal{C}$, then

1. the subcategory of *relative I-cell complexes* (also known as the subcategory of *regular I-cofibrations*) is the subcategory of maps that can be constructed as a transfinite composition (see Definition 12.2.2) of pushouts (see Definition 8.2.10) of elements of I,
2. an object is an *I-cell complex* if the map to it from the initial object of $\mathcal{C}$ is a relative I-cell complex, and
3. a map is an *inclusion of I-cell complexes* if it is a relative I-cell complex whose domain is an I-cell complex.

REMARK 12.4.7. Note that Definition 12.4.6 defines a relative I-cell complex to be a map that can be constructed as as transfinite composition of pushouts of elements of I, but it does not assume that there is any preferred such construction. In Definition 12.5.3 we define a *presented relative I-cell complex* to be a relative I-cell complex together with a choice of such a construction.

PROPOSITION 12.4.8. *If $\mathcal{C}$ is a category and I is a set of maps in $\mathcal{C}$, then every relative I-cell complex is an I-cofibration (see Definition* 12.4.1*).*

PROOF. This follows from Lemma 8.2.5 and Lemma 12.2.16. □

PROPOSITION 12.4.9. *If $\mathcal{M}$ is a category and I is a set of maps in $\mathcal{M}$, then a retract of a relative I-cell complex is an I-cofibration.*

PROOF. This follows from Proposition 12.4.8 and Lemma 8.2.7. □

DEFINITION 12.4.10. Let $\mathcal{M}$ be a cocomplete category and let I be a set of maps in $\mathcal{M}$.

1. If κ is a cardinal, then an object is *κ-small relative to I* if it is κ-small relative to the subcategory of relative I-cell complexes (see Definition 12.3.1 and Definition 12.4.6).
2. An object is *small relative to I* if it is κ-small relative to I for some cardinal κ.

DEFINITION 12.4.11. If $\mathcal{M}$ is a category and I is a set of maps in $\mathcal{M}$, then we will follow D. M. Kan and say that *I permits the small object argument* if the domains of the elements of I are small relative to I (see Definition 12.4.10 and Definition 12.4.6).

PROPOSITION 12.4.12 (The small object argument). *If $\mathcal{C}$ is a cocomplete category and I is a set of maps in $\mathcal{C}$ that permits the small object argument (see Definition* 12.4.11*), then there is a functorial factorization of every map in $\mathcal{C}$ into a relative I-cell complex (see Definition* 12.4.6*) followed by an I-injective (see Definition* 12.4.1*).*

PROOF. Lemma 12.3.6 implies that we can choose a regular cardinal λ such that every domain of an element of I is λ-small relative to the subcategory of relative I-cell complexes. If $g\colon X \to Y$ is a map in $\mathcal{C}$, then we will factor g as $X \xrightarrow{j} \mathrm{E}_I \xrightarrow{p} Y$, where j is the transfinite composition of a λ-sequence

$$\begin{tikzcd}
X = \mathrm{E}^0 \arrow[r] \arrow[drr, "p_0"] & \mathrm{E}^1 \arrow[r] \arrow[dr, "p_1"] & \mathrm{E}^2 \arrow[r] \arrow[d, "p_2"] & \cdots \arrow[r] & \mathrm{E}^\beta \arrow[r] \arrow[dll, "p_\beta"] & \cdots & (\beta < \lambda) \\
 & & Y & & & &
\end{tikzcd}$$

in which each $\mathrm{E}^\beta \to \mathrm{E}^{\beta+1}$ is a pushout of a coproduct of elements of I, each E^β comes with a map $p_\beta\colon \mathrm{E}^\beta \to Y$ such that all the triangles commute, and $p = \operatorname{colim}_{\beta<\lambda} p_\beta$.

We begin by letting $\mathrm{E}^0 = X$ and letting $p_0 \colon \mathrm{E}^0 \to Y$ equal g. Given E^β, we have the solid arrow diagram

$$\begin{tikzcd}
\displaystyle\coprod_{\substack{(A_i \to B_i) \in I \\ \mathcal{M}(A_i, \mathrm{E}^\beta) \times_{\mathcal{M}(A_i, Y)} \mathcal{M}(B_i, Y)}} A_i \arrow[r] \arrow[d] & \mathrm{E}^\beta \arrow[r, dotted] \arrow[d, "p_\beta"'] & \mathrm{E}^{\beta+1} \arrow[dl, dotted, "p_{\beta+1}"] \\
\displaystyle\coprod_{\substack{(A_i \to B_i) \in I \\ \mathcal{M}(A_i, \mathrm{E}^\beta) \times_{\mathcal{M}(A_i, Y)} \mathcal{M}(B_i, Y)}} B_i \arrow[r] \arrow[urr, dotted] & Y &
\end{tikzcd}$$

and we let $\mathrm{E}^{\beta+1}$ be the pushout $(\coprod B_i) \amalg_{(\coprod A_i)} \mathrm{E}^\beta$. If γ is a limit ordinal, we let $\mathrm{E}^\gamma = \operatorname{colim}_{\beta<\gamma} \mathrm{E}^\beta$, and we let $\mathrm{E}_I = \operatorname{colim}_{\beta<\lambda} \mathrm{E}^\beta$. The construction of the factorization $X \to \mathrm{E}_I \to Y$ makes it clear that it is functorial. Proposition 12.2.5, Lemma 8.2.11, and Lemma 12.2.11 imply that $X \to \mathrm{E}_I$ is a relative I-cell complex, and so it remains only to show that $\mathrm{E}_I \to Y$ is an I-injective.

Given an element $A \to B$ of I and a solid arrow diagram

(12.4.13)
$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathrm{E}_I \arrow[d] \\
B \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}$$

we must show that the dotted arrow exists. Since $\mathrm{E}_I = \operatorname{colim}_{\beta<\lambda} \mathrm{E}^\beta$ and A is λ-small relative to I, the natural map of sets $\operatorname{colim}_{\beta<\lambda} \mathcal{M}(A, \mathrm{E}^\beta) \to \mathcal{M}(A, \mathrm{E}_I)$ is an isomorphism. Thus, the map $A \to \mathrm{E}_I$ factors through $\mathrm{E}^\beta \to \mathrm{E}_I$ for some $\beta < \lambda$, and we have the solid arrow diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathrm{E}^\beta \arrow[r] \arrow[d] & \mathrm{E}^{\beta+1} \arrow[r] \arrow[dl] & \mathrm{E}_I \arrow[dll] \\
B \arrow[r] \arrow[urr, dotted] & Y & &
\end{tikzcd}$$

The construction of $\mathrm{E}^{\beta+1}$ implies that the dotted arrow exists, and this dotted arrow defines the dotted arrow in Diagram 12.4.13. $\square$

Definition 12.4.14. Let $\mathcal{C}$ be a cocomplete category, let I be a set of maps in $\mathcal{C}$, and let λ be an ordinal. If we apply the construction in the proof of Proposition 12.4.12 to a map $g \colon X \to Y$ using the set I and the ordinal λ to obtain the factorization $X \to E_I \to Y$, then we will call E_I the *object obtained by applying the small object factorization with the set I and the ordinal λ to the map g.*

Proposition 12.4.15. *Let $\mathcal{C}$ be a cocomplete category, let I be a set of maps in $\mathcal{C}$, and let λ be an ordinal. If the map $g \colon X \to Y$ is a retract of the map $\tilde{g} \colon \widetilde{X} \to \widetilde{Y}$ and we apply the small object factorization to both g and $\tilde{g}$ using the set I and the ordinal λ (see Definition 12.4.14), then the factorization $X \to E_I \to Y$ obtained from g is a retract of the factorization $\widetilde{X} \to \widetilde{E}_I \to \widetilde{Y}$ obtained from $\tilde{g}$.*

Proof. At each step in the construction of E_I and $\widetilde{E}_I$, the factorization $X \to E^\beta \to Y$ is a retract of the factorization $\widetilde{X} \to \widetilde{E}^\beta \to \widetilde{Y}$. $\square$

COROLLARY 12.4.16. *Let $\mathcal{C}$ be a cocomplete category and let I be a set of maps in $\mathcal{C}$. If κ is a regular cardinal such that the domains of the elements of I are κ-small relative to I, then there is a functorial factorization of every map in $\mathcal{C}$ into the composition of a κ-sequence of pushouts of coproducts of elements of I followed by an I-injective.*

PROOF. This follows from the proof of Proposition 12.4.12. ☐

COROLLARY 12.4.17. *Let $\mathcal{C}$ be a cocomplete category and let I be a set of maps in $\mathcal{C}$ that permits the small object argument (see Definition 12.4.11). If $g\colon X \to Y$ is a map with the left lifting property (see Definition 8.2.1) with respect to every I-injective (see Definition 12.4.1), then g is a retract of a relative I-cell complex.*

PROOF. If we apply the factorization of Proposition 12.4.12 to g, we obtain $X \xrightarrow{j} \mathrm{E}_I \xrightarrow{p} Y$ in which j is a relative I-cell complex and p is an I-injective. The result now follows from the retract argument (see Proposition 8.2.2). ☐

COROLLARY 12.4.18. *If $\mathcal{C}$ is a cocomplete category, I is a set of maps in $\mathcal{C}$ that permits the small object argument, and $g\colon X \to Y$ is an I-cofibration (see Definition 12.4.1), then g is a retract of a relative I-cell complex.*

PROOF. This follows from Corollary 12.4.17. ☐

LEMMA 12.4.19. *Let $\mathcal{C}$ be a cocomplete category, let I be a set of maps in $\mathcal{M}$ that permit the small object argument, and let κ be a regular cardinal such that the domain of every element of I is κ-small relative to I (see Lemma 12.3.6). If λ is an ordinal and $X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots$ ($\beta < \lambda$) is a λ-sequence of I-cofibrations, then there is a λ-sequence $\widetilde{X}_0 \to \widetilde{X}_1 \to \widetilde{X}_2 \to \cdots \to \widetilde{X}_\beta \to \cdots$ ($\beta < \lambda$) of relative I-cell complexes and maps of λ-sequences*

(12.4.20)
$$\begin{tikzcd}
X_0 \arrow[r, "\sigma_0"] \arrow[d, "i_0"'] & X_1 \arrow[r, "\sigma_1"] \arrow[d, "i_1"'] & X_2 \arrow[r, "\sigma_2"] \arrow[d, "i_2"'] & \cdots \arrow[r] & X_\beta \arrow[r, "\sigma_\beta"] \arrow[d, "i_\beta"'] & \cdots \\
\widetilde{X}_0 \arrow[r, "\tau_0"] \arrow[d, "r_0"'] & \widetilde{X}_1 \arrow[r, "\tau_1"] \arrow[d, "r_1"'] & \widetilde{X}_2 \arrow[r, "\tau_2"] \arrow[d, "r_2"'] & \cdots \arrow[r] & \widetilde{X}_\beta \arrow[r, "\tau_\beta"] \arrow[d, "r_\beta"'] & \cdots \\
X_0 \arrow[r, "\sigma_0"] & X_1 \arrow[r, "\sigma_1"] & X_2 \arrow[r, "\sigma_2"] & \cdots \arrow[r] & X_\beta \arrow[r, "\sigma_\beta"] & \cdots
\end{tikzcd}$$

such that, for every $\beta < \lambda$,

1. *the composition $r_\beta i_\beta$ is the identity map of X_β, and*
2. *the map $\tau_\beta\colon \widetilde{X}_\beta \to \widetilde{X}_{\beta+1}$ is the composition of a κ-sequence of pushouts of coproducts of elements of I.*

PROOF. We let $\widetilde{X}_0 = X_0$, and we let both i_0 and r_0 be the identity map of X_0. If β is an ordinal such that $\beta + 1 < \lambda$ and we've defined the sequence through $\widetilde{X}_\beta$, then we apply the factorization of Corollary 12.4.16 to the map $\sigma_\beta r_\beta\colon \widetilde{X}_\beta \to X_{\beta+1}$ to obtain $\widetilde{X}_\beta \xrightarrow{\tau_\beta} \widetilde{X}_{\beta+1} \xrightarrow{r_{\beta+1}} X_{\beta+1}$, in which τ_β is the composition of a κ-sequence of pushouts of coproducts of elements of I and $r_{\beta+1}$ is an I-injective.

Since $r_{\beta+1}\tau_\beta i_\beta = \sigma_\beta r_\beta i_\beta = \sigma_\beta$, we now have the solid arrow diagram

$$\begin{tikzcd} X_\beta \arrow[r, "\tau_\beta i_\beta"] \arrow[d, "\sigma_\beta"'] & \widetilde{X}_{\beta+1} \arrow[d, "r_{\beta+1}"] \\ X_{\beta+1} \arrow[r, equal] \arrow[ur, dotted, "i_{\beta+1}"] & X_{\beta+1} \end{tikzcd}$$

in which σ_β is an I-cofibration and $r_{\beta+1}$ is an I-injective, and so there exists a dotted arrow $i_{\beta+1}$ such that $i_{\beta+1}\sigma_\beta = \tau_\beta i_\beta$ and $r_{\beta+1}i_{\beta+1} = 1_{X_{\beta+1}}$.

For every limit ordinal γ such that $\gamma < \lambda$, we let $\widetilde{X}_\gamma = \operatorname{colim}_{\beta<\gamma}\widetilde{X}_\beta$, $i_\gamma = \operatorname{colim}_{\beta<\gamma} i_\beta$, and $r_\gamma = \operatorname{colim}_{\beta<\lambda} r_\beta$. □

THEOREM 12.4.21. *Let $\mathcal{C}$ be a cocomplete category and let I be a set of maps in $\mathcal{C}$ that permits the small object argument. If W is an object that is small relative to I, then it is small relative to the subcategory of all I-cofibrations.*

PROOF. Let μ be a cardinal such that W is μ-small relative to I. Lemma 12.3.6 implies that there is a cardinal κ such that the domain of every element of I is κ-small relative to I. If ν is the first cardinal greater than both μ and κ, then we will show that W is ν-small relative to the subcategory of I-cofibrations.

Let λ be a regular cardinal such that $\lambda \geq \nu$ and let $X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots$ $(\beta < \lambda)$ be a λ-sequence of I-cofibrations. Lemma 12.4.19 implies that there is a λ-sequence $\widetilde{X}_0 \to \widetilde{X}_1 \to \widetilde{X}_2 \to \cdots \to \widetilde{X}_\beta \to \cdots$ $(\beta < \lambda)$ of relative I-cell complexes and maps of λ-sequences as in Diagram 12.4.20 satisfying the conclusion of Lemma 12.4.19. Proposition 12.2.13 implies that, after interpolations, the λ-sequence $\widetilde{X}_0 \to \widetilde{X}_1 \to \widetilde{X}_2 \to \cdots \to \widetilde{X}_\beta \to \cdots$ $(\beta < \lambda)$ is a λ-sequence of relative I-cell complexes, and so Proposition 12.2.10 and Corollary 14.4.11 imply that the map of sets $\operatorname{colim}_{\beta<\lambda}\mathcal{M}(W,\widetilde{X}_\beta) \to \mathcal{M}(W, \operatorname{colim}_{\beta<\lambda}\widetilde{X}_\beta)$ is an isomorphism. Since the map of sets $\operatorname{colim}_{\beta<\lambda}\mathcal{M}(W,X_\beta) \to \mathcal{M}(W,\operatorname{colim}_{\beta<\lambda} X_\beta)$ is a retract of this isomorphism, it is also an isomorphism. □

12.5. Subcomplexes of relative I-cell complexes

If $\mathcal{M}$ is a cocomplete category and I is a set of maps in $\mathcal{M}$, then a relative I-cell complex is a map that can be constructed as a transfinite composition of pushouts of coproducts of elements of I (see Definition 12.4.6 and Proposition 12.2.12). To consider "subcomplexes" of a relative I-cell complex, we need to choose a "presentation" of it (see Definition 12.5.2), i.e., a particular such construction. In Definition 12.5.3, we define a *presented relative I-cell complex* to be a relative I-cell complex together with a chosen presentation.

12.5.1. Presentations of relative I-cell complexes.

DEFINITION 12.5.2. Let $\mathcal{M}$ be a cocomplete category and let I be a set of maps in $\mathcal{M}$. If $f\colon X \to Y$ is a relative I-cell complex (see Definition 12.4.6), then a *presentation* of f is a pair consisting of a λ-sequence

$$X = X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \qquad (\beta < \lambda)$$

(for some ordinal λ) and a sequence of ordered triples

$$\bigl\{(T^\beta, e^\beta, h^\beta)\bigr\}_{\beta<\lambda}$$

such that

1. the composition of the λ-sequence is isomorphic to f,
2. every T^β is a set,
3. every e^β is a function $e^\beta : T^\beta \to I$,
4. for every $\beta < \lambda$, if $i \in T^\beta$ and e_i^β is the element $C_i \to D_i$ of I, then h_i^β is a map $h_i^\beta : C_i \to X_\beta$, and
5. every $X_{\beta+1}$ is the pushout

$$\begin{tikzcd} \coprod_{T^\beta} C_i \arrow[r] \arrow[d, "\coprod h_i^\beta"'] & \coprod_{T^\beta} D_i \arrow[d] \\ X_\beta \arrow[r] & X_{\beta+1}. \end{tikzcd}$$

If the map $f: \emptyset \to Y$ (where $\emptyset$ is the initial object of $\mathcal{M}$) is a relative I-cell complex, then a presentation of f will also be called a *presentation of* Y.

Definition 12.5.3. If $\mathcal{M}$ is a cocomplete category and I is a set of maps in $\mathcal{M}$, then a *presented relative I-cell complex* is a relative I-cell complex $f: X \to Y$ together with a particular presentation $\bigl(X = X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \quad (\beta < \lambda), \{T^\beta, e^\beta, h^\beta\}_{\beta<\lambda}\bigr)$ of it (see Definition 12.5.2). A presented relative I-cell complex in which $X = \emptyset$ (the initial object of $\mathcal{M}$) will be called a *presented I-cell complex*.

Definition 12.5.4. Let $\mathcal{M}$ be a cocomplete category, let I be a set of maps in $\mathcal{M}$, and let $(f: X \to Y, X = X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \quad (\beta < \lambda), \{T^\beta, e^\beta, h^\beta\}_{\beta<\lambda})$ be a presented relative I-cell complex (see Definition 12.5.3).

1. The *presentation ordinal* of f is λ.
2. The *set of cells* of f is $\coprod_{\beta<\lambda} T^\beta$.
3. The *size* of f is the cardinal of the set of cells of f.
4. If e is a cell of f, the *presentation ordinal* of e is the ordinal β such that $e \in T^\beta$.
5. If $\beta < \lambda$, then the *β-skeleton* of f is X_β.

12.5.5. Subcomplexes of relative I-cell complexes.

Definition 12.5.6. If $\mathcal{M}$ is a cocomplete category, I is a set of maps in $\mathcal{M}$, and $(f: X \to Y, X = X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \quad (\beta < \lambda), \{T^\beta, e^\beta, h^\beta\}_{\beta<\lambda})$ is a presented relative I-cell complex, then a *subcomplex* of f consists of a sequence of ordered triples $\{(\widetilde{T}^\beta, \tilde{e}^\beta, \tilde{h}^\beta)\}_{\beta<\lambda}$ such that

1. every $\widetilde{T}^\beta$ is a subset of T^β, and $\tilde{e}^\beta$ is the restriction of e^β to $\widetilde{T}^\beta$,
2. there is a λ-sequence

$$X = \widetilde{X}_0 \to \widetilde{X}_1 \to \widetilde{X}_2 \to \cdots \to \widetilde{X}_\beta \to \cdots \qquad (\beta < \lambda)$$

(called the λ-sequence *associated* with the subcomplex) and a map of λ-sequences

$$\begin{tikzcd} \widetilde{X}_0 \arrow[r] \arrow[d] & \widetilde{X}_1 \arrow[r] \arrow[d] & \widetilde{X}_2 \arrow[r] \arrow[d] & \cdots \\ X_0 \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & \cdots \end{tikzcd}$$

such that, for every $\beta < \lambda$ and every $i \in \widetilde{T}^\beta$, the map $\tilde{h}_i^\beta \colon C_i \to \widetilde{X}_\beta$ is a factorization of the map $h_i^\beta \colon C_i \to X_\beta$ through the map $\widetilde{X}_\beta \to X_\beta$, and
3. every $\widetilde{X}_{\beta+1}$ is the pushout

$$\begin{array}{ccc} \coprod_{\widetilde{T}^\beta} C_i & \longrightarrow & \coprod_{\widetilde{T}^\beta} D_i \\ {\scriptstyle \amalg \tilde{h}_i^\beta}\downarrow & & \downarrow \\ \widetilde{X}_\beta & \longrightarrow & \widetilde{X}_{\beta+1}. \end{array}$$

REMARK 12.5.7. Although a subcomplex of a cell complex is defined to be a sequence of triples $\{(\widetilde{T}^\beta, \tilde{e}^\beta, \tilde{h}^\beta)\}_{\beta<\lambda}$ (see Definition 12.5.6), we will often abuse language and refer to the λ-sequence associated with the subcomplex, or the colimit of that λ-sequence, as the subcomplex.

12.5.8. The case of monomorphisms.

PROPOSITION 12.5.9. *If $\mathcal{M}$ is a cocomplete category and I is a set of maps in $\mathcal{M}$ such that relative I-cell complexes are monomorphisms, then a subcomplex of a presented relative I-cell complex is entirely determined by its set of cells $\{\widetilde{T}^\beta\}_{\beta<\lambda}$ (see Definition 12.5.6).*

PROOF. The definition of a subcomplex implies that the maps $\widetilde{X}_\beta \to X_\beta$ are all inclusions of subcomplexes. Since inclusions of subcomplexes are monomorphisms, there is at most one possible factorization $\tilde{h}_i^\beta$ of each h_i^β through $\widetilde{X}_\beta \to X_\beta$. $\square$

PROPOSITION 12.5.10. *Let $\mathcal{M}$ be a cocomplete category and let I be a set of maps in $\mathcal{M}$ such that relative I-cell complexes are monomorphisms. If $(f \colon X \to Y, X = X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \quad (\beta < \lambda), \{T^\beta, e^\beta, h^\beta\}_{\beta<\lambda})$ is a presented relative I-cell complex, then an arbitrary subcomplex of f can be constructed by the following inductive procedure:*

1. *Choose an arbitrary subset $\widetilde{T}^0$ of T^0.*
2. *If $\beta < \lambda$ and we have defined $\{\widetilde{T}^\gamma\}_{\gamma<\beta}$, then we have determined the object $\widetilde{X}_\beta$ and the map $\widetilde{X}_\beta \to X_\beta$ (where $\widetilde{X}_\beta$ is the object that appears in the λ-sequence associated to the subcomplex). Consider the set*

$$\{i \in T^\beta \mid h_i^\beta \colon C_i \to X_\beta \text{ factors through } \widetilde{X}_\beta \to X_\beta\}$$

Choose an arbitrary subset $\widetilde{T}^\beta$ of this set. For every $i \in \widetilde{T}^\beta$, there is a unique map $\tilde{h}_i^\beta \colon C_i \to \widetilde{X}_\beta$ that makes the diagram

$$\begin{array}{ccc} C_i & & \\ {\scriptstyle \tilde{h}_i^\beta}\downarrow & \searrow^{h_i^\beta} & \\ \widetilde{X}_\beta & \longrightarrow & X_\beta \end{array}$$

commute. We let $\widetilde{X}_{\beta+1}$ be the pushout

$$\begin{tikzcd} \coprod_{\widetilde{T}^\beta} C_i \arrow[r] \arrow[d, "\coprod \tilde{h}^\beta"'] & \coprod_{\widetilde{T}^\beta} D_i \arrow[d] \\ \widetilde{X}_\beta \arrow[r] & \widetilde{X}_{\beta+1} \end{tikzcd}$$

PROOF. This follows directly from the definitions. □

REMARK 12.5.11. If $\mathcal{M}$ is a cocomplete category, I is a set of maps in $\mathcal{M}$ such that relative I-cell complexes are monomorphisms, and $(f\colon X \to Y, X = X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \quad (\beta < \lambda), \{T^\beta, e^\beta, h^\beta\}_{\beta<\lambda})$ is a presented relative I-cell complex, then not every sequence $\{\widetilde{T}^\beta\}_{\beta<\lambda}$ of subsets of $\{T^\beta\}_{\beta<\lambda}$ determines a subcomplex of f. Given such a sequence $\{\widetilde{T}^\beta\}_{\beta<\lambda}$, it determines a subcomplex of f if and only if it satisfies the inductive conditions described in Proposition 12.5.10.

12.6. Compactness

DEFINITION 12.6.1. Let $\mathcal{M}$ be a cocomplete model category and let I be a set of maps in $\mathcal{M}$.

1. If κ is a cardinal, then an object W in $\mathcal{M}$ is *κ-compact relative to I* if, for every presented relative I-cell complex $f\colon X \to Y$ (see Definition 13.2.4), every map from W to Y factors through a subcomplex of f of size (see Definition 12.5.4) at most κ.
2. An object W in $\mathcal{M}$ is *compact relative to I* if it is κ-compact relative to I for some cardinal κ.

EXAMPLE 12.6.2. If $\mathcal{M} = \mathsf{SS}_{(*)}$ and I is the set of inclusions $\{\partial\Delta[n] \to \Delta[n] \mid n \geq 0\}$, then every finite simplicial set is ω-compact relative to I (where ω is the countable cardinal). If κ is an infinite cardinal and X is a simplicial set of size κ, then X is κ-compact relative to I.

EXAMPLE 12.6.3. If $\mathcal{M} = \mathsf{Top}_{(*)}$ and I is the set of inclusions $\{|\partial\Delta[n]| \to |\Delta[n]| \mid n \geq 0\}$, then Corollary 2.2.7 implies that every finite cell complex is ω-compact relative to I (where ω is the countable cardinal). If κ is an infinite cardinal and X is a cell complex of size κ, then Corollary 2.2.7 implies that X is κ-compact relative to I.

PROPOSITION 12.6.4. *Let $\mathcal{M}$ be a cocomplete category and let I be a set of maps in $\mathcal{M}$. If κ is a cardinal and an object W is κ-compact relative to I, then any retract of W is κ-compact relative to I.*

PROOF. Let $i\colon V \to W$ and $r\colon W \to V$ be maps such that $ri = 1_V$. If $f\colon X \to Y$ is a relative I-cell complex and $f\colon V \to Y$ is a map, then the map $fr\colon W \to Y$ must factor through some subcomplex Z of Y of size at most κ. Thus, $fri\colon V \to Y$ factors through Z, and $fri = f$. □

PROPOSITION 12.6.5. *Let $\mathcal{M}$ be a cocomplete category and let I be a set of maps in $\mathcal{M}$. If κ and λ are cardinals such that $\kappa < \lambda$, then any object that is κ-compact relative to I is also λ-compact relative to I.*

PROOF. This follows directly from the definitions. □

PROPOSITION 12.6.6. *If* $\mathcal{M}$ *is a cocomplete category,* I *is a set of maps in* $\mathcal{M}$*, and* S *is a set of objects that are compact relative to* I*, then there is a cardinal* κ *such that every element of* S *is* κ*-compact relative to* I*.*

PROOF. For each element X of S, let κ_X be a cardinal such that X is κ_X-compact relative to I. If κ is the cardinal of $\bigcup_{X \in S} \kappa_X$, then Proposition 12.6.5 implies that every element of S is κ-compact relative to I. □

PROPOSITION 12.6.7. *Let* $\mathcal{M}$ *be a cocomplete category and let* I *be a set of maps in* $\mathcal{M}$ *such that relative* I*-cell complexes are monomorphisms. If* κ *is a cardinal and* W *is an object that is* κ*-compact relative to* I *(see Definition* 12.6.1*), then* W *is* $(\kappa+1)$*-small relative to* I*.*

PROOF. Let λ be a regular cardinal such that $\lambda > \kappa$, and let $X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots$ $(\beta < \lambda)$ be a λ-sequence of inclusions of relative I-cell complexes. Since inclusions of relative I-cell complexes are monomorphisms, the map $\operatorname{colim}_{\beta<\lambda} \mathcal{M}(W, X_\beta) \to \mathcal{M}(W, \operatorname{colim}_{\beta<\lambda} X_\beta)$ is injective, and it remains only to show that it is surjective.

If $W \to \operatorname{colim}_{\beta<\lambda} X_\beta$ is a map, then (since W is κ-compact) there is a subcomplex K of $\operatorname{colim} X_\beta$, of size at most κ, such that the map factors through K. For each cell of K there is an ordinal $\beta < \lambda$ such that that cell contained in X_β. Since λ is a regular cardinal, the union μ of these β is less than λ, and K is contained in X_μ. □

12.7. Effective monomorphisms

DEFINITION 12.7.1. Let $\mathcal{M}$ be a category that is closed under pushouts. The map $f\colon A \to B$ is an *effective monomorphism* if f is the equalizer of the pair of natural inclusions $B \rightrightarrows B \amalg_A B$.

EXAMPLE 12.7.2. If $\mathcal{M}$ is the category of sets, then the class of effective monomorphisms is the class of injective maps.

PROPOSITION 12.7.3. *If* $\mathcal{M}$ *is a category that is closed under pushouts, then a map is an effective monomorphism if and only if it is the equalizer of some pair of parallel maps.*

PROOF. If $f\colon A \to B$ is an effective monomorphism, then it is defined to be the equalizer of a particular pair of maps. Conversely, if $f\colon A \to B$ is the equalizer of the maps $B \overset{g}{\underset{h}{\rightrightarrows}} W$, then the maps g and h factor as

$$B \overset{i_0}{\underset{i_1}{\rightrightarrows}} B \amalg_A B \xrightarrow{g \amalg h} W,$$

and we must show that f is the equalizer of i_0 and i_1. Since $(g \amalg h)i_0 = g$ and $(g \amalg h)i_1 = h$, this follows directly from the definitions. □

REMARK 12.7.4. An effective monomorphism is the dual to what Quillen has called an effective epimorphism (see [**46**, Part II, page 4.1]). Effective epimorphisms have also been called *regular epimorphisms* (see [**7**, Definition 4.3.1]).

PROPOSITION 12.7.5. *An effective monomorphism is a monomorphism.*

PROOF. Let $f\colon A \to B$ be an effective monomorphism, and let $g\colon W \to A$ and $h\colon W \to A$ be maps such that $fg = fh$. If i_0 and i_1 are the natural maps $B \to B \amalg_A B$, then $i_0 fg = i_0 fh$ and $i_0 fg = i_1 fh$, Since f is the equalizer of i_0 and i_1, this implies that $g = h$. $\square$

CHAPTER 13

Cofibrantly generated model categories

13.1. Introduction

A model category structure on a category consists of three classes of maps (weak equivalences, fibrations, and cofibrations) satisfying five axioms (see Definition 8.1.2). Any two of these classes determine the third, but there are other ways to determine the three classes of maps as well. For example, the fibrations are exactly the maps with the right lifting property (see Definition 8.2.1) with respect to all trivial cofibrations, and so the class of trivial cofibrations determines the class of fibrations. Similarly, the trivial fibrations are exactly the maps with the right lifting property with respect to all cofibrations, and so the class of cofibrations determines the class of trivial fibrations. Since the weak equivalences are exactly the maps that can be written as the composition of a trivial cofibration followed by a trivial fibration, this shows that the classes of cofibrations and of trivial cofibrations entirely determine the model category structure. In some model categories, this leads to a convenient description of the model category structure.

For example, the standard model category structure on the category of simplicial sets can be described as follows:

- A map is a cofibration if it is a retract of a transfinite composition (see Definition 12.2.2) of pushouts of the maps $\partial\Delta[n] \to \Delta[n]$ for all $n \geq 0$.
- A map is a trivial fibration if it has the right lifting property with respect to the maps $\partial\Delta[n] \to \Delta[n]$ for all $n \geq 0$.
- A map is a trivial cofibration if it is a retract of a transfinite composition (see Definition 12.2.2) of pushouts of the maps $\Lambda[n,k] \to \Delta[n]$ for all $n \geq 0$ and $0 \leq k \leq n$.
- A map is a fibration if it has the right lifting property with respect to the maps $\Lambda[n,k] \to \Delta[n]$ for all $n \geq 0$ and $0 \leq k \leq n$.
- A map is a weak equivalence if it is the composition of a trivial cofibration followed by a trivial fibration.

These ideas lead to the notion (due to D. M. Kan) of a *cofibrantly generated model category* (see Definition 13.2.1).

13.2. Cofibrantly generated model categories

DEFINITION 13.2.1. A *cofibrantly generated model category* is a model category $\mathcal{M}$ such that

1. there exists a set I of cofibrations (called a set of *generating cofibrations*) that permits the small object argument (see Definition 12.4.11) and such that a map is a trivial fibration if and only if it has the right lifting property with respect to every element of I, and

2. there exists a set J of trivial cofibrations (called a set of *generating trivial cofibrations*) that permits the small object argument and such that a map is a fibration if and only if it has the right lifting property with respect to every element of J.

REMARK 13.2.2. Although the set I of generating cofibrations is not part of the structure of a cofibrantly generated model category, we will often assume that some particular set I of generating cofibrations has been chosen.

PROPOSITION 13.2.3. *If $\mathcal{M}$ is a cofibrantly generated model category and I is a set of generating cofibrations for $\mathcal{M}$, then there is a regular cardinal κ such that the domain of every element of I is κ-small relative to I.*

PROOF. This follows from Lemma 12.3.6. □

DEFINITION 13.2.4. If $\mathcal{M}$ is a cofibrantly generated model category with generating cofibrations I, then a relative I-cell complex (see Definition 12.4.6) will be called a *relative cell complex*, and an I-cell complex (see Definition 12.4.6) will be called a *cell complex*. If $\emptyset \to X$ (where $\emptyset$ is the initial object of $\mathcal{M}$) is a finite composition of pushouts of elements of I, then X will be called a *finite cell complex*. If X is a cell complex and $g\colon X \to Y$ is a relative I-cell complex, then g will be called an *inclusion of a subcomplex*.

We will show in Proposition 13.2.9 that in a cofibrantly generated model category the class of cofibrations equals the class of retracts of relative cell complexes, and the class of trivial cofibrations equals the class of retracts of relative J-cell complexes.

EXAMPLE 13.2.5. The model category SS is cofibrantly generated. The generating cofibrations are the inclusions $\partial\Delta[n] \to \Delta[n]$ for $n \geq 0$, and the generating trivial cofibrations are the inclusions $\Lambda[n,k] \to \Delta[n]$ for $n > 0$ and $0 \leq k \leq n$.

EXAMPLE 13.2.6. The model category SS_* is cofibrantly generated. The generating cofibrations are the inclusions $\partial\Delta[n]^+ \to \Delta[n]^+$ for $n \geq 0$, and the generating trivial cofibrations are the inclusions $\Lambda[n,k]^+ \to \Delta[n]^+$ for $n > 0$ and $0 \leq k \leq n$.

EXAMPLE 13.2.7. The model category Top is cofibrantly generated. The generating cofibrations are the inclusions $\left|\partial\Delta[n]\right| \to \left|\Delta[n]\right|$ for $n \geq 0$, and the generating trivial cofibrations are the inclusions $\left|\Lambda[n,k]\right| \to \left|\Delta[n]\right|$ for $n > 0$ and $0 \leq k \leq n$.

EXAMPLE 13.2.8. The model category Top_* is cofibrantly generated. The generating cofibrations are the inclusions $\left|\partial\Delta[n]\right|^+ \to \left|\Delta[n]\right|^+$ for $n \geq 0$, and the generating trivial cofibrations are the inclusions $\left|\Lambda[n,k]\right|^+ \to \left|\Delta[n]\right|^+$ for $n > 0$ and $0 \leq k \leq n$.

PROPOSITION 13.2.9. *Let $\mathcal{M}$ be a cofibrantly generated model category (see Definition 13.2.1) with generating cofibrations I and generating trivial cofibrations J.*

1. *The class of cofibrations of $\mathcal{M}$ equals the class of retracts of relative I-cell complexes (see Definition 12.4.6).*
2. *The class of trivial fibrations of $\mathcal{M}$ equals the class of I-injectives (see Definition 12.4.1).*
3. *The class of trivial cofibrations of $\mathcal{M}$ equals the class of retracts of relative J-cell complexes.*

4. *The class of fibrations of $\mathcal{M}$ equals the class of J-injectives.*

PROOF. This follows from Proposition 8.2.3, Proposition 12.4.9, and Corollary 12.4.17. □

PROPOSITION 13.2.10. *Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations I. If W is an object that is small relative to I, then it is small relative to the subcategory of all cofibrations.*

PROOF. This follows from Theorem 12.4.21. □

COROLLARY 13.2.11. *Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations I. If the codomains of the elements of I are small relative to the I, then every cofibrant object of $\mathcal{M}$ is small relative to the subcategory of all cofibrations.*

PROOF. This follows from Corollary 12.3.9, Corollary 13.2.13, Proposition 12.3.7, and Proposition 13.2.10. □

COROLLARY 13.2.12. *Let $\mathcal{M}$ be a cofibrantly generated model category. If I is a set of generating cofibrations for $\mathcal{M}$ and κ is a regular cardinal such that the domain of every element of I is κ-small relative to I, then there is a functorial factorization of every map in $\mathcal{M}$ into a cofibration that is the composition of a κ-sequence of pushouts of coproducts of elements of I followed by a trivial fibration.*

PROOF. This follows from Corollary 12.4.16 and Proposition 13.2.9. □

COROLLARY 13.2.13. *If $\mathcal{M}$ is a cofibrantly generated model category with generating cofibrations I, then every cofibrant object in $\mathcal{M}$ is a retract of a cell complex (see Definition 13.2.4).*

PROOF. This follows from Proposition 13.2.9. □

PROPOSITION 13.2.14. *Let $\mathcal{M}$ be a cofibrantly generated model category, and let I be a set of generating cofibrations for $\mathcal{M}$. If J is a set of generating trivial cofibrations for $\mathcal{M}$, then there is a set $\widetilde{J}$ of generating trivial cofibrations for $\mathcal{M}$ such that*

1. *there is a bijection between the sets J and $\widetilde{J}$ under which corresponding elements have the same domain, and*
2. *the elements of $\widetilde{J}$ are relative I-cell complexes.*

PROOF. Factor each element $j\colon C \to D$ of J as $C \xrightarrow{\tilde{\jmath}} \widetilde{D} \xrightarrow{p} D$ where $\tilde{\jmath}$ is a relative I-cell complex and p is a trivial fibration (see Corollary 13.2.12). The retract argument (see Proposition 8.2.2) implies that j is a retract of $\tilde{\jmath}$. Since j and p are weak equivalences, $\tilde{\jmath}$ is also a weak equivalence, and so $\tilde{\jmath}$ is a trivial cofibration. Thus, if we let $\widetilde{J} = \{\tilde{\jmath}\}_{j\in J}$, then $\widetilde{J}$ satisfies conditions 1 and 2, and it remains only to show that $\widetilde{J}$ is a set of generating trivial cofibrations for $\mathcal{M}$. Proposition 12.4.5 implies that it is sufficient to show that the subcategory of $\widetilde{J}$-injectives equals the subcategory of J-injectives (i.e., of fibrations).

Since every $\tilde{\jmath}$ is a trivial cofibration, every J-injective is a $\widetilde{J}$-injective (see Proposition 8.2.3). Since every j is a retract of $\tilde{\jmath}$, Lemma 8.2.7 implies that every $\widetilde{J}$-injective is a J-injective. □

PROPOSITION 13.2.15. *If $\mathcal{M}$ is a cofibrantly generated model category with generating cofibrations I, then every object X has a fibrant cofibrant approximation $\tilde{\imath}\colon \widetilde{X} \to X$ such that $\widetilde{X}$ is a cell complex.*

PROOF. This follows from Proposition 12.4.12, Proposition 12.4.8, and Proposition 13.2.9. □

PROPOSITION 13.2.16. *If $\mathcal{M}$ is a cofibrantly generated model category with generating cofibrations I, then every map $g\colon X \to Y$ has a cofibrant approximation $\tilde{g}\colon \widetilde{X} \to \widetilde{Y}$ such that $\tilde{g}\colon \widetilde{X} \to \widetilde{Y}$ is an inclusion of a subcomplex and both $i_X : \widetilde{X} \to X$ and $i_Y : \widetilde{Y} \to Y$ are trivial fibrations.*

PROOF. Choose a cofibrant approximation $i_X : \widetilde{X} \to X$ such that $\widetilde{X}$ is a cell complex and i_X is a trivial fibration (see Proposition 13.2.15). We can then factor the composition $\widetilde{X} \xrightarrow{i_X} X \xrightarrow{g} Y$ as $\widetilde{X} \xrightarrow{\tilde{g}} \widetilde{Y} \xrightarrow{i_Y} Y$ where $\tilde{g}$ is a relative I-cell complex and i_Y is a trivial fibration (see Proposition 12.4.12). The result now follows from Proposition 12.4.8 and Proposition 13.2.9. □

13.3. Recognizing cofibrantly generated model categories

THEOREM 13.3.1 (D. M. Kan). *Let $\mathcal{M}$ be a category that is closed under small limits and colimits and let W be a class of maps in $\mathcal{M}$ that is closed under retracts and satisfies the "two out of three" axiom (axiom M2 of Definition 8.1.2). If I and J are sets of maps in $\mathcal{M}$ such that*

1. *both I and J permit the small object argument (see Definition 12.4.11),*
2. *every J-cofibration is both an I-cofibration and an element of W,*
3. *every I-injective is both a J-injective and an element of W, and*
4. *one of the following two conditions holds:*
    (a) *a map that is both an I-cofibration and an element of W is a J-cofibration, or*
    (b) *a map that is both a J-injective and an element of W is an I-injective,*

then there is a cofibrantly generated model category structure (see Definition 13.2.1) on $\mathcal{M}$ in which W is the class of weak equivalences, I is the set of generating cofibrations, and J is the set of generating trivial cofibrations.

PROOF. We define the weak equivalences to be the elements of W, the cofibrations to be the I-cofibrations, and the fibrations to be the J-injectives. We must show that axioms M1 through M5 are satisfied (see Definition 8.1.2).

Axioms M1 and M2 are part of our assumptions, and axiom M3 follows from the assumptions on W, the definition of I-cofibration (see Definition 12.4.1), and Lemma 8.2.7.

If we apply the small object argument (Proposition 12.4.12) to the set I, then assumption 3 implies that this satisfies axiom M5 part 1, and if we apply the small object argument to the set J, then assumption 2 implies that this satisfies axiom M5 part 2.

It remains only to show that axiom M4 is satisfied. The proof of axiom M4 depends on which part of assumption 4 is satisfied. If assumption 4a is satisfied, then axiom M4 part 2 is clear. For axiom M4 part 1, if $f\colon X \to Y$ is both a fibration and a weak equivalence, we can factor it as $X \xrightarrow{g} Z \xrightarrow{h} Y$ where g is an I-cofibration and h is an I-injective. Axiom M2 and assumption 3 imply that g

is also a weak equivalence, and so assumption 4a implies that g is a J-cofibration. Since f is a J-injective, the retract argument (Proposition 8.2.2) implies that f is a retract of h, and is thus an I-injective (see Lemma 8.2.7). This proves axiom M4 part 1, and so the proof in the case that assumption 4a is satisfied is complete. The proof in the case in which assumption 4b is satisfied is similar. □

13.4. Compactness

DEFINITION 13.4.1. Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations I.

1. If κ is a cardinal, then an object W in $\mathcal{M}$ is *κ-compact* if it is κ-compact relative to I (see Definition 12.6.1).
2. An object W in $\mathcal{M}$ is *compact* if there is a cardinal κ for which it is κ-compact.

EXAMPLE 13.4.2. If $\mathcal{M} = \mathsf{SS}_{(*)}$, then every finite simplicial set is ω-compact (where ω is the countable cardinal). If κ is an infinite cardinal and X is a simplicial set of size κ, then X is κ-compact.

EXAMPLE 13.4.3. If $\mathcal{M} = \mathsf{Top}_{(*)}$, then Corollary 2.2.7 implies that every finite cell complex is ω-compact (where ω is the countable cardinal). If κ is an infinite cardinal and X is a cell complex of size κ, then Corollary 2.2.7 implies that X is κ-compact.

PROPOSITION 13.4.4. *Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations I. if κ is a cardinal and an object W in $\mathcal{M}$ is κ-compact, then any retract of W is κ-compact.*

PROOF. This follows from Proposition 12.6.4. □

PROPOSITION 13.4.5. *Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations I. If κ and λ are cardinals such that $\kappa < \lambda$, then any object in $\mathcal{M}$ that is κ-compact is also λ-compact.*

PROOF. This follows directly from the definitions. □

PROPOSITION 13.4.6. *If $\mathcal{M}$ is a cofibrantly generated model category with generating cofibrations I and S is a set of objects that are compact, then there is a cardinal κ such that every element of S is κ-compact.*

PROOF. This follows from Proposition 12.6.6. □

CHAPTER 14

Diagrams in a cofibrantly generated model category

14.1. Free cell complexes

In Section 14.2, we will show that there is a model category structure on the category of diagrams in a cofibrantly generated model category in which the free cell complexes and their retracts are the cofibrant objects. The fact that the $\mathcal{C}$-diagram of simplicial sets $\mathrm{B}(\mathcal{C}\downarrow -)$ is a free cell complex (see Corollary 14.6.8) will imply that a map of diagrams that is a weak equivalence of fibrant spaces at each object of $\mathcal{C}$ induces a weak equivalence of the homotopy limits of the diagrams (see Theorem 20.6.10). We will also show that if a diagram of spaces is a free cell complex, then its homotopy colimit is weakly equivalent to its colimit (see Proposition 20.9.1).

14.1.1. Free diagrams of sets. In this section, we define *free diagrams of sets*. This will be used in the next section to define free diagrams in a category of diagrams, which will be used in Section 14.1.23 to define free cell complexes in a category of diagrams in a cofibrantly generated model category.

DEFINITION 14.1.2. If $\mathcal{C}$ is a small category and α is an object in $\mathcal{C}$, the *free $\mathcal{C}$-diagram of sets generated at position α* is the $\mathcal{C}$-diagram of sets $\boldsymbol{F}_*^\alpha$ for which

$$\boldsymbol{F}_*^\alpha(\beta) = \mathcal{C}(\alpha, \beta)$$

for every object β in $\mathcal{C}$. A *free $\mathcal{C}$-diagram of sets* is a $\mathcal{C}$-diagram of sets that is a coproduct of $\mathcal{C}$-diagrams of the form $\boldsymbol{F}_*^\alpha$.

PROPOSITION 14.1.3. *If $\mathcal{C}$ is a small category and α is an object in $\mathcal{C}$, then, for every object $\boldsymbol{S}$ of $\mathsf{Set}^{\mathcal{C}}$, there is a natural isomorphism*

$$\mathsf{Set}^{\mathcal{C}}(\boldsymbol{F}_*^\alpha, \boldsymbol{S}) \approx \boldsymbol{S}_\alpha.$$

PROOF. This is the Yoneda lemma (see, e.g., [**7**, page 11] or [**41**, page 61]). □

EXAMPLE 14.1.4. If $\mathcal{C}$ is a small category and S is a set, the *free $\mathcal{C}$-diagram of sets on the set S generated at position α* is the $\mathcal{C}$-diagram of sets $\boldsymbol{F}_S^\alpha = \coprod_S \boldsymbol{F}_*^\alpha$. Thus, for every object β in $\mathcal{C}$,

$$\boldsymbol{F}_S^\alpha(\beta) = \coprod_{s\in S} \mathcal{C}(\alpha, \beta).$$

EXAMPLE 14.1.5. The diagram of sets $A \to B$ is free if and only if the map $A \to B$ is an inclusion.

EXAMPLE 14.1.6. The diagram of sets $A \to C \leftarrow B$ is free if and only if the maps $A \to C$ and $B \to C$ are inclusions with disjoint images in C.

EXAMPLE 14.1.7. The diagram of sets $A_1 \to A_2 \to A_3 \to \cdots$ is free if and only if all of the maps in the diagram are inclusions.

EXAMPLE 14.1.8. The diagram of sets $A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow \cdots$ is free if and only if all of the maps are inclusions and the inverse limit of the diagram is empty.

EXAMPLE 14.1.9. If a discrete group G is considered to be a category with one object and morphisms equal to the elements of G, then a free G-diagram of sets is what classically is called a free G-set.

EXAMPLE 14.1.10. If $\mathcal{C}$ is a small category and $\boldsymbol{P}\colon \mathcal{C} \to \mathsf{Set}$ is the constant diagram at a point, then $\boldsymbol{P}$ is free if and only if each connected component of $\mathcal{C}$ has an initial object.

EXAMPLE 14.1.11. If $\boldsymbol{\Delta}$ is the simplicial category (see Definition 16.1.2) and $\mathcal{C} = \boldsymbol{\Delta}^{\mathrm{op}}$, then a $\mathcal{C}$-diagram of sets is a simplicial set. The free $\mathcal{C}$-diagram of sets generated at position $[n]$ is the simplicial set $\Delta[n]$. Thus, the set of k-simplices of $\Delta[n]$ equals the set $\boldsymbol{\Delta}\big([k],[n]\big)$.

EXAMPLE 14.1.12. If $\mathcal{C}$ is the category $\boldsymbol{\Delta}^{\mathrm{op}}$ so that $\boldsymbol{F}_*^{[n]}$ is $\Delta[n]$ (see Example 14.1.11), then Proposition 14.1.3 is the statement that, for every simplicial set X, the set $\mathsf{SS}(\Delta[n],X)$ is naturally isomorphic to the set of n-simplices of X.

PROPOSITION 14.1.13. *If $\mathcal{C}$ is a small category and α is an object in $\mathcal{C}$, then the functor $\boldsymbol{F}_-^{\alpha}\colon \mathsf{Set} \to \mathsf{Set}^{\mathcal{C}}$ (see Example 14.1.4) is left adjoint to the functor $\mathsf{Set}^{\mathcal{C}} \to \mathsf{Set}$ that evaluates at α, i.e., for every set S and every $\mathcal{C}$-diagram of sets $\boldsymbol{T}$ there is a natural isomorphism*

$$\mathsf{Set}^{\mathcal{C}}(\boldsymbol{F}_S^{\alpha}, \boldsymbol{T}) \approx \mathsf{Set}(S, \boldsymbol{T}_{\alpha}).$$

PROOF. This follows from Proposition 14.1.3. $\square$

14.1.14. Free diagrams. In this section, we define *free diagrams* in a category of diagrams (see Definition 14.1.17). In section Section 14.1.23, we will apply this to the generating cofibrations (see Definition 13.2.1) of a cofibrantly generated model category $\mathcal{M}$ to obtain the *free cells*, which are the generating cofibrations in the category of $\mathcal{C}$-diagrams in $\mathcal{M}$.

DEFINITION 14.1.15. Let $\mathcal{M}$ be a model category. If X is an object in $\mathcal{M}$ and S is a set, then by $X \otimes S$ we will mean the object in $\mathcal{M}$ obtained by taking the coproduct, indexed by S, of copies of X. Thus,

$$X \otimes S \approx \coprod_S X.$$

DEFINITION 14.1.16. If $\mathcal{M}$ is a model category, $\mathcal{C}$ is a small category, $\boldsymbol{S}\colon \mathcal{C} \to \mathsf{Set}$ is a diagram of sets, and X is an object in $\mathcal{M}$, then by $X \otimes \boldsymbol{S}\colon \mathcal{C} \to \mathcal{M}$ we will mean the $\mathcal{C}$-diagram in $\mathcal{M}$ such that

$$(X \otimes \boldsymbol{S})_{\alpha} = X \otimes \boldsymbol{S}_{\alpha}$$

for every object α in $\mathcal{C}$ (see Definition 14.1.15).

DEFINITION 14.1.17. If $\mathcal{M}$ is a model category, X is an object in $\mathcal{M}$, $\mathcal{C}$ is a small category, and α is an object in $\mathcal{C}$, then the *free diagram on X generated at α* is the $\mathcal{C}$-diagram $X \otimes \boldsymbol{F}_*^{\alpha}$ (see Definition 14.1.16 and Definition 14.1.2). At every object β of $\mathcal{C}$, this is $\coprod_{\mathcal{C}(\alpha,\beta)} X$.

PROPOSITION 14.1.18. *If $\mathcal{M}$ is a model category, $\mathcal{C}$ is a small category, and α is an object of $\mathcal{C}$, then the functor $- \otimes \boldsymbol{F}^{\alpha}_{*} \colon \mathcal{M} \to \mathcal{M}^{\mathcal{C}}$ (see Definition 14.1.17) is left adjoint to the functor $\mathcal{M}^{\mathcal{C}} \to \mathcal{M}$ that evaluates at α, i.e., for every object X in $\mathcal{M}$ and every diagram $\boldsymbol{Y}$ in $\mathcal{M}^{\mathcal{C}}$, there is a natural isomorphism*

$$\mathcal{M}^{\mathcal{C}}(X \otimes \boldsymbol{F}^{\alpha}_{*}, \boldsymbol{Y}) \approx \mathcal{M}(X, \boldsymbol{Y}_{\alpha}).$$

PROOF. This follows from Proposition 14.1.13. $\square$

DEFINITION 14.1.19. If $\mathcal{C}$ is a small category, let $\mathcal{C}^{\mathrm{disc}}$ be the discrete category with objects equal to the objects of $\mathcal{C}$. If $\mathcal{M}$ is a model category and $\boldsymbol{X}$ is an object in $\mathcal{M}^{(\mathcal{C}^{\mathrm{disc}})}$, we define an object $\boldsymbol{F}(\boldsymbol{X})$ in $\mathcal{M}^{\mathcal{C}}$ by

$$\boldsymbol{F}(\boldsymbol{X}) = \coprod_{\alpha \in \mathrm{Ob}(\mathcal{C})} \boldsymbol{X}_{\alpha} \otimes \boldsymbol{F}^{\alpha}_{*}$$

(see Definition 14.1.17), so that, for every object β in $\mathcal{C}$, we have

$$\big(\boldsymbol{F}(\boldsymbol{X})\big)_{\beta} = \coprod_{\alpha \in \mathrm{Ob}(\mathcal{C})} \coprod_{\mathcal{C}(\alpha,\beta)} \boldsymbol{X}_{\alpha}.$$

COROLLARY 14.1.20. *If $\mathcal{M}$ is a model category and $\mathcal{C}$ is a small category, then the functor $\boldsymbol{F} \colon \mathcal{M}^{(\mathcal{C}^{\mathrm{disc}})} \to \mathcal{M}^{\mathcal{C}}$ of Definition 14.1.19 is left adjoint to the forgetful functor $\boldsymbol{U} \colon \mathcal{M}^{\mathcal{C}} \to \mathcal{M}^{(\mathcal{C}^{\mathrm{disc}})}$, i.e., if $\boldsymbol{X}$ is an object in $\mathcal{M}^{(\mathcal{C}^{\mathrm{disc}})}$ and $\boldsymbol{Y}$ is an object in $\mathcal{M}^{\mathcal{C}}$, then there is a natural isomorphism*

$$\mathcal{M}^{\mathcal{C}}\big(\boldsymbol{F}(\boldsymbol{X}), \boldsymbol{Y}\big) \approx \mathcal{M}^{(\mathcal{C}^{\mathrm{disc}})}(\boldsymbol{X}, \boldsymbol{U}\boldsymbol{Y}).$$

PROOF. This follows from Proposition 14.1.18. $\square$

DEFINITION 14.1.21. If $\mathcal{C}$ is a small category, let $\mathcal{C}^{\mathrm{disc}}$ be the discrete category with objects equal to the objects of $\mathcal{C}$. If $\boldsymbol{S} \in \mathsf{Set}^{(\mathcal{C}^{\mathrm{disc}})}$, we define a $\mathcal{C}$-diagram of sets $\boldsymbol{F}(\boldsymbol{S})$ by

$$\boldsymbol{F}(\boldsymbol{S}) = \coprod_{\alpha \in \mathrm{Ob}(\mathcal{C})} \boldsymbol{F}^{\alpha}_{\boldsymbol{S}_{\alpha}}$$

(see Example 14.1.4), so that, for every object β in $\mathcal{C}$, we have

$$\big(\boldsymbol{F}(\boldsymbol{S})\big)_{\beta} = \coprod_{\alpha \in \mathrm{Ob}(\mathcal{C})} \coprod_{s \in \boldsymbol{S}_{\alpha}} \mathcal{C}(\alpha, \beta).$$

THEOREM 14.1.22. *The functor $\boldsymbol{F} \colon \mathsf{Set}^{(\mathcal{C}^{\mathrm{disc}})} \to \mathsf{Set}^{\mathcal{C}}$ of Definition 14.1.21 is left adjoint to the forgetful functor $\boldsymbol{U} \colon \mathsf{Set}^{\mathcal{C}} \to \mathsf{Set}^{(\mathcal{C}^{\mathrm{disc}})}$, i.e., if $\boldsymbol{S} \in \mathsf{Set}^{(\mathcal{C}^{\mathrm{disc}})}$ and $\boldsymbol{T} \in \mathsf{Set}^{\mathcal{C}}$, there is a natural isomorphism*

$$\mathsf{Set}^{\mathcal{C}}\big(\boldsymbol{F}(\boldsymbol{S}), \boldsymbol{T}\big) \approx \mathsf{Set}^{(\mathcal{C}^{\mathrm{disc}})}(\boldsymbol{S}, \boldsymbol{U}\boldsymbol{T}).$$

PROOF. This follows from Proposition 14.1.13. $\square$

14.1.23. Free cell complexes. Relative free cell complexes are the analogues for diagrams of topological spaces of relative cell complexes for topological spaces (see Definition 2.2.1). Relative free cell complexes and their retracts will be the cofibrations in the model category of $\mathcal{C}$-diagrams in a cofibrantly generated model category (see Theorem 14.2.1). We will first describe *free cells*, which will be the generating cofibrations in this model category (see Definition 13.2.1).

DEFINITION 14.1.24. Let $\mathcal{C}$ be a small category, and let α be an object in $\mathcal{C}$. If $\mathcal{M}$ is a cofibrantly generated model category with generating cofibrations I, then a *free cell generated at* α in $\mathcal{M}^{\mathcal{C}}$ is a map of the form

$$A \otimes \boldsymbol{F}_*^\alpha \to B \otimes \boldsymbol{F}_*^\alpha$$

(see Definition 14.1.17) where $A \to B$ is an element of I. At every object β in $\mathcal{C}$, this is the map

$$\coprod_{\mathcal{C}(\alpha,\beta)} A \to \coprod_{\mathcal{C}(\alpha\beta)} B.$$

EXAMPLE 14.1.25. Let $\mathcal{C}$ be a small category and let α be an object in $\mathcal{C}$.

- A *free cell generated at* α in $\mathsf{Top}^{\mathcal{C}}$ is a map of the form
$$\left|\partial\Delta[n]\right| \otimes \boldsymbol{F}_*^\alpha \to \left|\Delta[n]\right| \otimes \boldsymbol{F}_*^\alpha$$
for some $n \geq 0$.
- A *free cell generated at* α in $\mathsf{Top}_*^{\mathcal{C}}$ is a map of the form
$$\left|\partial\Delta[n]\right|^+ \otimes \boldsymbol{F}_*^\alpha \to \left|\Delta[n]\right|^+ \otimes \boldsymbol{F}_*^\alpha$$
for some $n \geq 0$.
- A *free cell generated at* α in $\mathsf{SS}^{\mathcal{C}}$ is a map of the form
$$\partial\Delta[n] \otimes \boldsymbol{F}_*^\alpha \to \Delta[n] \otimes \boldsymbol{F}_*^\alpha$$
for some $n \geq 0$.
- A *free cell generated at* α in $\mathsf{SS}_*^{\mathcal{C}}$ is a map of the form
$$\partial\Delta[n]^+ \otimes \boldsymbol{F}_*^\alpha \to \Delta[n]^+ \otimes \boldsymbol{F}_*^\alpha$$
for some $n \geq 0$.

DEFINITION 14.1.26. If $\mathcal{M}$ is a model category, J is a set of maps of $\mathcal{M}$, and $\mathcal{C}$ is a small category, then $J \otimes \mathcal{C}$ will denote the set of maps of $\mathcal{M}^{\mathcal{C}}$ of the form

$$C_j \otimes \boldsymbol{F}_*^\alpha \to D_j \otimes \boldsymbol{F}_*^\alpha$$

(see Definition 14.1.17) where $j\colon C_j \to D_j$ is a map in J and α is an object in $\mathcal{C}$.

PROPOSITION 14.1.27. *If $\mathcal{M}$ is a category, $\mathcal{C}$ is a small category, and J is a set of maps in $\mathcal{M}$, then the map $g\colon \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathcal{M}^{\mathcal{C}}$ is a $J{\otimes}\mathcal{C}$-injective (see Definition* 12.4.1*) if and only if $g_\alpha\colon \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ is a J-injective for every object α in $\mathcal{C}$.*

PROOF. This follows from Proposition 14.1.18. □

DEFINITION 14.1.28. If $\mathcal{M}$ is a model category and $\mathcal{C}$ is a small category, then a *relative free cell complex* in $\mathcal{M}^{\mathcal{C}}$ is a map that is a transfinite composition (see Definition 12.2.2) of pushouts (see Definition 8.2.10) of free cells (see Definition 14.1.24). A *free cell complex* in $\mathcal{M}^{\mathcal{C}}$ is a diagram $\boldsymbol{X}$ such that the map from the initial object of $\mathcal{M}^{\mathcal{C}}$ to $\boldsymbol{X}$ is a relative free cell complex. An *inclusion of free cell complexes* is a relative free cell complex whose domain is a free cell complex.

The relative free cell complexes and their retracts will be the cofibrations in the model category of $\mathcal{C}$-diagrams in a cofibrantly generated model category $\mathcal{M}$ (see Theorem 14.2.1).

14.2. The model category of $\mathcal{C}$-diagrams

THEOREM 14.2.1. *Let $\mathcal{C}$ be a small category, and let $\mathcal{M}$ be a cofibrantly generated model category (see Definition* 13.2.1*) with generating cofibrations I and generating trivial cofibrations J.*

1. *The category $\mathcal{M}^{\mathcal{C}}$ of diagrams $\boldsymbol{X}: \mathcal{C} \to \mathcal{M}$ is a cofibrantly generated model category in which a map $\boldsymbol{X} \to \boldsymbol{Y}$ is*
   - *a weak equivalence if $\boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ is a weak equivalence in $\mathcal{M}$ for every object α in $\mathcal{C}$,*
   - *a fibration if $\boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ is a fibration in $\mathcal{M}$ for every object α in $\mathcal{C}$, and*
   - *a cofibration if it is a retract of a transfinite composition of pushouts of elements of $I \otimes \mathcal{C}$.*

   The generating cofibrations of $\mathcal{M}^{\mathcal{C}}$ are the elements of $I \otimes \mathcal{C}$, and the generating trivial cofibrations are the elements of $J \otimes \mathcal{C}$.
2. *If $\mathcal{M}$ is a proper model category (see Definition* 11.1.1*), then the model category of part 1 is proper.*

PROOF. For part 1, let W be the class of maps $\boldsymbol{X} \to \boldsymbol{Y}$ such that $\boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ is a weak equivalence for all $\alpha \in \mathrm{Ob}(\mathcal{C})$. We will show that the class W and the sets $I \otimes \mathcal{C}$ and $J \otimes \mathcal{C}$ satisfy the hypotheses of Theorem 13.3.1.

Condition 1 follows from Proposition 14.1.18. Condition 2 holds because it holds for I and J in $\mathcal{M}$ and a transfinite composition of trivial cofibrations is a trivial cofibration (see Proposition 12.2.19). Proposition 14.1.27 implies condition 3 and condition 4b, and so the proof of part 1 is complete.

For part 2, since pushouts and pullbacks in $\mathcal{M}^{\mathcal{C}}$ are constructed objectwise, and both fibrations and weak equivalences are defined objectwise, part 2 of Definition 11.1.1 is clear. Since a map in $I \otimes \mathcal{C}$ is a cofibration at each object of $\mathcal{C}$, part 1 of Definition 11.1.1 is also clear, and the proof of part 2 is complete. $\square$

14.3. Diagrams in a simplicial model category

DEFINITION 14.3.1. Let $\mathcal{M}$ be a simplicial model category. If $\mathcal{C}$ is a small category, $\boldsymbol{X}: \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, and K is a simplicial set, then we define $\mathcal{C}$-diagrams $\boldsymbol{X} \otimes K$ and $\boldsymbol{X}^K$ in $\mathcal{M}$ by letting $(\boldsymbol{X} \otimes K)_\alpha = \boldsymbol{X}_\alpha \otimes K$ and $(\boldsymbol{X}^K)_\alpha = (\boldsymbol{X}_\alpha)^K$ for $\alpha \in \mathrm{Ob}(\mathcal{C})$ and, if $(\sigma: \alpha \to \alpha') \in \mathcal{C}$, then $(\boldsymbol{X} \otimes K)_\sigma = \boldsymbol{X}_\sigma \otimes 1_K$ and $(\boldsymbol{X}^K)_\sigma = \boldsymbol{X}_\sigma^{(1_K)}$.

DEFINITION 14.3.2. Let $\mathcal{M}$ be a simplicial model category. If $\mathcal{C}$ is a small category and $\boldsymbol{X}, \boldsymbol{Y}: \mathcal{C} \to \mathcal{M}$ are $\mathcal{C}$-diagrams in $\mathcal{M}$, then $\mathrm{Map}(\boldsymbol{X}, \boldsymbol{Y})$ is defined to be the simplicial set whose set of n-simplices is the set of maps of diagrams $\boldsymbol{X} \otimes \Delta[n] \to \boldsymbol{Y}$ (see Definition 14.3.1), and whose face and degeneracy maps are induced by the standard maps between the $\Delta[n]$.

THEOREM 14.3.3. *If $\mathcal{C}$ is a small category and $\mathcal{M}$ is a simplicial cofibrantly generated model category, then the model category structure of Theorem* 14.2.1 *with the simplicial structure of Definition* 14.3.1 *and Definition* 14.3.2 *makes $\mathcal{M}^{\mathcal{C}}$ a simplicial model category.*

PROOF. Definition 14.3.1 and Definition 14.3.2 satisfy axiom M6 (see Definition 10.1.2) because the constructions are all done objectwise and $\mathcal{M}$ is a simplicial model category. For axiom M7, Proposition 10.1.8 implies that it is sufficient to show that if $j\colon K \to L$ is a cofibration of simplicial sets and $p\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a fibration in $\mathcal{M}^{\mathcal{C}}$, then $\boldsymbol{X}^L \to \boldsymbol{X}^K \times_{\boldsymbol{Y}^K} \boldsymbol{Y}^L$ is a fibration that is also a weak equivalence if either j or p is a weak equivalence. Since both fibrations and weak equivalences in $\mathcal{M}^{\mathcal{C}}$ are defined objectwise, this follows from the assumption that $\mathcal{M}$ is a simplicial model category, and so the proof is complete. □

14.4. Overcategories and undercategories

The category of simplices of a simplicial set will be defined as an overcategory (see Definition 16.1.11). Overcategories and undercategories will also be used to define a model category structure on a category of diagrams in a model category indexed by a Reedy category (see Definition 16.3.2).

DEFINITION 14.4.1. If $\mathcal{C}$ and $\mathcal{D}$ are categories, $\mathrm{F}\colon \mathcal{C} \to \mathcal{D}$ is a functor, and α is an object of $\mathcal{D}$, then the category $(\mathrm{F}\downarrow\alpha)$ of *objects of* $\mathcal{C}$ *over* α is the category in which an object is a pair (β, σ) where β is an object of $\mathcal{C}$ and σ is a map $\mathrm{F}\beta \to \alpha$ in $\mathcal{D}$, and a morphism from the object (β, σ) to the object (β', σ') is a map $\tau\colon \beta \to \beta'$ in $\mathcal{C}$ such that the triangle

$$\begin{tikzcd} \mathrm{F}\beta \arrow[rr, "\mathrm{F}\tau"] \arrow[dr, "\sigma"'] & & \mathrm{F}\beta' \arrow[dl, "\sigma'"] \\ & \alpha & \end{tikzcd}$$

commutes.

If $\mathcal{C} = \mathcal{D}$ and F is the identity functor, then we use $(\mathcal{C}\downarrow\alpha)$ to denote the category $(\mathrm{F}\downarrow\alpha)$. An object of $(\mathcal{C}\downarrow\alpha)$ is a map $\beta \to \alpha$ in $\mathcal{C}$, and a morphism from $\beta \to \alpha$ to $\beta' \to \alpha$ is a map $\beta \to \beta'$ in $\mathcal{C}$ such that the triangle

$$\begin{tikzcd} \beta \arrow[rr] \arrow[dr] & & \beta' \arrow[dl] \\ & \alpha & \end{tikzcd}$$

commutes.

DEFINITION 14.4.2. If $\mathcal{C}$ and $\mathcal{D}$ are categories, $\mathrm{F}\colon \mathcal{C} \to \mathcal{D}$ is a functor, and α is an object of $\mathcal{D}$, then the category $(\alpha\downarrow\mathrm{F})$ of *objects of* $\mathcal{C}$ *under* α is the category in which an object is a pair (β, σ) where β is an object of $\mathcal{C}$ and σ is a map $\alpha \to \mathrm{F}\beta$ in $\mathcal{D}$, and a morphism from the object (β, σ) to the object (β', σ') is a map $\tau\colon \beta \to \beta'$ in $\mathcal{C}$ such that the triangle

$$\begin{tikzcd} & \alpha \arrow[dl, "\sigma"'] \arrow[dr, "\sigma'"] & \\ \mathrm{F}\beta \arrow[rr, "\mathrm{F}\tau"'] & & \mathrm{F}\beta' \end{tikzcd}$$

commutes. The opposite $(\alpha\downarrow\mathrm{F})^{\mathrm{op}}$ is the category in which an object is a pair (β, σ) where β is an object of $\mathcal{C}$ and σ is a map $\alpha \to \mathrm{F}\beta$ in $\mathcal{D}$, and a morphism from the object (β, σ) to the object (β', σ') is a map $\tau\colon \beta' \to \beta$ in $\mathcal{C}$ such that the triangle

$$\begin{tikzcd} & \alpha \arrow[dl, "\sigma"'] \arrow[dr, "\sigma'"] & \\ \mathrm{F}\beta & & \mathrm{F}\beta' \arrow[ll, "\mathrm{F}\tau"] \end{tikzcd}$$

commutes.

If $\mathcal{C} = \mathcal{D}$ and F is the identity functor, then we use $(\alpha \downarrow \mathcal{C})$ to denote the category $(\alpha \downarrow \mathrm{F})$. An object of $(\alpha \downarrow \mathcal{C})$ is a map $\alpha \to \beta$ in $\mathcal{C}$, and a morphism from $\alpha \to \beta$ to $\alpha \to \beta'$ is a map $\beta \to \beta'$ in $\mathcal{C}$ such that the triangle

$$\begin{tikzcd} & \alpha \arrow[dl] \arrow[dr] & \\ \beta \arrow[rr] & & \beta' \end{tikzcd}$$

commutes. The opposite $(\alpha \downarrow \mathcal{C})^{\mathrm{op}}$ is the category in which an object is a map $\alpha \to \beta$ in $\mathcal{C}$, and a morphism from $\alpha \to \beta$ to $\alpha \to \beta'$ is a map $\beta' \to \beta$ in $\mathcal{C}$ such that the triangle

$$\begin{tikzcd} & \alpha \arrow[dl] \arrow[dr] & \\ \beta & & \beta' \arrow[ll] \end{tikzcd}$$

commutes.

Proposition 14.4.3. *If $\mathcal{C}$ is a small category and α is an object of $\mathcal{C}$, then there is a natural isomorphism of categories*

$$(\alpha \downarrow \mathcal{C})^{\mathrm{op}} \approx (\mathcal{C}^{\mathrm{op}} \downarrow \alpha)$$

Proof. An object of $(\mathcal{C}^{\mathrm{op}} \downarrow \alpha)$ is a map $\alpha \to \beta$ in $\mathcal{C}$, and a morphism in $(\mathcal{C}^{\mathrm{op}} \downarrow \alpha)$ from $\alpha \to \beta$ to $\alpha \to \beta'$ is a map $\beta' \to \beta$ of $\mathcal{C}$ such that the triangle

$$\begin{tikzcd} & \alpha \arrow[dl] \arrow[dr] & \\ \beta & & \beta' \arrow[ll] \end{tikzcd}$$

commutes. An object of $(\alpha \downarrow \mathcal{C})$ is a map $\alpha \to \beta$ in $\mathcal{C}$, and a morphism in $(\alpha \downarrow \mathcal{C})$ from $\alpha \to \beta$ to $\alpha \to \beta'$ is a map $\beta \to \beta'$ in $\mathcal{C}$ such that the triangle

$$\begin{tikzcd} & \alpha \arrow[dl] \arrow[dr] & \\ \beta \arrow[rr] & & \beta' \end{tikzcd}$$

commutes. Thus, an object of $(\alpha \downarrow \mathcal{C})^{\mathrm{op}}$ is a map $\alpha \to \beta$ in $\mathcal{C}$ and a morphism in $(\alpha \downarrow \mathcal{C})^{\mathrm{op}}$ from $\alpha \to \beta$ to $\alpha \to \beta'$ is a map $\beta' \to \beta$ such that the triangle

$$\begin{tikzcd} & \alpha \arrow[dl] \arrow[dr] & \\ \beta & & \beta' \arrow[ll] \end{tikzcd}$$

commutes. □

14.4.4. Cofinal functors.

Definition 14.4.5. Let $\mathcal{C}$ and $\mathcal{D}$ be small categories and let $\mathrm{F}\colon \mathcal{C} \to \mathcal{D}$ be a functor.

- The functor F is *left cofinal* (or *initial*) if for every object α of $\mathcal{D}$ the space $\mathrm{B}(\mathrm{F}\downarrow\alpha)$ (see Definition 9.4.1 and Definition 14.4.1) is contractible.
- The functor F is *right cofinal* (or *terminal*) if for every object α of $\mathcal{D}$ the space $\mathrm{B}(\alpha\downarrow\mathrm{F})$ (see Definition 14.4.2) is contractible.

If $\mathcal{C}$ is a subcategory of $\mathcal{D}$ and F is the inclusion, then if F is left cofinal or right cofinal we will say that $\mathcal{C}$ is, respectively, a left cofinal subcategory or a right cofinal subcategory of $\mathcal{D}$.

We will show in Theorem 19.5.11 that these are the correct notions when considering homotopy limits and homotopy colimits.

Definition 14.4.6. Let $\mathcal{C}$ and $\mathcal{D}$ be small categories and let $\mathrm{F}\colon \mathcal{C} \to \mathcal{D}$ be a functor.

- The functor F is 0-*left cofinal* (or 0-*initial*) if for every object α of $\mathcal{D}$ the space $\mathrm{B}(\mathrm{F}\downarrow\alpha)$ (see Definition 14.4.1) is non-empty and connected.
- The functor F is 0-*right cofinal* (or 0-*terminal*) if for every object α of $\mathcal{D}$ the space $\mathrm{B}(\alpha\downarrow\mathrm{F})$ (see Definition 14.4.2) is non-empty and connected.

It is classical that these are the proper notions when considering limits and colimits.

Proposition 14.4.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be small categories, and let $\mathrm{F}\colon \mathcal{C} \to \mathcal{D}$ be a functor.*

1. *If* F *is left cofinal, then it is* 0*-left cofinal.*
2. *If* F *is right cofinal, then it is* 0*-right cofinal.*

Proof. This follows directly from the definitions. □

Theorem 14.4.8. *Let $\mathcal{M}$ be a category that is closed under limits. If $\mathcal{C}$ and $\mathcal{D}$ are small categories and $\mathrm{F}\colon \mathcal{C} \to \mathcal{D}$ is a* 0*-initial functor, then, for every $\mathcal{D}$-diagram $\boldsymbol{X}$ in $\mathcal{M}$, the natural map $\lim_{\mathcal{D}} \boldsymbol{X} \to \lim_{\mathcal{C}} \mathrm{F}^*\boldsymbol{X}$ is an isomorphism.*

Proof. The standard proof works. □

Corollary 14.4.9. *Let $\mathcal{M}$ be a category that is closed under limits. If $\mathcal{C}$ and $\mathcal{D}$ are small categories and $\mathrm{F}\colon \mathcal{C} \to \mathcal{D}$ is a left cofinal functor, then, for every $\mathcal{D}$-diagram $\boldsymbol{X}$ in $\mathcal{M}$, the natural map $\lim_{\mathcal{D}} \boldsymbol{X} \to \lim_{\mathcal{C}} \mathrm{F}^*\boldsymbol{X}$ is an isomorphism.*

Proof. This follows from Proposition 14.4.7 and Theorem 14.4.8. □

Theorem 14.4.10. *Let $\mathcal{M}$ be a category that is closed under limits. If $\mathcal{C}$ and $\mathcal{D}$ are small categories and $\mathrm{F}\colon \mathcal{C} \to \mathcal{D}$ is a* 0*-terminal functor, then, for every $\mathcal{D}$-diagram $\boldsymbol{X}$ in $\mathcal{M}$, the natural map $\mathrm{colim}_{\mathcal{C}} \mathrm{F}^*\boldsymbol{X} \to \mathrm{colim}_{\mathcal{D}} \boldsymbol{X}$ is an isomorphism.*

Proof. The standard proof works (see, e.g., [**41**, page 213]). □

Corollary 14.4.11. *Let $\mathcal{M}$ be a category that is closed under limits. If $\mathcal{C}$ and $\mathcal{D}$ are small categories and $\mathrm{F}\colon \mathcal{C} \to \mathcal{D}$ is a right cofinal functor, then, for every $\mathcal{D}$-diagram $\boldsymbol{X}$ in $\mathcal{M}$, the natural map $\mathrm{colim}_{\mathcal{C}} \mathrm{F}^*\boldsymbol{X} \to \mathrm{colim}_{\mathcal{D}} \boldsymbol{X}$ is an isomorphism.*

PROOF. This follows from Proposition 14.4.7 and Theorem 14.4.10. ☐

REMARK 14.4.12. The reader should be aware that there are conflicting uses of the above terms in the literature. Our definitions are those of Bousfield and Kan ([**15**, page 316]) and [**26**]. Heller ([**35**, page 54]) uses the terms *homotopically initial* and *homotopically final* for what we here call *initial* and *final*, while Mac Lane ([**41**, pages 213–214]) uses the terms *initial* and *final* for what we here call 0-*initial* and 0-*terminal*.

14.5. Diagrams of undercategories and overcategories

In this section, for every small category $\mathcal{C}$ we define a natural $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets $\mathrm{B}(-\downarrow \mathcal{C})^{\mathrm{op}}$ that will be used to define the homotopy colimit of a $\mathcal{C}$-diagram of spaces (see Definition 19.1.2), and a natural $\mathcal{C}$-diagram of simplicial sets $\mathrm{B}(\mathcal{C}\downarrow -)$ that will be used to define the homotopy limit of a $\mathcal{C}$-diagram of spaces (see Definition 19.1.10). We also derive a relation between them (see Corollary 14.5.11) that we will use to obtain a relation between the homotopy colimit and the homotopy limit functors (see Corollary 20.3.19).

14.5.1. Diagrams of undercategories.

DEFINITION 14.5.2. If $\mathcal{C}$ and $\mathcal{D}$ are small categories and $\mathrm{F}\colon \mathcal{C} \to \mathcal{D}$ is a functor, then, for each object α of $\mathcal{D}$, we have the category $(\alpha\downarrow \mathrm{F})^{\mathrm{op}}$, the opposite of the category of objects of $\mathcal{C}$ under α (see Definition 14.4.2). If $\sigma\colon \alpha \to \alpha'$ is a map in $\mathcal{D}$, then σ induces a functor $\sigma^*\colon (\alpha'\downarrow \mathrm{F})^{\mathrm{op}} \to (\alpha\downarrow \mathrm{F})^{\mathrm{op}}$, defined on objects by

$$\sigma^*(\alpha' \xrightarrow{\tau} \mathrm{F}\beta) = \alpha \xrightarrow{\tau\sigma} \mathrm{F}\beta.$$

If we take the classifying space of each undercategory (see Definition 9.4.1), we obtain the $\mathcal{D}^{\mathrm{op}}$-*diagram of simplicial sets* $\mathrm{B}(-\downarrow \mathrm{F})^{\mathrm{op}}$ which, on the object α of $\mathcal{D}$, takes the value $\mathrm{B}(\alpha\downarrow \mathrm{F})^{\mathrm{op}}$. Thus, an n-simplex of $\mathrm{B}(-\downarrow \mathrm{F})^{\mathrm{op}}(\alpha) = \mathrm{B}(\alpha\downarrow \mathrm{F})^{\mathrm{op}}$ is a commutative diagram in $\mathcal{D}$

$$\begin{tikzcd}
 & \alpha \arrow[dl] \arrow[d] \arrow[drrr] & & & \\
\mathrm{F}\beta_0 & \mathrm{F}\beta_1 \arrow[l, "\mathrm{F}\sigma_0"'] & \cdots \arrow[l, "\mathrm{F}\sigma_1"'] & & \mathrm{F}\beta_n \arrow[ll, "\mathrm{F}\sigma_{n-1}"']
\end{tikzcd}$$

with face and degeneracy maps defined as in (9.4.2).

As in Definition 14.4.2, if $\mathcal{C} = \mathcal{D}$ and F is the identity functor, then we use $\mathrm{B}(-\downarrow \mathcal{C})^{\mathrm{op}}$ to denote the diagram of the opposites of the undercategories, and an n-simplex of $B(-\downarrow \mathcal{C})^{\mathrm{op}}(\alpha) = \mathrm{B}(\alpha\downarrow \mathcal{C})^{\mathrm{op}}$ is a commutative diagram in $\mathcal{C}$

$$\begin{tikzcd}
 & \alpha \arrow[dl] \arrow[d] \arrow[drrr] & & & \\
\alpha_0 & \alpha_1 \arrow[l, "\sigma_0"'] & \cdots \arrow[l, "\sigma_1"'] & & \alpha_n \arrow[ll, "\sigma_{n-1}"']
\end{tikzcd}$$

with face and degeneracy maps defined as in (9.4.2).

LEMMA 14.5.3. *If $\mathcal{C}$ is a small category and α is an object of $\mathcal{C}$, then* $\mathrm{B}(\alpha\downarrow \mathcal{C})^{\mathrm{op}}$ *is contractible.*

PROOF. This follows from Proposition 9.4.4, since $(\alpha\downarrow \mathcal{C})^{\mathrm{op}}$ has the terminal object $1_\alpha\colon \alpha \to \alpha$. ☐

The $\mathcal{C}^{\mathrm{op}}$-diagram $\mathrm{B}(-\downarrow \mathcal{C})^{\mathrm{op}}$ will be used to define the homotopy colimit functor (see Definition 19.1.2). Lemma 14.5.3 implies that, in the model category of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets (see Theorem 14.2.1), the $\mathcal{C}^{\mathrm{op}}$-diagram $\mathrm{B}(-\downarrow \mathcal{C})^{\mathrm{op}}$ is weakly equivalent to the constant diagram at a point. We will show in Corollary 14.6.8 that $\mathrm{B}(-\downarrow \mathcal{C})^{\mathrm{op}}$ is also a free $\mathcal{C}^{\mathrm{op}}$-diagram, i.e., that $\mathrm{B}(-\downarrow \mathcal{C})^{\mathrm{op}}$ is a cofibrant approximation to the constant diagram at a point (see Definition 9.1.1). This will imply (in Theorem 20.8.4) that if we use a different cofibrant approximation to the constant diagram at a point in the definition of the homotopy colimit of a diagram, then, for a diagram of cofibrant spaces, we will get a space weakly equivalent to the homotopy colimit.

PROPOSITION 14.5.4. *If $\mathcal{C}$ and $\mathcal{D}$ are small categories and $\mathrm{F}\colon \mathcal{C} \to \mathcal{D}$ is a functor, then the colimit of the $\mathcal{D}^{\mathrm{op}}$-diagram of classifying spaces of undercategories* $\operatorname{colim}_{\mathcal{D}^{\mathrm{op}}} \mathrm{B}(-\downarrow \mathrm{F})$ *is naturally isomorphic to* $\mathrm{B}\mathcal{C}$.

PROOF. We define a map $\operatorname{colim}_{\mathcal{D}^{\mathrm{op}}} \mathrm{B}(-\downarrow \mathrm{F}) \to \mathrm{B}\mathcal{C}$ by taking the simplex $(\beta_0 \to \beta_1 \to \cdots \to \beta_n, \sigma\colon \alpha \to \mathrm{F}\beta_0)$ of $\mathrm{B}(-\downarrow \mathrm{F})$ to the simplex $\beta_0 \to \beta_1 \to \cdots \to \beta_n$ of $\mathrm{B}\mathcal{C}$. This map is onto because the simplex $\beta_0 \to \beta_1 \to \cdots \to \beta_n$ of $\mathrm{B}\mathcal{C}$ is in the image of $(\beta_0 \to \beta_1 \to \cdots \to \beta_n, 1_{\mathrm{F}\beta_0}\colon \mathrm{F}\beta_0 \to \mathrm{F}\beta_0)$, and it is one to one because the simplex $(\beta_0 \to \beta_1 \to \cdots \to \beta_n, \sigma\colon \alpha \to \mathrm{F}\beta_0)$ of $\mathrm{B}(\alpha\downarrow \mathrm{F})$ is identified with the simplex $(\beta_0 \to \beta_1 \to \cdots \to \beta_n, 1_{\mathrm{F}\beta_0}\colon \mathrm{F}\beta_0 \to \mathrm{F}\beta_0)$ of $\mathrm{B}(\mathrm{F}\beta_0 \downarrow \mathrm{F})$ in $\operatorname{colim} \mathrm{B}(-\downarrow \mathrm{F})$. $\square$

REMARK 14.5.5. We will show in Proposition 14.6.9 that the $\mathcal{D}^{\mathrm{op}}$-diagram $\mathrm{B}(-\downarrow \mathrm{F})$ is also a free cell complex (see Definition 14.1.28). It will then follow from Proposition 20.9.1 that the natural map $\operatorname{hocolim} \mathrm{B}(-\downarrow \mathrm{F}) \to \operatorname{colim} \mathrm{B}(-\downarrow \mathrm{F})$ is a weak equivalence, and so $\operatorname{hocolim} \mathrm{B}(-\downarrow \mathrm{F})$ is naturally weakly equivalent to $\mathrm{B}\mathcal{C}$.

14.5.6. Diagrams of overcategories.

DEFINITION 14.5.7. If $\mathcal{C}$ and $\mathcal{D}$ are small categories and $\mathrm{F}\colon \mathcal{C} \to \mathcal{D}$ is a functor, then, for each object α of $\mathcal{D}$, we have the category $(\mathrm{F}\downarrow \alpha)$, the category of objects of $\mathcal{C}$ over α (see Definition 14.4.1). If $\sigma\colon \alpha \to \alpha'$ is a map in $\mathcal{D}$, then σ induces a functor $\sigma_*\colon (\mathrm{F}\downarrow \alpha) \to (\mathrm{F}\downarrow \alpha')$, defined on objects by

$$\sigma_*(\mathrm{F}\beta \xrightarrow{\tau} \alpha) = \mathrm{F}\beta \xrightarrow{\sigma\tau} \alpha'.$$

If we take the classifying space of each overcategory (see Definition 9.4.1), we obtain the $\mathcal{D}$-*diagram of simplicial sets* $\mathrm{B}(\mathrm{F}\downarrow -)$ which, on the object α of $\mathcal{D}$, takes the value $\mathrm{B}(\mathrm{F}\downarrow \alpha)$. Thus, an n-simplex of $\mathrm{B}(\mathrm{F}\downarrow -)(\alpha) = \mathrm{B}(\mathrm{F}\downarrow \alpha)$ is a commutative diagram in $\mathcal{D}$

$$\begin{tikzcd}
\mathrm{F}\beta_0 \arrow[r, "\mathrm{F}\sigma_0"] \arrow[dr] & \mathrm{F}\beta_1 \arrow[r, "\mathrm{F}\sigma_1"] \arrow[d] & \cdots \arrow[r, "\mathrm{F}\sigma_{n-1}"] & \mathrm{F}\beta_n \arrow[dll] \\
& \alpha & &
\end{tikzcd}$$

with face and degeneracy maps defined as in (9.4.2).

As in Definition 14.5.2, if $\mathcal{C} = \mathcal{D}$ and F is the identity functor, then we use $\mathrm{B}(\mathcal{C}\downarrow -)$ to denote the diagram of overcategories, and an n-simplex of $\mathrm{B}(\mathcal{C}\downarrow -)(\alpha) =$

$\mathrm{B}(\mathcal{C}\downarrow\alpha)$ is a commutative diagram in $\mathcal{C}$

$$\begin{tikzcd} \alpha_0 \arrow[r, "\sigma_0"] \arrow[dr] & \alpha_1 \arrow[r, "\sigma_1"] \arrow[d] & \cdots \arrow[r, "\sigma_{n-1}"] & \alpha_n \arrow[dll] \\ & \alpha & & \end{tikzcd}$$

with face and degeneracy maps defined as in (9.4.2).

LEMMA 14.5.8. *If $\mathcal{C}$ is a small category and α is an object of $\mathcal{C}$, then $\mathrm{B}(\mathcal{C}\downarrow\alpha)$ is contractible.*

PROOF. This follows from Proposition 9.4.4, since $(\mathcal{C}\downarrow\alpha)$ has the terminal object $1_\alpha\colon \alpha \to \alpha$. □

The $\mathcal{C}$-diagram $\mathrm{B}(\mathcal{C}\downarrow -)$ will be used to define the homotopy limit functor (see Definition 19.1.10). Lemma 14.5.8 implies that in the model category of $\mathcal{C}$-diagrams of simplicial sets (see Theorem 14.2.1), the $\mathcal{C}$-diagram $\mathrm{B}(\mathcal{C}\downarrow -)$ is weakly equivalent to the constant diagram at a point. We will show in Corollary 14.6.8 that $\mathrm{B}(\mathcal{C}\downarrow -)$ is also a free $\mathcal{C}$-diagram, i.e., that $\mathrm{B}(\mathcal{C}\downarrow -)$ is a cofibrant approximation to the constant diagram at a point (see Definition 9.1.1). This will imply (in Theorem 20.8.1) that if we use a different cofibrant approximation to the constant diagram at a point in the definition of the homotopy limit of a diagram, then, for a diagram of fibrant objects, we will get an object weakly equivalent to the homotopy limit.

14.5.9. Relations.

PROPOSITION 14.5.10. *If $\mathcal{C}$ is a small category, then the isomorphism $(\alpha\downarrow\mathcal{C})^{\mathrm{op}} \approx (\mathcal{C}^{\mathrm{op}}\downarrow\alpha)$ of Proposition 14.4.3 is natural in the object α of $\mathcal{C}$.*

PROOF. This follows directly from the definitions. □

COROLLARY 14.5.11. *If $\mathcal{C}$ is a small category, then there is a natural isomorphism of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets*

$$\mathrm{B}(-\downarrow\mathcal{C})^{\mathrm{op}} \approx \mathrm{B}(\mathcal{C}^{\mathrm{op}}\downarrow -).$$

PROOF. This follows from Proposition 14.5.10. □

14.6. Recognizing free cell complexes

THEOREM 14.6.1. *If $\mathcal{C}$ is a small category and $\boldsymbol{X}$ is a $\mathcal{C}$-diagram of (pointed or unpointed) simplicial sets, then $\boldsymbol{X}$ is a free cell complex if and only if there is a sequence $\boldsymbol{S} = \{\boldsymbol{S}^0, \boldsymbol{S}^1, \boldsymbol{S}^2 \ldots\}$ of $\mathcal{C}^{\mathrm{disc}}$-diagrams of sets (where $\mathcal{C}^{\mathrm{disc}}$ is the discrete category with objects equal to the objects of $\mathcal{C}$) such that*

1. *For $n \ge 0$ and $\alpha \in \mathrm{Ob}(\mathcal{C})$, the set $\boldsymbol{S}^n_\alpha$ is a set of n-simplices of $\boldsymbol{X}_\alpha$.*
2. *For $n \ge 0$, $\alpha \in \mathrm{Ob}(\mathcal{C})$ and $0 \le i \le n$, we have $s_i(\boldsymbol{S}^n_\alpha) \subset \boldsymbol{S}^{n+1}_\alpha$ (i.e., $\boldsymbol{S}$ is closed under degeneracies).*
3. *For $n \ge 0$, the natural map $\boldsymbol{F}(\boldsymbol{S}^n) \to \boldsymbol{X}_n$ (see Theorem 14.1.22) is an isomorphism of $\mathcal{C}$-diagrams of sets (where $\boldsymbol{X}_n$ is the $\mathcal{C}$-diagram of n-simplices of $\boldsymbol{X}_\alpha$ for each $\alpha \in \mathrm{Ob}(\mathcal{C})$ and, if we are working in the category of* pointed *simplicial sets, $\boldsymbol{X}_n$ omits the basepoint and its degeneracies).*

REMARK 14.6.2. The reader should note the similarity between the *free cell complexes* among diagrams of simplicial sets and the *free simplicial groups* among simplicial groups (see, e.g., [**40**, Section 5]). Since a $\mathcal{C}$-diagram of simplicial sets is equivalently a simplicial object in the category of $\mathcal{C}$-diagrams of sets, we are comparing the definitions of free simplicial groups and free simplicial $\mathcal{C}$-diagrams of sets. This similarity can be made more precise by noting that a group is an algebra over the "underlying set of the free group" triple on the category of sets (see, e.g., [**4**, page 339] or [**42**, pages 176–177]), while a $\mathcal{C}$-diagram of sets is an algebra over the "underlying $\mathcal{C}^{\mathrm{disc}}$-diagram of sets on the free $\mathcal{C}$-diagram of sets" triple on the category of $\mathcal{C}^{\mathrm{disc}}$-diagrams of sets. The sequence $\boldsymbol{S}$ in Theorem 14.6.1 is the analogue for $\mathcal{C}$-diagrams of simplicial sets of a basis of a free simplicial group (see Definition 14.6.3). Free cell complexes are also free objects in the category of simplicial $\mathcal{C}$-diagrams of sets in the sense of [**38**, Definition 5.1].

PROOF OF THEOREM 14.6.1. We will prove the theorem in the category of *unpointed* simplicial sets; the proof for pointed simplicial sets is nearly identical.

We first assume that there is a sequence $\boldsymbol{S}^0$, $\boldsymbol{S}^1$, $\boldsymbol{S}^2 \ldots$ of $\mathcal{C}^{\mathrm{disc}}$-diagrams of sets satisfying conditions (1) through (3), and we will show that the n-skeleton $\boldsymbol{X}^n$ of $\boldsymbol{X}$ can be obtained from the $(n-1)$-skeleton $\boldsymbol{X}^{n-1}$ of $\boldsymbol{X}$ as a pushout of a coproduct of free cells. Proposition 12.2.5 and Lemma 12.2.11 will then imply that $\boldsymbol{X}$ is a free cell complex.

We begin by noting that $\boldsymbol{X}^0 = \Delta[0] \otimes \boldsymbol{F}(\boldsymbol{S}^0)$ (see Definition 14.1.21 and Definition 14.1.16). We now assume that n is a positive integer. For each $\alpha \in \mathrm{Ob}(\mathcal{C})$, let $\widetilde{\boldsymbol{S}}^n_\alpha \subset \boldsymbol{S}^n_\alpha$ be the subset of nondegenerate simplices. If $\sigma \in \widetilde{\boldsymbol{S}}^n_\alpha$, then all faces of σ are contained in $\boldsymbol{X}^{n-1}_\alpha$, and so σ defines a map $\partial\sigma \colon \partial\Delta[n] \to \boldsymbol{X}^{n-1}_\alpha$. Proposition 14.1.18 implies that this defines a map of $\mathcal{C}$-diagrams $\partial\sigma \otimes \boldsymbol{F}^\alpha_* \colon \partial\Delta[n] \otimes \boldsymbol{F}^\alpha_* \to \boldsymbol{X}^{n-1}$, and we can take the coproduct of these to obtain

$$\coprod_{\sigma\in\widetilde{\boldsymbol{S}}^n_\alpha} \partial\sigma \otimes \boldsymbol{F}^\alpha_* \colon \coprod_{\sigma\in\widetilde{\boldsymbol{S}}^n_\alpha} \partial\Delta[n] \otimes \boldsymbol{F}^\alpha_* = \partial\Delta[n] \otimes \boldsymbol{F}^\alpha_{\widetilde{\boldsymbol{S}}^n_\alpha} \to \boldsymbol{X}^{n-1}.$$

If we combine these for all $\alpha \in \mathrm{Ob}(\mathcal{C})$, we obtain the map

$$\coprod_{\alpha\in\mathrm{Ob}(\mathcal{C})} \partial\Delta[n] \otimes \boldsymbol{F}^\alpha_{\widetilde{\boldsymbol{S}}^n_\alpha} = \partial\Delta[n] \otimes \boldsymbol{F}(\widetilde{\boldsymbol{S}}^n) \to \boldsymbol{X}^{n-1}$$

(see Definition 14.1.21), and condition (3) implies that the square

$$\begin{tikzcd}
\partial\Delta[n] \otimes \boldsymbol{F}(\widetilde{\boldsymbol{S}}^n) \arrow[r] \arrow[d] & \boldsymbol{X}^{n-1} \arrow[d] \\
\Delta[n] \otimes \boldsymbol{F}(\widetilde{\boldsymbol{S}}^n) \arrow[r] & \boldsymbol{X}^n
\end{tikzcd}$$

is a pushout, which completes the first direction of the proof.

We now assume that $\boldsymbol{X}$ is a free cell complex. If γ is an ordinal and

$$\emptyset \to \boldsymbol{X}_1 \to \boldsymbol{X}_2 \to \cdots \to \boldsymbol{X}_\beta \to \cdots \qquad (\beta < \gamma)$$

is a presentation of $\boldsymbol{X}$ as a transfinite composition of pushouts of free cells, then for each $0 < \beta < \gamma$, we have a pushout diagram

$$\begin{tikzcd} \partial\Delta[n] \otimes \boldsymbol{F}^{\alpha}_{*} \arrow[r] \arrow[d] & \boldsymbol{X}_{\beta} \arrow[d] \\ \Delta[n] \otimes \boldsymbol{F}^{\alpha}_{*} \arrow[r] & \boldsymbol{X}_{\mathrm{Succ}(\beta)} \end{tikzcd}$$

(see Definition 12.1.11) for some integer $n \geq 0$ and some $\alpha \in \mathrm{Ob}(\mathcal{C})$. Let $\boldsymbol{S}_{\alpha}$ be the union (for each $0 \leq \beta < \gamma$ such that $\boldsymbol{X}_{\mathrm{Succ}(\beta)}$ is obtained from $\boldsymbol{X}_{\beta}$ by attaching a free cell based at α (see Definition 14.1.24)) of the images of $\delta_n \otimes 1_{\alpha}$ and its degeneracies (where δ_n is the nondegenerate n-simplex of $\Delta[n]$) in $\boldsymbol{X}$, and let $\boldsymbol{S}^{n}_{\alpha}$ be the set of n-simplices that occur in $\boldsymbol{S}_{\alpha}$. Since for each $0 \leq \beta < \gamma$ the diagram $\boldsymbol{X}$ is enlarged by adding the free diagram of simplices generated by the images of $\delta_n \otimes 1_{\alpha}$ and its degeneracies, it is clear that the sets $\boldsymbol{S}^{n}$ satisfy conditions (1) through (3), and so the proof is complete. ☐

Definition 14.6.3. If $\mathcal{C}$ is a small category and $\boldsymbol{X}$ is a $\mathcal{C}$-diagram of simplicial sets that is a free cell complex, then a sequence $\boldsymbol{S}^0, \boldsymbol{S}^1, \boldsymbol{S}^2, \ldots$ as in Theorem 14.6.1 will be called a *basis* for $\boldsymbol{X}$, and an element of an $\boldsymbol{S}^{n}_{\alpha}$ will be called a *generator* of the free cell complex $\boldsymbol{X}$. We will use $\boldsymbol{S}$ to denote the sequence $\boldsymbol{S}^0, \boldsymbol{S}^1, \boldsymbol{S}^2, \ldots$. We will let $\widetilde{\boldsymbol{S}}^{n}_{\alpha} \subset \boldsymbol{S}^{n}_{\alpha}$ be the subset of nondegenerate simplices, and we will call an element of an $\widetilde{\boldsymbol{S}}^{n}_{\alpha}$ a *nondegenerate generator* of $\boldsymbol{X}$. An element of an $\boldsymbol{S}^{n}_{\alpha} - \widetilde{\boldsymbol{S}}^{n}_{\alpha}$ will be called a *degenerate generator*.

Theorem 14.6.4. *Let $\mathcal{C}$ be a small category and let $\boldsymbol{X}\colon \mathcal{C} \to \mathrm{SS}_{(*)}$ be a $\mathcal{C}$-diagram of simplicial sets. If $\boldsymbol{S} = \{\boldsymbol{S}^0, \boldsymbol{S}^1, \boldsymbol{S}^2, \ldots\}$ is a sequence of $\mathcal{C}^{\mathrm{disc}}$-diagrams of sets, then $\boldsymbol{X}$ is a free cell complex with basis $\boldsymbol{S}$ if and only if:*

1. *For $n \geq 0$ and $\alpha \in \mathrm{Ob}(\mathcal{C})$, the set $\boldsymbol{S}^{n}_{\alpha}$ is a set of n-simplices of $\boldsymbol{X}_{\alpha}$.*
2. *For $n \geq 0$, $\alpha \in \mathrm{Ob}(\mathcal{C})$ and $0 \leq i \leq n$, we have $s_i(\boldsymbol{S}^{n}_{\alpha}) \subset \boldsymbol{S}^{n+1}_{\alpha}$ (i.e., $\boldsymbol{S}$ is closed under degeneracies).*
3. *If $n \geq 0$, $\beta \in \mathrm{Ob}(\mathcal{C})$ and τ is an n-simplex of $\boldsymbol{X}_{\beta}$ (where, if $\mathrm{SS}_{(*)} = \mathrm{SS}_{*}$, then τ is neither the basepoint nor one of its degeneracies), there exist unique $\alpha \in \mathrm{Ob}(\mathcal{C})$, $\sigma \in \boldsymbol{S}^{n}_{\alpha}$ and $\gamma\colon \alpha \to \beta$ in $\mathcal{C}$ such that $\boldsymbol{X}_{\gamma}(\sigma) = \tau$.*

Proof. This follows directly from Theorem 14.6.1 and Definition 14.1.21. ☐

Proposition 14.6.5. *If $\mathcal{C}$ and $\mathcal{D}$ are small categories and $F\colon \mathcal{C} \to \mathcal{D}$ is a functor, then the $\mathcal{D}^{\mathrm{op}}$-diagram of simplicial sets $\mathrm{B}(-\downarrow F)^{\mathrm{op}}$ (see Definition 14.5.2) and the $\mathcal{D}$-diagram of simplicial sets $\mathrm{B}(F\downarrow -)$ (see Definition 14.5.7) are both free cell complexes.*

Proof. If $\alpha \in \mathrm{Ob}(\mathcal{C})$, let $\boldsymbol{S}_{\alpha}$ be the set of simplices (of all dimensions) of $\mathrm{B}(F\alpha\downarrow F)^{\mathrm{op}}$ of the form

$$\begin{tikzcd} & F\alpha \arrow[dl] \arrow[d] \arrow[drrr, "1_{F\alpha}"] & & & \\ F\alpha_0 & F\alpha_1 \arrow[l, "F\sigma_0"] & \cdots \arrow[l, "F\sigma_1"] & & F\alpha \arrow[ll, "F\sigma_{n-1}"] \end{tikzcd} \tag{14.6.6}$$

and let $\boldsymbol{S}^{n}_{\alpha}$ be the set of n-simplices in $\boldsymbol{S}_{\alpha}$. The $\mathcal{C}^{\mathrm{disc}}$-diagrams of sets $\boldsymbol{S}^0, \boldsymbol{S}^1, \boldsymbol{S}^2, \ldots$ satisfy the conditions of Theorem 14.6.1, and so $\mathrm{B}(-\downarrow F)^{\mathrm{op}}$ is a free cell complex.

The proof for $\mathrm{B}(F\downarrow -)$ is similar, using the simplices of $\mathrm{B}(F\downarrow -)$ of the form

$$
\begin{tikzcd}
F\alpha_0 \arrow[r, "F\sigma_0"] \arrow[dr] & F\alpha_1 \arrow[r, "F\sigma_1"] \arrow[d] & \cdots \arrow[r, "F\sigma_{n-1}"] & F\alpha \arrow[dll, "1_{F\alpha}"] \\
& F\alpha & &
\end{tikzcd}
\tag{14.6.7}
$$

□

COROLLARY 14.6.8. *If $\mathcal{C}$ is a small category, the $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets $\mathrm{B}(-\downarrow\mathcal{C})^{\mathrm{op}}$ (see Definition 14.5.2) and the $\mathcal{C}$-diagram of simplicial sets $\mathrm{B}(\mathcal{C}\downarrow -)$ (see Definition 14.5.7) are both free cell complexes.*

PROPOSITION 14.6.9. *If $\mathcal{C}$ and $\mathcal{D}$ are small categories and $F\colon \mathcal{C}\to\mathcal{D}$ is a functor, then the $\mathcal{D}^{\mathrm{op}}$-diagram of simplicial sets $\mathrm{B}(-\downarrow F)$ is a free cell complex.*

PROOF. This is similar to the proof of Proposition 14.6.5. □

LEMMA 14.6.10. *If $\boldsymbol{X}\colon \mathcal{C}\to \mathsf{SS}$ is a free cell complex then $\boldsymbol{X}^{+}\colon \mathcal{C}\to\mathsf{SS}_*$ (defined by $\boldsymbol{X}^{+}_{\alpha} = (\boldsymbol{X}_{\alpha})^{+}$ for all $\alpha\in\mathrm{Ob}(\mathcal{C})$) is also a free cell complex.*

PROOF. This follows from Theorem 14.6.4. □

PROPOSITION 14.6.11. *If $\mathcal{C}$ is a small category and $\boldsymbol{X}\colon\mathcal{C}\to\mathsf{SS}_{(*)}$ is a free cell complex, then $|\boldsymbol{X}|\colon \mathcal{C}\to\mathsf{Top}_{(*)}$ (defined by $|\boldsymbol{X}|_{\alpha} = |\boldsymbol{X}_{\alpha}|$) is also a free cell complex.*

PROOF. This follows from the definition of free cell complex and the fact that if

$$
\begin{tikzcd}
\partial\Delta\otimes \boldsymbol{F}^{\alpha}_{*} \arrow[r] \arrow[d] & \boldsymbol{X}_{\beta} \arrow[d] \\
\Delta\otimes\boldsymbol{F}^{\alpha}_{*} \arrow[r] & \boldsymbol{X}_{\mathrm{Succ}(\beta)}
\end{tikzcd}
$$

is a pushout of $\mathcal{C}$-diagrams of simplicial sets, then

$$
\begin{tikzcd}
|\partial\Delta|\otimes \boldsymbol{F}^{\alpha}_{*} \arrow[r] \arrow[d] & |\boldsymbol{X}_{\beta}| \arrow[d] \\
|\Delta|\otimes\boldsymbol{F}^{\alpha}_{*} \arrow[r] & |\boldsymbol{X}_{\mathrm{Succ}(\beta)}|
\end{tikzcd}
$$

is a pushout of $\mathcal{C}$-diagrams of topological spaces. □

14.7. Maps from free cell complexes

PROPOSITION 14.7.1. *Let $\mathcal{C}$ be a small category and let $\boldsymbol{X}\colon\mathcal{C}\to\mathsf{SS}_{(*)}$ be a free cell complex with basis $\boldsymbol{S}=\{\boldsymbol{S}^0,\boldsymbol{S}^1,\boldsymbol{S}^2,\ldots\}$. If $n\ge 0$ and we let $\boldsymbol{Y}$ be the free cell complex with basis $\boldsymbol{T}=\{\boldsymbol{T}^0,\boldsymbol{T}^1,\boldsymbol{T}^2,\ldots\}$ where*

$$
\boldsymbol{T}^k = \begin{cases} \boldsymbol{S}^k & \text{if } k\le n \\ \boldsymbol{S}^k - \widetilde{\boldsymbol{S}}^k & \text{if } k > n \end{cases}
$$

(see Definition 14.6.3), then for each $\alpha\in\mathrm{Ob}(\mathcal{C})$, $\boldsymbol{Y}_{\alpha}$ is the n-skeleton of $\boldsymbol{X}_{\alpha}$.

PROOF. This follows from an examination of the proof of Theorem 14.6.1. □

PROPOSITION 14.7.2. *Let $\mathcal{C}$ be a small category, $\boldsymbol{X} \colon \mathcal{C} \to \mathsf{SS}_{(*)}$ a free cell complex with basis $\boldsymbol{S}^0, \boldsymbol{S}^1, \boldsymbol{S}^2, \dots$ and $\boldsymbol{X}^n \colon \mathcal{C} \to \mathsf{SS}_{(*)}$ the $\mathcal{C}$-diagram of n-skeletons of $\boldsymbol{X}$, i.e., $\boldsymbol{X}^n_\alpha$ is the n-skeleton of $\boldsymbol{X}_\alpha$ for all $\alpha \in \mathrm{Ob}(\mathcal{C})$. If $\boldsymbol{Y} \colon \mathcal{C} \to \mathsf{SS}_{(*)}$ is a $\mathcal{C}$-diagram of spaces and $g \colon \boldsymbol{X}^n \to \boldsymbol{Y}$ is a map of $\mathcal{C}$-diagrams, then extensions of g to the $(n+1)$-skeleton of $\boldsymbol{X}$ correspond to maps of $\mathcal{C}^{\mathrm{disc}}$-diagrams $h \colon \widetilde{\boldsymbol{S}}^{n+1} \to \boldsymbol{Y}_{n+1}$ such that $d_i h_\alpha = g_\alpha d_i$ for $\alpha \in \mathrm{Ob}(\mathcal{C})$ and $0 \le i \le n+1$.*

PROOF. This follows directly from the definitions. $\square$

CHAPTER 15

Cellular model categories

A cellular model category is a cofibrantly generated model category (see Definition 13.2.1) in which the cell complexes (see Definition 13.2.4) are well behaved (see Definition 15.1.1). I am not aware of any cofibrantly generated model categories that fail to be cellular model categories.

15.1. Cellular model categories

DEFINITION 15.1.1. A *cellular model category* is a cofibrantly generated (see Definition 13.2.1) model category $\mathcal{M}$ for which there are a set I of generating cofibrations and a set J of generating trivial cofibrations such that

1. both the domains and the codomains of the elements of I are compact (see Definition 13.4.1),
2. the domains of the elements of J are small relative to I (see Definition 12.4.10), and
3. inclusions of relative cell complexes (see Definition 13.2.4) are effective monomorphisms (see Definition 12.7.1).

REMARK 15.1.2. Although the sets I and J in Definition 15.1.1 are not part of the structure of a cellular model category, we will generally assume that some specific sets I and J satisfying the conditions of Definition 15.1.1 have been chosen.

15.1.3. Examples of cellular model categories. We still need to write out the proofs of the following propositions:

PROPOSITION 15.1.4. *The categories* SS, Top, SS_*, *and* Top_* *are cellular model categories.*

PROPOSITION 15.1.5. *If* $\mathcal{M}$ *is a cellular model category and* $\mathcal{C}$ *is a small category, then the diagram category* $\mathcal{M}^{\mathcal{C}}$ *is a cellular model category.*

PROPOSITION 15.1.6. *If* $\mathcal{M}$ *is a cellular model category and* Z *is an object of* $\mathcal{M}$, *then the overcategory* $(\mathcal{M}\downarrow Z)$ *is a cellular model category.*

PROPOSITION 15.1.7. *If* $\mathcal{M}$ *is a cellular model category and* $\mathcal{C}$ *is a small simplicial category, then the category* $\mathcal{M}^{\mathcal{C}}$ *of simplicial diagrams is a cellular model category.*

PROPOSITION 15.1.8. *If* $\mathcal{M}$ *is a pointed cellular model category with an action by pointed simplicial sets, then the category of spectra over* $\mathcal{M}$ *(as in* **[14]***) is a cellular model category.*

PROPOSITION 15.1.9. *If* $\mathcal{M}$ *is a pointed cellular model category with an action by pointed simplicial sets, then J. H. Smith's category of symmetric spectra over* $\mathcal{M}$ **[52, 36]** *is a cellular model category.*

15.1.10. Recognizing cellular model categories.

THEOREM 15.1.11. *If $\mathcal{M}$ is a model category, then $\mathcal{M}$ is a cellular model category if there are sets I and J of maps in $\mathcal{M}$ such that*

1. *a map is a trivial fibration if and only if it has the right lifting property with respect to every element of I,*
2. *a map is a fibration if and only if it has the right lifting property with respect to every element of J,*
3. *the domains and codomains of the elements of I are compact relative to I,*
4. *the domains of the elements of J are small relative to I, and*
5. *relative I-cell complexes are effective monomorphisms.*

PROOF. We need only show that I is a set of generating cofibrations for $\mathcal{M}$ and that J is a set of generating trivial cofibrations for $\mathcal{M}$. Proposition 12.6.7 and Proposition 12.7.5 imply that I permits the small object argument (see Definition 12.4.11), and so I is a set of generating cofibrations for $\mathcal{M}$. Proposition 13.2.10 now implies that J is a set of generating trivial cofibrations for $\mathcal{M}$. □

15.2. Subcomplexes in cellular model categories

PROPOSITION 15.2.1. *If $\mathcal{M}$ is a cellular model category, then a subcomplex of a presented relative cell complex is entirely determined by its set of cells (see Definition 12.5.4).*

PROOF. This follows from Proposition 12.5.9 and Proposition 12.5.10. □

Thus, if $f\colon X \to Y$ is a presented relative cell complex, then the union of a set of subcomplexes of f is well defined. The intersection of a family of subcomplexes is also well defined, but there is no guarantee that an intersection of subcomplexes exists (see, however, Proposition 15.2.3).

15.2.2. Intersections of subcomplexes. The main result of this section is Theorem 15.2.6, which asserts that the intersection of two subcomplexes of a presented cell complex always exists. We have not been able to determine whether an arbitrary intersection of subcomplexes must exist.

PROPOSITION 15.2.3. *Let $\mathcal{M}$ be a cellular model category and let X be a presented cell complex. If K and L are subcomplexes of X such that their intersection $K \cap L$ exists (see Remark 12.5.11), then the pushout square*

$$\begin{tikzcd} K \cap L \arrow[r] \arrow[d] & K \arrow[d, "v"] \\ L \arrow[r, "u"'] & K \cup L \end{tikzcd}$$

is a pullback square.

PROOF. If $f\colon W \to L$ and $g\colon W \to K$ are maps such that $vg = uf$, then we have the solid arrow diagram

$$\begin{tikzcd}
W \arrow[dr, dotted, "h"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] \\
& K \cap L \arrow[r, "t"] \arrow[d, "s"'] & K \arrow[r, shift left, "i_0"] \arrow[r, shift right, "i_1"'] \arrow[d, "v"'] & K \amalg_{K \cap L} L \arrow[d, "r"] \\
& L \arrow[r, "u"'] & K \cup L \arrow[r, shift left, "i'_0"] \arrow[r, shift right, "i'_1"'] & (K \cup L) \amalg_L (K \cup L)
\end{tikzcd}$$

in which the left hand square commutes, $ri_0 = i'_0 v$, and $ri_1 = i'_1 v$. We now have $ri_0 g = i'_0 vg = i'_0 uf = i'_1 uf = i'_1 vg = ri_1 g$; since r is an inclusion of a subcomplex, it is a monomorphism (see Proposition 12.7.5), and so $i_0 g = i_1 g$. Since t is an inclusion of a subcomplex (and, thus, an effective monomorphism), this implies that there is a unique map $h\colon W \to K \cap L$ such that $th = g$. Since $ush = vth = vg = uf$ and u is an inclusion of a subcomplex (and, thus, a monomorphism), we have $sh = f$, and the proof is complete. □

THEOREM 15.2.4. *Let $\mathcal{M}$ be a cellular model category and let $\bigl(X, \emptyset = X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \quad (\beta < \lambda), \{T^\beta, e^\beta, h^\beta\}_{\beta<\lambda}\bigr)$ be a presented cell complex. If $\{U^\beta\}_{\beta<\lambda}$ and $\{V^\beta\}_{\beta<\lambda}$ are subcomplexes of X (see Remark 12.5.11), then the sequence $\{\widetilde{T}^\beta\}_{\beta<\lambda}$ such that $\widetilde{T}^\beta = U^\beta \cap V^\beta$ for all $\beta < \lambda$ determines a subcomplex of X.*

PROOF. We must show that the sequence $\{\widetilde{T}^\beta\}_{\beta<\lambda}$ can be constructed by the inductive procedure of Proposition 12.5.10. Since Proposition 12.5.10 allows $\widetilde{T}^0$ to be any subset of T^0, the induction is begun.

Suppose now that β is an ordinal such that $\beta < \lambda$, and that the condition is satisfied for $\widetilde{T}^\gamma$ for all $\gamma < \beta$. We must show that, if $i \in \widetilde{T}^\beta$, then $h^\beta_i : C_i \to X_\beta$ factors through $\widetilde{X}_\beta \to X_\beta$. Since $\widetilde{T}^\beta = U^\beta \cap V^\beta$, this follows from Proposition 15.2.3, and so the proof is complete. □

DEFINITION 15.2.5. The subcomplex $\{\widetilde{T}^\beta\}_{\beta<\lambda}$ of Theorem 15.2.4 will be called the *intersection* of the subcomplexes $\{U^\beta\}_{\beta<\lambda}$ and $\{V^\beta\}_{\beta<\lambda}$.

THEOREM 15.2.6. *Let $\mathcal{M}$ be a cellular model category, and let X be a cell complex. If K and L are subcomplexes (see Remark 12.5.7) of X (relative to some presentation of X), then the subcomplex $K \cap L$ of X exists.*

PROOF. This follows from Theorem 15.2.4. □

15.3. Compactness in cellular model categories

PROPOSITION 15.3.1. *If $\mathcal{M}$ is a cellular model category then there is a cardinal σ such that if τ is a cardinal and X is a cell complex of size τ, then X is $\sigma\tau$-compact (see Definition 13.4.1).*

PROOF. Since the domains and codomains of the elements of I are compact, we can choose an infinite cardinal σ such that each of these domains and codomains is σ-compact (see Proposition 13.4.6).

If τ is a cardinal and X is a cell complex of size τ, then we can choose a presentation of X (see Definition 12.5.2), indexed by an ordinal λ whose cardinal is τ, that has no two cells with the same presentation ordinal (see Definition 12.5.4). Thus, we have a λ-sequence $\emptyset = X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots$ $(\beta < \lambda)$ whose colimit is X and such that every $X_{\beta+1}$ (for $\beta + 1 < \lambda$) is obtained as a pushout

$$\begin{tikzcd} A_{\beta+1} \arrow[r] \arrow[d] & B_{\beta+1} \arrow[d] \\ X_\beta \arrow[r] & X_{\beta+1} \end{tikzcd} \tag{15.3.2}$$

for some element $A_{\beta+1} \to B_{\beta+1}$ of I. If Y is a presented cell complex and $f\colon X \to Y$ is a map, then we must show that there is a subcomplex K of Y, of size at most $\sigma\tau$, through which f factors. We will show by induction on β that for every $\beta < \lambda$ the composition $X_\beta \to X \to Y$ factors through a subcomplex K_β of Y of size at most $\sigma\tau$. The map f will then factor through the union of the $\{K_\beta\}_{\beta<\lambda}$ (since the inclusion of that union into Y is a monomorphism; see Proposition 12.7.5), which is of size at most $(\sigma\tau)\tau = \sigma\tau$.

The induction is begun by noting that $X_0 = \emptyset$ (the initial object of $\mathcal{M}$). If $\beta + 1 < \lambda$ and the composition $X_\beta \to X \to Y$ factors through a subcomplex K_β of Y of size at most $\sigma\tau$, then the composition of the attaching map $B_{\beta+1} \to X_{\beta+1} \to X \to Y$ (see Diagram 15.3.2) also factors through a subcomplex of size at most $\sigma\tau$, and (since σ is infinite) the union of these subcomplexes will be of size at most $\sigma\tau$ (see Proposition 12.1.14). Finally, if β is a limit ordinal such that $\beta < \lambda$ and for every $\alpha < \beta$ the composition $X_\alpha \to X \to Y$ factors through a subcomplex K_α of Y of size at most $\sigma\tau$, then the composition $X_\beta \to X \to Y$ factors through the union $\bigcup_{\alpha<\beta} K_\alpha$, which is of size at most $\sigma\tau$. □

DEFINITION 15.3.3. If $\mathcal{M}$ is a cellular model category, then the smallest cardinal σ satisfying the conclusion of Proposition 15.3.1 will be called the *size of the cells* of $\mathcal{M}$.

15.4. Smallness in cellular model categories

The main result of this section is Theorem 15.4.3, which asserts that all cofibrant objects in a cellular model category are small relative to the subcategory of all cofibrations.

LEMMA 15.4.1. *If $\mathcal{M}$ is a cellular model category with generating cofibrations I, then every cell complex (see Definition 13.2.4) is small relative to I.*

PROOF. This follows from Proposition 12.6.7 and Corollary 12.3.9. □

LEMMA 15.4.2. *If $\mathcal{M}$ is a cellular model category with generating cofibrations I, then every cofibrant object of $\mathcal{M}$ is small relative to I.*

PROOF. This follows from Corollary 13.2.13, Proposition 12.3.7 and Lemma 15.4.1. □

THEOREM 15.4.3. *If $\mathcal{M}$ is a cellular model category, then every cofibrant object is small relative to the subcategory of cofibrations.*

PROOF. This follows from Lemma 15.4.2 and Proposition 13.2.10. □

THEOREM 15.4.4. *If $\mathcal{M}$ is a cellular model category and J is a set of generating trivial cofibrations for $\mathcal{M}$ as in Definition 15.1.1, then the domains of the elements of J are small relative to the subcategory of all cofibrations.*

PROOF. This follows from Definition 15.1.1 and Proposition 13.2.10. □

PROPOSITION 15.4.5. *Let $\mathcal{M}$ be a cellular model category. If S is a set of cofibrations with cofibrant domains and J is a set of generating trivial cofibrations for $\mathcal{M}$ as in Definition 15.1.1, then there is a functorial factorization of every map $X \to Y$ as $X \xrightarrow{p} W \xrightarrow{q} Y$ where p is a relative $(S \cup J)$-cell complex and q is an $(S \cup J)$-injective.*

PROOF. Theorem 15.4.3 and Theorem 15.4.4 imply that the domains of the elements of $S \cup J$ are small relative to $S \cup J$, and so the result follows from Proposition 12.4.12. □

PROPOSITION 15.4.6. *Let $\mathcal{M}$ be a left proper cellular model category, and let S be a set of inclusions of subcomplexes. If $X \to X'$ is the inclusion of a subcomplex and we apply a small object factorization using the set S and some ordinal λ (see Definition 12.4.14) to both of the maps $X \to *$ and $X' \to *$ to obtain the diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{E}_S \arrow[r] \arrow[d] & * \arrow[d] \\ X' \arrow[r] & \mathrm{E}'_S \arrow[r] & * \end{tikzcd}$$

then the map $\mathrm{E}_S \to \mathrm{E}'_S$ is the inclusion of a subcomplex.

PROOF. Using Proposition 12.7.5, one can check inductively that, at each stage in the construction of the factorization, the map $\mathrm{E}^\beta \to (\mathrm{E}^\beta)'$ is the inclusion of a subcomplex. □

15.5. Bounding the size of cell complexes

The main result of this section is Proposition 15.5.3, which asserts that if a small object factorization (see Definition 12.4.14) is applied to a map between "large enough" cell complexes, then the resulting cell complex is no larger than the ones with which you started.

PROPOSITION 15.5.1. *Let $\mathcal{M}$ be a cellular model category. If X is a cell complex (see Definition 13.2.4), then there is a cardinal η such that, if ν is a cardinal and Y is a cell complex of size ν, then the set $\mathcal{M}(X, Y)$ has cardinal at most $\eta\nu$.*

PROOF. Let σ be the size of the cells of $\mathcal{M}$ (see Definition 15.3.3), and let τ be the size of X. There is only a set of isomorphism classes of cell complexes of size at most $\sigma\tau$, and so we can choose a set $\{Y_\alpha\}_{\alpha \in A}$ of representatives of those isomorphism classes. We let η be the cardinal of the set $\coprod_{\alpha \in A} \mathcal{M}(X, Y_\alpha)$.

Let ν be a cardinal, and let Y be a cell complex of size ν. If $\nu \le \sigma\tau$, then Y is isomorphic to one of the Y_α, and so the cardinal of $\mathcal{M}(X, Y)$ is at most $\eta \le \eta\nu$. If $\nu > \sigma\tau$, then any map from X to Y must factor through a subcomplex of Y that is isomorphic to one of the Y_α (see Proposition 15.3.1). Since $\nu > \sigma\tau$, the set of such subcomplexes of Y has cardinal at most ν (see Lemma 12.1.17), and so the set $\mathcal{M}(X, Y)$ has cardinal at most $\eta\nu$. □

COROLLARY 15.5.2. *Let $\mathcal{M}$ be a cellular model category. If X is a cofibrant object, then there is a cardinal η such that, if ν is a cardinal and Y is a cell complex of size ν, then the set $\mathcal{M}(X, Y)$ has cardinal at most $\eta\nu$.*

PROOF. This follows from Proposition 15.5.1, Lemma 12.1.18, and Corollary 13.2.13. □

PROPOSITION 15.5.3. *Let $\mathcal{M}$ be a cellular model category with generating cofibrations I. If K is a set of relative I-cell complexes with cofibrant domains, then there is a cardinal κ such that, for every cardinal μ such that $\mu \geq \kappa$, if $g\colon X \to Y$ is a map of cell complexes of size at most μ and E_K is the object constructed by applying the small object factorization with the set K and an ordinal $\nu \leq \mu$ to the map g (see Definition 12.4.14), then E_K is a cell complex of size at most μ.*

PROOF. Let κ be an infinite cardinal at least as large as each of the following cardinals:

- for each domain of an element of K, the cardinal η as in Corollary 15.5.2,
- for each codomain of an element of K, the cardinal η as in Corollary 15.5.2,
- for each relative I-cell complex in K, the cardinal of the set of cells in that relative I-cell complex, and
- the cardinal of the set K.

If μ is a cardinal such that $\mu \geq \kappa$ and ν is an ordinal such that $\nu \leq \mu$, let $g\colon X \to Y$ be a map of cell complexes of size at most μ, and let $X = X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots$ $(\beta < \nu)$ be the ν-sequence constructed by applying the small object factorization with the set K and the ordinal ν to g. We will show by transfinite induction that, for $\beta < \nu$, the complex X_β has size at most μ. Since $\mathrm{Succ}(\mu)$ (see Definition 12.1.11) is a regular cardinal (see Proposition 12.1.15), this will imply the proposition.

We begin the induction by noting that $X_0 = X$. If we now assume that $\beta < \nu$ and that X_β has size at most μ, then the domain of each element of K has at most $\kappa\mu = \mu$ maps to X_β, the codomain has at most $\kappa\mu = \mu$ maps to Y, and there are at most μ elements of K. Thus, $X_{\beta+1}$ is built from X_β by pushing out at most $\mu \times \mu \times \mu = \mu$ maps, each of which attaches at most μ cells to X_β, and so $X_{\beta+1}$ has size at most μ.

If β is a limit ordinal, then X_β is a colimit of complexes of size at most μ. Since $\beta < \nu \leq \mu$, this implies that X_β is of size at most μ, and the proof is complete. □

DEFINITION 15.5.4. Let $\mathcal{M}$ be a cellular model category with generating cofibrations I, and let ρ be the smallest regular cardinal such that

1. the domains of the elements of I are ρ-small relative to I (see Definition 12.4.10), and
2. ρ is at least as great as the smallest infinite cardinal κ satisfying the conclusion of Proposition 15.5.3 for the set I.

We define a *natural cylinder object* (see Definition 8.3.2) $X \amalg X \to \mathrm{Cyl}^{\mathcal{M}}(X) \to X$ on $\mathcal{M}$ by applying the small object factorization with the set I and the ordinal ρ to the fold map $1_X \amalg 1_X : X \amalg X \to X$ (see Definition 12.4.14).

PROPOSITION 15.5.5. *Let $\mathcal{M}$ be a cellular model category. If ρ is as in Definition 15.5.4, μ is a cardinal such that $\mu \geq \rho$, and X is a cell complex of size at most μ, then the natural cylinder object $\mathrm{Cyl}^{\mathcal{M}}(X)$ (see Definition 15.5.4) is of size at most μ.*

PROOF. This follows from Proposition 15.5.3. $\square$

CHAPTER 16

The Reedy model category structure

The Reedy model category structure will be defined for diagrams in a model category indexed by a Reedy category (see Definition 16.3.2). The main examples of Reedy categories are the cosimplicial and simplicial indexing categories, and, more generally, categories of simplices of simplicial sets (see Definition 16.1.11) and their opposites. The standard model category structures on categories of simplicial (or cosimplicial) objects in a model category are examples of Reedy model category structures.

The Reedy model category structure on the category of cosimplicial spaces differs from the one defined by using the cofibrantly generated model category structure on spaces (see Theorem 14.2.1) in that, although it has the same weak equivalences, it has more cofibrations. A cosimplicial object in a model category will be Reedy cofibrant if for every $n \geq 0$ the map from the colimit of objects of lower degree to the object of degree n is a cofibration (see Definition 16.3.2). Thus, the cosimplicial standard simplex (see Definition 16.1.9) is a Reedy cofibrant diagram of simplicial sets (see Corollary 16.4.10).

The Reedy model category structure was defined first (in [**15**, Chapter X]) for the category of cosimplicial spaces. It was then defined for the category of simplicial objects in a model category in [**50**, Section 1] (see also [**14**, Theorem B.6]) and the category of cosimplicial objects in a model category (see [**30**, Section 2.4]). The common generalization of these indexing categories is due to D. M. Kan, and is called a *Reedy category* (see Definition 16.2.2).

16.1. The category of simplices of a simplicial set

If X is a simplicial set, we will define a category ΔX whose objects are the simplices of X and whose morphisms from the simplex σ to the simplex τ are the simplicial operators that take τ to σ (see Definition 16.1.11). Note the reversal of direction: If $\partial_i \tau = \sigma$, then ∂_i corresponds to a morphism that takes σ to τ. This is because a simplicial set is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathsf{Set}$, while ΔX is defined as an overcategory using a covariant functor $\mathbf{\Delta} \to \mathsf{SS}$. A diagram indexed by ΔX is a sort of generalized cosimplicial object, and a diagram indexed by $\Delta^{\mathrm{op}} X$ is a sort of generalized simplicial object (see Example 16.1.13, Definition 16.1.5, Definition 16.1.7, and Proposition 20.10.2). Categories of the form ΔX or $\Delta^{\mathrm{op}} X$ (for a simplicial set X) are the most important examples of *Reedy categories* (see Definition 16.2.2).

16.1.1. The simplicial category.

Definition 16.1.2 (The simplicial category). If n is a nonnegative integer, we let $[n]$ denote the ordered set $(0, 1, 2, \ldots, n)$. The category $\mathbf{\Delta}$ is the category with objects the $[n]$ for $n \geq 0$ and with morphisms $\mathbf{\Delta}([n],[k])$ the weakly monotone functions $[n] \to [k]$, i.e., the functions $\sigma\colon [n] \to [k]$ such that $\sigma(i) \le \sigma(j)$ for $0 \le i \le j \le n$.

REMARK 16.1.3. The simplicial category $\mathbf{\Delta}$ (see Definition 16.1.2) is a skeletal subcategory of the category whose objects are the finite ordered sets and whose morphisms are the weakly monotone maps.

EXAMPLE 16.1.4. A simplicial set is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathsf{Set}$.

DEFINITION 16.1.5. A simplicial space is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathsf{Spc}_{(*)}$.

NOTATION 16.1.6. If $\boldsymbol{X}$ is a simplicial object, we will usually denote the object $\boldsymbol{X}_{[n]}$ by $\boldsymbol{X}_n$.

DEFINITION 16.1.7. A cosimplicial space is a functor $\mathbf{\Delta} \to \mathsf{Spc}_{(*)}$.

NOTATION 16.1.8. If $\boldsymbol{X}$ is a cosimplicial object, we will usually denote the object $\boldsymbol{X}_{[n]}$ by $\boldsymbol{X}^n$.

DEFINITION 16.1.9. The *cosimplicial standard simplex* is the cosimplicial simplicial set $\Delta\colon \mathbf{\Delta} \to \mathsf{SS}$ (see Definition 16.1.2) that takes the object $[n]$ of $\mathbf{\Delta}$ to the standard n-simplex $\Delta[n]$. The simplicial set $\Delta[n]$ has as k-simplices the weakly monotone functions $[k] \to [n]$, i.e., $\Delta[n]_k = \mathbf{\Delta}([k],[n])$.

16.1.10. Categories of simplices.

DEFINITION 16.1.11. Let $\mathbf{\Delta}$ be the simplicial category (see Definition 16.1.2), and let $\mathrm{F}\colon \mathbf{\Delta} \to \mathsf{SS}$ be the functor that takes $[n]$ to $\Delta[n]$. If K is a simplicial set, then ΔK, the *category of simplices of* K, is defined to be the overcategory $(\mathrm{F}\downarrow K)$ (see Definition 14.4.1). Thus, ΔK is the category with objects the simplicial maps $\Delta[n] \to K$ (for some $n \ge 0$) and with morphisms from $\sigma\colon \Delta[n] \to K$ to $\tau\colon \Delta[k] \to K$ the commutative triangles of simplicial maps

$$\begin{tikzcd} \Delta[n] \arrow[rr] \arrow[dr, "\sigma"'] & & \Delta[k] \arrow[dl, "\tau"] \\ & K & \end{tikzcd}$$

PROPOSITION 16.1.12. *If K is a simplicial set, then there is a natural isomorphism of sets $\mathrm{Ob}(\Delta K) \approx \coprod_{n\ge 0} K_n$. If τ is an n-simplex (for some $n > 0$), k is an integer satisfying $0 \le k \le n$, and $\partial_k \tau = \sigma$, then ∂_k corresponds under this isomorphism to a morphism from $\chi_\sigma\colon \Delta[n-1] \to K$ to $\chi_\tau\colon \Delta[n] \to K$ (where the characteristic map χ_τ of an n-simplex τ is the unique map $\Delta[n] \to K$ that takes the non-degenerate n-simplex of $\Delta[n]$ to τ).*

PROOF. This follows from the one to one correspondence between n-simplices of K and maps of simplicial sets $\Delta[n] \to K$ (see Example 14.1.12). $\square$

EXAMPLE 16.1.13. If K is the one point simplicial set (i.e., $K_n = *$ for all $n \ge 0$), then ΔK is the simplicial category $\mathbf{\Delta}$ (see Definition 16.1.2).

PROPOSITION 16.1.14. *If K is a simplicial set and $\mathrm{G}\colon \Delta K \to \mathsf{SS}$ is the ΔK-diagram of simplicial sets that takes the object $\sigma\colon \Delta[n] \to K$ of ΔK to $\Delta[n]$, then there is a natural isomorphism $\mathrm{colim}_{\Delta K}\,\mathrm{G} \approx K$.*

PROOF. The objects $\sigma\colon \Delta[n] \to K$ of ΔK come with natural maps $\mathrm{G}(\sigma) \to K$ that commute with the structure maps of G, and so there is a natural map $\operatorname{colim}_{\Delta K} \mathrm{G} \to K$. Since every n-simplex σ of K defines an object $\chi_\sigma\colon \Delta[n] \to K$ of ΔK for which the image of the natural map $\mathrm{G}(\chi_\sigma) \to K$ contains σ, the map $\operatorname{colim}_{\Delta K} \mathrm{G} \to K$ is surjective.

To show that the map $\operatorname{colim}_{\Delta K} \mathrm{G} \to K$ is injective, assume that there are objects $\sigma\colon \Delta[m] \to K$ and $\tau\colon \Delta[n] \to K$ of ΔK together with a k-simplex η of $\Delta[m]$ and a k-simplex μ of $\Delta[n]$ such that the image in K of η under $\mathrm{G}(\sigma) \to K$ equals the image in K of μ under $\mathrm{G}(\tau) \to K$. This implies that there is a commutative square in SS

$$\begin{tikzcd}
\Delta[k] \arrow[r, "\chi_\mu"] \arrow[d, "\chi_\eta"'] & \Delta[n] \arrow[d, "\tau"] \\
\Delta[m] \arrow[r, "\sigma"'] & K
\end{tikzcd}$$

which we can regard as a diagram in ΔK. This diagram in ΔK implies that the image of η in $\operatorname{colim}_{\Delta K} \mathrm{G}$ equals the image of μ in $\operatorname{colim}_{\Delta K} \mathrm{G}$, and so the natural surjection $\operatorname{colim}_{\Delta K} \mathrm{G} \to K$ is a natural isomorphism. $\square$

16.2. Reedy categories and their diagram categories

16.2.1. Reedy categories.

DEFINITION 16.2.2. A *Reedy Category* is a small category $\mathcal{C}$ together with two subcategories $\overrightarrow{\mathcal{C}}$ (the *direct subcategory*) and $\overleftarrow{\mathcal{C}}$ (the *inverse subcategory*), both of which contain all the objects of $\mathcal{C}$, in which every object can be assigned a nonnegative integer (called its *degree*) such that

1. Every non-identity morphism of $\overrightarrow{\mathcal{C}}$ raises degree.
2. Every non-identity morphism of $\overleftarrow{\mathcal{C}}$ lowers degree.
3. Every morphism g in $\mathcal{C}$ has a unique factorization $g = \overrightarrow{g}\overleftarrow{g}$ where $\overrightarrow{g}$ is in $\overrightarrow{\mathcal{C}}$ and $\overleftarrow{g}$ is in $\overleftarrow{\mathcal{C}}$.

REMARK 16.2.3. Definition 16.2.2 implies that a Reedy category consists of a category and two subcategories, subject to certain conditions. The function that assigns to each object its degree is not a part of the structure, but we will often implicitly assume that a degree function has been chosen.

EXAMPLE 16.2.4. The simplicial category (see Definition 16.1.2) is a Reedy category in which the object $[n]$ has degree n, the direct subcategory consists of the injective maps, and the inverse subcategory consists of the surjective maps.

EXAMPLE 16.2.5. If X is a simplicial set, then the category ΔX of simplices of X (see Definition 16.1.11) is a Reedy category in which the degree of an object is the dimension of the simplex of X to which it corresponds, the direct subcategory consists of the morphisms corresponding to iterated face maps in X, and the inverse subcategory consists of the morphisms corresponding to iterated degeneracy maps of X. Note that Example 16.2.4 is a special case of this example (see Example 16.1.13).

PROPOSITION 16.2.6. *If $\mathcal{C}$ is a Reedy category, then $\mathcal{C}^{\mathrm{op}}$ is a Reedy category in which the degrees of the objects are unchanged, $\overrightarrow{\mathcal{C}^{\mathrm{op}}} = (\overleftarrow{\mathcal{C}})^{\mathrm{op}}$, and $\overleftarrow{\mathcal{C}^{\mathrm{op}}} = (\overrightarrow{\mathcal{C}})^{\mathrm{op}}$.*

PROOF. This follows directly from the definitions. $\square$

DEFINITION 16.2.7. If $\mathcal{C}$ is a Reedy category with a degree function (see Remark 16.2.3) and n is a nonnegative integer, the *n-filtration* $\mathrm{F}^n\,\mathcal{C}$ is the full subcategory of $\mathcal{C}$ whose objects are the objects of $\mathcal{C}$ of degree less than or equal to n.

EXAMPLE 16.2.8. If $\mathcal{C}$ is a Reedy category, then the 0-filtration of $\mathcal{C}$ is a category with no non-identity maps.

EXAMPLE 16.2.9. If X is a simplicial set and $\mathcal{C} = \Delta X$ (see Example 16.2.5), then the n-filtration of $\mathcal{C}$ is *not* the same as $\Delta(X^n)$, the category of simplices of the n-skeleton of the simplicial set X. This is because $\mathrm{F}^n\,\mathcal{C}$ has no objects of degree greater than n, while $\Delta(X^n)$ has among its objects the high dimensional simplices of X that are degeneracies of simplices of dimension less than or equal to n. The relationship between $\mathrm{F}^n\,\mathcal{C}$ and $\Delta(X^n)$ is that $\Delta(X^n)$ is the left Kan extension (see [**41**, page 232]) of $\mathrm{F}^n\,\mathcal{C}$ along the inclusion $\mathrm{F}^n\,\mathcal{C} \to \mathcal{C}$ (see also [**3**, page 6]).

PROPOSITION 16.2.10. *If $\mathcal{C}$ is a Reedy category, then each of its filtrations (see Definition* 16.2.7*) is a Reedy category with the obvious structure, and $\mathcal{C}$ equals the union of the increasing sequence of subcategories* $\mathrm{F}^0\,\mathcal{C} \subset \mathrm{F}^1\,\mathcal{C} \subset \mathrm{F}^2\,\mathcal{C} \subset \cdots$.

PROOF. This follows directly from the definitions. $\square$

16.2.11. Diagrams indexed by a Reedy category. Diagrams indexed by a Reedy category and maps of such diagrams are most naturally analyzed inductively on the degree of the object. In this section, we assume that we have a Reedy category with a degree function (see Remark 16.2.3), and we describe how to define a diagram indexed by the Reedy category inductively on the degree of the object in the Reedy category. In Remark 16.2.19, we summarize this description in terms of the latching objects and matching objects of the diagram, which are defined in Definition 16.2.17. In Section 16.2.20, we will describe how to define a map between two such diagrams. We will use this analysis in Section 16.3 to define a model category structure on a category of diagrams in a model category indexed by a Reedy category.

Since the 0-filtration of a Reedy category contains no non-identity maps, we can define a diagram $\boldsymbol{X}\colon \mathrm{F}^0\,\mathcal{C} \to \mathcal{M}$ by choosing an object $\boldsymbol{X}_\alpha$ of $\mathcal{M}$ for each object α of $\mathcal{C}$ of degree 0.

Suppose that we have a diagram $\boldsymbol{X}\colon \mathrm{F}^{n-1}\,\mathcal{C} \to \mathcal{M}$ indexed by the $(n-1)$-filtration of a Reedy category $\mathcal{C}$, and we wish to extend it to a diagram $\boldsymbol{X}\colon \mathrm{F}^n\,\mathcal{C} \to \mathcal{M}$. We begin by choosing an object $\boldsymbol{X}_\alpha$ in $\mathcal{M}$ for each object α of $\mathcal{C}$ of degree n. For each object β of $\mathrm{F}^{n-1}\,\mathcal{C}$ and map $\beta \to \alpha$ in $\mathrm{F}^n\,\mathcal{C}$, we must choose a map $\boldsymbol{X}_\beta \to \boldsymbol{X}_\alpha$ in $\mathcal{M}$. We must do this so that if $\beta \to \beta'$ is a map in $\mathrm{F}^{n-1}\,\mathcal{C}$ and

$$\begin{tikzcd} \beta \arrow[rr] \arrow[dr] & & \beta' \arrow[dl] \\ & \alpha & \end{tikzcd}$$

is a commutative triangle in $\mathrm{F}^n\,\mathcal{C}$, then the triangle in $\mathcal{M}$

$$\begin{tikzcd} \boldsymbol{X}_\beta \arrow[rr] \arrow[dr] & & \boldsymbol{X}_{\beta'} \arrow[dl] \\ & \boldsymbol{X}_\alpha & \end{tikzcd}$$

commutes. If $I^n\colon \mathrm{F}^{n-1}\,\mathcal{C} \to \mathrm{F}^n\,\mathcal{C}$ is the inclusion functor, then this is equivalent to choosing a map $\operatorname{colim}_{(I^n\downarrow\alpha)} \boldsymbol{X} \to \boldsymbol{X}_\alpha$. (The object $\operatorname{colim}_{(I^n\downarrow\alpha)} \boldsymbol{X}$ is the value on α of the left Kan extension of $\boldsymbol{X}\colon \mathrm{F}^{n-1}\,\mathcal{C} \to \mathcal{M}$ along the inclusion $\mathrm{F}^{n-1}\,\mathcal{C} \to \mathrm{F}^n\,\mathcal{C}$ (see [**41**, page 233]).)

Similarly, for each object γ of $\mathrm{F}^{n-1}\,\mathcal{C}$ and map $\alpha \to \gamma$ in $\mathrm{F}^n\,\mathcal{C}$, we must choose a map $\boldsymbol{X}_\alpha \to \boldsymbol{X}_\gamma$, such that if $\gamma \to \gamma'$ is a map in $\mathrm{F}^{n-1}\,\mathcal{C}$ and

$$\begin{tikzcd} & \alpha \arrow[dl] \arrow[dr] & \\ \gamma \arrow[rr] & & \gamma' \end{tikzcd}$$

is a commutative triangle in $\mathrm{F}^n\,\mathcal{C}$, then the triangle in $\mathcal{M}$

$$\begin{tikzcd} & \boldsymbol{X}_\alpha \arrow[dl] \arrow[dr] & \\ \boldsymbol{X}_\gamma \arrow[rr] & & \boldsymbol{X}_{\gamma'} \end{tikzcd}$$

commutes. This is equivalent to choosing a map $\boldsymbol{X}_\alpha \to \lim_{(\alpha\downarrow I^n)} \boldsymbol{X}$. (The object $\lim_{(\alpha\downarrow I^n)} \boldsymbol{X}$ is the value on α of the right Kan extension of $\boldsymbol{X}\colon \mathrm{F}^{n-1}\,\mathcal{C} \to \mathcal{M}$ along the inclusion $\mathrm{F}^{n-1}\,\mathcal{C} \to \mathrm{F}^n\,\mathcal{C}$ (see [**41**, page 233]).)

The maps $\operatorname{colim}_{(I^n\downarrow\alpha)} \boldsymbol{X} \to \boldsymbol{X}_\alpha$ and $\boldsymbol{X}_\alpha \to \lim_{(\alpha\downarrow I^n)} \boldsymbol{X}$ cannot be totally arbitrary. If $\beta \to \gamma$ is a map in $\mathrm{F}^{n-1}\,\mathcal{C}$ and

$$\begin{tikzcd} & \alpha \arrow[dr] & \\ \beta \arrow[ur] \arrow[rr] & & \gamma \end{tikzcd}$$

is a commutative triangle in $\mathrm{F}^n\,\mathcal{C}$, then the triangle in $\mathcal{M}$

$$\begin{tikzcd} & \boldsymbol{X}_\alpha \arrow[dr] & \\ \boldsymbol{X}_\beta \arrow[ur] \arrow[rr] & & \boldsymbol{X}_\gamma \end{tikzcd}$$

must commute. This is equivalent to requiring that the composition

$$\operatorname*{colim}_{(I^n\downarrow\alpha)} \boldsymbol{X} \to \boldsymbol{X}_\alpha \to \lim_{(\alpha\downarrow I^n)} \boldsymbol{X}$$

be a factorization of the natural map

$$\operatorname*{colim}_{(I^n\downarrow\alpha)} \boldsymbol{X} \to \lim_{(\alpha\downarrow I^n)} \boldsymbol{X}.$$

We will now show that the definition of a Reedy category implies that this last condition suffices to construct an extension of $\boldsymbol{X}$ from $\mathrm{F}^{n-1}\,\mathcal{C}$ to $\mathrm{F}^n\,\mathcal{C}$.

THEOREM 16.2.12. *Let $\mathcal{C}$ be a Reedy category, let $\mathcal{M}$ be a category closed under limits and colimits, let n be a positive integer, and let $\boldsymbol{X}\colon \mathrm{F}^{n-1}\,\mathcal{C} \to \mathcal{M}$ be a diagram. If for each object α of $\mathcal{C}$ of degree n we choose an object $\boldsymbol{X}_\alpha$ of $\mathcal{M}$ and a factorization $\operatorname{colim}_{(I^n\downarrow\alpha)} \boldsymbol{X} \to \boldsymbol{X}_\alpha \to \lim_{(\alpha\downarrow I^n)} \boldsymbol{X}$ of the natural map $\operatorname{colim}_{(I^n\downarrow\alpha)} \boldsymbol{X} \to \lim_{(\alpha\downarrow I^n)} \boldsymbol{X}$, then this uniquely determines an extension $\boldsymbol{X}\colon \mathrm{F}^n\,\mathcal{C} \to \mathcal{M}$ of the diagram $\boldsymbol{X}$.*

PROOF. The discussion above explains why our choices determine everything except the maps $\boldsymbol{X}_\alpha \to \boldsymbol{X}_{\alpha'}$ for a map $\alpha \to \alpha'$ in $\mathrm{F}^n\,\mathcal{C}$ between objects of degree n. Given such a map, if $\alpha \xrightarrow{\overleftarrow{g}} \beta \xrightarrow{\overrightarrow{g}} \alpha'$ is the factorization described in Definition 16.2.2, then we must define $\boldsymbol{X}_\alpha \to \boldsymbol{X}_{\alpha'}$ to be the composition $\boldsymbol{X}_\alpha \to \boldsymbol{X}_\beta \to \boldsymbol{X}_{\alpha'}$. It remains only to show that, if $\alpha \to \alpha' \to \alpha''$ are composable maps in $\mathrm{F}^n\,\mathcal{C}$ between objects of degree n, then the triangle

$$\begin{tikzcd} & \boldsymbol{X}_\alpha \arrow[dl] \arrow[dr] & \\ \boldsymbol{X}_{\alpha'} \arrow[rr] & & \boldsymbol{X}_{\alpha''} \end{tikzcd}$$

commutes.

Let $\alpha \xrightarrow{\overleftarrow{g}} \beta \xrightarrow{\overrightarrow{g}} \alpha'$ and $\alpha' \xrightarrow{\overleftarrow{h}} \beta' \xrightarrow{\overrightarrow{h}} \alpha''$ be the factorization of Definition 16.2.2 applied to $\alpha \to \alpha'$ and $\alpha' \to \alpha''$, respectively. If the factorization of Definition 16.2.2 applied to $\overleftarrow{h}\,\overrightarrow{g}\,\overleftarrow{g}\colon \alpha \to \beta'$ is $\alpha \xrightarrow{\overleftarrow{k}} \beta'' \xrightarrow{\overrightarrow{k}} \beta'$, then we have the commutative diagram

$$\begin{tikzcd} & & \alpha \arrow[dl, "\overleftarrow{g}"'] \arrow[d, "\overleftarrow{k}"] & \\ & \beta \arrow[dl, "\overrightarrow{g}"'] & \beta'' \arrow[d, "\overrightarrow{k}"] & \\ \alpha' \arrow[rr, "\overleftarrow{h}"] & & \beta' \arrow[r, "\overrightarrow{h}"] & \alpha''. \end{tikzcd}$$

Since $\overrightarrow{h}\,\overrightarrow{k}\,\overleftarrow{k} = \overrightarrow{h}\,\overleftarrow{h}\,\overrightarrow{g}\,\overleftarrow{g}$ and $\overrightarrow{h}\,\overrightarrow{k}$ is in $\overrightarrow{\mathcal{C}}$, the factorization $\alpha \xrightarrow{\overleftarrow{k}} \beta'' \xrightarrow{\overrightarrow{h}\,\overrightarrow{k}} \alpha''$ must be the factorization of $\alpha \to \alpha''$ described in Definition 16.2.2. Thus, it is sufficient to show that the composition $\boldsymbol{X}_\alpha \xrightarrow{\overleftarrow{k}_*} \boldsymbol{X}_{\beta''} \xrightarrow{\overrightarrow{k}_*} \boldsymbol{X}_{\beta'}$ equals the composition $\boldsymbol{X}_\alpha \xrightarrow{\overleftarrow{g}_*} \boldsymbol{X}_\beta \xrightarrow{\overrightarrow{g}_*} \boldsymbol{X}_{\alpha'} \xrightarrow{\overleftarrow{h}_*} \boldsymbol{X}_{\beta'}$.

Since both of the maps $\boldsymbol{X}_\alpha \to \boldsymbol{X}_{\beta''}$ and $\boldsymbol{X}_\alpha \to \boldsymbol{X}_\beta$ are defined as the composition of our map $\boldsymbol{X}_\alpha \to \lim_{(\alpha\downarrow I^n)} \boldsymbol{X}$ with a projection from the limit, the first of these maps equals the composition

$$\boldsymbol{X}_\alpha \to \lim_{(\alpha\downarrow I^n)} \boldsymbol{X} \to \boldsymbol{X}_{\beta''} \xrightarrow{\overrightarrow{k}_*} \boldsymbol{X}_{\beta'}$$

while the second equals the composition

$$\boldsymbol{X}_\alpha \to \lim_{(\alpha\downarrow I^n)} \boldsymbol{X} \to \boldsymbol{X}_\beta \xrightarrow{\overrightarrow{g}_*} \boldsymbol{X}_{\alpha'} \xrightarrow{\overleftarrow{h}_*} \boldsymbol{X}_{\beta'}.$$

The universal property of the limit implies that these are equal, and so the proof is complete. $\square$

16.2.13. Latching objects and matching objects. In this section, we show that the colimits and limits used in Section 16.2.11 to construct diagrams indexed by a Reedy category and those used in Section 16.2.20 to construct maps of such diagrams have a particularly convenient form. These colimits and limits are the *latching objects* and *matching objects* (see Definition 16.2.17). We continue to assume that we have chosen a degree function for our Reedy category (see Remark 16.2.3).

DEFINITION 16.2.14. Let $\mathcal{C}$ be a Reedy category and let α be an object of $\mathcal{C}$. If α is of degree n, then we let $\overrightarrow{I}^n\colon \mathrm{F}^{n-1}\overrightarrow{\mathcal{C}} \to \mathrm{F}^n\overrightarrow{\mathcal{C}}$ and $\overleftarrow{I}^n\colon \mathrm{F}^{n-1}\overleftarrow{\mathcal{C}} \to \mathrm{F}^n\overleftarrow{\mathcal{C}}$ be the inclusion functors.

1. The *latching category* $\big((\overrightarrow{\mathcal{C}}\downarrow\alpha)-1_\alpha\big)$ of $\mathcal{C}$ at α is the overcategory $(\overrightarrow{I}^n\downarrow\alpha)$. This is the largest subcategory of $(\overrightarrow{\mathcal{C}}\downarrow\alpha)$ that does not contain the identity map of α.
2. The *matching category* $\big((\alpha\downarrow\overleftarrow{\mathcal{C}})-1_\alpha\big)$ of $\mathcal{C}$ at α is the undercategory $(\alpha\downarrow\overleftarrow{I}^n)$. This is the largest largest subcategory of $(\alpha\downarrow\overleftarrow{\mathcal{C}})$ that does not contain the identity map of α.

PROPOSITION 16.2.15. *Let $\mathcal{C}$ be a Reedy category and let α be an object of $\mathcal{C}$.*

1. *The opposite of the latching category of $\mathcal{C}$ at α is naturally isomorphic to the matching category of $\mathcal{C}^{\mathrm{op}}$ at α.*
2. *The opposite of the matching category of $\mathcal{C}$ at α is naturally isomorphic to the latching category of $\mathcal{C}^{\mathrm{op}}$ at α.*

PROOF. This follows from Proposition 16.2.6 and Proposition 14.4.3. □

PROPOSITION 16.2.16. *Let $\mathcal{C}$ be a Reedy category, let α be an object of $\mathcal{C}$ of degree n, and let $I^n\colon \mathrm{F}^{n-1}\mathcal{C}\to\mathrm{F}^n\mathcal{C}$ be the inclusion functor.*

1. *The category $\big((\overrightarrow{\mathcal{C}}\downarrow\alpha)-1_\alpha\big)$ is a right cofinal subcategory (see Definition 14.4.5) of both $(I^n\downarrow\alpha)$ and $\big((\mathrm{F}^n\mathcal{C})\downarrow\alpha\big)$.*
2. *The category $\big((\alpha\downarrow\overleftarrow{\mathcal{C}})-1_\alpha\big)$ is a left cofinal subcategory of both $(\alpha\downarrow I^n)$ and $\big(\alpha\downarrow(\mathrm{F}^n\mathcal{C})\big)$.*

PROOF. We will prove part 1; the proof of part 2 is similar.

If $\beta\to\alpha$ is an object of $(I^n\downarrow\alpha)$, then we can factor it as $\beta\xrightarrow{\overleftarrow{g}}\bar\beta\xrightarrow{\overrightarrow{g}}\alpha$ where $\overleftarrow{g}\in\overleftarrow{\mathcal{C}}$ and $\overrightarrow{g}\in\overrightarrow{\mathcal{C}}$. This gives us the object

$$\begin{tikzcd} \beta \arrow[rr, "\overleftarrow{g}"] \arrow[dr] & & \bar\beta \arrow[dl, "\overrightarrow{g}"] \\ & \alpha & \end{tikzcd}$$

of $\big((\beta\to\alpha)\downarrow\big((\overrightarrow{\mathcal{C}}\downarrow\alpha)-1_\alpha\big)\big)$. The uniqueness of the factorization in Definition 16.2.2 implies that this object is initial in $\big((\beta\to\alpha)\downarrow\big((\overrightarrow{\mathcal{C}}\downarrow\alpha)-1_\alpha\big)\big)$, and so Proposition 9.4.4 implies that $\big((\alpha\downarrow\overleftarrow{\mathcal{C}})-1_\alpha\big)$ is right cofinal in $(I^n\downarrow\alpha)$. The proof that $\big((\alpha\downarrow\overleftarrow{\mathcal{C}})-1_\alpha\big)$ is right cofinal in $\big((\mathrm{F}^n\mathcal{C})\downarrow\alpha\big)$ is identical to this, and so the proof of the proposition is complete. □

DEFINITION 16.2.17. Let $\mathcal{C}$ be a Reedy category, let $\mathcal{M}$ be a model category, let $\boldsymbol{X}\in\mathrm{Ob}(\mathcal{M}^{\mathcal{C}})$ be a diagram, and let α be an object of $\mathcal{C}$. We use $\boldsymbol{X}$ to denote also the induced $\bigl((\overrightarrow{\mathcal{C}}\downarrow\alpha) - 1_\alpha\bigr)$-diagram (defined on objects by $\boldsymbol{X}_{(\beta\to\alpha)} = \boldsymbol{X}_\beta$) and the induced $\bigl((\alpha\downarrow\overleftarrow{\mathcal{C}}) - 1_\alpha\bigr)$-diagram (defined on objects by $\boldsymbol{X}_{(\alpha\to\beta)} = \boldsymbol{X}_\beta$).

1. The *latching object* of $\boldsymbol{X}$ at α is $\mathrm{L}_\alpha\boldsymbol{X} = \operatorname{colim}_{((\overrightarrow{\mathcal{C}}\downarrow\alpha)-1_\alpha)}\boldsymbol{X}$ and the *latching map* of $\boldsymbol{X}$ at α is the natural map $\mathrm{L}_\alpha\boldsymbol{X}\to\boldsymbol{X}_\alpha$.
2. The *matching object* of $\boldsymbol{X}$ at α is $\mathrm{M}_\alpha\boldsymbol{X} = \lim_{((\alpha\downarrow\overleftarrow{\mathcal{C}})-1_\alpha)}\boldsymbol{X}$ and the *matching map* of $\boldsymbol{X}$ at α is the natural map $\boldsymbol{X}_\alpha\to\mathrm{M}_\alpha\boldsymbol{X}$.

The objects $\operatorname{colim}_{(I^n\downarrow\alpha)}\boldsymbol{X}$ and $\lim_{(\alpha\downarrow I^n)}\boldsymbol{X}$ (where $\boldsymbol{X}$ is a diagram defined on the $(n-1)$-filtration of a Reedy category) were used in Section 16.2.11 to construct diagrams indexed by a Reedy category. The objects $\operatorname{colim}_{((\mathrm{F}^n\,\mathcal{C})\downarrow\alpha)}\boldsymbol{X}$ and $\lim_{(\alpha\downarrow(\mathrm{F}^n\,\mathcal{C}))}\boldsymbol{X}$ will be used in Section 16.2.20 to analyze maps between such diagrams. Corollary 16.2.18 shows that all of these colimits are latching objects of $\boldsymbol{X}$ and all of these limits are matching objects of $\boldsymbol{X}$.

COROLLARY 16.2.18. *Let $\mathcal{C}$ be a Reedy category, let $\mathcal{M}$ be a model category, let α be an object of $\mathcal{C}$ of degree n, and let $\boldsymbol{X}\in\mathrm{Ob}(\mathcal{M}^{\mathcal{C}})$ be a diagram. If $I^n\colon \mathrm{F}^{n-1}\,\mathcal{C}\to\mathrm{F}^n\,\mathcal{C}$ is the inclusion functor, then*

$$\operatorname*{colim}_{(I^n\downarrow\alpha)}\boldsymbol{X}\approx\mathrm{L}_\alpha\boldsymbol{X}\approx\operatorname*{colim}_{((\mathrm{F}^n\,\mathcal{C})\downarrow\alpha)}\boldsymbol{X}\quad\textit{and}\quad\lim_{(\alpha\downarrow I^n)}\boldsymbol{X}\approx\mathrm{M}_\alpha\boldsymbol{X}\approx\lim_{(\alpha\downarrow(\mathrm{F}^n\,\mathcal{C}))}\boldsymbol{X}$$

(see Definition 16.2.17).

PROOF. This follows from Proposition 16.2.16 and Corollary 14.4.9. □

REMARK 16.2.19. In light of Definition 16.2.17 and Corollary 16.2.18, the discussion in Section 16.2.11 can be summarized as follows: If $\mathcal{C}$ is a Reedy category, $\mathcal{M}$ is a model category, $\boldsymbol{X}\colon\mathrm{F}^{n-1}\,\mathcal{C}\to\mathcal{M}$ is a diagram indexed by the $(n-1)$-filtration of $\mathcal{C}$, and α is an object of $\mathcal{C}$ of degree n, then there is a natural map $\mathrm{L}_\alpha\boldsymbol{X}\to\mathrm{M}_\alpha\boldsymbol{X}$ from the latching object of $\boldsymbol{X}$ at α to the matching object of $\boldsymbol{X}$ at α. Extending $\boldsymbol{X}$ to a diagram $\mathrm{F}^n\,\mathcal{C}\to\mathcal{M}$ is equivalent to choosing, for every object α of degree n, an object $\boldsymbol{X}_\alpha$ and a factorization $\mathrm{L}_\alpha\boldsymbol{X}\to\boldsymbol{X}_\alpha\to\mathrm{M}_\alpha\boldsymbol{X}$ of that natural map, and this can be done independently for each of the objects of degree n.

16.2.20. Maps between diagrams. Maps between diagrams indexed by a Reedy category are most naturally analyzed inductively on the degree of the objects. We assume that we have chosen a degree function for our Reedy category (see Remark 16.2.3).

Let $\mathcal{C}$ be a Reedy category, let $\mathcal{M}$ be a model category, and let $\boldsymbol{X},\boldsymbol{Y}\colon\mathcal{C}\to\mathcal{M}$ be $\mathcal{C}$-diagrams in $\mathcal{M}$. Since the 0-filtration of a Reedy category contains no non-identity maps, a map $f\colon\boldsymbol{X}|_{\mathrm{F}^0\,\mathcal{C}}\to\boldsymbol{Y}|_{\mathrm{F}^0\,\mathcal{C}}$ is determined by choosing a map $\boldsymbol{X}_\alpha\to\boldsymbol{Y}_\alpha$ for every object α of degree 0.

Suppose that $f\colon\boldsymbol{X}|_{\mathrm{F}^{n-1}\,\mathcal{C}}\to\boldsymbol{Y}|_{\mathrm{F}^{n-1}\,\mathcal{C}}$ is a map of the restrictions of the diagrams to the $(n-1)$-filtration of $\mathcal{C}$. For every object α of $\mathcal{C}$ of degree n we have the solid arrow diagram

$$\begin{tikzcd}
\operatorname*{colim}_{((\mathrm{F}^n\,\mathcal{C})\downarrow\alpha)}\boldsymbol{X} \arrow[r] \arrow[d] & \boldsymbol{X}_\alpha \arrow[r] \arrow[d, dotted] & \lim_{(\alpha\downarrow(\mathrm{F}^n\,\mathcal{C}))}\boldsymbol{X} \arrow[d] \\
\operatorname*{colim}_{((\mathrm{F}^n\,\mathcal{C})\downarrow\alpha)}\boldsymbol{Y} \arrow[r] & \boldsymbol{Y}_\alpha \arrow[r] & \lim_{(\alpha\downarrow(\mathrm{F}^n\,\mathcal{C}))}\boldsymbol{Y}
\end{tikzcd}$$

and extensions of f to the n-filtration of $\mathcal{C}$ correspond to a choice, for every object α of degree n, of a dotted arrow that makes both squares commute. Corollary 16.2.18 implies that this diagram is isomorphic to the diagram

$$\begin{tikzcd} \mathrm{L}_\alpha \boldsymbol{X} \arrow[r] \arrow[d] & \boldsymbol{X}_\alpha \arrow[r] \arrow[d, dotted] & \mathrm{M}_\alpha \boldsymbol{X} \arrow[d] \\ \mathrm{L}_\alpha \boldsymbol{Y} \arrow[r] & \boldsymbol{Y}_\alpha \arrow[r] & \mathrm{M}_\alpha \boldsymbol{Y} \end{tikzcd}$$

Thus, if $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{X}$, and $\boldsymbol{Y}$ are objects in $\mathcal{M}^{\mathcal{C}}$ and we have a diagram

(16.2.21)
$$\begin{tikzcd} \boldsymbol{A} \arrow[r] \arrow[d] & \boldsymbol{X} \arrow[d] \\ \boldsymbol{B} \arrow[r] \arrow[ur, dotted, "h"] & \boldsymbol{Y} \end{tikzcd}$$

in which the dotted arrow h is defined only on the restriction of $\boldsymbol{B}$ to the $(n-1)$-filtration of $\mathcal{C}$, then for every object α of $\mathcal{C}$ of degree n we have an induced solid arrow diagram

$$\begin{tikzcd} \mathrm{L}_\alpha \boldsymbol{B} \amalg_{\mathrm{L}_\alpha \boldsymbol{A}} \boldsymbol{A}_\alpha \arrow[r] \arrow[d] & \boldsymbol{X}_\alpha \arrow[d] \\ \boldsymbol{B}_\alpha \arrow[r] \arrow[ur, dotted] & \boldsymbol{Y}_\alpha \times_{\mathrm{M}_\alpha \boldsymbol{Y}} \mathrm{M}_\alpha \boldsymbol{X} \end{tikzcd}$$

and there is a map $\boldsymbol{B}_\alpha \to \boldsymbol{X}_\alpha$ for every object α of degree n that makes both triangles commute if and only if h can be extended over the restriction of $\boldsymbol{B}$ to the n-filtration of $\mathcal{C}$ so that both triangles in Diagram 16.2.21 commute. This is the motivation for the definitions of the relative latching map and relative latching map, and their appearance in the definitions of *Reedy cofibration* and *Reedy fibration* (see Definition 16.3.2).

Definition 16.2.22. Let $\mathcal{C}$ be a Reedy category, let $\mathcal{M}$ be a model category, let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be $\mathcal{C}$-diagrams in $\mathcal{M}$, and let $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ be a map of $\mathcal{C}$-diagrams.

1. If α is an object in $\mathcal{C}$, then the *relative latching map of f at α* is the map $\mathrm{L}_\alpha \boldsymbol{Y} \amalg_{\mathrm{L}_\alpha \boldsymbol{X}} \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ (see Definition 16.2.17).
2. If α is an object in $\mathcal{C}$, then the *relative matching map of f at α* is the map $\boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha \times_{\mathrm{M}_\alpha \boldsymbol{Y}} \mathrm{M}_\alpha \boldsymbol{X}$.

16.3. The Reedy model category structure

If $\mathcal{C}$ is a Reedy category and $\mathcal{M}$ is a model category, we will define a model category structure on $\mathcal{M}^{\mathcal{C}}$, the category of $\mathcal{C}$-diagrams in $\mathcal{M}$, called the *Reedy model category structure*. If $\mathcal{M}$ is a *simplicial* model category, then we will show that the simplicial structure of Definition 14.3.1 and Definition 14.3.2 makes the Reedy model category structure on $\mathcal{M}^{\mathcal{C}}$ a simplicial model category.

If $\mathcal{M}$ is a cofibrantly generated model category, then the Reedy model category structure will have the same weak equivalences as the model category structure of Theorem 14.2.1, but it will have a larger class of cofibrations (see Proposition 16.4.1). Thus, free cell complexes and their retracts will be cofibrant in the Reedy model category structure, as will some diagrams that are not retracts of free cell complexes.

16.3.1. Statement of the theorem.

DEFINITION 16.3.2. Let $\mathcal{C}$ be a Reedy category, let $\mathcal{M}$ be a model category, and let $\boldsymbol{X}, \boldsymbol{Y}\colon \mathcal{C} \to \mathcal{M}$ be $\mathcal{C}$-diagrams in $\mathcal{M}$.

1. A map of diagrams $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a *Reedy weak equivalence* if, for every object α of $\mathcal{C}$, the map $f_\alpha\colon \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ is a weak equivalence in $\mathcal{M}$.
2. A map of diagrams $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a *Reedy cofibration* if, for every object α of $\mathcal{C}$, the relative latching map (see Definition 16.2.22)
$$\mathrm{L}_\alpha \boldsymbol{Y} \amalg_{\mathrm{L}_\alpha \boldsymbol{X}} \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$$
is a cofibration in $\mathcal{M}$.
3. A map of diagrams $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a *Reedy fibration* if, for every object α of $\mathcal{C}$, the relative matching map (see Definition 16.2.22)
$$\boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha \times_{\mathrm{M}_\alpha \boldsymbol{Y}} \mathrm{M}_\alpha \boldsymbol{X}$$
is a fibration in $\mathcal{M}$.

THEOREM 16.3.3. *Let $\mathcal{C}$ be a Reedy category and let $\mathcal{M}$ be a model category.*

1. *The category $\mathcal{M}^{\mathcal{C}}$ of $\mathcal{C}$-diagrams in $\mathcal{M}$ with the Reedy weak equivalences, Reedy cofibrations, and Reedy fibrations (see Definition 16.3.2) is a model category.*
2. *If $\mathcal{M}$ is a left proper, right proper, or proper model category (see Definition 11.1.1), then the model category of part 1 is, respectively, left proper, right proper, or proper.*
3. *If $\mathcal{M}$ is a simplicial model category (see Definition 10.1.2), then the model category of part 1 with the simplicial structure defined in Definition 14.3.1 and Definition 14.3.2, is a simplicial model category.*

The proof of Theorem 16.3.3 is in Section 16.3.11.

16.3.4. Trivial cofibrations and trivial fibrations. It is not obvious how to identify those maps of diagrams that are both Reedy cofibrations and Reedy weak equivalences, or those maps that are both Reedy fibrations and Reedy weak equivalences. In this section, we will show that f is both a Reedy cofibration and a Reedy weak equivalence if and only if each of the maps $\mathrm{L}_\alpha \boldsymbol{Y} \amalg_{\mathrm{L}_\alpha \boldsymbol{X}} \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ is a trivial cofibration in $\mathcal{M}$, and that f is both a Reedy fibration and a Reedy weak equivalence if and only if each of the maps $\boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha \times_{\mathrm{M}_\alpha \boldsymbol{Y}} \mathrm{M}_\alpha \boldsymbol{X}$ is a trivial fibration in $\mathcal{M}$ (see Theorem 16.3.10). We will use this theorem in Section 16.3.11 to prove Theorem 16.3.3.

LEMMA 16.3.5. *Let $\mathcal{C}$ be a Reedy category, let $\mathcal{M}$ be a model category, let $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ be a map of $\mathcal{C}$-diagrams in $\mathcal{M}$, let α be an object in $\mathcal{C}$, and let S be a class of maps in $\mathcal{M}$.*

1. *If for every object β in $\mathcal{C}$ whose degree is less than that of α the relative latching map*
$$\mathrm{L}_\beta \boldsymbol{Y} \amalg_{\mathrm{L}_\beta \boldsymbol{X}} \boldsymbol{X}_\beta \to \boldsymbol{Y}_\beta$$
has the left lifting property (see Definition 8.2.1) with respect to every element of S, then the induced map of latching objects $\mathrm{L}_\alpha \boldsymbol{X} \to \mathrm{L}_\alpha \boldsymbol{Y}$ has the left lifting property with respect to every element of S.

2. *If for every object β in $\mathcal{C}$ whose degree is less than that of α the relative matching map*
$$\boldsymbol{X}_\beta \to \boldsymbol{Y}_\beta \times_{\mathrm{M}_\beta \boldsymbol{Y}} \mathrm{M}_\beta \boldsymbol{X}$$
has the right lifting property (see Definition 8.2.1) with respect to every element of S, then the induced map of matching objects $\mathrm{M}_\alpha \boldsymbol{X} \to \mathrm{M}_\alpha \boldsymbol{Y}$ has the right lifting property with respect to every element of S.

PROOF. We will prove part 1; the proof of part 2 is dual. We assume that we have chosen a degree function for $\mathcal{C}$ (see Remark 16.2.3).

There is a filtration of the category $\big((\overrightarrow{\mathcal{C}}\downarrow\alpha) - 1_\alpha\big)$ in which $\mathrm{F}^k\big((\overrightarrow{\mathcal{C}}\downarrow\alpha) - 1_\alpha\big)$ is the full subcategory of $\big((\overrightarrow{\mathcal{C}}\downarrow\alpha) - 1_\alpha\big)$ whose objects are the maps $\beta \to \alpha$ in $\overrightarrow{\mathcal{C}}$ such that the degree of β is less than or equal to k. Thus, $\mathrm{F}^0\big((\overrightarrow{\mathcal{C}}\downarrow\alpha) - 1_\alpha\big)$ has no non-identity maps, and $\mathrm{F}^{\deg(\alpha)-1}\big((\overrightarrow{\mathcal{C}}\downarrow\alpha) - 1_\alpha\big) = \big((\overrightarrow{\mathcal{C}}\downarrow\alpha) - 1_\alpha\big)$. If $E \to B$ is an element of S and we have the solid arrow diagram
$$\begin{tikzcd} \mathrm{L}_\alpha \boldsymbol{X} \arrow[r] \arrow[d] & E \arrow[d] \\ \mathrm{L}_\alpha \boldsymbol{Y} \arrow[r] \arrow[ur, dotted, "h"] & B \end{tikzcd}$$
then we will define the map h by defining it inductively over $\operatorname{colim}_{\mathrm{F}^k((\overrightarrow{\mathcal{C}}\downarrow\alpha)-1_\alpha)} \boldsymbol{Y}$.

For objects $\beta \to \alpha$ of $(\overrightarrow{\mathcal{C}}\downarrow\alpha)$ such that β is of degree zero, the latching objects $\mathrm{L}_\beta \boldsymbol{X}$ and $\mathrm{L}_\beta \boldsymbol{Y}$ are the initial object of $\mathcal{M}$, and so the map $\boldsymbol{X}_\beta \to \boldsymbol{Y}_\beta$ equals the map $\mathrm{L}_\beta \boldsymbol{Y} \amalg_{\mathrm{L}_\beta \boldsymbol{X}} \boldsymbol{X}_\beta \to \boldsymbol{Y}_\beta$, which we have assumed has the left lifting property with respect to $E \to B$. Thus, there exists a dotted arrow h that makes both triangles commute in the diagram
$$\begin{tikzcd} \boldsymbol{X}_\beta \arrow[r] \arrow[d] & E \arrow[d] \\ \boldsymbol{Y}_\beta \arrow[r] \arrow[ur, dotted, "h"] & B \end{tikzcd}$$
Since $\mathrm{F}^0\big((\overrightarrow{\mathcal{C}}\downarrow\alpha) - 1_\alpha\big)$ has no non-identity maps, this defines h on $\mathrm{F}^0\big((\overrightarrow{\mathcal{C}}\downarrow\alpha) - 1_\alpha\big)$.

For the inductive step, we assume that $0 < k < \deg(\alpha)$ and that the map has been defined on $\operatorname{colim}_{\mathrm{F}^{k-1}((\overrightarrow{\mathcal{C}}\downarrow\alpha)-1_\alpha)} \boldsymbol{Y}$. Let $\beta \to \alpha$ be an object of $\big((\overrightarrow{\mathcal{C}}\downarrow\alpha) - 1_\alpha\big)$ such that β is of degree k. The map $\beta \to \alpha$ defines a functor $\big((\overrightarrow{\mathcal{C}}\downarrow\beta) - 1_\beta\big) \to \mathrm{F}^{k-1}\big((\overrightarrow{\mathcal{C}}\downarrow\alpha) - 1_\alpha\big)$ which, defines the map h on $\mathrm{L}_\beta \boldsymbol{Y}$. Thus, we have the commutative diagram
$$\begin{tikzcd} \mathrm{L}_\beta \boldsymbol{Y} \amalg_{\mathrm{L}_\beta \boldsymbol{X}} \boldsymbol{X}_\beta \arrow[r] \arrow[d] & E \arrow[d] \\ \boldsymbol{Y}_\beta \arrow[r] & B \end{tikzcd}$$
and the vertical map on the left is assumed to have the left lifting property with respect to $E \to B$. This implies that the map h can be defined on $\boldsymbol{Y}_\beta$, and the discussion in Section 16.2.20 explains why this can be done independently for the various objects of degree k. This completes the induction, and the proof. $\square$

LEMMA 16.3.6. *Let $\mathcal{C}$ be a Reedy category, let $\mathcal{M}$ be a model category, let $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ be a map of $\mathcal{C}$-diagrams in $\mathcal{M}$, and let S be a class of maps in $\mathcal{M}$.*

1. *If for every object α in $\mathcal{C}$ the relative latching map*
$$\mathrm{L}_\alpha \boldsymbol{Y} \amalg_{\mathrm{L}_\alpha \boldsymbol{X}} \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$$
has the left lifting property with respect to every element of S, then for every object α in $\mathcal{C}$ the map $f_\alpha\colon \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ has the left lifting property with respect to every element of S.
2. *If for every object α in $\mathcal{C}$ the relative matching map*
$$\boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha \times_{\mathrm{M}_\alpha \boldsymbol{Y}} \mathrm{M}_\alpha \boldsymbol{X}$$
has the right lifting property with respect to every element of S, then for every object α in $\mathcal{C}$ the map $f_\alpha\colon \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ has the right lifting property with respect to every element of S.

PROOF. We will prove part 1; the proof of part 2 is dual.

The relative latching map $\boldsymbol{X}_\alpha \to \mathrm{L}_\alpha \boldsymbol{Y} \amalg_{\mathrm{L}_\alpha \boldsymbol{X}} \boldsymbol{X}_\alpha$ is a pushout of the map $\mathrm{L}_\alpha \boldsymbol{X} \to \mathrm{L}_\alpha \boldsymbol{Y}$, and so Lemma 16.3.5 implies that it has the left lifting property with respect to every element of S. Since the composition of this map with the map $\mathrm{L}_\alpha \boldsymbol{Y} \amalg_{\mathrm{L}_\alpha \boldsymbol{X}} \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ is the map $f_\alpha\colon \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$, the proof is complete. □

PROPOSITION 16.3.7. *Let $\mathcal{C}$ be a Reedy category, let $\mathcal{M}$ be a model category, and let $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ be a map of $\mathcal{C}$-diagrams in $\mathcal{M}$.*

1. *If f is a Reedy cofibration, then for every object α in $\mathcal{C}$ both the map $f_\alpha\colon \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ and the induced map of latching objects $\mathrm{L}_\alpha \boldsymbol{X} \to \mathrm{L}_\alpha \boldsymbol{Y}$ are cofibrations in $\mathcal{M}$.*
2. *If f is a Reedy fibration, then for every object α in $\mathcal{C}$ both the map $f_\alpha\colon \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ and the induced map of matching objects $\mathrm{M}_\alpha \boldsymbol{X} \to \mathrm{M}_\alpha \boldsymbol{Y}$ are fibrations in $\mathcal{M}$.*

PROOF. This follows from Lemma 16.3.5, Lemma 16.3.6, and Proposition 8.2.3. □

PROPOSITION 16.3.8. *Let $\mathcal{C}$ be a Reedy category, let $\mathcal{M}$ be a model category, and let $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ be a map of $\mathcal{C}$-diagrams in $\mathcal{M}$.*

1. *If for every object α of $\mathcal{C}$ the relative latching map $\mathrm{L}_\alpha \boldsymbol{Y} \amalg_{\mathrm{L}_\alpha \boldsymbol{X}} \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ is a trivial cofibration, then for every object α in $\mathcal{C}$ both the map $f_\alpha\colon \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ and the induced map of latching objects $\mathrm{L}_\alpha \boldsymbol{X} \to \mathrm{L}_\alpha \boldsymbol{Y}$ are trivial cofibrations.*
2. *If for every object α in $\mathcal{C}$ the relative matching map $\boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha \times_{\mathrm{M}_\alpha \boldsymbol{Y}} \mathrm{M}_\alpha \boldsymbol{X}$ is a trivial fibration, then for every object α in $\mathcal{C}$ both the map $f_\alpha\colon \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ and the induced map of matching objects $\mathrm{M}_\alpha \boldsymbol{X} \to \mathrm{M}_\alpha \boldsymbol{Y}$ are trivial fibrations.*

PROOF. This follows from Lemma 16.3.5, Lemma 16.3.6, and Proposition 8.2.3. □

PROPOSITION 16.3.9. *Let $\mathcal{C}$ be a Reedy category, let $\mathcal{M}$ be a model category, and let $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ be a map of $\mathcal{C}$-diagrams in $\mathcal{M}$.*

1. *If f is both a Reedy cofibration and a Reedy weak equivalence, then for every object α in $\mathcal{C}$ the map $f_\alpha\colon \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$, the induced map of latching*

objects $\mathrm{L}_\alpha\boldsymbol{X} \to \mathrm{L}_\alpha\boldsymbol{Y}$*, and the relative latching map* $\mathrm{L}_\alpha\boldsymbol{Y} \amalg_{\mathrm{L}_\alpha\boldsymbol{X}} \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ *are trivial cofibrations.*

2. *If* f *is both a Reedy fibration and a Reedy weak equivalence, then for every object* α *in* $\mathcal{C}$ *the map* $f_\alpha\colon \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$*, the induced map of matching objects* $\mathrm{M}_\alpha\boldsymbol{X} \to \mathrm{M}_\alpha\boldsymbol{Y}$*, and the relative matching map* $\boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha \times_{\mathrm{M}_\alpha\boldsymbol{Y}} \mathrm{M}_\alpha\boldsymbol{X}$ *are trivial fibrations.*

PROOF. We will prove part 1; the proof of part 2 is dual. We assume that we have chosen a degree function for $\mathcal{C}$ (see Remark 16.2.3).

Proposition 16.3.7 implies that f_α is a cofibration for every object α in $\mathcal{C}$. Since f is a Reedy weak equivalence, this implies that f_α is a trivial cofibration for every object α in $\mathcal{C}$.

We will prove that the maps $\mathrm{L}_\alpha\boldsymbol{X} \to \mathrm{L}_\alpha\boldsymbol{Y}$ and $\mathrm{L}_\alpha\boldsymbol{Y} \amalg_{\mathrm{L}_\alpha\boldsymbol{X}} \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ are trivial cofibrations for every object α of $\mathcal{C}$ by induction on the degree of α. If $\mathrm{L}_\alpha\boldsymbol{X} \to \mathrm{L}_\alpha\boldsymbol{Y}$ is a trivial cofibration in $\mathcal{M}$ for some particular object α of $\mathcal{C}$, then, since $\boldsymbol{X}_\alpha \to \mathrm{L}_\alpha\boldsymbol{Y} \amalg_{\mathrm{L}_\alpha\boldsymbol{X}} \boldsymbol{X}_\alpha$ is a pushout of $\mathrm{L}_\alpha\boldsymbol{X} \to \mathrm{L}_\alpha\boldsymbol{Y}$, this map is also a trivial cofibration. Since the weak equivalence $f_\alpha\colon \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ equals the composition $\boldsymbol{X}_\alpha \to \mathrm{L}_\alpha\boldsymbol{Y} \amalg_{\mathrm{L}_\alpha\boldsymbol{X}} \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$, this implies that the cofibration $\mathrm{L}_\alpha\boldsymbol{Y} \amalg_{\mathrm{L}_\alpha\boldsymbol{X}} \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ is actually a trivial cofibration.

If α is of degree 0, then $\mathrm{L}_\alpha\boldsymbol{X}$ and $\mathrm{L}_\alpha\boldsymbol{Y}$ are both the initial object of $\mathcal{M}$, and so $\mathrm{L}_\alpha\boldsymbol{X} \to \mathrm{L}_\alpha\boldsymbol{Y}$ is the identity map, which is certainly a trivial cofibration.

We now assume that n is a positive integer, $\mathrm{L}_\beta\boldsymbol{X} \to \mathrm{L}_\beta\boldsymbol{Y}$ is a trivial cofibration for all objects β of degree less than n, and α is an object of degree n. The discussion above explains why our inductive hypothesis implies that $\mathrm{L}_\beta\boldsymbol{Y} \amalg_{\mathrm{L}_\beta\boldsymbol{X}} \boldsymbol{X}_\beta \to \boldsymbol{Y}_\beta$ is a trivial cofibration for all objects β of degree less than n, and so Lemma 16.3.5 and Proposition 8.2.3 imply that $\mathrm{L}_\alpha\boldsymbol{X} \to \mathrm{L}_\alpha\boldsymbol{Y}$ is a trivial cofibration. □

THEOREM 16.3.10. *Let* $\mathcal{C}$ *be a Reedy category, let* $\mathcal{M}$ *be a model category, and let* $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ *be a map of* $\mathcal{C}$*-diagrams in* $\mathcal{M}$*.*

1. *The map* f *is both a Reedy cofibration and a Reedy weak equivalence if and only if for every object* α *in* $\mathcal{C}$ *the relative latching map* $\mathrm{L}_\alpha\boldsymbol{Y} \amalg_{\mathrm{L}_\alpha\boldsymbol{X}} \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ *is a trivial cofibration in* $\mathcal{M}$*.*
2. *The map* f *is both a Reedy fibration and a Reedy weak equivalence if and only if for every object* α *in* $\mathcal{C}$ *the relative matching map* $\boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha \times_{\mathrm{M}_\alpha\boldsymbol{Y}} \mathrm{M}_\alpha\boldsymbol{X}$ *is a trivial fibration in* $\mathcal{M}$*.*

PROOF. This follows from Proposition 16.3.8 and Proposition 16.3.9. □

16.3.11. Proof of Theorem 16.3.3. For part 1, we must show that axioms M1 through M5 of Definition 8.1.2 are satisfied. Axioms M1 and M2 follow from the fact that limits, colimits, and weak equivalences of diagrams are all defined objectwise.

Axiom M3 follows from the observation that if the map $g\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a retract of the map $h\colon \boldsymbol{W} \to \boldsymbol{Z}$, then, for each object α of $\mathcal{C}$, the relative latching map $\mathrm{L}_\alpha\boldsymbol{Y} \amalg_{\mathrm{L}_\alpha\boldsymbol{X}} \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ is a retract of the relative latching map $\mathrm{L}_\alpha\boldsymbol{Z} \amalg_{\mathrm{L}_\alpha\boldsymbol{W}} \boldsymbol{W}_\alpha \to \boldsymbol{Z}_\alpha$ and the relative matching map $\boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha \times_{\mathrm{M}_\alpha\boldsymbol{Y}} \mathrm{M}_\alpha\boldsymbol{X}$ is a retract of the relative matching map $\boldsymbol{W}_\alpha \to \boldsymbol{Z}_\alpha \times_{\mathrm{M}_\alpha\boldsymbol{Z}} \mathrm{M}_\alpha\boldsymbol{W}$.

If we choose a degree function for $\mathcal{C}$ (see Remark 16.2.3), then the maps required by axiom M4 are constructed inductively on the degree of the objects of $\mathcal{C}$, using Theorem 16.3.10 (see the discussion in Section 16.2.20).

The factorizations required by axiom M5 are also constructed inductively on the degree of the objects of $\mathcal{C}$. For axiom M5 part 1, if $g\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a map in $\mathcal{M}^{\mathcal{C}}$, then, for every object α of degree zero of $\mathcal{C}$, we have a functorial factorization of g_α in $\mathcal{M}$ as $\boldsymbol{X}_\alpha \xrightarrow{i} \boldsymbol{Z}_\alpha \xrightarrow{h} \boldsymbol{Y}_\alpha$ with i a cofibration and h a trivial fibration. If we now assume that g has been factored on all objects of degree less than n and that α is an object of degree n, then we have the induced map $\mathrm{L}_\alpha \boldsymbol{Z} \amalg_{\mathrm{L}_\alpha \boldsymbol{X}} \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha \times_{\mathrm{M}_\alpha \boldsymbol{Y}} \mathrm{M}_\alpha \boldsymbol{Z}$. We can now factor this map (functorially) in $\mathcal{M}$ as

$$\mathrm{L}_\alpha \boldsymbol{Z} \amalg_{\mathrm{L}_\alpha \boldsymbol{X}} \boldsymbol{X}_\alpha \xrightarrow{i} \boldsymbol{Z}_\alpha \xrightarrow{h} \boldsymbol{Y}_\alpha \times_{\mathrm{M}_\alpha \boldsymbol{Y}} \mathrm{M}_\alpha \boldsymbol{Z}$$

with i a cofibration and h a trivial fibration to obtain $\boldsymbol{Z}_\alpha$. This completes the construction, and Theorem 16.3.10 implies that it has the required properties. The proof for axiom M5 part 2 is similar, and so $\mathcal{M}^{\mathcal{C}}$ is a model category, and the proof of part 1 is complete.

For part 2, Proposition 16.3.7 implies that a Reedy cofibration is an objectwise cofibration and a Reedy fibration is an objectwise fibration. Since weak equivalences are defined objectwise and both pushouts and pullbacks are constructed objectwise, the conditions of Definition 11.1.1 follow if they hold in $\mathcal{M}$.

For part 3, if $\mathcal{M}$ is a simplicial model category, then axiom M6 of Definition 10.1.2 follows because the constructions are all done objectwise and $\mathcal{M}$ is a simplicial model category, and so it remains only to show that axiom M7 follows as well. Proposition 10.1.8 implies that it is sufficient to show that if $i\colon \boldsymbol{A} \to \boldsymbol{B}$ is a Reedy cofibration and $j\colon K \to L$ is a cofibration of simplicial sets, then $\boldsymbol{A} \otimes L \amalg_{\boldsymbol{A} \otimes K} \boldsymbol{B} \otimes K \to \boldsymbol{B} \otimes L$ is a Reedy cofibration that is also a weak equivalence if either i or j is a weak equivalence. Thus, we must show that, for every object α of $\mathcal{C}$, the map

$$\mathrm{L}_\alpha(\boldsymbol{B} \otimes L) \amalg_{\mathrm{L}_\alpha(\boldsymbol{A} \otimes L \amalg_{\boldsymbol{A} \otimes K} \boldsymbol{B} \otimes K)} (\boldsymbol{A} \otimes L \amalg_{\boldsymbol{A} \otimes K} \boldsymbol{B} \otimes K)_\alpha \to (\boldsymbol{B} \otimes L)_\alpha$$

is a cofibration in $\mathcal{M}$ that is also a weak equivalence if either i or j is a weak equivalence. Since each latching object is a colimit, Lemma 10.2.3 implies that this map is isomorphic to the map

$$\big((\mathrm{L}_\alpha \boldsymbol{B} \amalg_{\mathrm{L}_\alpha \boldsymbol{A}} \boldsymbol{A}_\alpha) \otimes L\big) \amalg_{(\mathrm{L}_\alpha \boldsymbol{B} \amalg_{\mathrm{L}_\alpha \boldsymbol{A}} \boldsymbol{A}_\alpha) \otimes K} \boldsymbol{B}_\alpha \otimes K \to \boldsymbol{B}_\alpha \otimes L.$$

Since $i\colon A \to B$ is a Reedy cofibration and $\mathcal{M}$ is a simplicial model category, this map is a cofibration that is a weak equivalence if either i or j is a weak equivalence, and so the proof is complete.

16.4. Reedy cofibrant diagrams

PROPOSITION 16.4.1. *Let $\mathcal{C}$ be a Reedy category, let $\mathcal{M}$ be a cofibrantly generated model category (see Definition* 13.2.1*), and let $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{M}^{\mathcal{C}}$ be $\mathcal{C}$-diagrams in $\mathcal{M}$.*

1. *If the map $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a Reedy fibration (see Definition* 16.3.2*), then it is also a fibration in the cofibrantly generated model category structure on $\mathcal{M}^{\mathcal{C}}$ (see Theorem* 14.2.1*).*
2. *If the map $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a cofibration in the cofibrantly generated model category structure on $\mathcal{M}^{\mathcal{C}}$, then it is a Reedy cofibration.*

PROOF. Part 1 follows from Proposition 16.3.7.

Part 2 follows from part 1 and Proposition 8.2.3, since the weak equivalences are the same in both model category structures. $\square$

COROLLARY 16.4.2. *If $\mathcal{C}$ is a Reedy category, $\mathcal{M}$ is a cofibrantly generated model category, $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{M}^{\mathcal{C}}$ are $\mathcal{C}$-diagrams in $\mathcal{M}$, and $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a relative free cell complex (see Definition 14.1.28), then f is a Reedy cofibration.*

PROOF. This follows from Theorem 14.2.1 and Proposition 16.4.1. □

COROLLARY 16.4.3. *If $\mathcal{C}$ is a Reedy category, $\mathcal{M}$ is a cofibrantly generated model category, and $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a free cell complex (see Definition 14.1.28), then $\boldsymbol{X}$ is Reedy cofibrant.*

PROOF. This follows from Corollary 16.4.2. □

COROLLARY 16.4.4. *If $\mathcal{C}$ is a Reedy category, then the $\mathcal{C}^{\mathrm{op}}$-diagram $\mathrm{B}(-\downarrow \mathcal{C})^{\mathrm{op}}$ and the $\mathcal{C}$-diagram $\mathrm{B}(\mathcal{C}\downarrow -)$ (see Section 14.5) are both Reedy cofibrant.*

PROOF. This follows from Corollary 14.6.8 and Corollary 16.4.3. □

LEMMA 16.4.5. *Let $\mathcal{C}$ be a Reedy category, $\boldsymbol{X} \in \mathsf{Spc}_{(*)}^{\mathcal{C}}$ a $\mathcal{C}$-diagram of spaces, and Y an object of $\mathsf{Spc}_{(*)}$. If $\boldsymbol{X}$ is Reedy cofibrant in $\mathsf{Spc}_{(*)}^{\mathcal{C}}$ and Y is fibrant in $\mathsf{Spc}_{(*)}$, then $Y^{\boldsymbol{X}}$ is Reedy fibrant in $\mathsf{Spc}_{(*)}^{\mathcal{C}^{\mathrm{op}}}$ and $\mathrm{Map}(\boldsymbol{X}, Y)$ is Reedy fibrant in $\mathsf{SS}^{\mathcal{C}^{\mathrm{op}}}$ (see Proposition 16.2.6).*

PROOF. If α is an object of $\mathcal{C}$ and $\mathrm{L}_\alpha \boldsymbol{X}$ is the latching object of $\boldsymbol{X}$ at α (see Definition 16.2.17), then Proposition 16.2.15 implies that

$$Y^{\mathrm{L}_\alpha \boldsymbol{X}} = Y^{\mathrm{colim}_{((\overrightarrow{\mathcal{C}}\downarrow\alpha)-1_\alpha)} \boldsymbol{X}} = \lim_{((\overrightarrow{\mathcal{C}}\downarrow\alpha)-1_\alpha)^{\mathrm{op}}} Y^{\boldsymbol{X}} = \lim_{((\alpha\downarrow\overleftarrow{\mathcal{C}^{\mathrm{op}}})-1_\alpha)} Y^{\boldsymbol{X}} = \mathrm{M}_\alpha\big(Y^{\boldsymbol{X}}\big),$$

i.e., that $Y^{\mathrm{L}_\alpha \boldsymbol{X}}$ is the matching object at α of the $\mathcal{C}^{\mathrm{op}}$-diagram $Y^{\boldsymbol{X}}$. Since the latching map $\mathrm{L}_\alpha \boldsymbol{X} \to \boldsymbol{X}_\alpha$ is a cofibration and Y is fibrant, this implies that the matching map $Y^{\boldsymbol{X}_\alpha} \to \mathrm{M}_\alpha\big(Y^{\boldsymbol{X}}\big)$ is a fibration, and so $Y^{\boldsymbol{X}}$ is a Reedy fibrant $\mathcal{C}^{\mathrm{op}}$-diagram. Since the total singular complex functor is a right adjoint, it commutes with limits, and so Proposition 1.1.7 now implies that the matching map $\mathrm{Map}(\boldsymbol{X}_\alpha, Y) \to \mathrm{M}_\alpha\big(\mathrm{Map}(\boldsymbol{X}, Y)\big)$ is also a fibration, and so $\mathrm{Map}(\boldsymbol{X}, Y)$ is a Reedy fibrant $\mathcal{C}^{\mathrm{op}}$-diagram, and the proof is complete. □

PROPOSITION 16.4.6. *A simplicial space is Reedy cofibrant if, for every integer $n \geq 0$, the map from the colimit of the diagram of lower degree spaces and degeneracy maps to the nth space is a cofibration.*

PROOF. This follows from Definition 16.3.2. □

COROLLARY 16.4.7. *A simplicial object in $\mathsf{SS}_{(*)}$ is always Reedy cofibrant.*

PROOF. The latching map (see Definition 16.2.17) of a simplicial object is always an inclusion, and an inclusion in $\mathsf{SS}_{(*)}$ is a cofibration. Thus, the result follows from Proposition 16.4.6. □

PROPOSITION 16.4.8. *Let $\mathcal{C}$ be a Reedy category, let $\boldsymbol{X}, \boldsymbol{Y}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ be $\mathcal{C}$-diagrams of spaces, and let $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ be a map of $\mathcal{C}$-diagrams. If the restriction of f to the direct subcategory (see Definition 16.2.2) of $\mathcal{C}$ is a relative free cell complex (see Definition 14.1.28), then f is a Reedy cofibration.*

PROOF. The hypotheses imply that, for each object α of $\mathcal{C}$, the map $\mathrm{L}_\alpha \boldsymbol{Y} \amalg_{\mathrm{L}_\alpha \boldsymbol{X}} \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ is a relative cell complex. □

PROPOSITION 16.4.9. *A cosimplicial space is Reedy cofibrant if and only if for every integer $n \geq 0$ the map from the colimit of the diagram of lower degree spaces and coface maps to the nth space is a cofibration.*

PROOF. This follows from Definition 16.3.2. □

COROLLARY 16.4.10. *The cosimplicial standard simplex (see Definition 16.1.9) is Reedy cofibrant.*

PROOF. Each of the latching maps (see Definition 16.2.17) is the inclusion of the boundary of a simplex into that simplex. □

16.5. Bisimplicial sets

DEFINITION 16.5.1. Let $\mathcal{C}$ be a small category. If $\mathrm{F}\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{SS}$ and $\mathrm{G}\colon \mathcal{C} \to \mathsf{SS}$ are diagrams of simplicial sets, then the *tensor product* $\mathrm{F} \otimes_{\mathcal{C}} \mathrm{G}$ of F and G is the simplicial set that is the coequalizer of the diagram

$$\coprod_{(\sigma\colon \alpha \to \alpha') \in \mathcal{C}} \mathrm{F}(\alpha') \times \mathrm{F}(\alpha) \underset{\psi}{\overset{\phi}{\rightrightarrows}} \coprod_{\alpha \in \mathrm{Ob}(\mathcal{C})} \mathrm{F}(\alpha) \times \mathrm{G}(\alpha)$$

where the map ϕ on the summand $\sigma\colon \alpha \to \alpha'$ is $\mathrm{F}(1_{\alpha'}) \times \mathrm{G}(\sigma)\colon \mathrm{F}(\alpha') \times \mathrm{G}(\alpha) \to \mathrm{F}(\alpha') \times \mathrm{G}(\alpha')$ and the map ψ on the summand $\sigma\colon \alpha \to \alpha'$ is $\mathrm{F}(\sigma) \times \mathrm{G}(1_{\alpha})\colon \mathrm{F}(\alpha') \times \mathrm{G}(\alpha) \to \mathrm{F}(\alpha) \times \mathrm{G}(\alpha)$.

REMARK 16.5.2. The tensor product of functors (see Definition 16.5.1) is a special case of a coend of a functor $\mathrm{H}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{SS}$, where $\mathrm{H}(K, L) = K \times L$ (see Remark 19.2.4). We use the name "tensor product" because of the similarity to the case in which a ring R is viewed as an additive category (with one object, and with morphisms equal to the elements of R). In this case, a left R-module is just an additive functor $\mathrm{G}\colon R \to \mathcal{A}$ from R to the category of abelian groups, and a right R-module is an additive functor $\mathrm{F}\colon R^{\mathrm{op}} \to \mathcal{A}$. If $\mathrm{H}\colon R^{\mathrm{op}} \times R \to \mathcal{A}$ is defined by $\mathrm{H}(\alpha, \alpha) = \mathrm{F}(\alpha) \otimes \mathrm{G}(\alpha)$, then $\mathrm{F} \otimes_R \mathrm{G}$ is the usual tensor product of a right R-module with a left R-module.

DEFINITION 16.5.3. If $\boldsymbol{X}$ is a bisimplicial set, i.e., an object of $\mathsf{SS}^{\boldsymbol{\Delta}^{\mathrm{op}}}$, then the *realization* of $\boldsymbol{X}$ is the simplicial set $|\boldsymbol{X}| = \boldsymbol{X} \otimes_{\boldsymbol{\Delta}} \Delta$ (see Definition 16.5.1 and Definition 16.1.9).

THEOREM 16.5.4. *If $\boldsymbol{X}$ is a bisimplicial set, then the realization of $\boldsymbol{X}$ is naturally isomorphic to the diagonal simplicial set of $\boldsymbol{X}$.*

PROOF. See, e.g., [**49**, page 94]. □

THEOREM 16.5.5 (A. K. Bousfield and E. M. Friedlander, [**14**]). *If $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a map of bisimplicial sets such that*

1. *as a map of horizontal simplicial objects in the category of simplicial sets (i.e., $(\boldsymbol{X}_n)_k = \boldsymbol{X}_{n,k}$), f is a Reedy fibration, and*
2. *as a map of vertical simplicial objects in the category of simplicial sets (i.e., $(\boldsymbol{X}_n)_k = \boldsymbol{X}_{k,n}$), f is an objectwise fibration (i.e., every induced map $\boldsymbol{X}_{*,n} \to \boldsymbol{Y}_{*,n}$ is a fibration of simplicial sets),*

then the induced map of diagonals $\operatorname{diag} f\colon \operatorname{diag} \boldsymbol{X} \to \operatorname{diag} \boldsymbol{Y}$ *is a fibration of simplicial sets.*

PROOF. This is [**14**, Lemma B.9]. □

DEFINITION 16.5.6. If $\boldsymbol{X}$ is a bisimplicial set, i.e., an object of $\mathsf{SS}^{\boldsymbol{\Delta}^{\mathrm{op}}}$, and Y is a simplicial set, then $\mathrm{Map}(\boldsymbol{X}, Y)$ is the cosimplicial simplicial set given by $\mathrm{Map}(\boldsymbol{X}, Y)^n = \mathrm{Map}(\boldsymbol{X}_n, Y)$, with coface and codegeneracy maps induced by the face and degeneracy maps of $\boldsymbol{X}$.

THEOREM 16.5.7. *If $\boldsymbol{X}\colon \boldsymbol{\Delta}^{\mathrm{op}} \to \mathsf{SS}$ is a bisimplicial set, $\boldsymbol{Y}\colon \boldsymbol{\Delta} \to \mathsf{SS}$ is a cosimplicial simplicial set, and Z is a simplicial set, then there is a natural isomorphism of simplicial sets*

$$\mathrm{Map}(\boldsymbol{X} \otimes_{\boldsymbol{\Delta}} \boldsymbol{Y}, Z) \approx \mathrm{Map}\bigl(\boldsymbol{Y}, \mathrm{Map}(\boldsymbol{X}, Z)\bigr).$$

PROOF. We have the coequalizer diagram of simplicial sets

$$\coprod_{(\sigma\colon [n]\to[m])\in\boldsymbol{\Delta}} \boldsymbol{X}_m \times \boldsymbol{Y}^n \underset{\psi}{\overset{\phi}{\rightrightarrows}} \coprod_{n\geq 0} \boldsymbol{X}_n \times \boldsymbol{Y}^n \quad\to\quad \boldsymbol{X} \otimes_{\boldsymbol{\Delta}} \boldsymbol{Y}.$$

Since the functor $-\times \Delta[k]\colon \mathsf{SS} \to \mathsf{SS}$ is a left adjoint, the diagram

$$\coprod_{(\sigma\colon [n]\to[m])\in\boldsymbol{\Delta}} \boldsymbol{X}_m \times \boldsymbol{Y}^n \times \Delta[k] \quad\rightrightarrows\quad \coprod_{n\geq 0} \boldsymbol{X}_n \times \boldsymbol{Y}^n \times \Delta[k] \to (\boldsymbol{X} \otimes_{\boldsymbol{\Delta}} \boldsymbol{Y}) \times \Delta[k]$$

is also a coequalizer diagram, and so we have the equalizer diagram

$$\begin{aligned}\mathsf{SS}\bigl((\boldsymbol{X} \otimes_{\boldsymbol{\Delta}} \boldsymbol{Y}) \times \Delta[k], Z\bigr) \to \prod_{n\geq 0} \mathsf{SS}(\boldsymbol{X}_n \times \boldsymbol{Y}^n \times \Delta[k], Z)&\\ \rightrightarrows \prod_{(\sigma\colon [n]\to[m])\in\boldsymbol{\Delta}} \mathsf{SS}(\boldsymbol{X}_m \times \boldsymbol{Y}^n \times \Delta[k], Z)&\end{aligned}$$

which is isomorphic to the diagram

$$\begin{aligned}\mathsf{SS}\bigl((\boldsymbol{X} \otimes_{\boldsymbol{\Delta}} \boldsymbol{Y}) \times \Delta[k], Z\bigr) \to \prod_{n\geq 0} \mathsf{SS}\bigl(\boldsymbol{Y}^n \times \Delta[k], \mathrm{Map}(\boldsymbol{X}_n, Z)\bigr)&\\ \rightrightarrows \prod_{(\sigma\colon [n]\to[m])\in\boldsymbol{\Delta}} \mathsf{SS}\bigl(\boldsymbol{Y}^n \times \Delta[k], \mathrm{Map}(\boldsymbol{X}_m, Z)\bigr).&\end{aligned}$$

This implies that the diagram

$$\begin{aligned}\mathrm{Map}(\boldsymbol{X} \otimes_{\boldsymbol{\Delta}} \boldsymbol{Y}, Z) \to \prod_{n\geq 0} \mathrm{Map}\bigl(\boldsymbol{Y}^n, \mathrm{Map}(\boldsymbol{X}_n, Z)\bigr)&\\ \rightrightarrows \prod_{(\sigma\colon [n]\to[m])\in\boldsymbol{\Delta}} \mathrm{Map}\bigl(\boldsymbol{Y}^n, \mathrm{Map}(\boldsymbol{X}_m, Z)\bigr)&\end{aligned}$$

is an equalizer diagram, from which the result follows. □

THEOREM 16.5.8. *If $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a map of bisimplicial sets, such that $f_n\colon \boldsymbol{X}_n \to \boldsymbol{Y}_n$ is a weak equivalence of simplicial sets for every $n \geq 0$, then the induced map of realizations $|f|\colon |\boldsymbol{X}| \to |\boldsymbol{Y}|$ is a weak equivalence of simplicial sets.*

PROOF. It is sufficient to show that if Z is a fibrant simplicial set, the the induced map $|f|^*\colon \mathrm{Map}\bigl(|\boldsymbol{Y}|, Z\bigr) \to \mathrm{Map}\bigl(|\boldsymbol{X}|, Z\bigr)$ is a weak equivalence (see Corollary 10.5.5).

Corollary 16.4.7 implies that $\boldsymbol{X}$ and $\boldsymbol{Y}$ are Reedy cofibrant. Since Z is fibrant, Lemma 16.4.5 implies that the map $\mathrm{Map}(\boldsymbol{Y}, Z) \to \mathrm{Map}(\boldsymbol{X}, Z)$ is a map of Reedy fibrant cosimplicial simplicial sets, and Corollary 10.2.2 implies that it is a Reedy weak equivalence of cosimplicial simplicial sets. Since Δ (see Definition 16.1.9) is a cofibrant cosimplicial simplicial set (see Corollary 16.4.10), the map $\mathrm{Map}\bigl(\Delta, \mathrm{Map}(\boldsymbol{Y}, Z)\bigr) \to \mathrm{Map}\bigl(\Delta, \mathrm{Map}(\boldsymbol{X}, Z)\bigr)$ is a weak equivalence of simplicial sets (see Corollary 10.2.2 and Theorem 16.3.3). This is isomorphic to the map $\mathrm{Map}(\boldsymbol{Y} \otimes_{\boldsymbol{\Delta}} \Delta, Z) \to \mathrm{Map}(\boldsymbol{X} \otimes_{\boldsymbol{\Delta}} \Delta, Z)$ (see Theorem 16.5.7), which is the definition of the map $\mathrm{Map}\bigl(|\boldsymbol{Y}|, Z\bigr) \to \mathrm{Map}\bigl(|\boldsymbol{X}|, Z\bigr)$ (see Definition 16.5.3). $\square$

COROLLARY 16.5.9. *If $\boldsymbol{X}\colon \boldsymbol{\Delta}^{\mathrm{op}} \to \mathsf{SS}$ is a bisimplicial set such that the natural map $\mathrm{s}(\boldsymbol{X}_0) \to \boldsymbol{X}$ from the constant simplicial simplicial set to $\boldsymbol{X}$ is a weak equivalence, then the natural map $\boldsymbol{X}_0 \to |\boldsymbol{X}|$ is a weak equivalence.*

PROOF. This follows from Theorem 16.5.8. $\square$

16.6. Quillen functors

PROPOSITION 16.6.1. *Let $\mathcal{C}$ be a Reedy category and let $\mathcal{M}$ and $\mathcal{N}$ be model categories.*

1. *If $\mathrm{F}\colon \mathcal{M} \rightleftarrows \mathcal{N} :\mathrm{U}$ is a Quillen pair (see Definition 9.8.1), then the induced functors $\mathrm{F}^{\mathcal{C}}\colon \mathcal{M}^{\mathcal{C}} \rightleftarrows \mathcal{N}^{\mathcal{C}} :\mathrm{U}^{\mathcal{C}}$ form a Quillen pair.*
2. *If (F, U) is a pair of Quillen equivalences, then so is the induced pair $(\mathrm{F}^{\mathcal{C}}, \mathrm{U}^{\mathcal{C}})$.*

PROOF. The induced functors $\mathrm{F}^{\mathcal{C}}$ and $\mathrm{U}^{\mathcal{C}}$ are adjoint (see, e.g., [**7**, page 107]), and so for part 1 is is sufficient to show that $\mathrm{F}^{\mathcal{C}}$ preserves both cofibrations and trivial cofibrations (see Proposition 9.8.2). If $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a cofibration or a trivial cofibration in $\mathcal{M}^{\mathcal{C}}$, then for every object α in $\mathcal{C}$ the relative latching map $\mathrm{L}_\alpha \boldsymbol{B} \amalg_{\mathrm{L}_\alpha \boldsymbol{A}} \boldsymbol{A}_\alpha \to \boldsymbol{B}_\alpha$ is, respectively, a cofibration or a trivial cofibration in $\mathcal{M}$ (see Theorem 16.3.10). Since the latching objects $\mathrm{L}_\alpha \boldsymbol{A}$ and $\mathrm{L}_\alpha \boldsymbol{B}$ are defined as colimits (see Definition 16.2.17) and left adjoints commute with colimits, the relative latching map $\mathrm{L}_\alpha \mathrm{F}\boldsymbol{B} \amalg_{\mathrm{L}_\alpha \mathrm{F}\boldsymbol{A}} \mathrm{F}\boldsymbol{A}_\alpha \to \mathrm{F}\boldsymbol{B}_\alpha$ is isomorphic to the map $\mathrm{F}(\mathrm{L}_\alpha \boldsymbol{B} \amalg_{\mathrm{L}_\alpha \boldsymbol{A}} \boldsymbol{A}_\alpha) \to \mathrm{F}\boldsymbol{B}_\alpha$, and is thus, respectively, a cofibration or a trivial cofibration in $\mathcal{N}$. Thus, $\mathrm{F}^{\mathcal{C}}$ is a left Quillen functor. Part 2 follows immediately, since weak equivalences in $\mathcal{M}^{\mathcal{C}}$ and $\mathcal{N}^{\mathcal{C}}$ are defined objectwise in $\mathcal{C}$. $\square$

COROLLARY 16.6.2. *Let $\mathcal{C}$ be a Reedy category, let $\mathcal{M}$ and $\mathcal{N}$ be model categories, and let $\mathrm{F}\colon \mathcal{M} \rightleftarrows \mathcal{N} :\mathrm{U}$ be a Quillen pair.*

1. *If $\boldsymbol{B}\colon \mathcal{C} \to \mathcal{M}$ is a cofibrant $\mathcal{C}$-diagram in $\mathcal{M}$, then $\mathrm{F}\boldsymbol{B}\colon \mathcal{C} \to \mathcal{N}$ is a cofibrant $\mathcal{C}$-diagram in $\mathcal{N}$.*
2. *If $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{N}$ is a fibrant $\mathcal{C}$-diagram in $\mathcal{N}$, then $\mathrm{U}\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a fibrant $\mathcal{C}$-diagram in $\mathcal{M}$.*

PROOF. This follows from Proposition 16.6.1. $\square$

CHAPTER 17

Homotopy function complexes

If $\mathcal{C}$ is a category and $\mathcal{W}$ is a subcategory of $\mathcal{C}$ (the maps of which we call "weak equivalences"), then W. G. Dwyer and D. M. Kan define the *simplicial localization* $\mathrm{sL}_{\mathcal{W}}\mathcal{C}$ of $\mathcal{C}$ with respect to $\mathcal{W}$ to be the derived functor of localization of $\mathcal{C}$ with respect to $\mathcal{W}$ (see [**29, 27, 28**]). Thus, $\mathrm{sL}_{\mathcal{W}}\mathcal{C}$ is a *simplicial category*, i.e., a category enriched over simplicial sets, and they show that for every pair of objects (X, Y) in $\mathcal{C}$ the set $\pi_0\mathrm{sL}_{\mathcal{W}}\mathcal{C}(X, Y)$ of components of the simplicial set $\mathrm{sL}_{\mathcal{W}}\mathcal{C}(X, Y)$ is isomorphic to the set of maps from X to Y in the localization of $\mathcal{C}$ with respect to $\mathcal{W}$, i.e., the set of maps from X to Y in the homotopy category of $\mathcal{C}$ (see Definition 9.6.2). They also show that if $\mathcal{M}$ is a simplicial model category and $\mathcal{W}$ is its subcategory of weak equivalences, then when X is cofibrant and Y is fibrant the simplicial set $\mathrm{Map}(X, Y)$ that is part of the simplicial structure of $\mathcal{M}$ is naturally weakly equivalent to $\mathrm{sL}_{\mathcal{W}}\mathcal{M}(X, Y)$. Since a weak equivalence $Y \to Z$ in $\mathcal{M}$ always induces a weak equivalence $\mathrm{sL}_{\mathcal{W}}\mathcal{M}(X, Y) \cong \mathrm{sL}_{\mathcal{W}}\mathcal{M}(X, Z)$, while the map $\mathrm{Map}(X, Y) \to \mathrm{Map}(X, Z)$ is guaranteed to be a weak equivalence only when X is cofibrant and both Y and Z are fibrant (and a similar statement is true for weak equivalences of the first argument), this implies that the simplicial set $\mathrm{sL}_{\mathcal{W}}\mathcal{M}(X, Y)$ is the "correct" function complex of maps from X to Y.

Dwyer and Kan show that if $\mathcal{M}$ is a model category and if $\mathcal{W}$ is the subcategory of weak equivalences in $\mathcal{M}$, then these function complexes can be computed (up to weak equivalence) using resolutions in the model category $\mathcal{M}$ (see [**28**]). In this chapter, we define a *homotopy function complex* to be a function complex obtained from the Dwyer-Kan construction in the model category $\mathcal{M}$ (see Definition 17.2.2). We present a self-contained development of the properties of these homotopy function complexes, with no explicit reference to the more general construction of the simplicial localization of Dwyer and Kan.

17.1. Resolutions

In this section, we define cosimplicial and simplicial resolutions of objects in a model category. These will be used in Section 17.2 to define homotopy function complexes between objects in a model category (see Definition 17.2.2).

NOTATION 17.1.1. Let $\mathcal{M}$ be a model category.

- The category of cosimplicial objects in $\mathcal{M}$ will be denoted $\mathcal{M}^{\boldsymbol{\Delta}}$, and the category of simplicial objects in $\mathcal{M}$ will be denoted $\mathcal{M}^{\boldsymbol{\Delta}^{\mathrm{op}}}$.
- If X is an object in $\mathcal{M}$, then the constant cosimplicial object at X will be denoted $\mathrm{c}X$, and constant simplicial object at X will be denoted $\mathrm{s}X$.

DEFINITION 17.1.2. Let $\mathcal{M}$ be a model category, and let X be an object in $\mathcal{M}$.

- A *cosimplicial resolution* of X is a cofibrant approximation (see Definition 9.1.1) $\widetilde{\boldsymbol{X}} \to \mathrm{c}X$ to $\mathrm{c}X$ (see Notation 17.1.1) in the Reedy model category structure (see Definition 16.3.2) on $\mathcal{M}^{\mathbf{\Delta}}$. A *fibrant cosimplicial resolution* is a cosimplicial resolution in which the weak equivalence $\widetilde{\boldsymbol{X}} \to \mathrm{c}X$ is a Reedy trivial fibration. We will sometimes use the term *cosimplicial resolution* to refer to the object $\widetilde{\boldsymbol{X}}$ without explicitly mentioning the weak equivalence $\widetilde{\boldsymbol{X}} \to \mathrm{c}X$.

- A *simplicial resolution* of X is a fibrant approximation $\mathrm{s}X \to \widehat{\boldsymbol{X}}$ to $\mathrm{s}X$ in the Reedy model category structure on $\mathcal{M}^{\mathbf{\Delta}^{\mathrm{op}}}$. A *cofibrant simplicial resolution* is a simplicial resolution in which the weak equivalence $\mathrm{s}X \to \widehat{\boldsymbol{X}}$ is a Reedy trivial cofibration. We will sometimes use the term *simplicial resolution* to refer to the object $\widehat{\boldsymbol{X}}$ without explicitly mentioning the weak equivalence $\mathrm{s}X \to \widehat{\boldsymbol{X}}$.

PROPOSITION 17.1.3. *If $\mathcal{M}$ is a model category, then every object has a natural fibrant cosimplicial resolution and a natural cofibrant simplicial resolution.*

PROOF. This follows from Proposition 9.1.2. □

PROPOSITION 17.1.4. *Let $\mathcal{M}$ be a simplicial model category.*

1. *If X is an object in $\mathcal{M}$ and $W \to X$ is a cofibrant approximation to X, then the cosimplicial object $\widetilde{\boldsymbol{W}}$ in which $\widetilde{\boldsymbol{W}}^n = W \otimes \Delta[n]$ is a cosimplicial resolution of X.*
2. *If Y is an object in $\mathcal{M}$ and $Y \to Z$ is a fibrant approximation to Y, then the simplicial object $\widehat{\boldsymbol{Z}}$ in which $\widehat{\boldsymbol{Z}}_n = Z^{\Delta[n]}$ is a simplicial resolution of Y.*

PROOF. We will prove part 1; the proof of part 2 is similar.

Since all of the inclusions $\Delta[0] \to \Delta[n]$ are trivial cofibrations and W is cofibrant, all of the maps $W \approx W \otimes \Delta[0] \to W \otimes \Delta[n]$ are trivial cofibrations. Thus, $\widetilde{\boldsymbol{W}}$ is weakly equivalent to $\mathrm{c}X$. Since each $\partial\Delta[n] \to \Delta[n]$ is a cofibration and W is cofibrant, each latching map $W \otimes \partial\Delta[n] \to W \otimes \Delta[n]$ is a cofibration, and so $\widetilde{\boldsymbol{W}}$ is cofibrant. □

COROLLARY 17.1.5. *Let $\mathcal{M}$ be a simplicial model category.*

1. *If X is a cofibrant object in $\mathcal{M}$, then the cosimplicial object $\widetilde{\boldsymbol{X}}$ in which $\widetilde{\boldsymbol{X}}^n = X \otimes \Delta[n]$ is a cosimplicial resolution of X.*
2. *If Y is a fibrant object in $\mathcal{M}$, then the simplicial object $\widehat{\boldsymbol{Y}}$ in which $\widehat{\boldsymbol{Y}}_n = Y^{\Delta[n]}$ is a simplicial resolution of Y.*

PROOF. This follows from Proposition 17.1.4. □

PROPOSITION 17.1.6. *Let $\mathcal{M}$ be a model category, and let X be an object in $\mathcal{M}$.*

1. *If $\widetilde{\boldsymbol{X}} \to \mathrm{c}X$ is a cosimplicial resolution of X (see Definition 17.1.2), then $\widetilde{\boldsymbol{X}}^0 \to X$ is a cofibrant approximation to X. If $\widetilde{\boldsymbol{X}} \to \mathrm{c}X$ is a fibrant cosimplicial resolution of X, then $\widetilde{\boldsymbol{X}}^0 \to X$ is a fibrant cofibrant approximation to X.*
2. *If $\mathrm{s}X \to \widehat{\boldsymbol{X}}$ is a simplicial resolution of X, then $X \to \widehat{\boldsymbol{X}}_0$ is a fibrant approximation to X. If $\mathrm{s}X \to \widehat{\boldsymbol{X}}$ is a cofibrant simplicial resolution of X, then $X \to \widehat{\boldsymbol{X}}_0$ is a cofibrant fibrant approximation to X.*

PROOF. This follows from Proposition 16.3.7. □

DEFINITION 17.1.7. Let $\mathcal{M}$ be a model category.

1. If $\widetilde{\boldsymbol{X}} \xrightarrow{i} \mathrm{c}X$ and $\widetilde{\boldsymbol{X}}' \xrightarrow{i'} \mathrm{c}X$ are cosimplicial resolutions of X, then a *map of cosimplicial resolutions* from $(\widetilde{\boldsymbol{X}}, i)$ to $(\widetilde{\boldsymbol{X}}', i')$ is a map $g\colon \widetilde{\boldsymbol{X}} \to \widetilde{\boldsymbol{X}}'$ such that $i'g = i$.
2. If $\mathrm{s}X \xrightarrow{j} \widehat{\boldsymbol{X}}$ and $\mathrm{s}X \xrightarrow{j'} \widehat{\boldsymbol{X}}'$ are simplicial resolutions of X, then a *map of simplicial resolutions* from $(\widehat{\boldsymbol{X}}, j)$ to $(\widehat{\boldsymbol{X}}', j')$ is a map $g\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{X}}'$ such that $gj = j'$.

LEMMA 17.1.8. *Let $\mathcal{M}$ be a model category.*

1. *If $(\widetilde{\boldsymbol{X}}, i)$ and $(\widetilde{\boldsymbol{X}}', i')$ are cosimplicial resolutions of X and $g\colon \widetilde{\boldsymbol{X}} \to \widetilde{\boldsymbol{X}}'$ is a map of cosimplicial resolutions, then g is a weak equivalence.*
2. *If $(\widehat{\boldsymbol{X}}, j)$ and $(\widehat{\boldsymbol{X}}', j')$ are simplicial resolutions of X and $g\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{X}}'$ is a map of simplicial resolutions, then g is a weak equivalence.*

PROOF. This follows from Lemma 9.1.4. □

PROPOSITION 17.1.9. *Let $\mathcal{M}$ be a model category.*

1. *If $\widetilde{\boldsymbol{X}} \to \mathrm{c}X$ is cosimplicial resolution of X and $\widetilde{\boldsymbol{X}}' \to \mathrm{c}X$ is a fibrant cosimplicial resolution of X, then there is a map $\widetilde{\boldsymbol{X}} \to \widetilde{\boldsymbol{X}}'$ of cosimplicial resolutions, unique up to homotopy over $\mathrm{c}X$, and any such map is a weak equivalence.*
2. *If $\mathrm{s}X \to \widehat{\boldsymbol{X}}$ is a simplicial resolution of X and $\mathrm{s}X \to \widehat{\boldsymbol{X}}'$ is a cofibrant simplicial resolution of X, then there is a map $\widehat{\boldsymbol{X}}' \to \widehat{\boldsymbol{X}}$ of simplicial resolutions, unique up to homotopy under $\mathrm{s}X$, and any such map is a weak equivalence.*

PROOF. This follows from Proposition 9.1.6. □

DEFINITION 17.1.10. Let $\mathcal{M}$ be a model category, and let $g\colon X \to Y$ be a map in $\mathcal{M}$.

1. A *cosimplicial resolution of g* consists of a cosimplicial resolution $\widetilde{\boldsymbol{X}} \to \mathrm{c}X$ of X, a cosimplicial resolution $\widetilde{\boldsymbol{Y}} \to \mathrm{c}Y$ of Y, and a map $\tilde{g}\colon \widetilde{\boldsymbol{X}} \to \widetilde{\boldsymbol{Y}}$ that makes the square
$$\begin{tikzcd} \widetilde{\boldsymbol{X}} \arrow[r, "\tilde{g}"] \arrow[d] & \widetilde{\boldsymbol{Y}} \arrow[d] \\ \mathrm{c}X \arrow[r] & \mathrm{c}Y \end{tikzcd}$$
commute.
2. A *simplicial resolution of g* consists of a simplicial resolution $\mathrm{s}X \to \widehat{\boldsymbol{X}}$ of X, a simplicial resolution $\mathrm{s}Y \to \widehat{\boldsymbol{Y}}$ of Y, and a map $\hat{g}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ that makes the square
$$\begin{tikzcd} \mathrm{s}X \arrow[r] \arrow[d] & \mathrm{s}Y \arrow[d] \\ \widehat{\boldsymbol{X}} \arrow[r, "\hat{g}"'] & \widehat{\boldsymbol{Y}} \end{tikzcd}$$
commute.

REMARK 17.1.11. The effect of Definition 17.1.2 and Definition 17.1.10 is that

- a cosimplicial resolution of an object or map in a model category is exactly a Reedy cofibrant approximation to a constant cosimplicial object or map, and
- a simplicial resolution of an object or map in a model category is exactly a Reedy fibrant approximation to a constant simplicial object or map.

This is the explanation of the terminology "fibrant cosimplicial resolution" and "cofibrant simplicial resolution".

PROPOSITION 17.1.12. *Let $\mathcal{M}$ be a model category, and let $g\colon X \to Y$ be a map in $\mathcal{M}$.*

1. *There exists a natural cosimplicial resolution $\tilde{g}\colon \widetilde{\boldsymbol{X}} \to \widetilde{\boldsymbol{Y}}$ of g such that $\widetilde{\boldsymbol{X}}$ and $\widetilde{\boldsymbol{Y}}$ are fibrant cosimplicial resolutions of, respectively, X and Y, and $\tilde{g}$ is a Reedy cofibration.*
2. *There exists a natural simplicial resolution $\hat{g}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ of g such that $\widehat{\boldsymbol{X}}$ and $\widehat{\boldsymbol{Y}}$ are cofibrant simplicial resolutions of, respectively, X and Y, and $\hat{g}$ is a Reedy fibration.*

PROOF. This follows from Proposition 9.1.9. □

PROPOSITION 17.1.13. *Let $\mathcal{M}$ be a model category, and let $g\colon X \to Y$ be a map in $\mathcal{M}$.*

1. *If $\widetilde{\boldsymbol{X}} \to \mathrm{c}X$ is a cosimplicial resolution of X and $\widetilde{\boldsymbol{Y}} \to \mathrm{c}Y$ is a fibrant cosimplicial resolution of Y, then there exists a resolution $\tilde{g}\colon \widetilde{\boldsymbol{X}} \to \widetilde{\boldsymbol{Y}}$ of g, and $\tilde{g}$ is unique up to homotopy in $(\mathcal{M}^{\boldsymbol{\Delta}} \downarrow \mathrm{c}Y)$.*
2. *If $\mathrm{s}Y \to \widehat{\boldsymbol{Y}}$ is a simplicial resolution of Y and $\mathrm{s}X \to \widehat{\boldsymbol{X}}$ is a cofibrant simplicial resolution of X, then there exists a resolution $\hat{g}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ of g, and $\hat{g}$ is unique up to homotopy in $(\mathrm{s}X \downarrow \mathcal{M}^{\boldsymbol{\Delta}^{\mathrm{op}}})$.*

PROOF. This follows from Proposition 9.1.10. □

PROPOSITION 17.1.14. *If $\mathcal{M}$ is a model category and $g\colon X \to Y$ is a weak equivalence in $\mathcal{M}$, then every cosimplicial resolution of g and every simplicial resolution of g are Reedy weak equivalences.*

PROOF. This follows from the "two out of three" axiom for weak equivalences. □

17.1.15. Recognizing resolutions.

DEFINITION 17.1.16. Let $\mathcal{M}$ be a model category.

1. If $\widetilde{\boldsymbol{X}}$ is a cosimplicial object in $\mathcal{M}$, then we will say that $\widetilde{\boldsymbol{X}}$ is a *cosimplicial resolution* if there is an object X in $\mathcal{M}$ and a map $\widetilde{\boldsymbol{X}} \to \mathrm{c}X$ that is a cosimplicial resolution of X (see Definition 17.1.2).
2. If $\widehat{\boldsymbol{Y}}$ is a simplicial object in $\mathcal{M}$, then we will say that $\widehat{\boldsymbol{Y}}$ is a *simplicial resolution* if there is an object Y in $\mathcal{M}$ and a map $\mathrm{s}Y \to \widehat{\boldsymbol{Y}}$ that is a simplicial resolution of Y.

PROPOSITION 17.1.17. *Let $\mathcal{M}$ be a model category.*

1. *If $\boldsymbol{X}$ is a cosimplicial object in $\mathcal{M}$, then $\boldsymbol{X}$ is a cosimplicial resolution (see Definition 17.1.16) if and only if $\boldsymbol{X}$ is Reedy cofibrant and all of the coface and codegeneracy operators of $\boldsymbol{X}$ are weak equivalences.*

2. *If $\boldsymbol{Y}$ is a simplicial object in $\mathcal{M}$, then $\boldsymbol{Y}$ is a simplicial resolution if and only if $\boldsymbol{Y}$ is Reedy fibrant and all of the face and degeneracy operators of $\boldsymbol{Y}$ are weak equivalences.*

PROOF. We will prove part 1; the proof of part 2 is dual.

If $\boldsymbol{X}$ is a cosimplicial resolution, then it follows directly from the definitions that $\boldsymbol{X}$ is Reedy cofibrant and all of the coface and codegeneracy operators of $\boldsymbol{X}$ are weak equivalences. For the converse, the map $\boldsymbol{X} \to \mathrm{c}\boldsymbol{X}^0$ defined on $\boldsymbol{X}^n$ as any n-fold iterated coface map is a cosimplicial resolution of $\boldsymbol{X}^0$. □

LEMMA 17.1.18. *Let $\mathcal{M}$ be a model category.*

1. *If $i\colon \boldsymbol{A} \to \boldsymbol{B}$ is a weak equivalence of cosimplicial resolutions in $\mathcal{M}$, then there is a natural factorization of i as $\boldsymbol{A} \xrightarrow{q} \boldsymbol{C} \xrightarrow{r} \boldsymbol{B}$ such that $\boldsymbol{C}$ is a cosimplicial resolution in $\mathcal{M}$, q is a Reedy trivial cofibration, and r has a right inverse that is a Reedy trivial cofibration.*
2. *If $p\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a weak equivalence of simplicial resolutions in $\mathcal{M}$, then there is a natural factorization of p as $\boldsymbol{X} \xrightarrow{q} \boldsymbol{Z} \xrightarrow{r} \boldsymbol{Y}$ such that $\boldsymbol{Z}$ is a simplicial resolution in $\mathcal{M}$, r is a Reedy trivial fibration, and q has a left inverse that is a Reedy trivial fibration.*

PROOF. This follows from Lemma 8.5.1 and Proposition 17.1.17. □

17.1.19. Frames. Proposition 17.1.6 shows how a cosimplicial resolution of an object in a model category yields a cofibrant approximation to that object (and a similar statement is true for simplicial resolutions and fibrant approximations). Frames (see Definition 17.1.20) allow us to discuss the reverse operation (see Proposition 17.1.27).

DEFINITION 17.1.20. Let $\mathcal{M}$ be a model category, and let X be an object in $\mathcal{M}$.

- A *cosimplicial frame* on X is a cosimplicial object $\widetilde{\boldsymbol{X}}$ in $\mathcal{M}$ together with a weak equivalence $\widetilde{\boldsymbol{X}} \to \mathrm{c}X$ (see Notation 17.1.1) in the Reedy model category structure (see Definition 16.3.2) on $\mathcal{M}^{\boldsymbol{\Delta}}$ such that
  1. the induced map $\widetilde{\boldsymbol{X}}^0 \to X$ is an isomorphism, and
  2. if X is a cofibrant object in $\mathcal{M}$, then $\widetilde{\boldsymbol{X}}$ is a cofibrant object in $\mathcal{M}^{\boldsymbol{\Delta}}$.

  We will sometimes refer to $\widetilde{\boldsymbol{X}}$ as a cosimplicial frame on X, without explicitly mentioning the map $\widetilde{\boldsymbol{X}} \to \mathrm{c}X$.
- A *simplicial frame* on X is a simplicial object $\widehat{\boldsymbol{X}}$ in $\mathcal{M}$ together with a weak equivalence $\mathrm{s}X \to \widehat{\boldsymbol{X}}$ in the Reedy model category structure on $\mathcal{M}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ such that
  1. the induced map $X \to \widehat{\boldsymbol{X}}^0$ is an isomorphism, and
  2. if X is a fibrant object in $\mathcal{M}$, then $\widehat{\boldsymbol{X}}$ is a fibrant object in $\mathcal{M}^{\boldsymbol{\Delta}^{\mathrm{op}}}$.

  We will sometimes refer to $\widehat{\boldsymbol{X}}$ as a simplicial frame on X, without explicitly mentioning the map $\mathrm{s}X \to \widehat{\boldsymbol{X}}$.

REMARK 17.1.21. Note that Definition 17.1.20 does not require cosimplicial frames on non-cofibrant objects to be cofibrant or simplicial frames on non-fibrant objects to be fibrant. This was done in order to make Proposition 17.1.25 true.

PROPOSITION 17.1.22. *Let $\mathcal{M}$ be a model category, and let X be an object in $\mathcal{M}$.*

1. *If X is cofibrant, then any cosimplicial frame on X is a cosimplicial resolution of X.*
2. *If X is fibrant, then any simplicial frame on X is a simplicial resolution of X.*

PROOF. This follows directly from the definitions. □

LEMMA 17.1.23. *If $n \geq 0$, then the inclusion of $\Delta[0]$ into $\Delta[n]$ as the initial vertex is the inclusion of a simplicial strong deformation retract.*

PROPOSITION 17.1.24. *If $\mathcal{M}$ is a simplicial model category, X is an object of $\mathcal{M}$, and $n \geq 0$, then the maps $X \otimes \Delta[0] \to X \otimes \Delta[n]$ and $X^{\Delta[n]} \to X^{\Delta[0]}$ induced by the inclusion of $\Delta[0]$ as the initial vertex of $\Delta[n]$ are weak equivalences.*

PROOF. Lemma 17.1.23 implies that these maps are simplicial homotopy equivalences. □

PROPOSITION 17.1.25. *If $\mathcal{M}$ is a simplicial model category and X is an object in $\mathcal{M}$, then the cosimplicial object $\widetilde{\boldsymbol{X}}$ in which $\widetilde{\boldsymbol{X}}^n = X \otimes \Delta[n]$ is a cosimplicial frame on X, and the simplicial object $\widehat{\boldsymbol{Y}}$ in which $\widehat{\boldsymbol{Y}}_n = X^{\Delta[n]}$ is a simplicial frame on X.*

PROOF. This follows from Proposition 17.1.24 and Proposition 17.1.4. □

DEFINITION 17.1.26. If $\mathcal{M}$ is a simplicial model category and X is an object in $\mathcal{M}$, then the cosimplicial frame on X of Proposition 17.1.25 will be called the *standard cosimplicial frame on X*, and the simplicial frame on X of Proposition 17.1.25 will be called the *standard simplicial frame on X*.

PROPOSITION 17.1.27. *Let $\mathcal{M}$ be a model category.*

1. *If X is an object in $\mathcal{M}$, $\widetilde{X} \to X$ is a cofibrant approximation to X, and $\widetilde{\boldsymbol{X}}' \to \mathrm{c}\widetilde{X}$ is a cosimplicial frame on $\widetilde{X}$, then the induced map $\widetilde{\boldsymbol{X}}' \to \mathrm{c}X$ is a cosimplicial resolution of X, and every cosimplicial resolution of X can be constructed in this way.*
2. *If X is an object in $\mathcal{M}$, $X \to \widehat{X}$ is a fibrant approximation to X, and $\mathrm{s}\widehat{X} \to \widehat{\boldsymbol{X}}'$ is a simplicial frame on $\widehat{X}$, then the induced map $\mathrm{s}X \to \widehat{\boldsymbol{X}}'$ is a simplicial resolution of X, and every simplicial resolution of X can be constructed in this way.*

PROOF. This follows from Proposition 17.1.6. □

THEOREM 17.1.28. *If $\mathcal{M}$ is a model category, then there exists a functorial cosimplicial frame on every object in $\mathcal{M}$ and a functorial simplicial frame on every object in $\mathcal{M}$.*

PROOF. We will construct a functorial cosimplicial frame on $\mathcal{M}$; the construction of a functorial simplicial frame is dual.

For every object X of $\mathcal{M}$, we will construct a functorial factorization $\emptyset \to \widetilde{\boldsymbol{X}} \to \mathrm{c}X$ of the map in $\mathcal{M}^{\boldsymbol{\Delta}}$ from the initial object to $\mathrm{c}X$ such that

1. the map $\widetilde{\boldsymbol{X}} \to \mathrm{c}X$ is a Reedy trivial fibration,
2. the induced map $\widetilde{\boldsymbol{X}}^0 \to X$ is the identity, and such that
3. if X is cofibrant in $\mathcal{M}$, then the map $\emptyset \to \widetilde{\boldsymbol{X}}$ is a Reedy cofibration.

We will construct $\widetilde{\boldsymbol{X}}$ and the map $\widetilde{\boldsymbol{X}} \to \mathrm{c}X$ inductively, and we begin by letting $\widetilde{\boldsymbol{X}}^0 = X$. If $n > 0$ and we have constructed $\widetilde{\boldsymbol{X}} \to \mathrm{c}X$ in degrees less than n, then we have the induced map $\mathrm{L}_n\widetilde{\boldsymbol{X}} \to (\mathrm{c}X)^n \times_{\mathrm{M}_n\mathrm{c}X} \mathrm{M}_n\widetilde{\boldsymbol{X}}$. We can factor this map functorially in $\mathcal{M}$ as

$$\mathrm{L}_n\widetilde{\boldsymbol{X}} \xrightarrow{i} \widetilde{\boldsymbol{X}}^n \xrightarrow{p} (\mathrm{c}X)^n \times_{\mathrm{M}_n\mathrm{c}X} \mathrm{M}_n\widetilde{\boldsymbol{X}}$$

with i a cofibration and p a trivial fibration. This complete the construction, and Theorem 16.3.10 implies that the map $\widetilde{\boldsymbol{X}} \to \mathrm{c}X$ is always a Reedy trivial fibration. If X is cofibrant, then $\mathrm{L}_n\widetilde{\boldsymbol{X}} \to \widetilde{\boldsymbol{X}}^n$ is a cofibration for all $n \geq 0$, and so $\widetilde{\boldsymbol{X}}$ is Reedy cofibrant. $\square$

Definition 17.1.29. Let $\mathcal{M}$ be a model category, and let $g\colon X \to Y$ be a map in $\mathcal{M}$.

1. A *cosimplicial frame on g* consists of a cosimplicial frame $\widetilde{\boldsymbol{X}} \to \mathrm{c}X$ on X, a cosimplicial frame $\widetilde{\boldsymbol{Y}} \to \mathrm{c}Y$ on Y, and a map $\tilde{g}\colon \widetilde{\boldsymbol{X}} \to \widetilde{\boldsymbol{Y}}$ that makes the square
$$\begin{tikzcd} \widetilde{\boldsymbol{X}} \arrow[r, "\tilde{g}"] \arrow[d] & \widetilde{\boldsymbol{Y}} \arrow[d] \\ \mathrm{c}X \arrow[r] & \mathrm{c}Y \end{tikzcd}$$
commute.
2. A *simplicial frame on g* consists of a simplicial frame $\mathrm{s}X \to \widehat{\boldsymbol{X}}$ on X, a simplicial frame $\mathrm{s}Y \to \widehat{\boldsymbol{Y}}$ on Y, and a map $\hat{g}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ that makes the square
$$\begin{tikzcd} \mathrm{s}X \arrow[r] \arrow[d] & \mathrm{s}Y \arrow[d] \\ \widehat{\boldsymbol{X}} \arrow[r, "\hat{g}"'] & \widehat{\boldsymbol{Y}} \end{tikzcd}$$
commute.

Example 17.1.30. Let $\mathcal{M}$ be a simplicial model category.

1. If $i\colon A \to B$ is a map in $\mathcal{M}$, let $\widetilde{\boldsymbol{A}}$ and $\widetilde{\boldsymbol{B}}$ be the cosimplicial objects in $\mathcal{M}$ such that $\widetilde{A}^n = A \otimes \Delta[n]$ and $\widetilde{B}^n = B \otimes \Delta[n]$, and let $\tilde{\imath}\colon \widetilde{A} \to \widetilde{B}$ be the obvious map. Proposition 17.1.25 implies that $\tilde{\imath}$ is a cosimplicial frame on i, and Proposition 10.1.8 implies that $\tilde{\imath}$ is a Reedy cofibration if i is a cofibration in $\mathcal{M}$.
2. If $p\colon X \to Y$ is a map in $\mathcal{M}$, let $\widehat{\boldsymbol{X}}$ and $\widehat{\boldsymbol{Y}}$ be the simplicial objects in $\mathcal{M}$ such that $\widehat{\boldsymbol{X}}_n = X^{\Delta[n]}$ and $\widehat{\boldsymbol{Y}}_n = Y^{\Delta[n]}$, and let $\hat{p}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ be the obvious map. Proposition 17.1.25 implies that $\hat{p}$ is a simplicial frame on p, and Proposition 10.1.8 implies that $\hat{p}$ is a Reedy fibration if p is a fibration in $\mathcal{M}$.

Proposition 17.1.31. *Let $\mathcal{M}$ be a model category, and let $g\colon X \to Y$ be a map in $\mathcal{M}$.*

1. *There is a natural cosimplicial frame $\tilde{g}\colon \widetilde{\boldsymbol{X}} \to \widetilde{\boldsymbol{Y}}$ on g that is a Reedy cofibration if g is a cofibration.*

2. *There is a natural simplicial frame* $\hat{g}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ *on* g *that is a Reedy fibration if* g *is a fibration.*

PROOF. We will prove part 1; the proof of part 2 is dual.

We begin by constructing a natural cosimplicial frame $\widetilde{\boldsymbol{X}} \to \mathrm{c}X$ on X as in the proof of Theorem 17.1.28.

We will define $\widetilde{\boldsymbol{Y}}$ and $\tilde{g}$ inductively. We let $\widetilde{\boldsymbol{Y}}_0 = Y$. If $n > 0$ and we have constructed $\widetilde{\boldsymbol{Y}}$ and $\tilde{g}$ in degrees less than n, then we have the induced map $\mathrm{L}_n\widetilde{\boldsymbol{Y}} \amalg_{\mathrm{L}_n\widetilde{\boldsymbol{X}}} \widetilde{\boldsymbol{X}}^n \to (\mathrm{c}Y)^n \times_{\mathrm{M}_n\mathrm{c}Y} \mathrm{M}_n\widetilde{\boldsymbol{Y}}$. We factor this map functorially in $\mathcal{M}$ as

$$\mathrm{L}_n\widetilde{\boldsymbol{Y}} \amalg_{\mathrm{L}_n\widetilde{\boldsymbol{X}}} \widetilde{\boldsymbol{X}}^n \xrightarrow{i} \widetilde{\boldsymbol{Y}}^n \xrightarrow{p} (\mathrm{c}Y)^n \times_{\mathrm{M}_n\mathrm{c}Y} \mathrm{M}_n\widetilde{\boldsymbol{Y}}$$

with i a cofibration and p a trivial fibration. This completes the construction, and Theorem 16.3.10 implies that the map $\widetilde{\boldsymbol{Y}} \to \mathrm{c}Y$ is always a Reedy trivial fibration. Since $\mathrm{L}_n\widetilde{\boldsymbol{X}} \to \widetilde{\boldsymbol{X}}^n$ was constructed to be a cofibration for all $n > 0$, and $\mathrm{L}_n\widetilde{\boldsymbol{Y}} \to \mathrm{L}_n\widetilde{\boldsymbol{Y}} \amalg_{\mathrm{L}_n\widetilde{\boldsymbol{X}}} \widetilde{\boldsymbol{X}}^n$ is a pushout of that cofibration, the composition $\mathrm{L}_n\widetilde{\boldsymbol{Y}} \to \mathrm{L}_n\widetilde{\boldsymbol{Y}} \amalg_{\mathrm{L}_n\widetilde{\boldsymbol{X}}} \widetilde{\boldsymbol{X}}^n \to \widetilde{\boldsymbol{Y}}^n$ is a cofibration for all $n > 0$. Thus, if Y is cofibrant, then $\widetilde{\boldsymbol{Y}}$ is Reedy cofibrant. Finally, if g is a cofibration, then $\mathrm{L}_n\widetilde{\boldsymbol{Y}} \amalg_{\mathrm{L}_n\widetilde{\boldsymbol{X}}} \widetilde{\boldsymbol{X}}^n \to \widetilde{\boldsymbol{Y}}^n$ is a cofibration for all $n \geq 0$, and so $\tilde{g}$ is a Reedy cofibration. □

17.1.32. Framed model categories.

DEFINITION 17.1.33. A *framed model category* is a model category $\mathcal{M}$ together with

1. a functorial cosimplicial frame (see Definition 17.1.20) $\widetilde{\boldsymbol{X}}$ on every object X in $\mathcal{M}$, and
2. a functorial simplicial frame $\widehat{\boldsymbol{X}}$ on every object X in $\mathcal{M}$.

PROPOSITION 17.1.34. *If* $\mathcal{M}$ *is a model category, then there exists a framed model category structure on* $\mathcal{M}$.

PROOF. This follows from Theorem 17.1.28. □

PROPOSITION 17.1.35. *If* $\mathcal{M}$ *is a simplicial model category, then there is a natural framing on* $\mathcal{M}$ *(called the* standard framing*) defined on objects* X *in* $\mathcal{M}$ *by* $\widetilde{\boldsymbol{X}}^n = X \otimes \Delta[n]$ *and* $\widehat{\boldsymbol{X}}_n = X^{\Delta[n]}$.

PROOF. This follows from Proposition 17.1.25. □

REMARK 17.1.36. If $\mathcal{M}$ is a simplicial model category, and we make reference to $\mathcal{M}$ in a context that calls for a framed model category, then we will consider $\mathcal{M}$ as a framed model category using the standard framing of Proposition 17.1.35.

17.2. Homotopy function complexes

NOTATION 17.2.1. Let $\mathcal{M}$ be a model category.

1. If $\boldsymbol{B}$ is a cosimplicial object in $\mathcal{M}$ and X is an object in $\mathcal{M}$, then $\mathcal{M}(\boldsymbol{B}, X)$ will denote the simplicial set, natural in both $\boldsymbol{B}$ and X, defined by $\mathcal{M}(\boldsymbol{B}, X)_n = \mathcal{M}(\boldsymbol{B}^n, X)$, with face and degeneracy maps induced by the coface and codegeneracy maps in $\boldsymbol{B}$.
2. If B is an object in $\mathcal{M}$ and $\boldsymbol{X}$ is a simplicial object in $\mathcal{M}$, then $\mathcal{M}(B, \boldsymbol{X})$ will denote the simplicial set, natural in both B and $\boldsymbol{X}$, defined by $\mathcal{M}(B, \boldsymbol{X})_n = \mathcal{M}(B, \boldsymbol{X}_n)$, with face and degeneracy maps induced by those in $\boldsymbol{X}$.

3. If $\boldsymbol{B}$ is a cosimplicial object in $\mathcal{M}$ and $\boldsymbol{X}$ is a simplicial object in $\mathcal{M}$, then $\mathcal{M}(\boldsymbol{B},\boldsymbol{X})$ will denote the bisimplicial set, natural in both $\boldsymbol{B}$ and $\boldsymbol{X}$, defined by $\mathcal{M}(\boldsymbol{B},\boldsymbol{X})_{n,k} = \mathcal{M}(\boldsymbol{B}^k,\boldsymbol{X}_n)$, with face and degeneracy maps induced by the coface and codegeneracy maps in $\boldsymbol{B}$ and the face and degeneracy maps in $\boldsymbol{X}$.
4. If $\boldsymbol{B}$ is a cosimplicial object in $\mathcal{M}$ and $\boldsymbol{X}$ is a simplicial object in $\mathcal{M}$, then $\operatorname{diag}\mathcal{M}(\boldsymbol{B},\boldsymbol{X})$ will denote the simplicial set, natural in both $\boldsymbol{B}$ and $\boldsymbol{X}$, defined by $\bigl(\operatorname{diag}\mathcal{M}(\boldsymbol{B},\boldsymbol{X})\bigr)_n = \mathcal{M}(\boldsymbol{B}^n,\boldsymbol{X}_n)$, with face and degeneracy maps induced by the coface and codegeneracy maps in $\boldsymbol{B}$ and the face and degeneracy maps in $\boldsymbol{X}$.

DEFINITION 17.2.2. Let $\mathcal{M}$ be a model category, and let B and X be objects of $\mathcal{M}$.

- A *left homotopy function complex* from B to X is a simplicial set of the form $\mathcal{M}(\widetilde{\boldsymbol{B}},\widehat{X})$ (see Notation 17.2.1) for some cosimplicial resolution $\widetilde{\boldsymbol{B}} \to \mathrm{c}B$ of B (see Definition 17.1.2) and some fibrant approximation $X \to \widehat{X}$ to X.
- A *right homotopy function complex* from B to X is a simplicial set of the form $\mathcal{M}(\widetilde{B},\widehat{\boldsymbol{X}})$ for some cofibrant approximation $\widetilde{B} \to B$ to B and some simplicial resolution $\mathrm{s}X \to \widehat{\boldsymbol{X}}$ of X.
- a *two-sided homotopy function complex* from B to X is a simplicial set of the form $\operatorname{diag}\mathcal{M}(\widetilde{\boldsymbol{B}},\widehat{\boldsymbol{X}})$ for some cosimplicial resolution $\widetilde{\boldsymbol{B}} \to \mathrm{c}B$ of B and some simplicial resolution $\mathrm{s}X \to \widehat{\boldsymbol{X}}$ of X.
- A *homotopy function complex* from B to X is either a left homotopy function complex from B to X, a right homotopy function complex from B to X, or a two-sided homotopy function complex from B to X.

EXAMPLE 17.2.3. If $\mathcal{M}$ is a simplicial model category, B is a cofibrant object in $\mathcal{M}$, and X is a fibrant object in $\mathcal{M}$, then Corollary 17.1.5 implies that $\operatorname{Map}(B,X)$ (i.e., the simplicial set that is part of the simplicial structure of $\mathcal{M}$) is both a left homotopy function complex from B to X and a right homotopy function complex from B to X.

DEFINITION 17.2.4. Let $\mathcal{M}$ be a model category, let W, X, Y, and Z be objects in $\mathcal{M}$, and let $g\colon X \to Y$ be a map.

1. A *map of left homotopy function complexes induced by g* will mean either
   (a) the map $\hat{g}_*\colon \mathcal{M}(\widetilde{\boldsymbol{W}},\widehat{X}) \to \mathcal{M}(\widetilde{\boldsymbol{W}},\widehat{Y})$ where $\widetilde{\boldsymbol{W}}$ is a cosimplicial resolution of W and $\hat{g}\colon \widehat{X} \to \widehat{Y}$ is a fibrant approximation to g (see Definition 9.1.8), or
   (b) the map $\tilde{g}^*\colon \mathcal{M}(\widetilde{\boldsymbol{Y}},\widehat{Z}) \to \mathcal{M}(\widetilde{\boldsymbol{X}},\widehat{Z})$ where $\tilde{g}\colon \widetilde{\boldsymbol{X}} \to \widetilde{\boldsymbol{Y}}$ is a cosimplicial resolution of g (see Definition 17.1.10), and $\widehat{Z}$ is a fibrant approximation to Z.
2. A *map of right homotopy function complexes induced by g* will mean either
   (a) the map $\hat{g}_*\colon \mathcal{M}(\widetilde{W},\widehat{\boldsymbol{X}}) \to \mathcal{M}(\widetilde{W},\widehat{\boldsymbol{Y}})$ where $\widetilde{W}$ is a cofibrant approximation to W and $\hat{g}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ is a simplicial resolution of g, or
   (b) the map $\tilde{g}^*\colon \mathcal{M}(\widetilde{Y},\widehat{\boldsymbol{Z}}) \to \mathcal{M}(\widetilde{X},\widehat{\boldsymbol{Z}})$ where $\tilde{g}$ is a cofibrant approximation to g (see Definition 9.1.8) and $\widehat{\boldsymbol{Z}}$ is a simplicial resolution of Z.
3. A *map of two-sided homotopy function complexes induced by g* will mean either

(a) the map $\operatorname{diag}\hat{g}_*\colon \operatorname{diag}\mathcal{M}(\widetilde{\boldsymbol{W}},\widehat{\boldsymbol{X}}) \to \operatorname{diag}\mathcal{M}(\widetilde{\boldsymbol{W}},\widehat{\boldsymbol{Y}})$ where $\widetilde{\boldsymbol{W}}$ is a cosimplicial resolution of W and $\hat{g}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ is a simplicial resolution of g, or

(b) the map $\operatorname{diag}\tilde{g}^*\colon \operatorname{diag}\mathcal{M}(\widetilde{\boldsymbol{Y}},\widehat{\boldsymbol{Z}}) \to \operatorname{diag}\mathcal{M}(\widetilde{\boldsymbol{X}},\widehat{\boldsymbol{Z}})$ where $\tilde{g}\colon \widetilde{\boldsymbol{X}} \to \widetilde{\boldsymbol{Y}}$ is a cosimplicial resolution of g and $\widehat{\boldsymbol{Z}}$ is a simplicial resolution of Z.

4. A *map of homotopy function complexes induced by g* will mean either a map of left homotopy function complexes induced by g, a map of right homotopy function complexes induced by g, or a map of two-sided homotopy function complexes induced by g.

EXAMPLE 17.2.5. Let $\mathcal{M}$ be a model category.

1. If $\mathcal{M}\big(\widetilde{\boldsymbol{C}}(-),\widehat{F}(-)\big)$ is a left homotopy function complex on $\mathcal{M}$, then for objects W, X, Y, and Z in $\mathcal{M}$, a map $g\colon X \to Y$ induces maps of left homotopy function complexes $\mathcal{M}\big(\widetilde{\boldsymbol{C}}(Y),\widehat{F}(Z)\big) \to \mathcal{M}\big(\widetilde{\boldsymbol{C}}(X),\widehat{F}(Z)\big)$ and $\mathcal{M}\big(\widetilde{\boldsymbol{C}}(W),\widehat{F}(X)\big) \to \mathcal{M}\big(\widetilde{\boldsymbol{C}}(W),\widehat{F}(Y)\big)$.
2. If $\mathcal{M}\big(\widetilde{C}(-),\widehat{\boldsymbol{F}}(-)\big)$ is a right homotopy function complex on $\mathcal{M}$, then for objects W, X, Y, and Z in $\mathcal{M}$, a map $g\colon X \to Y$ induces maps of right homotopy function complexes $\mathcal{M}\big(\widetilde{C}(W),\widehat{\boldsymbol{F}}(X)\big) \to \mathcal{M}\big(\widetilde{C}(W),\widehat{\boldsymbol{F}}(Y)\big)$ and $\mathcal{M}\big(\widetilde{C}(Y),\widehat{\boldsymbol{F}}(Z)\big) \to \mathcal{M}\big(\widetilde{C}(X),\widehat{\boldsymbol{F}}(Z)\big)$.
3. If $\operatorname{diag}\mathcal{M}\big(\widetilde{\boldsymbol{C}}(-),\widehat{\boldsymbol{F}}(-)\big)$ is a two-sided homotopy function complex on $\mathcal{M}$, then for objects W, X, Y, and Z in $\mathcal{M}$, a map $g\colon X \to Y$ induces maps of two-sided homotopy function complexes $\operatorname{diag}\mathcal{M}\big(\widetilde{\boldsymbol{C}}(W),\widehat{\boldsymbol{F}}(X)\big) \to \operatorname{diag}\mathcal{M}\big(\widetilde{\boldsymbol{C}}(W),\widehat{\boldsymbol{F}}(Y)\big)$ and $\operatorname{diag}\mathcal{M}\big(\widetilde{\boldsymbol{C}}(Y),\widehat{\boldsymbol{F}}(Z)\big) \to \operatorname{diag}\mathcal{M}\big(\widetilde{\boldsymbol{C}}(X),\widehat{\boldsymbol{F}}(Z)\big)$.

DEFINITION 17.2.6. Let $\mathcal{M}$ be a model category.

1. A *left homotopy function complex* on $\mathcal{M}$ is a functor from some subcategory of $\mathcal{M}^{\mathrm{op}} \times \mathcal{M}$ to SS that is a left homotopy function complex (see Definition 17.2.2) on every object in its domain and is a map of left homotopy function complexes (see Definition 17.2.4) on every morphism in its domain.
2. A *right homotopy function complex* on $\mathcal{M}$ is a functor from some subcategory of $\mathcal{M}^{\mathrm{op}} \times \mathcal{M}$ to SS that is a right homotopy function complex (see Definition 17.2.2) on every object in its domain and is a map of right homotopy function complexes on every morphism in its domain.
3. A *two-sided homotopy function complex* on $\mathcal{M}$ is a functor from some subcategory of $\mathcal{M}^{\mathrm{op}} \times \mathcal{M}$ to SS that is a two-sided homotopy function complex (see Definition 17.2.2) on every object in its domain and is a map of two-sided homotopy function complexes on every morphism in its domain.
4. A *homotopy function complex* on $\mathcal{M}$ is either a left homotopy function complex on $\mathcal{M}$, a right homotopy function complex on $\mathcal{M}$, or a two-sided homotopy function complex on $\mathcal{M}$.

PROPOSITION 17.2.7. *If $\mathcal{M}$ is a model category, then there exist left homotopy function complexes defined on all of $\mathcal{M}^{\mathrm{op}} \times \mathcal{M}$, right homotopy function complexes defined on all of $\mathcal{M}^{\mathrm{op}} \times \mathcal{M}$, and two-sided homotopy function complexes defined on all of $\mathcal{M}^{\mathrm{op}} \times \mathcal{M}$.*

PROOF. This follows from Proposition 17.1.3 and Proposition 9.1.2. □

EXAMPLE 17.2.8. If $\mathcal{M}$ is a model category and $\widetilde{\boldsymbol{C}}(X)$ is a natural cosimplicial frame on X (see Definition 17.1.20), then $\mathcal{M}\big(\widetilde{\boldsymbol{C}}(X),Y\big)$ is a left homotopy function complex defined on the full subcategory of $\mathcal{M}^{\mathrm{op}} \times \mathcal{M}$ determined by the objects (X, Y) such that X is a cofibrant object of $\mathcal{M}$ and Y is a fibrant object of $\mathcal{M}$.

Similarly, if $\mathcal{M}$ is a model category and $\widehat{\boldsymbol{F}}(X)$ is a natural simplicial frame on X, then $\mathcal{M}\big(X, \widehat{\boldsymbol{F}}(Y)\big)$ is a right homotopy function complex defined on the full subcategory of $\mathcal{M}^{\mathrm{op}} \times \mathcal{M}$ determined by the objects (X, Y) such that X is a cofibrant object of $\mathcal{M}$ and Y is a fibrant object of $\mathcal{M}$.

Example 17.2.9. If $\mathcal{M}$ is a model category, $\widetilde{\boldsymbol{X}} \to X$ is a cosimplicial resolution of X, and $Y \to \widehat{Y}$ is a fibrant approximation to Y, then $\mathcal{M}(\widetilde{\boldsymbol{X}}, \widehat{Y})$ is a left homotopy function complex defined on the subcategory of $\mathcal{M}^{\mathrm{op}} \times \mathcal{M}$ consisting of the one object (X, Y) and the identity map.

Example 17.2.10. If $\mathcal{M}$ is a model category, $\widetilde{W} \to W$ is a cofibrant approximation to W, $f \colon X \to Y$ is a map and $\hat{f} \colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ is a simplicial resolution of f, then the diagram $\hat{f}_* \colon \mathcal{M}(\widetilde{W}, \widehat{\boldsymbol{X}}) \to \mathcal{M}(\widetilde{W}, \widehat{\boldsymbol{Y}})$ defines a right homotopy function complex defined on the subcategory of $\mathcal{M}^{\mathrm{op}} \times \mathcal{M}$ with the two objects (W, X) and (W, Y) and the single non-identity map $(1_W^{\mathrm{op}}, f)$.

Notation 17.2.11. If $\mathcal{M}$ is a model category and X and Y are objects in $\mathcal{M}$, then we will use $\mathrm{map}(X, Y)$ to denote some unspecified homotopy function complex (see Definition 17.2.2) from X to Y.

17.3. Realizations

This section contains a number of technical results needed for the homotopy lifting extension theorems of Section 17.4.

Definition 17.3.1. Let $\mathcal{M}$ be a model category.

1. If $\boldsymbol{X}$ is a cosimplicial object in $\mathcal{M}$ and K is a simplicial set, then the object $\boldsymbol{X} \otimes K$ in $\mathcal{M}$ is defined to be the colimit of the (ΔK)-diagram in $\mathcal{M}$ (see Definition 16.1.11) that takes each n-simplex of K to $\boldsymbol{X}^n$.
2. If $\boldsymbol{Y}$ is a simplicial object in $\mathcal{M}$ and K is a simplicial set, then the object $\boldsymbol{Y}^K$ in $\mathcal{M}$ is defined to be the limit of the $(\Delta^{\mathrm{op}} K)$-diagram in $\mathcal{M}$ that takes each n-simplex of K to $\boldsymbol{Y}_n$.

Proposition 17.3.2. *If $\mathcal{M}$ is a model category, then the constructions of Definition 17.3.1 are natural in $\boldsymbol{X}$, $\boldsymbol{Y}$ and K.*

Proof. This follows directly from the definitions. □

Proposition 17.3.3. *If $\mathcal{M} = \mathsf{SS}$, the cosimplicial object $\boldsymbol{X}$ is the cosimplicial standard simplex (see Definition 16.1.9), and K is a simplicial set, then $\boldsymbol{X} \otimes K$ is naturally isomorphic to K.*

Proof. This is a restatement of Proposition 16.1.14. □

Example 17.3.4. If $\mathcal{M} = \mathsf{Top}$, the cosimplicial object $\boldsymbol{X}$ is the geometric realization of the cosimplicial standard simplex (i.e., $\boldsymbol{X}^n = \big|\Delta[n]\big|$), and K is a simplicial set, then $\boldsymbol{X} \otimes K$ is the usual geometric realization of K.

Lemma 17.3.5. *Let $\mathcal{M}$ be a model category.*

1. *If $\boldsymbol{B}$ is a cosimplicial object in $\mathcal{M}$ and $n \geq 0$, then $\boldsymbol{B} \otimes \Delta[n]$ is naturally isomorphic to $\boldsymbol{B}^n$.*

2. *If $\boldsymbol{X}$ is a simplicial object in $\mathcal{M}$ and $n \geq 0$, then $\boldsymbol{X}^{\Delta[n]}$ is naturally isomorphic to $\boldsymbol{X}_n$.*

PROOF. The nondegenerate n-simplex of $\Delta[n]$ is a terminal object of $\Delta(\Delta[n])$ and an initial object of $\Delta^{\mathrm{op}}(\Delta[n])$. □

LEMMA 17.3.6. *Let $\mathcal{M}$ be a model category.*

1. *If $\boldsymbol{B}$ is a cosimplicial object in $\mathcal{M}$ and $n \geq 0$, then $\boldsymbol{B} \otimes \partial\Delta[n]$ is naturally isomorphic to $\mathrm{L}_n\boldsymbol{B}$, the latching object of $\boldsymbol{B}$ at $[n]$ (see Definition 16.2.17).*
2. *If $\boldsymbol{X}$ is a simplicial object in $\mathcal{M}$ and $n \geq 0$, then $\boldsymbol{X}^{\partial\Delta[n]}$ is naturally isomorphic to $\mathrm{M}_n\boldsymbol{X}$, the matching object of $\boldsymbol{X}$ at $[n]$.*

PROOF. We will prove part 1; the proof of part 2 is dual.

If $n \geq 0$, then the latching object of $\boldsymbol{B}$ at n is

$$\mathrm{L}_n\boldsymbol{B} = \operatorname*{colim}_{((\overrightarrow{\boldsymbol{\Delta}}\downarrow[n])-1_{[n]})} \boldsymbol{B} = \operatorname*{colim}_{\substack{k<n \\ \boldsymbol{\Delta}([k],[n])}} \boldsymbol{B}$$

(see Definition 16.2.17). Since $\boldsymbol{\Delta}([k],[n])$ is naturally isomorphic to the set of k-simplices of $\Delta[n]$, this is the colimit of the diagram with one copy of $\boldsymbol{B}^k$ for every k-simplex of $\Delta[n]$ for $k < n$. The result now follows from Definition 17.3.1. □

PROPOSITION 17.3.7. *Let $\mathcal{M}$ be a model category.*

1. *If $\boldsymbol{B}$ is a cosimplicial object in $\mathcal{M}$ and $n \geq 0$, then the latching map (see Definition 16.2.17) of $\boldsymbol{B}$ at $[n]$ is naturally isomorphic to the map $\boldsymbol{B} \otimes \partial\Delta[n] \to \boldsymbol{B} \otimes \Delta[n]$.*
2. *If $\boldsymbol{X}$ is a simplicial object in $\mathcal{M}$ and $n \geq 0$, then the matching map of $\boldsymbol{X}$ at $[n]$ is naturally isomorphic to the map $\boldsymbol{X}^{\Delta[n]} \to \boldsymbol{X}^{\partial\Delta[n]}$.*

PROOF. This follows from Lemma 17.3.5 and Lemma 17.3.6. □

THEOREM 17.3.8. *Let $\mathcal{M}$ be a model category.*

1. *If $\boldsymbol{A}$ is a cosimplicial object in $\mathcal{M}$, X is an object in $\mathcal{M}$, and K is a simplicial set, then there is a natural isomorphism of sets*

$$\mathsf{SS}\big(K, \mathcal{M}(\boldsymbol{A}, X)\big) \approx \mathcal{M}(\boldsymbol{A} \otimes K, X).$$

2. *If B is an object in $\mathcal{M}$, $\boldsymbol{Y}$ is a simplicial object in $\mathcal{M}$, and K is a simplicial set, then there is a natural isomorphism of sets*

$$\mathsf{SS}\big(K, \mathcal{M}(B, \boldsymbol{Y})\big) \approx \mathcal{M}(B, \boldsymbol{Y}^K).$$

PROOF. We will prove part 1; the proof of part 2 is similar.

Since $\boldsymbol{A} \otimes K$ is the colimit of a (ΔK)-diagram, a map in $\mathcal{M}$ from $\boldsymbol{A} \otimes K$ to X corresponds to a coherent set of maps from each object in the diagram to X. Thus, each map $\boldsymbol{A} \otimes K \to X$ is defined by a map $\boldsymbol{A}^n \to X$ for each n-simplex of K that commute with the simplicial operators. This is also a description of a map of simplicial sets from K to $\mathcal{M}(\boldsymbol{A}, X)$. □

PROPOSITION 17.3.9. *Let $\mathcal{M}$ be a model category.*

1. *If $\boldsymbol{A}$ is a cosimplicial object in $\mathcal{M}$, $\mathcal{C}$ is a small category, and $\boldsymbol{K} \colon \mathcal{C} \to \mathsf{SS}$ is a $\mathcal{C}$-diagram of simplicial sets, then the natural map $\operatorname{colim}_{\mathcal{C}}(\boldsymbol{A} \otimes \boldsymbol{K}) \to \boldsymbol{A} \otimes (\operatorname{colim}_{\mathcal{C}} \boldsymbol{K})$ is an isomorphism.*

2. *If $\boldsymbol{X}$ is a simplicial object in $\mathcal{M}$, $\mathcal{C}$ is a small category, and $\boldsymbol{K}\colon \mathcal{C} \to \mathsf{SS}$ is a $\mathcal{C}$-diagram of simplicial sets, then the natural map $\boldsymbol{X}^{(\operatorname{colim}_{\mathcal{C}} \boldsymbol{K})} \to \lim_{\mathcal{C}^{\mathrm{op}}}(\boldsymbol{X}^{\boldsymbol{K}})$ is an isomorphism.*

PROOF. This follows from the adjointness relations of Theorem 17.3.8. $\square$

PROPOSITION 17.3.10. *Let $\mathcal{M}$ be a simplicial model category.*

1. *If X is an object in $\mathcal{M}$, $\widetilde{\boldsymbol{X}}$ is the standard cosimplicial frame on X (see Proposition 17.1.35), and K is a simplicial set, then $\widetilde{\boldsymbol{X}} \otimes K$ is naturally isomorphic to $X \otimes K$.*
2. *If X is an object in $\mathcal{M}$, $\widehat{\boldsymbol{X}}$ is the standard simplicial frame on X, and K is a simplicial set, then $\widehat{\boldsymbol{X}}^K$ is naturally isomorphic to X^K.*

PROOF. This follows from Proposition 17.3.9 and Proposition 16.1.14. $\square$

LEMMA 17.3.11. *Let $\mathcal{M}$ be a model category, and let (K, L) be a pair of simplicial sets.*

1. *If $\boldsymbol{A}$ is a Reedy cofibrant cosimplicial object in $\mathcal{M}$, then the map $\boldsymbol{A} \otimes L \to \boldsymbol{A} \otimes K$ is a cofibration in $\mathcal{M}$.*
2. *If $\boldsymbol{X}$ is a Reedy fibrant simplicial object in $\mathcal{M}$, then the map $\boldsymbol{X}^K \to \boldsymbol{X}^L$ is a fibration in $\mathcal{M}$.*

PROOF. Since an inclusion $L \to K$ of simplicial sets is a transfinite composition of pushouts of the maps $\partial\Delta[n] \to \Delta[n]$ for $n \geq 0$, the map $\boldsymbol{A} \otimes L \to \boldsymbol{A} \otimes K$ is a transfinite composition of pushouts of the maps $\boldsymbol{A} \otimes \partial\Delta[n] \to \boldsymbol{A} \otimes \Delta[n]$ for $n \geq 0$, and so part 1 follows from Proposition 17.3.9, Proposition 17.3.7, and Proposition 12.2.19. The proof of part 2 is similar. $\square$

PROPOSITION 17.3.12. *Let $\mathcal{M}$ be a model category.*

1. *If $\boldsymbol{A} \to \boldsymbol{B}$ is a Reedy trivial cofibration of cosimplicial objects in $\mathcal{M}$ and $n \geq 0$, then the induced map $\boldsymbol{A} \otimes \Delta[n] \amalg_{\boldsymbol{A} \otimes \partial\Delta[n]} \boldsymbol{B} \otimes \partial\Delta[n] \to \boldsymbol{B} \otimes \Delta[n]$ is a trivial cofibration in $\mathcal{M}$.*
2. *If $\boldsymbol{X} \to \boldsymbol{Y}$ is a Reedy trivial fibration of simplicial objects in $\mathcal{M}$ and $n \geq 0$, then the induced map $\boldsymbol{X}^{\Delta[n]} \to \boldsymbol{Y}^{\Delta[n]} \times_{\boldsymbol{Y}^{\partial\Delta[n]}} \boldsymbol{X}^{\partial\Delta[n]}$ is a trivial fibration in $\mathcal{M}$.*

PROOF. This follows from Proposition 17.3.7 and Theorem 16.3.10. $\square$

PROPOSITION 17.3.13. *Let $\mathcal{M}$ be a model category.*

1. *If $\boldsymbol{A} \to \boldsymbol{B}$ is a Reedy cofibration of cosimplicial objects in $\mathcal{M}$ and $n \geq 0$, then the induced map $\boldsymbol{A} \otimes \Delta[n] \amalg_{\boldsymbol{A} \otimes \partial\Delta[n]} \boldsymbol{B} \otimes \partial\Delta[n] \to \boldsymbol{B} \otimes \Delta[n]$ is a cofibration.*
2. *If $\boldsymbol{X} \to \boldsymbol{Y}$ is a Reedy fibration of simplicial objects in $\mathcal{M}$ and $n \geq 0$, then the induced map $\boldsymbol{X}^{\Delta[n]} \to \boldsymbol{Y}^{\Delta[n]} \times_{\boldsymbol{Y}^{\partial\Delta[n]}} \boldsymbol{X}^{\partial\Delta[n]}$ is a fibration.*

PROOF. This follows from Proposition 17.3.7. $\square$

PROPOSITION 17.3.14. *Let $\mathcal{M}$ be a model category.*

1. *If $i\colon \boldsymbol{A} \to \boldsymbol{B}$ is a map of cosimplicial objects in $\mathcal{M}$, $p\colon X \to Y$ is a map in $\mathcal{M}$, and (K, L) is a pair of simplicial sets, then the following are equivalent:*

(a) *The dotted arrow exists in every solid arrow diagram of the form*

$$\begin{tikzcd} L \arrow[r] \arrow[d] & \mathcal{M}(\boldsymbol{B}, X) \arrow[d] \\ K \arrow[r] \arrow[ur, dotted] & \mathcal{M}(\boldsymbol{A}, X) \times_{\mathcal{M}(\boldsymbol{A}, Y)} \mathcal{M}(\boldsymbol{B}, Y). \end{tikzcd}$$

(b) *The dotted arrow exists in every solid arrow diagram of the form*

$$\begin{tikzcd} \boldsymbol{A} \otimes K \amalg_{\boldsymbol{A} \otimes L} \boldsymbol{B} \otimes L \arrow[r] \arrow[d] & X \arrow[d] \\ \boldsymbol{B} \otimes K \arrow[r] \arrow[ur, dotted] & Y. \end{tikzcd}$$

2. *If $i\colon A \to B$ is a map in $\mathcal{M}$, $p\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a map of simplicial objects in $\mathcal{M}$, and (K, L) is a pair of simplicial sets, then the following are equivalent:*
   (a) *The dotted arrow exists in every solid arrow diagram of the form*

$$\begin{tikzcd} L \arrow[r] \arrow[d] & \mathcal{M}(B, \boldsymbol{X}) \arrow[d] \\ K \arrow[r] \arrow[ur, dotted] & \mathcal{M}(A, \boldsymbol{X}) \times_{\mathcal{M}(A, \boldsymbol{Y})} \mathcal{M}(B, \boldsymbol{Y}). \end{tikzcd}$$

   (b) *The dotted arrow exists in every solid arrow diagram of the form*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \boldsymbol{X}^K \arrow[d] \\ B \arrow[r] \arrow[ur, dotted] & \boldsymbol{X}^L \times_{\boldsymbol{Y}^L} \boldsymbol{Y}^K. \end{tikzcd}$$

PROOF. This follows from Theorem 17.3.8. □

PROPOSITION 17.3.15. *Let $\mathcal{M}$ be a model category.*

1. *If $i\colon \boldsymbol{A} \to \boldsymbol{B}$ is a Reedy cofibration of cosimplicial objects in $\mathcal{M}$, $p\colon X \to Y$ is a fibration in $\mathcal{M}$, and at least one of i and p is also a weak equivalence, then the map of simplicial sets*

$$\mathcal{M}(\boldsymbol{B}, X) \to \mathcal{M}(\boldsymbol{A}, X) \times_{\mathcal{M}(\boldsymbol{A}, Y)} \mathcal{M}(\boldsymbol{B}, Y)$$

   is a trivial fibration.
2. *If $i\colon A \to B$ is a cofibration in $\mathcal{M}$, $p\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a Reedy fibration of simplicial objects in $\mathcal{M}$, and at least one of i and p is also a weak equivalence, then the map of simplicial sets*

$$\mathcal{M}(B, \boldsymbol{X}) \to \mathcal{M}(A, \boldsymbol{X}) \times_{\mathcal{M}(A, \boldsymbol{Y})} \mathcal{M}(B, \boldsymbol{Y})$$

   is a trivial fibration.

PROOF. A map of simplicial sets is a trivial fibration if and only if it has the right lifting property with respect to the maps $\partial\Delta[n] \to \Delta[n]$ for $n \geq 0$, and so the result follows from Proposition 17.3.14, Proposition 17.3.12, and Proposition 17.3.13. □

Proposition 17.3.15 may seem to be incomplete, in that it does not assert the full homotopy lifting extension theorem. We will show in Theorem 17.4.1 that if the cosimplicial and simplicial objects are assumed to be cosimplicial and simplicial *resolutions* (see Definition 17.1.16), then the full homotopy lifting extension theorem does hold. We now give an example that shows that it does not hold without the assumption that the cosimplicial or simplicial objects are resolutions.

EXAMPLE 17.3.16. Let $\mathcal{M}$ be the category SS_* of pointed simplicial sets. Let $\boldsymbol{B}$ be the cosimplicial object in $\mathcal{M}$ that is the free diagram on S^1 generated at $[1]$ (see Definition 14.1.17 and Definition 16.1.2), so that $\boldsymbol{B}^n = \bigvee_{\boldsymbol{\Delta}([1],[n])} \mathrm{S}^1$ (where $\boldsymbol{\Delta}([1],[n])$ is the set of 1-simplices of $\Delta[n]$). Corollary 16.4.3 implies that $\boldsymbol{B}$ is a Reedy cofibrant cosimplicial object.

Let $p\colon X \to Y$ be any fibration of fibrant pointed simplicial sets for which the induced homomorphism of fundamental groups $p_*\colon \pi_1 X \to \pi_1 Y$ is *not* surjective. We will show that the map of simplicial sets $\mathcal{M}(\boldsymbol{B}, X) \to \mathcal{M}(\boldsymbol{B}, Y)$ is not a fibration.

$\boldsymbol{B}^1$ is the wedge of three copies of S^1 (indexed by $[0,0]$, $[1,1]$, and $[0,1]$), $\boldsymbol{B}^0$ is a single copy of S^1, and the maps $d^0, d^1\colon \boldsymbol{B}^0 \to \boldsymbol{B}^1$ take the S^1 in $\boldsymbol{B}^0$ to the summand indexed by, respectively, $[0,0]$ and $[1,1]$. Thus, we can define a 1-simplex of $\mathcal{M}(\boldsymbol{B}, Y)$ by sending the summands of $\boldsymbol{B}^1$ corresponding to $[0,0]$ and $[1,1]$ to the basepoint of Y and sending the summand S^1 of $\boldsymbol{B}^1$ corresponding to $[0,1]$ to some 1-simplex of Y that represents an element of $\pi_1 Y$ that is *not* in the image of $p_*\colon \pi_1 X \to \pi_1 Y$. If we define a 0-simplex of $\mathcal{M}(\boldsymbol{B}, X)$ by sending $\boldsymbol{B}^0$ to the basepoint of X, then we have a solid arrow diagram

$$\begin{tikzcd}
\Delta[0] \arrow[r] \arrow[d] & \mathcal{M}(\boldsymbol{B}, X) \arrow[d] \\
\Delta[1] \arrow[r] \arrow[ur, dotted] & \mathcal{M}(\boldsymbol{B}, Y)
\end{tikzcd}$$

for which there is no dotted arrow making the triangles commute.

LEMMA 17.3.17. *Let $\mathcal{M}$ be a model category.*

1. *If $\boldsymbol{A} \to \boldsymbol{B}$ is a Reedy cofibration of cosimplicial objects in $\mathcal{M}$, $n \geq 1$, and $n \geq k \geq 0$, then the induced map $\boldsymbol{A} \otimes \Delta[n] \amalg_{\boldsymbol{A} \otimes \Lambda[n,k]} \boldsymbol{B} \otimes \Lambda[n,k] \to \boldsymbol{B} \otimes \Delta[n]$ is a cofibration.*
2. *If $\boldsymbol{X} \to \boldsymbol{Y}$ is a Reedy fibration of simplicial objects in $\mathcal{M}$, $n \geq 1$, and $n \geq k \geq 0$, then the induced map $\boldsymbol{X}^{\Delta[n]} \to \boldsymbol{Y}^{\Delta[n]} \times_{\boldsymbol{Y}^{\Lambda[n,k]}} \boldsymbol{X}^{\Lambda[n,k]}$ is a fibration.*

PROOF. We will prove part 1; the proof of part 2 is similar.

We have the diagram

$$\begin{tikzcd}
\boldsymbol{A} \otimes \Delta[n-1] \amalg_{\boldsymbol{A} \otimes \partial\Delta[n-1]} \boldsymbol{B} \otimes \partial\Delta[n-1] \arrow[r] \arrow[d] & \boldsymbol{A} \otimes \Delta[n] \amalg_{\boldsymbol{A} \otimes \Lambda[n,k]} \boldsymbol{B} \otimes \Lambda[n,k] \arrow[d] \\
\boldsymbol{B} \otimes \Delta[n-1] \arrow[r] & \boldsymbol{A} \otimes \Delta[n] \amalg_{\boldsymbol{A} \otimes \partial\Delta[n]} \boldsymbol{B} \otimes \partial\Delta[n] \arrow[d] \\
 & \boldsymbol{B} \otimes \Delta[n]
\end{tikzcd}$$

in which the square is a pushout, and so Proposition 17.3.13 implies that all of the vertical maps are cofibrations. Our map is thus the composition of two cofibrations. □

LEMMA 17.3.18. *If $n > 1$ and $n \geq k \geq 0$, then there is a finite sequence of inclusions of simplicial sets*

$$\Delta[0] = K_0 \to K_1 \to K_2 \to \cdots \to K_p = \Lambda[n,k]$$

where each map $K_i \to K_{i+1}$ for $i < p$ is constructed as a pushout

$$\begin{tikzcd} \Lambda[m_i, l_i] \arrow[r] \arrow[d] & K_i \arrow[d, dotted] \\ \Delta[m_i] \arrow[r, dotted] & K_{i+1} \end{tikzcd}$$

with $m_i < n$.

PROOF. We let $\Delta[0] = K_0$ be vertex k of $\Delta[n]$. We can then add in all the 1-simplices of $\Lambda[n,k]$ that contain that vertex, followed by the 2-simplices of $\Lambda[n,k]$ that contain that vertex, etc., until we've added in all of $\Lambda[n,k]$. □

LEMMA 17.3.19. *Let $\mathcal{M}$ be a model category.*

1. *If $\boldsymbol{A}$ is a cosimplicial resolution in $\mathcal{M}$, $n \geq 1$, and $n \geq k \geq 0$, then the natural map $\boldsymbol{A} \otimes \Lambda[n,k] \to \boldsymbol{A} \otimes \Delta[n]$ is a weak equivalence.*
2. *If $\boldsymbol{X}$ is a simplicial resolution in $\mathcal{M}$, $n \geq 1$, and $n \geq k \geq 0$, then the natural map $\boldsymbol{X}^{\Delta[n]} \to \boldsymbol{X}^{\Lambda[n,k]}$ is a weak equivalence.*

PROOF. We will prove part 1; the proof of part 2 is similar.

We will prove the lemma by induction on n. If $n = 1$, then the result follows from Lemma 17.3.5 and Proposition 17.1.17.

We now assume that $\boldsymbol{A} \otimes \Lambda[m,l] \to \boldsymbol{A} \otimes \Delta[m]$ is a weak equivalence for $l \leq m < n$. Lemma 17.3.18 implies that there is a finite sequence of maps in $\mathcal{M}$

$$\boldsymbol{A} \otimes \Delta[0] = \boldsymbol{A} \otimes K_0 \to \boldsymbol{A} \otimes K_1 \to \boldsymbol{A} \otimes K_2 \to \cdots \to \boldsymbol{A} \otimes K_p = \boldsymbol{A} \otimes \Lambda[n,k]$$

where each $\boldsymbol{A} \otimes K_i \to \boldsymbol{A} \otimes K_{i+1}$ for $i < p$ is constructed as a pushout

$$\begin{tikzcd} \boldsymbol{A} \otimes \Lambda[m_i, l_i] \arrow[r] \arrow[d] & \boldsymbol{A} \otimes K_i \arrow[d, dotted] \\ \boldsymbol{A} \otimes \Delta[m_i] \arrow[r, dotted] & \boldsymbol{A} \otimes K_{i+1} \end{tikzcd}$$

with $m_i < n$. Lemma 17.3.11 and the induction hypothesis imply that each of these maps is a trivial cofibration, and so $\boldsymbol{A} \otimes \Delta[0] \to \boldsymbol{A} \otimes \Lambda[n,k]$ is a trivial cofibration. Since $\boldsymbol{A} \otimes \Delta[0] \to \boldsymbol{A} \otimes \Delta[n]$ is a weak equivalence, the "two out of three" property of weak equivalences implies the result. □

PROPOSITION 17.3.20. *Let $\mathcal{M}$ be a model category.*

1. *If $\boldsymbol{A} \to \boldsymbol{B}$ is a Reedy cofibration of cosimplicial resolutions in $\mathcal{M}$, $n > 1$, and $n \geq k \geq 0$, then the map $\boldsymbol{A} \otimes \Delta[n] \amalg_{\boldsymbol{A} \otimes \Lambda[n,k]} \boldsymbol{B} \otimes \Lambda[n,k] \to \boldsymbol{B} \otimes \Delta[n]$ is a trivial cofibration.*
2. *If $\boldsymbol{X} \to \boldsymbol{Y}$ is a Reedy fibration of simplicial resolutions in $\mathcal{M}$, $n > 1$, and $n \geq k \geq 0$, then the map $\boldsymbol{X}^{\Delta[n]} \to \boldsymbol{Y}^{\Delta[n]} \times_{\boldsymbol{Y}^{\Lambda[n,k]}} \boldsymbol{X}^{\Lambda[n,k]}$ is a trivial fibration.*

Proof. We will prove part 1; the proof of part 2 is similar.

Lemma 17.3.17 implies that our map is a cofibration, and so it remains only to show that it is a weak equivalence. Lemma 17.3.19 and Lemma 17.3.11 imply that $\boldsymbol{A}\otimes\Lambda[n,k]\to\boldsymbol{A}\otimes\Delta[n]$ is a trivial cofibration. Since the diagram

$$\begin{tikzcd}
\boldsymbol{A}\otimes\Lambda[n,k] \arrow[r] \arrow[d] & \boldsymbol{B}\otimes\Lambda[n,k] \arrow[d] \\
\boldsymbol{A}\otimes\Delta[n] \arrow[r] & \boldsymbol{A}\otimes\Delta[n]\amalg_{\boldsymbol{A}\otimes\Lambda[n,k]}\boldsymbol{B}\otimes\Lambda[n,k]
\end{tikzcd}$$

is a pushout, the map $\boldsymbol{B}\otimes\Lambda[n,k]\to\boldsymbol{A}\otimes\Delta[n]\amalg_{\boldsymbol{A}\otimes\Lambda[n,k]}\boldsymbol{B}\otimes\Lambda[n,k]$ is also a trivial cofibration. Since Lemma 17.3.19 implies that the map $\boldsymbol{B}\otimes\Lambda[n,k]\to\boldsymbol{B}\otimes\Delta[n]$ is a weak equivalence, the result follows from the "two out of three" property of weak equivalences. □

17.4. Homotopy lifting extension theorems

Theorem 17.4.1 (The one-sided homotopy lifting extension theorem). *Let $\mathcal{M}$ be a model category.*

1. *If $i\colon\boldsymbol{A}\to\boldsymbol{B}$ is a Reedy cofibration of cosimplicial resolutions in $\mathcal{M}$ and $p\colon X\to Y$ is a fibration in $\mathcal{M}$, then the map of simplicial sets*
$$\mathcal{M}(\boldsymbol{B},X)\to\mathcal{M}(\boldsymbol{A},X)\times_{\mathcal{M}(\boldsymbol{A},Y)}\mathcal{M}(\boldsymbol{B},Y)$$
is a fibration that is a trivial fibration if at least one of i and p is also a weak equivalence.
2. *If $i\colon A\to B$ is a cofibration in $\mathcal{M}$ and $p\colon\boldsymbol{X}\to\boldsymbol{Y}$ is a Reedy fibration of simplicial resolutions in $\mathcal{M}$, then the map of simplicial sets*
$$\mathcal{M}(B,\boldsymbol{X})\to\mathcal{M}(A,\boldsymbol{X})\times_{\mathcal{M}(A,\boldsymbol{Y})}\mathcal{M}(B,\boldsymbol{Y})$$
is a fibration that is a trivial fibration if at least one of i and p is also a weak equivalence.

Proof. A map of simplicial sets is a fibration if and only if it has the right lifting property with respect to the maps $\Lambda[n,k]\to\Delta[n]$ for $n>0$ and $n\geq k\geq 0$, and so the result follows from Proposition 17.3.14, Proposition 17.3.20, and Proposition 17.3.15. □

Corollary 17.4.2. *Let $\mathcal{M}$ be a model category.*

1. *If $i\colon\boldsymbol{A}\to\boldsymbol{B}$ is a Reedy cofibration of cosimplicial resolutions in $\mathcal{M}$ and X is a fibrant object in $\mathcal{M}$, then the map $i^*\colon\mathcal{M}(\boldsymbol{B},X)\to\mathcal{M}(\boldsymbol{A},X)$ is a fibration of simplicial sets.*
2. *If $\boldsymbol{A}$ is a cosimplicial resolution in $\mathcal{M}$ and $p\colon X\to Y$ is a fibration in $\mathcal{M}$, then the map $p_*\colon\mathcal{M}(\boldsymbol{A},X)\to\mathcal{M}(\boldsymbol{A},Y)$ is a fibration of simplicial sets.*
3. *If $i\colon A\to B$ is a cofibration in $\mathcal{M}$ and $\boldsymbol{X}$ is a simplicial resolution in $\mathcal{M}$, then the map $i^*\colon\mathcal{M}(B,\boldsymbol{X})\to\mathcal{M}(A,\boldsymbol{X})$ is a fibration of simplicial sets.*
4. *If A is a cofibrant object in $\mathcal{M}$ and $p\colon\boldsymbol{X}\to\boldsymbol{Y}$ is a Reedy fibration of simplicial resolutions in $\mathcal{M}$, then the map $p_*\colon\mathcal{M}(A,\boldsymbol{X})\to\mathcal{M}(A,\boldsymbol{Y})$ is a fibration of simplicial sets.*

Proof. This follows from Theorem 17.4.1. □

Corollary 17.4.3. *Let $\mathcal{M}$ be a model category.*

1. *If $i\colon \boldsymbol{A} \to \boldsymbol{B}$ is a Reedy trivial cofibration of cosimplicial resolutions in $\mathcal{M}$ and X is a fibrant object in $\mathcal{M}$, then the map $i^*\colon \mathcal{M}(\boldsymbol{B}, X) \to \mathcal{M}(\boldsymbol{A}, X)$ is a trivial fibration of simplicial sets.*
2. *If $\boldsymbol{A}$ is a cosimplicial resolution in $\mathcal{M}$ and $p\colon X \to Y$ is a trivial fibration in $\mathcal{M}$, then the map $p_*\colon \mathcal{M}(\boldsymbol{A}, X) \to \mathcal{M}(\boldsymbol{A}, Y)$ is a trivial fibration of simplicial sets.*
3. *If $i\colon A \to B$ is a trivial cofibration in $\mathcal{M}$ and $\boldsymbol{X}$ is a simplicial resolution in $\mathcal{M}$, then the map $i^*\colon \mathcal{M}(B, \boldsymbol{X}) \to \mathcal{M}(A, \boldsymbol{X})$ is a trivial fibration of simplicial sets.*
4. *If A is a cofibrant object in $\mathcal{M}$ and $p\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a Reedy trivial fibration of simplicial resolutions in $\mathcal{M}$, then the map $p_*\colon \mathcal{M}(A, \boldsymbol{X}) \to \mathcal{M}(A, \boldsymbol{Y})$ is a trivial fibration of simplicial sets.*

PROOF. This follows from Theorem 17.4.1. □

PROPOSITION 17.4.4. *Let $\mathcal{M}$ be a model category.*

1. *If $i\colon \boldsymbol{A} \to \boldsymbol{B}$ is a Reedy cofibration of cosimplicial resolutions in $\mathcal{M}$ and $j\colon L \to K$ is a cofibration of simplicial sets, then the map $\boldsymbol{A} \otimes K \amalg_{\boldsymbol{A}\otimes L} \boldsymbol{B} \otimes L \to \boldsymbol{B} \otimes K$ is a cofibration in $\mathcal{M}$ that is a trivial cofibration if either i or j is a weak equivalence.*
2. *If $p\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a Reedy fibration of simplicial resolutions in $\mathcal{M}$ and $j\colon L \to K$ is a cofibration of simplicial sets, then the map $\boldsymbol{X}^K \to \boldsymbol{X}^L \times_{\boldsymbol{Y}^L} \boldsymbol{Y}^K$ is a fibration in $\mathcal{M}$ that is a trivial fibration if either p or j is a weak equivalence.*

PROOF. This follows from Proposition 8.2.3, Proposition 17.3.14, and Theorem 17.4.1. □

DEFINITION 17.4.5. Let $\mathcal{M}$ be a model category. If $\boldsymbol{B}$ is a cosimplicial object in $\mathcal{M}$ and $\boldsymbol{X}$ is a simplicial object in $\mathcal{M}$, then the bisimplicial set $\mathcal{M}(\boldsymbol{B}, \boldsymbol{X})$ (for which $\mathcal{M}(\boldsymbol{B}, \boldsymbol{X})_{n,k} = \mathcal{M}(\boldsymbol{B}^k, \boldsymbol{X}_n)$) can be considered a simplicial object in the category of simplicial objects in $\mathcal{M}$ in two ways. We define the *horizontal simplicial object* to be the one whose object in degree n is $\mathcal{M}(\boldsymbol{B}, \boldsymbol{X}_n)$ (see Notation 17.2.1), and the *vertical simplicial object* to be the one whose object in degree k is $\mathcal{M}(\boldsymbol{B}^k, \boldsymbol{X})$.

LEMMA 17.4.6. *Let $\mathcal{M}$ be a model category, let $\boldsymbol{B}$ be a cosimplicial object in $\mathcal{M}$, and let $\boldsymbol{X}$ be a simplicial object in $\mathcal{M}$.*

1. *If we consider $\mathcal{M}(\boldsymbol{B}, \boldsymbol{X})$ as a horizontal simplicial object, then for every $n \geq 0$ there is a natural isomorphism of simplicial objects (see Definition 16.2.17) $\mathrm{M}_n\mathcal{M}(\boldsymbol{B}, \boldsymbol{X}) \approx \mathcal{M}(\boldsymbol{B}, \mathrm{M}_n\boldsymbol{X})$.*
2. *If we consider $\mathcal{M}(\boldsymbol{B}, \boldsymbol{X})$ as a vertical simplicial object, then for every $n \geq 0$ there is a natural isomorphism of simplicial objects $\mathrm{M}_n\mathcal{M}(\boldsymbol{B}, \boldsymbol{X}) \approx \mathcal{M}(\mathrm{L}_n\boldsymbol{B}, \boldsymbol{X})$.*

PROOF. Since the matching object M_n is defined as a limit, part 1 follows from the universal mapping property of the limit. Since the latching object L_n is defined as a colimit, part 2 follows from Proposition 16.2.15 and the universal mapping property of the colimit. □

LEMMA 17.4.7. *Let $\mathcal{M}$ be a model category, let $\boldsymbol{A} \to \boldsymbol{B}$ be a map of cosimplicial objects in $\mathcal{M}$, and let $\boldsymbol{X} \to \boldsymbol{Y}$ be a map of simplicial objects in $\mathcal{M}$.*

1. *If all bisimplicial sets are considered horizontal simplicial objects, then for every $n \geq 0$ there is a natural isomorphism of simplicial sets (see Definition* 16.2.17*)*

$$\mathrm{M}_n\bigl(\mathcal{M}(\boldsymbol{A},\boldsymbol{X}) \times_{\mathcal{M}(\boldsymbol{A},\boldsymbol{Y})} \mathcal{M}(\boldsymbol{B},\boldsymbol{Y})\bigr) \approx \mathcal{M}(\boldsymbol{A},\mathrm{M}_n\boldsymbol{X}) \times_{\mathcal{M}(\boldsymbol{A},\mathrm{M}_n\boldsymbol{Y})} \mathcal{M}(\boldsymbol{B},\mathrm{M}_n\boldsymbol{Y}).$$

2. *If all bisimplicial sets are considered vertical simplicial objects, then for every $n \geq 0$ there is a natural isomorphism of simplicial sets*

$$\mathrm{M}_n\bigl(\mathcal{M}(\boldsymbol{A},\boldsymbol{X}) \times_{\mathcal{M}(\boldsymbol{A},\boldsymbol{Y})} \mathcal{M}(\boldsymbol{B},\boldsymbol{Y})\bigr) \approx \mathcal{M}(\mathrm{L}_n\boldsymbol{A},\boldsymbol{X}) \times_{\mathcal{M}(\mathrm{L}_n\boldsymbol{A},\boldsymbol{Y})} \mathcal{M}(\mathrm{L}_n\boldsymbol{B},\boldsymbol{Y}).$$

PROOF. This follows from Lemma 17.4.6. □

THEOREM 17.4.8 (The bisimplicial homotopy lifting extension theorem). *Let $\mathcal{M}$ be a model category. If $i\colon \boldsymbol{A} \to \boldsymbol{B}$ is a Reedy cofibration of cosimplicial resolutions in $\mathcal{M}$ and $p\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a Reedy fibration of simplicial resolutions in $\mathcal{M}$, then, for both the horizontal simplicial object structure (see Definition* 17.4.5*) and the vertical simplicial object structure, the induced map of bisimplicial sets*

$$\mathcal{M}(\boldsymbol{B},\boldsymbol{X}) \to \mathcal{M}(\boldsymbol{A},\boldsymbol{X}) \times_{\mathcal{M}(\boldsymbol{A},\boldsymbol{Y})} \mathcal{M}(\boldsymbol{B},\boldsymbol{Y})$$

is a Reedy fibration of simplicial objects that is a Reedy trivial fibration if at least one of i and p is a weak equivalence.

PROOF. We will prove this for the horizontal structure; the proof for the vertical structure is similar.

Theorem 16.3.10 implies that it is sufficient to show that for every $n \geq 0$ the map

$$\begin{aligned}&\mathcal{M}(\boldsymbol{B},\boldsymbol{X})_n\\ &\quad\to \bigl(\mathcal{M}(\boldsymbol{A},\boldsymbol{X}) \times_{\mathcal{M}(\boldsymbol{A},\boldsymbol{Y})} \mathcal{M}(\boldsymbol{B},\boldsymbol{Y})\bigr)_n \times_{\mathrm{M}_n(\mathcal{M}(\boldsymbol{A},\boldsymbol{X})\times_{\mathcal{M}(\boldsymbol{A},\boldsymbol{Y})}\mathcal{M}(\boldsymbol{B},\boldsymbol{Y}))} \mathrm{M}_n\mathcal{M}(\boldsymbol{B},\boldsymbol{X})\end{aligned}$$

is a fibration of simplicial sets that is a trivial fibration if either of i and p is a weak equivalence. Lemma 17.4.6 and Lemma 17.4.7 imply that this map is isomorphic to the map

$$\begin{aligned}&\mathcal{M}(\boldsymbol{B},\boldsymbol{X})\\ &\to \bigl(\mathcal{M}(\boldsymbol{A},\boldsymbol{X}_n)\times_{\mathcal{M}(\boldsymbol{A},\boldsymbol{Y}_n)}\mathcal{M}(\boldsymbol{B},\boldsymbol{Y}_n)\bigr)\times_{(\mathcal{M}(\boldsymbol{A},\mathrm{M}_n\boldsymbol{X})\times_{\mathcal{M}(\boldsymbol{A},\mathrm{M}_n\boldsymbol{Y})}\mathcal{M}(\boldsymbol{B},\mathrm{M}_n\boldsymbol{Y}))}\mathcal{M}(\boldsymbol{B},\mathrm{M}_n\boldsymbol{X})\end{aligned}$$

The codomain of this map is the limit of the diagram

$$\begin{tikzcd}
\mathcal{M}(\boldsymbol{A},\boldsymbol{X}_n) \arrow[r] \arrow[d] & \mathcal{M}(\boldsymbol{A},\boldsymbol{Y}_n) \arrow[d] & \mathcal{M}(\boldsymbol{B},\boldsymbol{Y}_n) \arrow[l] \arrow[d] \\
\mathcal{M}(\boldsymbol{A},\mathrm{M}_n\boldsymbol{X}) \arrow[r] & \mathcal{M}(\boldsymbol{A},\mathrm{M}_n\boldsymbol{Y}) & \mathcal{M}(\boldsymbol{B},\mathrm{M}_n\boldsymbol{Y}) \arrow[l] \\
& \mathcal{M}(\boldsymbol{B},\mathrm{M}_n\boldsymbol{X}) \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

and so our map is isomorphic to the map

$$\begin{aligned}&\mathcal{M}(\boldsymbol{B},\boldsymbol{X}_n)\\ &\to \mathcal{M}(\boldsymbol{A},\boldsymbol{X}_n)\times_{(\mathcal{M}(\boldsymbol{A},\boldsymbol{Y}_n)\times_{\mathcal{M}(\boldsymbol{A},\mathrm{M}_n\boldsymbol{Y})}\mathcal{M}(\boldsymbol{A},\mathrm{M}_n\boldsymbol{X}))}\bigl(\mathcal{M}(\boldsymbol{B},\boldsymbol{Y}_n)\times_{\mathcal{M}(\boldsymbol{B},\mathrm{M}_n\boldsymbol{Y})}\mathcal{M}(\boldsymbol{B},\mathrm{M}_n\boldsymbol{X})\bigr)\end{aligned}$$

Since p is a Reedy fibration, the map $\boldsymbol{X}_n \to \boldsymbol{Y}_n \times_{\mathrm{M}_n \boldsymbol{Y}} \mathrm{M}_n \boldsymbol{X}$ is a fibration of simplicial sets, and so the result now follows from Theorem 17.4.1 and Theorem 16.3.10. □

THEOREM 17.4.9 (The two-sided homotopy lifting extension theorem). *Let $\mathcal{M}$ be a model category. If $i\colon \boldsymbol{A} \to \boldsymbol{B}$ is a Reedy cofibration of cosimplicial resolutions in $\mathcal{M}$ and $p\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a Reedy fibration of simplicial resolutions in $\mathcal{M}$, then the induced map of simplicial sets*

$$\operatorname{diag}\mathcal{M}(\boldsymbol{B},\boldsymbol{X}) \to \operatorname{diag}\mathcal{M}(\boldsymbol{A},\boldsymbol{X}) \times_{\operatorname{diag}\mathcal{M}(\boldsymbol{A},\boldsymbol{Y})} \operatorname{diag}\mathcal{M}(\boldsymbol{B},\boldsymbol{Y})$$

is a fibration of fibrant simplicial sets that is a trivial fibration if at least one of i and p is a weak equivalence.

PROOF. This follows from Theorem 17.4.8, Proposition 16.3.7, Theorem 16.5.5, Proposition 16.3.8, Theorem 16.5.8, and Theorem 16.5.4. □

COROLLARY 17.4.10. *If $\mathcal{M}$ is a model category and X and Y are objects in $\mathcal{M}$, then all homotopy function complexes (see Definition 17.2.2) from X to Y in $\mathcal{M}$ are fibrant simplicial sets.*

PROOF. This follows from Theorem 17.4.1 and Theorem 17.4.9. □

COROLLARY 17.4.11. *Let $\mathcal{M}$ be a model category.*

1. *If $i\colon \boldsymbol{A} \to \boldsymbol{B}$ is a Reedy cofibration of cosimplicial resolutions in $\mathcal{M}$ and $\boldsymbol{X}$ is a simplicial resolution in $\mathcal{M}$, then the induced map of two-sided homotopy function complexes* $\operatorname{diag} i^*\colon \operatorname{diag}\mathcal{M}(\boldsymbol{B},\boldsymbol{X}) \to \operatorname{diag}\mathcal{M}(\boldsymbol{A},\boldsymbol{X})$ *is a fibration of fibrant simplicial sets.*
2. *If $\boldsymbol{A}$ is a cosimplicial resolution in $\mathcal{M}$ and $p\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a Reedy fibration of simplicial resolutions in $\mathcal{M}$, then the induced map of two-sided homotopy function complexes* $\operatorname{diag} p_*\colon \operatorname{diag}\mathcal{M}(\boldsymbol{A},\boldsymbol{X}) \to \operatorname{diag}\mathcal{M}(\boldsymbol{A},\boldsymbol{Y})$ *is a fibration of fibrant simplicial sets.*

PROOF. This follows from Theorem 17.4.9 and Corollary 17.4.10. □

COROLLARY 17.4.12. *Let $\mathcal{M}$ be a model category.*

1. *If $i\colon \boldsymbol{A} \to \boldsymbol{B}$ is a Reedy trivial cofibration of cosimplicial resolutions in $\mathcal{M}$ and $\boldsymbol{X}$ is a simplicial resolution in $\mathcal{M}$, then the induced map of two-sided homotopy function complexes* $\operatorname{diag} i^*\colon \operatorname{diag}\mathcal{M}(\boldsymbol{B},\boldsymbol{X}) \to \operatorname{diag}\mathcal{M}(\boldsymbol{A},\boldsymbol{X})$ *is a trivial fibration.*
2. *If $\boldsymbol{A}$ is a cosimplicial resolution in $\mathcal{M}$ and $p\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a Reedy trivial fibration of simplicial resolutions in $\mathcal{M}$, then the induced map of two-sided homotopy function complexes* $\operatorname{diag} p_*\colon \operatorname{diag}\mathcal{M}(\boldsymbol{A},\boldsymbol{X}) \to \operatorname{diag}\mathcal{M}(\boldsymbol{A},\boldsymbol{Y})$ *is a trivial fibration.*

PROOF. This follows from Theorem 16.5.5, Theorem 17.4.8, Proposition 16.3.8, Theorem 16.5.8, and Theorem 16.5.4. □

17.5. Homotopy invariance

THEOREM 17.5.1. *Let $\mathcal{M}$ be a model category.*

1. *If $i\colon \boldsymbol{A} \to \boldsymbol{B}$ is a weak equivalence of cosimplicial resolutions in $\mathcal{M}$ and X is a fibrant object in $\mathcal{M}$, then the map of left homotopy function complexes $i^*\colon \mathcal{M}(\boldsymbol{B},X) \to \mathcal{M}(\boldsymbol{A},X)$ is a weak equivalence of fibrant simplicial sets.*

2. *If $\boldsymbol{A}$ is a cosimplicial resolution in $\mathcal{M}$ and $p\colon X \to Y$ is a weak equivalence of fibrant objects in $\mathcal{M}$, then the map of left homotopy function complexes $p_*\colon \mathcal{M}(\boldsymbol{A}, X) \to \mathcal{M}(\boldsymbol{A}, Y)$ is a weak equivalence of fibrant simplicial sets.*
3. *If $i\colon A \to B$ is a weak equivalence of cofibrant objects in $\mathcal{M}$ and $\boldsymbol{X}$ is a simplicial resolution in $\mathcal{M}$, then the map of right homotopy function complexes $i^*\colon \mathcal{M}(B, \boldsymbol{X}) \to \mathcal{M}(A, \boldsymbol{X})$ is a weak equivalence of fibrant simplicial sets.*
4. *If A is a cofibrant object in $\mathcal{M}$ and $p\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a weak equivalence of simplicial resolutions in $\mathcal{M}$, then the map of right homotopy function complexes $p_*\colon \mathcal{M}(A, \boldsymbol{X}) \to \mathcal{M}(A, \boldsymbol{Y})$ is a weak equivalence of fibrant simplicial sets.*
5. *If $i\colon \boldsymbol{A} \to \boldsymbol{B}$ is a weak equivalence of cosimplicial resolutions in $\mathcal{M}$ and $\boldsymbol{X}$ is a simplicial resolution in $\mathcal{M}$, then the induced map of two-sided homotopy function complexes $\operatorname{diag} i^*\colon \operatorname{diag}\mathcal{M}(\boldsymbol{B}, \boldsymbol{X}) \to \operatorname{diag}\mathcal{M}(\boldsymbol{A}, \boldsymbol{X})$ is a weak equivalence of fibrant simplicial sets.*
6. *If $\boldsymbol{A}$ is a cosimplicial resolution in $\mathcal{M}$ and $p\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a weak equivalence of simplicial resolutions in $\mathcal{M}$, then the induced map of two-sided homotopy function complexes $\operatorname{diag} p_*\colon \operatorname{diag}\mathcal{M}(\boldsymbol{A}, \boldsymbol{X}) \to \operatorname{diag}\mathcal{M}(\boldsymbol{A}, \boldsymbol{Y})$ is a weak equivalence of fibrant simplicial sets.*

PROOF. This follows from Corollary 17.4.3, Corollary 17.4.12, Lemma 17.1.18, Lemma 8.5.1, and Corollary 17.4.10. $\square$

THEOREM 17.5.2. *Let $\mathcal{M}$ be a model category and let W, X, Y, and Z be objects in $\mathcal{M}$. If $g\colon X \to Y$ is a weak equivalence, then*

1. *any map of homotopy function complexes $g_*\colon \operatorname{map}(W, X) \to \operatorname{map}(W, Y)$ induced by g (see Definition 17.2.4) is a weak equivalence of fibrant simplicial sets, and*
2. *any map of homotopy function complexes $g^*\colon \operatorname{map}(Y, Z) \to \operatorname{map}(X, Z)$ induced by g is a weak equivalence of fibrant simplicial sets.*

PROOF. This follows from Theorem 17.5.1 and Proposition 17.1.14. $\square$

PROPOSITION 17.5.3. *Let $\mathcal{M}$ be a model category, let $\boldsymbol{B}$ be a cosimplicial resolution in $\mathcal{M}$, and let $\boldsymbol{X}$ be a simplicial resolution in $\mathcal{M}$.*

1. *If we consider the bisimplicial set $\mathcal{M}(\boldsymbol{B}, \boldsymbol{X})$ as a horizontal simplicial object (see Definition 17.4.5) in the category of simplicial sets (so that in simplicial degree n we have the simplicial set $\mathcal{M}(\boldsymbol{B}, \boldsymbol{X}_n)$), then $\mathcal{M}(\boldsymbol{B}, \boldsymbol{X})$ is a simplicial resolution of the simplicial set $\mathcal{M}(\boldsymbol{B}, \boldsymbol{X}_0)$.*
2. *If we consider the bisimplicial set $\mathcal{M}(\boldsymbol{B}, \boldsymbol{X})$ as a vertical simplicial object in the category of simplicial sets (so that in simplicial degree n we have the simplicial set $\mathcal{M}(\boldsymbol{B}^n, \boldsymbol{X})$), then $\mathcal{M}(\boldsymbol{B}, \boldsymbol{X})$ is a simplicial resolution of the simplicial set $\mathcal{M}(\boldsymbol{B}^0, \boldsymbol{X})$.*

PROOF. We will prove part 1; the proof of part 2 is similar.

Theorem 17.4.8 implies that $\mathcal{M}(\boldsymbol{B}, \boldsymbol{X})$ is a fibrant simplicial object, and Theorem 17.5.1 implies that, for every $n > 0$, the natural map $\mathcal{M}(\boldsymbol{B}, \boldsymbol{X}_0) \to \mathcal{M}(\boldsymbol{B}, \boldsymbol{X}_n)$ is a weak equivalence. $\square$

17.6. Uniqueness of homotopy function complexes

THEOREM 17.6.1. *Let $\mathcal{M}$ be a model category. If $\widetilde{\boldsymbol{X}}$ is a cosimplicial resolution in $\mathcal{M}$ and $\widehat{\boldsymbol{Y}}$ is a simplicial resolution in $\mathcal{M}$, then there is a natural diagram of* *homotopy equivalences of fibrant simplicial sets* $\mathcal{M}(\widetilde{\boldsymbol{X}}, \widehat{\boldsymbol{Y}}_0) \xrightarrow{\cong} \operatorname{diag}\mathcal{M}(\widetilde{\boldsymbol{X}}, \widehat{\boldsymbol{Y}}) \xleftarrow{\cong} \mathcal{M}(\widetilde{\boldsymbol{X}}^0, \widehat{\boldsymbol{Y}})$.

Proof. Theorem 17.5.1 implies that the bisimplicial set $\mathcal{M}(\widetilde{\boldsymbol{X}}, \widehat{\boldsymbol{Y}})$ satisfies the hypotheses of Corollary 16.5.9, and so Theorem 16.5.4 implies that there is a natural weak equivalence $\mathcal{M}(\widetilde{\boldsymbol{X}}^0, \widehat{\boldsymbol{Y}}) \to \operatorname{diag}\mathcal{M}(\widetilde{\boldsymbol{X}}, \widehat{\boldsymbol{Y}})$. If we reverse the indices of the bisimplicial set $\mathcal{M}(\widetilde{\boldsymbol{X}}, \widehat{\boldsymbol{Y}})$, we obtain a natural weak equivalence $\mathcal{M}(\widetilde{\boldsymbol{X}}, \widehat{\boldsymbol{Y}}_0) \to \operatorname{diag}\mathcal{M}(\widetilde{\boldsymbol{X}}, \widehat{\boldsymbol{Y}})$. Corollary 17.4.10 implies that all these simplicial sets are fibrant, and so these natural weak equivalences are natural homotopy equivalences. $\square$

Theorem 17.6.2. *Let $\mathcal{M}$ be a model category. If map_1 and map_2 are homotopy function complexes on $\mathcal{M}$ (see Definition 17.2.6), then there is a natural zig-zag of weak equivalences from map_1 to map_2, unique up to an equivalence of such zig-zags (see Definition 9.5.3), on the intersection of the domains of definition of map_1 and map_2.*

Proof. **Rewrite this, and fill in the proof!!** $\square$

Proposition 17.6.3. *Let $\mathcal{M}$ be a model category.*

1. (a) *If $\widetilde{\boldsymbol{B}}$ is a cosimplicial resolution in $\mathcal{M}$ and $\hat{f}, \hat{g}\colon \widehat{X} \to \widehat{Y}$ are left homotopic, right homotopic, or homotopic maps of fibrant objects in $\mathcal{M}$, then the induced maps of left homotopy function complexes $\hat{f}_*, \hat{g}_*\colon \mathcal{M}(\widetilde{\boldsymbol{B}}, \widehat{X}) \to \mathcal{M}(\widetilde{\boldsymbol{B}}, \widehat{Y})$ are homotopic.*
   (b) *If $\tilde{f}, \tilde{g}\colon \widetilde{\boldsymbol{A}} \to \widetilde{\boldsymbol{B}}$ are left homotopic, right homotopic, or homotopic maps of cosimplicial resolutions in $\mathcal{M}$ and $\widehat{X}$ is a fibrant object in $\mathcal{M}$, then the induced maps of left homotopy function complexes $\tilde{f}^*, \tilde{g}^*\colon \mathcal{M}(\widetilde{\boldsymbol{B}}, \widehat{X}) \to \mathcal{M}(\widetilde{\boldsymbol{A}}, \widehat{X})$ are homotopic.*
2. (a) *If $\widetilde{B}$ is a cofibrant object in $\mathcal{M}$ and $\hat{f}, \hat{g}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ are left homotopic, right homotopic, or homotopic maps of simplicial resolutions in $\mathcal{M}$, then the induced maps of right homotopy function complexes $\hat{f}_*, \hat{g}_*\colon \mathcal{M}(\widetilde{B}, \widehat{\boldsymbol{X}}) \to \mathcal{M}(\widetilde{B}, \widehat{\boldsymbol{Y}})$ are homotopic.*
   (b) *If $\tilde{f}, \tilde{g}\colon \widetilde{A} \to \widetilde{B}$ are left homotopic, right homotopic, or homotopic maps of cofibrant objects in $\mathcal{M}$ and $\widehat{\boldsymbol{X}}$ is a simplicial resolution in $\mathcal{M}$, then the induced maps of right homotopy function complexes $\tilde{f}^*, \tilde{g}^*\colon \mathcal{M}(\widetilde{B}, \widehat{\boldsymbol{X}}) \to \mathcal{M}(\widetilde{A}, \widehat{\boldsymbol{X}})$ are homotopic.*
3. (a) *If $\widetilde{\boldsymbol{B}}$ is a cosimplicial resolution in $\mathcal{M}$ and $\hat{f}, \hat{g}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ are left homotopic, right homotopic, or homotopic maps of simplicial resolutions in $\mathcal{M}$, then the induced maps of two-sided homotopy function complexes $\operatorname{diag}\hat{f}_*, \operatorname{diag}\hat{g}_*\colon \operatorname{diag}\mathcal{M}(\widetilde{\boldsymbol{B}}, \widehat{\boldsymbol{X}}) \to \operatorname{diag}\mathcal{M}(\widetilde{\boldsymbol{B}}, \widehat{\boldsymbol{Y}})$ are homotopic.*
   (b) *If $\tilde{f}, \tilde{g}\colon \widetilde{\boldsymbol{A}} \to \widetilde{\boldsymbol{B}}$ are left homotopic, right homotopic, or homotopic maps of cosimplicial resolutions in $\mathcal{M}$ and $\widehat{\boldsymbol{X}}$ is a simplicial resolution in $\mathcal{M}$, then the induced maps of two-sided homotopy function complexes $\operatorname{diag}\tilde{f}^*, \operatorname{diag}\tilde{g}^*\colon \operatorname{diag}\mathcal{M}(\widetilde{\boldsymbol{B}}, \widehat{\boldsymbol{X}}) \to \operatorname{diag}\mathcal{M}(\widetilde{\boldsymbol{A}}, \widehat{\boldsymbol{X}})$ are homotopic.*

Proof. We will prove part 1(a); the proofs of the other parts are similar.

If $\hat{f}$ and $\hat{g}$ are left homotopic, then Proposition 8.3.20 implies that there is a cylinder object $\widehat{X} \amalg \widehat{X} \to \operatorname{Cyl}(\widehat{X}) \xrightarrow{p} \widehat{X}$ for $\widehat{X}$ such that p is a trivial fibration and a left homotopy $H\colon \operatorname{Cyl}(\widehat{X}) \to \widehat{Y}$ from $\hat{f}$ to $\hat{g}$. Corollary 17.4.3 implies that the map

$\mathcal{M}\bigl(\widetilde{\boldsymbol{B}}, \mathrm{Cyl}(\widehat{X})\bigr) \to \mathcal{M}(\widetilde{\boldsymbol{B}}, \widehat{X})$ is a weak equivalence, and so Proposition 8.3.4 implies that $\hat{f}_*$ and $\hat{g}_*$ are left homotopic. Corollary 17.4.10 and Proposition 8.3.18 now imply that $\hat{f}_*$ and $\hat{g}_*$ are homotopic.

If $\hat{f}$ and $\hat{g}$ are right homotopic and if $\widehat{Y} \to \mathrm{Path}(\widehat{Y}) \to \widehat{Y} \times \widehat{Y}$ is a path object for $\widehat{Y}$ and $H\colon \widehat{X} \to \mathrm{Path}(\widehat{Y})$ is a right homotopy from $\hat{f}$ to $\hat{g}$, then Theorem 17.5.1 implies that the map $\mathcal{M}(\widetilde{\boldsymbol{B}}, \widehat{Y}) \to \mathcal{M}\bigl(\widetilde{\boldsymbol{B}}, \mathrm{Path}(\widehat{Y})\bigr)$ is a weak equivalence. Thus, Proposition 8.3.4 implies that $\hat{f}_*$ and $\hat{g}_*$ are right homotopic. Corollary 17.4.10 and Proposition 8.3.18 now imply that $\hat{f}_*$ and $\hat{g}_*$ are homotopic. $\square$

PROPOSITION 17.6.4. *Let $\mathcal{M}$ be a model category, let $\widetilde{\boldsymbol{C}}(X) \to \mathrm{c}X$ be a natural fibrant cosimplicial resolution of every object X in $\mathcal{M}$, and let $\mathrm{s}Y \to \widehat{\boldsymbol{F}}(Y)$ be a natural cofibrant simplicial resolution of every object Y in $\mathcal{M}$.*

1. *If $\widetilde{\boldsymbol{C}}'(X) \to \mathrm{c}X$ is a natural cosimplicial resolution of X defined on some subcategory of $\mathcal{M}$ and $Y \to \widehat{F}'(Y)$ is a natural fibrant approximation to Y defined on some subcategory of $\mathcal{M}$, then there is a homotopy equivalence*
$$\operatorname{diag} \mathcal{M}\bigl(\widetilde{\boldsymbol{C}}(X), \widehat{\boldsymbol{F}}(Y)\bigr) \cong \mathcal{M}\bigl(\widetilde{\boldsymbol{C}}'(X), \widehat{F}'(Y)\bigr),$$
defined up to homotopy and natural up to homotopy, wherever the homotopy function complex on the right is defined.
2. *If $\widetilde{C}'(X) \to X$ is a natural cofibrant approximation to X defined on some subcategory of $\mathcal{M}$ and $\mathrm{s}Y \to \widehat{\boldsymbol{F}}'(Y)$ is a natural simplicial resolution of Y defined on some subcategory of $\mathcal{M}$, then there is a homotopy equivalence*
$$\operatorname{diag} \mathcal{M}\bigl(\widetilde{\boldsymbol{C}}(X), \widehat{\boldsymbol{F}}(Y)\bigr) \cong \mathcal{M}\bigl(\widetilde{C}'(X), \widehat{\boldsymbol{F}}'(Y)\bigr),$$
defined up to homotopy and natural up to homotopy, wherever the homotopy function complex on the right is defined.
3. *If $\widetilde{\boldsymbol{C}}'(X) \to \mathrm{c}X$ is a natural cosimplicial resolution of X defined on some subcategory of $\mathcal{M}$ and $\mathrm{s}Y \to \widehat{\boldsymbol{F}}'(Y)$ is a natural simplicial resolution of Y defined on some subcategory of $\mathcal{M}$, then there is a homotopy equivalence*
$$\operatorname{diag} \mathcal{M}\bigl(\widetilde{\boldsymbol{C}}(X), \widehat{\boldsymbol{F}}(Y)\bigr) \cong \operatorname{diag} \mathcal{M}\bigl(\widetilde{\boldsymbol{C}}'(X), \widehat{\boldsymbol{F}}'(Y)\bigr),$$
defined up to homotopy and natural up to homotopy, wherever the homotopy function complex on the right is defined.

PROOF. For part 1, Proposition 17.1.6 implies that $Y \to \widehat{\boldsymbol{F}}(Y)_0$ is a cofibrant fibrant approximation to Y, and so Proposition 9.1.6 implies that there is a weak equivalence of fibrant approximations $\widehat{\boldsymbol{F}}(Y)_0 \to \widehat{F}'(Y)$, unique up to homotopy under Y. Proposition 9.1.10 implies that this weak equivalence is natural up to homotopy. Proposition 17.1.9 implies that there is a weak equivalence of resolutions $\widetilde{\boldsymbol{C}}'(X) \to \widetilde{\boldsymbol{C}}(X)$, unique up to homotopy over $\mathrm{c}X$, and Proposition 17.1.13 implies that this weak equivalence is natural up to homotopy. Theorem 17.5.1 implies that these weak equivalences induce a weak equivalence $\mathcal{M}\bigl(\widetilde{\boldsymbol{C}}(X), \widehat{\boldsymbol{F}}(Y)_0\bigr) \to \mathcal{M}\bigl(\widetilde{\boldsymbol{C}}'(X), \widehat{F}'(Y)\bigr)$, and Proposition 17.6.3 implies that this weak equivalence is well defined up to homotopy and that it is natural up to homotopy. Since all of these simplicial sets are fibrant (see Corollary 17.4.10), this weak equivalence is a homotopy equivalence. If we compose this with a homotopy inverse to the natural homotopy equivalence $\mathcal{M}\bigl(\widetilde{\boldsymbol{C}}(X), \widehat{\boldsymbol{F}}(Y)_0\bigr) \to \operatorname{diag} \mathcal{M}\bigl(\widetilde{\boldsymbol{C}}(X), \widehat{\boldsymbol{F}}(Y)\bigr)$ of Theorem 17.6.1, then this completes the proof of part 1.

The proof of part 2 is similar to that of part 1.

For part 3, Proposition 17.1.9 implies that there are weak equivalences of resolutions $\widetilde{\boldsymbol{C}}'(X) \to \widetilde{\boldsymbol{C}}(X)$ (unique up to homotopy over $\mathrm{c}X$) and $\widehat{\boldsymbol{F}}(Y) \to \widehat{\boldsymbol{F}}'(Y)$ (unique up to homotopy under $\mathrm{s}Y$), and Proposition 17.1.13 implies that each of these is natural up to homotopy. Theorem 17.5.1 implies that these induce a weak equivalence $\operatorname{diag}\mathcal{M}\big(\widetilde{\boldsymbol{C}}(X), \widehat{\boldsymbol{F}}(Y)\big) \to \operatorname{diag}\mathcal{M}\big(\widetilde{\boldsymbol{C}}'(X), \widehat{\boldsymbol{F}}'(Y)\big)$, and Proposition 17.6.3 implies that the homotopy class of this weak equivalence is independent of the choices of the weak equivalences of resolutions, and that it is natural up to homotopy. Since these homotopy function complexes are fibrant simplicial sets, this weak equivalence is a homotopy equivalence. □

THEOREM 17.6.5. *Let $\mathcal{M}$ be a model category. If $\operatorname{map}_1(X,Y)$ and $\operatorname{map}_2(X,Y)$ are homotopy function complexes on $\mathcal{M}$ (see Definition 17.2.6), then there is a homotopy equivalence $h_{1,2}\colon \operatorname{map}_1(X,Y) \to \operatorname{map}_2(X,Y)$, defined up to homotopy and natural up to homotopy, such that if $\operatorname{map}_3(X,Y)$ is a third homotopy function complex and $h_{1,3}\colon \operatorname{map}_1(X,Y) \to \operatorname{map}_3(X,Y)$ and $h_{2,3}\colon \operatorname{map}_2(X,Y) \to \operatorname{map}_3(X,Y)$ are the corresponding homotopy equivalences, then $h_{2,3}h_{1,2} \simeq h_{1,3}$.*

PROOF. Choose a natural fibrant cosimplicial resolution $\widetilde{\boldsymbol{C}}(X) \to \mathrm{c}X$ for every object X in $\mathcal{M}$ and a natural cofibrant simplicial resolution $\mathrm{s}Y \to \widehat{\boldsymbol{F}}(Y)$ for every object Y in $\mathcal{M}$ (see Proposition 17.1.3), and let $\operatorname{map}(X,Y) = \operatorname{diag}\mathcal{M}\big(\widetilde{\boldsymbol{C}}(X), \widehat{\boldsymbol{F}}(Y)\big)$. Let $h_1\colon \operatorname{map}(X,Y) \to \operatorname{map}_1(X,Y)$ be the homotopy equivalence (defined up to homotopy and natural up to homotopy) of Proposition 17.6.4, and let $h_2\colon \operatorname{map}(X,Y) \to \operatorname{map}_2(X,Y)$ and $h_3\colon \operatorname{map}(X,Y) \to \operatorname{map}_3(X,Y)$ be defined similarly. We can now let $h_{1,2} = h_2h_1^{-1}$, $h_{1,3} = h_3h_1^{-1}$, and $h_{2,3} = h_3h_2^{-1}$. □

THEOREM 17.6.6. *Let $\mathcal{M}$ be a model category.*

1. *If B is an object in $\mathcal{M}$ and $g\colon X \to Y$ is a map for which there is some map of homotopy function complexes $g_*\colon \operatorname{map}(B,X) \to \operatorname{map}(B,Y)$ induced by g that is a weak equivalence, then every such map of homotopy function complexes induced by g is a weak equivalence.*
2. *If X is an object in $\mathcal{M}$ and $f\colon A \to B$ is a map for which there is some map of homotopy function complexes (see Definition 17.2.4) $f^*\colon \operatorname{map}(B,X) \to \operatorname{map}(A,X)$ induced by f that is a weak equivalence, then every such map of homotopy function complexes induced by f is a weak equivalence.*

PROOF. This follows from Theorem 17.6.5, Proposition 8.5.6, and the "two out of three" axiom (see Definition 8.1.2). □

PROPOSITION 17.6.7. *If $\mathcal{M}$ is a model category, then the homotopy equivalences of Theorem 17.6.5 are independent (up to homotopy) of the choices of resolutions made in the proof.*

PROOF. If $\widetilde{\boldsymbol{C}}_1(X) \to \mathrm{c}X$ and $\widetilde{\boldsymbol{C}}_2(X) \to \mathrm{c}X$ are natural fibrant cosimplicial resolutions of every object X in $\mathcal{M}$, then Proposition 17.1.9 implies that there are weak equivalences $\widetilde{\boldsymbol{C}}_1(X) \to \widetilde{\boldsymbol{C}}_2(X)$ and $\widetilde{\boldsymbol{C}}_2(X) \to \widetilde{\boldsymbol{C}}_1(X)$, unique up to homotopy over $\mathrm{c}X$ and natural up to homotopy, and that these are inverse homotopy equivalences over $\mathrm{c}X$. Similarly, if $\mathrm{s}Y \to \widehat{\boldsymbol{F}}_1(Y)$ and $\mathrm{s}Y \to \widehat{\boldsymbol{F}}_2(Y)$ are natural cofibrant simplicial resolutions of every object Y in $\mathcal{M}$, then there are weak equivalences $\widehat{\boldsymbol{F}}_1(Y) \to \widehat{\boldsymbol{F}}_2(Y)$ and $\widehat{\boldsymbol{F}}_2(Y) \to \widehat{\boldsymbol{F}}_1(Y)$, unique up to homotopy under $\mathrm{s}Y$

and natural up to homotopy, and these are inverse homotopy equivalences under $\mathrm{s}Y$.

The uniqueness clause of Proposition 17.1.9 implies that if $\widetilde{\boldsymbol{C}}'(X) \to \mathrm{c}X$ is a cosimplicial resolution of X, then the map $\widetilde{\boldsymbol{C}}'(X) \to \widetilde{\boldsymbol{C}}_1(X)$ (see the proof of Proposition 17.6.4) is homotopic to the composition $\widetilde{\boldsymbol{C}}'(X) \to \widetilde{\boldsymbol{C}}_2(X) \to \widetilde{\boldsymbol{C}}_1(X)$. Similarly, if $\mathrm{s}Y \to \widehat{\boldsymbol{F}}'(Y)$ is a simplicial resolution of Y, then the map $\widehat{\boldsymbol{F}}_1(Y) \to \widehat{\boldsymbol{F}}'(Y)$ is homotopic to the composition $\widehat{\boldsymbol{F}}_1(Y) \to \widehat{\boldsymbol{F}}_2(Y) \to \widehat{\boldsymbol{F}}'(Y)$. Thus, if $\mathrm{map}_1(X,Y)$ and $\mathrm{map}_2(X,Y)$ are two-sided homotopy function complexes on $\mathcal{M}$ and

$$\begin{aligned}
h_1^1 &\colon \operatorname{diag} \mathcal{M}\big(\widetilde{\boldsymbol{C}}_1(X), \widehat{\boldsymbol{F}}_1(Y)\big) \to \mathrm{map}_1(X,Y),\\
h_1^2 &\colon \operatorname{diag} \mathcal{M}\big(\widetilde{\boldsymbol{C}}_2(X), \widehat{\boldsymbol{F}}_2(Y)\big) \to \mathrm{map}_1(X,Y),\\
h_2^1 &\colon \operatorname{diag} \mathcal{M}\big(\widetilde{\boldsymbol{C}}_1(X), \widehat{\boldsymbol{F}}_1(Y)\big) \to \mathrm{map}_2(X,Y), \quad \text{and}\\
h_2^2 &\colon \operatorname{diag} \mathcal{M}\big(\widetilde{\boldsymbol{C}}_2(X), \widehat{\boldsymbol{F}}_2(Y)\big) \to \mathrm{map}_2(X,Y)
\end{aligned}$$

are the homotopy equivalences constructed in the proof of Proposition 17.6.4, then $h_2^1(h_1^1)^{-1} \simeq h_2^2(h_1^2)^{-1}$. Similar remarks apply to the cases of left homotopy function complexes and right homotopy function complexes. □

THEOREM 17.6.8. *Let $\mathcal{M}$ be a model category, and let X and Y be objects in $\mathcal{M}$. If $\mathrm{map}_1(X,Y)$ and $\mathrm{map}_2(X,Y)$ are homotopy function complexes and $h\colon \mathrm{map}_1(X,Y) \cong \mathrm{map}_2(X,Y)$ is a homotopy equivalence that is a composition of*

1. *homotopy equivalences of left homotopy function complexes induced by maps of cosimplicial resolutions of X or by maps of fibrant approximations to Y (see Lemma* 17.1.8*, Lemma* 9.1.4*, and Theorem* 17.5.1*),*
2. *homotopy equivalences of right homotopy function complexes induced by maps of cofibrant approximations to X or by maps of simplicial resolutions of Y,*
3. *homotopy equivalences of two-sided homotopy function complexes induced by maps of cosimplicial resolutions of X or by maps of simplicial resolutions of Y,*
4. *the homotopy equivalences of Theorem* 17.6.1*,*

or a homotopy inverse to one of these, then h is homotopic to the homotopy equivalence $h_{1,2}$ of Theorem 17.6.5.

PROOF. It is sufficient to show that any of the homotopy equivalences of homotopy function complexes listed above is homotopic to the corresponding homotopy equivalence of Theorem 17.6.5. We will consider the case in which there is a cosimplicial resolution $\widetilde{\boldsymbol{X}}$ of X and a map $g\colon \widehat{Y} \to \widehat{Y}'$ of fibrant approximations to Y such that h is the homotopy equivalence $g_*\colon \mathcal{M}(\widetilde{\boldsymbol{X}}, \widehat{Y}) \to \mathcal{M}(\widetilde{\boldsymbol{X}}, \widehat{Y}')$. The proofs in the other cases are similar.

Let $\widetilde{\boldsymbol{C}}(X)$ and $\widehat{\boldsymbol{F}}(Y)$ be the natural fibrant cosimplicial resolution of X and the natural cofibrant simplicial resolution of Y chosen in the proof of Theorem 17.6.5. Proposition 17.1.6 implies that $Y \to \widehat{\boldsymbol{F}}(Y)_0$ is a cofibrant fibrant approximation to Y, and so Proposition 9.1.6 implies that the composition of the weak equivalence $\widehat{\boldsymbol{F}}(Y)_0 \to \widehat{Y}$ used in the proof of Proposition 17.6.4 with the map of fibrant approximations $g\colon \widehat{Y} \to \widehat{Y}'$ is homotopic under Y to the weak equivalence $\widehat{\boldsymbol{F}}(Y)_0 \to \widehat{Y}'$

used in the proof of Proposition 17.6.4. The result now follows from Proposition 17.6.3. $\square$

CHAPTER 18

Applications of homotopy function complexes

18.1. Homotopy classes of maps

LEMMA 18.1.1. *Let $\mathcal{M}$ be a model category.*

1. *If $\widetilde{\boldsymbol{A}}$ is a cosimplicial resolution in $\mathcal{M}$, then $\widetilde{\boldsymbol{A}}^0 \amalg \widetilde{\boldsymbol{A}}^0 \xrightarrow{d^0 \amalg d^1} \widetilde{\boldsymbol{A}}^1 \xrightarrow{s^0} \widetilde{\boldsymbol{A}}^0$ is a cylinder object (see Definition 8.3.2) for $\widetilde{\boldsymbol{A}}^0$.*
2. *If $\widehat{\boldsymbol{X}}$ is a simplicial resolution in $\mathcal{M}$, then $\widehat{\boldsymbol{X}}_0 \xrightarrow{s_0} \widehat{\boldsymbol{X}}_1 \xrightarrow{d_0 \times d_1} \widehat{\boldsymbol{X}}_0 \times \widehat{\boldsymbol{X}}_0$ is a path object for $\widehat{\boldsymbol{X}}_0$.*

PROOF. This follows directory from the definitions. $\square$

PROPOSITION 18.1.2. *Let $\mathcal{M}$ be a model category.*

1. *If $\widetilde{\boldsymbol{B}}$ is a cosimplicial resolution in $\mathcal{M}$ and X is a fibrant object in $\mathcal{M}$, then the set $\pi_0\mathcal{M}(\widetilde{\boldsymbol{B}}, X)$ is naturally isomorphic to the set of homotopy classes of maps from $\widetilde{\boldsymbol{B}}^0$ to X.*
2. *If B is a cofibrant object in $\mathcal{M}$ and $\widehat{\boldsymbol{X}}$ is a simplicial resolution in $\mathcal{M}$, then the set $\pi_0\mathcal{M}(B, \widehat{\boldsymbol{X}})$ is naturally isomorphic to the set of homotopy classes of maps from B to $\widehat{\boldsymbol{X}}_0$.*

PROOF. We will prove part 1; the proof of part 2 is similar.

The set of vertices of $\mathcal{M}(\widetilde{\boldsymbol{B}}, X)$ is the set of maps from $\widetilde{\boldsymbol{B}}^0$ to X, and Lemma 18.1.1 implies that if two vertices of $\mathcal{M}(\widetilde{\boldsymbol{B}}, X)$ represent the same element of $\pi_0\mathcal{M}(\widetilde{\boldsymbol{B}}, X)$, then those vertices (i.e., maps) from $\widetilde{\boldsymbol{B}}_0$ to X are homotopic. Finally, if two maps from $\widetilde{\boldsymbol{B}}^0$ to X are homotopic, then Proposition 8.3.16 and Lemma 18.1.1 imply that there is a 1-simplex of $\mathcal{M}(\widetilde{\boldsymbol{B}}, X)$ whose vertices are those maps. $\square$

LEMMA 18.1.3. *Let $\mathcal{M}$ be a model category.*

1. *If $\widetilde{\boldsymbol{B}}$ is a cosimplicial resolution in $\mathcal{M}$ and $p\colon X \to Y$ is a map of fibrant objects in $\mathcal{M}$ that induces a weak equivalence of simplicial sets $p_*\colon \mathcal{M}(\widetilde{\boldsymbol{B}}, X) \cong \mathcal{M}(\widetilde{\boldsymbol{B}}, Y)$, then p induces an isomorphism of the sets of homotopy classes of maps $p_*\colon \pi(\widetilde{\boldsymbol{B}}^0, X) \approx \pi(\widetilde{\boldsymbol{B}}^0, Y)$.*
2. *If $\widehat{\boldsymbol{X}}$ is a simplicial resolution in $\mathcal{M}$ and $i\colon A \to B$ is a map of cofibrant objects in $\mathcal{M}$ that induces a weak equivalence of simplicial sets $i^*\colon \mathcal{M}(B, \widehat{\boldsymbol{X}}) \cong \mathcal{M}(A, \widehat{\boldsymbol{X}})$, then i induces an isomorphism of the sets of homotopy classes of maps $i^*\colon \pi(B, \widehat{\boldsymbol{X}}_0) \approx \pi(A, \widehat{\boldsymbol{X}}_0)$.*

PROOF. This follows from Proposition 18.1.2. $\square$

PROPOSITION 18.1.4. *Let $\mathcal{M}$ be a model category.*

1. *If B is cofibrant and $p\colon X \to Y$ is a map of fibrant objects that induces a weak equivalence of homotopy function complexes $p_*\colon \operatorname{map}(B, X) \to$*

map(B, Y), *then* p *induces an isomorphism of the sets of homotopy classes of maps* $p_*: \pi(B, X) \approx \pi(B, Y)$.

2. *If* X *is fibrant and* $i: A \to B$ *is a map of cofibrant objects that induces a weak equivalence of homotopy function complexes* $i^*: \mathrm{map}(B, X) \to \mathrm{map}(A, X)$, *then* i *induces an isomorphism of the sets of homotopy classes of maps* $i^*: \pi(B, X) \approx \pi(A, X)$.

PROOF. We will prove part 1; the proof of part 2 is dual.

If $\widetilde{\boldsymbol{B}}$ is a cosimplicial resolution of B, then p induces a weak equivalence $p_*: \mathcal{M}(\widetilde{\boldsymbol{B}}, X) \to \mathcal{M}(\widetilde{\boldsymbol{B}}, Y)$ (see Theorem 17.6.6), and so Lemma 18.1.3 implies that p induces an isomorphism $p_*: \pi(\widetilde{\boldsymbol{B}}^0, X) \approx \pi(\widetilde{\boldsymbol{B}}^0, Y)$. Since $\widetilde{\boldsymbol{B}}^0 \to B$ is a weak equivalence of cofibrant objects, the result now follows from Corollary 8.5.4. $\square$

PROPOSITION 18.1.5. *If* $\mathcal{M}$ *is a model category, then a map* $g: X \to Y$ *is a weak equivalence if either of the following two conditions is satisfied:*

1. *The map* g *induces weak equivalences of homotopy function complexes*
$$g_*: \mathrm{map}(X, X) \cong \mathrm{map}(X, Y) \quad \textit{and} \quad g_*: \mathrm{map}(Y, X) \cong \mathrm{map}(Y, Y).$$
2. *The map* g *induces weak equivalences of homotopy function complexes*
$$g^*: \mathrm{map}(Y, X) \cong \mathrm{map}(X, X) \quad \textit{and} \quad g^*: \mathrm{map}(Y, Y) \cong \mathrm{map}(X, Y).$$

PROOF. We will prove this using condition 1; the proof using condition 2 is similar.

If $\tilde{g}: \widetilde{X} \to \widetilde{Y}$ is a cofibrant approximation to g, then Theorem 17.5.2 implies that $\tilde{g}$ induces weak equivalences of homotopy function complexes $\tilde{g}_*: \mathrm{map}(\widetilde{X}, \widetilde{X}) \cong \mathrm{map}(\widetilde{X}, \widetilde{Y})$ and $\tilde{g}_*: \mathrm{map}(\widetilde{Y}, \widetilde{X}) \cong \mathrm{map}(\widetilde{Y}, \widetilde{Y})$. If $\hat{g}: \widehat{X} \to \widehat{Y}$ is a cofibrant fibrant approximation to $\tilde{g}$, then $\hat{g}$ is a map of cofibrant-fibrant objects, and Theorem 17.5.2 implies that $\hat{g}$ induces weak equivalences of homotopy function complexes $\hat{g}_*: \mathrm{map}(\widehat{X}, \widehat{X}) \cong \mathrm{map}(\widehat{X}, \widehat{Y})$ and $\hat{g}_*: \mathrm{map}(\widehat{Y}, \widehat{X}) \cong \mathrm{map}(\widehat{Y}, \widehat{Y})$. Proposition 18.1.4 now implies that $\hat{g}$ induces isomorphisms of the sets of homotopy classes of maps $\hat{g}_*: \pi(\widehat{X}, \widehat{X}) \approx \pi(\widehat{X}, \widehat{Y})$ and $\hat{g}_*: \pi(\widehat{Y}, \widehat{X}) \approx \pi(\widehat{Y}, \widehat{Y})$, and so Proposition 8.3.28 implies that $\hat{g}$ is a homotopy equivalence. Thus, $\hat{g}$ is a weak equivalence, and so $\tilde{g}$ is a weak equivalence, and so g is a weak equivalence. $\square$

THEOREM 18.1.6. *If* $\mathcal{M}$ *is a model category and* $g: X \to Y$ *is a map in* $\mathcal{M}$, *then the following are equivalent:*

1. *The map* g *is a weak equivalence.*
2. *For every object* W *in* $\mathcal{M}$ *the map* g *induces a weak equivalence of homotopy function complexes* $g_*: \mathrm{map}(W, X) \cong \mathrm{map}(W, Y)$.
3. *For every cofibrant object* W *in* $\mathcal{M}$ *the map* g *induces a weak equivalence of homotopy function complexes* $g_*: \mathrm{map}(W, X) \cong \mathrm{map}(W, Y)$.
4. *For every object* Z *in* $\mathcal{M}$ *the map* g *induces a weak equivalence of homotopy function complexes* $g^*: \mathrm{map}(Y, Z) \cong \mathrm{map}(X, Z)$.
5. *For every fibrant object* Z *in* $\mathcal{M}$ *the map* g *induces a weak equivalence of homotopy function complexes* $g^*: \mathrm{map}(Y, Z) \cong \mathrm{map}(X, Z)$.

PROOF. This follows from Theorem 17.5.2, Proposition 18.1.5, and Proposition 9.1.2. $\square$

18.2. Homotopic maps of homotopy function complexes

LEMMA 18.2.1. *If $\mathcal{M}$ is a model category and $f, g\colon X \to Y$ are left homotopic, right homotopic, or homotopic, then both the induced maps of constant cosimplicial objects $\mathrm{c}f, \mathrm{c}g\colon \mathrm{c}X \to \mathrm{c}Y$ and the induced maps of constant simplicial objects $\mathrm{s}f, \mathrm{s}g\colon \mathrm{s}X \to \mathrm{s}Y$ are, respectively, left homotopic, right homotopic, or homotopic.*

PROOF. The constant cosimplicial and constant simplicial objects obtained from either a cylinder object for X or a path object for Y satisfy the conditions of Proposition 8.3.4. □

PROPOSITION 18.2.2. *Let $\mathcal{M}$ be a model category, and let W, X, Y, and Z be objects in $\mathcal{M}$.*

1. *If $f, g\colon X \to Y$ are left homotopic, right homotopic, or homotopic, and if $f_*, g_*\colon \mathrm{map}(W,X) \to \mathrm{map}(W,Y)$ are maps of homotopy function complexes induced by, respectively, f and g, then f_* and g_* are homotopic.*
2. *If $f, g\colon X \to Y$ are left homotopic, right homotopic, or homotopic, and if $f^*, g^*\colon \mathrm{map}(Z,W) \to \mathrm{map}(Z,W)$ are maps of homotopy function complexes induced by, respectively, f and g, then f^* and g^* are homotopic.*

PROOF. We will prove part 1 in the case in which f_* and g_* are maps of left homotopy function complexes; the proof in the other cases (and of part 2) are similar.

Let $\widetilde{\boldsymbol{W}}$ be a cosimplicial resolution of W and let $\hat{f}, \hat{g}\colon \widehat{X} \to \widehat{Y}$ be fibrant approximations to, respectively, f and g, such that the maps f_* and g_* are, respectively, the maps $\hat{f}_*\colon \mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{X}) \to \mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{Y})$ and $\hat{g}_*\colon \mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{X}) \to \mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{Y})$. If we factor the weak equivalences $X \to \widehat{X}$ and $Y \to \widehat{Y}$ as, respectively, $X \xrightarrow{i_X} \widehat{X}' \xrightarrow{p_X} \widehat{X}$ and $Y \xrightarrow{i_Y} \widehat{Y}' \xrightarrow{p_Y} \widehat{Y}$ such that i_X and i_Y are trivial cofibrations and p_X and p_Y are fibrations, then the "two out of three" axiom implies that p_X and p_Y are trivial fibrations.

The dotted arrow exists in the solid arrow diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_X"'] & Y \arrow[r, "i_Y"] & \widehat{Y}' \arrow[d, "p_Y"] \\ \widehat{X}' \arrow[r, "p_X"'] \arrow[urr, dotted, "\hat{f}'"] & \widehat{X} \arrow[r, "\hat{f}"'] & \widehat{Y} \end{tikzcd}$$

and a similar diagram implies that the corresponding map $\hat{g}'\colon \widehat{X}' \to \widehat{Y}'$ exists. Thus, $\hat{f}'$ and $\hat{g}'$ are cofibrant fibrant approximations to, respectively, f and g, and we have the diagram

$$\begin{tikzcd} \widehat{X}' \arrow[r, shift left, "\hat{f}'"] \arrow[r, shift right, "\hat{g}'"'] \arrow[d, "p_X"'] & \widehat{Y}' \arrow[d, "p_Y"] \\ \widehat{X} \arrow[r, shift left, "\hat{f}"] \arrow[r, shift right, "\hat{g}"'] & \widehat{Y} \end{tikzcd}$$

in which $p_Y \hat{f}' = \hat{f} p_X$ and $p_Y \hat{g}' = \hat{g} p_X$. Lemma 18.2.1 and Proposition 9.2.4 imply that if f and g are left homotopic, right homotopic, or homotopic, then $\hat{f}'$ and $\hat{g}'$ are, respectively, left homotopic, right homotopic, or homotopic. In any of these cases, Proposition 17.6.3 implies that the maps $\hat{f}'_*\colon \mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{X}') \to \mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{Y}')$ and $\hat{g}'_*\colon \mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{X}') \to \mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{Y}')$ are homotopic. Since p_X and p_Y are weak equivalences of fibrant objects, Theorem 17.5.1 implies that the maps $\mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{X}') \to \mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{X})$ and $\mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{Y}') \to \mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{Y})$ are homotopy equivalences of fibrant simplicial sets, and this implies that $\hat{f}_*\colon \mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{X}) \to \mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{Y})$ and $\hat{g}_*\colon \mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{X}) \to \mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{Y})$ are homotopic. $\square$

18.3. Homotopy orthogonal maps

If $\mathcal{M}$ is a simplicial model category and if $i\colon A \to B$ and $p\colon X \to Y$ are maps such that either

1. i is a trivial cofibration and p is a fibration, or
2. i is a cofibration and p is a trivial fibration,

then the map of function complexes $\mathrm{Map}(B, X) \to \mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, Y)$ is a trivial fibration (see axiom M7 of Definition 10.1.2). If we also assume that

1. A and B are cofibrant, and
2. X and Y are fibrant,

then the maps $\mathrm{Map}(A, X) \to \mathrm{Map}(A, Y)$ and $\mathrm{Map}(B, Y) \to \mathrm{Map}(A, Y)$ are fibrations, and so the pullback $\mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, Y)$ is weakly equivalent to the homotopy pullback (see Corollary 11.2.8). It is only in this case (A and B cofibrant, X and Y fibrant) that these function complexes are homotopy function complexes, and in this case the "orthogonality" condition is equivalent to saying that the square

$$\begin{array}{ccc} \mathrm{Map}(B,X) & \longrightarrow & \mathrm{Map}(B,Y) \\ \downarrow & & \downarrow \\ \mathrm{Map}(A,X) & \longrightarrow & \mathrm{Map}(A,Y) \end{array}$$

is a homotopy fiber square (see Definition 11.2.12). Proposition 18.3.1 shows that this condition on a pair of maps is independent of the choice of homotopy function complex, and then Definition 18.3.3 defines what it means for a pair of maps (i, p) to be *homotopy orthogonal*.

PROPOSITION 18.3.1. *Let $\mathcal{M}$ be a model category, and let $i\colon A \to B$ and $p\colon X \to Y$ be maps in $\mathcal{M}$. If there is some homotopy function complex (see Definition 17.2.6) $\mathrm{map}(-,-)$ on $\mathcal{M}$ such that the square*

$$\begin{array}{ccc} \mathrm{map}(B,X) & \longrightarrow & \mathrm{map}(B,Y) \\ \downarrow & & \downarrow \\ \mathrm{map}(A,X) & \longrightarrow & \mathrm{map}(A,Y) \end{array} \tag{18.3.2}$$

is a homotopy fiber square of simplicial sets (see Definition 11.2.12), then Diagram 18.3.2 for any other homotopy function complex on $\mathcal{M}$ is also a homotopy fiber square.

PROOF. If $\mathrm{map}_1(-,-)$ and $\mathrm{map}_2(-,-)$ are homotopy function complexes on $\mathcal{M}$, then Theorem 17.6.5 implies that there is a homotopy equivalence $\mathrm{map}_1(-,-) \cong$

$\mathrm{map}_2(-,-)$ that is natural up to homotopy. If we can alter these homotopy equivalences by homotopies to get maps from Diagram 18.3.2 for map_1 to Diagram 18.3.2 for map_2 that commute on the nose, then the result will follow from Proposition 11.2.13. If the maps $\mathrm{map}_2(A,X) \to \mathrm{map}_2(A,Y)$, $\mathrm{map}_2(B,Y) \to \mathrm{map}_2(A,Y)$, and $\mathrm{map}_2(B,X) \to \mathrm{map}_2(A,X) \times_{\mathrm{map}_2(A,Y)} \mathrm{map}_2(B,Y)$ are fibrations, then we can use the homotopy lifting property (see Proposition 8.3.8) to alter the homotopy equivalences from map_1 to map_2 in our diagrams by homotopies so that we do get a map of diagrams. Thus, it is sufficient to show that for any homotopy function complex, Diagram 18.3.2 maps to one with fibrations as described. We will do this for left homotopy function complexes; the proofs for right and two-sided homotopy function complexes are similar.

If map is a left homotopy function complex defined by the cosimplicial resolution $\tilde{\imath}\colon \widetilde{\boldsymbol{A}} \to \widetilde{\boldsymbol{B}}$ to i and the fibrant approximation $\hat{p}\colon \widehat{X} \to \widehat{Y}$ to p, then we can factor $\tilde{\imath}$ into a cofibration followed by a trivial fibration $\widetilde{\boldsymbol{A}} \to \widetilde{\boldsymbol{B}}' \to \widetilde{\boldsymbol{B}}$ and factor $\hat{p}$ into a trivial cofibration followed by a fibration $\widehat{X} \to \widehat{X}' \to \widehat{Y}$. This yields a diagram

$$\begin{tikzcd}
\mathcal{M}(\widetilde{\boldsymbol{B}},\widehat{X}) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{M}(\widetilde{\boldsymbol{B}},\widehat{Y}) \arrow[dr] \arrow[dd] & \\
& \mathcal{M}(\widetilde{\boldsymbol{B}}',\widehat{X}') \arrow[rr] \arrow[dd] & & \mathcal{M}(\widetilde{\boldsymbol{B}}',\widehat{Y}) \arrow[dd] \\
\mathcal{M}(\widetilde{\boldsymbol{A}},\widehat{X}) \arrow[rr] \arrow[dr] & & \mathcal{M}(\widetilde{\boldsymbol{A}},\widehat{Y}) \arrow[dr] & \\
& \mathcal{M}(\widetilde{\boldsymbol{A}},\widehat{X}') \arrow[rr] & & \mathcal{M}(\widetilde{\boldsymbol{A}},\widehat{Y})
\end{tikzcd}$$

in which all four maps from the back square to the front square are weak equivalences (see Theorem 17.5.1), and Corollary 17.4.2 and Theorem 17.4.1 imply that the front square has the fibrations required. □

DEFINITION 18.3.3. If $\mathcal{M}$ is a model category and $i\colon A \to B$ and $p\colon X \to Y$ are maps in $\mathcal{M}$, then we will say that

1. *(i,p) is a homotopy orthogonal pair,*
2. *i is left homotopy orthogonal to p, and*
3. *p is right homotopy orthogonal to i*

if the square

$$\begin{tikzcd}
\mathrm{map}(B,X) \arrow[r] \arrow[d] & \mathrm{map}(B,Y) \arrow[d] \\
\mathrm{map}(A,X) \arrow[r] & \mathrm{map}(A,Y)
\end{tikzcd}$$

is a homotopy fiber square (see Definition 11.2.12). (Proposition 18.3.1 implies that this is independent of the choice of homotopy function complex.)

PROPOSITION 18.3.4. *If $\mathcal{M}$ is a model category and $g\colon X \to Y$ is a weak equivalence in $\mathcal{M}$, then g is both left homotopy orthogonal and right homotopy orthogonal to every map in $\mathcal{M}$.*

PROOF. This follows from Proposition 17.1.14 and Theorem 17.4.1. □

PROPOSITION 18.3.5. *Let $\mathcal{M}$ be a model category.*

1. *If $i\colon A \to B$ is a map in $\mathcal{M}$ and $p\colon X \to *$ is the map to the terminal object of $\mathcal{M}$, then (i,p) is a homotopy orthogonal pair if and only if i induces a weak equivalence of homotopy function complexes $i^*\colon \operatorname{map}(B,X) \cong \operatorname{map}(A,X)$.*
2. *If $p\colon X \to Y$ is a map in $\mathcal{M}$ and $i\colon \emptyset \to B$ is the map from the initial object of $\mathcal{M}$, then (i,p) is a homotopy orthogonal pair if and only if p induces a weak equivalence of homotopy function complexes $p_*\colon \operatorname{map}(B,X) \cong \operatorname{map}(A,X)$.*

Proof. This follows directly from the definitions. □

Proposition 18.3.6. *Let $\mathcal{M}$ be a model category.*

1. *If $p\colon X \to Y$ is a map in $\mathcal{M}$ and we have a square*

$$\begin{tikzcd} A \arrow[r, "\cong"] \arrow[d, "i"'] & A' \arrow[d, "i'"] \\ B \arrow[r, "\cong"'] & B' \end{tikzcd}$$

in which the horizontal maps are weak equivalences, then (i,p) is a homotopy orthogonal pair if and only if (i',p) is one.

2. *If $i\colon A \to B$ is a map in $\mathcal{M}$ and we have a square*

$$\begin{tikzcd} X \arrow[r, "\cong"] \arrow[d, "p"'] & X' \arrow[d, "p'"] \\ Y \arrow[r, "\cong"'] & Y' \end{tikzcd}$$

in which the horizontal maps are weak equivalences, then (i,p) is a homotopy orthogonal pair if and only if (i,p') is one.

Proof. This follows from Proposition 11.2.13 and Theorem 17.5.2. □

Theorem 18.3.7. *Let $\mathcal{M}$ be a model category. If $i\colon A \to B$ and $p\colon X \to Y$ are maps in $\mathcal{M}$, then the following are equivalent:*

1. *(i,p) is a homotopy orthogonal pair.*
2. *For some cosimplicial resolution $\tilde{\imath}\colon \widetilde{\boldsymbol{A}} \to \widetilde{\boldsymbol{B}}$ of i such that $\tilde{\imath}$ is a Reedy cofibration and some fibrant approximation $\hat{p}\colon \widehat{X} \to \widehat{Y}$ to p such that $\hat{p}$ is a fibration, the map of simplicial sets*
$$\mathcal{M}(\widetilde{\boldsymbol{B}},\widehat{X}) \to \mathcal{M}(\widetilde{\boldsymbol{A}},\widehat{X}) \times_{\mathcal{M}(\widetilde{\boldsymbol{A}},\widehat{Y})} \mathcal{M}(\widetilde{\boldsymbol{B}},\widehat{Y})$$
is a trivial fibration.
3. *For every cosimplicial resolution $\tilde{\imath}\colon \widetilde{\boldsymbol{A}} \to \widetilde{\boldsymbol{B}}$ of i such that $\tilde{\imath}$ is a Reedy cofibration and every fibrant approximation $\hat{p}\colon \widehat{X} \to \widehat{Y}$ to p such that $\hat{p}$ is a fibration, the map of simplicial sets*
$$\mathcal{M}(\widetilde{\boldsymbol{B}},\widehat{X}) \to \mathcal{M}(\widetilde{\boldsymbol{A}},\widehat{X}) \times_{\mathcal{M}(\widetilde{\boldsymbol{A}},\widehat{Y})} \mathcal{M}(\widetilde{\boldsymbol{B}},\widehat{Y})$$
is a trivial fibration.
4. *For some cofibrant approximation $\tilde{\imath}\colon \widetilde{A} \to \widetilde{B}$ to i such that $\tilde{\imath}$ is a cofibration and some simplicial resolution $\hat{p}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ to p such that $\hat{p}$ is a Reedy fibration, the map of simplicial sets*
$$\mathcal{M}(\widetilde{B},\widehat{\boldsymbol{X}}) \to \mathcal{M}(\widetilde{A},\widehat{\boldsymbol{X}}) \times_{\mathcal{M}(\widetilde{A},\widehat{\boldsymbol{Y}})} \mathcal{M}(\widetilde{B},\widehat{\boldsymbol{Y}})$$

is a trivial fibration.

5. *For every cofibrant approximation* $\tilde{\imath}\colon \widetilde{A} \to \widetilde{B}$ *to* i *such that* $\tilde{\imath}$ *is a cofibration and every simplicial resolution* $\hat{p}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ *to* p *such that* $\hat{p}$ *is a Reedy fibration, the map of simplicial sets*
$$\mathcal{M}(\widetilde{B}, \widehat{\boldsymbol{X}}) \to \mathcal{M}(\widetilde{A}, \widehat{\boldsymbol{X}}) \times_{\mathcal{M}(\widetilde{A}, \widehat{\boldsymbol{Y}})} \mathcal{M}(\widetilde{B}, \widehat{\boldsymbol{Y}})$$
is a trivial fibration.
6. *For some cosimplicial resolution* $\tilde{\imath}\colon \widetilde{\boldsymbol{A}} \to \widetilde{\boldsymbol{B}}$ i *such that* $\tilde{\imath}$ *is a Reedy cofibration and some simplicial resolution* $\hat{p}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ *to* p *such that* $\hat{p}$ *is a Reedy fibration, the map of simplicial sets*
$$\operatorname{diag} \mathcal{M}(\widetilde{\boldsymbol{B}}, \widehat{\boldsymbol{X}}) \to \operatorname{diag} \mathcal{M}(\widetilde{\boldsymbol{A}}, \widehat{\boldsymbol{X}}) \times_{\operatorname{diag} \mathcal{M}(\widetilde{\boldsymbol{A}}, \widehat{\boldsymbol{Y}})} \operatorname{diag} \mathcal{M}(\widetilde{\boldsymbol{B}}, \widehat{\boldsymbol{Y}})$$
is a trivial fibration.
7. *For every cosimplicial resolution* $\tilde{\imath}\colon \widetilde{\boldsymbol{A}} \to \widetilde{\boldsymbol{B}}$ i *such that* $\tilde{\imath}$ *is a Reedy cofibration and every simplicial resolution* $\hat{p}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ *to* p *such that* $\hat{p}$ *is a Reedy fibration, the map of simplicial sets*
$$\operatorname{diag} \mathcal{M}(\widetilde{\boldsymbol{B}}, \widehat{\boldsymbol{X}}) \to \operatorname{diag} \mathcal{M}(\widetilde{\boldsymbol{A}}, \widehat{\boldsymbol{X}}) \times_{\operatorname{diag} \mathcal{M}(\widetilde{\boldsymbol{A}}, \widehat{\boldsymbol{Y}})} \operatorname{diag} \mathcal{M}(\widetilde{\boldsymbol{B}}, \widehat{\boldsymbol{Y}})$$
is a trivial fibration.

PROOF. This follows from Proposition 18.3.1, Theorem 17.4.1, and Theorem 17.4.9. □

PROPOSITION 18.3.8. *Let* $\mathcal{M}$ *be a model category. If* $i\colon A \to B$ *and* $p\colon X \to Y$ *are maps in* $\mathcal{M}$*, then the following are equivalent:*

1. (i, p) *is a homotopy orthogonal pair.*
2. *For some cosimplicial resolution* $\tilde{\imath}\colon \widetilde{\boldsymbol{A}} \to \widetilde{\boldsymbol{B}}$ *of* i *such that* $\tilde{\imath}$ *is a Reedy cofibration, some fibrant approximation* $\hat{p}\colon \widehat{X} \to \widehat{Y}$ *to* p *such that* $\hat{p}$ *is a fibration, and every* $n \geq 0$*, the dotted arrow exists in every solid arrow diagram of the form*
$$\begin{tikzcd} \widetilde{\boldsymbol{A}} \otimes \Delta[n] \amalg_{\widetilde{\boldsymbol{A}} \otimes \partial\Delta[n]} \widetilde{\boldsymbol{B}} \otimes \partial\Delta[n] \arrow[r] \arrow[d] & \widehat{X} \arrow[d] \\ \widetilde{\boldsymbol{B}} \otimes \Delta[n] \arrow[r] \arrow[ur, dotted] & \widehat{Y} \end{tikzcd}$$
3. *For every cosimplicial resolution* $\tilde{\imath}\colon \widetilde{\boldsymbol{A}} \to \widetilde{\boldsymbol{B}}$ *of* i *such that* $\tilde{\imath}$ *is a Reedy cofibration, every fibrant approximation* $\hat{p}\colon \widehat{X} \to \widehat{Y}$ *to* p *such that* $\hat{p}$ *is a fibration, and every* $n \geq 0$*, the dotted arrow exists in every solid arrow diagram of the form*
$$\begin{tikzcd} \widetilde{\boldsymbol{A}} \otimes \Delta[n] \amalg_{\widetilde{\boldsymbol{A}} \otimes \partial\Delta[n]} \widetilde{\boldsymbol{B}} \otimes \partial\Delta[n] \arrow[r] \arrow[d] & \widehat{X} \arrow[d] \\ \widetilde{\boldsymbol{B}} \otimes \Delta[n] \arrow[r] \arrow[ur, dotted] & \widehat{Y} \end{tikzcd}$$
4. *For some cofibrant approximation* $\tilde{\imath}\colon \widetilde{A} \to \widetilde{B}$ *to* i *such that* $\tilde{\imath}$ *is a cofibration, some simplicial resolution* $\hat{p}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ *to* p *such that* $\hat{p}$ *is a Reedy fibration,*

and every $n \geq 0$, the dotted arrow exists in every solid arrow diagram of the form

$$\begin{tikzcd} \widetilde{A} \arrow[r] \arrow[d] & \widehat{\boldsymbol{X}}^{\Delta[n]} \arrow[d] \\ \widetilde{B} \arrow[r] \arrow[ur, dotted] & \widehat{\boldsymbol{Y}}^{\Delta[n]} \times_{\widehat{\boldsymbol{Y}}^{\partial\Delta[n]}} \widehat{\boldsymbol{X}}^{\partial\Delta[n]} \end{tikzcd}$$

5. *For every cofibrant approximation $\tilde{\imath}\colon \widetilde{A} \to \widetilde{B}$ to i such that $\tilde{\imath}$ is a cofibration, every simplicial resolution $\hat{p}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ to p such that $\hat{p}$ is a Reedy fibration, and every $n \geq 0$, the dotted arrow exists in every solid arrow diagram of the form*

$$\begin{tikzcd} \widetilde{A} \arrow[r] \arrow[d] & \widehat{\boldsymbol{X}}^{\Delta[n]} \arrow[d] \\ \widetilde{B} \arrow[r] \arrow[ur, dotted] & \widehat{\boldsymbol{Y}}^{\Delta[n]} \times_{\widehat{\boldsymbol{Y}}^{\partial\Delta[n]}} \widehat{\boldsymbol{X}}^{\partial\Delta[n]} \end{tikzcd}$$

PROOF. Since a map of simplicial sets is a trivial fibration if and only if it has the right lifting property with respect to the map $\partial\Delta[n] \to \Delta[n]$ for every $n \geq 0$, this follows from Theorem 18.3.7 and Proposition 17.3.14. $\square$

PROPOSITION 18.3.9. *Let $\mathcal{M}$ be a model category. If $i\colon A \to B$ is a cofibration between cofibrant objects, $p\colon X \to Y$ is a fibration between fibrant objects, and (i, p) is a homotopy orthogonal pair, then (i, p) is a lifting-extension pair.*

PROOF. Proposition 17.1.31 implies that there is a cosimplicial frame $\tilde{\imath}\colon \widetilde{\boldsymbol{A}} \to \widetilde{\boldsymbol{B}}$ on i such that $\tilde{\imath}$ is a Reedy cofibration. Proposition 18.3.8 now implies that $\widetilde{\boldsymbol{A}} \otimes \Delta[0] \to \widetilde{\boldsymbol{B}} \otimes \Delta[0]$ has the left lifting property with respect to p, and Lemma 17.3.5 implies that $\widetilde{\boldsymbol{A}} \otimes \Delta[0] \to \widetilde{\boldsymbol{B}} \otimes \Delta[0]$ is isomorphic to the map i. $\square$

PROPOSITION 18.3.10. *Let $\mathcal{M}$ be a model category.*

1. *If $i\colon A \to B$ is a cofibration between cofibrant objects and $p\colon X \to Y$ is a map such that i is left homotopy orthogonal to p, then any pushout of i is left homotopy orthogonal to p.*
2. *If $p\colon X \to Y$ is a fibration between fibrant objects and $i\colon A \to B$ is a map such that p is right homotopy orthogonal to i, then any pullback of p is right homotopy orthogonal to i.*

PROOF. We will prove part 1; the proof of part 2 is dual.

If we choose a simplicial resolution $\hat{p}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ of p such that $\hat{p}$ is a Reedy fibration (see Proposition 17.1.12), then Proposition 18.3.8 implies that i has the left lifting property with respect to the map $\widehat{\boldsymbol{X}}^{\Delta[n]} \to \widehat{\boldsymbol{Y}}^{\Delta[n]} \times_{\widehat{\boldsymbol{Y}}^{\partial\Delta[n]}} \widehat{\boldsymbol{X}}^{\partial\Delta[n]}$ for every $n \geq 0$. Since any pushout of i is also a cofibration between cofibrant objects, the result follows from Lemma 8.2.5 and Proposition 18.3.8. $\square$

COROLLARY 18.3.11. *Let $\mathcal{M}$ be a model category.*

1. *If X is an object of $\mathcal{M}$ and $i\colon A \to B$ is a cofibration between cofibrant objects that induces a weak equivalence of homotopy function complexes $i^*\colon \operatorname{map}(B, X) \cong \operatorname{map}(A, X)$, then any pushout of i also induces a weak equivalence of homotopy function complexes to X.*

2. *If B is an object of $\mathcal{M}$ and $p\colon X \to Y$ is a fibration between fibrant objects that induces a weak equivalence of homotopy function complexes $p_*\colon \mathrm{map}(B,X) \cong \mathrm{map}(B,Y)$, then any pullback of p also induces a weak equivalence of homotopy function complexes from B.*

PROOF. This follows from Proposition 18.3.5 and Proposition 18.3.10. $\square$

PROPOSITION 18.3.12. *Let $\mathcal{M}$ be a model category.*

1. *If $i\colon A \to B$, $j\colon B \to C$, and $p\colon X \to Y$ are maps in $\mathcal{M}$ such that (i,p) is a homotopy orthogonal pair, then (j,p) is a homotopy orthogonal pair if and only if (ji,p) is one.*
2. *If $i\colon A \to B$, $p\colon X \to Y$, and $q\colon Y \to Z$ are maps in $\mathcal{M}$ such that (i,q) is a homotopy orthogonal pair, then (i,p) is a homotopy orthogonal pair if and only if (i,qp) is one.*

PROOF. This follows from Proposition 11.2.15. $\square$

PROPOSITION 18.3.13. *Let $\mathcal{M}$ be a model category, and let $i\colon A \to B$ and $p\colon X \to Y$ be maps in $\mathcal{M}$ such that (i,p) is a homotopy orthogonal pair.*

1. *If $\tilde{\imath}\colon \widetilde{\boldsymbol{A}} \to \widetilde{\boldsymbol{B}}$ is a cosimplicial resolution of i such that $\tilde{\imath}$ is a Reedy cofibration, then for every $n \geq 0$ the map $\widetilde{\boldsymbol{A}} \otimes \Delta[n] \amalg_{\widetilde{\boldsymbol{A}} \otimes \partial\Delta[n]} \widetilde{\boldsymbol{B}} \otimes \partial\Delta[n] \to \widetilde{\boldsymbol{B}} \otimes \Delta[n]$ is left homotopy orthogonal to p.*
2. *If $\hat{p}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ is a simplicial resolution of p such that $\hat{p}$ is a Reedy fibration, then for every $n \geq 0$ the map $\widehat{\boldsymbol{X}}^{\Delta[n]} \to \widehat{\boldsymbol{Y}}^{\Delta[n]} \times_{\widehat{\boldsymbol{Y}}^{\partial\Delta[n]}} \widehat{\boldsymbol{X}}^{\partial\Delta[n]}$ is right homotopy orthogonal to i.*

PROOF. We will prove part 1; the proof of part 2 is dual.

Proposition 17.3.7 and Proposition 16.3.7 imply that for every $n \geq 0$ the map $\sigma_n\colon \widetilde{\boldsymbol{A}} \otimes \Delta[n] \amalg_{\widetilde{\boldsymbol{A}} \otimes \partial\Delta[n]} \widetilde{\boldsymbol{B}} \otimes \partial\Delta[n] \to \widetilde{\boldsymbol{B}} \otimes \Delta[n]$ is a cofibration between cofibrant objects. Thus, Proposition 18.3.8 implies that if $\hat{p}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ is a simplicial resolution of p such that $\hat{p}$ is a Reedy fibration, then it is sufficient to show that σ_n has the left lifting property with respect to the map $\tau_k\colon \widehat{\boldsymbol{X}}^{\Delta[k]} \to \widehat{\boldsymbol{Y}}^{\Delta[k]} \times_{\widehat{\boldsymbol{Y}}^{\partial\Delta[k]}} \widehat{\boldsymbol{X}}^{\partial\Delta[k]}$ for every $k \geq 0$. We will do this by induction on n.

Lemma 17.3.5 and Proposition 18.3.6 imply that for every $n \geq 0$ the map $\widetilde{\boldsymbol{A}} \otimes \Delta[n] \to \widetilde{\boldsymbol{B}} \otimes \Delta[n]$ is left homotopy orthogonal to p. Since the map σ_0 is the map $\widetilde{\boldsymbol{A}} \otimes \Delta[0] \to \widetilde{\boldsymbol{B}} \otimes \Delta[0]$, the induction is begun.

We now assume that $n > 0$ and that the result is true for all lesser values of n. Lemma 16.3.5 now implies that $\mathrm{L}_n\widetilde{\boldsymbol{A}} \to \mathrm{L}_n\widetilde{\boldsymbol{B}}$ has the left lifting property with respect to τ_k for every $k \geq 0$. Proposition 16.3.7 implies that $\mathrm{L}_n\widetilde{\boldsymbol{A}} \to \mathrm{L}_n\widetilde{\boldsymbol{B}}$ is a cofibration between cofibrant objects, and so Proposition 18.3.8 and Proposition 18.3.10 imply that any pushout of $\mathrm{L}_n\widetilde{\boldsymbol{A}} \to \mathrm{L}_n\widetilde{\boldsymbol{B}}$ is left homotopy orthogonal to p. Since Lemma 17.3.6 implies that the map $\widetilde{\boldsymbol{A}} \otimes \Delta[n] \to \widetilde{\boldsymbol{B}} \otimes \Delta[n]$ is isomorphic to the map $\mathrm{L}_n\widetilde{\boldsymbol{A}} \to \mathrm{L}_n\widetilde{\boldsymbol{B}}$ and the map $\widetilde{\boldsymbol{A}} \otimes \Delta[n] \to \widetilde{\boldsymbol{A}} \otimes \Delta[n] \amalg_{\widetilde{\boldsymbol{A}} \otimes \partial\Delta[n]} \widetilde{\boldsymbol{B}} \otimes \partial\Delta[n]$ is a pushout of it, this last map is left homotopy orthogonal to p. Since the composition $\widetilde{\boldsymbol{A}} \otimes \Delta[n] \to \widetilde{\boldsymbol{A}} \otimes \Delta[n] \amalg_{\widetilde{\boldsymbol{A}} \otimes \partial\Delta[n]} \widetilde{\boldsymbol{B}} \otimes \partial\Delta[n] \to \widetilde{\boldsymbol{B}} \otimes \Delta[n]$ is also left homotopy orthogonal to p, Proposition 18.3.12 completes the inductive step. $\square$

18.4. Sequential colimits

PROPOSITION 18.4.1. *If $\mathcal{M}$ is a model category, λ is an ordinal, and*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "g_0"'] & X_1 \arrow[r] \arrow[d, "g_1"'] & X_2 \arrow[r] \arrow[d, "g_2"'] & \cdots \\ Y_0 \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r] & \cdots \end{tikzcd}$$

is a map of λ-sequences in $\mathcal{M}$ such that

1. *each of the maps $g_\alpha \colon X_\alpha \to Y_\alpha$ (for $\alpha < \lambda$) is a weak equivalence of cofibrant objects, and*
2. *each of the maps $X_\alpha \to X_{\alpha+1}$ and $Y_\alpha \to Y_{\alpha+1}$ (for $\alpha < \lambda$) is a cofibration,*

then the induced map of colimits $(\operatorname{colim} g_\alpha)\colon \operatorname{colim} X_\alpha \to \operatorname{colim} Y_\alpha$ *is a weak equivalence.*

PROOF. If Z is an object of $\mathcal{M}$ and $\mathrm{s}Z \to \widehat{\boldsymbol{Z}}$ is a simplicial resolution of Z, then Theorem 18.1.6 implies that it is sufficient to show that the map $\mathcal{M}(\operatorname{colim} Y_\alpha, \widehat{\boldsymbol{Z}}) \to \mathcal{M}(\operatorname{colim} X_\alpha, \widehat{\boldsymbol{Z}})$ is a weak equivalence of simplicial sets.

Theorem 17.5.1 implies that the map $g^*\colon \mathcal{M}(Y_\alpha, \widehat{\boldsymbol{Z}}) \to \mathcal{M}(X_\alpha, \widehat{\boldsymbol{Z}})$ is a weak equivalence of fibrant simplicial sets for every $\alpha < \lambda$, and so the diagram

$$\begin{tikzcd} \cdots \arrow[r] & \mathcal{M}(Y_2, \widehat{\boldsymbol{Z}}) \arrow[r] \arrow[d] & \mathcal{M}(Y_1, \widehat{\boldsymbol{Z}}) \arrow[r] \arrow[d] & \mathcal{M}(Y_0, \widehat{\boldsymbol{Z}}) \arrow[d] \\ \cdots \arrow[r] & \mathcal{M}(X_2, \widehat{\boldsymbol{Z}}) \arrow[r] & \mathcal{M}(X_1, \widehat{\boldsymbol{Z}}) \arrow[r] & \mathcal{M}(X_0, \widehat{\boldsymbol{Z}}) \end{tikzcd}$$

is a weak equivalence of towers of fibrations of fibrant simplicial sets. Thus, the induced map $\lim \mathcal{M}(Y_\alpha, \widehat{\boldsymbol{Z}}) \to \lim \mathcal{M}(X_\alpha, \widehat{\boldsymbol{Z}})$ is a weak equivalence. Since this map is isomorphic to the map $\mathcal{M}(\operatorname{colim} Y_\alpha, \widehat{\boldsymbol{Z}}) \to \mathcal{M}(\operatorname{colim} X_\alpha, \widehat{\boldsymbol{Z}})$, the proof is complete. $\square$

18.5. Properness

18.5.1. Sequential colimits.

PROPOSITION 18.5.2. *Let $\mathcal{M}$ be a left proper model category (see Definition 11.1.1). If λ is an ordinal and*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "g_0"'] & X_1 \arrow[r] \arrow[d, "g_1"'] & X_2 \arrow[r] \arrow[d, "g_2"'] & \cdots \\ Y_0 \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r] & \cdots \end{tikzcd}$$

is a map of λ-sequences in $\mathcal{M}$ such that

1. *each of the maps $X_\alpha \to X_{\alpha+1}$ and $Y_\alpha \to Y_{\alpha+1}$ (for $\alpha < \lambda$) is a cofibration;*
2. *each of the maps $g_\alpha \colon X_\alpha \to Y_\alpha$ (for $\alpha < \lambda$) is a weak equivalence;*

then the induced map $(\operatorname{colim} g_\alpha)\colon \operatorname{colim} X_\alpha \to \operatorname{colim} Y_\alpha$ *is a weak equivalence.*

PROOF. This is identical to the proof of Proposition 11.1.21, except that we use Proposition 18.4.1 in place of Proposition 10.5.6. $\square$

PROPOSITION 18.5.3. *Let $\mathcal{M}$ be a left proper model category. If λ is an ordinal and*

$$X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \qquad (\beta < \lambda)$$

is a λ-sequence in $\mathcal{M}$ such that $X_\beta \to X_{\beta+1}$ is a cofibration for every $\beta < \lambda$, then there is a λ-sequence

$$\widetilde{X}_0 \to \widetilde{X}_1 \to \widetilde{X}_2 \to \cdots \to \widetilde{X}_\beta \to \cdots \qquad (\beta < \lambda)$$

and a map of λ-sequences

$$\begin{tikzcd} \widetilde{X}_0 \arrow[r] \arrow[d, "g_0"'] & \widetilde{X}_1 \arrow[r] \arrow[d, "g_1"'] & \widetilde{X}_2 \arrow[r] \arrow[d, "g_2"'] & \cdots \arrow[r] & \widetilde{X}_\beta \arrow[r] \arrow[d, "g_\beta"'] & \cdots & (\beta < \lambda) \\ X_0 \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & \cdots \arrow[r] & X_\beta \arrow[r] & \cdots & (\beta < \lambda) \end{tikzcd}$$

such that

1. *every $\widetilde{X}_\beta$ is cofibrant,*
2. *every $g_\beta : \widetilde{X}_\beta \to X_\beta$ is a weak equivalence,*
3. *every $\widetilde{X}_\beta \to \widetilde{X}_{\beta+1}$ is a cofibration, and*
4. *the map $\operatorname{colim}_{\beta<\lambda} \widetilde{X}_\beta \to \operatorname{colim}_{\beta<\lambda} X_\beta$ is a weak equivalence.*

PROOF. This is identical to the proof of Proposition 11.1.22, except that we use Proposition 18.5.2 in place of Proposition 11.1.21. □

18.5.4. Homotopy orthogonal maps.

PROPOSITION 18.5.5. *Let $\mathcal{M}$ be a left proper model category, and let $p\colon X \to Y$ be a map in $\mathcal{M}$. If $i\colon A \to B$ is left homotopy orthogonal to p, the diagram*

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & C \arrow[d, "k"] \\ B \arrow[r] & D \end{tikzcd}$$

is a pushout, and at least one of i and j is a cofibration, then k is left homotopy orthogonal to p.

PROOF. Let $\tilde{\imath}\colon \widetilde{A} \to \widetilde{B}$ be a cofibrant approximation to i such that $\tilde{\imath}$ is a cofibration (see Proposition 9.1.9). Proposition 18.3.6 implies that $\tilde{\imath}$ is left homotopy orthogonal to p, and so Proposition 18.3.10 implies that any pushout of $\tilde{\imath}$ is left homotopy orthogonal to p. Since Proposition 11.3.2 implies that k has a cofibrant approximation that is a pushout of $\tilde{\imath}$, the result follows from Proposition 18.3.6. □

PROPOSITION 18.5.6. *Let $\mathcal{M}$ be a right proper model category, and let $i\colon A \to B$ be a map in $\mathcal{M}$. If $p\colon X \to Y$ is right homotopy orthogonal to i, the diagram*

$$\begin{tikzcd} W \arrow[r] \arrow[d, "r"'] & X \arrow[d, "p"] \\ Z \arrow[r, "q"'] & Y \end{tikzcd}$$

is a pullback, and at least one of p and q is a fibration, then r is right homotopy orthogonal to i.

PROOF. This is dual to Proposition 18.5.5 (see Remark 8.1.7). □

18.6. Quillen functors and resolutions

PROPOSITION 18.6.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories and let* $\mathrm{F}\colon \mathcal{M} \rightleftarrows \mathcal{N} :\mathrm{U}$ *be a Quillen pair (seeDefinition 9.8.1).*

1. *If $\widetilde{\boldsymbol{B}}$ is a cosimplicial resolution in $\mathcal{M}$ (see Definition 17.1.16), then $\mathrm{F}\widetilde{\boldsymbol{B}}$ is a cosimplicial resolution in $\mathcal{N}$.*
2. *If $\widehat{\boldsymbol{X}}$ is a simplicial resolution in $\mathcal{N}$, then $\mathrm{U}\widehat{\boldsymbol{X}}$ is a simplicial resolution in $\mathcal{M}$.*

PROOF. We will prove part 1; the proof of part 2 is dual.

Corollary 16.6.2 implies that $\mathrm{F}\widetilde{\boldsymbol{B}}$ is Reedy cofibrant. Since $\widetilde{\boldsymbol{B}}$ is Reedy cofibrant, all of the objects $\widetilde{\boldsymbol{B}}^n$ are cofibrant in $\mathcal{M}$, and so the coface and codegeneracy operators are weak equivalences of cofibrant objects. Thus, Corollary 8.5.2 implies that all of the coface and codegeneracy operators of $\mathrm{F}\widetilde{\boldsymbol{B}}$ are weak equivalences. □

PROPOSITION 18.6.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories and let* $\mathrm{F}\colon \mathcal{M} \rightleftarrows \mathcal{N} :\mathrm{U}$ *be a Quillen pair (see Definition 9.8.1).*

1. *If B is a cofibrant object in $\mathcal{M}$ and $\widetilde{\boldsymbol{B}}$ is a cosimplicial resolution of B, then $\mathrm{F}\widetilde{\boldsymbol{B}}$ is a cosimplicial resolution of $\mathrm{F}B$.*
2. *If X is a fibrant object in $\mathcal{N}$ and $\widehat{\boldsymbol{X}}$ is a simplicial resolution of X, then $\mathrm{U}\widehat{\boldsymbol{X}}$ is a simplicial resolution of $\mathrm{U}X$.*

PROOF. This follows from Proposition 18.6.1 and Corollary 8.5.2. □

COROLLARY 18.6.3. *Let $\mathcal{C}$ be a Reedy category, let $\mathcal{M}$ and $\mathcal{N}$ be small categories, and let* $\mathrm{F}\colon \mathcal{M} \rightleftarrows \mathcal{N} :\mathrm{U}$ *be a Quillen pair.*

1. *If $i\colon A \to B$ is a map of cofibrant objects in $\mathcal{M}$ and $\tilde{\imath}\colon \widetilde{\boldsymbol{A}} \to \widetilde{\boldsymbol{B}}$ is a cosimplicial resolution of i such that $\tilde{\imath}$ is a Reedy cofibration, then $\mathrm{F}\tilde{\imath}\colon \mathrm{F}\widetilde{\boldsymbol{A}} \to \mathrm{F}\widetilde{\boldsymbol{B}}$ is a cosimplicial resolution of $\mathrm{F}i$ and $\mathrm{F}\tilde{\imath}$ is a Reedy cofibration.*
2. *If $p\colon X \to Y$ is a map of fibrant objects in $\mathcal{N}$ and $\hat{p}\colon \widehat{\boldsymbol{X}} \to \widehat{\boldsymbol{Y}}$ is a simplicial resolution of p such that $\hat{p}$ is a Reedy fibration, then $\mathrm{U}\hat{p}$ is a simplicial resolution of $\mathrm{U}p$ and $\mathrm{U}\hat{p}$ is a Reedy fibration.*

PROOF. This follows from Proposition 16.6.1 and Proposition 18.6.2. □

CHAPTER 19

Homotopy colimits and homotopy limits

The main references for this chapter are [**15**, Chapters X through XII], [**18**], and [**31**]. Our definitions for diagrams of simplicial sets are essentially those of [**15**] (see Remark 19.1.15). Our definitions for diagrams in a general model category are due to D. M. Kan, who also established their properties using methods different from the ones used here.

19.1. Homotopy colimits and homotopy limits

19.1.1. Homotopy colimits.

Definition 19.1.2. Let $\mathcal{M}$ be a framed model category (see Definition 17.1.33), and let $\mathcal{C}$ be a small category. If $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, then the *homotopy colimit* $\operatorname{hocolim} \boldsymbol{X}$ of $\boldsymbol{X}$ is defined to be the coequalizer of the maps

$$\coprod_{(\sigma\colon \alpha \to \alpha') \in \mathcal{C}} \widetilde{\boldsymbol{X}}_\alpha \otimes \mathrm{B}(\alpha' \downarrow \mathcal{C})^{\mathrm{op}} \overset{\phi}{\underset{\psi}{\rightrightarrows}} \coprod_{\alpha \in \mathrm{Ob}(\mathcal{C})} \widetilde{\boldsymbol{X}}_\alpha \otimes \mathrm{B}(\alpha \downarrow \mathcal{C})^{\mathrm{op}}$$

(see Definition 17.3.1, Definition 9.4.1, and Definition 14.4.2) where $\widetilde{\boldsymbol{X}}_\alpha$ is the natural cosimplicial frame on $\boldsymbol{X}_\alpha$, the map ϕ on the summand $\sigma\colon \alpha \to \alpha'$ is the composition of the map

$$\sigma_* \otimes 1_{\mathrm{B}(\alpha' \downarrow \mathcal{C})}\colon \widetilde{\boldsymbol{X}}_\alpha \otimes \mathrm{B}(\alpha' \downarrow \mathcal{C})^{\mathrm{op}} \to \widetilde{\boldsymbol{X}}_{\alpha'} \otimes \mathrm{B}(\alpha' \downarrow \mathcal{C})^{\mathrm{op}}$$

with the natural injection into the coproduct, and the map ψ on the summand $\sigma\colon \alpha \to \alpha'$ is the composition of the map

$$1_{\widetilde{\boldsymbol{X}}_\alpha} \otimes \mathrm{B}(\sigma^*)\colon \widetilde{\boldsymbol{X}}_\alpha \otimes \mathrm{B}(\alpha' \downarrow \mathcal{C})^{\mathrm{op}} \to \widetilde{\boldsymbol{X}}_\alpha \otimes \mathrm{B}(\alpha \downarrow \mathcal{C})^{\mathrm{op}}$$

(where $\sigma^*\colon (\alpha' \downarrow \mathcal{C})^{\mathrm{op}} \to (\alpha \downarrow \mathcal{C})^{\mathrm{op}}$; see Definition 14.5.2) with the natural injection into the coproduct.

For a discussion of the relation of our definition of the homotopy colimit to that of [**15**], see Remark 19.1.15.

Remark 19.1.3. If $\mathcal{M}$ is a simplicial model category, then for every object X in $\mathcal{M}$ and every simplicial set K, the object $\widetilde{\boldsymbol{X}} \otimes K$ (where $\widetilde{\boldsymbol{X}}$ is the standard cosimplicial frame on X; see Proposition 17.1.35) is naturally isomorphic to $X \otimes K$ (see Proposition 17.3.10). Thus, if $\mathcal{C}$ is a small category and $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, then $\operatorname{hocolim} \boldsymbol{X}$ is naturally isomorphic to the coequalizer of the maps

$$\coprod_{(\sigma\colon \alpha \to \alpha') \in \mathcal{C}} \boldsymbol{X}_\alpha \otimes \mathrm{B}(\alpha' \downarrow \mathcal{C})^{\mathrm{op}} \overset{\phi}{\underset{\psi}{\rightrightarrows}} \coprod_{\alpha \in \mathrm{Ob}(\mathcal{C})} \boldsymbol{X}_\alpha \otimes \mathrm{B}(\alpha \downarrow \mathcal{C})^{\mathrm{op}}$$

For example, with the standard framings on the simplicial model categories SS, SS_*, Top, and Top_* (see Notation 1.1.2),

$$\widetilde{\boldsymbol{X}}_\alpha \otimes \mathrm{B}(\alpha\downarrow\mathcal{C})^{\mathrm{op}} \approx \begin{cases} \boldsymbol{X}_\alpha \times \mathrm{B}(\alpha\downarrow\mathcal{C})^{\mathrm{op}} & \text{if } \mathcal{M} = \mathsf{SS} \\ \boldsymbol{X}_\alpha \wedge \left(\mathrm{B}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}\right)^+ & \text{if } \mathcal{M} = \mathsf{SS}_* \\ \boldsymbol{X}_\alpha \times \left|\mathrm{B}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}\right| & \text{if } \mathcal{M} = \mathsf{Top} \\ \boldsymbol{X}_\alpha \wedge \left|\mathrm{B}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}\right|^+ & \text{if } \mathcal{M} = \mathsf{Top}_* \end{cases}$$

(see Definition 1.1.11).

Example 19.1.4. If $g\colon X \to Y$ is a map in $\mathsf{Spc}_{(*)}$ (see Notation 1.1.2), then the homotopy colimit of this diagram is the mapping cylinder of g.

Example 19.1.5. If $Z \xleftarrow{h} X \xrightarrow{g} Y$ are maps in $\mathsf{Spc}_{(*)}$ (see Notation 1.1.2), then the homotopy colimit of this diagram is the double mapping cylinder of g and h.

Proposition 19.1.6. *If $\mathcal{C}$ is a small category and $\boldsymbol{P}\colon \mathcal{C} \to \mathsf{SS}$ is the diagram of simplicial sets in which $\boldsymbol{P}_\alpha$ is a single point for every object α in $\mathcal{C}$, then there is a natural isomorphism* $\operatorname{hocolim} \boldsymbol{P} \approx \mathrm{B}\mathcal{C}^{\mathrm{op}}$.

Proof. Remark 19.1.3 implies that $\operatorname{hocolim} \boldsymbol{P}$ is naturally isomorphic to the coequalizer of the maps

$$\coprod_{(\sigma\colon \alpha\to\alpha')\in\mathcal{C}} \mathrm{B}(\alpha'\downarrow\mathcal{C})^{\mathrm{op}} \underset{\psi}{\overset{\phi}{\rightrightarrows}} \coprod_{\alpha\in\mathrm{Ob}(\mathcal{C})} \mathrm{B}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$$

where the map ϕ is the identity map and the map ψ on the summand $\sigma\colon \alpha \to \alpha'$ is the composition of the map $\mathrm{B}(\sigma^*)\colon \mathrm{B}(\alpha'\downarrow\mathcal{C})^{\mathrm{op}} \to \mathrm{B}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$ with the natural injection into the coproduct. We define a map $\mathrm{B}(\alpha\downarrow\mathcal{C})^{\mathrm{op}} \to \mathrm{B}\mathcal{C}^{\mathrm{op}}$ by sending the simplex

$$\begin{tikzcd} & \alpha \arrow[dl] \arrow[d] \arrow[drr] & & \\ \sigma_0 & \sigma_1 \arrow[l] & \cdots \arrow[l] & \sigma_n \arrow[l] \end{tikzcd}$$

of $\mathrm{B}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$ to the simplex $\sigma_0 \leftarrow \sigma_1 \leftarrow \cdots \leftarrow \sigma_n$ of $\mathrm{B}\mathcal{C}^{\mathrm{op}}$. This defines a surjective map $\operatorname{hocolim} \boldsymbol{P} \to \mathrm{B}\mathcal{C}^{\mathrm{op}}$ which is also injective because every simplex of $\coprod_{\sigma\in\mathrm{Ob}(\mathcal{C})} \mathrm{B}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$ that is mapped to $\sigma_0 \leftarrow \sigma_1 \leftarrow \cdots \leftarrow \sigma_n$ is equal (in $\operatorname{hocolim} \boldsymbol{P}$) to the simplex

$$\begin{tikzcd} & \sigma_n \arrow[dl] \arrow[d] \arrow[drr, "1_{\sigma_n}"] & & \\ \sigma_0 & \sigma_1 \arrow[l] & \cdots \arrow[l] & \sigma_n \arrow[l] \end{tikzcd}$$

□

Definition 19.1.7. Let $\mathcal{M}$ be a framed model category, let $\mathcal{C}$ and $\mathcal{D}$ be small categories, and let $\mathrm{F}\colon \mathcal{C} \to \mathcal{D}$ be a functor. If $\boldsymbol{X}\colon \mathcal{D} \to \mathcal{M}$ is a $\mathcal{D}$-diagram in $\mathcal{M}$, then composition with F defines a $\mathcal{C}$-diagram $\mathrm{F}^*\boldsymbol{X}$ in $\mathcal{M}$ which we will call the $\mathcal{C}$-diagram *induced* by F. If α is an object in $\mathcal{C}$, then $(\mathrm{F}^*\boldsymbol{X})_\alpha = \boldsymbol{X}_{\mathrm{F}\alpha}$, and if $\sigma\colon \alpha \to \alpha'$ in $\mathcal{C}$, then $(\mathrm{F}^*\boldsymbol{X})_\sigma = \boldsymbol{X}_{\mathrm{F}\sigma}\colon \boldsymbol{X}_{\mathrm{F}\alpha} \to \boldsymbol{X}_{\mathrm{F}\alpha'}$.

PROPOSITION 19.1.8. *Let $\mathcal{M}$ be a framed model category. If $\mathcal{C}$ and $\mathcal{D}$ are small categories, $\mathrm{F}\colon \mathcal{C} \to \mathcal{D}$ is a functor, and $\boldsymbol{X}\colon \mathcal{D} \to \mathcal{M}$ is a $\mathcal{D}$-diagram in $\mathcal{M}$, then there is a natural map*

$$\operatorname*{hocolim}_{\mathcal{C}} \mathrm{F}^*\boldsymbol{X} \to \operatorname*{hocolim}_{\mathcal{D}} \boldsymbol{X}$$

defined by sending $\widetilde{\mathrm{F}^*\boldsymbol{X}}_\alpha \otimes \mathrm{B}(\alpha\downarrow\mathcal{C})^{\mathrm{op}} = \widetilde{\boldsymbol{X}}_{\mathrm{F}\alpha} \otimes \mathrm{B}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$ *to* $\widetilde{\boldsymbol{X}}_{\mathrm{F}\alpha} \otimes \mathrm{B}(\mathrm{F}\alpha\downarrow\mathcal{D})^{\mathrm{op}}$.

PROOF. This follows directly from the definitions. $\square$

It is often of interest to know conditions on a functor F that ensure that the natural map of Proposition 19.1.8 is a weak equivalence for all $\mathcal{D}$-diagrams of cofibrant objects. For this, see Theorem 19.5.11.

19.1.9. Homotopy limits.

DEFINITION 19.1.10. Let $\mathcal{M}$ be a framed model category (see Definition 17.1.33) and let $\mathcal{C}$ be a small category. If $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, then the *homotopy limit* $\operatorname{holim}\boldsymbol{X}$ of $\boldsymbol{X}$ is defined to be the equalizer of the maps

$$\prod_{\alpha\in\mathrm{Ob}(\mathcal{C})} (\widehat{\boldsymbol{X}}_\alpha)^{\mathrm{B}(\mathcal{C}\downarrow\alpha)} \overset{\phi}{\underset{\psi}{\rightrightarrows}} \prod_{(\sigma\colon \alpha\to\alpha')\in\mathcal{C}} (\widehat{\boldsymbol{X}}_{\alpha'})^{\mathrm{B}(\mathcal{C}\downarrow\alpha)}$$

(see Definition 17.3.1, Definition 9.4.1, and Definition 14.4.1) where $\widehat{\boldsymbol{X}}_\alpha$ is the standard simplicial frame on $\boldsymbol{X}_\alpha$, the projection of the map ϕ on the factor $\sigma\colon \alpha \to \alpha'$ is the composition of a natural projection from the product with the map

$$\sigma_*^{1_{\mathrm{B}(\mathcal{C}\downarrow\alpha)}}\colon (\widehat{\boldsymbol{X}}_\alpha)^{\mathrm{B}(\mathcal{C}\downarrow\alpha)} \to (\widehat{\boldsymbol{X}}_{\alpha'})^{\mathrm{B}(\mathcal{C}\downarrow\alpha)}$$

and the projection of the map ψ on the factor $\sigma\colon \alpha \to \alpha'$ is the composition of a natural projection from the product with the map

$$(1_{\widehat{\boldsymbol{X}}_{\alpha'}})^{\mathrm{B}(\sigma_*)}\colon (\widehat{\boldsymbol{X}}_{\alpha'})^{\mathrm{B}(\mathcal{C}\downarrow\alpha')} \to (\widehat{\boldsymbol{X}}_{\alpha'})^{\mathrm{B}(\mathcal{C}\downarrow\alpha)}$$

(where $\sigma_*\colon (\mathcal{C}\downarrow\alpha) \to (\mathcal{C}\downarrow\alpha')$; see Definition 14.5.7).

For a discussion of the relation of our definition of the homotopy limit to that of [**15**], see Remark 19.1.15.

If $\mathcal{C}$ is a small category and $\mathcal{M}$ is a category of spaces (i.e., one of SS, SS_*, Top, or Top_*), then the homotopy limit of a $\mathcal{C}$-diagram in $\mathcal{M}$ can be described as a space of maps between diagrams. If X is a space and K is a simplicial set, then the space $\widehat{\boldsymbol{X}}^K$ (where $\widehat{\boldsymbol{X}}$ is the standard simplicial frame on X; see Proposition 17.1.35) is naturally isomorphic to

$$\widehat{\boldsymbol{X}}^K \approx \begin{cases} \operatorname{Map}(K, X) & \text{if } \mathcal{M} = \mathsf{SS} \\ \operatorname{Map}_*(K^+, X) & \text{if } \mathcal{M} = \mathsf{SS}_* \\ \operatorname{map}(|K|, X) & \text{if } \mathcal{M} = \mathsf{Top} \\ \operatorname{map}_*(|K|^+, X) & \text{if } \mathcal{M} = \mathsf{Top}_* \end{cases}$$

(see Proposition 17.3.10 and Definition 1.1.11). Thus, in these cases, Definition 19.1.10 defines $\operatorname{holim} \boldsymbol{X}$ as the equalizer of the maps

$$\prod_{\alpha \in \mathrm{Ob}(\mathcal{C})} \mathrm{Map}\big(\mathrm{B}(\mathcal{C}\downarrow\alpha), \boldsymbol{X}_\alpha\big) \rightrightarrows \prod_{(\sigma\colon \alpha\to\alpha')\in\mathcal{C}} \mathrm{Map}\big(\mathrm{B}(\mathcal{C}\downarrow\alpha), \boldsymbol{X}_{\alpha'}\big) \qquad \text{if } \mathcal{M} = \mathsf{SS}$$

$$\prod_{\alpha \in \mathrm{Ob}(\mathcal{C})} \mathrm{Map}_*\big(\mathrm{B}(\mathcal{C}\downarrow\alpha)^+, \boldsymbol{X}_\alpha\big) \rightrightarrows \prod_{(\sigma\colon \alpha\to\alpha')\in\mathcal{C}} \mathrm{Map}_*\big(\mathrm{B}(\mathcal{C}\downarrow\alpha)^+, \boldsymbol{X}_{\alpha'}\big) \qquad \text{if } \mathcal{M} = \mathsf{SS}_*$$

$$\prod_{\alpha \in \mathrm{Ob}(\mathcal{C})} \mathrm{map}\big(\big|\mathrm{B}(\mathcal{C}\downarrow\alpha)\big|, \boldsymbol{X}_\alpha\big) \rightrightarrows \prod_{(\sigma\colon \alpha\to\alpha')\in\mathcal{C}} \mathrm{map}\big(\big|\mathrm{B}(\mathcal{C}\downarrow\alpha)\big|, \boldsymbol{X}_{\alpha'}\big) \qquad \text{if } \mathcal{M} = \mathsf{Top}$$

$$\prod_{\alpha \in \mathrm{Ob}(\mathcal{C})} \mathrm{map}_*\big(\big|\mathrm{B}(\mathcal{C}\downarrow\alpha)\big|^+, \boldsymbol{X}_\alpha\big) \rightrightarrows \prod_{(\sigma\colon \alpha\to\alpha')\in\mathcal{C}} \mathrm{map}_*\big(\big|\mathrm{B}(\mathcal{C}\downarrow\alpha)\big|^+, \boldsymbol{X}_{\alpha'}\big) \qquad \text{if } \mathcal{M} = \mathsf{Top}_*$$

This is exactly the definition of the space of maps

$$\begin{array}{lll} \text{from } \mathrm{B}(\mathcal{C}\downarrow -) & \text{to } \boldsymbol{X} \text{ in } \mathsf{SS}^{\mathcal{C}}, & \text{if } \mathcal{M} = \mathsf{SS} \\ \text{from } \mathrm{B}(\mathcal{C}\downarrow -)^+ & \text{to } \boldsymbol{X} \text{ in } \mathsf{SS}_*^{\mathcal{C}}, & \text{if } \mathcal{M} = \mathsf{SS}_* \\ \text{from } \big|\mathrm{B}(\mathcal{C}\downarrow -)\big| & \text{to } \boldsymbol{X} \text{ in } \mathsf{Top}^{\mathcal{C}}, & \text{if } \mathcal{M} = \mathsf{Top} \\ \text{from } \big|\mathrm{B}(\mathcal{C}\downarrow -)\big|^+ & \text{to } \boldsymbol{X} \text{ in } \mathsf{Top}_*^{\mathcal{C}}, & \text{if } \mathcal{M} = \mathsf{Top}_* \end{array}$$

PROPOSITION 19.1.11. *If $\mathcal{M}$ is a category of spaces (i.e., one of SS, SS_*, Top, or Top_*), $\mathcal{C}$ is a small category, and $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram of spaces, then* $\operatorname{holim} \boldsymbol{X}$ *is naturally isomorphic to the space of maps*

$$\begin{array}{ll} \mathrm{Map}\big(\mathrm{B}(\mathcal{C}\downarrow -), \boldsymbol{X}\big), & \textit{if } \mathcal{M} = \mathsf{SS} \\ \mathrm{Map}_*\big(\mathrm{B}(\mathcal{C}\downarrow -)^+, \boldsymbol{X}\big), & \textit{if } \mathcal{M} = \mathsf{SS}_* \\ \mathrm{map}\big(\big|\mathrm{B}(\mathcal{C}\downarrow -)\big|, \boldsymbol{X}\big), & \textit{if } \mathcal{M} = \mathsf{Top} \\ \mathrm{map}_*\big(\big|\mathrm{B}(\mathcal{C}\downarrow -)\big|^+, \boldsymbol{X}\big), & \textit{if } \mathcal{M} = \mathsf{Top}_* \end{array}$$

(see Definition 1.1.6*).*

PROOF. This follows from the discussion immediately preceding the proposition. □

EXAMPLE 19.1.12. If $g\colon X \to Y$ is a map in $\mathsf{Spc}_{(*)}$(*seeNotation* 1.1.2), then the homotopy limit of this diagram is the mapping path space of g.

REMARK 19.1.13. When considering diagrams of spaces, there is a fundamental difference between the homotopy colimit and the homotopy limit regarding the significance of working in a category of *pointed* spaces. In Section 20.7, we will show that if $\boldsymbol{X}$ is a diagram of pointed spaces, then the homotopy colimit of $\boldsymbol{X}$ formed in the category of pointed spaces is not, in general weakly equivalent to the homotopy colimit formed in the category of unpointed spaces after forgetting the basepoints of the spaces in the diagram $\boldsymbol{X}$. However, the homotopy limit of $\boldsymbol{X}$ formed in the category of pointed spaces is isomorphic (or homeomorphic) to the space obtained by forgetting the basepoints of the spaces in $\boldsymbol{X}$ and forming the homotopy limit in the category of unpointed spaces. This is because if X is an unpointed space and Y is a pointed space, then the space of pointed maps $\mathrm{map}_*(X^+, Y)$ is isomorphic (or homeomorphic) to the space of unpointed maps $\mathrm{map}(X, Y)$ (except that in the first case we've kept track of the basepoint of the space of maps).

PROPOSITION 19.1.14. *Let $\mathcal{M}$ be a framed model category. If $\mathcal{C}$ and $\mathcal{D}$ are small categories, $\mathrm{F}\colon \mathcal{C} \to \mathcal{D}$ is a functor, and $\boldsymbol{X}\colon \mathcal{D} \to \mathcal{M}$ is a $\mathcal{D}$-diagram in $\mathcal{M}$, then there* *is a natural map*

$$\operatorname*{holim}_{\mathcal{D}} \boldsymbol{X} \to \operatorname*{holim}_{\mathcal{C}} \mathrm{F}^* \boldsymbol{X}$$

(see Definition 19.1.7*) induced by the natural map* $\mathrm{B}(\mathcal{C}\downarrow\alpha) \to \mathrm{B}(\mathcal{D}\downarrow\mathrm{F}\alpha)$*.*

Proof. This follows directly from the definitions. □

It is often of interest to know conditions on a functor F that ensure that the natural map of Proposition 19.1.14 is a weak equivalence for all $\mathcal{D}$-diagrams of fibrant objects. For this, see Theorem 19.5.11.

Remark 19.1.15. There are two respects in which our definitions of the homotopy colimit and the homotopy limit differ from those of [**15**] (which uses the term *homotopy direct limit* for the homotopy colimit). First, we use the diagrams of simplicial sets $\mathrm{B}(-\downarrow\mathcal{C})^{\mathrm{op}}$ and $\mathrm{B}(\mathcal{C}\downarrow-)$ (see Definition 19.1.2 and Definition 19.1.10) where [**15**] uses the diagrams $\mathrm{B}(-\downarrow\mathcal{C})$ and $\mathrm{B}(\mathcal{C}\downarrow-)$ (see [**15**, Chapter XII, Paragraph 2.1 and Chapter XI, Paragraph 3.2]. Since both $\mathrm{B}(-\downarrow\mathcal{C})^{\mathrm{op}}$ and $\mathrm{B}(-\downarrow\mathcal{C})$ are cofibrant approximations to the constant $\mathcal{C}^{\mathrm{op}}$-diagram at a point (see Corollary 14.6.8), these two choices give definitions that are naturally weakly equivalent for $\mathcal{C}$-diagrams of cofibrant spaces (see Theorem 20.8.4), but our definition was chosen to make Corollary 20.3.19 true. It is incorrectly stated in [**15**, Chapter XII, Proposition 4.1] that this is true for the definitions used in [**15**]; this is due to an error in the proof of [**15**, Chapter XII, Proposition 4.1]. This error is a minor one, since the spaces claimed there to be isomorphic are in fact naturally weakly equivalent, which is all that was needed.

The second difference between our definitions and those of [**15**] is that the definition of the classifying space (i.e., the nerve) of a category used in [**15**] is “opposite” to our definition (see Definition 9.4.1 and [**15**, Chapter XI, Paragraph 2.1]), i.e., if $\mathcal{C}$ is a small category, then the definition of $\mathrm{B}\mathcal{C}$ used in [**15**] (which is called there the *underlying space* of the category) is isomorphic to our definition of $\mathrm{B}\mathcal{C}^{\mathrm{op}}$.

The combined effect of the above two differences is that our definition of the homotopy colimit is isomorphic to that of [**15**], but our definition of the homotopy limit is different. Since the $\mathcal{C}$-diagrams of simplicial sets $\mathrm{B}(\mathcal{C}\downarrow-)$ and $\mathrm{B}(\mathcal{C}\downarrow-)^{\mathrm{op}}$ are both free cell complexes (see Definition 14.1.28), these two definitions of the homotopy limit are naturally weakly equivalent for diagrams of fibrant spaces (see Theorem 20.8.1).

19.2. Adjointness

19.2.1. Coends and ends.

Definition 19.2.2. Let $\mathcal{M}$ be a framed model category (see Definition 17.1.33) and let $\mathcal{C}$ be a small category.

1. If $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$ and $\boldsymbol{K}\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{SS}$ is a $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets, then $\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}$ is defined to be the object of $\mathcal{M}$ that is the coequalizer of the maps

$$\coprod_{(\sigma\colon \alpha\to\alpha')\in\mathcal{C}} \widetilde{\boldsymbol{X}}_{\alpha} \otimes \boldsymbol{K}_{\alpha'} \overset{\phi}{\underset{\psi}{\rightrightarrows}} \coprod_{\alpha\in\mathrm{Ob}(\mathcal{C})} \widetilde{\boldsymbol{X}}_{\alpha} \otimes \boldsymbol{K}_{\alpha}$$

(see Definition 17.3.1) where $\widetilde{\boldsymbol{X}}_\alpha$ is the natural cosimplicial frame on $\boldsymbol{X}_\alpha$, the map ϕ on the summand $\sigma\colon \alpha \to \alpha'$ is the composition of the map

$$\sigma_* \otimes 1_{\boldsymbol{K}_{\alpha'}} \colon \widetilde{\boldsymbol{X}}_\alpha \otimes \boldsymbol{K}_{\alpha'} \to \widetilde{\boldsymbol{X}}_{\alpha'} \otimes \boldsymbol{K}_{\alpha'}$$

(where $\sigma_*\colon \widetilde{\boldsymbol{X}}_\alpha \to \widetilde{\boldsymbol{X}}_{\alpha'}$) with the natural injection into the coproduct, and the map ψ on the summand $\sigma\colon \alpha \to \alpha'$ is the composition of the map

$$1_{\widetilde{\boldsymbol{X}}_\alpha} \otimes \sigma^* \colon \widetilde{\boldsymbol{X}}_\alpha \otimes \boldsymbol{K}_{\alpha'} \to \widetilde{\boldsymbol{X}}_\alpha \otimes \boldsymbol{K}_\alpha$$

(where $\sigma^*\colon \boldsymbol{K}_{\alpha'} \to \boldsymbol{K}_\alpha$) with the natural injection into the coproduct.

2. If $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$ and $\boldsymbol{K}\colon \mathcal{C} \to \mathsf{SS}$ is a $\mathcal{C}$-diagram of simplicial sets, then $\hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X})$ is defined to be the equalizer of the maps

$$\prod_{\alpha \in \mathrm{Ob}(\mathcal{C})} (\widehat{\boldsymbol{X}}_\alpha)^{\boldsymbol{K}_\alpha} \underset{\psi}{\overset{\phi}{\rightrightarrows}} \prod_{(\sigma\colon \alpha \to \alpha') \in \mathcal{C}} (\widehat{\boldsymbol{X}}_{\alpha'})^{\boldsymbol{K}_\alpha}$$

(see Definition 17.3.1) where $\widehat{\boldsymbol{X}}_\alpha$ is the natural simplicial frame on $\boldsymbol{X}_\alpha$, the projection of the map ϕ on the factor $\sigma\colon \alpha \to \alpha'$ is the composition of a natural projection from the product with the map

$$\sigma_*^{1_{\boldsymbol{K}_\alpha}} \colon (\widehat{\boldsymbol{X}}_\alpha)^{\boldsymbol{K}_\alpha} \to (\widehat{\boldsymbol{X}}_{\alpha'})^{\boldsymbol{K}_\alpha}$$

(where $\sigma_*\colon \widehat{\boldsymbol{X}}_\alpha \to \widehat{\boldsymbol{X}}_{\alpha'}$) and the projection of the map ψ on the factor $\sigma\colon \alpha \to \alpha'$ is the composition of a natural projection from the product with the map

$$(1_{\widehat{\boldsymbol{X}}_{\alpha'}})^{\boldsymbol{K}_{\sigma_*}} \colon (\widehat{\boldsymbol{X}}_{\alpha'})^{\boldsymbol{K}_{\alpha'}} \to (\widehat{\boldsymbol{X}}_{\alpha'})^{\boldsymbol{K}_\alpha}$$

(where $\sigma_*\colon (\mathcal{C}\downarrow\alpha) \to (\mathcal{C}\downarrow\alpha')$; see Definition 14.5.7).

Example 19.2.3. Let $\mathcal{M}$ be a framed model category and let $\mathcal{C}$ be a small category.

1. If $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, then $\boldsymbol{X} \otimes_{\mathcal{C}} \mathrm{B}(-\downarrow\mathcal{C})^{\mathrm{op}}$ is the homotopy colimit of $\boldsymbol{X}$ (see Definition 19.1.2).
2. If $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, then $\hom^{\mathcal{C}}(\mathrm{B}(\mathcal{C}\downarrow -), \boldsymbol{X})$ is the homotopy limit $\operatorname{holim}\boldsymbol{X}$ of $\boldsymbol{X}$ (see Definition 19.1.10).

Remark 19.2.4. Let $\mathcal{M}$ be a framed model category and let $\mathcal{C}$ be a small category.

1. The construction of the object $\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}$ in $\mathcal{M}$ from the functor $\widetilde{\boldsymbol{X}} \otimes K\colon \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{M}$ is an example of the general construction known as a *coend* (see [**41**, pages 222–223]). In the notation of [**41**], $\boldsymbol{X} \otimes_{\mathcal{C}} K = \int^\alpha \widetilde{\boldsymbol{X}}_\alpha \otimes \boldsymbol{K}_\alpha$.
2. The construction of the object $\hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X})$ of $\mathcal{M}$ from the functor $\widehat{\boldsymbol{X}}^{\boldsymbol{K}}\colon \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{M}$ is an example of a general construction known as an *end* (see [**41**, pages 218–223] or [**8**, page 329]). In the notation of [**41**], $\hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X}) = \int_\alpha (\widehat{\boldsymbol{X}}_\alpha)^{\boldsymbol{K}_\alpha}$.

Proposition 19.2.5. *Let $\mathcal{M}$ be a framed model category and let $\mathcal{C}$ be a small category.*

1. *If $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, and $\boldsymbol{P}\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{SS}$ is a single point for every object α in $\mathcal{C}$, then $\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{P}$ is naturally isomorphic to* $\operatorname{colim}\boldsymbol{X}$.
2. *If $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, and $\boldsymbol{P}\colon \mathcal{C} \to \mathsf{SS}$ is a single point for every object α in $\mathcal{C}$, then $\hom^{\mathcal{C}}(\boldsymbol{P}, \boldsymbol{X})$ is naturally isomorphic to* $\lim\boldsymbol{X}$.

PROOF. For part 1, $\boldsymbol{P}_\alpha$ is naturally isomorphic to $\Delta[0]$ for every object α in $\mathcal{C}^{\mathrm{op}}$, and so we have natural isomorphisms

$$\widetilde{\boldsymbol{X}}_\alpha \otimes \boldsymbol{P}_\alpha \approx \widetilde{\boldsymbol{X}}_\alpha \otimes \Delta[0] \approx (\widetilde{\boldsymbol{X}}_\alpha)^0 \approx \boldsymbol{X}_\alpha$$

(see Lemma 17.3.5). Under these isomorphisms, the map ϕ of Definition 19.2.2 is defined by $\sigma_*\colon \boldsymbol{X}_\alpha \to \boldsymbol{X}_{\alpha'}$ and the map ψ is the identity.

For part 2, $\boldsymbol{P}_\alpha$ is naturally isomorphic to $\Delta[0]$ for every object α in $\mathcal{C}$, and so we have natural isomorphisms

$$\widehat{\boldsymbol{X}}_\alpha^{\boldsymbol{P}_\alpha} \approx \widehat{\boldsymbol{X}}_\alpha^{\Delta[0]} \approx (\widehat{\boldsymbol{X}}_\alpha)_0 \approx \boldsymbol{X}_\alpha$$

(see Lemma 17.3.5). Under these isomorphisms, the map ϕ of Definition 19.2.2 is defined by $\sigma_*\colon \boldsymbol{X}_\alpha \to \boldsymbol{X}_{\alpha'}$ and the map ψ is the identity. $\square$

EXAMPLE 19.2.6. Let $\mathcal{M}$ be a framed model category and let $\mathcal{C}$ be a small category.

1. If $\boldsymbol{P}\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{SS}$ is a single point for every object α in $\mathcal{C}^{\mathrm{op}}$, then the unique map of $\mathcal{C}^{\mathrm{op}}$-diagrams $\mathrm{B}(-\downarrow \mathcal{C})^{\mathrm{op}} \to \boldsymbol{P}$ induces a natural map
$$\operatorname{hocolim} \boldsymbol{X} = \widehat{\boldsymbol{X}} \otimes_{\mathcal{C}} \mathrm{B}(\alpha \downarrow \mathcal{C})^{\mathrm{op}} \to \widehat{\boldsymbol{X}} \otimes_{\mathcal{C}} \boldsymbol{P} = \operatorname{colim} \boldsymbol{X}$$
for all $\mathcal{C}$-diagrams $\boldsymbol{X}$ in $\mathcal{M}$ (see Example 19.2.3 and Proposition 19.2.5).
2. If $\boldsymbol{P}\colon \mathcal{C} \to \mathsf{SS}$ is a single point for every object α in $\mathcal{C}$, then the unique map of $\mathcal{C}$-diagrams $\mathrm{B}(\mathcal{C}\downarrow -) \to \boldsymbol{P}$ induces a natural map
$$\lim \boldsymbol{X} = \hom^{\mathcal{C}}(\boldsymbol{P}, \widehat{\boldsymbol{X}}) \to \hom^{\mathcal{C}}\bigl(\mathrm{B}(\mathcal{C}\downarrow \alpha), \widehat{\boldsymbol{X}}\bigr) = \operatorname{holim} \boldsymbol{X}$$
for all $\mathcal{C}$-diagrams $\boldsymbol{X}$ in $\mathcal{M}$.

19.2.7. Adjointness.

PROPOSITION 19.2.8. *Let $\mathcal{M}$ be a framed model category and let $\mathcal{C}$ be a small category.*

1. *If $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, $\boldsymbol{K}\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{SS}$ is a $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets, and Z is an object in $\mathcal{M}$, then there is a natural isomorphism of sets*
$$\mathcal{M}(\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}, Z) \approx \mathsf{SS}^{\mathcal{C}^{\mathrm{op}}}\bigl(\boldsymbol{K}, \mathcal{M}(\widetilde{\boldsymbol{X}}, Z)\bigr)$$
where $\widetilde{\boldsymbol{X}}$ is the natural cosimplicial frame on $\boldsymbol{X}$ and $\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}$ is as in Definition 19.2.2.
2. *If $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, $\boldsymbol{K}\colon \mathcal{C} \to \mathsf{SS}$ is a $\mathcal{C}$-diagram of simplicial sets, and W is an object in $\mathcal{M}$, then there is a natural isomorphism of sets*
$$\mathcal{M}\bigl(W, \hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X})\bigr) \approx \mathsf{SS}^{\mathcal{C}}\bigl(\boldsymbol{K}, \mathcal{M}(W, \widehat{\boldsymbol{X}})\bigr)$$
where $\widehat{\boldsymbol{X}}$ is the natural simplicial frame on $\boldsymbol{X}$ and $\hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X})$ is as in Definition 19.2.2.

PROOF. For part 1, $\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}$ is defined as a colimit, and so $\mathcal{M}(\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}, Z)$ is naturally isomorphic to the limit of the diagram

$$\prod_{\alpha \in \mathrm{Ob}(\mathcal{C})} \mathcal{M}(\widetilde{\boldsymbol{X}}_\alpha \otimes \boldsymbol{K}_\alpha, Z) \underset{\psi^*}{\overset{\phi^*}{\rightrightarrows}} \prod_{(\sigma\colon \alpha \to \alpha') \in \mathcal{C}} \mathcal{M}(\widetilde{\boldsymbol{X}}_\alpha \otimes \boldsymbol{K}_{\alpha'}, Z)$$

Theorem 17.3.8 implies that this limit is naturally isomorphic to the limit of the diagram

$$\prod_{\alpha \in \mathrm{Ob}(\mathcal{C})} \mathsf{SS}(\boldsymbol{K}_\alpha, \mathcal{M}(\widetilde{\boldsymbol{X}}_\alpha, Z)) \overset{\phi^*}{\underset{\psi^*}{\rightrightarrows}} \prod_{(\sigma\colon \alpha \to \alpha') \in \mathcal{C}} \mathsf{SS}(\boldsymbol{K}_{\alpha'}, \mathcal{M}(\widetilde{\boldsymbol{X}}_\alpha, Z))$$

which is the definition of $\mathsf{SS}^{\mathcal{C}^{\mathrm{op}}}\bigl(\boldsymbol{K}, \mathcal{M}(\widetilde{\boldsymbol{X}}, Z)\bigr)$.

For part 2, $\hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X})$ is defined as a limit, and so $\mathcal{M}\bigl(W, \hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X})\bigr)$ is naturally isomorphic to the limit of the diagram

$$\prod_{\alpha \in \mathrm{Ob}(\mathcal{C})} \mathcal{M}(W, (\widehat{\boldsymbol{X}}_\alpha)^{\boldsymbol{K}_\alpha}) \overset{\phi}{\underset{\psi}{\rightrightarrows}} \prod_{(\sigma\colon \alpha \to \alpha') \in \mathcal{C}} \mathcal{M}(W, (\widehat{\boldsymbol{X}}_{\alpha'})^{\boldsymbol{K}_\alpha})$$

Theorem 17.3.8 implies that this limit is naturally isomorphic to the limit of the diagram

$$\prod_{\alpha \in \mathrm{Ob}(\mathcal{C})} \mathsf{SS}(\boldsymbol{K}_\alpha, \mathcal{M}(W, \widehat{\boldsymbol{X}}_\alpha)) \overset{\phi}{\underset{\psi}{\rightrightarrows}} \prod_{(\sigma\colon \alpha \to \alpha') \in \mathcal{C}} \mathsf{SS}(\boldsymbol{K}_\alpha, \mathcal{M}(W, \widehat{\boldsymbol{X}}_{\alpha'}))$$

which is the definition of $\mathsf{SS}^{\mathcal{C}}\bigl(\boldsymbol{K}, \mathcal{M}(W, \widehat{\boldsymbol{X}})\bigr)$. $\square$

PROPOSITION 19.2.9. *Let $\mathcal{M}$ be a framed model category and let $\mathcal{C}$ be a small category.*

1. *If $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$ such that $\boldsymbol{X}_\alpha$ is cofibrant for every object α in $\mathcal{C}$ and $\boldsymbol{K}\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{SS}$ is a $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets that is a cofibrant object in $\mathsf{SS}^{\mathcal{C}^{\mathrm{op}}}$, then $\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}$ is cofibrant.*
2. *If $\boldsymbol{K}\colon \mathcal{C} \to \mathsf{SS}$ is a $\mathcal{C}$-diagram of simplicial sets that is a cofibrant object in $\mathsf{SS}^{\mathcal{C}}$ and $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$ such that $\boldsymbol{X}_\alpha$ is fibrant for every object α in $\mathcal{C}$, then $\hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X})$ is fibrant.*

PROOF. For part 1, Proposition 8.2.3 implies that it is sufficient to show that if $p\colon Y \to Z$ is a trivial fibration in $\mathcal{M}$, then the dotted arrow exists in every solid arrow diagram of the form

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ \boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K} \arrow[r] \arrow[ur, dotted] & Z \end{tikzcd}$$

(where $\emptyset$ is the initial object of $\mathcal{M}$). Proposition 19.2.8 implies that this is equivalent to showing that the dotted arrow exists in every solid arrow diagram in $\mathsf{SS}^{\mathcal{C}^{\mathrm{op}}}$ of the form

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \mathcal{M}(\widetilde{\boldsymbol{X}}, Y) \arrow[d, "p_*"] \\ \boldsymbol{K} \arrow[r] \arrow[ur, dotted] & \mathcal{M}(\widetilde{\boldsymbol{X}}, Z) \end{tikzcd}$$

(where $\emptyset$ denotes the initial object of $\mathsf{SS}^{\mathcal{C}^{\mathrm{op}}}$). Corollary 17.4.3 implies that the map $p_*\colon \mathcal{M}(\widetilde{\boldsymbol{X}}_\alpha, Y) \to \mathcal{M}(\widetilde{\boldsymbol{X}}_\alpha, Z)$ is a trivial fibration in SS for every object α in $\mathcal{C}^{\mathrm{op}}$, and so p_* is a trivial fibration in $\mathrm{SS}^{\mathcal{C}^{\mathrm{op}}}$ (see Theorem 14.2.1). Since $\boldsymbol{K}$ is cofibrant in $\mathrm{SS}^{\mathcal{C}^{\mathrm{op}}}$, the result follows.

For part 2, Proposition 8.2.3 implies that it is sufficient to show that if $i\colon A \to B$ is a trivial cofibration in $\mathcal{M}$, then the dotted arrow exists in every solid arrow diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & \hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X}) \arrow[d] \\ B \arrow[r] \arrow[ur, dotted] & * \end{tikzcd}$$

(where $*$ denotes the terminal object of $\mathcal{M}$). Proposition 19.2.8 implies that this is equivalent to showing that the dotted arrow exists in every solid arrow diagram in $\mathrm{SS}^{\mathcal{C}}$ of the form

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \mathcal{M}(B, \widehat{\boldsymbol{X}}) \arrow[d, "i^*"] \\ \boldsymbol{K} \arrow[r] \arrow[ur, dotted] & \mathcal{M}(A, \widehat{\boldsymbol{X}}) \end{tikzcd}$$

(where $\emptyset$ denotes the initial object of $\mathrm{SS}^{\mathcal{C}}$). Corollary 17.4.3 implies that $i^*_\alpha\colon \mathcal{M}(B, \widehat{\boldsymbol{X}}_a) \to \mathcal{M}(A, \widehat{\boldsymbol{X}}_\alpha)$ is a trivial fibration in $\mathrm{SS}^{\mathcal{C}}$ for every object α in $\mathcal{C}$, and so i^* is a trivial fibration in $\mathrm{SS}^{\mathcal{C}}$ (see Theorem 14.2.1). Since $\boldsymbol{K}$ is cofibrant in $\mathrm{SS}^{\mathcal{C}}$, the result follows. $\square$

Proposition 19.2.10. *Let $\mathcal{M}$ be a framed model category and let $\mathcal{C}$ be a small category.*

1. *If $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram of cofibrant objects in $\mathcal{M}$ and $f\colon \boldsymbol{K} \to \boldsymbol{K}'$ is a weak equivalence of cofibrant $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets, then the induced map $f_*\colon \boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K} \to \boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}'$ is a weak equivalence of cofibrant objects in $\mathcal{M}$.*
2. *If $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram of fibrant objects in $\mathcal{M}$ and $f\colon \boldsymbol{K} \to \boldsymbol{K}'$ is a weak equivalence of cofibrant $\mathcal{C}$-diagrams of simplicial sets, then the induced map $f^*\colon \hom^{\mathcal{C}}(\boldsymbol{K}', \boldsymbol{X}) \to \hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X})$ is a weak equivalence of fibrant objects in $\mathcal{M}$.*

Proof. For part 1, Theorem 18.1.6 and Proposition 19.2.9 imply that it is sufficient to show that if $\widehat{\boldsymbol{Z}}$ is a simplicial frame on a fibrant object Z in $\mathcal{M}$, then the map $\mathcal{M}(\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}', \widehat{\boldsymbol{Z}}) \to \mathcal{M}(\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}, \widehat{\boldsymbol{Z}})$ is a weak equivalence of simplicial sets.

Proposition 19.2.8 implies that, for every $n \geq 0$, the map $\mathcal{M}(\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}', \widehat{\boldsymbol{Z}}_n) \to \mathcal{M}(\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}, \widehat{\boldsymbol{Z}}_n)$ is isomorphic to the map $\mathrm{SS}^{\mathcal{C}^{\mathrm{op}}}\big(\boldsymbol{K}', \mathcal{M}(\widetilde{\boldsymbol{X}}, \widehat{\boldsymbol{Z}}_n)\big) \to \mathrm{SS}^{\mathcal{C}^{\mathrm{op}}}\big(\boldsymbol{K}, \mathcal{M}(\widetilde{\boldsymbol{X}}, \widehat{\boldsymbol{Z}}_n)\big)$. Proposition 17.5.3 implies that $\mathcal{M}(\widetilde{\boldsymbol{X}}, \widehat{\boldsymbol{Z}})$ is a simplicial resolution of $\mathcal{M}(\widetilde{\boldsymbol{X}}, Z)$, which is a fibrant object in $\mathrm{SS}^{\mathcal{C}^{\mathrm{op}}}$. Since $f\colon \boldsymbol{K} \to \boldsymbol{K}'$ is a weak equivalence of cofibrant objects in $\mathrm{SS}^{\mathcal{C}^{\mathrm{op}}}$, Theorem 17.5.2 implies that the map $\mathrm{SS}^{\mathcal{C}^{\mathrm{op}}}\big(\boldsymbol{K}', \mathcal{M}(\widetilde{\boldsymbol{X}}, \widehat{\boldsymbol{Z}})\big) \to \mathrm{SS}^{\mathcal{C}^{\mathrm{op}}}\big(\boldsymbol{K}, \mathcal{M}(\widetilde{\boldsymbol{Z}}, \widehat{\boldsymbol{Z}})\big)$ is a weak equivalence of simplicial sets. Since this map is isomorphic to the map $\mathcal{M}(\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}', \widehat{\boldsymbol{Z}}) \to \mathcal{M}(\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}, \widehat{\boldsymbol{Z}})$, the result follows.

For part 2, Theorem 18.1.6 and Proposition 19.2.9 imply that it is sufficient to show that if $\widetilde{\boldsymbol{W}}$ is a cosimplicial frame on a cofibrant object W in $\mathcal{M}$, then

$f_*\colon \mathcal{M}\bigl(\widetilde{\boldsymbol{W}}, \hom^{\mathcal{C}}(\boldsymbol{K}', \boldsymbol{X})\bigr) \to \mathcal{M}\bigl(\widetilde{\boldsymbol{W}}, \hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X})\bigr)$ is a weak equivalence of simplicial sets. Proposition 19.2.8 implies that for every $n \geq 0$ the map of sets $\mathcal{M}\bigl(\widetilde{\boldsymbol{W}}^n, \hom^{\mathcal{C}}(\boldsymbol{K}', \boldsymbol{X})\bigr) \to \mathcal{M}\bigl(\widetilde{\boldsymbol{W}}^n, \hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X})\bigr)$ is isomorphic to the map $\mathsf{SS}^{\mathcal{C}}\bigl(\boldsymbol{K}', \mathcal{M}(\widetilde{\boldsymbol{W}}^n, \widehat{\boldsymbol{X}})\bigr) \to \mathsf{SS}^{\mathcal{C}}\bigl(\boldsymbol{K}, \mathcal{M}(\widetilde{\boldsymbol{W}}^n, \widehat{\boldsymbol{X}})\bigr)$. Proposition 17.5.3 implies that $\mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{\boldsymbol{X}})$ is a simplicial resolution of the fibrant object $\mathcal{M}(W, \widehat{\boldsymbol{X}})$ in $\mathsf{SS}^{\mathcal{C}}$. Since $f\colon \boldsymbol{K} \to \boldsymbol{K}'$ is a weak equivalence of cofibrant objects in $uSS^{\mathcal{C}}$, Theorem 17.5.2 implies that the map $uSS^{\mathcal{C}}\bigl(\boldsymbol{K}' m\mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{\boldsymbol{X}})\bigr) \to \mathsf{SS}^{\mathcal{C}}\bigl(\boldsymbol{K}, \mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{\boldsymbol{X}})\bigr)$ is a weak equivalence of simplicial sets. Since this map is isomorphic to the map $\mathcal{M}\bigl(\widetilde{\boldsymbol{W}}, \hom^{\mathcal{C}}(\boldsymbol{K}', \boldsymbol{X})\bigr) \to \mathcal{M}\bigl(\widetilde{\boldsymbol{W}}, \hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X})\bigr)$, the result follows. $\square$

PROPOSITION 19.2.11. *Let $\mathcal{M}$ be a model category and let $\mathcal{C}$ be a small category.*

1. *If $\boldsymbol{K}\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{SS}$ is a cofibrant $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets and $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ is an objectwise weak equivalence of $\mathcal{C}$-diagrams of cofibrant objects in $\mathcal{M}$, then the induced map $f_*\colon \boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K} \to \boldsymbol{Y} \otimes_{\mathcal{C}} \boldsymbol{K}$ is a weak equivalence.*
2. *If $\boldsymbol{K}\colon \mathcal{C} \to \mathsf{SS}$ is a cofibrant $\mathcal{C}$-diagram of simplicial sets and $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ is an objectwise weak equivalence of $\mathcal{C}$-diagrams of fibrant objects in $\mathcal{M}$, then the induced map $f_*\colon \hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X}) \to \hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{Y})$ is a weak equivalence.*

PROOF. For part 1, Theorem 18.1.6 and Theorem 19.3.1 imply that it is sufficient to show that if $\widehat{\boldsymbol{Z}}$ is a simplicial frame on a fibrant object Z in $\mathcal{M}$, then $f^*\colon \mathcal{M}(\boldsymbol{Y} \otimes_{\mathcal{C}} \boldsymbol{K}, \widehat{\boldsymbol{Z}}) \to \mathcal{M}(\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}, \widehat{\boldsymbol{Z}})$ is a weak equivalence of simplicial sets.

Proposition 19.2.8 implies that, for every $n \geq 0$, the map of sets $\mathcal{M}(\boldsymbol{Y} \otimes_{\mathcal{C}} \boldsymbol{K}, \widehat{\boldsymbol{Z}}_n) \to \mathcal{M}(\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}, \widehat{\boldsymbol{Z}}_n)$ is isomorphic to the map $\mathsf{SS}^{\mathcal{C}^{\mathrm{op}}}\bigl(\boldsymbol{K}, \mathcal{M}(\widetilde{\boldsymbol{Y}}, \widehat{\boldsymbol{Z}}_n)\bigr) \to \mathsf{SS}^{\mathcal{C}^{\mathrm{op}}}\bigl(\boldsymbol{K}, \mathcal{M}(\widetilde{\boldsymbol{X}}, \widehat{\boldsymbol{Z}}_n)\bigr)$. Theorem 17.5.2 implies that for every object α in $\mathcal{C}$ the map $\mathcal{M}(\widetilde{\boldsymbol{Y}}_\alpha, Z) \to \mathcal{M}(\widetilde{\boldsymbol{X}}_\alpha, Z)$ is a weak equivalence of fibrant simplicial sets. Thus, the map $\mathcal{M}(\widetilde{\boldsymbol{Y}}, Z) \to \mathcal{M}(\widetilde{\boldsymbol{X}}, Z)$ is a weak equivalence of fibrant objects in $\mathsf{SS}^{\mathcal{C}^{\mathrm{op}}}$ (see Theorem 14.2.1). Since Proposition 17.5.3 implies that the map $\mathcal{M}(\widetilde{\boldsymbol{Y}}, \widehat{\boldsymbol{Z}}) \to \mathcal{M}(\widetilde{\boldsymbol{X}}, \widehat{\boldsymbol{Z}})$ is a simplicial resolution of the map $\mathcal{M}(\widetilde{\boldsymbol{Y}}, Z) \to \mathcal{M}(\widetilde{\boldsymbol{X}}, Z)$ and $\boldsymbol{K}$ is a cofibrant object in $\mathsf{SS}^{\mathcal{C}^{\mathrm{op}}}$, Theorem 17.5.2 implies that the map $\mathsf{SS}^{\mathcal{C}^{\mathrm{op}}}\bigl(\boldsymbol{K}, \mathcal{M}(\widetilde{\boldsymbol{Y}}, \widehat{\boldsymbol{Z}})\bigr) \to \mathsf{SS}^{\mathcal{C}^{\mathrm{op}}}\bigl(\boldsymbol{K}, \mathcal{M}(\widetilde{\boldsymbol{X}}, \widehat{\boldsymbol{Z}})\bigr)$ is a weak equivalence of simplicial sets. Since this map is isomorphic to the map $\mathcal{M}(\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}, \widehat{\boldsymbol{Z}}) \to \mathcal{M}(\boldsymbol{Y} \otimes_{\mathcal{C}} \boldsymbol{K}, \widehat{\boldsymbol{Z}})$, the result follows.

For part 2, Theorem 18.1.6 and Theorem 19.3.1 imply that it is sufficient to show that if $\widetilde{\boldsymbol{W}}$ is a cosimplicial frame on a cofibrant object W in $\mathcal{M}$, then $f_*\colon \mathcal{M}(\widetilde{\boldsymbol{W}}, \hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X}) \to \mathcal{M}(\widetilde{\boldsymbol{W}}, \hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{Y})$ is a weak equivalence of simplicial sets. Proposition 19.2.8 implies that for every $n \geq 0$ the map $\mathcal{M}(\widetilde{\boldsymbol{W}}^n, \hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X}) \to \mathcal{M}(\widetilde{\boldsymbol{W}}^n, \hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{Y})$ is isomorphic to the map $\mathsf{SS}^{\mathcal{C}}\bigl(\boldsymbol{K}, \mathcal{M}(\widetilde{\boldsymbol{W}}^n, \widehat{\boldsymbol{X}})\bigr) \to \mathsf{SS}^{\mathcal{C}}\bigl(\boldsymbol{K}, \mathcal{M}(\widetilde{\boldsymbol{W}}^n, \widehat{\boldsymbol{Y}})\bigr)$. Thus, it is sufficient to show that the map $\mathsf{SS}^{\mathcal{C}}\bigl(\boldsymbol{K}, \mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{\boldsymbol{X}})\bigr) \to \mathsf{SS}^{\mathcal{C}}\bigl(\boldsymbol{K}, \mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{\boldsymbol{Y}})\bigr)$ is a weak equivalence of simplicial sets.

Theorem 17.5.2 implies that for every object α in $\mathcal{C}$ the map $\mathcal{M}(W, \widehat{\boldsymbol{X}}_\alpha) \to \mathcal{M}(W, \widehat{\boldsymbol{Y}}_\alpha)$ is a weak equivalence of fibrant simplicial sets. Thus, the map $\mathcal{M}(W, \widehat{\boldsymbol{X}}) \to \mathcal{M}(W, \widehat{\boldsymbol{Y}})$ is a weak equivalence of fibrant objects in $\mathsf{SS}^{\mathcal{C}}$ (see Theorem 14.2.1). Since Proposition 17.5.3 implies that the map $\mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{\boldsymbol{X}}) \to \mathcal{M}(\widetilde{\boldsymbol{W}}, \widehat{\boldsymbol{Y}})$ is a simplicial resolution of the map $\mathcal{M}(W, \widehat{\boldsymbol{X}}) \to \mathcal{M}(W, \widehat{\boldsymbol{Y}})$ and $\boldsymbol{K}$ is a cofibrant object in $\mathsf{SS}^{\mathcal{C}}$, Theorem 17.5.2 implies the result. $\square$

19.3. Homotopy invariance

THEOREM 19.3.1. *Let $\mathcal{M}$ be a framed model category, and let $\mathcal{C}$ be a small category.*

1. *If $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$ such that $\boldsymbol{X}_\alpha$ is cofibrant for every object α in $\mathcal{C}$, then* $\operatorname{hocolim} \boldsymbol{X}$ *is cofibrant.*
2. *If $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$ such that $\boldsymbol{X}_\alpha$ is fibrant for every object α in $\mathcal{C}$, then* $\operatorname{holim} \boldsymbol{X}$ *is fibrant.*

PROOF. This follows from Proposition 19.2.9 and Corollary 14.6.8. □

THEOREM 19.3.2. *Let $\mathcal{M}$ be a framed model category, and let $\mathcal{C}$ be a small category.*

1. *If $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a map of $\mathcal{C}$-diagrams in $\mathcal{M}$ such that $f_\alpha\colon \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ is a weak equivalence of cofibrant objects for every object α in $\mathcal{C}$, then the induced map of homotopy colimits $f_*\colon \operatorname{hocolim} \boldsymbol{X} \to \operatorname{hocolim} \boldsymbol{Y}$ is a weak equivalence.*
2. *If $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a map of $\mathcal{C}$-diagrams in $\mathcal{M}$ such that $f_\alpha\colon \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ is a weak equivalence of fibrant objects for every object α in $\mathcal{C}$, then the induced map of homotopy limits $f_*\colon \operatorname{holim} \boldsymbol{X} \to \operatorname{holim} \boldsymbol{Y}$ is a weak equivalence.*

PROOF. This follows from Proposition 19.2.11. □

19.4. Homotopy pullbacks and homotopy limits

If $\mathcal{M}$ is a *right proper* framed model category, then the diagram $X \to Z \leftarrow Y$ has both a homotopy pullback (see Definition 11.2.2) and a homotopy limit (see Definition 19.1.10). We will show that for *fibrant* X, Y, and Z, the homotopy pullback of a diagram $X \to Z \leftarrow Y$ is naturally weakly equivalent to the homotopy limit of that diagram (see Proposition 19.4.3). We begin by showing that, for a map of fibrant objects, the "classical" method of converting a map into a fibration does provide a factorization into a weak equivalence followed by a fibration.

LEMMA 19.4.1. *Let $\mathcal{M}$ be a framed model category, and let $g\colon X \to Z$ be a map of fibrant objects. If $\mathrm{ev}_0\colon \widehat{\boldsymbol{Z}}^{\Delta[1]} \to Z$ is the composition $\widehat{\boldsymbol{Z}}^{\Delta[1]} \xrightarrow{(d^1)^*} \widehat{\boldsymbol{Z}}^{\Delta[0]} \approx \widehat{\boldsymbol{Z}}_0 \approx Z$ (see Lemma 17.3.5) and the square*

$$\begin{tikzcd} W \arrow[r, "\tilde{g}"] \arrow[d, "k"'] & \widehat{\boldsymbol{Z}}^{\Delta[1]} \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r, "g"] & Z \end{tikzcd}$$

is a pullback, then

1. *the map $\mathrm{ev}_1 \tilde{g}\colon W \to Z$ is a fibration (where $\mathrm{ev}_1\colon \widehat{\boldsymbol{Z}}^{\Delta[1]} \to Z$ is the composition $\widehat{\boldsymbol{Z}}^{\Delta[1]} \xrightarrow{(d^0)^*} \widehat{\boldsymbol{Z}}^{\Delta[0]} \approx \widehat{\boldsymbol{Z}}_0 \approx Z$),*
2. *the map $j\colon X \to W$ (which is defined by the requirements that $kj = 1_X$ and $\tilde{g}j\colon X \to \widehat{\boldsymbol{Z}}^{\Delta[1]}$ is the map $X \xrightarrow{g} Z \approx \widehat{\boldsymbol{Z}}_0 \approx \widehat{\boldsymbol{Z}}^{\Delta[0]} \xrightarrow{(s^0)^*} \widehat{\boldsymbol{Z}}^{\Delta[1]}$) is a weak equivalence, and*
3. $(\mathrm{ev}_1 \tilde{g}) \circ j = g$.

PROOF. Since Z is fibrant, $\widehat{\boldsymbol{Z}}$ is Reedy fibrant, and so ev_0 is a trivial fibration. Thus, k is a trivial fibration. Since $kj = 1_X$, this implies that j is a weak equivalence. Since the composition $\widehat{\boldsymbol{Z}}^{\Delta[0]} \xrightarrow{(s^0)^*} \widehat{\boldsymbol{Z}}^{\Delta[1]} \xrightarrow{(d^0)^*} \widehat{\boldsymbol{Z}}^{\Delta[0]}$ is the identity map, it follows that $(\mathrm{ev}_1 \tilde{g})j = g$, and so it remains only to show that $\mathrm{ev}_1 \tilde{g}$ is a fibration.

Proposition 8.2.3 implies that it is sufficient to show that for every trivial cofibration $A \to B$ in $\mathcal{M}$ and every solid arrow diagram

(19.4.2)
$$\begin{tikzcd} A \arrow[r, "r"] \arrow[d] & W \arrow[d, "\mathrm{ev}_1 \tilde{g}"] \\ B \arrow[r, "s"] \arrow[ur, dotted] & Z \end{tikzcd}$$

there exists a dotted arrow making both triangles commute. We first note that, since X is fibrant, the map $kr\colon A \to X$ can be extended over B. Since W is defined as a pullback, it remains only to find an appropriate map $B \to \widehat{\boldsymbol{Z}}^{\Delta[1]}$.

If we compost our map $B \to X$ with the composition $X \xrightarrow{g} Z \approx \widehat{\boldsymbol{Z}}_0 \approx \widehat{\boldsymbol{Z}}^{\Delta[0]} \to \widehat{\boldsymbol{Z}}^{\partial\Delta[1]}$ (where that last map is induced by the projection of $\partial\Delta[1] \approx \Delta[0] \times \Delta[0]$ onto the second factor), then we have a map $B \to \widehat{\boldsymbol{Z}}^{\partial\Delta[1]}$ that makes Thus, we have the solid arrow diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \widehat{\boldsymbol{Z}}^{\Delta[1]} \arrow[d] \\ B \arrow[r] \arrow[ur, dotted] & \widehat{\boldsymbol{Z}}^{\partial\Delta[1]} \end{tikzcd}$$

commute. Since $\widehat{\boldsymbol{Z}}$ is Reedy fibrant, Proposition 17.3.7 implies that $\widehat{\boldsymbol{Z}}^{\Delta[1]} \to \widehat{\boldsymbol{Z}}^{\partial\Delta[1]}$ is a fibration, and so the dotted arrow exists in this diagram. This dotted arrow combines with the map $B \to X$ to define the dotted arrow in Diagram 19.4.2. $\square$

PROPOSITION 19.4.3. *Let $\mathcal{M}$ be a right proper framed model category. If X, Y, and Z are fibrant objects, then the homotopy pullback (see Definition* 11.2.2*) of the diagram $X \xrightarrow{g} Z \xleftarrow{h} Y$ is naturally weakly equivalent to the homotopy limit (see Definition* 19.1.10*) of that diagram.*

PROOF. If K is the simplicial set that is the union of two copies of $\Delta[1]$ with vertex 1 of both copies identified to a single point, then the homotopy limit of the diagram $X \xrightarrow{g} Z \xleftarrow{h} Y$ is naturally isomorphic to the limit of the diagram

$$\begin{tikzcd}[column sep=small] X \arrow[dr, "g"] & & \widehat{\boldsymbol{Z}}^{T} \arrow[dl] \arrow[dr] & & Y \arrow[dl, "h"'] \\ & Z & & Z & \end{tikzcd}$$

where the two maps with domain $\widehat{\boldsymbol{Z}}^K$ are defined by evaluation on vertex 0 of the two copies of $\Delta[1]$ (see Definition 19.1.10). The limit of this last diagram is naturally isomorphic to the limit of the diagram

(19.4.4)
$$\begin{tikzcd}[column sep=small] X \arrow[dr, "g"] & & \widehat{\boldsymbol{Z}}^{\Delta[1]} \arrow[dl, "\mathrm{ev}_0"'] \arrow[dr, "\mathrm{ev}_1"] & & \widehat{\boldsymbol{Z}}^{\Delta[1]} \arrow[dl, "\mathrm{ev}_1"'] \arrow[dr, "\mathrm{ev}_0"] & & Y \arrow[dl, "h"'] \\ & Z & & Z & & Z & \end{tikzcd}$$

(see Proposition 17.3.9). If W_g is the pullback of the diagram $X \xrightarrow{g} Z \xleftarrow{\mathrm{ev}_0} \widehat{\boldsymbol{Z}}^{\Delta[1]}$ and W_h is the pullback of the diagram $Y \xrightarrow{h} Z \xleftarrow{\mathrm{ev}_0} \widehat{\boldsymbol{Z}}^{\Delta[1]}$, then the limit of Diagram 19.4.4 is naturally isomorphic to the pullback of the diagram $W_g \to Z \leftarrow W_h$. Lemma 19.4.1 implies that the maps $W_g \to Z$ and $W_h \to Z$ arise as factorizations of, respectively, g and h into a weak equivalence followed by a fibration, and so the result follows from Proposition 11.2.7. $\square$

19.5. Cofinality

In this section, we characterize those functors between small categories that induce weak equivalences of homotopy limits for all diagrams of fibrant objects, and those that induce weak equivalences of homotopy colimits for all diagrams of cofibrant objects.

PROPOSITION 19.5.1. *Let $\mathcal{M}$ be a framed model category, let $\mathcal{C}$ and $\mathcal{D}$ be small categories, let $\mathrm{F}\colon \mathcal{C} \to \mathcal{D}$ be a functor, and let $\boldsymbol{X}\colon \mathcal{D} \to \mathcal{M}$ be a $\mathcal{D}$-diagram in $\mathcal{M}$.*

1. *There is a natural isomorphism of objects in $\mathcal{M}$*

$$\operatorname*{hocolim}_{\mathcal{C}} \mathrm{F}^*\boldsymbol{X} \approx \boldsymbol{X} \otimes_{\mathcal{D}} \mathrm{B}(-\downarrow \mathrm{F})^{\mathrm{op}}$$

(see Definition 19.1.7 and Definition 14.5.2).

2. *There is a natural isomorphism of objects in $\mathcal{M}$*

$$\operatorname*{holim}_{\mathcal{C}} \mathrm{F}^*\boldsymbol{X} \approx \hom^{\mathcal{D}}\bigl(\mathrm{B}(\mathrm{F}\downarrow -), \boldsymbol{X}\bigr)$$

(see Definition 19.2.2 and Definition 14.5.7).

PROOF. For part 1, we will show that for every object Z in $\mathcal{M}$ there is a natural isomorphism of sets

$$\mathcal{M}\bigl(\boldsymbol{X} \otimes_{\mathcal{D}} \mathrm{B}(-\downarrow \mathrm{F})^{\mathrm{op}}, Z\bigr) \approx \mathcal{M}(\operatorname*{hocolim}_{\mathcal{C}} \mathrm{F}^*\boldsymbol{X}, Z);$$

the Yoneda lemma (see, e.g., [**7**, page 11], [**41**, page 61], or [**5**, pages 26–28]) will then imply that that isomorphism is induced by an isomorphism $\operatorname{hocolim}_{\mathcal{C}} \mathrm{F}^*\boldsymbol{X} \approx \boldsymbol{X} \otimes_{\mathcal{D}} \mathrm{B}(-\downarrow \mathrm{F})^{\mathrm{op}}$.

Example 19.2.3 implies that $\operatorname{hocolim}_{\mathcal{C}} \mathrm{F}^*\boldsymbol{X}$ is naturally isomorphic to $(\mathrm{F}^*\boldsymbol{X}) \otimes_{\mathcal{C}} \mathrm{B}(-\downarrow \mathcal{C})^{\mathrm{op}}$, and so Proposition 19.2.8 implies that there are natural isomorphisms

$$\mathcal{M}\bigl(\boldsymbol{X} \otimes_{\mathcal{D}} \mathrm{B}(-\downarrow \mathrm{F})^{\mathrm{op}}, Z\bigr) \approx \mathrm{SS}^{\mathcal{D}^{\mathrm{op}}}\bigl(\mathrm{B}(-\downarrow \mathrm{F})^{\mathrm{op}}, \mathcal{M}(\widetilde{\boldsymbol{X}}, Z)\bigr) \tag{19.5.2}$$

$$\mathcal{M}(\operatorname*{hocolim}_{\mathcal{C}} \mathrm{F}^*\boldsymbol{X}, Z) \approx \mathrm{SS}^{\mathcal{C}^{\mathrm{op}}}\bigl(\mathrm{B}(-\downarrow \mathcal{C})^{\mathrm{op}}, \mathcal{M}(\widetilde{\mathrm{F}^*\boldsymbol{X}}, Z)\bigr) \tag{19.5.3}$$

(where $\widetilde{\boldsymbol{X}}$ is the natural cosimplicial frame on $\boldsymbol{X}$). Proposition 14.6.5 implies that the $\mathcal{D}^{\mathrm{op}}$-diagram of simplicial sets $\mathrm{B}(-\downarrow \mathrm{F})^{\mathrm{op}}$ is a free cell complex with basis equal to the set of simplices described in Diagram 14.6.6, and that the $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets $\mathrm{B}(-\downarrow \mathcal{C})^{\mathrm{op}}$ is a free cell complex with basis equal to the set of simplices of the form

$$\begin{tikzcd}
 & \alpha \arrow[dl] \arrow[d] \arrow[drr, "1_\alpha"] & & \\
\alpha_0 & \alpha_1 \arrow[l, "\sigma_0"] & \cdots \arrow[l, "\sigma_1"] & \alpha \arrow[l, "\sigma_n-1"]
\end{tikzcd}$$

Thus, there is a natural one-to-one correspondence between the bases, and we can now use Proposition 14.7.2 to show by induction on the skeleta of the domains that the set of maps (19.5.2) is naturally isomorphic to the set of maps (19.5.3).

For part 2, we will show that for every object W in $\mathcal{M}$ there is a natural isomorphism of sets

$$\mathcal{M}(W, \operatorname*{holim}_{\mathcal{C}} \mathrm{F}^*\boldsymbol{X}) \approx \mathcal{M}\big(W, \hom^{\mathcal{D}}(\mathrm{B}(\mathrm{F}\downarrow -), \boldsymbol{X})\big);$$

the Yoneda lemma will then imply that that isomorphism is induced by an isomorphism $\operatorname{holim}_{\mathcal{C}} \mathrm{F}^*\boldsymbol{X} \approx \hom^{\mathcal{D}}\big(\mathrm{B}(\mathrm{F}\downarrow -), \boldsymbol{X}\big)$.

Example 19.2.3 implies that $\operatorname{holim}_{\mathcal{C}} \mathrm{F}^*\boldsymbol{X}$ is naturally isomorphic to $\hom^{\mathcal{C}}\big(\mathrm{B}(\mathcal{C}\downarrow -), \boldsymbol{X}\big)$, and so Proposition 19.2.8 implies that there are natural isomorphisms

$$\mathcal{M}(W, \operatorname*{holim}_{\mathcal{C}} \mathrm{F}^*\boldsymbol{X}) \approx \mathsf{SS}^{\mathcal{C}}\big(\mathrm{B}(\mathcal{C}\downarrow -), \mathcal{M}(W, \widehat{\mathrm{F}^*\boldsymbol{X}})\big) \tag{19.5.4}$$

$$\mathcal{M}\big(W, \hom^{\mathcal{D}}(\mathrm{B}(\mathrm{F}\downarrow -), \boldsymbol{X})\big) \approx \mathsf{SS}^{\mathcal{D}}\big(\mathrm{B}(\mathrm{F}\downarrow -), \mathcal{M}(W, \widehat{\boldsymbol{X}})\big) \tag{19.5.5}$$

(where $\widehat{\boldsymbol{X}}$ is the natural simplicial frame on $\boldsymbol{X}$). Proposition 14.6.5 implies that the $\mathcal{D}$-diagram of simplicial sets $\mathrm{B}(\mathrm{F}\downarrow -)$ is a free cell complex with basis equal to the set of simplices described in Diagram 14.6.7, and that the $\mathcal{C}$-diagram of simplicial sets $\mathrm{B}(\mathcal{C}\downarrow -)$ is a free cell complex with basis equal to the set of simplices of the form

$$\begin{tikzcd}
\alpha_0 \arrow[r, "\sigma_0"] \arrow[dr] & \alpha_1 \arrow[r, "\sigma_1"] \arrow[d] & \cdots \arrow[r, "\sigma_{n-1}"] & \alpha \arrow[dll, "1_\alpha"] \\
& \alpha & &
\end{tikzcd}$$

Thus, there is a natural one-to-one correspondence between the bases, and we can now use Proposition 14.7.2 to show by induction on the dimension of the skeleta of the domains that the set of maps (19.5.4) is naturally isomorphic to the set of maps (19.5.5). □

THEOREM 19.5.6. *Let $\mathcal{M}$ be a framed model category, let $\mathcal{C}$ and $\mathcal{D}$ be small categories, and let $\mathrm{F}\colon \mathcal{C} \to \mathcal{D}$ be a functor.*

1. *If F is right cofinal (see Definition 14.4.5), then for every $\mathcal{D}$-diagram $\boldsymbol{X}\colon \mathcal{D} \to \mathcal{M}$ in $\mathcal{M}$ such that $\boldsymbol{X}_\alpha$ is cofibrant for every object α in $\mathcal{D}$, the natural map of homotopy colimits (see Proposition 19.1.8)*
$$\operatorname*{hocolim}_{\mathcal{C}} \mathrm{F}^*\boldsymbol{X} \to \operatorname*{hocolim}_{\mathcal{D}} \boldsymbol{X}$$
is a weak equivalence.
2. *If F is left cofinal (see Definition 14.4.5), then for every $\mathcal{D}$-diagram $\boldsymbol{X}\colon \mathcal{D} \to \mathcal{M}$ in $\mathcal{M}$ such that $\boldsymbol{X}_\alpha$ is fibrant for every object α in $\mathcal{D}$, the natural map of homotopy limits (see Proposition 19.1.14)*
$$\operatorname*{holim}_{\mathcal{D}} \boldsymbol{X} \to \operatorname*{holim}_{\mathcal{C}} \mathrm{F}^*\boldsymbol{X}$$
is a weak equivalence.

PROOF. We will prove part 1; the proof of part 2 is similar.

Proposition 19.5.1 and Example 19.2.3 imply that our map of homotopy colimits is isomorphic to the map

$$\boldsymbol{X} \otimes_{\mathcal{D}} \mathrm{B}(-\downarrow \mathrm{F})^{\mathrm{op}} \to \boldsymbol{X} \otimes_{\mathcal{D}} \mathrm{B}(-\downarrow \mathcal{D})^{\mathrm{op}}.$$

Proposition 14.6.5 and Corollary 14.6.8 imply that both of the $\mathcal{D}^{\mathrm{op}}$-diagrams of simplicial sets $\mathrm{B}(-\downarrow \mathrm{F})^{\mathrm{op}}$ and $\mathrm{B}(-\downarrow \mathcal{D})^{\mathrm{op}}$ are free cell complexes, and are thus cofibrant objects in $\mathsf{SS}^{\mathcal{D}^{\mathrm{op}}}$. Lemma 14.5.3 implies that $\mathrm{B}(\alpha\downarrow \mathcal{D})^{\mathrm{op}}$ is contractible for every object α in $\mathcal{D}$, and so F is right cofinal if and only if the map $\mathrm{B}(-\downarrow \mathrm{F})^{\mathrm{op}} \to \mathrm{B}(-\downarrow \mathcal{D})^{\mathrm{op}}$ is a weak equivalence of cofibrant objects in $\mathsf{SS}^{\mathcal{D}^{\mathrm{op}}}$. The result now follows from Proposition 19.2.10. ☐

We are indebted to W. G. Dwyer for the following proposition.

Proposition 19.5.7. *Let $\mathcal{C}$ be a small category, let α be an object in $\mathcal{C}$, and let $\boldsymbol{F}^{\alpha}_{*}$ be the free $\mathcal{C}$-diagram of sets generated at α (see Definition 14.1.2), regarded as a $\mathcal{C}$-diagram of discrete simplicial sets.*

1. *If $\boldsymbol{K}\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{SS}$ is a $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets, then the natural map*
$$\boldsymbol{K}_{\alpha} \approx \{1_{\alpha}\} \times \boldsymbol{K}_{\alpha} \subset (\boldsymbol{F}^{\alpha}_{*})_{\alpha} \times \boldsymbol{K}_{\alpha} \to \boldsymbol{F}^{\alpha}_{*} \otimes_{\mathcal{C}} \boldsymbol{K}$$
(see Definition 19.2.2) is an isomorphism.
2. *If $\boldsymbol{K}\colon \mathcal{C} \to \mathsf{SS}$ is a $\mathcal{C}$-diagram of simplicial sets, then the natural map*
$$\hom^{\mathcal{C}}(\boldsymbol{F}^{\alpha}_{*}, \boldsymbol{K}) \to \boldsymbol{K}_{\alpha}^{(\boldsymbol{F}^{\alpha}_{*})_{\alpha}} \xrightarrow{1_{\boldsymbol{K}_{\alpha}}^{(1_{\alpha} \subset (\boldsymbol{F}^{\alpha}_{*})_{\alpha})}} \boldsymbol{K}_{\alpha}$$
is an isomorphism.

Proof. For part 1, we will show that for every simplicial set Z our natural map induces an isomorphism of the sets of maps
$$\mathsf{SS}(\boldsymbol{F}^{\alpha}_{*} \otimes_{\mathcal{C}} \boldsymbol{K}, Z) \approx \mathsf{SS}(\boldsymbol{K}_{\alpha}, Z);$$
the result will then follow from the Yoneda lemma. Proposition 19.2.8 and Proposition 17.3.10 imply that there are natural isomorphisms of sets
$$\mathsf{SS}(\boldsymbol{F}^{\alpha}_{*} \otimes_{\mathcal{C}} \boldsymbol{K}, Z) \approx \mathsf{SS}(\boldsymbol{K} \otimes_{\mathcal{C}} \boldsymbol{F}^{\alpha}_{*}, Z) \approx \mathsf{SS}^{\mathcal{C}}\big(\boldsymbol{F}^{\alpha}_{*}, \mathsf{SS}(\boldsymbol{K}, Z)\big)$$
and Proposition 14.1.3 implies that this last set is naturally isomorphic to $\mathsf{SS}(\boldsymbol{K}_{\alpha}, Z)$.

For part 2, we will show that for every simplicial set W our natural map induces an isomorphism of the sets of maps
$$\mathsf{SS}\big(W, \hom^{\mathcal{C}}(\boldsymbol{F}^{\alpha}_{*}, \boldsymbol{K})\big) \to \mathsf{SS}(W, \boldsymbol{K}_{\alpha});$$
the result will then follow from the Yoneda lemma. Proposition 19.2.8 implies that there are natural isomorphisms of sets
$$\mathsf{SS}\big(W, \hom^{\mathcal{C}}(\boldsymbol{F}^{\alpha}_{*}, \boldsymbol{K})\big) \approx \mathsf{SS}^{\mathcal{C}}\big(\boldsymbol{F}^{\alpha}_{*}, \mathsf{SS}(W, \boldsymbol{K})\big) \approx \mathsf{SS}(W, \boldsymbol{K}_{\alpha}).$$
☐

Corollary 19.5.8. *If $\mathcal{C}$ is a small category and α is an object of $\mathcal{C}$, then* $\operatorname{hocolim} \boldsymbol{F}^{\alpha}_{*}$ *(see Definition 14.1.2) is naturally isomorphic to* $\mathrm{B}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$*, and* $\operatorname{holim} \boldsymbol{F}^{\alpha}_{*}$ *is naturally isomorphic to* $\mathrm{B}(\mathcal{C}\downarrow\alpha)$.

Proof. This follows from Proposition 19.5.7 and Example 19.2.3. ☐

Corollary 19.5.9. *If $\mathcal{C}$ and $\mathcal{D}$ are small categories, $\mathrm{F}\colon \mathcal{C} \to \mathcal{D}$ is a functor, and α is an object in $\mathcal{D}$, then there are natural isomorphisms*
$$\boldsymbol{F}^{\alpha}_{*} \otimes_{\mathcal{D}} \mathrm{B}(-\downarrow F)^{\mathrm{op}} \approx \mathrm{B}(\alpha\downarrow F)^{\mathrm{op}}$$
$$\hom^{\mathcal{D}}\big(\boldsymbol{F}^{\alpha}_{*}, \mathrm{B}(F\downarrow -)\big) \approx \mathrm{B}(F\downarrow\alpha)$$

(where $\boldsymbol{F}_*^\alpha$ is the $\mathcal{D}$-diagram of Definition 14.1.2, regarded as a diagram of discrete simplicial sets).

PROOF. This follows from Proposition 19.5.7. □

PROPOSITION 19.5.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be small categories, and let $\mathrm{F}\colon \mathcal{C} \to \mathcal{D}$ be a functor.*

1. *If for every $\mathcal{D}$-diagram $\boldsymbol{X}\colon \mathcal{D} \to \mathsf{SS}$ of cofibrant simplicial sets the induced map of homotopy colimits*
$$\operatorname*{hocolim}_{\mathcal{C}} \mathrm{F}^*\boldsymbol{X} \to \operatorname*{hocolim}_{\mathcal{D}} \boldsymbol{X}$$
is a weak equivalence, then F is a right cofinal functor.
2. *If for every $\mathcal{D}$-diagram $\boldsymbol{X}\colon \mathcal{D} \to \mathsf{SS}$ of fibrant simplicial sets the induced map of homotopy limits*
$$\operatorname*{holim}_{\mathcal{D}} \boldsymbol{X} \to \operatorname*{holim}_{\mathcal{C}} \mathrm{F}^*\boldsymbol{X}$$
is a weak equivalence, then F is a left cofinal functor.

PROOF. For part 1, if α is an object of $\mathcal{D}$, we can let $\boldsymbol{X} = \boldsymbol{F}_*^\alpha$ (see Definition 14.1.2), and regard it as a diagram of discrete simplicial sets. Proposition 19.5.1, Corollary 19.5.8, and Corollary 19.5.9 imply that $\mathrm{B}(\alpha\downarrow F)$ and $\mathrm{B}(\alpha\downarrow \mathcal{D})$ are weakly equivalent. Since $\mathrm{B}(\alpha\downarrow\mathcal{D})^{\mathrm{op}}$ is always contractible (see Lemma 14.5.3), Proposition 9.4.5 implies that F is right cofinal.

For part 2, Proposition 19.5.1 and Proposition 19.1.11 imply that our natural map of homotopy limits is isomorphic to the map
$$\mathrm{Map}\big(\mathrm{B}(\mathcal{D}\downarrow -), \boldsymbol{X}\big) \to \mathrm{Map}\big(\mathrm{B}(F\downarrow -), \boldsymbol{X}\big).$$
The $\mathcal{D}$-diagrams of simplicial sets $\mathrm{B}(F\downarrow -)$ and $\mathrm{B}(\mathcal{D}\downarrow -)$ are always free cell complexes (see Proposition 14.6.5 and Corollary 14.6.8), and are thus cofibrant $\mathcal{D}$-diagrams (see Theorem 14.2.1). Since $\mathrm{B}(\mathcal{D}\downarrow -)$ is a diagram of contractible simplicial sets (see Lemma 14.5.8), the map $\mathrm{B}(F\downarrow -) \to \mathrm{B}(\mathcal{D}\downarrow -)$ is a weak equivalence of $\mathcal{D}$-diagrams if and only if the functor F is left cofinal. Since a $\mathcal{D}$-diagram of simplicial sets is fibrant exactly when it is a diagram of fibrant simplicial sets (see Theorem 14.2.1), Corollary 20.4.7 implies that we are trying to prove that a map of cofibrant diagrams is a weak equivalence if and only if it induces a weak equivalence of simplicial mapping spaces to an arbitrary fibrant object. This follows from Corollary 10.5.5, and so the proof is complete. □

THEOREM 19.5.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be small categories, and let $\mathrm{F}\colon \mathcal{C} \to \mathcal{D}$ be a functor.*

1. *F is right cofinal (see Definition 14.4.5) if and only if for every framed model category $\mathcal{M}$ and every $\mathcal{D}$-diagram $\boldsymbol{X}\colon \mathcal{D} \to \mathcal{M}$ in $\mathcal{M}$ of cofibrant objects, the natural map*
$$\operatorname*{hocolim}_{\mathcal{C}} F^*\boldsymbol{X} \to \operatorname*{hocolim}_{\mathcal{D}} \boldsymbol{X}$$
(see Proposition 19.1.8) is a weak equivalence.
2. *F is left cofinal if and only if for every framed model category $\mathcal{M}$ and every $\mathcal{D}$-diagram $\boldsymbol{X}\colon \mathcal{D} \to \mathcal{M}$ in $\mathcal{M}$ of fibrant objects, the natural map*
$$\operatorname*{holim}_{\mathcal{D}} \boldsymbol{X} \to \operatorname*{holim}_{\mathcal{C}} \mathrm{F}^*\boldsymbol{X}$$

(see Proposition 19.1.14*) is a weak equivalence.*

PROOF. This follows from Theorem 19.5.6 and Proposition 19.5.10. □

COROLLARY 19.5.12. *Let $\mathcal{M}$ be a framed model category and let $\mathcal{C}$ be a small category.*

1. *If α is an initial object of $\mathcal{C}$ and $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram of fibrant objects in $\mathcal{M}$, then the natural map* $\operatorname{holim} \boldsymbol{X} \to \boldsymbol{X}_\alpha$ *is a weak equivalence.*
2. *If α is a terminal object of $\mathcal{C}$ and $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram of cofibrant objects in $\mathcal{M}$, then the natural map* $\boldsymbol{X}_\alpha \to \operatorname{hocolim} \boldsymbol{X}$ *is a weak equivalence.*

PROOF. This follows from Theorem 19.5.6. □

As a corollary, we obtain Quillen's "Theorem A" (see [**49**, Page 93]).

THEOREM 19.5.13 (Quillen). *If $\mathcal{C}$ and $\mathcal{D}$ are small categories and $F\colon \mathcal{C} \to \mathcal{D}$ is a right cofinal functor, then F induces a weak equivalence of classifying spaces* $\mathrm{B}\mathcal{C} \cong \mathrm{B}\mathcal{D}$.

PROOF. This follows from Theorem 19.5.11, Proposition 19.1.6, and Proposition 9.4.5. □

PROPOSITION 19.5.14. *Let $\mathcal{M}$ be a framed model category. If the object X is a retract of the cofibrant object Y (with inclusion $i\colon X \to Y$ and retraction $r\colon Y \to X$), then X is weakly equivalent to the homotopy colimit of the diagram*

$$Y \xrightarrow{ir} Y \xrightarrow{ir} Y \xrightarrow{ir} \cdots$$

PROOF. We have an ω-sequence (where ω is the first infinite ordinal)

$$X \xrightarrow{i} Y \xrightarrow{r} X \xrightarrow{i} Y \xrightarrow{r} X \xrightarrow{i} Y \cdots$$

which has the two subdiagrams

$$X \xrightarrow{1_X} X \xrightarrow{1_X} X \xrightarrow{1_X} \cdots$$

and

$$Y \xrightarrow{ir} Y \xrightarrow{ir} Y \xrightarrow{ir} \cdots$$

Both of the subdiagrams are right cofinal, because all of the undercategories have an initial object (see Proposition 9.4.4). Thus, Theorem 19.5.11 implies that the homotopy colimits of the three diagrams are all weakly equivalent. Since X is a retract of a cofibrant object, it is cofibrant, and so the second diagram is a Reedy cofibrant diagram. Thus, the homotopy colimit of this diagram is weakly equivalent to its colimit (**Fill in a reference!!**), which is isomorphic to X. □

CHAPTER 20

Leftovers on homotopy colimits and homotopy limits

20.1. Frames on diagrams

DEFINITION 20.1.1. Let $\mathcal{M}$ be a model category, let $\mathcal{C}$ be a small category, and let $\boldsymbol{X}: \mathcal{C} \to \mathcal{M}$ be a $\mathcal{C}$-diagram in $\mathcal{M}$.

1. A *cosimplicial frame on* $\boldsymbol{X}$ is a diagram $\widetilde{\boldsymbol{X}}: \mathcal{C} \to \mathcal{M}^{\boldsymbol{\Delta}}$ of cosimplicial objects in $\mathcal{M}$ together with a map of diagrams $i: \widetilde{\boldsymbol{X}} \to \mathrm{c}\boldsymbol{X}$ to the diagram of constant cosimplicial objects such that, for every object α in $\mathcal{C}$, the map $i_\alpha: \widetilde{\boldsymbol{X}}_\alpha \to \mathrm{c}\boldsymbol{X}_\alpha$ is a cosimplicial frame on $\boldsymbol{X}_\alpha$ (see Definition 17.1.20).
2. A *simplicial frame on* $\boldsymbol{X}$ is a diagram $\widehat{\boldsymbol{X}}: \mathcal{C} \to \mathcal{M}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ of simplicial objects in $\mathcal{M}$ together with a map of diagrams $j: \mathrm{s}\boldsymbol{X} \to \widehat{\boldsymbol{X}}$ from the diagram of constant simplicial objects such that, for every object α in $\mathcal{C}$, the map $j_\alpha: \mathrm{s}\boldsymbol{Z}_\alpha \to \widehat{\boldsymbol{X}}_\alpha$ is a simplicial frame on $\boldsymbol{X}_\alpha$.

EXAMPLE 20.1.2. If $\mathcal{M}$ is a framed model category (see Definition 17.1.33), $\mathcal{C}$ is a small category, and $\boldsymbol{X}: \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, then there is a cosimplicial frame $\widetilde{\boldsymbol{X}}: \mathcal{C} \to \mathcal{M}$ on $\boldsymbol{X}$ and a simplicial frame $\widehat{\boldsymbol{X}}: \mathcal{C} \to \mathcal{M}$ on $\boldsymbol{X}$ where $\widetilde{\boldsymbol{X}}_\alpha$ and $\widehat{\boldsymbol{X}}_\alpha$ are defined by the frame on $\mathcal{M}$ for every object α in $\mathcal{C}$.

DEFINITION 20.1.3. Let $\mathcal{M}$ be a model category, let $\mathcal{C}$ be a Reedy category, and let $\boldsymbol{:}\ \mathcal{C} \to \mathcal{M}$ be a $\mathcal{C}$-diagram in $\mathcal{M}$.

1. A *Reedy cosimplicial frame on* $\boldsymbol{X}$ is a cosimplicial frame $\widetilde{\boldsymbol{X}}: \mathcal{C} \to \mathcal{M}^{\boldsymbol{\Delta}}$ on $\boldsymbol{X}$ (see Definition 20.1.1) such that if $\boldsymbol{X}$ is a Reedy cofibrant diagram in $\mathcal{M}$ (see Definition 16.3.2), then $\widetilde{\boldsymbol{X}}$ is a Reedy cofibrant diagram in $\mathcal{M}^{\boldsymbol{\Delta}}$.
2. A *Reedy simplicial frame on* $\boldsymbol{X}$ is a simplicial frame $\widehat{\boldsymbol{X}}: \mathcal{C} \to \mathcal{M}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ on $\boldsymbol{X}$ such that if $\boldsymbol{X}$ is a Reedy fibrant diagram in $\mathcal{M}$, then $\widehat{\boldsymbol{X}}$ is a Reedy fibrant diagram in $\mathcal{M}^{\boldsymbol{\Delta}^{\mathrm{op}}}$.

PROPOSITION 20.1.4. *Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a Reedy category. If $\boldsymbol{X}: \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, then*

1. *the cosimplicial frame on $\boldsymbol{X}$ defined by the standard frame on $\mathcal{M}$ (see Definition 17.1.26) is a Reedy cosimplicial frame on $\boldsymbol{X}$, and*
2. *the simplicial frame on $\boldsymbol{X}$ defined by the standard frame on $\mathcal{M}$ is a Reedy simplicial frame on $\boldsymbol{X}$.*

PROOF. We will prove part 1; the proof of part 2 is dual.

Let $\boldsymbol{X}$ be Reedy cofibrant, and let $\widetilde{\boldsymbol{X}}: \mathcal{C} \to \mathcal{M}^{\boldsymbol{\Delta}}$ be the cosimplicial frame on $\boldsymbol{X}$ defined by the standard frame on $\mathcal{M}$. For every object α in $\mathcal{C}$, let $\mathrm{L}_\alpha^{\mathcal{M}}\boldsymbol{X} \to \boldsymbol{X}_\alpha$ denote the latching map of $\boldsymbol{X}$ in $\mathcal{M}$, and let $\mathrm{L}_\alpha^{\mathcal{M}^{\boldsymbol{\Delta}}}\widetilde{\boldsymbol{X}} \to \widetilde{\boldsymbol{X}}$ denote the latching map of $\widetilde{\boldsymbol{X}}$ in $\mathcal{M}^{\boldsymbol{\Delta}}$. For every object α in $\mathcal{C}$, $\mathrm{L}_\alpha^{\mathcal{M}}\boldsymbol{X} \to \boldsymbol{X}_\alpha$ is a cofibration in $\mathcal{M}$, and we must show that $\mathrm{L}_\alpha^{\mathcal{M}^{\mathbf{\Delta}}}\widetilde{\boldsymbol{X}} \to \widetilde{\boldsymbol{X}}_\alpha$ is a cofibration in $\mathcal{M}^{\mathbf{\Delta}}$. Thus, Proposition 17.3.7 implies that we must show that, for every $n \geq 0$, the relative latching map

$$\widetilde{\boldsymbol{X}}_\alpha \otimes \partial\Delta[n] \amalg_{(\mathrm{L}_\alpha^{\mathcal{M}^{\mathbf{\Delta}}}\widetilde{\boldsymbol{X}})\otimes\partial\Delta[n]} (\mathrm{L}_\alpha^{\mathcal{M}^{\mathbf{\Delta}}}\widetilde{\boldsymbol{X}}) \otimes \Delta[n] \to \widetilde{\boldsymbol{X}}_\alpha \otimes \Delta[n] \tag{20.1.5}$$

(see Proposition 17.3.7) is a cofibration in $\mathcal{M}$. Since the latching object $\mathrm{L}_\alpha^{\mathcal{M}^{\mathbf{\Delta}}}\widetilde{\boldsymbol{X}}$ is defined as a colimit (see Definition 16.2.17), Proposition 17.3.10 and Lemma 10.2.3 imply that the map (??) is isomorphic to the map

$$\boldsymbol{X}_\alpha \otimes \partial\Delta[n] \amalg_{(\mathrm{L}_\alpha^{\mathcal{M}}\boldsymbol{X})\otimes\partial\Delta[n]} (\mathrm{L}_\alpha^{\mathcal{M}}\boldsymbol{X}) \otimes \Delta[n] \to \boldsymbol{X}_\alpha \otimes \Delta[n].$$

Since $\mathrm{L}_\alpha^{\mathcal{M}}\boldsymbol{X} \to \boldsymbol{X}_\alpha$ is a cofibration in the simplicial model category $\mathcal{M}$, Proposition 10.1.8 implies that this is a cofibration. □

PROPOSITION 20.1.6. *If $\mathcal{M}$ is a model category and $\mathcal{C}$ is a Reedy category, then*

1. *there is a functorial Reedy cosimplicial frame on every $\mathcal{C}$-diagram in $\mathcal{M}$, and*
2. *there is a functorial Reedy simplicial frame on every $\mathcal{C}$-diagram in $\mathcal{M}$.*

PROOF. **Fill this in!! It follows from the equivalence of the two Reedy model category structures on $\mathcal{C} \times \mathbf{\Delta}$-diagrams in $\mathcal{M}$.** □

PROPOSITION 20.1.7. *Let $\mathcal{M}$ be a model category and let $\mathcal{C}$ be a Reedy category. If $\boldsymbol{B}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$ that is Reedy cofibrant and $\boldsymbol{X}$ is a simplicial resolution in $\mathcal{M}$, then the diagram $\mathcal{M}(\boldsymbol{B}, \boldsymbol{X})$ (which on an object α in $\mathcal{C}$ is $\mathcal{M}(\boldsymbol{B}_\alpha, \boldsymbol{X})$) is a Reedy fibrant $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets.*

PROOF. If α is an object in $\mathcal{C}$ and $\mathrm{L}_\alpha\boldsymbol{B}$ is the latching object of $\boldsymbol{B}$ at α (see Definition 16.2.17), then Proposition 16.2.15 implies that

$$\begin{aligned}
\mathcal{M}(\mathrm{L}_\alpha\boldsymbol{B}, \boldsymbol{X}) &= \mathcal{M}\Big(\operatorname*{colim}_{((\overrightarrow{\mathcal{C}}\downarrow\alpha)-1_\alpha)} \boldsymbol{B}, \boldsymbol{X}\Big) \\
&\approx \lim_{((\overrightarrow{\mathcal{C}}\downarrow\alpha)-1_\alpha)^{\mathrm{op}}} \mathcal{M}(\boldsymbol{B}, \boldsymbol{X}) \\
&\approx \lim_{((\alpha\downarrow\overleftarrow{\mathcal{C}^{\mathrm{op}}})-1_\alpha)} \mathcal{M}(\boldsymbol{B}, \boldsymbol{X}) \\
&\approx \mathrm{M}_\alpha\mathcal{M}(\boldsymbol{B}, \boldsymbol{X})
\end{aligned}$$

and so $\mathcal{M}(\mathrm{L}_\alpha\boldsymbol{B}, \boldsymbol{X})$ is naturally isomorphic to the matching object at α of the $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets $\mathcal{M}(\boldsymbol{B}, \boldsymbol{X})$. Since the latching map $\mathrm{L}_\alpha\boldsymbol{B} \to \boldsymbol{B}_\alpha$ is a cofibration, Corollary 17.4.2 implies that the matching map $\mathcal{M}(\boldsymbol{B}_\alpha, \boldsymbol{X}) \to \mathrm{M}_\alpha\mathcal{M}(\boldsymbol{B}, \boldsymbol{X})$ is a fibration, and so $\mathcal{M}(\boldsymbol{B}, \boldsymbol{X})$ is a Reedy fibrant diagram. □

20.2. Realizations and total spaces

20.2.1. The realization of a simplicial object.

DEFINITION 20.2.2. **We need to rephrase this to use a Reedy frame on the diagram category, but first we've got to type up the definition of Reedy frame!!** If $\mathcal{M}$ is a framed model category and $\boldsymbol{X} : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{M}$ is a simplicial object in $\mathcal{M}$, the *realization* $|\boldsymbol{X}|$ of $\boldsymbol{X}$ is the coequalizer of the maps

$$\coprod_{(\sigma\colon [n]\to[k])\in\mathbf{\Delta}} \widetilde{\boldsymbol{X}}_n \otimes \Delta[k] \overset{\phi}{\underset{\psi}{\rightrightarrows}} \coprod_{[n]\in\mathrm{Ob}(\mathbf{\Delta})} \widetilde{\boldsymbol{X}}_n \otimes \Delta[n]$$

where $\widetilde{\boldsymbol{X}}_n$ is the cosimplicial frame on $\boldsymbol{X}_n$, the map ϕ on the summand $\sigma\colon [n] \to [k]$ is the composition of the map

$$\sigma_* \otimes 1_{\Delta[k]} \colon \widetilde{\boldsymbol{X}}_n \otimes \Delta[k] \to \widetilde{\boldsymbol{X}}_k \otimes \Delta[k]$$

(where $\sigma_* \colon \widetilde{\boldsymbol{X}}_n \to \widetilde{\boldsymbol{X}}_k$) with the natural injection into the coproduct, and the map ψ on the summand $\sigma\colon [n] \to [k]$ is the composition of the map

$$1_{\widetilde{\boldsymbol{X}}_n} \otimes \sigma^* \colon \widetilde{\boldsymbol{X}}_n \otimes \Delta[k] \to \widetilde{\boldsymbol{X}}_n \otimes \Delta[n]$$

(where $\sigma^* \colon \Delta[k] \to \Delta[n]$) with the natural injection into the coproduct.

Remark 20.2.3. Since each standard simplex $\Delta[n]$ is a contractible simplicial set, the map from each $\Delta[n]$ to a point is a weak equivalence. Thus, the cosimplicial standard simplex is a diagram of simplicial sets weakly equivalent to a point. We will show in Corollary 16.4.10 that the cosimplicial standard simplex is also a cofibrant diagram in the Reedy model category structure on cosimplicial spaces. This will imply that the cosimplicial standard simplex is a Reedy cofibrant approximation to the constant diagram at a point (see Definition 9.1.1), as is the diagram of opposites of undercategories $\mathrm{B}(-\downarrow \boldsymbol{\Delta})^{\mathrm{op}}$ (see Corollary 16.4.4), which will imply that that realization of a simplicial space that is cofibrant in each degree is naturally weakly equivalent to the homotopy colimit of the simplicial space (see Theorem 20.11.6). **This is all rearranged enough so that we can just prove all this right now!!**

20.2.4. The total space of a cosimplicial space. The principal reference for cosimplicial spaces and their total spaces is [**15**, Chapter X].

Definition 20.2.5. **Rewrite this to use a Reedy frame!!** If $\mathcal{M}$ is a framed model category and $\boldsymbol{X}\colon \boldsymbol{\Delta} \to \mathcal{M}$ is a cosimplicial object in $\mathcal{M}$ (see Definition 16.1.7), the *total object* $\operatorname{Tot}\boldsymbol{X}$ of the cosimplicial object $\boldsymbol{X}$ is the equalizer of the maps

$$\prod_{[n]\in \mathrm{Ob}(\boldsymbol{\Delta})} (\widehat{\boldsymbol{X}}^n)^{\Delta[n]} \overset{\phi}{\underset{\psi}{\rightrightarrows}} \prod_{(\sigma\colon [n]\to[k])\in\boldsymbol{\Delta}} (\widehat{\boldsymbol{X}}^k)^{\Delta[n]}$$

where $\widehat{\boldsymbol{X}}^n$ is the natural simplicial frame on $\boldsymbol{X}^n$, the projection of the map ϕ on the factor $\sigma\colon [n] \to [k]$ is the composition of a projection from the product with the map

$$\sigma_*^{(1_{\Delta[n]})} \colon (\widehat{\boldsymbol{X}}^n)^{\Delta[n]} \to (\widehat{\boldsymbol{X}}^k)^{\Delta[n]}$$

and the projection of the map ψ on the factor $\sigma\colon [n] \to [k]$ is the composition of a projection from the product with the map

$$(1_{\widehat{\boldsymbol{X}}^k})^{\sigma_*} \colon (\widehat{\boldsymbol{X}}^k)^{\Delta[k]} \to (\widehat{\boldsymbol{X}}^k)^{\Delta[n]}$$

(where $\sigma_*\colon \Delta[n] \to \Delta[k]$).

Example 20.2.6. If $\mathcal{M}$ is a framed model category, $\boldsymbol{X}\colon \boldsymbol{\Delta} \to \mathcal{M}$ is a cosimplicial object in $\mathcal{M}$, and Δ is the cosimplicial standard simplex (see Definition 16.1.9), then $\hom^{\mathcal{C}}(\Delta, \widehat{\boldsymbol{X}})$ is the total space of $\boldsymbol{X}$ (see Definition 20.2.5).

REMARK 20.2.7. If $\mathcal{M}$ is a category of spaces, then the space $(X^n)^{\Delta[n]}$ takes a different form in each of the categories in which we work (see Definition 1.1.11):

$$(X^n)^{\Delta[n]} = \begin{cases} \operatorname{map}(|\Delta[n]|, \boldsymbol{X}^n) & \text{if } \mathsf{Spc}_{(*)} = \mathsf{Top} \\ \operatorname{map}_*(|\Delta[n]|^+, \boldsymbol{X}^n) & \text{if } \mathsf{Spc}_{(*)} = \mathsf{Top}_* \\ \operatorname{Map}(\Delta[n], \boldsymbol{X}^n) & \text{if } \mathsf{Spc}_{(*)} = \mathsf{SS} \\ \operatorname{Map}_*(\Delta[n]^+, \boldsymbol{X}^n) & \text{if } \mathsf{Spc}_{(*)} = \mathsf{SS}_* \end{cases}$$

Thus, in each case the homotopy limit is constructed by first taking the codegreewise mapping space of the cosimplicial space $|\Delta|$ (or $|\Delta|^+$, or Δ, or Δ^+) and the cosimplicial space $\boldsymbol{X}$, and then taking a subspace of the product of these mapping spaces. The total space is actually an example of a space of maps between diagrams (see Example 20.4.3).

REMARK 20.2.8. Since each standard simplex $\Delta[n]$ is a contractible space, the map from each $\Delta[n]$ to a point is a weak equivalence. Thus, the cosimplicial standard simplex is a diagram of spaces weakly equivalent to a point. We will show in Corollary 16.4.10 that the cosimplicial standard simplex is also a cofibrant diagram in the Reedy model category structure on cosimplicial spaces. This will imply that the cosimplicial standard simplex is a Reedy cofibrant approximation to the constant diagram at a point (see Definition 9.1.1), as is the diagram of overcategories $\mathrm{B}(\boldsymbol{\Delta}\downarrow -)$ (see Corollary 16.4.4), which will imply that that total space of a Reedy fibrant cosimplicial space is weakly equivalent to its homotopy limit (see Theorem 20.11.5).

20.3. Leftovers on coends and ends

20.3.1. Coends.

PROPOSITION 20.3.2. *If $\mathcal{C}$ is a small category, $\boldsymbol{X}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ a diagram of spaces, $\boldsymbol{K}\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{SS}$ a diagram of simplicial sets, and $F\colon \mathsf{Spc}_{(*)} \to \mathsf{Spc}_{(*)}$ a functor that is a left adjoint, then there is a natural isomorphism (or homeomorphism)*

$$F\Big(\int^{\alpha} \boldsymbol{X}_\alpha \otimes \boldsymbol{K}_\alpha\Big) \approx \int^{\alpha} (F\boldsymbol{X}_\alpha) \otimes \boldsymbol{K}_\alpha .$$

PROOF. The coend $\int^{\alpha} \boldsymbol{X}_\alpha \otimes \boldsymbol{K}_\alpha$ is the coequalizer of the diagram

$$\coprod_{(\sigma\colon \alpha\to\alpha')\in\mathcal{C}} \boldsymbol{X}_\alpha \otimes \boldsymbol{K}_{\alpha'} \rightrightarrows \coprod_{\alpha\in\mathrm{Ob}(\mathcal{C})} \boldsymbol{X}_\alpha \otimes \boldsymbol{K}_\alpha$$

As a functor of $\boldsymbol{X}$, this is a composition of functors that commute with left adjoints, and so it commutes with left adjoints. $\square$

COROLLARY 20.3.3. *If $\mathcal{C}$ is a small category, $\boldsymbol{X}\colon \mathcal{C} \to \mathsf{Spc}$ a diagram of unpointed spaces, A an unpointed space, $\boldsymbol{Y}\colon \mathcal{C} \to \mathsf{Spc}_*$ a diagram of pointed spaces, and B a pointed space, then there are natural isomorphisms (or homeomorphisms)*

$$A \times \operatorname{hocolim} \boldsymbol{X} \approx \operatorname{hocolim}(A \times \boldsymbol{X})$$
$$B \wedge \operatorname{hocolim} \boldsymbol{Y} \approx \operatorname{hocolim}(B \wedge \boldsymbol{Y})$$
$$A^+ \wedge \operatorname{hocolim} \boldsymbol{Y} \approx \operatorname{hocolim}(A^+ \wedge \boldsymbol{Y})$$
$$B \wedge (\operatorname{hocolim} \boldsymbol{X})^+ \approx \operatorname{hocolim}(B \wedge \boldsymbol{X}^+).$$

REMARK 20.3.4. The assertion that $B \wedge (-)^{+} : \mathsf{Spc} \to \mathsf{Spc}_{*}$ commutes with taking the homotopy colimit is really an assertion about the homotopy colimit in two different categories: the *pointed* homotopy colimit in Spc_{*} and the *unpointed* homotopy colimit in Spc. More specifically, for any pointed space B and diagram of unpointed spaces $\boldsymbol{X} : \mathcal{C} \to \mathsf{Spc}$, we have an isomorphism (or homeomorphism)

$$B \wedge (\operatorname{hocolim} \boldsymbol{X})^{+} \approx \operatorname{hocolim}_{*}(B \wedge \boldsymbol{Y}^{+})$$

in Spc_{*} where (in this remark) $\operatorname{hocolim} \boldsymbol{X}$ means the homotopy colimit in the category of unpointed spaces and $\operatorname{hocolim}_{*}(B \wedge \boldsymbol{Y}^{+})$ means the homotopy colimit in the category of pointed spaces. Similar remarks apply to the assertion about $A^{+} \wedge - : \mathsf{Spc}_{*} \to \mathsf{Spc}$.

PROOF OF COROLLARY 20.3.3. This follows from Proposition 20.3.2, a deleted example, the standard adjunctions

$$\mathsf{Top}(A \times W, Z) \approx \mathsf{Top}\big(W, \mathsf{Top}(A, Z)\big)$$
$$\mathsf{Top}_{*}(B \wedge U, V) \approx \mathsf{Top}_{*}\big(U, \mathsf{Top}_{*}(B, V)\big)$$

and the analogous formulas for simplicial sets. □

COROLLARY 20.3.5. *If $\boldsymbol{X}$ is a simplicial unpointed space, A an unpointed space, $\boldsymbol{Y}$ a simplicial pointed space, and B a pointed space, then there are natural isomorphisms (or homeomorphisms)*

$$A \times |\boldsymbol{X}| \approx |A \times \boldsymbol{X}|$$
$$B \wedge |\boldsymbol{Y}| \approx |B \wedge \boldsymbol{Y}|$$
$$A^{+} \wedge |\boldsymbol{Y}| \approx |A^{+} \wedge \boldsymbol{Y}|$$
$$B \wedge |\boldsymbol{X}|^{+} \approx |B \wedge \boldsymbol{X}^{+}|.$$

PROOF. This is similar to the proof of Corollary 20.3.3, using a deleted example. □

PROPOSITION 20.3.6. *If $\mathcal{C}$ is a small category and $\boldsymbol{X} : \mathcal{C} \to \mathsf{SS}_{(*)}$ is a diagram of simplicial sets, then there is a natural homeomorphism $|\operatorname{hocolim} \boldsymbol{X}| \approx \operatorname{hocolim}|\boldsymbol{X}|$.*

PROOF. Since the geometric realization functor is a left adjoint and $|\boldsymbol{X}_{\beta} \otimes \boldsymbol{K}_{\alpha}| \approx |\boldsymbol{X}_{\beta}| \otimes \boldsymbol{K}_{\alpha}$, this is similar to the proof of Proposition 20.3.2, using a deleted example. □

PROPOSITION 20.3.7. *If $\boldsymbol{X} : \boldsymbol{\Delta}^{\mathrm{op}} \to \mathsf{SS}_{(*)}$ is a simplicial simplicial set (i.e., a simplicial object in the category of simplicial sets), then there is a natural homeomorphism from the geometric realization of the simplicial set $|\boldsymbol{X}|$ to the realization of the simplicial topological space $|\boldsymbol{X}|$.*

PROOF. This is similar to the proof of Proposition 20.3.6. □

PROPOSITION 20.3.8. *If $\boldsymbol{X}$ is the diagram of spaces $C \leftarrow A \to B$ and the map $A \to B$ is a cofibration, then the natural map $\operatorname{hocolim} \boldsymbol{X} \to \operatorname{colim} \boldsymbol{X}$ (see Example 19.2.6) is a weak equivalence.*

PROOF. If $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$ (in which every space is fibrant) we can use the homotopy extension property of $A \to B$ to define a map $\operatorname{colim} \boldsymbol{X} \to \operatorname{hocolim} \boldsymbol{X}$ that is a homotopy inverse to the natural map $\operatorname{hocolim} \boldsymbol{X} \to \operatorname{colim} \boldsymbol{X}$, and so these maps are homotopy equivalences. If $\mathsf{Spc}_{(*)} = \mathsf{SS}_{(*)}$ then Proposition 20.3.6 implies that the geometric realization of the natural map is a homotopy equivalence, and so the natural map is a weak equivalence. □

20.3.9. Ends.

PROPOSITION 20.3.10. *If $\mathcal{C}$ is a small category, $\boldsymbol{X}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ a diagram of spaces, $\boldsymbol{K}\colon \mathcal{C} \to \mathsf{SS}$ a diagram of simplicial sets, and $F\colon \mathsf{Spc}_{(*)} \to \mathsf{Spc}_{(*)}$ a functor that is a right adjoint, then there is a natural isomorphism (or homeomorphism)*

$$F\left(\int_\alpha \boldsymbol{X}_\alpha^{\boldsymbol{K}_\alpha}\right) \approx \int_\alpha (F\boldsymbol{X}_\alpha)^{\boldsymbol{K}_\alpha}.$$

PROOF. This is similar to the proof of Proposition 20.3.2. □

COROLLARY 20.3.11. *If $\mathcal{C}$ is a small category, $\boldsymbol{X}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ a diagram of spaces, and $A \in \mathrm{Ob}(\mathsf{Spc}_{(*)})$ is a space, then there is a natural isomorphism (or homeomorphism)*

$$(\operatorname{holim} \boldsymbol{X})^A \approx \operatorname{holim}(\boldsymbol{X}^A)$$

(see Definition 1.1.6) (where $\boldsymbol{X}^A\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ is the diagram in which $(\boldsymbol{X}^A)_\alpha = (\boldsymbol{X}_\alpha)^A$ for all $\alpha \in \mathrm{Ob}(\mathcal{C})$).

PROOF. This is similar to the proof of Corollary 20.3.3. □

COROLLARY 20.3.12. *If $\boldsymbol{X}\colon \boldsymbol{\Delta} \to \mathsf{Spc}_{(*)}$ is a cosimplicial space and $A \in \mathrm{Ob}(\mathsf{Spc}_{(*)})$ is a space, then there is a natural isomorphism (or homeomorphism)*

$$(\operatorname{Tot} \boldsymbol{X})^A \approx \operatorname{Tot}(\boldsymbol{X}^A)$$

(see Definition 1.1.6) (where $\boldsymbol{X}^A$ is the cosimplicial space in which $(\boldsymbol{X}^A)^n = (\boldsymbol{X}^n)^A$).

PROOF. This is similar to the proof of Corollary 20.3.11. □

PROPOSITION 20.3.13. *If $\mathcal{C}$ is a small category and $\boldsymbol{X}\colon \mathcal{C} \to \mathsf{Top}_{(*)}$ is a diagram of topological spaces, then there is a natural isomorphism* $\operatorname{Sing}\operatorname{holim}\boldsymbol{X} \approx \operatorname{holim}\operatorname{Sing}\boldsymbol{X}$.

PROOF. Since the total singular complex functor is a right adjoint and $\operatorname{Sing}(\boldsymbol{X}_\beta^{\boldsymbol{K}_\alpha}) \approx (\operatorname{Sing}\boldsymbol{X}_\beta)^{\boldsymbol{K}_\alpha}$, this is similar to the proof of Proposition 20.3.10, using a deleted example.. □

PROPOSITION 20.3.14. *If $\boldsymbol{X}\colon \boldsymbol{\Delta} \to \mathsf{Top}_{(*)}$ is a simplicial topological space, then there is a natural isomorphism* $\operatorname{Sing}\operatorname{Tot}\boldsymbol{X} \approx \operatorname{Tot}\operatorname{Sing}\boldsymbol{X}$.

PROOF. This is similar to the proof of Proposition 20.3.13, using a deleted example. □

PROPOSITION 20.3.15. *If $\boldsymbol{X}$ is the diagram of spaces $C \to B \leftarrow A$ and the map $A \to B$ is a fibration, then the natural map $\lim \boldsymbol{X} \to \operatorname{holim}\boldsymbol{X}$ is a weak equivalence.*

PROOF. This is similar to the proof of Proposition 20.3.8. □

20.3.16. Adjointness.

PROPOSITION 20.3.17. *If $\mathcal{C}$ is a small category, $\boldsymbol{X}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ a $\mathcal{C}$-diagram of spaces, $\boldsymbol{K}\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{SS}$ a $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets, and $Y \in \mathsf{Spc}_{(*)}$ a space, then there are natural isomorphisms (or homeomorphisms)*

$$\begin{aligned}\mathrm{Map}\Big(\int^{\alpha} \boldsymbol{X}_{\alpha} \otimes \boldsymbol{K}_{\alpha}, Y\Big) &\approx \int_{\alpha} \mathrm{Map}\big(\boldsymbol{K}_{\alpha}, \mathrm{Map}(\boldsymbol{X}_{\alpha}, Y)\big)\\ \mathrm{Map}\Big(\int^{\alpha} \boldsymbol{X}_{\alpha} \otimes \boldsymbol{K}_{\alpha}, Y\Big) &\approx \int_{\alpha} \mathrm{Map}(\boldsymbol{X}_{\alpha}, Y^{\boldsymbol{K}_{\alpha}})\\ Y^{(\int^{\alpha} \boldsymbol{X}_{\alpha}\otimes \boldsymbol{K}_{\alpha})} &\approx \int_{\alpha} \big(Y^{\boldsymbol{X}_{\alpha}}\big)^{\boldsymbol{K}_{\alpha}}.\end{aligned}$$

PROOF. We will establish the first isomorphism; the others are similar.

A deleted definition describes $\int^{\alpha} \boldsymbol{X}_{\alpha}\otimes\boldsymbol{K}_{\alpha}$ as a quotient of $\coprod_{\alpha\in\mathrm{Ob}(\mathcal{C})} \boldsymbol{X}_{\alpha}\otimes\boldsymbol{K}_{\alpha}$. For each $\alpha \in \mathrm{Ob}(\mathcal{C})$, we have a natural isomorphism

$$\mathrm{Map}\big(\boldsymbol{X}_{\alpha} \otimes \boldsymbol{K}_{\alpha}, Y\big) \approx \mathrm{Map}\big(\boldsymbol{K}_{\alpha}, \mathrm{Map}(\boldsymbol{X}_{\alpha}, Y)\big)$$

(see Definition 1.1.11) and so we have natural isomorphisms

$$\begin{aligned}\mathrm{Map}\Big(\coprod_{\alpha\in\mathrm{Ob}(\mathcal{C})} \boldsymbol{X}_{\alpha} \otimes \boldsymbol{K}_{\alpha}, Y\Big) &\approx \prod_{\alpha\in\mathrm{Ob}(\mathcal{C})} \mathrm{Map}\big(\boldsymbol{X}_{\alpha} \otimes \boldsymbol{K}_{\alpha}, Y\big)\\ &\approx \prod_{\alpha\in\mathrm{Ob}(\mathcal{C})} \mathrm{Map}\big(\boldsymbol{K}_{\alpha}, \mathrm{Map}(\boldsymbol{X}_{\alpha}, Y)\big).\end{aligned} \tag{20.3.18}$$

The relations imposed on $\coprod_{\alpha\in\mathrm{Ob}(\mathcal{C})} \boldsymbol{X}_{\alpha} \otimes \boldsymbol{K}_{\alpha}$ in the definition of $\int^{\alpha} \boldsymbol{X}_{\alpha} \otimes \boldsymbol{K}_{\alpha}$ are exactly the relations that must be respected by an element of the right hand side of (20.3.18) for it to be an element of $\int_{\alpha} \mathrm{Map}\big(\boldsymbol{K}_{\alpha}, \mathrm{Map}(\boldsymbol{X}_{\alpha}, Y)\big)$, and so the proposition follows. □

COROLLARY 20.3.19. *If $\mathcal{C}$ is a small category, $\boldsymbol{X}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ a $\mathcal{C}$-diagram of spaces and $Y \in \mathsf{Spc}_{(*)}$ a space, then $Y^{\boldsymbol{X}}$ is a $\mathcal{C}^{\mathrm{op}}$-diagram of spaces, $\mathrm{Map}(\boldsymbol{X}, Y)$ is a $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets, and there are natural isomorphisms (or homeomorphisms)*

$$\begin{aligned}Y^{\mathrm{hocolim}_{\mathcal{C}} \boldsymbol{X}} &\approx \operatorname*{holim}_{\mathcal{C}^{\mathrm{op}}}(Y^{\boldsymbol{X}})\\ \mathrm{Map}(\operatorname*{hocolim}_{\mathcal{C}} \boldsymbol{X}, Y) &\approx \operatorname*{holim}_{\mathcal{C}^{\mathrm{op}}} \mathrm{Map}(\boldsymbol{X}, Y).\end{aligned}$$

PROOF. This follows from Proposition 20.3.17, Corollary 14.5.11, Example ?? and Example ??. □

20.4. Mapping spaces

20.4.1. The internal mapping space. If $\mathcal{C}$ is a small category and $\boldsymbol{X}$ and $\boldsymbol{Y}$ are $\mathcal{C}$-diagrams of unpointed or pointed topological spaces, then $\boldsymbol{Y}^{\boldsymbol{X}}$ will be the (unpointed or pointed) topological space of maps of diagrams $\boldsymbol{X} \to \boldsymbol{Y}$, topologized as a subspace of the product $\prod_{\alpha\in\mathrm{Ob}(\mathcal{C})} \boldsymbol{Y}_{\alpha}^{\boldsymbol{X}_{\alpha}}$. If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are diagrams of simplicial sets, then $\boldsymbol{Y}^{\boldsymbol{X}}$ will be the (unpointed or pointed) simplicial set with n-simplices the simplicial maps $\boldsymbol{X} \otimes \Delta[n] \to \boldsymbol{Y}$ (see Definition 14.3.1). All of these mapping spaces can be described concisely as ends (see Definition ??) of functors constructed from the internal mapping space functors of Definition 1.1.6.

DEFINITION 20.4.2. Let $\mathcal{C}$ be a small category.

- If $\boldsymbol{X}, \boldsymbol{Y}\colon \mathcal{C} \to \mathsf{Top}$, then $\boldsymbol{Y}^{\boldsymbol{X}}$ is the unpointed topological space
$$\boldsymbol{Y}^{\boldsymbol{X}} = \int_\alpha \boldsymbol{Y}_\alpha^{\boldsymbol{X}_\alpha} = \int_\alpha \operatorname{map}(\boldsymbol{X}_\alpha, \boldsymbol{Y}_\alpha).$$
- If $\boldsymbol{X}, \boldsymbol{Y}\colon \mathcal{C} \to \mathsf{Top}_*$, then $\boldsymbol{Y}^{\boldsymbol{X}}$ is the pointed topological space
$$\boldsymbol{Y}^{\boldsymbol{X}} = \int_\alpha \boldsymbol{Y}_\alpha^{\boldsymbol{X}_\alpha} = \int_\alpha \operatorname{map}_*(\boldsymbol{X}_\alpha, \boldsymbol{Y}_\alpha).$$
- If $\boldsymbol{X}, \boldsymbol{Y}\colon \mathcal{C} \to \mathsf{SS}$, then $\boldsymbol{Y}^{\boldsymbol{X}}$ is the unpointed simplicial set
$$\boldsymbol{Y}^{\boldsymbol{X}} = \int_\alpha \boldsymbol{Y}_\alpha^{\boldsymbol{X}_\alpha} = \int_\alpha \operatorname{Map}(\boldsymbol{X}_\alpha, \boldsymbol{Y}_\alpha).$$
- If $\boldsymbol{X}, \boldsymbol{Y}\colon \mathcal{C} \to \mathsf{SS}_*$, then $\boldsymbol{Y}^{\boldsymbol{X}}$ is the pointed simplicial set
$$\boldsymbol{Y}^{\boldsymbol{X}} = \int_\alpha \boldsymbol{Y}_\alpha^{\boldsymbol{X}_\alpha} = \int_\alpha \operatorname{Map}_*(\boldsymbol{X}_\alpha, \boldsymbol{Y}_\alpha).$$

EXAMPLE 20.4.3. If $\boldsymbol{X}$ is a cosimplicial space (see Definition 16.1.7), then the total space (see Definition 20.2.5) of $\boldsymbol{X}$ is
$$\operatorname{Tot} \boldsymbol{X} = \boldsymbol{X}^{\boldsymbol{\Delta}}$$
(see Definition ??).

EXAMPLE 20.4.4. More generally, if $\mathcal{C}$ is a small category, $\boldsymbol{X}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ is a $\mathcal{C}$-diagram of spaces and $\boldsymbol{P}\colon \mathcal{C} \to \mathsf{SS}$ is a $\mathcal{C}$-diagram of simplicial sets, then the end (see Definition ??) of $(\boldsymbol{X}_\alpha)^{\boldsymbol{P}_\alpha}$ is
$$\int_\alpha (\boldsymbol{X}_\alpha)^{\boldsymbol{P}_\alpha} = \boldsymbol{X}^{\boldsymbol{P}}$$
(see Definition ??).

LEMMA 20.4.5. *If $\mathcal{C}$ is a small category and $\boldsymbol{X}, \boldsymbol{Y} : \mathcal{C} \to \mathsf{Top}$ are diagrams, then there is a natural isomorphism of simplicial sets*
$$\operatorname{Sing}(\boldsymbol{Y}^{\boldsymbol{X}}) \approx \operatorname{Map}(\boldsymbol{X}, \boldsymbol{Y})$$
(see Definition 14.3.2 and Definition 20.4.2).

PROOF. This is similar to the proof of Proposition 20.3.10. The space $\boldsymbol{Y}^{\boldsymbol{X}}$ is the end $\int_\alpha \boldsymbol{Y}_\alpha^{\boldsymbol{X}_\alpha}$, i.e., the limit of the diagram
$$\prod_{\alpha \in \operatorname{Ob}(\mathcal{C})} \boldsymbol{Y}_\alpha^{\boldsymbol{X}_\alpha} \rightrightarrows \prod_{(\sigma\colon \alpha \to \alpha') \in \mathcal{C}} \boldsymbol{Y}_{\alpha'}^{\boldsymbol{X}_\alpha}$$
(see Definition ??). Since the total singular complex functor is a right adjoint, it commutes with all limits, and so the result follows from the natural isomorphism $\operatorname{Sing}(\boldsymbol{Y}_{\alpha'}^{\boldsymbol{X}_\alpha}) \approx \operatorname{Map}(\boldsymbol{X}_\alpha, \boldsymbol{Y}_{\alpha'})$ (see Proposition 1.1.7). □

PROPOSITION 20.4.6. *If $\mathcal{C}$ is a small category and $\boldsymbol{X}, \boldsymbol{Y}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ are $\mathcal{C}$-diagrams of spaces, then the internal mapping spaces Y^X of Definition 20.4.2 are related to the simplicial mapping spaces $\operatorname{Map}(X, Y)$ of Definition 14.3.2 as follows:*

- *If $\mathsf{Spc}_{(*)} = \mathsf{Top}$, then the simplicial set $\operatorname{Map}(\boldsymbol{X}, \boldsymbol{Y})$ is the total singular complex of $\boldsymbol{Y}^{\boldsymbol{X}}$.*

- *If* $\mathsf{Spc}_{(*)} = \mathsf{Top}_*$, *the simplicial set* $\mathrm{Map}(\boldsymbol{X},\boldsymbol{Y})$ *is the total singular complex of the unpointed space obtained from* $\boldsymbol{Y}^{\boldsymbol{X}}$ *by forgetting the basepoint.*
- *If* $\mathsf{Spc}_{(*)} = \mathsf{SS}$, *then* $\mathrm{Map}(\boldsymbol{X},\boldsymbol{Y})$ *equals* $\boldsymbol{Y}^{\boldsymbol{X}}$.
- *If* $\mathsf{Spc}_{(*)} = \mathsf{SS}_*$, *then* $\mathrm{Map}(\boldsymbol{X},\boldsymbol{Y})$ *is obtained from* $\boldsymbol{Y}^{\boldsymbol{X}}$ *by forgetting the basepoint.*

PROOF. This follows from Proposition 1.1.7 and Lemma 20.4.5. □

COROLLARY 20.4.7. *If* $\mathcal{C}$ *is a small category,* $\boldsymbol{W},\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}\colon \mathcal{C}\to \mathsf{Spc}_{(*)}$ *are* $\mathcal{C}$*-diagrams of spaces, and* $g\colon \boldsymbol{W}\to\boldsymbol{X}$ *and* $h\colon \boldsymbol{Y}\to\boldsymbol{Z}$ *are maps of* $\mathcal{C}$*-diagrams, then* $h_*\colon \boldsymbol{Y}^{\boldsymbol{X}}\to\boldsymbol{Z}^{\boldsymbol{X}}$ *is a weak equivalence (of topological spaces (if* $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$*) or of simplicial sets (if* $\mathsf{Spc}_{(*)} = \mathsf{SS}_{(*)}$*)) if and only if* $h_*\colon\ \mathrm{Map}(\boldsymbol{X},\boldsymbol{Y})\to\mathrm{Map}(\boldsymbol{X},\boldsymbol{Z})$ *is a weak equivalence of simplicial sets, and* $g^*\colon \boldsymbol{Y}^{\boldsymbol{X}}\to\boldsymbol{Y}^{\boldsymbol{W}}$ *is a weak equivalence (of topological spaces or simplicial sets) if and only if* $g^*\colon\ \mathrm{Map}(\boldsymbol{X},\boldsymbol{Y})\to\mathrm{Map}(\boldsymbol{W},\boldsymbol{Y})$ *is a weak equivalence of simplicial sets.*

PROOF. Since a map of pointed spaces is a weak equivalence if and only if it is a weak equivalence of unpointed spaces after forgetting the basepoint, and a map of topological spaces is a weak equivalence if and only if its total singular complex is a weak equivalence of simplicial sets, this follows from Proposition 20.4.6. □

PROPOSITION 20.4.8. *If* $\mathcal{C}$ *is a small category,* $\boldsymbol{X}\colon\mathcal{C}\to\mathsf{Spc}_{(*)}$ *a* $\mathcal{C}$*-diagram of spaces,* $\boldsymbol{Y}\colon\mathcal{C}^{\mathrm{op}}\to\mathsf{Spc}_{(*)}$ *a* $\mathcal{C}^{\mathrm{op}}$*-diagram of spaces, and* $Z\in\mathrm{Ob}(\mathsf{Spc}_{(*)})$ *a space, then*

$$\left(Z^{\boldsymbol{Y}}\right)^{\boldsymbol{X}} \approx Z^{\left(\int^{\alpha}\boldsymbol{X}_\alpha\otimes\boldsymbol{Y}_\alpha\right)}$$

PROOF. This is similar to the proof of Proposition 20.3.17. □

COROLLARY 20.4.9. *If* $\boldsymbol{Y}\colon\boldsymbol{\Delta}^{\mathrm{op}}\to\mathsf{Spc}_{(*)}$ *is a simplicial space and* $Z\in\mathrm{Ob}(\mathsf{Spc}_{(*)})$ *is a space, then* $Z^{\boldsymbol{Y}}$ *is a cosimplicial space, and there is a natural homeomorphism (if* $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$*) or isomorphism (if* $\mathsf{Spc}_{(*)} = \mathsf{SS}_{(*)}$*)*

$$\mathrm{Tot}\left(Z^{\boldsymbol{Y}}\right)\approx Z^{|\boldsymbol{Y}|}.$$

PROOF. This follows from Proposition 20.4.8, Example 20.4.3, and Example ??. □

PROPOSITION 20.4.10. *If* $\mathcal{C}$ *is a small category,* $\boldsymbol{X},\boldsymbol{Y}\colon\mathcal{C}\to\mathsf{Spc}_{(*)}$ $\mathcal{C}$*-diagrams of spaces, and* $Z\in\mathrm{Ob}(\mathsf{Spc}_{(*)})$ *a space, then there is a natural isomorphism (if* $\mathsf{Spc}_{(*)} = \mathsf{SS}_{(*)}$*) or homeomorphism (if* $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$*)*

$$\boldsymbol{Y}^{\boldsymbol{X}\otimes Z}\approx\left(\boldsymbol{Y}^{\boldsymbol{X}}\right)^{Z}\approx\left(\boldsymbol{Y}^{Z}\right)^{\boldsymbol{X}}.$$

COROLLARY 20.4.11. *If* $\mathcal{C}$ *is a small category,* $\boldsymbol{X}\colon\mathcal{C}\to\mathsf{Spc}_{(*)}$ *a* $\mathcal{C}$*-diagram of spaces and* $W\in\mathrm{Ob}(\mathsf{Spc}_{(*)})$ *a space, then there is a natural isomorphism (if* $\mathsf{Spc}_{(*)} = \mathsf{SS}_{(*)}$*) or homeomorphism (if* $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$*)*

$$(\operatorname{holim}\boldsymbol{X})^{W}\approx\operatorname{holim}(\boldsymbol{X}^{W}).$$

COROLLARY 20.4.12. *If* $\boldsymbol{X}\colon\boldsymbol{\Delta}\to\mathsf{Spc}_{(*)}$ *is a cosimplicial space and* $W\in\mathrm{Ob}(\mathsf{Spc}_{(*)})$ *is a space, then there is a natural isomorphism (if* $\mathsf{Spc}_{(*)} = \mathsf{SS}_{(*)}$*) or homeomorphism (if* $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$*)*

$$(\mathrm{Tot}\,\boldsymbol{X})^{W}\approx\mathrm{Tot}(\boldsymbol{X}^{W}).$$

LEMMA 20.4.13. *If $\mathcal{C}$ is a small category, $\boldsymbol{S}\colon \mathcal{C} \to \mathsf{Set}$ a $\mathcal{C}$-diagram of sets, $\boldsymbol{Y}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ a $\mathcal{C}$-diagram of spaces, and $X \in \mathrm{Ob}(\mathsf{Spc}_{(*)})$ a space, then there is a natural isomorphism (or homeomorphism, if $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$)*

$$\boldsymbol{Y}^{(X\otimes \boldsymbol{S})} \approx \left(\boldsymbol{Y}^X\right)^{\boldsymbol{S}}$$

(see Definition 1.1.11).

PROPOSITION 20.4.14. *If $\mathcal{C}$ is a small category, $\boldsymbol{Y}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ a $\mathcal{C}$-diagram of spaces, $X \in \mathrm{Ob}(\mathsf{Spc}_{(*)})$ a space and α an object of $\mathcal{C}$, then there is a natural isomorphism (or homeomorphism, if $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$)*

$$\boldsymbol{Y}^{(X\otimes \boldsymbol{F}^{\alpha}_{*})} \approx (\boldsymbol{Y}_{\alpha})^X$$

(see Definition 14.1.2).

PROOF. We will discuss the case $\mathsf{Spc}_{(*)} = \mathsf{Top}_*$; the other cases are similar. We have natural homeomorphisms

$$\begin{aligned}
\boldsymbol{Y}^{(X\otimes \boldsymbol{F}^{\alpha}_{*})} &\approx \mathrm{map}_*(X \otimes \boldsymbol{F}^{\alpha}_{*}, \boldsymbol{Y})\\
&\approx \mathrm{map}_*\big((\boldsymbol{F}^{\alpha}_{*})^{+}, \mathrm{map}_*(X, \boldsymbol{Y})\big)\\
&\approx \mathrm{map}_*\big(*^{+}, \mathrm{map}_*(X, \boldsymbol{Y}_{\alpha})\big)\\
&\approx \mathrm{map}_*(X \otimes *, \boldsymbol{Y}_{\alpha})\\
&\approx \mathrm{map}_*(X, \boldsymbol{Y}_{\alpha})\\
&\approx (\boldsymbol{Y}_{\alpha})^X
\end{aligned}$$

(see Lemma 20.4.13) where $*^{+}$ denotes the space with one point plus an adjoined basepoint. ☐

THEOREM 20.4.15. *If $\mathcal{C}$ is a small category, $\boldsymbol{Y}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ a $\mathcal{C}$-diagram of spaces and $\boldsymbol{X}\colon \mathcal{C}^{\mathrm{disc}} \to \mathsf{Spc}_{(*)}$ a discrete diagram of spaces, then there is a natural isomorphism (or homeomorphism, if $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$) of mapping spaces*

$$\boldsymbol{Y}^{\boldsymbol{F}\boldsymbol{X}} \approx (\boldsymbol{U}\boldsymbol{Y})^{\boldsymbol{X}}$$

(where $\boldsymbol{U}$ is the forgetful functor $\boldsymbol{U}\colon \mathsf{Spc}^{\mathcal{C}}_{(*)} \to \mathsf{Spc}^{(\mathcal{C}^{\mathrm{disc}})}_{(*)}$).

PROOF. This follows from Proposition 20.4.14. ☐

20.5. Topological spaces and simplicial sets

We proved in Proposition 20.3.6 that the geometric realization functor commutes with the homotopy colimit functor up to a natural homeomorphism, and in Proposition 20.3.13 that the total singular complex functor commutes with the homotopy limit functor up to a natural isomorphism. In this section, we show that, for a diagram of cofibrant topological spaces, the total singular complex functor commutes with the homotopy colimit functor up to a natural weak equivalence and that, for a diagram of fibrant simplicial sets, the geometric realization functor commutes with the homotopy limit functor up to a natural weak equivalence.

PROPOSITION 20.5.1. *If $\mathcal{C}$ is a small category and $\boldsymbol{X}\colon \mathcal{C} \to \mathsf{Top}_{(*)}$ is a diagram of cofibrant topological spaces, then there is a natural weak equivalence* $\operatorname{hocolim} \operatorname{Sing} \boldsymbol{X} \to \operatorname{Sing} \operatorname{hocolim} \boldsymbol{X}$.

PROOF. The natural map of diagrams $\left|\operatorname{Sing}\boldsymbol{X}\right| \to \boldsymbol{X}$ induces a natural weak equivalence $\operatorname{hocolim}\left|\operatorname{Sing}\boldsymbol{X}\right| \to \operatorname{hocolim}\boldsymbol{X}$ (see Theorem 20.6.11). Proposition 20.3.6 implies that this is isomorphic to a natural weak equivalence $\left|\operatorname{hocolim}\operatorname{Sing}\boldsymbol{X}\right| \to \operatorname{hocolim}\boldsymbol{X}$, which corresponds (under the standard adjunction) to a natural weak equivalence $\operatorname{hocolim}\operatorname{Sing}\boldsymbol{X} \to \operatorname{Sing}\operatorname{hocolim}\boldsymbol{X}$. □

PROPOSITION 20.5.2. *If $\mathcal{C}$ is a small category and $\boldsymbol{X}\colon \mathcal{C} \to \mathsf{SS}_{(*)}$ is a diagram of fibrant simplicial sets, then there is a natural weak equivalence $\left|\operatorname{holim}\boldsymbol{X}\right| \to \operatorname{holim}\left|\boldsymbol{X}\right|$.*

PROOF. This is similar to the proof of Proposition 20.5.1. □

20.6. Mapping spaces and homotopy invariance

PROPOSITION 20.6.1. *Let $\mathcal{C}$ be a small category, and let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{X}, \boldsymbol{Y}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ be diagrams. If $i\colon \boldsymbol{A} \to \boldsymbol{B}$ is a cofibration and $p\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a fibration, then the map*

$$\boldsymbol{X}^{\boldsymbol{B}} \to \boldsymbol{X}^{\boldsymbol{A}} \times_{\boldsymbol{Y}^{\boldsymbol{A}}} \boldsymbol{Y}^{\boldsymbol{B}}$$

is a fibration in $\mathsf{Spc}_{(*)}$ that is also a trivial fibration if either i or p is a weak equivalence.

PROOF. Since both the total singular complex functor and the functor that forgets the basepoint of a pointed space commute with limits, this follows from Theorem 14.2.1 and Proposition 20.4.6. □

COROLLARY 20.6.2. *If $\mathcal{C}$ is a small category, $i\colon \boldsymbol{A} \to \boldsymbol{B}$ a cofibration in $\mathsf{Spc}_{(*)}^{\mathcal{C}}$, and $\boldsymbol{X}$ a diagram of fibrant spaces, then the map $\boldsymbol{X}^{\boldsymbol{B}} \to \boldsymbol{X}^{\boldsymbol{A}}$ is a fibration in $\mathsf{Spc}_{(*)}$ that is a trivial fibration if i is a weak equivalence.*

PROOF. This follows from Proposition 20.6.1. □

COROLLARY 20.6.3. *If $\mathcal{C}$ is a small category, $p\colon \boldsymbol{X} \to \boldsymbol{Y}$ a fibration in $\mathsf{Spc}_{(*)}^{\mathcal{C}}$, and $\boldsymbol{A}$ a cofibrant diagram in $\mathsf{Spc}_{(*)}^{\mathcal{C}}$, then the map $\boldsymbol{X}^{\boldsymbol{A}} \to \boldsymbol{Y}^{\boldsymbol{A}}$ is a fibration in $\mathsf{Spc}_{(*)}$ that is a trivial fibration if p is a weak equivalence.*

PROOF. This follows from Proposition 20.6.1. □

COROLLARY 20.6.4. *If $\mathcal{C}$ is a small category and $p\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a fibration in $\mathsf{Spc}_{(*)}^{\mathcal{C}}$, then the map $\operatorname{holim}\boldsymbol{X} \to \operatorname{holim}\boldsymbol{Y}$ is a fibration in $\mathsf{Spc}_{(*)}$ that is a trivial fibration if p is a weak equivalence.*

PROOF. This follows from Corollary 20.6.3, **something deleted**, Corollary 14.6.8, Lemma 14.6.10, and Proposition 14.6.11. □

COROLLARY 20.6.5. *If $\mathcal{C}$ is a small category and $\boldsymbol{X}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ is a diagram of fibrant spaces, then $\operatorname{holim}\boldsymbol{X}$ is a fibrant space.*

PROOF. This follows from Corollary 20.6.4. □

PROPOSITION 20.6.6. *Let $\mathcal{C}$ be a small category, $i\colon \boldsymbol{A} \to \boldsymbol{B}$ a map in $\mathsf{Spc}_{(*)}^{\mathcal{C}}$ and $\boldsymbol{K}\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{SS}$ a free cell complex (see Definition 14.1.28). If $i_\alpha\colon \boldsymbol{A}_\alpha \to \boldsymbol{B}_\alpha$*

is a cofibration (resp., trivial cofibration) in $\mathsf{Spc}_{(*)}$ *for every* $\alpha \in \mathrm{Ob}(\mathcal{C})$*, then the induced map of coends*

$$\int^{\alpha} \boldsymbol{A}_{\alpha} \otimes \boldsymbol{K}_{\alpha} \to \int^{\alpha} \boldsymbol{B}_{\alpha} \otimes \boldsymbol{K}_{\alpha}$$

(see Definition ?? and Definition ??) is a cofibration (resp., trivial cofibration) in $\mathsf{Spc}_{(*)}$.

Proof. Proposition 10.3.5 implies that it is sufficient to show that if $p\colon X \to Y$ is a trivial fibration (resp., fibration) in $\mathsf{Spc}_{(*)}$, then the map of simplicial sets

$$\begin{aligned}\mathrm{Map}\Big(\int^{\alpha} \boldsymbol{B}_{\alpha} \otimes \boldsymbol{K}_{\alpha}, X\Big) \\ \to \mathrm{Map}\Big(\int^{\alpha} \boldsymbol{A}_{\alpha} \otimes \boldsymbol{K}_{\alpha}, X\Big) \times_{\mathrm{Map}(\int^{\alpha} \boldsymbol{A}_{\alpha}\otimes \boldsymbol{K}_{\alpha}, Y)} \mathrm{Map}\Big(\int^{\alpha} \boldsymbol{B}_{\alpha} \otimes \boldsymbol{K}_{\alpha}, Y\Big)\end{aligned}$$

is a trivial fibration. Proposition 20.3.17 and Definition 14.3.2 imply that this is isomorphic to the map

$$\begin{aligned}\mathrm{Map}\big(\boldsymbol{K}, \mathrm{Map}(\boldsymbol{B}, X)\big) \\ \to \mathrm{Map}\big(\boldsymbol{K}, \mathrm{Map}(\boldsymbol{A}, X)\big) \times_{\mathrm{Map}(\boldsymbol{K}, \mathrm{Map}(\boldsymbol{A}, Y))} \mathrm{Map}\big(\boldsymbol{K}, \mathrm{Map}(\boldsymbol{B}, Y)\big)\end{aligned}$$

(where, e.g., $\mathrm{Map}(\boldsymbol{B}, X)$ is the $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets in which $\big(\mathrm{Map}(\boldsymbol{B}, X)\big)_{\alpha} = \mathrm{Map}(\boldsymbol{B}_{\alpha}, X)$ for all $\alpha \in \mathrm{Ob}(\mathcal{C})$), and this is isomorphic to the map

$$\mathrm{Map}\big(\boldsymbol{K}, \mathrm{Map}(\boldsymbol{B}, X)\big) \to \mathrm{Map}\big(\boldsymbol{K}, \mathrm{Map}(\boldsymbol{A}, X) \times_{\mathrm{Map}(\boldsymbol{A}, Y)} \mathrm{Map}(\boldsymbol{B}, Y)\big).$$

Since $X \to Y$ is a trivial fibration (resp., fibration) and $\boldsymbol{A}_{\alpha} \to \boldsymbol{B}_{\alpha}$ is a cofibration (resp.. trivial cofibration) for every object α of $\mathcal{C}$, the map

$$\mathrm{Map}(\boldsymbol{B}_{\alpha}, X) \to \mathrm{Map}(\boldsymbol{A}_{\alpha}, X) \times_{\mathrm{Map}(\boldsymbol{A}_{\alpha}, Y)} \mathrm{Map}(\boldsymbol{B}_{\alpha}, Y)$$

is a trivial fibration for every object α of $\mathcal{C}$. Since $\boldsymbol{K}$ is a free cell complex, it is a cofibrant object of $\mathsf{SS}^{\mathcal{C}^{\mathrm{op}}}$, and so the result follows from the simplicial model category structure in $\mathsf{SS}^{\mathcal{C}^{\mathrm{op}}}$. ☐

Theorem 20.6.7. *If* $\mathcal{C}$ *is a small category and* $g\colon \boldsymbol{X} \to \boldsymbol{Y}$ *is a map in* $\mathsf{Spc}_{(*)}^{\mathcal{C}}$ *such that* $g_{\alpha}\colon \boldsymbol{X}_{\alpha} \to \boldsymbol{Y}_{\alpha}$ *is a cofibration (resp., trivial cofibration) in* $\mathsf{Spc}_{(*)}$ *for every* $\alpha \in \mathrm{Ob}(\mathcal{C})$*, then the induced map* $\mathrm{hocolim}\, g\colon \mathrm{hocolim}\, \boldsymbol{X} \to \mathrm{hocolim}\, \boldsymbol{Y}$ *is a cofibration (resp., trivial cofibration) in* $\mathsf{Spc}_{(*)}$.

Proof. This follows from Proposition 20.6.6, Example ??, and Corollary 14.6.8. ☐

Corollary 20.6.8. *If* $\mathcal{C}$ *is a small category and* $\boldsymbol{B}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ *is a diagram such that* $\boldsymbol{B}_{\alpha}$ *is a cofibrant space for every* $\alpha \in \mathrm{Ob}(\mathcal{C})$*, then* $\mathrm{hocolim}\, \boldsymbol{B}$ *is a cofibrant space.*

Proof. This follows from Theorem 20.6.7. ☐

Theorem 20.6.9. *If* $\mathcal{C}$ *is a small category and* $g\colon \boldsymbol{X} \to \boldsymbol{Y}$ *is a map of* $\mathcal{C}$*-diagrams of spaces such that* $g_{\alpha}\colon \boldsymbol{X}_{\alpha} \to \boldsymbol{Y}_{\alpha}$ *is a trivial fibration for every* $\alpha \in \mathrm{Ob}(\mathcal{C})$*, then* g *induces a trivial fibration* $g_{*}\colon \mathrm{holim}\, \boldsymbol{X} \cong \mathrm{holim}\, \boldsymbol{Y}$ *(see Definition 19.1.10).*

PROOF. This follows from Corollary 20.6.4. □

THEOREM 20.6.10. *If $\mathcal{C}$ is a small category and $g\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a map of $\mathcal{C}$-diagrams of spaces such that $g_\alpha\colon \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ is a weak equivalence of fibrant spaces for every $\alpha \in \mathrm{Ob}(\mathcal{C})$, then g induces a weak equivalence $g_*\colon \operatorname{holim} \boldsymbol{X} \cong \operatorname{holim} \boldsymbol{Y}$ (see Definition 19.1.10).*

PROOF. This follows from Corollary 20.6.4 and Corollary 8.5.2. □

THEOREM 20.6.11. *If $\mathcal{C}$ is a small category and $g\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a map of $\mathcal{C}$-diagrams of spaces such that $g_\alpha\colon \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ is a weak equivalence of cofibrant spaces for every $\alpha \in \mathrm{Ob}(\mathcal{C})$, then g induces a weak equivalence $\operatorname{hocolim} \boldsymbol{X} \to \operatorname{hocolim} \boldsymbol{Y}$.*

PROOF. It is sufficient to show that if W is a fibrant space, then the induced map $\mathrm{Map}(\operatorname{hocolim} \boldsymbol{Y}, W) \to \mathrm{Map}(\operatorname{hocolim} \boldsymbol{X}, W)$ is a weak equivalence (see Proposition 10.5.1). This follows from Corollary 20.3.19 and Theorem 20.6.10. □

20.7. Pointed and unpointed homotopy colimits

Given a small category $\mathcal{C}$ and a $\mathcal{C}$-diagram of pointed spaces $\boldsymbol{X}$, we can take the homotopy limit of the diagram in the category of pointed spaces, or we can forget the basepoints of the spaces in the diagram and take the homotopy limit in the category of unpointed spaces, and these two homotopy limits will be isomorphic (or homeomorphic) after we forget the basepoint of the pointed homotopy limit (see Remark 19.1.13). On the other hand, the homotopy colimit of $\boldsymbol{X}$ will generally have different homotopy types when taken in the categories of pointed and unpointed spaces (see Proposition 20.7.4). In this section, we describe the difference between the pointed and unpointed homotopy colimit.

NOTATION 20.7.1. In this section, if $\boldsymbol{X}$ is a diagram of pointed spaces, then $\operatorname{hocolim}_* \boldsymbol{X}$ will denote the homotopy colimit formed in the category of pointed spaces, and $\operatorname{hocolim} \boldsymbol{X}$ will denote the homotopy colimit formed in the category of unpointed spaces after forgetting the basepoints of the spaces in the diagram.

DEFINITION 20.7.2. A pointed space X is *well pointed* if the inclusion of the basepoint into the space is a cofibration in the model category Spc_*. Since the one point space is the initial object in Spc_*, a pointed space X is well pointed if and only if it is a cofibrant space.

PROPOSITION 20.7.3. *If $\mathsf{Spc}_* = \mathsf{SS}_*$, then every pointed space is well pointed.*

PROOF. Every inclusion of simplicial sets is a cofibration. □

The following proposition is due to E. Dror Farjoun ([**22**]).

PROPOSITION 20.7.4. *Let $\mathcal{C}$ be a small category and let $\boldsymbol{X}$ be a $\mathcal{C}$-diagram of pointed spaces.*

- *If $\mathsf{Spc}_* = \mathsf{SS}_*$, then there is a natural cofibration $\mathrm{B}\mathcal{C}^{\mathrm{op}} \to \operatorname{hocolim} \boldsymbol{X}$, and a natural isomorphism $(\operatorname{hocolim} \boldsymbol{X})/(\mathrm{B}\mathcal{C}^{\mathrm{op}}) \approx \operatorname{hocolim}_* \boldsymbol{X}$*
- *If $\mathsf{Spc}_* = \mathsf{Top}_*$, then there is a natural inclusion $\left|\mathrm{B}\mathcal{C}^{\mathrm{op}}\right| \to \operatorname{hocolim} \boldsymbol{X}$ which is a cofibration if $\boldsymbol{X}$ is a diagram of well pointed spaces, and a natural homeomorphism $(\operatorname{hocolim} \boldsymbol{X})/\left(\left|\mathrm{B}\mathcal{C}^{\mathrm{op}}\right|\right) \approx \operatorname{hocolim}_* \boldsymbol{X}$*

(see Notation 20.7.1) where $\mathrm{B}\mathcal{C}^{\mathrm{op}}$ *is the classifying space of the category* $\mathcal{C}^{\mathrm{op}}$ *(see Definition 9.4.1).*

PROOF. This follows from the definition of the homotopy colimit (see Definition 19.1.2), Remark ??, Proposition 19.1.6, and Theorem 20.6.7. $\square$

COROLLARY 20.7.5. *If* $\mathcal{C}$ *is a small category and* $\boldsymbol{X}\colon \mathcal{C} \to \mathsf{Spc}_*$ *is a diagram of well pointed spaces such that, for every object* α *in* $\mathcal{C}$*, the space* $\boldsymbol{X}_\alpha$ *is contractible, then* $\operatorname{hocolim}_* \boldsymbol{X}$ *is contractible (see Notation 20.7.1).*

PROOF. We will prove this in the case $\mathsf{Spc}_* = \mathsf{Top}_*$; the case $\mathsf{Spc}_* = \mathsf{SS}_*$ is similar.

Proposition 20.7.4, Proposition 19.1.6 and Theorem 20.6.11 imply that the map $\left|\mathrm{B}\mathcal{C}^{\mathrm{op}}\right| \to \operatorname{hocolim} \boldsymbol{X}$ is a trivial cofibration. Since the quotient space $(\operatorname{hocolim} \boldsymbol{X})/(\left|\mathrm{B}\mathcal{C}^{\mathrm{op}}\right|)$ is naturally homeomorphic to the pushout of the diagram $* \leftarrow \left|\mathrm{B}\mathcal{C}^{\mathrm{op}}\right| \to \operatorname{hocolim} \boldsymbol{X}$, this implies that the map $* \to \operatorname{hocolim}_* \boldsymbol{X}$ (see Notation 20.7.1) is a trivial cofibration. $\square$

PROPOSITION 20.7.6. *If the classifying space of the small category* $\mathcal{C}$ *is contractible, then, for any* $\mathcal{C}$*-diagram of well pointed spaces* $\boldsymbol{X}$*, the natural map (see Proposition 20.7.4)* $\operatorname{hocolim} \boldsymbol{X} \to \operatorname{hocolim}_* \boldsymbol{X}$ *is a weak equivalence (see Notation 20.7.1).*

PROOF. We will prove this in the case $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$; the case $\mathsf{Spc}_{(*)} = \mathsf{SS}_{(*)}$ is similar.

The quotient space $(\operatorname{hocolim} \boldsymbol{X})/(\left|\mathrm{B}\mathcal{C}^{\mathrm{op}}\right|)$ is naturally homeomorphic to the pushout of the diagram $* \leftarrow \left|\mathrm{B}\mathcal{C}^{\mathrm{op}}\right| \to \operatorname{hocolim} \boldsymbol{X}$. Since Spc is a proper model category (see Theorem 11.1.16), the result now follows from Proposition 20.7.4. $\square$

EXAMPLE 20.7.7. If $\mathcal{C}$ is the category $\cdot \leftarrow \cdot \rightarrow \cdot$ then the homotopy colimit of a $\mathcal{C}$-diagram of well pointed spaces has the same weak homotopy type whether formed in the category of pointed spaces or in the category of unpointed spaces.

EXAMPLE 20.7.8. If $\mathcal{C}$ is the category $\cdot \to \cdot \to \cdot \to \cdots$, then the homotopy colimit of a $\mathcal{C}$-diagram of well pointed spaces has the same weak homotopy type whether formed in the category of pointed spaces or in the category of unpointed spaces.

EXAMPLE 20.7.9. The homotopy colimit of a diagram indexed by a discrete group does not, in general, have the same weak homotopy type when formed in the category of pointed spaces as it does when formed in the category of unpointed spaces.

20.8. The significance of overcategories and undercategories

THEOREM 20.8.1. *If* $\mathcal{C}$ *is a small category,* $\boldsymbol{X}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ *a* $\mathcal{C}$*-diagram of fibrant spaces, and* $\boldsymbol{P}\colon \mathcal{C} \to \mathsf{SS}$ *a cofibrant approximation (see Definition 9.1.1) to the constant* $\mathcal{C}$*-diagram at a point, then* $\int_\alpha (\boldsymbol{X}_\alpha)^{\boldsymbol{P}_\alpha}$ *(see Definition ??) is naturally weakly equivalent to* $\operatorname{holim} \boldsymbol{X}$*.*

PROOF. If we choose a fibrant cofibrant approximation $\boldsymbol{Q}$ (see Definition 9.1.1 and Proposition 9.1.2) to the constant $\mathcal{C}$-diagram at a point, then there are weak equivalences

$$\boldsymbol{P} \to \boldsymbol{Q} \leftarrow \mathrm{B}(\mathcal{C}\downarrow -)$$

(see Proposition 9.1.6). These somewhat arbitrary (see Remark 20.8.3) weak equivalences induce natural transformations

$$\int_\alpha (\boldsymbol{X}_\alpha)^{\boldsymbol{P}_\alpha} \leftarrow \int_\alpha (\boldsymbol{X}_\alpha)^{\boldsymbol{Q}_\alpha} \rightarrow \int_\alpha (\boldsymbol{X}_\alpha)^{\mathrm{B}(\mathcal{C}\downarrow -)} = \operatorname{holim} \boldsymbol{X}$$

which, for a diagram $\boldsymbol{X}$ of fibrant spaces, are weak equivalences (see **something deleted**, if $\mathsf{Spc}_{(*)} = \mathsf{Top}_{(*)}$ then Proposition 14.6.11, if $\mathsf{Spc}_{(*)} = \mathsf{Spc}_*$ then Lemma 14.6.10, Corollary 20.4.7, Theorem 14.2.1 and Corollary 10.2.2), and so the proof is complete. □

COROLLARY 20.8.2. *If $\mathcal{C}$ is a small category, $\boldsymbol{X}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ a $\mathcal{C}$-diagram of fibrant spaces and $\boldsymbol{P}\colon \mathcal{C} \to \mathsf{SS}$ a cofibrant approximation (see Definition* 9.1.1*) to the constant $\mathcal{C}$-diagram at a point, then* $\operatorname{holim} \boldsymbol{X}$ *is naturally weakly equivalent to $\boldsymbol{X}^{\boldsymbol{P}}$ (see Definition* 1.1.11*).*

PROOF. This follows from Theorem 20.8.1 and Example 20.4.4. □

REMARK 20.8.3. The natural weak equivalences constructed in the proof of Theorem 20.8.1 depended on the arbitrary choice of a fibrant cofibrant approximation to the constant diagram at a point and then the arbitrary choice of two weak equivalences. The purpose of this remark is to point out the essential equivalence of the various natural chains of weak equivalences resulting from these choices.

Each of the choices of a weak equivalence connecting a cofibrant approximation to the fibrant cofibrant approximation was actually unique up to simplicial homotopy (see Proposition 9.1.6), and so the weak equivalences of ends that it induced was also unique up to simplicial homotopy (see Example 20.4.4, Corollary 20.4.7 and Proposition 10.4.22).

If we were to make a different choice of fibrant cofibrant approximation, then the two choices would be connected by a unique simplicial homotopy class of simplicial homotopy equivalences (see Corollary 9.1.7). Furthermore, the composition of any map in this simplicial homotopy class of simplicial homotopy equivalences connecting two fibrant cofibrant approximations with the chosen weak equivalence from either of our cofibrant approximations to the fibrant cofibrant approximation is a map that is simplicially homotopic to the weak equivalence we chose to the other fibrant cofibrant approximation. Thus, the natural isomorphism in the homotopy category that is induced by our natural chain of weak equivalences is unique.

In addition, if we used a longer "zig-zag" of weak equivalences of cofibrant approximations, then similar arguments would imply that the isomorphism in the homotopy category that we obtained would still be independent of our chosen "zig-zag" of weak equivalences. Thus, if we use these methods to construct isomorphisms (in the homotopy category) between different ends of functors constructed as above from cofibrant approximations to the constant $\mathcal{C}$-diagram at a point, then any chain of isomorphisms that we construct would equal any other such chain, i.e., these methods construct a unique isomorphism in the homotopy category between any two such functors.

THEOREM 20.8.4. *If $\mathcal{C}$ is a small category, $\boldsymbol{X}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ a $\mathcal{C}$-diagram of cofibrant spaces, and $\boldsymbol{P}\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{SS}$ a cofibrant approximation to the constant $\mathcal{C}^{\mathrm{op}}$-diagram at a point, then $\int^\alpha \boldsymbol{X}_\alpha \otimes \boldsymbol{P}_\alpha$ (see Definition ??) is naturally weakly equivalent to* $\operatorname{hocolim} \boldsymbol{X}$.

PROOF. Since $\operatorname{hocolim} \boldsymbol{X} = \int^{\alpha} \boldsymbol{X}_{\alpha} \otimes \mathrm{B}(-\downarrow \mathcal{C})^{\mathrm{op}}$ (see Example ??), we must show that this is weakly equivalent to $\int^{\alpha} \boldsymbol{X}_{\alpha} \otimes \boldsymbol{P}_{\alpha}$. The proof of this fact is similar to the proof of Theorem 20.8.1, using Proposition 19.2.10. The discussion in Remark 20.8.3 applies to this natural weak equivalence as well. $\square$

20.9. Colimits, homotopy colimits, and total derived functors

PROPOSITION 20.9.1. *If $\mathcal{C}$ is a small category and $\boldsymbol{X}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ is a cofibrant $\mathcal{C}$-diagram of spaces, then the natural map* $\operatorname{hocolim} \boldsymbol{X} \to \operatorname{colim} \boldsymbol{X}$ *is a weak equivalence.*

PROOF. It is sufficient to show that if $W \in \mathsf{Spc}_{(*)}$ is a fibrant space, then $\operatorname{Map}(\operatorname{colim} \boldsymbol{X}, W) \to \operatorname{Map}(\operatorname{hocolim} \boldsymbol{X}, W)$ is a weak equivalence (see Proposition 10.5.1). Since the natural map $\operatorname{hocolim} \boldsymbol{X} \to \operatorname{colim} \boldsymbol{X}$ is isomorphic to the map

$$\int^{\alpha} \boldsymbol{X}_{\alpha} \otimes \mathrm{B}(\alpha \downarrow \mathcal{C})^{\mathrm{op}} \to \int^{\alpha} \boldsymbol{X}_{\alpha} \otimes \boldsymbol{P}_{\alpha}$$

where $\boldsymbol{P}\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{SS}$ is the constant diagram at a point (see Example ?? and Example ??), it is sufficient to show that the map

$$\int_{\alpha} \operatorname{Map}(\boldsymbol{X}_{\alpha}, W^{\boldsymbol{P}_{\alpha}}) \to \int_{\alpha} \operatorname{Map}(\boldsymbol{X}_{\alpha}, W^{\mathrm{B}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}})$$

is a weak equivalence (see Proposition 20.3.17). This last map is exactly the map

$$\operatorname{Map}(\boldsymbol{X}, W^{\boldsymbol{P}}) \to \operatorname{Map}(\boldsymbol{X}, W^{\mathrm{B}(-\downarrow\mathcal{C})^{\mathrm{op}}})$$

(see Definition 14.3.2). Since $\boldsymbol{X}$ is a cofibrant $\mathcal{C}$-diagram and $W^{\boldsymbol{P}} \to W^{\mathrm{B}(-\downarrow\mathcal{C})^{\mathrm{op}}}$ is a weak equivalence of fibrant $\mathcal{C}$-diagrams, the proposition follows from Corollary 10.2.2 and Theorem 14.2.1. $\square$

THEOREM 20.9.2. *If $\mathcal{C}$ is a small category, $\boldsymbol{X}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ a $\mathcal{C}$-diagram of cofibrant spaces and $i\colon \widetilde{\boldsymbol{X}} \to \boldsymbol{X}$ a cofibrant approximation to $\boldsymbol{X}$ (i.e., $\widetilde{\boldsymbol{X}}$ is a cofibrant $\mathcal{C}$-diagram and $i_{\alpha}\colon \widetilde{\boldsymbol{X}}_{\alpha} \to \boldsymbol{X}_{\alpha}$ is a weak equivalence for every $\alpha \in \operatorname{Ob}(\mathcal{C})$), then* $\operatorname{colim} \widetilde{\boldsymbol{X}}$ *is weakly equivalent to* $\operatorname{hocolim} \boldsymbol{X}$.

PROOF. This follows from Theorem 20.6.11 and Proposition 20.9.1. $\square$

COROLLARY 20.9.3. *If $\mathcal{C}$ is a small category, $\boldsymbol{X}, \boldsymbol{Y}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ cofibrant $\mathcal{C}$-diagrams of spaces and $g\colon \boldsymbol{X} \to \boldsymbol{Y}$ a map of diagrams such that for every $\alpha \in \operatorname{Ob}(\mathcal{C})$ the map $g_{\alpha}\colon \boldsymbol{X}_{\alpha} \to \boldsymbol{Y}_{\alpha}$ is a weak equivalence, then $g_*\colon$* $\operatorname{colim} \boldsymbol{X} \to \operatorname{colim} \boldsymbol{Y}$ *is a weak equivalence.*

PROOF. Since a cofibrant $\mathcal{C}$-diagram of spaces is also an $\mathcal{C}$-diagram of cofibrant spaces, this follows from Theorem 20.9.2 and Theorem 20.6.11. $\square$

REMARK 20.9.4. Corollary 20.9.3 implies that the *total left derived functor*

$$\mathbf{L}\operatorname{colim}\colon \operatorname{Ho}\bigl(\mathsf{Spc}_{(*)}^{\mathcal{C}}\bigr) \to \operatorname{Ho}\bigl(\mathsf{Spc}_{(*)}\bigr)$$

(see [**46**, Chapter I, Section 4, Definition 2]) of the functor $\operatorname{colim}\colon \mathsf{Spc}_{(*)}^{\mathcal{C}} \to \mathsf{Spc}_{(*)}$ exists (see [**46**, Chapter I, Section 4, Proposition 1]) and that if $\boldsymbol{X}$ is a cofibrant $\mathcal{C}$-diagram, then $\mathbf{L}\operatorname{colim} \boldsymbol{X}$ is weakly equivalent to $\operatorname{colim} \boldsymbol{X}$. (Note that although there is a natural functor $\operatorname{Ho}\bigl(\mathsf{Spc}_{(*)}^{\mathcal{C}}\bigr) \to \bigl(\operatorname{Ho} \mathsf{Spc}_{(*)}\bigr)^{\mathcal{C}}$ it is not, in general, an equivalence.) Thus, if $\boldsymbol{X}$ is a diagram of cofibrant spaces, then $\operatorname{hocolim} \boldsymbol{X}$ represents $\mathbf{L}\operatorname{colim} \boldsymbol{X}$.

For a discussion of homotopy pushouts and homotopy pullbacks from this point of view, see [**31**].

PROPOSITION 20.9.5. *If $\mathcal{C}$ is a small category and $\boldsymbol{X}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ is a $\mathcal{C}$-diagram of cofibrant spaces, then we define a new $\mathcal{C}$-diagram of spaces $\boldsymbol{X}^c$ as follows: For every $\alpha \in \mathrm{Ob}(\mathcal{C})$, we have the functor $i_\alpha\colon (\mathcal{C}\downarrow\alpha) \to \mathcal{C}$ that takes $(\beta \to \alpha) \in (\mathcal{C}\downarrow\alpha)$ to β, and so we have the induced diagram $i_\alpha^*\boldsymbol{X}$ on $(\mathcal{C}\downarrow\alpha)$ defined on objects by*

$$(i_\alpha^*\boldsymbol{X})_{(\beta\to\alpha)} = \boldsymbol{X}_{(i_\alpha(\beta\to\alpha))} = \boldsymbol{X}_\beta$$

We let $\boldsymbol{X}^c_\alpha = \mathrm{hocolim}_{(\mathcal{C}\downarrow\alpha)}\, i_\alpha^*\boldsymbol{X}$.

1. *$\boldsymbol{X}^c$ is a cofibrant $\mathcal{C}$-diagram.*
2. *The map $\boldsymbol{X}^c \to \boldsymbol{X}$ that for each $\alpha \in \mathrm{Ob}(\mathcal{C})$ is the natural map*
$$\boldsymbol{X}^c_\alpha = \operatorname*{hocolim}_{(\mathcal{C}\downarrow\alpha)} i_\alpha^*\boldsymbol{X} \to (i_\alpha^*\boldsymbol{X})_{1_\alpha} \approx \boldsymbol{X}_\alpha$$
is a weak equivalence of $\mathcal{C}$-diagrams.
3. *There is a natural isomorphism $\mathrm{colim}_{\mathcal{C}}\, \boldsymbol{X}^c \approx \mathrm{hocolim}_{\mathcal{C}}\, \boldsymbol{X}$.*

In particular, $\boldsymbol{X}^c \to \boldsymbol{X}$ is a cofibrant approximation to $\boldsymbol{X}$.

EXAMPLE 20.9.6. If the discrete group G is considered to be a category with one object and X is a G-space, then the construction of Proposition 20.9.5 is known classically as the *Borel construction.*

DEFINITION 20.9.7. A category $\mathcal{C}$ is *right filtering* if $\mathcal{C}$ is non-empty and

1. If α and β are objects of $\mathcal{C}$, then there exists an object γ of $\mathcal{C}$ and morphisms $\alpha \to \gamma$ and $\beta \to \gamma$.
2. If $f, g\colon \alpha \to \beta$ are morphisms of $\mathcal{C}$, then there exists a morphism $h\colon \beta \to \gamma$ such that $hf = hg$.

PROPOSITION 20.9.8. *If $\mathcal{C}$ is a small category that is right filtering (see Definition 20.9.7) and $\boldsymbol{X}: \mathcal{C} \to \mathsf{SS}_{(*)}$ is an $\mathcal{C}$-diagram of simplicial sets, then the natural map (see Example 19.2.6) $\mathrm{hocolim}\,\boldsymbol{X} \to \mathrm{colim}\,\boldsymbol{X}$ is a weak equivalence.*

PROOF. If $f\colon \overline{\boldsymbol{X}} \to \boldsymbol{X}$ is a cofibrant approximation to $\boldsymbol{X}$ (see Definition 9.1.1), then we have the commutative diagram

$$\begin{tikzcd}
\mathrm{hocolim}\,\overline{\boldsymbol{X}} \arrow[r, "\bar{p}"] \arrow[d, "\mathrm{hocolim}\, f"'] & \mathrm{colim}\,\overline{\boldsymbol{X}} \arrow[d, "\mathrm{colim}\, f"] \\
\mathrm{hocolim}\,\boldsymbol{X} \arrow[r, "p"] & \mathrm{colim}\,\boldsymbol{X}
\end{tikzcd}$$

Since $f_\alpha\colon \overline{\boldsymbol{X}}_\alpha \to \boldsymbol{X}_\alpha$ is a weak equivalence of cofibrant spaces for all $\alpha \in \mathrm{Ob}(\mathcal{C})$, Theorem 20.6.11 implies that $\mathrm{hocolim}\, f$ is a weak equivalence. Since $\mathcal{C}$ is right filtering, $\pi_n\, \mathrm{colim}\,\boldsymbol{X} \approx \mathrm{colim}\,\pi_n\boldsymbol{X}$ for every $\mathcal{C}$-diagram $\boldsymbol{X}$, and so $\mathrm{colim}\, f$ is also a weak equivalence. Proposition 20.9.1 implies that $\bar{p}$ is a weak equivalence, and so p is a weak equivalence and the proof is complete. $\square$

PROPOSITION 20.9.9. *If λ is an ordinal and*

$$X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \qquad (\beta < \lambda)$$

is a λ-sequence (see Definition 12.2.1) of relative cell complexes, then the natural map $\mathrm{hocolim}_{\beta<\lambda} X_\beta \to \mathrm{colim}_{\beta<\lambda} X_\beta$ is a weak equivalence.

PROOF. This is identical to the proof of Theorem 20.9.8, since for a λ-sequence of relative cell complexes $\boldsymbol{X}$, we have $\pi_n \operatorname{colim} \boldsymbol{X} \approx \operatorname{colim} \pi_n \boldsymbol{X}$. □

PROPOSITION 20.9.10. *If λ is an ordinal and*

$$\cdots \to X_\beta \to \cdots \to X_1 \to X_0 \qquad (\beta < \lambda)$$

is a tower of fibrant spaces such that each $X_{\beta+1} \to X_\beta$ is a weak equivalence, then the map $\operatorname{holim}_{\beta<\lambda} X_\beta \to X_0$ is a weak equivalence.

PROOF. If we dualize the construction of Proposition 20.9.5 and construct a new tower

$$\cdots \to Y_\beta \to \cdots \to Y_1 \to Y_0 \qquad (\beta < \lambda)$$

in which $Y_\beta = \operatorname{holim}_{\alpha<\beta} X_\alpha$, then for each β the natural map $X_\beta \to Y_\beta$ is a weak equivalence and $\lim_{\beta<\lambda} Y_\beta \approx \operatorname{holim}_{\beta<\lambda} X_\beta$. The tower Y_β is now a tower of trivial fibrations, and so its inverse limit is weakly equivalent to each space in the tower. □

20.10. The category of simplices of a simplicial set

EXAMPLE 20.10.1. If $p\colon E \to B$ is a map of simplicial sets, we will decompose E into a (ΔB)-diagram of simplicial sets $\tilde{p}$. If σ is an n-simplex of B, then the characteristic map of σ is the unique map $\chi_\sigma\colon \Delta[n] \to B$ that takes the non-degenerate n-simplex of $\Delta[n]$ to σ, and we let $\tilde{p}(\sigma)$ be the pullback of the diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "\chi_\sigma"'] & B \end{tikzcd}$$

If $\delta\colon B_n \to B_k$ is a simplicial operator, then δ corresponds to a map $\Delta[k] \to \Delta[n]$, and so we get a map $\tilde{p}(\sigma,\delta)\colon \tilde{p}\big(\delta(\sigma)\big) \to \tilde{p}(\sigma)$. For each simplex σ in B there is an obvious map $\tilde{p}(\sigma) \to E$, and these induce an isomorphism of simplicial sets $\operatorname{colim}_{\sigma\in\operatorname{Ob}(\Delta B)} \tilde{p}(\sigma) \approx E$. We will show in Corollary 20.11.12 that the natural map $\operatorname{hocolim}_{\sigma\in\operatorname{Ob}(\Delta B)} \tilde{p}(\sigma) \to \operatorname{colim}_{\sigma\in\operatorname{Ob}(\Delta B)} \tilde{p}(\sigma) \approx E$ is a weak equivalence.

PROPOSITION 20.10.2. *Let X be a simplicial set, and let ΔX be the category of simplices of X (see Definition 16.1.11).*

1. *If $\boldsymbol{Y}\colon \Delta^{\mathrm{op}}X \to \mathsf{Spc}_{(*)}$ is a diagram, then $\operatorname{hocolim}\boldsymbol{Y}$ (see Definition 19.1.2) is naturally isomorphic to the homotopy colimit of the simplicial space $\boldsymbol{Z}$ (see Definition 16.1.5) for which $\boldsymbol{Z}_n = \coprod_{\sigma\in X_n} \boldsymbol{Y}_\sigma$.*
2. *If $\boldsymbol{Y}\colon \Delta X \to \mathsf{Spc}_{(*)}$ is a diagram, then $\operatorname{holim}\boldsymbol{Y}$ (see Definition 19.1.10) is naturally isomorphic to the homotopy limit of the cosimplicial space $\boldsymbol{Z}$ (see Definition 16.1.7) for which $\boldsymbol{Z}^n = \prod_{\sigma\in X_n} \boldsymbol{Y}_\sigma$.*

PROOF. We will prove part 1; the proof of part 2 is similar.

Definition 19.1.2 describes $\operatorname{hocolim}\boldsymbol{Y}$ as the coequalizer of the diagram

$$\coprod_{(\sigma\to\sigma')\in\Delta^{\mathrm{op}}X} \boldsymbol{Y}_\sigma \otimes \mathrm{B}\big(\sigma'\downarrow(\Delta^{\mathrm{op}}X)\big)^{\mathrm{op}} \rightrightarrows \coprod_{\sigma\in\operatorname{Ob}(\Delta^{\mathrm{op}}X)} \boldsymbol{Y}_\sigma \otimes \mathrm{B}\big(\sigma\downarrow(\Delta^{\mathrm{op}}X)\big)^{\mathrm{op}}.$$

We have the natural isomorphisms

$$\begin{aligned}
\coprod_{(\sigma\to\sigma')\in\Delta^{\mathrm{op}}X} \boldsymbol{Y}_\sigma \otimes \mathrm{B}\big(\sigma'\downarrow(\Delta^{\mathrm{op}}X)\big)^{\mathrm{op}} &\approx \coprod_{\substack{n\geq 0\\ k\geq 0}} \coprod_{\substack{\sigma\in X_n\\ \delta\colon X_n\to X_k}} \boldsymbol{Y}_\sigma \otimes \mathrm{B}\big([k]\downarrow\boldsymbol{\Delta}^{\mathrm{op}}\big)^{\mathrm{op}}\\
&\approx \coprod_{\substack{n\geq 0\\ k\geq 0}} \coprod_{\delta\colon X_n\to X_k} \Big(\coprod_{\sigma\in X_n} \boldsymbol{Y}_\sigma\Big) \otimes \mathrm{B}\big([k]\downarrow\boldsymbol{\Delta}^{\mathrm{op}}\big)^{\mathrm{op}}\\
&\approx \coprod_{\substack{n\geq 0\\ k\geq 0}} \coprod_{\boldsymbol{\Delta}^{\mathrm{op}}([n],[k])} \boldsymbol{Z}_n \otimes \mathrm{B}\big([k]\downarrow\boldsymbol{\Delta}^{\mathrm{op}}\big)^{\mathrm{op}}
\end{aligned}$$

and the natural isomorphisms

$$\begin{aligned}
\coprod_{\sigma\in\mathrm{Ob}(\Delta^{\mathrm{op}}X)} \boldsymbol{Y}_\sigma \otimes \mathrm{B}\big(\sigma\downarrow(\Delta^{\mathrm{op}}X)\big)^{\mathrm{op}} &\approx \coprod_{n\geq 0}\Big(\coprod_{\sigma\in X_n} \boldsymbol{Y}_\sigma\Big) \otimes \mathrm{B}\big(\sigma\downarrow(\Delta^{\mathrm{op}}X)\big)^{\mathrm{op}}\\
&\approx \coprod_{n\geq 0} \boldsymbol{Z}_n \otimes \mathrm{B}\big([n]\downarrow\boldsymbol{\Delta}^{\mathrm{op}}\big)^{\mathrm{op}}
\end{aligned}$$

and so $\operatorname{hocolim}\boldsymbol{Y}$ is naturally isomorphic to the coequalizer of the diagram

$$\coprod_{\substack{n\geq 0\\ k\geq 0}} \coprod_{\boldsymbol{\Delta}^{\mathrm{op}}([n],[k])} \boldsymbol{Z}_n \otimes \mathrm{B}\big([k]\downarrow\boldsymbol{\Delta}^{\mathrm{op}}\big)^{\mathrm{op}} \quad\rightrightarrows\quad \coprod_{[n]\in\mathrm{Ob}(\boldsymbol{\Delta}^{\mathrm{op}})} \boldsymbol{Z}_n \otimes \mathrm{B}\big([n]\downarrow\boldsymbol{\Delta}^{\mathrm{op}}\big)^{\mathrm{op}}$$

which is exactly the definition of $\operatorname{hocolim}\boldsymbol{Z}$. □

20.11. Homotopy invariance

THEOREM 20.11.1. *If $g\colon \boldsymbol{X}\to\boldsymbol{Y}$ is a map of Reedy fibrant cosimplicial spaces such that $g_n\colon \boldsymbol{X}^n\to\boldsymbol{Y}^n$ is a weak equivalence for all $n\geq 0$, then g induces a weak equivalence $g_*\colon \operatorname{Tot}\boldsymbol{X}\to\operatorname{Tot}\boldsymbol{Y}$ (see Definition 20.2.5).*

PROOF. This follows from Theorem 16.3.3, Corollary 16.4.10, Corollary 10.2.2, and Corollary 1.1.9. □

THEOREM 20.11.2. *If $\mathcal{C}$ is a Reedy category and $g\colon \boldsymbol{X}\to\boldsymbol{Y}$ is a weak equivalence of Reedy fibrant diagrams in $\mathsf{Spc}_{(*)}^{\mathcal{C}}$, then the induced map $g_*\colon \operatorname{holim}\boldsymbol{X}\to\operatorname{holim}\boldsymbol{Y}$ is a weak equivalence.*

PROOF. This follows from Theorem 20.6.10 and Proposition 16.4.1. □

THEOREM 20.11.3. *If $\mathcal{C}$ is a Reedy category and $g\colon \boldsymbol{X}\to\boldsymbol{Y}$ is a weak equivalence of Reedy cofibrant diagrams in $\mathsf{Spc}_{(*)}^{\mathcal{C}}$, then the induced map $g_*\colon \operatorname{hocolim}\boldsymbol{X}\to\operatorname{hocolim}\boldsymbol{Y}$ is a weak equivalence.*

PROOF. This is similar to the proof of Theorem 20.6.11, using Theorem 20.11.2 and Lemma 16.4.5. □

COROLLARY 20.11.4. *If $\mathcal{C}$ is a Reedy category and $f\colon \boldsymbol{X}\to\boldsymbol{Y}$ is a weak equivalence of Reedy cofibrant diagrams, then the induced map $f_*\colon \operatorname{colim}\boldsymbol{X}\to\operatorname{colim}\boldsymbol{Y}$ is a weak equivalence.*

PROOF. This follows from Theorem 20.11.10 and Theorem 20.11.3. □

THEOREM 20.11.5. *The total space of a Reedy fibrant cosimplicial space is naturally weakly equivalent to its homotopy limit.*

PROOF. This follows from Corollary 16.4.10 and Theorem 20.11.13. □

THEOREM 20.11.6. *The realization of a Reedy cofibrant simplicial space is naturally weakly equivalent to its homotopy colimit.*

PROOF. This follows from Corollary 16.4.10 and Theorem 20.11.14. □

COROLLARY 20.11.7. *The diagonal of a bisimplicial set (i.e., a simplicial object in $\mathsf{SS}_{(*)}$) is naturally weakly equivalent to its homotopy colimit.*

PROOF. This follows from Theorem 20.11.6, Corollary 16.4.7, and Theorem 16.5.4. □

THEOREM 20.11.8. *If $g\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a map of Reedy cofibrant simplicial spaces such that $g_n\colon \boldsymbol{X}_n \to \boldsymbol{Y}_n$ is a weak equivalence for all $n \geq 0$, then the induced map of realizations $g_*\colon |\boldsymbol{X}| \to |\boldsymbol{Y}|$ is a weak equivalence.*

PROOF. It is sufficient to show that if W is a fibrant space, then the induced map $\mathrm{Map}(|\boldsymbol{Y}|, W) \to \mathrm{Map}(|\boldsymbol{X}|, W)$ is a weak equivalence (see Proposition 10.5.1). This follows from Corollary 20.4.9 and Theorem 20.11.1. □

COROLLARY 20.11.9. *If $\mathsf{Spc}_{(*)} = \mathsf{SS}_{(*)}$ and $g\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a map of simplicial spaces such that $g_n\colon \boldsymbol{X}_n \to \boldsymbol{Y}_n$ is a weak equivalence for all $n \geq 0$, then the induced map of realizations $g_*\colon |\boldsymbol{X}| \to |\boldsymbol{Y}|$ is a weak equivalence.*

PROOF. This follows from Theorem 20.11.8 and Corollary 16.4.7. □

THEOREM 20.11.10. *If $\mathcal{C}$ is a Reedy category and $\boldsymbol{X}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ is a Reedy cofibrant diagram of spaces, then the natural map* $\operatorname{hocolim} \boldsymbol{X} \to \operatorname{colim} \boldsymbol{X}$ *is a weak equivalence.*

PROOF. If W is a space and $\boldsymbol{P}\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{SS}$ is the constant diagram at a point, then the matching maps (see Definition 16.2.17) of the $\mathcal{C}$-diagram $W^{\boldsymbol{P}}$ are either the identity map or the constant map to a point. Thus, if W is a fibrant space, then $W^{\boldsymbol{P}}$ is a Reedy fibrant $\mathcal{C}$-diagram. Corollary 16.4.4 and Lemma 16.4.5 imply that if W is a fibrant space, then the $\mathcal{C}$-diagram $W^{\mathrm{B}(-\downarrow\mathcal{C})^{\mathrm{op}}}$ is also Reedy fibrant. The theorem now follows as in the proof of Proposition 20.9.1. □

PROPOSITION 20.11.11. *If $p\colon E \to B$ is a map of simplicial sets, then the (ΔB)-diagram of simplicial sets constructed in Example* 20.10.1 *is Reedy cofibrant.*

PROOF. The latching map at the n-simplex σ is the inclusion of the part of $\tilde{p}(\sigma)$ above $\partial\Delta[n]$ into $\tilde{p}(\sigma)$. □

The following corollary is a theorem of J. F. Jardine ([**37**, Lemma 2.7]).

COROLLARY 20.11.12. *If $p\colon E \to B$ is a map of simplicial sets and $\tilde{p}\colon \Delta B \to \mathsf{SS}$ is the diagram constructed in Example* 20.10.1, *then the natural map* $\operatorname{hocolim} \tilde{p} \to E$ *is a weak equivalence.*

PROOF. This follows from Theorem 20.11.10 and Proposition 20.11.11. □

THEOREM 20.11.13. *If $\mathcal{C}$ is a Reedy category, $\boldsymbol{X}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ a Reedy fibrant $\mathcal{C}$-diagram of spaces, and $\boldsymbol{P}\colon \mathcal{C} \to \mathsf{Spc}_{(*)}$ a Reedy cofibrant approximation (see Definition* 9.1.1*) to the constant $\mathcal{C}$-diagram at a point, then $\int_\alpha (\boldsymbol{X}_\alpha)^{\boldsymbol{P}_\alpha}$ (see Definition ??) is naturally weakly equivalent to* $\operatorname{holim} \boldsymbol{X}$.

PROOF. This is similar to the proof of Theorem 20.8.1. □

THEOREM 20.11.14. *If $\mathcal{C}$ is a Reedy category, $\boldsymbol{X}: \mathcal{C} \to \mathsf{Spc}_{(*)}$ a Reedy cofibrant $\mathcal{C}$-diagram of spaces, and $\boldsymbol{P}: \mathcal{C}^{\mathrm{op}} \to \mathsf{Spc}_{(*)}$ a Reedy cofibrant approximation (see Definition 9.1.1) to the constant $\mathcal{C}^{\mathrm{op}}$-diagram at a point, then $\int^{\alpha} \boldsymbol{X}_\alpha \otimes \boldsymbol{P}_\alpha$ (see Definition ??) is naturally weakly equivalent to* $\operatorname{hocolim} \boldsymbol{X}$.

PROOF. This is similar to the proof of Theorem 20.8.4. □

20.12. Realizations and homotopy colimits

PROPOSITION 20.12.1. *If X is a simplicial set, ΔX is the category of simplices of X (see Definition 16.1.11), and $\boldsymbol{P}: (\Delta^{\mathrm{op}} X) \to \mathsf{SS}$ is the diagram in which $\boldsymbol{P}_\sigma$ is a single point for all $\sigma \in \mathrm{Ob}(\Delta^{\mathrm{op}} X)$, then* $\operatorname{hocolim} \boldsymbol{P}$ *is naturally weakly equivalent to X.*

PROOF. Proposition 20.10.2 implies that $\operatorname{hocolim} \boldsymbol{P}$ is naturally isomorphic to $\operatorname{hocolim} \boldsymbol{Z}$ where $\boldsymbol{Z}: \boldsymbol{\Delta}^{\mathrm{op}} \to \mathsf{SS}$ is the bisimplicial set (i.e., simplicial simplicial set) such that

$$\boldsymbol{Z}_n = \coprod_{\sigma \in X_n} \boldsymbol{P}_\sigma = \coprod_{\sigma \in X_n} * = X_n$$

(where we view the set X_n as a constant (i.e., discrete) simplicial set). Since the diagonal of $\boldsymbol{Z}$ is naturally isomorphic to the original simplicial set X, the theorem follows from Corollary 20.11.7. □

THEOREM 20.12.2. *If X is a simplicial set and ΔX is the category of simplices of X (see Definition 16.1.11), then $\mathrm{B}(\Delta X)$ is naturally weakly equivalent to X.*

PROOF. Proposition 19.1.6 implies that if $\boldsymbol{P}: (\Delta^{\mathrm{op}} X) \to \mathsf{SS}$ is the diagram in which $\boldsymbol{P}_\sigma$ is a single point for all $\sigma \in \mathrm{Ob}(\Delta^{\mathrm{op}} X)$, then $\mathrm{B}(\Delta X)$ is naturally isomorphic to $\operatorname{hocolim} \boldsymbol{P}$. Proposition 20.12.1 implies that this homotopy colimit is naturally weakly equivalent to X, and so the proof is complete. □

20.13. Topological spaces and simplicial sets

We proved in Proposition 20.3.7 that the geometric realization functor commutes with the realization functor up to a natural homeomorphism, and in Proposition 20.3.14 that the total singular complex functor commutes with the total space functor up to a natural isomorphism. In this section, we show that, for a Reedy cofibrant simplicial topological space, the total singular complex functor commutes with the realization functor up to a natural weak equivalence, and that, for a cosimplicial simplicial set, the geometric realization functor commutes with the total space functor up to a natural weak equivalence.

PROPOSITION 20.13.1. *If $\boldsymbol{X}: \boldsymbol{\Delta}^{\mathrm{op}} \to \mathsf{Top}_{(*)}$ is a Reedy cofibrant simplicial topological space, then there is a natural weak equivalence from the simplicial set that is the realization of the simplicial simplicial set* $\operatorname{Sing} \boldsymbol{X}$ *to* $\operatorname{Sing}|\boldsymbol{X}|$.

PROOF. This is similar to the proof of Proposition 20.5.1. □

PROPOSITION 20.13.2. *If $\boldsymbol{X}: \boldsymbol{\Delta} \to \mathsf{SS}_{(*)}$ is a cosimplicial simplicial set, then there is a natural weak equivalence* $|\operatorname{Tot} \boldsymbol{X}| \to \operatorname{Tot}|\boldsymbol{X}|$.

PROOF. This is similar to the proof of Proposition 20.5.1. □

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