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V. Molotkov

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INFINITE-DIMENSIONAL $\mathbb{Z}_2^k$-SUPERMANIFOLDS *

V. Molotkov **

International Centre for Theoretical Physics, Trieste, Italy.

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** Permanent address: Institute of Nuclear Research and Nuclear Energetics, Sofia, Bulgaria.

ABSTRACT

In this paper the theory of finite-dimensional supermanifolds of Berezin, Leites and Kostant is generalized in two directions.

First, we introduce infinite-dimensional supermanifolds "locally isomorphic" to arbitrary Banach (or, more generally, locally convex) superspaces. This is achieved by considering supermanifolds as functors (equipped with some additional structure) from the category of finite-dimensional Grassman superalgebras into the category of the corresponding smooth manifolds (Banach or locally convex). As examples, flag supermanifolds of Banach superspaces as well as unitary supergroups of Hilbert superspaces are constructed.

Second, we define "generalized" supermanifolds, graded by Abelian groups $\mathbb{Z}_2^k$, instead of the group $\mathbb{Z}_2$ ($\mathbb{Z}_2^k$-supermanifolds). The corresponding superfields, describing, potentially, particles with more general statistics than Bose + Fermi, generally speaking, turn out to have an infinite number of components.

0. Introduction

0.1. One of the purposes of this paper is to extend the theory of finite-dimensional supermanifolds of Berezin-Leites-Kostant /1-5/ so as to include infinite-dimensional supermanifolds "locally isomorphic", in a sense, to arbitrary Banach (or, more generally, locally convex) superspaces.

The other purpose is to construct "$\mathbb{Z}_2^k$-graded" supermanifolds, related in the same manner to $\mathbb{Z}_2^k$-graded superalgebras as ordinary supermanifolds are related to ordinary (i.e. $\mathbb{Z}_2$-graded) superalgebras. In particular, we want to have the correspondence ("super" Lie functor):

$$\mathbb{Z}_2^k\text{-Lie supergroups} \xrightarrow{\text{SLie}} \mathbb{Z}_2^k\text{-graded Lie superalgebras}.$$

0.2. An evident obstacle we are faced with trying to define infinite-dimensional supermanifold is that the language of topological spaces with sheaves of superalgebras of superfields on them is inadequate for infinite dimensions. Hence, to extend the theory of Berezin-Leites-Kostant we are, simultaneously, to reformulate it.

This aim is achieved here by considering, say, Banach supermanifolds (the category of which is denoted further as **SMan**) as functors from the category **Gr** of finite-dimensional real Grassman superalgebras into the category **Man** of smooth Banach manifolds, equipped with some additional structure, whereas supersmooth morphisms of Banach supermanifolds are defined as functor morphisms preserving this structure.

The corresponding structure admits a simple characterization in terms of linear algebra and topology in the functor category $\mathbf{Man}^{\mathbf{Gr}}$. Namely, Banach supermanifolds could be defined locally (as Banach superregions) as open subfunctors of some "linear supermanifolds" constructed out of Banach superspaces (Sect. 2 and 3), whereas supersmooth morphisms of Banach superregions are just those functor morphisms, whose "weak derivative" morphism is a family of linear morphisms (in the sense of Sect.1.3). Globally-supermanifolds could be defined as functors of the category $\underline{\mathrm{Man}}^{\mathrm{Gr}}$, equipped with some supersmooth atlas of Banach superregions.

The arising neglecting functor $\underline{\mathrm{SMan}} \xrightarrow{N} \underline{\mathrm{Man}}^{\mathrm{Gr}}$ could be interpreted as a "geometrization" of the Yoneda point functor for supermanifolds composed with the functor of "restriction to finite-dimensional superpoints" (see Sect. 8.1).

0.3. The category $\underline{\mathrm{Man}}$ of Banach manifolds imbeds in the category $\underline{\mathrm{SMan}}$ of Banach supermanifolds through a generalization of Berezin's "Grassman analytic continuation"[2] (see Sect. 4.2); the natural isomorphism of the subcategory of locally finite-dimensional supermanifolds with the category of supermanifolds of Berezin-Leites-Kostant is established in Sect.4.7.

0.4. Sections 5 to 7 of the present paper are devoted to a development of the theory of Banach supermanifolds along such standard lines as vector bundles (Sect.5), inverse function theorem and related topics (Sect.6), and Lie supergroups (Sect.7). From the author's point of view, the main result here, shedding some additional light on the nature and metaphisics of supermanifolds, is Th.4.4.1 and its Coroll.4.4.2 stating that Banach superalgebras (of any given type) "are" algebras (of the same type) in the category $\underline{\mathrm{SMan}}$ and vice verse.

Among the variety of possible examples of Banach supermanifolds and Banach Lie supergroups there were chosen such as flag supermanifolds (Sect.4.6) and unitary supergroups of Hilbert superspaces (Sect.7.1).

0.5. In Sect.8 is constructed, for any Banach supermanifold $\mathcal{M}$ its super-group of superdiffeomorphisms $\widehat{\mathrm{SDiff}}(\mathcal{M})$ (which is not, generally speaking, a Lie supergroup); besides, for any vector bundle $\mathcal{E}$ in the category $\underline{\mathrm{SMan}}$ there is defined the functor $\hat{\Gamma}(\mathcal{E})$ of its "supersections" which, the author supposes, will play an important role in the theory of infinite-dimensional representations of Lie supergroups due to the fact that actions of Lie supergroups on the vector bundle $\mathcal{E}$ induce linear actions on the functor $\hat{\Gamma}(\mathcal{E})$.

0.6. In Sect.9 the definition of the category $\underline{\mathrm{SMan}}$ is iterated to produce the category $S^k\underline{\mathrm{Man}}$ of "$(\mathrm{super})^k$manifolds", which could be defined as functors of the functor category $S^{k-1}\underline{\mathrm{Man}}$ (equipped with some additional structure) or,

equivalently, as functors of the functor category $\underline{\mathrm{Man}}^{\mathrm{Gr}^k}$. The main result here is Th.9.2.1 stating that the category of $\mathbb{Z}_2^k$-graded Banach superalgebras of any type is equivalent to the category of "ordinary" algebras of the same type in the category $S^k\underline{\mathrm{Man}}$. Besides, a big part of elementary differential geometry (inverse function theorem, Lie theory, etc.) generalizes literally to the case of the category $S^k\underline{\mathrm{Man}}$.

0.7. To conclude with, note that one can extend as well the theory of locally convex and tame Frèchet smooth manifolds[19] defining locally convex (resp. tame Frèchet) supermanifolds. The counterpart of Nash-Moser inverse function theorem is valid for tame Frèhet supermanifolds as well. This generalization will be considered in a separate publication.

Notations and conventions. Throughout the paper $\underline{\mathrm{Set}}$, $\underline{\mathrm{Top}}$, $\underline{\mathrm{Man}}$, $\underline{\mathrm{VBun}}$ denote the category of sets, topological spaces, smooth Banach manifolds and smooth real Banach vector bundles, respectively; $\underline{\mathrm{Gr}}$ will denote the full subcategory of the category of real Grassman finite-dimensional superalgebras, containing exactly one Grassman superalgebra Λ_i with i odd generators (in particular, $\Lambda_{\bar{0}} = \mathbb{R}$).

The category of $\mathcal{D}$-valued functors defined on the category $\mathcal{C}$ is denoted $\mathcal{D}^{\mathcal{C}}$; the class of objects of the category $\mathcal{D}$ is denoted here as $|\mathcal{D}|$, whereas the set of morphisms of an object $X \in |\mathcal{D}|$ into an object $Y \in |\mathcal{D}|$ will be denoted, as a rule, as $\mathcal{D}(X,Y)$.

If a category $\mathcal{D}$ is a category with a terminal object, the latter will be denoted as p and all morphisms of the type $p \longrightarrow X$ will be called points of X.

All vector spaces and superspaces are over the field $K = \mathbb{R}$ or $K = \mathbb{C}$.

For $i \in \mathbb{Z}$ denote $\bar{i}$ the element $i(\mathrm{mod}\,2) \in \mathbb{Z}_2$; $(-1)^{\bar{0}} = 1$ and $(-1)^{\bar{1}} = -1$.

The variables Λ, Λ', etc. will run over the set of objects of the category $\underline{\mathrm{Gr}}$ (the sole exception being Sect. 9, where it is permitted for Λ to run over "generalized" Grassman superalgebras).

1. Linear Algebra in Categories

This section deals with such things as algebras, superalgebras, polylinear morphisms, etc., in categories with finite products.

Throughout the section $\mathcal{D}$ will be some fixed category with finite products.

The most compact way to define an algebraic structure of some type $\mathcal{T}$ on an object X belonging to $\mathcal{D}$ is to use the Yoneda imbedding $\mathcal{D} \xrightarrow{H_*} \underline{Set}^{\mathcal{D}^\circ}$ (see, e.g., Ref. /6/). In what follows, $\mathcal{D}$ will be identified with its image in the functor category $\underline{Set}^{\mathcal{D}^\circ}$. The fact that H_* commutes with products permits one to define a structure of type $\mathcal{T}$ on an object X pointwise, reducing it to the case $\mathcal{D} = \underline{Set}$ (see §11 of Ref. /6/).

1.1. Rings in categories. For example, an object R of $\mathcal{D}$ together with morphisms $R \times R \xrightarrow{+} R$, $R \times R \xrightarrow{\bullet} R$ and $p \xrightarrow{e} R$, where p is the terminal object in $\mathcal{D}$, is said to be the (commutative) ring with unity in the category $\mathcal{D}$ if for every object Y of the category $\mathcal{D}$ the triple $(R(Y), \bullet_Y, +_Y)$ is a (commutative) ring and $e_Y(p)$ is the unity of this ring. Recall that we have identified the object R with the functor $H_*(R) = H_R$, whereas morphisms e, + and $\bullet$ with the corresponding functor morphisms $e = \{e_Y\}_{Y\in|\mathcal{D}|}$, $+ = \{+_Y\}_{Y\in|\mathcal{D}|}$ and $\bullet = \{\bullet_Y\}_{Y\in|\mathcal{D}|}$, respectively.

To the end of section 1 R will be some fixed commutative ring with unity in the category $\mathcal{D}$.

1.2. R-Modules. An object V of $\mathcal{D}$ together with a morphism $R \times V \xrightarrow{\rho} V$ is called an R-module if for every $Y \in |\mathcal{D}|$ the pair $(V(Y), \rho_Y)$ is an R(Y)-module (all modules over commutative rings with unity in the category Set are supposed to be unitary). Given two R-modules V and V' a functor morphism $f: V \longrightarrow V'$ is morphism of R-modules if for every $Y \in |\mathcal{D}|$ the map $f_Y: V(Y) \longrightarrow V'(Y)$ is morphism of R(Y)-modules.

The category $\underline{Mod}_R(\mathcal{D})$ of R-modules of the category $\mathcal{D}$ is an additive category; it has, in particular, direct sums and zero object.

The category $\underline{Mod}_K(\underline{Top})$ coincides, evidently, with the category of topological vector spaces over the field K, whereas the category

$\underline{Mod}_K(\underline{Man})$ is the category of Banach spaces over K.

1.3. Polylinear morphisms. Let $V_1, \ldots, V_n$, V be R-modules and Z be some object of $\mathcal{D}$. A morphism $f: Z \times V_1 \times \ldots \times V_n \longrightarrow V$ will be called a Z-family of R-n-linear morphisms if for every $Y \in |\mathcal{D}|$ the map $f_Y: Z(Y) \times V_1(Y) \times \ldots \times V_n(Y) \longrightarrow V(Y)$ is Z(Y)-family of R(Y)-n-linear maps, i.e. if for every $z \in Z(Y)$ the partial map $f_Y(z, \cdot, \ldots, \cdot)$ sending $V_1(Y) \times \ldots \times V_n(Y)$ into $V(Y)$ is R(Y)-n-linear. The set $L^n_R(Z; V_1, \ldots, V_n; V)$ of Z-families of R-n-linear morphisms of $V_1 \times \ldots \times V_n$ into V is canonically equipped with the structure of Abelian group ($(f+f')_Y = f_Y + f'_Y$).

In particular, a morphism $f: V_1 \times \ldots \times V_n \longrightarrow V$ is called R-n-linear if it is a p-family of R-n-linear morphisms for the terminal object p. The corresponding Abelian group of R-n-linear morphisms will be denoted as $L^n_R(V_1, \ldots, V_n; V)$ or, simply, as $L_R(V_1, \ldots, V_n; V)$.

Note that the correspondence $f \longmapsto \rho \circ (1_R \times f)$, where f belongs to $L_R(V_1, \ldots, V_n, V)$ and $\rho: R \times V \longrightarrow V$ is the R-module structure of V, defines the natural isomorphism

$$L_R(V_1, \ldots, V_n; V) \underset{\approx}{\longrightarrow} L_R(R, V_1, \ldots, V_n; V) \qquad (1.3.1)$$

of Abelian groups.

1.4. R-Algebras. 1.4.1. An R-module A together with an R-bilinear morphism $A \times A \xrightarrow{\mu} A$ is called an R-algebra; R-algebra A is said to be (anti)commutative, resp. associative, resp. Lie, resp. Jordan algebra, if for every $Y \in |\mathcal{D}|$ the R(Y)-algebra $(A(Y), \rho_Y)$ is (anti)-commutative, resp. associative, etc..

If A is associative (resp. Lie) R-algebra, then a pair $(V, A \times V \xrightarrow{\rho} V)$ is called left A-module if V is an R-module and ρ is an R-bilinear morphism such that for every $Y \in |\mathcal{D}|$ the pair $(V(Y), \rho_Y)$ is the left A(Y)-module. Morphisms of R-algebras and of left modules are defined in an obvious way.

1.4.2. We leave it to the reader to define the general notion of R-algebra of a type $\mathcal{T}$ as a sequence $V_1, \ldots, V_n$ of R-modules ("ground objects") equipped with a sequence $f_1, \ldots, f_i, \ldots$ of R-polylinear morphisms defined on them ("ground operations"), satisfying some set of "laws" of the type $g = 0$, where g is an R-polylinear morphism constructed in a finite number of steps from the ground operations by means of compositions like $h\circ(h_1\times\ldots\times h_m)$ with R-polylinear h, $h_1,\ldots,h_m$, addition of R-polylinear morphisms, as well as compositions of R-polylinear morphisms with canonical isomorphisms of the type $V\times V'\overset{\varepsilon}{\approx}V'\times V$ and $(V\times V')\times V''\approx V\times(V'\times V'')$ arising from the commutativity and associativity of products; the number n of ground objects, the "spectrum" of ground operations as well as "laws" - all depending on the type $\mathcal{T}$. Morphisms of algebras of type $\mathcal{T}$ can be defined as families of R-linear morphisms sending every ground object of one algebra into the corresponding ground object of another and commuting with every ground operation.

The category of R-algebras of the type $\mathcal{T}$ in the category $\mathcal{D}$ will be denoted as $\mathcal{T}_R(\mathcal{D})$.

1.4.3. Example. Let the type $\mathcal{T}$ be "left modules over Lie algebras". Then there are two ground objects A and V, two ground operations $A\times A\xrightarrow{\mu}A$ and $A\times V\xrightarrow{\rho}V$, and three "laws": threelinear Jacobi identity and bilinear anticommutativity law for μ, as well as threelinear identity stating that V is left A-module. The Jacobi identity, for example, can be written, up to canonical isomorphisms of associativity of products, as $\sum\mu\circ(1_A\times\mu)\circ\sigma=0$, where the sum runs over "even" permutation isomorphisms $A\times A\times A\xrightarrow{\sigma}A\times A\times A$, arising from the commutativity of products.

1.4.4. Remark. Another, more invariant and consistent (but more involved at the same time), way to define R-algebras of type $\mathcal{T}$ is, following some ideas of Lowvere[7] (see also §18 of Ref.[6]), to define "type" $\mathcal{T}$ as an additive strict monoidal category with some additional structures, whereas R-algebras of type $\mathcal{T}$ in the category $\mathcal{D}$ to define as functors (preserving all of the structures involved) from the category $\mathcal{T}$ into the "category of R-polylinear morphisms" of the category $\mathcal{D}$. We asume here more naive point of view on "universal polylinear algebra" in categories, hoping to present constructions a là Lowvere in a more complete version of this work.

For the reader unsatisfied of "do-it- yourself" prescriptions in the "definition" of R-algebras of the type $\mathcal{T}$ the author should note that for all practical purposes of the present work the variable $\mathcal{T}$ of type could be assumed to run over the following finite set: "modules", "algebras", "commutative (resp. associative, resp. Lie) algebras" and "modules over associative (or Lie) algebras".

1.5. Internal functors of polylinear morphisms. Let n be a natural number. The functor $\mathcal{L}^n_R$: $(\underline{\mathrm{Mod}}_R(\mathcal{D})^{\circ})^n\times\underline{\mathrm{Mod}}_R(\mathcal{D})\longrightarrow\underline{\mathrm{Mod}}_R(\mathcal{D})$ such that there exists the functor isomorphism

$$L_R(W,\mathcal{L}^n_R(V_1,\ldots,V_n;V))\xrightarrow{\rho_n}L^{n+1}_R(W,V_1,\ldots,V_n;V)\ , \qquad (1.5.1)$$

will be called internal L^n_R-functor. Of course, the functors $\mathcal{L}^n_R$ not necessarily exist (excepting the trivial case $n=0$, when $\mathcal{L}^0_R=\mathrm{Id}$).

Let for a given n the functor $\mathcal{L}^n_R$ exists. Setting in the Eq.(1.5.1) $W=\mathcal{L}^n_R(V_1,\ldots,V_n;V)$ define the R-(n+1)-linear evaluation morphism

$$ev_n\colon\mathcal{L}^n_R(V_1,\ldots,V_n;V)\times V_1\ \ldots\times V_n\longrightarrow V \qquad (1.5.2)$$

as follows: $ev_n:=\rho_n(\mathrm{Id}_{\mathcal{L}^n_R(V_1,\ldots V_n;V)})$. The Yoneda lemma (see, e.g., Ref.[6]) implies that for every $f\in L_R(W,\mathcal{L}^n_R(V_1,\ldots,V_n;V))$ the identity

$$\rho_n(f)=ev_n\circ(f\times\mathrm{Id}_{V_1}\times\ldots\times\mathrm{Id}_{V_n}) \qquad (1.5.3)$$

holds.

The r.h.s. of (1.5.3) is defined when f is an arbitrary morphism with codomain $\mathcal{L}^n_R(V_1,\ldots,V_n;V)$ generating thus (due to the R-polylinearity of ev_n) the morphisms

$$L^m_R(W_1,\ldots,W_m;\mathcal{L}^n_R(V_1,\ldots,V_n;V))\xrightarrow{{}_m\rho_n}L^{m+n}_R(W_1,\ldots,V_n;V) \qquad (1.5.4)$$

and

$$\mathcal{D}(Z,\mathcal{L}^n_R(V_1,\ldots,V_n;V))\xrightarrow{\tilde{\rho}_n}L^n_R(Z;V_1,\ldots,V_n;V) \qquad (1.5.5)$$

which, evidently, are natural on all of the arguments.

The functor $\mathcal{L}^n_R$ will be called algebraically coherent if the functor morphisms ${}_m\rho_n$ are isomorphisms for all m; it will be called coherent if, in addition, ρ_n is an isomorphism. The category $\mathcal{D}$ will be said to have (algebraically) coherent $\mathcal{L}_R$-functors if for every $n \in \mathbb{N}$ there exists the functor $\mathcal{L}_R^n$ which is (algebraically) coherent.

If $\mathcal{D}$ has algebraically coherent $\mathcal{L}_R$-functors, one can easily construct the functor isomorphisms

$$\mathcal{L}_R^m(W_1,\dots,W_m;\mathcal{L}_R^n(V_1,\dots,V_n;V)) \xrightarrow{{}_n\hat{\rho}_n} \mathcal{L}_R^{n+m}(W_1,\dots,V_n;V) \qquad (1.5.6)$$

"internalizing" the isomorphisms (1.5.4).

Moreover, one can define in this case the R-bilinear <u>internal composition</u> morphism

$$\text{comp}: \mathcal{L}_R(V',V'') \times \mathcal{L}_R(V,V') \longrightarrow \mathcal{L}_R(V,V'') \qquad (1.5.7)$$

as the inverse image by ${}_2\rho_1$ of the morphism

$$\mathcal{L}_R(V',V'') \times \mathcal{L}_R(V,V') \times V \xrightarrow{Id \times ev_1} \mathcal{L}_R(V',V'') \times V' \xrightarrow{ev_1} V'' . \qquad (1.5.8)$$

Taking in (1.5.7) $V = V' = V''$ one can verify that the multiplication $\text{comp}: \mathcal{L}_R(V,V)^2 \longrightarrow \mathcal{L}_R(V,V)$ turns $\mathcal{L}_R(V,V)$ into an associative algebra with unity

$$R \xrightarrow{e} \mathcal{L}_R(V,V) \qquad (1.5.9)$$

defined as the image of Id_V by the isomorphism

$$L_R(V,V) \xrightarrow{\sim} L_R(R,V;V) \xrightarrow{\rho_1} L_R(R,\mathcal{L}_R(V,V)) . \qquad (1.5.10)$$

The reader could verify that the existence of algebraically coherent $\mathcal{L}_R^1$-functor implies the existence of algebraically coherent $\mathcal{L}_R^n$-functors for every $n \in \mathbb{N}$.

Let now $\mathcal{D}$ have coherent $\mathcal{L}_R$-functors. Taking $Z = p$ (the terminal object in $\mathcal{D}$) in (1.5.4) we obtain the canonical isomorphism which permits us to identify the Abelian group of points of $\mathcal{L}_R^n(V_1,\dots,V_n;V)$ with the Abelian group $L_R^n(V_1,\dots,V_n;V)$.

Let us reinterpret the functor morphism $\tilde{\rho}_n$ defined by (1.5.4) in terms of R-modules in the functor category $\hat{\mathcal{D}} := \underline{Set}^{\mathcal{D}^\circ}$. Equip the set $L_R^n(Z;V_1,\dots,V_n;V)$ of all Z-families of R-polylinear morphisms of $V_1 \times \dots \times V_n$ into V with the structure of R(Z)-module, defining the multiplication of a morphism $f: Z \times V_1 \times \dots \times V_n \longrightarrow V$ belonging to $L_R^n(Z;V_1,\dots,V_n;V)$ on some morphism $r: Z \longrightarrow R$ in $R(Z)$ by means of Yoneda imbedding $\mathcal{D} \hookrightarrow \hat{\mathcal{D}}$ as follows:

$$(rf)_Y(z,v_1,\dots,v_n) = r_Y(z) f_Y(z,v_1,\dots v_n), \ z \in Z(Y), \ v_i \in V_i(Y) \qquad (1.5.11)$$

Then the functor $\hat{\mathcal{L}}_R^n(V_1,\dots,V_n;V)$ in the functor category $\hat{\mathcal{D}}$ defined by the equation

$$\hat{\mathcal{L}}_R^n(V_1,\dots,V_n;V)(Z) := L_R^n(Z;V_1,\dots,V_n;V) \qquad (1.5.12)$$

turns actually into an R-module in the functor category $\hat{\mathcal{D}}$. If there exists in $\mathcal{D}$ the functor $\mathcal{L}_R^n$, the morphisms ρ_n defined by (1.5.4) turn out to be R(Z)-linear, producing together (when Z runs in $\mathcal{D}$) some morphism

$$\mathcal{L}_R^n(V_1,\dots,V_n;V) \xrightarrow{\hat{\rho}_n} \hat{\mathcal{L}}_R^n(V_1,\dots,V_n;V) \qquad (1.5.13)$$

of R-modules in $\hat{\mathcal{D}}$.

The existence of coherent $\mathcal{L}_R$-functors in $\mathcal{D}$ implies, hence, that R-modules $\hat{\mathcal{L}}_R^n(V_1,\dots,V_n;V)$ are representable. The inverse is not true as shaws Example 3 below.

<u>Examples</u>. 1) In the category <u>Man</u> of smooth Banach manifolds there exist coherent $\mathcal{L}_K$-functors: $\mathcal{L}_K^n(V_1,\dots,V_n;V)$ is $L_K^n(V_1,\dots,V_n;V)$ equipped with the topology of uniform convergence on bounded sets/8/.

2) In the category of Banach manifolds of class C°(i.e. continuous) there exist algebraically coherent $\mathcal{L}_K$-functors, defined as in example 1 above, which are not coherent.

3) For every functor category $\hat{\mathcal{C}} := \underline{Set}^{\mathcal{C}^\circ}$ and every R-modules $V_1,\dots,V_n,V$ in $\hat{\mathcal{C}}$ the R-module $\hat{\mathcal{L}}_R^n(V_1,\dots,V_n;V)$ in $\hat{\hat{\mathcal{C}}} := \underline{SET}^{\hat{\mathcal{C}}^\circ}$, defined by (1.5.12), is representable by the functor in $\hat{\mathcal{C}}$ obtained by restriction of the argument Z in (1.5.12) to the subcategory $\mathcal{C}$ of $\hat{\mathcal{C}}$. Nevertheless, the category $\underline{Set}^{\underline{Gr}^{(1)}}$ gives an example of a topos with no internal L_R^n-functors, excepting the trivial $\mathcal{L}_R^o$, if one takes, say, R to be the constant ring $R(\Lambda) = \mathbb{R}$ (the category $\underline{Gr}^{(1)}$ is defined in Sect. 3.4 below).

1.6. <u>Tensor product</u>. The category $\mathcal{D}$ will be said to have <u>tensor products</u> over R if for every R-modules $V_1,\dots,V_n$, there exists an R-module $V_1 \otimes_R \dots \otimes_R V_n$ and a natural isomorphism

$$L_R(V_1 \otimes_R \dots \otimes_R V_n;W) \xrightarrow{\sim} L_R^n(V_1,\dots,V_n;W) \ . \qquad (1.6.1)$$

In close analogy with construction of functor isomorphisms ${}_m\rho_n$ of the preceeding section we can define the functor morphisms

$$L_R(V_1\otimes_R\dots\otimes_R V_n, V_{n+1},\dots,V_{n+m};W)\longrightarrow L_R(V_1,\dots,V_{n+m};W) \quad (1.6.2)$$

The category $\mathcal{D}$ will be said to have coherent tensor products over the ring R if all of the morphisms (1.6.2) are isomorphisms.

If $\mathcal{D}$ has coherent tensor products over R then canonical isomorphisms (1.3.1) generate natural (on V) isomorphisms

$$R\otimes_R V \overset{\lambda_V}{\simeq} V \overset{\rho_V}{\simeq} V\otimes_R R \quad (1.6.3)$$

and the tensor product can (and will) be chosen in such a way that $\lambda_V = \rho_V = Id_V$ for every R-module V .

Examples. 1) The category Man has coherent tensor products over the field K (completion of the algebraic tensor product w.r.t. the projective topology[8]).

2) For every category $\mathcal{C}$ the corresponding category $\hat{\mathcal{C}}$ of Set-valued functors has coherent tensor products which can be defined pointwise:

$$(V\otimes_R V')(X) = V(X)\otimes_{R(X)} V'(X) \ . \quad (1.6.4)$$

Note that the Yoneda imbedding $\mathcal{C}\hookrightarrow\hat{\mathcal{C}}$ does not commute, in general, with tensor products (counterexample: $\mathcal{C}$ = Man and $R = \mathbb{R}$).

1.7. R-Supermodules. An R-module V in the category $\mathcal{D}$ together with a fixed direct sum decomposition

$$V = {}_{\bar 0}V + {}_{\bar 1}V \qquad (\ \bar{i} := i(\mathrm{mod}2)\in \mathbb{Z}_2\ ) \quad (1.7.1)$$

will be called an R-supermodule. The submodule ${}_{\bar 0}V$ (resp. ${}_{\bar 1}V$) is even (resp. odd) submodule of V . Morphisms of R-supermodules are defined as morphisms of underying R-modules commuting with the corresponding direct sum decompositions.

The category of R-supermodules in $\mathcal{D}$ will be denoted $\mathrm{SMod}_R(\mathcal{D})$.

Let $V_1,\dots,V_n$ be R-supermodules. The fact that the functor $L^n_R(Z;\dots;.)$ commutes with (finite) direct sums permits one canonically equip $L^n_R(Z;V_1,\dots,V_n;V)$ with the structure of R(Z)-supermodule as follows:

$${}_{\tau}L^n_R(Z;V_1,\dots,V_n;V) := \bigoplus L^n_R(Z;{}_{\varepsilon_1}V_1,\dots,{}_{\varepsilon_n}V_n;{}_{\varepsilon}V) \ \ (\textstyle\sum\varepsilon_i+\varepsilon=\tau) \quad (1.7.2)$$

In particular, $V(Z) = L_R(Z;V)$ is an R-supermodule if V is.

Algebraically coherent $\mathcal{L}_R$-functors and/or coherent tensor products, if they exist , commute with direct sums which permits one to define the structure of R-supermodules on $\mathcal{L}^n_R(V_1,\dots,V_n;V)$ and $V_1\otimes_R\dots\otimes_R V_n$ by means of direct sum decompositions similar to (1.7.2).

Noting that the set of morphisms of an R-supermodule V into an R-supermodule V' is naturally isomorphic to ${}_{\bar 0}L(V,V')$ and that the natural isomorphisms (1.5.1) and (1.6.1) are actually morphisms of R(p)-supermodules, one can see that the R-supermodule $V_1\otimes_R\dots\otimes_R V_n$ (resp. $\mathcal{L}^n_R(V_1,\dots,V_n;V)$) represents (resp. corepresents) the corresponding functor of even polylinear morphisms. In particular, the canonical evaluation and internal composition morphisms, defined by eqs.(1.5.2) and (1.5.7), respectively, are even polylinear morphisms of R-supermodules.

1.8. Change of parity functor. Define the functor

$$\Pi:\ \mathrm{SMod}_R(\mathcal{D}) \longrightarrow \mathrm{SMod}_R(\mathcal{D}) \quad (1.8.1)$$

(called the change of parity functor) as follows:

$${}_{\varepsilon}(\Pi V) = {}_{\bar 1+\varepsilon}V\ ; \qquad \Pi(f) = f\ . \quad (1.8.2)$$

The fact that every (even) R-polylinear morphism $f:\ V_1\times\dots\times V_n\longrightarrow V$ "is" at the same time an (even) R-polylinear morphism $f: V\times\dots\times\Pi V_n\to\Pi V$ permits one to construct the natural isomorphism

$$\mathcal{L}^n_R(V_1,\dots,V_n;V)\longrightarrow\mathcal{L}^n_R(V_1,\dots,\Pi V_n;\Pi V) \quad (1.8.3)$$

using the isomorphisms (1.5.1).

1.9. R-Superalgebras. Let $V_1,\dots,V_n$ be R-supermodules and $\mathfrak{S}_n$ be the permutation group of the set $\{1,\dots,n\}$. On the union $\bigcup_{\sigma\in\mathfrak{S}_n} L^n_R(V_{\sigma 1},\dots,V_{\sigma n};V)$ the "graded" right action of the permutation group $\mathfrak{S}_n$ can be defined in such a way that for the generator σ_j := = (j,j+1) belonging to $\mathfrak{S}_n$ this action is determined by the equations

$$(f.\sigma_j)_Y(v_1,\dots,v_n) = \sum_{\varepsilon,\varepsilon'\in\mathbb{Z}_2}(-1)^{\varepsilon\varepsilon'} f_Y(v_1,\dots,{}_{\varepsilon}v_{j+1},{}_{\varepsilon'}v_j,\dots,v_n) \quad (1.9.1)$$

Here again $f = \{f_Y\}_{Y\in\mathcal{D}}$ is identified with the corresponding natural transformation of functors through Yoneda imbedding, v_i are arbitrary elements of the R(Y)-supermodule $V_i(Y)$, whereas ${}_{\bar 0}v_i$ (resp. ${}_{\bar 1}v_i$) denotes the even (resp. odd) part of the element v_i (i.e. $v_i = {}_{\bar 0}v_i + {}_{\bar 1}v_i;\ {}_{\varepsilon}v_i \in {}_{\varepsilon}V(Y)$)

An R-n-linear morphism f belonging to an R(p)-supermodule $L^n_R(V;V') := L^n_R(V,\dots,V;V')$ will be called supersymmetric if it is invariant w.r.to the action of $\mathfrak{S}_n$ defined above. Denote $\mathrm{Sym}^n_R(V;V')$ the set of all supersymmetric morphisms of V^n into V' ; it is in fact an R(p)-subsupermodule of $L^n_R(V;V')$.

Replacing now in the definition of R-algebras of type $\mathcal{T}$ ground objects by R-supermodules, ground operations by even polylinear morphisms, and replacing in every "law" $g=0$ every composition $f\circ\sigma$ of an R-polylinear morphism f with canonical isomorphism σ of commutativity of products by its "Z_2-graded" counterpart $f;\sigma$, defined by eq.(1.9.1), we will arrive to the definition of R-superalgebras of the type $\mathcal{T}$ in the category $\mathcal{D}$; the corresponding category will be denoted as $S\mathcal{T}_R(\mathcal{D})$.

1.10.Z^k-graded R-superalgebras. To define $\mathbb{Z}_2^k$-graded R-supermodules we have simply to replace the direct sum decomposition (1.7.1) by the decomposition $V = \bigoplus_{\varepsilon\in\mathbb{Z}_2^k} {}_\varepsilon V$. Given $\mathbb{Z}_2^k$-graded R-supermodules $V_1,\dots,V_n,V$, the modules $L^n_R(V_1,\dots,V_n;V)$, $\mathcal{L}^n_R(V_1,\dots,V_n;V)$ and $V_1\otimes_R\dots\otimes_R V_n$ could be canonically turned into $\mathbb{Z}_2^k$-graded supermodules just as it was made in subsect.1.7 for the case k=1. In order to define $\mathbb{Z}_2^k$-graded superalgebras of some type $\mathcal{T}$ we need as well to introduce the $\mathbb{Z}_2^k$-graded action of permutation groups on R-polylinear morphisms. This is done by setting in the counterpart of eq.(1.9.1) for the case of arbitrary k, the indices ε and ε' to belong to $\mathbb{Z}_2^k$, and defining the factor $(-1)^{\varepsilon\varepsilon'}$ for $\varepsilon=(\varepsilon_1,\dots,\varepsilon_k)$ and $\varepsilon'=(\varepsilon'_1,\dots,\varepsilon'_k)$ as $(-1)^{\varepsilon\varepsilon'}=\prod_{i=1}^{k}(-1)^{\varepsilon_i\varepsilon'_i}$.

1.11.Linear algebra in functor categories. Let $\mathcal{D}=\mathcal{C}^{\mathcal{C}'}$ be the functor category. R-(super)algebras of a given type $\mathcal{T}$ in $\mathcal{C}^{\mathcal{C}'}$ could be reduced to R-(super)algebras of the same type $\mathcal{T}$ in the category $\mathcal{C}$ just in the same way as algebras in arbitrary categories are reduced to algebras in Set by means of Yoneda imbedding. For example, an algebra A of a type $\mathcal{T}$ in $\mathcal{C}^{\mathcal{C}'}$ defines for every $Y\in|\mathcal{C}'|$ the R(Y)-algebra of the type $\mathcal{T}$ in $\mathcal{C}$.

2. Superrepresentable Modules in Functor Categories.

In this section some classes of "vector spaces" in the functor categories $\underline{\mathrm{Set}}^{\mathrm{Gr}}$, $\underline{\mathrm{Top}}^{\mathrm{Gr}}$ and $\underline{\mathrm{Man}}^{\mathrm{Gr}}$ are introduced, playing the crucial part in the theory of supermanifolds. All the definitions and results are given here, for the case of the category $\underline{\mathrm{Set}}^{\mathrm{Gr}}$,but could be literally applied to the categories $\underline{\mathrm{Top}}^{\mathrm{Gr}}$ and $\underline{\mathrm{Man}}^{\mathrm{Gr}}$ as well.

Define in the functor category $\underline{\mathrm{Set}}^{\mathrm{Gr}}$ the functor $\overline{\mathbb{R}}$ as follows:

$$\overline{\mathbb{R}}(\Lambda) := {}_{\bar 0}(\Lambda\otimes_{\mathbb{R}} V) = {}_{\bar 0}\Lambda , \quad \overline{\mathbb{R}}(\varphi)(\lambda) = \varphi(\lambda) , \tag{2.1}$$

where $\varphi:\Lambda\longrightarrow\Lambda'$ is a morphism of Grassman superalgebras and $\lambda\in{}_{\bar 0}\Lambda$.

The ring structures on each ${}_{\bar 0}\Lambda$ generate the structure of a commutative ring with unity on $\overline{\mathbb{R}}$.

Let now V be some real vector superspace ($=\mathbb{R}$-supermodule in Set). Define the functor $\overline{V}$ as follows:

$$\overline{V}(\Lambda) := {}_{\bar 0}(\Lambda\otimes_{\mathbb{R}} V) = \bigoplus_{\varepsilon\in Z_2} {}_\varepsilon\Lambda\otimes_{\mathbb{R}} {}_\varepsilon V , \quad \overline{V}(\varphi) := \varphi\otimes \mathrm{Id}_V|\overline{V}(\Lambda) \tag{2.2}$$

for $\varphi:\Lambda\longrightarrow\Lambda'$. The canonical ${}_{\bar 0}\Lambda$-module structures on each $\overline{V}(\Lambda)$ turn the functor V into an $\overline{\mathbb{R}}$-module.

At last, let $f: V_1\times\dots\times V_n\longrightarrow V$ be an even $\mathbb{R}$-n-linear map of vector superspaces. Define the functor morphism $\bar f:\overline{V}_1\times\dots\times\overline{V}_n\to\overline{V}$ such that every component $\bar f_\Lambda: V_1(\Lambda)\times\dots\times V_n(\Lambda)\longrightarrow V(\Lambda)$ of $\bar f$ is the ${}_{\bar 0}\Lambda$-n-linear map uniquelly determined by the equations

$$\bar f_\Lambda(\lambda_1\otimes v_1,\dots,\lambda_n\otimes v_n) = \lambda_n\dots\lambda_1\otimes f(v_1,\dots,v_n) . \tag{2.3}$$

for every $\lambda_i\otimes v_i\in\overline{V}_i(\Lambda)$.

The functor morphism $\bar f$ is $\overline{\mathbb{R}}$-n-linear.

If V is a complex vector superspace then $\overline{V}$ turns out to be a $\mathbb{C}$-module, where the ring $\mathbb{C}(\Lambda)={}_{\bar 0}\Lambda\otimes\mathbb{C}={}_{\bar 0}(\Lambda_{\mathbb{C}})$ is the complexification of the ring ${}_{\bar 0}\Lambda$; if f is $\mathbb{C}$-n-linear then $\bar f$ is $\overline{\mathbb{C}}$-n-linear.

The main properties of the correspondence $V\longmapsto\overline{V}$ and $f\longmapsto\bar f$ are summarized in the following proposition.

Proposition 2.1. Let $V_1,\dots,V_n,V$ be vector superspaces over the field K. Then

a) The map ${}_{\mathfrak{z}}L_K(V_1,\dots,V_n;V) \xrightarrow{-} L_{\overline{K}}(\overline{V}_1,\dots,\overline{V}_n;\overline{V})$ is an isomorphism of K-modules (taking into account that the set $\overline{K}(p)$ of points of $\overline{K}$ coincides with K);

b) If $g_1,\dots,g_n$ are K-polylinear maps such that for every i the codomain of g_i is V_i then for every $f\in L_K(V_1,\dots,V_n;V)$ the identity $\overline{f\circ(g_1\times\dots\times g_n)} = \overline{f}\circ(\overline{g}_1\times\dots\times\overline{g}_n)$ holds;

c) If $f\in L_K(V_1,\dots,V_n;V)$ and σ belongs to the permutation group $\mathfrak{S}_n$ then the identity $\overline{f.\sigma} = \overline{f}\circ\sigma$ holds, where $f.\sigma$ is the "graded" action of σ on f and $\overline{f}\circ\sigma$ is the "ordinary" composition (in $\underline{Set}^{Gr}$) of f with permutation isomorphism σ.

Corollary 2.2. The correspondence $V\mapsto\overline{V}$ and $f\mapsto\overline{f}$ defines full and faithful functor $\underline{SMod}_K(\underline{Set}) \xrightarrow{-} \underline{Mod}_{\overline{K}}(\underline{Set}^{Gr})$ respecting finite direct sums; more generally, it generates, for every type T of polylinear algebraic structure, the full and faithful functor $S\mathsf{T}_K(\underline{Set}) \xrightarrow{-} \mathsf{T}_{\overline{K}}(\underline{Set}^{Gr})$.

An $\overline{K}$-module (or, more generally, $\overline{K}$-algebra of some type T) $\mathcal{V}$ in the functor category $\underline{Set}^{Gr}$ will be called superrepresentable if it is isomorphic to $\overline{V}$ for some K-supermodule (resp. K-superalgebra of the type T) V in $\underline{Set}$.

In $\underline{Set}^{Gr}$ there exist, of course, $\overline{K}$-modules which are not superrepresentable. For example, if V is some vector superspace over K, then the K-module $\overline{V}^{nil}$, defined as $\overline{V}^{nil}(\Lambda) := {}_{\mathfrak{z}}(\Lambda^{nil}\otimes_{K} V)$, where Λ^{nil} is the ideal of nilpotents of the Grassman algebra Λ, is not superrepresentable. Note that for every Λ the identity

$$\overline{V}(\Lambda) = {}_{\mathfrak{z}}V\oplus\overline{V}^{nil}(\Lambda) \qquad (2.4)$$

holds.

In conclusion, we note that a $\overline{K}$-module in $\underline{Top}^{Gr}$ or in $\underline{Man}^{Gr}$ is superrepresentable exactly when $\mathcal{V}$ it is superrepresentable considered as a $\overline{K}$-module in $\underline{Set}^{Gr}$.

3. Banach superregions

3.1. Topology on the functor category $\underline{Top}^{Gr}$. Let $\mathcal{F}'$ and $\mathcal{F}$ be some functors in $\underline{Top}^{Gr}$. The functor $\mathcal{F}'$ is called a subfunctor of the functor $\mathcal{F}$ if for every $\Lambda\in|\underline{Gr}|$ the topological space $\mathcal{F}'(\Lambda)$ is the topological subspace of the topological space $\mathcal{F}(\Lambda)$ and, besides, if the family $\{\mathcal{F}'(\Lambda)\subset\mathcal{F}(\Lambda)\}_{\Lambda\in|\underline{Gr}|}$ forms a functor morphism (denoted further as $\mathcal{F}'\subset\mathcal{F}$); the subfunctor $\mathcal{F}'$ of the functor $\mathcal{F}$ is called open if every $\mathcal{F}'(\Lambda)$ is open in $\mathcal{F}(\Lambda)$.

Given two open subfunctors $\mathcal{F}'$ and $\mathcal{F}''$ of the functor $\mathcal{F}$ one can define the open subfunctor $\mathcal{F}'\cap\mathcal{F}''$ of $\mathcal{F}$ pointwise as follows: $(\mathcal{F}'\cap\mathcal{F}'')(\Lambda) := \mathcal{F}'(\Lambda)\cap\mathcal{F}''(\Lambda)$. Similarly, given a family $\{\mathcal{F}_\alpha\}$ of open subfunctors of $\mathcal{F}$ one can define the open subfunctor $\bigcup\mathcal{F}_\alpha$ of $\mathcal{F}$ by the equalities $(\bigcup\mathcal{F}_\alpha)(\Lambda) = \bigcup\mathcal{F}_\alpha(\Lambda)$. Besides, the initial functor ϕ is an open subfunctor of every functor $\mathcal{F}$ in $\underline{Top}^{Gr}$.

The topologies just defined on functors in $\underline{Top}^{Gr}$ incorporate themselves to produce some Grothendieck pretopology (see, e.g., Ref. /6/) on the functor category $\underline{Top}^{Gr}$.

Namely, call a functor morphism open if it can be represented as $\mathcal{F}''\xrightarrow{f}\mathcal{F}'\subset\mathcal{F}$, where $\mathcal{F}''\xrightarrow{f}\mathcal{F}$ is an isomorphism and $\mathcal{F}'$ is an open subfunctor of $\mathcal{F}$. A family $\{\mathcal{U}_\alpha\xrightarrow{u_\alpha}\mathcal{F}\}$ of functor morphisms will be called an open covering of the functor $\mathcal{F}$ if each u_α is an open morphism and, besides, if for every $\Lambda\in\underline{Gr}$ the family $\{u_{\alpha\Lambda}(\mathcal{U}_\alpha(\Lambda))\}$ is an open covering (in the usual sense) of the topological space $\mathcal{F}(\Lambda)$. It is elementary to verify that the class of open coverings defined here really is a (Grothendieck) pretopology on the category $\underline{Top}^{Gr}$.

Note that the obvious neglecting functor $\underline{Man}^{Gr}\longrightarrow\underline{Top}^{Gr}$ induces some pretopology on the category $\underline{Man}^{Gr}$: a family $\{\mathcal{U}_\alpha\xrightarrow{u_\alpha}\mathcal{F}\}$ of morphisms in $\underline{Man}^{Gr}$ is an open covering of the functor $\mathcal{F}$ iff it is an open covering of $\mathcal{F}$ considered as the family of morphisms in $\underline{Top}^{Gr}$.

Throughout the rest of the paper the categories $\underline{Top}^{Gr}$ and $\underline{Man}^{Gr}$ are supposed to be equipped with the pretopologies defined above.

To give an example of open subfunctors consider an arbitrary functor $\mathcal{F}$ in $\underline{\mathrm{Top}}^{\mathrm{Gr}}$. Let U be an open subset in $\mathcal{F}:=\mathcal{F}(\mathbb{R})$ (the base of the functor $\mathcal{F}$) and for every Λ let $\varepsilon_\Lambda : \Lambda \to \mathbb{R}$ be the only morphism of Grassman superalgebras. The family $\{\mathcal{F}(\varepsilon_\Lambda)^{-1}(U) \subset \mathcal{F}(\Lambda)\}_{\Lambda\in \underline{\mathrm{Gr}}}$ defines an open subfunctor of the functor $\mathcal{F}$ which will be denoted as $\mathcal{F}|_U$. It turns out that if the functor $\mathcal{F}$ is locally isomorphic to locally convex superrepresentable modules then all its open subfunctors are of this type.

In more details, a $\bar{K}$-module $\mathcal{V}$ in $\underline{\mathrm{Top}}^{\mathrm{Gr}}$ will be called locally convex (resp. Banach, resp. Fréchet,etc.) $\bar{K}$-module if for every Λ the topological vector space $\mathcal{V}(\Lambda)$ is locally convex (resp. Banach, resp. Fréchet,etc.). An open subfunctor of a superrepresentable locally convex (resp. Banach, etc.) $\bar{K}$-module will be called locally convex (resp. Banach, etc.) superregion, real or complex depending on whether $K=\mathbb{R}$ or $K=\mathbb{C}$. The functor $\mathcal{F}$ will be said to be locally isomorphic to locally convex superregions if there exists an open covering $\{\mathcal{U}_\alpha \to \mathcal{F}\}$ of $\mathcal{F}$ such that each $\mathcal{U}_\alpha$ is a locally convex superregion.

Proposition 3.1.1. If a functor $\mathcal{F}$ in $\underline{\mathrm{Top}}^{\mathrm{Gr}}$ is locally isomorphic to locally convex superregions, then every open subfunctor $\mathcal{F}'$ of the functor $\mathcal{F}$ coincides with the functor $\mathcal{F}|_{\underline{\mathcal{F}}'}$, where $\underline{\mathcal{F}}' = \mathcal{F}'(\mathbb{R})$ is the base of the functor $\mathcal{F}'$.

3.2. Supersmooth morphisms of Banach superregions. In what follows, $\mathcal{V}$, $\mathcal{V}'$, $\mathcal{W}$, etc. will denote (Banach) superrepresentable modules.

Given two real B(anach) superregions $\mathcal{V}|_U$ and $\mathcal{V}'|_{U'}$, the functor morphism $f: \mathcal{V}|_U \to \mathcal{V}'|_{U'}$ will be called supersmooth if for every Λ the map $f_\Lambda : \mathcal{V}|_U(\Lambda) \longrightarrow \mathcal{V}'|_{U'}(\Lambda)$ is smooth and, besides, for every $u\in \mathcal{V}|_U(\Lambda)$ the derivative map $Df_\Lambda(u): \mathcal{V}(\Lambda) \longrightarrow \mathcal{V}'(\Lambda)$ is ${}_{\bar{0}}\Lambda$-linear.

The latter condition is equivalent, in turn, to the following one: the "weak superderivative" morphism $\mathcal{D}^w f: \mathcal{V}|_U \times \mathcal{V} \to \mathcal{V}'$, defined by equalities $(\mathcal{D}^w f)(u,v) = Df_\Lambda(u).v$ for $u\in \mathcal{V}|_U(\Lambda)$ and $v\in \mathcal{V}(\Lambda)$,

is a $\mathcal{V}|_U$-family of $\bar{\mathbb{R}}$-linear morphisms.

It is evident that a composition of supersmooth morphisms is again supersmooth, hence B.superregions and supersmooth morphisms between them define a category, which will be called the category of supersmooth Banach superregions and denoted $\underline{\mathrm{SReg}}$.

Given a B.superregion $\mathcal{U} = \mathcal{V}|_U$, every open subfunctor $\mathcal{U}'$ of $\mathcal{U}$ (equal according to Prop.3.1.1 to some superregion $\mathcal{V}|_{U'}$, with U' being open in U) will be called an open subsuperregion of $\mathcal{U}$. The inclusion morphism $\mathcal{U}' \subset \mathcal{U}$ is, obviously, supersmooth. Hence, one can define the pretopology on the category $\underline{\mathrm{SReg}}$, induced by that on the category $\underline{\mathrm{Man}}^{\mathrm{Gr}}$ along the obvious neglecting functor

$$\underline{\mathrm{SReg}} \xrightarrow{N} \underline{\mathrm{Man}}^{\mathrm{Gr}} . \qquad (3.2.1)$$

The category $\underline{\mathrm{SReg}}$ will be asumed further to be equipped with this induced pretopology.

Remark. It is quite evident how one can define the category of real superanalytic superregions. As to the complex analytic case, there arise two evident possibilities, namely, using complex Banach superregions in the functor category $\underline{\mathrm{Top}}^{\mathrm{Gr}}$, or using instead from the very beginning the category $\underline{\mathrm{Gr}}_{\mathbb{C}}$ of complex finite-dimensional Grassman superalgebras and copying preceeding constructions for the functor category $\underline{\mathrm{Top}}^{\mathrm{Gr}}$ on the place of $\underline{\mathrm{Top}}^{\mathrm{Gr}}$. Nevertheless, it follows from Prop.2.1 for $K=\mathbb{C}$, that the two arising categories of complex superanalytic superregions are equivalent.

We will restrict ourselves in this paper only with the supersmooth case, but most of the results of this work (if not all) are valid, with obvious changes, for the K-superanalytic case as well.

3.3. The structure of supersmooth morphisms. Here is given some characterization of supersmooth morphisms which, being rather technical one, turns out to be, nevertheless, a very useful tool in various proofs and constructions.

Let $f\colon \bar V|_U \longrightarrow \bar V'|_{U'}$ be a natural transformation of B.superregions. The family $\{f_i\}_{i\in\mathbb{N}}$ will be called skeleton of f if the following conditions are satisfied:

i) $f_0 = f_{\mathbb{R}}\colon U\longrightarrow U'$ and $f_i\colon U\to {}_{\bar 0}\mathcal{L}({}_{\bar 1}V;V')$ for $i\geqslant 1$ are smooth maps such that for every $u\in U$ the $\mathbb{R}$-i-linear map $f_i(u)$ is supersymmetric in the sense of Sect.1.9; here ${}_{\bar 1}V$ is consedered as purely odd Banach superspace;

ii) for every Grassman superalgebra Λ and every $u\in U$, $\lambda_0\in {}_{\bar 0}\bar V^{nil}(\Lambda)$, $\lambda_1\in {}_{\bar 1}V^{nil}(\Lambda)$ the identity

$$f\,(u+\lambda_0+\lambda_1) = \sum \frac{1}{k!m!}\,\overline{D^k f_m(u)}_\Lambda \lambda_0^k\,\lambda_1^m \qquad (3.3.1)$$

holds. In the latter expression $u+\lambda_0+\lambda_1$ is considered as an element of the B.region $\bar V|_U(\Lambda)$ in accord with the canonical decompositions (2.4) and

$$\bar V^{nil}(\Lambda) = {}_{\bar 0}\bar V^{nil}(\Lambda)\oplus {}_{\bar 1}\bar V^{nil}(\Lambda)\ , \qquad (3.3.2)$$

$D^k f_m(u)$ is identified with an element of ${}_{\bar 0}L^{k+m}({}_{\bar 0}V^k, {}_{\bar 1}V^m;V')$ via the canonical isomorphism of the type (1.5.4), and the $\bar{\mathbb{R}}$-polylinear morphism $\overline{D^k f_m(u)}$ in Top^{Gr} is defined by eq.(2.3), which shaws that the sum in (3.3.1) is actually finite (for the Grassman algebra Λ_i with i odd generators only terms with $2k+m\leqslant i$ could be non-zero).

Proposition 3.3.1. a) A skeleton of f, if it exists, is uniquelly determined. b) Every family $\{f_i\}_{i\in\mathbb{N}}$ of smooth maps such that $f_0\colon U\to U'$ and $f_i\colon U \; {}_{\bar 0}\mathcal{L}({}_{\bar 1}V;V')$ for $i\geqslant 1$ is the sceleton of some functor morphism f.

Now, Prop.2.1 permits one to prove the following important result.

Theorem 3.3.2. The following conditions on the functor morphism $f\colon \bar V|_U \longrightarrow \bar V'|_{U'}$ of Banach superregions in Top^{Gr} are equivalent:

i) f is supersmooth;

ii) Each component f_Λ of f is smooth and the derivative $Df_\Lambda(x)\colon \bar V(\Lambda)\to \bar V'(\Lambda)$ is ${}_{\bar 0}\Lambda$-linear for every x of the form $x=V(\iota_\Lambda)(u)$, where $\iota_\Lambda\colon \mathbb{R}\to\Lambda$ is the initial morphism of Grassman superalgebras and $u\in U$;

iii) f has a skeleton.

For a number of applications it is necessary to know the expression of the skeleton $\{(g\circ f)_i\}$ of the composition $(g\circ f)$ of supersmooth morphisms g and f in terms of skeletons of g and f. A bit of combinatorics produces the following result.

Proposition 3.3.3. Let $f\colon \bar V|_U\to \bar V'|_{U'}$ and $g\colon \bar V'|_{U'}\to \bar V''|_{U''}$ be supersmooth morphisms of B.superregions with the skeletons $\{f_i\}$ and $\{g_i\}$, respectively. The skeleton $\{(g\circ f)_i\}$ of the composition $g\circ f$ is determined then by the expression

$$(g\circ f)_n(u)=\sum \frac{n!}{l!\,m!\,\prod(\alpha_i!)\prod(\beta_j!)}\, S\Big\{(D^l g_m(f_0(u)))\circ\Big(\mathop{\times}_{i=1}^{l} f_{\alpha_i}(u)\times \mathop{\times}_{j=1}^{m} f_{\beta_j}(u)\Big)\Big\} \qquad (3.3.3)$$

for every $u\in U$, where the sum runs over l, m, even α_i and odd β_j such that $\sum\alpha_i+\sum\beta_j=n$ and S is the projection operator of supersymmetrization of $\mathbb{R}$-n-linear maps defined as

$$S.h = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} h.\sigma\ , \qquad (3.3.4)$$

the action $h.\sigma$ being defined by eq.(1.9.1).

In what follows, the supersmooth morphism f of B.superregions will be identified either with the family $\{f_\Lambda\}_{\Lambda\in Gr}$ of its components or with its skeleton $\{f_i\}_{i\in\mathbb{N}}$, depending on the circumstances.

Remarks. 1) The skeleton $\{f_i\}$ of a supersmooth morphism $f=\{f_\Lambda\}$, codes in itself the naturality properties of the family $\{f_\Lambda\}$, permitting one to give a direct (i.e. doing without functors) description of B.superregions and supermanifolds as $\mathbb{N}$-graded manifolds (=objects of the category $\mathrm{Man}^{\mathbb{N}}$) equipped with some structure. From this point of view Theorem 3.3.2 gives an explicite description of the neglecting functor

$$N'\colon \mathrm{SReg} \xrightarrow{N} \mathrm{Man}^{Gr} \xrightarrow{\subset^*} \mathrm{Man}^{\mathbb{N}}, \qquad (3.3.5)$$

where $\subset^*(\mathcal{F}):=\{\mathcal{F}(\Lambda_i)\}_{i\in\mathbb{N}}$ and $\subset^*(f):=\{f_{\Lambda_i}\}_{i\in\mathbb{N}}$.

One can prolong the "path of conceptual simplification" (3.3.5) adding to its end one functor more, namely, the functor $\underline{\mathrm{Man}}^{N} \xrightarrow{\sqcup} \underline{\mathrm{Man}}$ sending $\{M_i\}_{i\in N}$ into $\bigsqcup_{i\in N} M_i$. This permits one to visualize supermanifolds and their morphisms as ordinary manifolds and smooth maps between them writing $\mathcal{U} = \bigsqcup_{i\in N} \mathcal{U}(\Lambda_i)$ and $f = \bigsqcup_{i\in N} f_{\Lambda_i}$. Note that the functor $\sqcup$ does not commute with products which implies, e.g., that Lie supergroups (= groups of the category S<u>Man</u>) are not groups at all (considered in <u>Set</u>).

2) Theorem (3.3.2) together with Prop.(3.3.3) permits one as well to give an equivalent "abstract" definition of supermanifolds doing without both functors and Grassman algebras. Namely, one can define Banach superregions as pairs $(U \subset {}_0V, {}_1V)$,where U is an open region in a Banach space ${}_0V$ and ${}_1V$ is some Banach space; morphisms here are "abstract skeletons" $\{f_i\}_{i\in N}$ and the composition of morphisms is then to be defined by eq.(3.3.3).

This was just this definition that was used in the author's work[9] devoted to an extension to Banach supermanifolds of results of M.Batchelor[10] and V.Palamodov[11] on the structure of finite-dimensional supermanifolds.

3.4. <u>The categories</u> S<u>Reg</u>$^{(m)}$. Besides the category S<u>Reg</u> of supersmooth Banach superregions one can as well construct a family of categories, "approximating" in a sense the category S<u>Reg</u>.

Namely, denote as <u>Gr</u>$^{(m)}$ the full subcategory of the category <u>Gr</u> , consisting of all Grassman algebras with not more than m odd generators. In the functor category $\underline{\mathrm{Top}}^{\underline{\mathrm{Gr}}}$ one can define the ring $\overline{\mathbb{R}}^{(m)}$, superrepresentable $\overline{\mathbb{R}}^{(m)}$-modules, topology and superregions in close analogy with the preceeding case, with obvious changes. For example, the skeleton of supersmooth morphism f is now a family $\{f_i\}_{i\leq m}$, satisfying the corresponding conditions. The corresponding category of supersmooth superregions will be denoted S<u>Reg</u>$^{(m)}$, whereas objects of it will be called m-<u>cut superregions</u> or , simply, m-<u>superregions</u>. We will write sometimes S<u>Reg</u>$^{(\infty)}$ instead of S<u>Reg</u> in order to unify notations.

Note that the counterparts of all of the results of the present section, in particular, Theorem 3.3.2 remain , valid for the category S<u>Reg</u>$^{(m)}$ with arbitrary m, though p. a) of Prop.(2.1) fails: the map $f \mapsto \overline{f}$ is bijective only for n-linear morphisms with $n \leq m$.

There exist obvious functors π^n_m: S<u>Reg</u>$^{(n)} \longrightarrow$ S<u>Reg</u>$^{(m)}$ for $0 \leq m \leq \leq n \leq \infty$,induced by inclusion functor <u>Gr</u>$^{(m)} \hookrightarrow$ <u>Gr</u>$^{(n)}$ (on skeleton's language $\pi^n_m\{f_i\}_{i\leq n} = \{f_i\}_{i\leq m}$). Obviously, the category S<u>Reg</u>$^{(0)}$ is naturally equivalent to (and will be identified with) the category <u>Reg</u> of smooth Banach regions, whereas Theorem 3.3.2 and Prop. 3.3.3 imply that the category S<u>Reg</u>$^{(1)}$ is naturally equivalent to the category <u>VBun</u>$_x$ of smooth trivial vector bundles over Banach regions (to an 1-superregion $\overline{V}^{(1)}_U$ corresponds the vector bundle $U \times {}_1V \to U$). The same Theorem 3.3.2 and Prop. 3.3.3 imply, moreover, the existence of functors ζ^0_m: <u>Reg</u>$^{(0)} \to$ S<u>Reg</u>$^{(m)}$ and ζ^1_m:S<u>Reg</u>$^{(1)} \longrightarrow$ S<u>Reg</u>$^{(m)}$ for $m \geq 1$ ($f \mapsto (f,0,0,\ldots)$,resp. $(f_0,f_1) \mapsto (f_0,f_1,0,\ldots)$ on skeleton's language), ζ^1_m being faithful, whereas ζ^0_m being full and faithful and left ajoint to the functor π^m_0 . In particular, the following result is valid.

<u>Proposition</u> 3.4.1. <u>The category</u> <u>Reg</u> <u>of smooth regions in Banach spaces could be identified</u> (through the functor ζ^0_m) <u>with full subcategory of</u> S<u>Reg</u>$^{(m)}$ <u>and for every</u> m-<u>superregion</u> $\mathcal{V}|_U$ <u>there exists the canonical monomorphism</u> $\zeta^0_m(U) \to \mathcal{V}|_U$, <u>being the component of the natural transformation</u> $\zeta^0_m \cdot \pi^m_0 \to \mathrm{Id}_{S\underline{Reg}}$ <u>defined by the ajunction</u>.

One can observe that the correspondence sending a smooth map $f\colon U \to U'$ of B.regions to the smooth map $\zeta^0_m(f)_\Lambda$, defined by the "Tailor expansion" (3.3.1) with the skeleton $(f,0,0,\ldots)$, is just the infinite-dimensional counterpart of Berezin's "Grassman analytic continuation"[2].

Note in conclusion that m-supermanifolds with finite m (glued out of m-superregions) play an important part in construction of invariants of Banach supermanifolds/9/ being the counterparts (on the functor's language) of "m-th infinitesimal neighbourhoods" of supermanifolds exploited in Palamodov's work/11/.

4. Banach Supermanifolds.

4.1. The definition of the category SMan. We can define now Banach supermanifolds by means of atlases on functors of the category $\underline{\mathrm{Man}}^{\underline{\mathrm{Gr}}}$.

Let $\mathcal{F}$ be a functor in $\underline{\mathrm{Man}}^{\underline{\mathrm{Gr}}}$. An open covering $\mathcal{A}=\{\mathcal{U}_\alpha \xrightarrow{i_\alpha} \mathcal{F}\}$ of the functor $\mathcal{F}$ is called a (supersmooth) atlas on $\mathcal{F}$ if every $\mathcal{U}_\alpha$ is a B.superregion and every pullback

$$\begin{array}{ccc} \mathcal{U}_{\alpha\beta} & \xrightarrow{\pi_\beta} & \mathcal{U}_\beta \\ \pi_\alpha\downarrow & & \downarrow i_\beta \\ \mathcal{U}_\alpha & \xrightarrow{i_\alpha} & \mathcal{F} \end{array} \tag{4.1.1}$$

can be chosen in such a way, that $\mathcal{U}_{\alpha\beta}$ be a B.superregion and the pullback projections be supersmooth. Two atlases $\mathcal{A}$ and $\mathcal{A}'$ on $\mathcal{F}$ are said to be equivalent if $\mathcal{A}\cup\mathcal{A}'$ is an atlas as well; it is an equivalence relation on the class of atlases on $\mathcal{F}$.

A Banach supermanifold is a functor $\mathcal{M}$ in $\underline{\mathrm{Man}}^{\underline{\mathrm{Gr}}}$ together with an equivalence class of atlases on it; elements of every atlas of the corresponding equivalence class will be called charts of the supermanifold $\mathcal{M}$. We will not distinguish in notations between a supermanifold and its underlying functor.

Let $\mathcal{M}$ and $\mathcal{M}'$ be B.supermanifolds. A functor morphism $f:\mathcal{M}\to\mathcal{M}'$ will be said to be supersmooth if for every charts $\mathcal{U}\xrightarrow{i}\mathcal{M}$ and $\mathcal{U}'\xrightarrow{i'}\mathcal{M}'$ of $\mathcal{M}$ and $\mathcal{M}'$, respectively, the pullback

$$\begin{array}{ccccc} \mathcal{U}\prod_{\mathcal{M}'}\mathcal{U}' & & \xrightarrow{\pi'} & & \mathcal{U}' \\ \pi\downarrow & & & & \downarrow i' \\ \mathcal{U} & \xrightarrow{i} & \mathcal{M} & \xrightarrow{f} & \mathcal{M}' \end{array} \tag{4.1.2}$$

could be chosen in such a way, that $\mathcal{U}\prod_{\mathcal{M}'}\mathcal{U}'$ be a B.superregion and the pullback projections π and π' be supersmooth.

Composition of two supersmooth morphisms is again supersmooth, which permits one to define correctly the category SMan of Banach supermanifolds. The set of morphisms of a supermanifold $\mathcal{M}$ into the supermanifold $\mathcal{M}'$ will be denoted $SC^\infty(\mathcal{M},\mathcal{M}')$.

Let $\mathcal{M}$ be a B.supermanifold and let $\mathcal{M}'$ be an open subfunctor of $\mathcal{M}$. There exists on the functor $\mathcal{M}'$ the only structure of the supermanifold such that the inclusion $\mathcal{M}'\subset\mathcal{M}$ is supersmooth morphism. The functor $\mathcal{M}'$ equipped with this structure will be called an open subsupermanifold of $\mathcal{M}$. Note that in accord with Prop.3.1.1 every open subsupermanifold $\mathcal{M}'$ of the supermanifold $\mathcal{M}$ is of the form $\mathcal{M}|_U$ for some open subset U of the base manifold $\underline{\mathcal{M}}:=\mathcal{M}(\mathbb{R})$ of $\mathcal{M}$. Inclusions of open supermanifolds generate in a standard way (cf. Sect.3.1) some pretopology on the category SMan. This pretopology is induced by the canonical pretopology on the category $\underline{\mathrm{Man}}^{\underline{\mathrm{Gr}}}$ along the neglecting functor

$$\underline{\mathrm{SMan}} \xrightarrow{N} \underline{\mathrm{Man}}^{\underline{\mathrm{Gr}}}, \tag{4.1.3}$$

continuing the functor (3.2.1) and denoted by the same letter (the category SReg is, of course, asumed to be imbedded into SMan by means of attaching to a superregion $\mathcal{U}$ the trivial atlas $\mathrm{Id}_{\mathcal{U}}$). The category SMan will be asumed to be equipped with the pretopology just defined.

Remark. In the definition of supermanifolds one can use as well the neglecting functor $\underline{\mathrm{SReg}} \xrightarrow{N'} \underline{\mathrm{Set}}^{\underline{\mathrm{Gr}}}$ instead of the functor (3.2.1). The definition of atlases on Set-valued functors and supersmooth morphisms follows closely that given above for Man-valued functors, with some changes caused by the fact that the pretopology on SReg is not induced by that on $\underline{\mathrm{Set}}^{\underline{\mathrm{Gr}}}$ (where open coverings are defined as such families $\{\mathcal{U}_\alpha \xrightarrow{u_\alpha} \mathcal{F}\}$ that for every Λ the family $\{\mathcal{U}_\alpha(\Lambda) \xrightarrow{u_{\alpha\Lambda}} \mathcal{F}(\Lambda)\}$ is an epi family of monos). This changes are the following ones: we are to demand that pullback projections π_α and π_β in the pullback (4.1.1) as well as projection π in the pullback (4.1.2) are open, considered as morphisms of SReg. We will arrive as a result to the category SMan' of supermanifolds as Set-valued functors on Gr with atlases on them.

It turns out that the functor SMan $\longrightarrow$ SMan', generated by the neglecting functor $\mathrm{Man}^{\mathrm{Gr}} \longrightarrow \mathrm{Set}^{\mathrm{Gr}}$, is not only a natural equivalence of categories but even an isomorphism of them, permitting one to identify this two categories.

In practice, both categories SMan and SMan' will be used, depending on circumstances: whereas general definitions look simpler taken "modulo manifolds", some concrete supermanifolds (e.g., Grassmanians) arise naturally first as Set-valued functors.

4.2. The categories $\mathrm{SMan}^{(m)}$. One can define the categories $\mathrm{SMan}^{(m)}$ of m-supermanifolds starting from the categories SReg and repeating almost literally the definitions of the preceeding subsection. Besides, each category $\mathrm{SMan}^{(m)}$ will be equipped with the pretopology induced by that on the category $\mathrm{Man}^{\mathrm{Gr}(m)}$ along the neglecting functor $\mathrm{SMan}^{(m)} \xrightarrow{N^{(m)}} \mathrm{Man}^{\mathrm{Gr}^{(m)}}$.

If $\mathcal{D}$ and $\mathcal{D}'$ are categories with pretopologies on them, the functor $F: \mathcal{D} \to \mathcal{D}'$ is called continuous if it respects open coverings and pullbacks of open (=belonging to some open covering) morphisms. For example, the neglecting functor $N^{(m)}$ as well as the functors π^n_m, ζ^o_m and ζ^1_m, defined in Sect.3.4 are continuous.

Proposition 4.2.1. a) The category $\mathrm{SMan}^{(o)}$ of o-supermanifolds is continuosly naturally equivalent to the category Man of B.manifolds; b) the category $\mathrm{SMan}^{(1)}$ of 1-supermanifolds is naturally equivalent to the category VBun of smooth Banach real vector bundles (continuously, if one equips VBun with the pretopology generated by open inclusions of vector bundles); c) the functors π^n_m, ζ^o_m and ζ^1_m have continuous extensions (denoted here by the same letters)

$$\pi^n_m: \mathrm{SMan}^{(n)} \longrightarrow \mathrm{SMan}^{(m)}, \ \zeta^o_m: \mathrm{Man} \longrightarrow \mathrm{SMan}^{(m)}, \ \zeta^1_m: \mathrm{VBun} \longrightarrow \mathrm{SMan}^{(m)}, \qquad (4.2.1)$$

such that ζ^o_m is full and faithful and left ajoint to π^m_o, whereas ζ^1_m is faithful functor such that $\pi^m_1 \circ \zeta^1_m \approx \mathrm{Id}_{\mathrm{VBun}}$.

In particular, for every supermanifold $\mathcal{M}$ there exists the canonical monomorphism $\zeta^o_m(\underline{\mathcal{M}}) \hookrightarrow \mathcal{M}$.

4.3. Products of supermanifolds. Let $\mathcal{M}$ and $\mathcal{M}'$ be supermanifolds with atlases $\{\mathcal{U}_\alpha \xrightarrow{i_\alpha} \mathcal{M}\}$ and $\{\mathcal{U}_\beta \xrightarrow{i_\beta} \mathcal{M}'\}$ on $\mathcal{M}$ and $\mathcal{M}'$, respectively. The family $\{\mathcal{U}_\alpha \times \mathcal{U}_\beta \xrightarrow{i_\alpha \times i_\beta} \mathcal{M} \times \mathcal{M}'\}$ is an atlas on the functor $\mathcal{M} \times \mathcal{M}'$ turning it into a supermanifold such that the corresponding projections are supersmooth. This is, in fact, the product of the supermanifolds $\mathcal{M}$ and $\mathcal{M}'$.

Let $\mathcal{P}$ be a B.superregion isomorphic to a B.superregion $\overline{V}$ for some purely odd (i.e. such that ${}_{\bar{o}}V = 0$) B.superspace V; such $\mathcal{P}$ will be called a superpoint. It follows from Prop.3.1.1 and the definition of superrepresentable modules (see eq.(2.2)) that every B.superregion $\mathcal{U}$ is isomorphic to a product $\zeta^o_\infty(\underline{\mathcal{U}}) \times \mathcal{P}$ of "ordinary" manifold $\zeta^o_\infty(\underline{\mathcal{U}})$ and some superpoint $\mathcal{P}$. The supermanifold is called simple if it is isomorphic to a product $\zeta^o_\infty(M) \times \mathcal{P}$ for some B.manifold M and some superpoint $\mathcal{P}$.

4.4. Linear algebra in the category of supermanifolds. Let $f: V_1 \times \ldots \times V_n \longrightarrow V$ be an even K-n-linear map of B.superspaces. Then, evidently, the $\overline{K}$-n-linear functor morphism $\overline{f}: \overline{V}_1 \times \ldots \times \overline{V}_n \longrightarrow \overline{V}$ is supersmooth. Hence, we have due to Coroll.2.2 the full and faithful functor

$$S\mathcal{T}_K(\mathrm{Man}) \xrightarrow{\;-\;} \mathcal{T}_{\overline{K}}(\mathrm{SMan}) \qquad (4.4.1)$$

for every polylinear type $\mathcal{T}$ of algebraic structure.

Theorem 4.4.1. The functor

$$\mathrm{SMod}_K(\mathrm{Man}) \xrightarrow{\;-\;} \mathrm{Mod}_{\overline{K}}(\mathrm{SMan}) \qquad (4.4.2)$$

is a natural equivalence of categories.

Corollary 4.4.2. For every polylinear type $\mathcal{T}$ of algebraic structure the functor (4.4.2) establishes a natural equivalence of the category of K-superalgebras of the type $\mathcal{T}$ in Man with the category of $\overline{K}$-algebras of the same type $\mathcal{T}$ in SMan.

Corollary 4.4.3. In the category SMan of supermanifolds there exist coherent tensor products over $\overline{K}$ as well as coherent internal $L_{\overline{K}}$-functors (see Sects. 1.5 and 1.6 for the definition).

We choose the functors $\mathcal{L}_{\overline{K}}$ and $\otimes_{\overline{K}}$ in such a way that for every B.superspaces $V_1,\dots,V_n,V$ the identities

$$\mathcal{L}_{\overline{K}}(\overline{V}_1,\dots,\overline{V}_n;\overline{V}) = \overline{\mathcal{L}_K(V_1,\dots,V_n;V)} \text{ and } \overline{V}_1\otimes_{\overline{K}}\overline{V}_2=\overline{V_1\otimes_K V_2} \quad (4.4.3)$$

hold.

Note that, generally speaking, the dinatural (on Λ) morphism

$$\mathcal{L}_{\overline{R}}(\overline{V};\overline{V}')(\Lambda) \xrightarrow{\varphi_\Lambda} L_{{}_\delta\Lambda}(V(\Lambda);V'(\Lambda)) \quad (4.4.4)$$

defined by equations

$$\varphi_\Lambda(\lambda\otimes f)(\lambda'\otimes v) = \lambda'\lambda\otimes f(v) \quad (4.4.5)$$

for every $\lambda\otimes f\in\mathcal{L}_{\overline{R}}(\overline{V};\overline{V}')(\Lambda)$ and $\lambda'\otimes v\in V(\Lambda)$, is not an isomorphism. Similarly, $(\overline{V}_1\otimes_{\overline{R}}\overline{V}_2)(\Lambda)$ is not isomorphic, in general, to $V_1(\Lambda)\otimes_{{}_\delta\Lambda} V(\Lambda)$.

An important role plays as well the image of the change of parity functor Π along the natural isomorphism (4.4.2). It will be denoted $\overline{\Pi}$: $\underline{\mathrm{Mod}}_{\overline{K}}(\mathrm{S}\underline{\mathrm{Man}})\longrightarrow\underline{\mathrm{Mod}}_{\overline{K}}(\mathrm{S}\underline{\mathrm{Man}})$ and will be chosen in such a way that

$$\overline{\Pi}(\overline{V})= \overline{\Pi(V)} \text{ and } \overline{\Pi}(\overline{f})= \overline{\Pi(f)}. \quad (4.4.6)$$

At last, choose and fix, for every type $\mathcal{T}$ of polylinear algebraic structure some functor

$$S:\ \mathcal{T}_{\overline{K}}(\mathrm{S}\underline{\mathrm{Man}}) \longrightarrow S\mathcal{T}_K(\underline{\mathrm{Man}}) \quad (4.4.7)$$

quasiinverse to the functor (4.4.1). It seems that there is no canonical choise for this "superization" functor S .

4.5. Linear algebra in $\mathrm{S}\underline{\mathrm{Man}}^{(m)}$. The counterpart of Th.4.4.1 is valid as well for the category $\mathrm{S}\underline{\mathrm{Man}}^{(m)}$, if $1\le m<\infty$, but the Corollaries 4.4.2 and 4.4.3 fail to be true for this case. Nevertheless, if a polylinear type $\mathcal{T}$ of algebraic structure is such that all its ground operations and laws are not more than m-linear then the category of K-superalgebras of the type $\mathcal{T}$ in Man is equivalent to the category of K-algebras of the same type in $\mathrm{S}\underline{\mathrm{Man}}^{(m)}$. For example:

Proposition 4.5.1. Let $m\geqslant 3$. Then the category of Banach Lie superalgebras (resp. modules over Lie superalgebras) over K is naturally equivalent to the category of Lie algebras (resp. modules over Lie algebras) over K in the category $\mathrm{S}\underline{\mathrm{Man}}^{(m)}$.

4.6. Example: GrassManians and flag supermanifolds. Here is constructed the supermanifold $Fl_n(\mathcal{V})$ of flags of any given length n for any $\mathbb{R}$-module $\mathcal{V}$ in the category SMan (the complex case could be treated quite similarly). The definition of $Fl_n(\mathcal{V})$ considered as a set-valued functor is, essentially, that given by Yu.Manin /12/ in the context of the algebraic supergeometry for the finite-dimensional case. As to the supersmooth structure on $Fl_n(\mathcal{V})$, here is used a superized and "analytically continued on n" version of "coordinate free" atlases for ordinary Grassmanians (see, e.g., Ref. /13/), what makes things look a bit more transparent.

A Banach Λ-supermodule $\mathcal{V}$ will be called free if it is isomorphic to a Λ-supermodule $\Lambda\otimes_{\mathbb{R}}V$ for some real B.superspace V. A Banach Λ-subsupermodule E' of E will be called direct if there exists a Banach Λ-supermodule E" such that $E\approx E'\oplus E''$.

Proposition 4.6.1. Let V be a real B.superspace and E be a free direct Λ-subsupermodule of $\Lambda\otimes_{\mathbb{R}}V$. Then for every morphism $\varphi:\Lambda\longrightarrow\Lambda'$ of Grassman superalgebras the Λ'-subsupermodule of $\Lambda'\otimes_{\mathbb{R}}V$, generated by the real subsuperspace $\mathrm{Im}(\varphi\otimes \mathrm{Id}_V)$ of $\Lambda'\otimes_{\mathbb{R}}V$, is free and direct.

This implies that one can correctly define, for a given $\mathbb{R}$-module $\mathcal{V}$ and any natural number n , the functor $Fl_n(\mathcal{V})$ in $\underline{\mathrm{Set}}^{\underline{\mathrm{Gr}}}$ such that $Fl_n(\mathcal{V})(\Lambda)$ is the set of all sequences $E_1\subset E_2\subset\dots\subset E_{n+1}=\mathcal{V}(\Lambda)$ of Λ-supermodules with E_i being a free direct Λ-subsupermodule of E_{i+1} for every $i\le n$.

Define now in a canonical way some supersmooth structure on the functor $Fl_n(\mathcal{V})$.

Consider first the case of a GrassManian $Fl_1(\overline{V})$, where V is some B.superspace.

Let E' and E" be Λ-supermodules. Denoting the set (and, actually the ${}_\delta\Lambda$-module) of morphisms of E' into E" as $\mathrm{Hom}_\Lambda(E',E'')$, define the morphism of ${}_\delta\Lambda$-modules (setting $E'=\Lambda\otimes V'$ and $E''=\Lambda\otimes V''$)

$$\gamma(\Lambda\otimes\mathcal{L}_{\mathbb{R}}(V';V''))\xrightarrow{I_\Lambda}\mathrm{Hom}_\Lambda(\Lambda\otimes V',\Lambda\otimes V'') \qquad (4.6.2)$$

by means of the equations

$$I_\Lambda(\lambda\otimes f)(\lambda'\otimes v)=\lambda'\lambda\otimes f(v) \ . \qquad (4.6.3)$$

The morphism I_Λ is, in fact, an isomorphism; besides, for every $f\in\mathrm{Hom}_\Lambda(\Lambda\otimes V',\Lambda\otimes V'')$ the graph of f is a free direct Λ-subsupermodule of $\Lambda\otimes(V'+V'')\approx\Lambda\otimes V'\oplus\Lambda\otimes V''$. Taking, hence, the composition of the map I_Λ with the map $f\mapsto\mathrm{graph}(f)$ we obtain as a result some function

$$\varphi_{V',V'',\Lambda}:\mathcal{L}_{\bar{\mathbb{R}}}(\bar V';\bar V'')(\Lambda)\longrightarrow Fl_1(\bar V'\oplus\bar V'')(\Lambda). \qquad (4.6.4)$$

Proposition 4.6.2. a) Let V' and V'' be subsuperspaces of a real Banach superspace V, such that $V=V'\oplus V''$. Then the family $\{\varphi_{V',V'',\Lambda}\}_{\Lambda\in|Gr|}$ defines some functor morphism

$$\varphi_{V',V''}:\mathcal{L}_{\mathbb{R}}(\bar V';\bar V'')\longrightarrow Fl_1(\bar V) \ . \qquad (4.6.5)$$

b) The family $\{\varphi_{V',V''}\}_{V'\oplus V''=V}$ is a supersmooth atlas on the functor $Fl_1(\bar V)$.

Consider now the functor $Fl_n(\bar V)$ for arbitrary n.

Define first, for every Λ and every decomposition $V=V'\oplus V''$, the map $\varphi^n_{V',V'',}:\mathcal{L}_{\mathbb{R}}(\bar V';\bar V'')(\Lambda)\times Fl_{n-1}(\bar V)(\Lambda)\longrightarrow Fl_n(\bar V)(\Lambda)$ as such a map, which sends every pair $(f,E_1\subset\ldots\subset E_{n-1}\subset V'(\Lambda))$ to the flag $E_1'\subset\ldots\subset E_{n-1}'\subset E_n\subset V(\Lambda)$, where $E_n=\varphi_{V',V'',\Lambda}(f)$ and E_i' is the inverse image of E_i w.r.t. the restriction $\pi|_E : E_n \xrightarrow{\sim} V'(\Lambda)$ of the canonical projection $V(\Lambda)\xrightarrow{\pi}V'(\Lambda)$.

As Λ runs in Gr the maps $\varphi^n_{V',V'',\Lambda}$ determine some functor morphism

$$\varphi^n_{V',V''}:\mathcal{L}_{\mathbb{R}}(\bar V';\bar V'')\times Fl_{n-1}(\bar V)\longrightarrow Fl_n(\bar V) \ . \qquad (4.6.6)$$

At last, define inductively canonical charts on $Fl_n(\bar V)$ as all functor morphisms of the form $\varphi^n_{V',V''}(\mathrm{Id}\times\varphi)$, where $V'\oplus V''=V$ and φ is any canonical chart on $Fl_{n-1}(\bar V')$, asuming, of course, that canonical charts on $Fl_1(\bar V)$ are just $\varphi_{V',V''}$.

Proposition 4.6.3. Canonical charts form a supersmooth atlas on the functor $Fl_n(\bar V)$.

If $\mathcal{V}$ is an arbitrary $\bar{\mathbb{R}}$-module, then there exists due to Theorem 4.4.1 an isomorphism $\bar V\xrightarrow{J}\mathcal{V}$ for some real B.superspace V; it induces, obviously, an isomorphism $Fl_n(\bar V)\xrightarrow{J'}Fl_n(\mathcal{V})$ for every natural number n. Define a supersmooth structure on $Fl_n(\mathcal{V})$ as the image of the supersmooth structure on $Fl_n(\bar V)$ defined above. This structure does not depend, in fact, on the choise of an isomorphism J.

4.7. Connection with Berezin-Leites-Kostant theory. Define an $\bar{\mathbb{R}}$-superalgebra $\mathcal{R}$ in the category SMan as the functor

$$\mathcal{R}(\Lambda):=\Lambda \ , \quad \mathcal{R}(\varphi):=\varphi \quad \text{for} \quad \varphi:\Lambda\longrightarrow\Lambda' \ , \qquad (4.7.1)$$

with an $\bar{\mathbb{R}}$-superalgebra structure on it generated by Λ-superalgebra structures on every Λ when Λ runs in Gr. The reader could verify that $\mathcal{R}$, considered as an $\mathbb{R}$-algebra in SMan is isomorphic to the $\bar{\mathbb{R}}$-algebra $\bar{\mathbb{C}}^s$, where the real superalgebra $\mathbb{C}^s$ coincides as an $\mathbb{R}$-algebra with $\mathbb{C}$, but is not trivial as a superspace: $\mathbb{C}^s_{\bar 0}=\mathbb{R}$ and $\mathbb{C}^s_{\bar 1}=i\mathbb{R}$.

The $\bar{\mathbb{R}}$-superalgebra $\mathcal{R}$ is commutative. It plays the role of coordinate ring for supermanifolds.

Let $\mathcal{M}$ be a supermanifold. In accord with Sect.1 the set $SC^\infty(\mathcal{M}):=SC^\infty(\mathcal{M},\mathcal{R})$ is canonically equipped with the structure of commutative superalgebra over $SC^\infty(\mathcal{M},\bar{\mathcal{R}})$. Moreover, it is evident that for every $r\in\mathbb{R}$ the functor morphism $f_r:\mathcal{M}\to\mathcal{R}$, defined by equations $(f_r)_\Lambda(m)=r\in\Lambda$, is supersmooth. The corresponding imbedding $\mathbb{R}\hookrightarrow SC^\infty(\mathcal{M})$ equips, canonically, the set $SC^\infty(\mathcal{M})$ with the structure of an $\mathbb{R}$-superalgebra.

Elements of the superalgebra $SC^\infty(\mathcal{M})$ will be called superfields on the supermanifold $\mathcal{M}$.

Example. Let $\mathcal{U}\subset\overline{\mathbb{R}^{n|m}}$ be a finite-dimensional superregion. Let $x_i:\mathcal{U}\xrightarrow{\bar\pi_i|\mathcal{U}}\bar{\mathbb{R}}^{1|0}\hookrightarrow\mathcal{R}$ for $i=1,\ldots,n$, and $\theta_j:\mathcal{U}\xrightarrow{\bar\pi_{n+j}|\mathcal{U}}\bar{\mathbb{R}}^{0|1}\hookrightarrow\mathcal{R}$ for $j=1,\ldots,m$, where π_i (resp. π_{n+j}) are projections of $\overline{\mathbb{R}^{n|m}}$ onto even (resp. odd) coordinates. Then

$$SC^\infty(\mathcal{M})\approx C^\infty(x_1,\ldots,x_n)\otimes\Lambda(\theta_1,\ldots,\theta_m) \ , \qquad (4.7.2)$$

where $C^\infty(x_1,\ldots,x_n)\approx C^\infty(\underline{\mathcal{U}})$ and $\Lambda(\theta_1,\ldots,\theta_m)$ is the Grassman superalgebra with generators $\theta_1, \ldots, \theta_m$.

Let $\mathcal{M}$ be a B.supermanifold. The correspondence $U \mapsto SC^\infty(\mathcal{M}|_U)$, where U runs over all open subsets in the base manifold $\underline{\mathcal{M}}$ of $\mathcal{M}$, defines a sheaf of $\mathbb{R}$-superalgebras on $\underline{\mathcal{M}}$. Denote the corresponding sheaved space $Sh(\mathcal{M})$. Every morphism $f: \mathcal{M} \to \mathcal{M}'$ of B.supermanifolds generates, in an evident manner, some morphism $Sh(f): Sh(\mathcal{M}) \to Sh(\mathcal{M}')$ of spaces sheaved with $\mathbb{R}$-superalgebras. This defines the functor Sh from the category SMan to the category of topological spaces sheaved with $\mathbb{R}$-superalgebras.

A Banach supermanifold $\mathcal{M}$ will be called locally finite-dimensional if there exists an atlas $\{V_\alpha|_{U_\alpha} \to \mathcal{M}\}$ on such that every Banach superspace V_α is finite-dimensional. Let SManfin be the full subcategory of SMan whose objects are just locally finite-dimensional supermanifolds.

Proposition 4.7.1. The functor Sh establishes an equivalence of the category SManfin with the category of supermanifolds of Berezin-Leites-Kostant.

4.8. BLK-supermanifolds as variable Λ-supermanifolds. Here are clarified some relations between supermanifolds of Berezin-Leites-Kostant (BLK-supermanifolds, for brevity) and various types of "supermanifolds over finite-dimensional Grassman algebras"/14-16/.

In what follows, the category BLK-SMan of BLK-supermanifolds will be identified with the category SManfin through the functor Sh defined in Sect.4.7.

Denote, for any $\Lambda \in |\underline{Gr}|$, the category of G^∞-supermanifolds over Λ of Alice Rogers/15/ as Λ-RMan ; the category of H -supermanifolds/14-15/ as Λ-BMan ; the category of supermanifolds over Λ in the sense of Ref./16/ as Λ-JPMan .

One has the inclusions of categories

$$\Lambda\text{-BMan} \subset \Lambda\text{-RMan} \subset \Lambda\text{-JPMan} . \qquad (4.8.1)$$

Note, that the category Λ-RMan does not coincide, generally speaking, with the category Λ-JPMan (for example, ${}_{\bar{0}}\Lambda$-linearity of derivative maps imposes no restrictions at all in the case of $\Lambda = \Lambda_1$).

One can see immediately from the definition of the Jadczyk-Pilch supersmoothness (= C^∞-smoothness + ${}_{\bar{0}}\Lambda$-linearity of derivarives), that "evaluation at point Λ" ($\mathcal{M} \mapsto \mathcal{M}(\Lambda)$, $f \mapsto f_\Lambda$) defines for every $\Lambda \in |\underline{Gr}|$ some functor

$$\text{BLK-SMan} \xrightarrow{\overline{\pi}_\Lambda} \Lambda\text{-JPMan} . \qquad (4.8.2)$$

The functor $\overline{\pi}_\Lambda$ is, for any Λ , neither full nor faithful.

When Λ runs in Gr we obtain, hence, for every BLK-supermanifold $\mathcal{M}$ (resp. for every morphism f of supermanifolds) some "object section" $\Lambda \mapsto \mathcal{M}(\Lambda)$ (resp. some "morphism section" $\Lambda \mapsto f_\Lambda$) of the "bundle" $\coprod_{\Lambda \in |\underline{Gr}|} \Lambda\text{-JPMan} \longrightarrow |\underline{Gr}|$, which permits us to consider BLK-supermanifolds as "variable" Jadchyk-Pilch supermanifols depending on a discrete parameter Λ .

Now one can formulate the relation between the category of BLK-supermanifolds and the supermanifolds of Jadczyk-Pilch in the following tautological motto: BLK-supermanifolds (and their morphisms) are just those sections of the "bundle" $\coprod_{\Lambda \in |\underline{Gr}|} \Lambda\text{-JPMan} \longrightarrow |\underline{Gr}|$, which are "analytic" (=functorial) on the discrete parameter Λ .

Moreover, Theorem 3.3.2 implies that every functorial on Λ section of the bundle $\coprod_{\Lambda \in |\underline{Gr}|} \Lambda\text{-JPMan} \longrightarrow |\underline{Gr}|$ belongs, in fact, to the subbundle $\coprod_{\Lambda \in |\underline{Gr}|} \Lambda\text{-BMan} \longrightarrow \underline{Gr}$.

Note that the category Λ_L-BMan contains the category $\mathcal{SM}_L$ of M.Batchelor/14/ as a full subcategory and is equivalent to our category $\text{SManfin}^{(L)}$ of locally finite-dimensional L-supermanifolds. In particular, for every $L' \geqslant L$ there is defined a "projection" functor $\overline{\pi}^{L'}_L : \Lambda_{L'}\text{-BMan} \longrightarrow \Lambda_L\text{-BMan}$ (see Prop.4.2.1), and BLK-supermanifolds can be characterized in terms of projective limites as

$$\text{BLK-SMan} \approx \text{Proj lim}\{\Lambda_L\text{-BMan}\} , \qquad (4.8.3)$$

in addition to M.Batchelor's characterization of them as inductive limite of her categories $\mathcal{SM}_L$.

To conclude with, the author hopes the reader could see now, that pretentious declarations of A.Rogers stating that her definition of G^{∞}-supermanifolds "embraces" that of BLK-supermanifolds /15/, is exactly as reasonable as, say, the statement that the "definition of complex numbers embraces that of analytic functions".

5. Vector Bundles in the Category of Supermanifolds.

5.1. The definition. The triple $(\mathcal{M}\times\mathcal{V}, \mathcal{M}, \pi_{\mathcal{M}})$ will be called a trivial real vector bundle in the category SMan or, simply, trivial s.vector bundle, if $\mathcal{M}$ is a supermanifold (called base of a given s.vector bundle), $\mathcal{V}$ is an $\bar{\mathbb{R}}$-module and $\pi_{\mathcal{M}}: \mathcal{M}\times\mathcal{V}\to\mathcal{M}$ is a canonical projection. We will often write simply $\mathcal{M}\times\mathcal{V}$ instead of $(\mathcal{M}\times\mathcal{V}, \mathcal{M}, \pi_{\mathcal{M}})$. Morphism of a trivial s.vector bundle $\mathcal{M}\times\mathcal{V}$ into a trivial s.vector bundle $\mathcal{M}'\times\mathcal{V}'$ is a pair $(f: \mathcal{M}\times\mathcal{V}\to\mathcal{M}'\times\mathcal{V}', g: \mathcal{M}\to\mathcal{M}')$ such that $\pi_{\mathcal{M}'}\circ f = g\circ\pi_{\mathcal{M}}$ and $\pi_{\mathcal{V}'}\circ f: \mathcal{M}\times\mathcal{V}\to\mathcal{V}'$ is an $\mathcal{M}$-family of $\bar{\mathbb{R}}$-linear morphisms (see Section 1.3 for the definition).

Open s.vector subbundles and the corresponding pretopology on the category of trivial s.vector bundles are defined in an evident way; besides, one has an obvious neglecting functor sending trivial s.vector bundles into the functor category $\underline{\mathrm{VBun}}^{\mathrm{Gr}}$. This permits one to define the category SVBun of (Banach) vector superbundles by means of atlases on functors in $\underline{\mathrm{VBun}}^{\mathrm{Gr}}$ just in the same way as we have defined supermanifolds, with obvious changes (for the abstract theory of glueing, atlases, etc. see the authors paper /17/). In particular, there is defined the canonical neglecting functor

$$\underline{\mathrm{SVBun}} \xrightarrow{N_{VB}} \underline{\mathrm{VBun}}^{\mathrm{Gr}} . \tag{5.1.1}$$

Besides, there is defined the functor $\underline{\mathrm{SVBun}} \longrightarrow \underline{\mathrm{SMan}}$ sending any s.vector bundle $\mathcal{E}\xrightarrow{\pi}\mathcal{M}$ to its base $\mathcal{M}$.

Note that due to Coroll.4.4.3 s.vector bundles could be constructed by means of cocicles, i.e. families of morphisms of supermanifolds of the type $\{\mathcal{N}_\alpha\cap\mathcal{N}_\beta\xrightarrow{\theta_{\alpha\beta}}\mathcal{L}_{\bar{\mathbb{R}}}(\mathcal{V},\mathcal{V})\}$ where $\{\mathcal{N}_\alpha\hookrightarrow\mathcal{M}\}$ is some open covering of a supermanifold $\mathcal{M}$ by open subsupermanifolds and the family $\{\theta_{\alpha\beta}\}$ satisfies cocicle conditions

$$\theta_{\alpha\beta}\theta_{\beta\alpha} = 1 , \quad \theta_{\alpha\beta}\theta_{\beta\gamma}\theta_{\gamma\alpha} = 1 . \tag{5.1.2}$$

The products in the l.h.s. of eqs.(5.1.2) are defined just because $\mathcal{L}_{\bar{\mathbb{R}}}(\mathcal{V},\mathcal{V})$ is an $\bar{\mathbb{R}}$-algebra (see Sect.1); in fact, the corresponding "functions" $\theta_{\alpha\beta}$ "take values" in the Lie supergroup (= group in the category SMan) $\mathcal{GL}_{\mathbb{R}}(\mathcal{V})$ defined in Sect. 7.1 below.

5.2. Inverse images. Let $\mathcal{E}\xrightarrow{\pi}\mathcal{M}$ be a s.vector bundle with a base $\mathcal{M}$ and let $f: \mathcal{M}'\to\mathcal{M}$ be some morphism of supermanifolds. Define in the category $\underline{\mathrm{VBun}}^{\mathrm{Gr}}$ a functor $f^*\mathcal{E}$ pointwise as $(f^*\mathcal{E})(\Lambda) = f_\Lambda^*\mathcal{E}(\Lambda)$. The functor $f^*\mathcal{E}$ can be canonically equipped with the structure of s.vector bundle in such a way that it becomes an inverse image of of s.vector bundle $\mathcal{E}$ along the morphism f with all usual properties of inverse images. The bundle $f^*\mathcal{E}\to\mathcal{M}'$ is, as a matter of fact, the pullback projection of the pullback of $\mathcal{E}\xrightarrow{\pi}\mathcal{M}$ along f.

In particular, if $p\xrightarrow{x}\mathcal{M}$ is some point of $\mathcal{M}$, define the fiber $\mathcal{E}_x$ of the s.vector bundle $\mathcal{E}$ at point x as follows: $\mathcal{E}_x := x^*\mathcal{E}$; the fiber $\mathcal{E}_x$ is canonically equipped with the structure of $\bar{\mathbb{R}}$-module. If, besides, $f: \mathcal{E}\to\mathcal{E}'$ is some morphism of s.vector bundles, then there is defined, due to the properties of inverse images, the canonical morphism $f_x: \mathcal{E}_x\longrightarrow\mathcal{E}'_{f\circ x}$.

5.3. The tangent functor and superderivative morphisms. Define, for every supermanifold $\mathcal{M}$, the functor $\mathcal{T}\mathcal{M}$ in $\underline{\mathrm{VBun}}^{\mathrm{Gr}}$ pointwise as follows: $\mathcal{T}\mathcal{M}(\Lambda) = T(\mathcal{M}(\Lambda))$. If $f: \mathcal{M}\to\mathcal{M}'$ is morphism of supermanifolds, define the functor morphism $\mathcal{T}f: \mathcal{T}\mathcal{M}\to\mathcal{T}\mathcal{M}'$ as $(\mathcal{T}f)_\Lambda = Tf_\Lambda$. This determines some functor $\underline{\mathrm{SMan}}\xrightarrow{\mathcal{T}}\underline{\mathrm{VBun}}^{\mathrm{Gr}}$ which actually lifts to the functor (denoted here by the same letter $\mathcal{T}$)

$$\underline{\mathrm{SMan}} \xrightarrow{\mathcal{T}} \underline{\mathrm{SVBun}} \tag{5.3.1}$$

along the neglecting functor (5.1.1). The functor (5.3.1) will be called the tangent functor.

If $p\xrightarrow{x}\mathcal{M}$ is a point of $\mathcal{M}$ and $f: \mathcal{M}\to\mathcal{M}'$ is some morphism of supermanifolds, we will write $\mathcal{T}_x\mathcal{M}$ in-stead of $(\mathcal{T}\mathcal{M})_x$ and $\mathcal{T}_x f$ instead of $(\mathcal{T}f)_x$.

Given a B.superregion $\mathcal{V}|_U$ one can identify the tangent bundle $\mathcal{T}(\mathcal{V}|_U)$ with the trivial s.vector bundle $\mathcal{V}|_U\times\mathcal{V}$; if $f: \mathcal{V}|_U\to\mathcal{V}'|_{U'}$ is a supersmooth morphism of B.superregions, then the morphism $\pi_{\mathcal{V}'}\circ\mathcal{T}f: \mathcal{V}|_U\times\mathcal{V}\to\mathcal{V}'$ is just the weak superderivative morphism $\mathcal{D}^w f$,defined in Sect.3.2 as $(\mathcal{D}^w f)_\Lambda(u,v) = Df_\Lambda(u)v$.

In accord with Coroll.4.4.3 there exists the only morphism $\mathcal{D}f\colon \mathcal{V}|_U \to \mathcal{L}_{\bar{\mathbb{R}}}(\mathcal{V},\mathcal{V}')$ (the super derivative morphism of f) such that $\mathcal{D}^W f = ev\circ(\mathcal{D}f \times Id_{\mathcal{V}})$.

Superderivative morphisms possess many of the properties of ordinary derivative maps. It is left to the reader to formulate, say, the chain rule (using morphism comp of Sect.1.5) reflecting the functoriality of $\mathcal{T}$.

5.4. Vector bundles in the categories $\underline{SMan}^{(m)}$. One can define the category $\underline{SVBun}^{(m)}$ of vector bundles in the category $\underline{SMan}^{(m)}$ of m-supermanifolds with finite m , repeating literally the corresponding definitions of the case $m=\infty$; one can define as well the tangent functor $\underline{SMan}^{(m)} \xrightarrow{\mathcal{T}} \underline{SVBun}^{(m)}$.

Note, nevertheless, that, generally speaking, vector bundles in $\underline{SMan}^{(m)}$ (with finite $m\neq 0$) could not be constructed by means of cocicles; besides, the superderivative map $\mathcal{D}f$ (contrary to the tangent map $\mathcal{T}f$) for a morphism f in $SMan^{(m)}$ with finite $m\neq 0$, could be uniquelly determined only as morphism in $\underline{SMan}^{(m-1)}$.

5.5. Vector functors. The definition and the main properties of vector functors for the category $\underline{SMan}$ is similar to that for the "non-super" case (see, e.g. Ref./13/).

In particular, for given s.vector bundles $\mathcal{E}$ and $\mathcal{E}'$ over one and the same base $\mathcal{M}$ one can define s.vector bundles $\mathcal{E}\oplus\mathcal{E}'$ and $\mathcal{L}(\mathcal{E},\mathcal{E}')$ in such a way that locally (= for trivial s.vector bundles) $(\mathcal{M}\times\mathcal{V})\oplus(\mathcal{M}\times\mathcal{V}') = \mathcal{M}\times(\mathcal{V}\oplus\mathcal{V}')$ and $\mathcal{L}(\mathcal{M}\times\mathcal{V},\mathcal{M}\times\mathcal{V}') = \mathcal{M}\times\mathcal{L}_{\bar{\mathbb{R}}}(\mathcal{V},\mathcal{V}')$. Define as well $\mathcal{E}^*$ as $\mathcal{E}^* := \mathcal{L}(\mathcal{E},\bar{\mathbb{R}}_{\mathcal{M}})$, where $\mathcal{V}_{\mathcal{M}}$ denotes the trivial s.vector bundle $\mathcal{M}\times\mathcal{V}\longrightarrow\mathcal{M}$.

One should note that whereas the functors of evaluation at point Λ commute with $\mathcal{E}\oplus\mathcal{E}'$ (i.e. $(\mathcal{E}\oplus\mathcal{E}')(\Lambda) = \mathcal{E}(\Lambda)\oplus\mathcal{E}'(\Lambda)$), this is not the case for $\mathcal{L}(\mathcal{E},\mathcal{E}')$.

5.6. Change of parity functor for s.vector bundles. An important role in the theory of supermanifolds plays the natural extension $\bar{\Pi}\colon \underline{SVBun} \longrightarrow \underline{SVBun}$ of the change of parity functor $\bar{\Pi}\colon \underline{Mod}_{\bar{\mathbb{R}}}(\underline{SMan}) \longrightarrow \underline{Mod}_{\bar{\mathbb{R}}}(\underline{SMan})$ defined in Sect.4.4.

To define the functor $\bar{\Pi}$ for trivial s.vector bundles note first of all that the natural isomorphism

$$\mathcal{L}_{\bar{\mathbb{R}}}(\mathcal{V};\mathcal{V}') \overset{\bar{\Pi}}{\simeq} \mathcal{L}_{\bar{\mathbb{R}}}(\bar{\Pi}\mathcal{V};\bar{\Pi}\mathcal{V}') \qquad (5.6.1)$$

extends, as a consequence of Coroll.4.4.3, to the natural isomorphism

$$L_{\bar{\mathbb{R}}}(\mathcal{M};\mathcal{V};\mathcal{V}') \overset{\bar{\Pi}}{\simeq} L_{\bar{\mathbb{R}}}(\mathcal{M};\bar{\Pi}\mathcal{V};\bar{\Pi}\mathcal{V}') , \qquad (5.6.2)$$

which sends an $\mathcal{M}$-family of $\bar{\mathbb{R}}$-linear morphisms $f\colon \mathcal{M}\times\mathcal{V}\to\mathcal{V}'$ to an $\mathcal{M}$-family of $\bar{\mathbb{R}}$-linear morphisms

$$\bar{\Pi}f\colon \mathcal{M}\times\bar{\Pi}\mathcal{V} \xrightarrow{\tilde{P}_1(f)\times Id} \mathcal{L}_{\bar{\mathbb{R}}}(\mathcal{V};\mathcal{V}')\times\bar{\Pi}\mathcal{V} \simeq \mathcal{L}_{\bar{\mathbb{R}}}(\bar{\Pi}\mathcal{V};\bar{\Pi}\mathcal{V}')\times\bar{\Pi}\mathcal{V} \xrightarrow{ev} \bar{\Pi}\mathcal{V}', \qquad (5.6.3)$$

where $\tilde{P}_1$ is defined in Sect.1.5 (see eq.(1.5.5)).

If now $\mathcal{M}\times\mathcal{V}$ is a trivial s.vector bundle, put $\bar{\Pi}(\mathcal{M}\times\mathcal{V}) = \mathcal{M}\times\bar{\Pi}\mathcal{V}$. Besides, one can define the action of $\bar{\Pi}$ on morphisms of trivial s.vector bundles using the isomorphisms (5.6.2) and the fact that the set of all morphisms (f,g) of a trivial s.vector bundle $\mathcal{M}\times\mathcal{V}$ into a trivial s.vector bundle $\mathcal{M}\times\mathcal{V}'$ over a fixed morphism $f\colon \mathcal{M}\times\mathcal{M}'$ of bases is in an evident one-to-one correspondence with the set $L_{\bar{\mathbb{R}}}(\mathcal{M};\mathcal{V};\mathcal{V}')$.

In fact, the action $\bar{\Pi}$ thus defined, is an extension of the functor $\bar{\Pi}$ of Sect. 4.4 to the category of trivial s.vector bundles; this extension is, obviously, continuous functor, which permits one to construct automatically the desired functor

$$\bar{\Pi}\colon \underline{SVBun} \longrightarrow \underline{SVBun} \qquad (5.6.4)$$

by means of "completion of functors by continuity" procedure, described in Ref./17/ .

5.7. The functor $\mathcal{R}\otimes_{\bar{\mathbb{R}}}\cdot$. Let $\mathcal{V}$ be an $\bar{\mathbb{R}}$-module. Define an $\bar{\mathbb{R}}$-module $\mathcal{V}_{\mathcal{R}}$ as

$$\mathcal{V}_{\mathcal{R}} := \mathcal{R}\otimes_{\bar{\mathbb{R}}}\mathcal{V} , \qquad (5.7.1)$$

where $\mathcal{R}$ is the "coordinate ring" defined in Sect.4.7 . If, further, $f\colon\mathcal{V}\to\mathcal{V}'$ is a morphism of $\bar{\mathbb{R}}$-modules, define the morphism $f_{\mathcal{R}}\colon \mathcal{V}_{\mathcal{R}}\longrightarrow\mathcal{V}'_{\mathcal{R}}$ as $f_{\mathcal{R}} = Id_{\mathcal{R}}\otimes f$.

The correspondence $\mathcal{V}\longmapsto\mathcal{V}_{\mathcal{R}}$ and $f\longmapsto f_{\mathcal{R}}$ is, in fact, a functor, and there exists an evident functor isomorphism

$$\mathcal{V}_{\mathcal{R}} \approx \mathcal{V}\oplus\bar{\Pi}\mathcal{V} . \qquad (5.7.2)$$

Besides, if $\bar{V}\xrightarrow{I}\mathcal{V}$ is some isomorphism of $\bar{\mathbb{R}}$-modules, it generates for every Λ an isomorphism

$$\Lambda\otimes_{\mathbb{R}} V \approx \bar{V}_{\mathcal{R}}(\Lambda) \overset{I_\Lambda}{\simeq} \mathcal{V}(\Lambda) , \qquad (5.7.3)$$

permitting one to equipe $\mathcal{V}_{\mathcal{R}}$ with the structure of an $\mathcal{R}$-module. This structure does not depend, in fact, on the choise of an isomorphism I, and for every morphism f of $\bar{\mathbb{R}}$-modules the morphism $f_{\mathcal{R}}$ turns out to be a morphism of $\mathcal{R}$-modules.

We have defined thus the functor $\mathcal{R}\otimes\cdot$ as the functor from the category of $\bar{\mathbb{R}}$-modules to the category of $\mathcal{R}$-modules.

The functor $\mathcal{R}\otimes\cdot$ is, in fact, a covariant supersmooth vector functor, so that one can extend it to the whole category of s.vector bundles. Bearing in mind the canonical isomorphism (5.7.2), one can as well simply define $\mathcal{E}_{\mathcal{R}}$ as

$$\mathcal{E}_{\mathcal{R}} := \mathcal{E} \oplus \Pi \mathcal{E} , \tag{5.7.4}$$

for every s.vector bundle $\mathcal{E}$.

Moreover, for every s.vector bundle $\mathcal{E}$ the s.vector bundle $\mathcal{E}_{\mathcal{R}}$ can be canonically equipped with the structure of a bundle of $\mathcal{R}$-modules (to define the latter, replace $\bar{\mathbb{R}}$ by $\mathcal{R}$ in the definition of s.vector bundles). The details are left to the reader.

5.8. Extended sections of s.vector bundles. Let $\mathcal{E} \xrightarrow{\pi} \mathcal{M}$ be an s.vector bundle. Morphism $s: \mathcal{M} \to \mathcal{E}$ of supermanifolds is called a section of s.vector bundle $\mathcal{E}$ if $\pi \circ s = \mathrm{Id}_{\mathcal{M}}$. Denote $\Gamma(\mathcal{E})$ the set of sections of $\mathcal{E}$. Sections of $\mathcal{E}_{\mathcal{R}}$ will be called extended sections of $\mathcal{E}$ and we will write $\Gamma_{\mathcal{R}}(\mathcal{E})$ instead of $\Gamma(\mathcal{E}_{\mathcal{R}})$.

Extended sections of the tangent bundle $\mathcal{T}\mathcal{M}$ of a supermanifold $\mathcal{M}$ will be called vector fields on $\mathcal{M}$; extended sections of $(\mathcal{T}\mathcal{M})^*$ will be called differential 1-forms (or covector fields) on $\mathcal{M}$.

Let $\mathcal{M}\times\mathcal{V} \to \mathcal{M}$ be a trivial s.vector bundle and $s \in \Gamma(\mathcal{M}\times\mathcal{V})$ be its section. One can see that $s = (\mathrm{Id}_{\mathcal{M}}, s')$,where $s' = \bar{\pi}_{\mathcal{V}} \circ s$ is the principal part of the section s . The correspondence $s \mapsto s'$ determines a bijection

$$\Gamma(\mathcal{M}\times\mathcal{V}) \xrightarrow{\sim} SC^{\infty}(\mathcal{M},\mathcal{V}) . \tag{5.8.1}$$

If, besides, $\mathcal{V}$ is an $\mathcal{R}$-module, then the bijection (5.8.1) permits one to equip the set of sections $\Gamma(\mathcal{M}\times\mathcal{V})$ of the bundle of $\mathcal{R}$-modules $\mathcal{M}\times\mathcal{V}$ with the structure of an $SC^{\infty}(\mathcal{M})$-module.

More generally, if $\mathcal{E} \xrightarrow{\pi} \mathcal{M}$ is an arbitrary bundle of $\mathcal{R}$-modules, the set $\Gamma(\mathcal{E})$ of sections of $\mathcal{E}$ could be naturally equipped with the structure of an $SC^{\infty}(\mathcal{M})$-module in such a way that for every atlas $\{\mathcal{E}_\alpha \xrightarrow{i_\alpha} \mathcal{E}\}$ (with $\mathcal{E}_\alpha$ being trivial bundles of $\mathcal{R}$-modules) all induced maps $\Gamma(\mathcal{E}) \xrightarrow{i_\alpha^*} \Gamma(\mathcal{E}_\alpha)$ ~~are morphisms~~ of modules over commutative associative $\bar{\mathbb{R}}$-superalgebras with unity.

5.9. Differential of a superfield. If $\mathcal{V}|_U$ is a B.superregion, one can identify its tangent bundle $\mathcal{T}(\mathcal{V}|_U)$ with the trivial bundle $\mathcal{V}|_U \times \mathcal{V}$ and cotangent bundle $\mathcal{T}(\mathcal{V}|_U)^*$ with the trivial s.vector bundle $\mathcal{V}|_U \times \mathcal{L}_{\bar{\mathbb{R}}}(\mathcal{V},\bar{\mathbb{R}})$. Let now $f: \mathcal{V}|_U \to \mathcal{R}$ be a superfield on $\mathcal{V}|_U$.

Noting that the evident natural isomorphism $\mathcal{L}_{\mathbb{R}}(V;\Pi V') \simeq \Pi\mathcal{L}_{\mathbb{R}}(V;V')$ of $\mathbb{R}$-supermodules generates the natural isomorphism

$$\mathcal{L}_{\bar{\mathbb{R}}}(\mathcal{V},\mathcal{V}'_{\mathcal{R}}) \approx \mathcal{L}_{\bar{\mathbb{R}}}(\mathcal{V},\mathcal{V}')_{\mathcal{R}} \tag{5.9.1}$$

of $\bar{\mathbb{R}}$-modules, define the differential df of the superfield f as a covector field

on $\mathcal{V}|_U$ with principal part $(df)'$ defined as a composition

$$\mathcal{V}|_U \xrightarrow{\mathcal{D}f} \mathcal{L}_{\bar{\mathbb{R}}}(\mathcal{V},\mathcal{R}) \approx \mathcal{L}_{\bar{\mathbb{R}}}(\mathcal{V},\bar{\mathbb{R}}_{\mathcal{R}}) \xrightarrow{\sim} \mathcal{L}_{\bar{\mathbb{R}}}(\mathcal{V},\bar{\mathbb{R}})_{\mathcal{R}} . \tag{5.9.2}$$

Let now $\mathcal{M}$ be an arbitrary supermanifold and $A = \{\mathcal{U}_\alpha \xrightarrow{i_\alpha} \mathcal{M}\}$ be an atlas of $\mathcal{M}$. The atlas A generates, in an evident way, some atlas $\{(\mathcal{T}\mathcal{U}_\alpha)^*_{\mathcal{R}} \xrightarrow{i'_\alpha} (\mathcal{T}\mathcal{M})^*_{\mathcal{R}}\}$ on $(\mathcal{T}\mathcal{M})^*_{\mathcal{R}}$. Let $f: \mathcal{M} \to \mathcal{R}$ be a superfield on $\mathcal{M}$. Then there exists the only covector field df on $\mathcal{M}$ such that $(df)\circ i_\alpha = i'_\alpha \circ d(f\circ i_\alpha)$ for every α . The covector field df thus defined does not depend on the choise of an atlas A .

5.10. Action of vector fields on superfields and the Lie bracket. Let first $\mathcal{V}|_U$ be a B.superregion, $f: \mathcal{V}|_U \to \mathcal{R}$ be a superfield on $\mathcal{V}|_U$ and ξ be some vector field on $\mathcal{V}|_U$ with principal part $\xi': \mathcal{V}|_U \to \mathcal{V}$. Define the superfield $\xi.f$ on $\mathcal{V}|_U$ as a composition

$$\xi.f: \mathcal{V}|_U \xrightarrow{((df)',\xi')} \mathcal{L}_{\bar{\mathbb{R}}}(\mathcal{V},\bar{\mathbb{R}})_{\mathcal{R}} \times \mathcal{V}_{\mathcal{R}} \xrightarrow{ev_{\mathcal{R}}} \bar{\mathbb{R}}_{\mathcal{R}} \approx \mathcal{R} . \tag{5.10.1}$$

Let now $\mathcal{M}$ be an arbitrary supermanifold, f be a superfield on $\mathcal{M}$ and ξ be a vector field on $\mathcal{M}$. Then there exists the only superfield ξf on $\mathcal{M}$ such that for every chart $\mathcal{U} \to \mathcal{M}$ on $\mathcal{M}$ the identity $(\xi.f)|_{\mathcal{U}} = \xi|_{\mathcal{U}} . f|_{\mathcal{U}}$ holds.

For every vector field ξ the map $f \mapsto \xi.f$ is a superderivation of the $\mathcal{R}$-superalgebra $SC^{\infty}(\mathcal{M})$.

If ξ_1 and ξ_2 are two vector fields on $\mathcal{M}$,then there exists the only vector field $[\xi_1,\xi_2]$ on $\mathcal{M}$ such that for every superfield f on $\mathcal{M}$ the identity

$$[\xi_1,\xi_2].f = \xi_1.(\xi_2.f) - \sum_{\varepsilon_1,\varepsilon_2 \in \mathbb{Z}_2} (-)^{\varepsilon_1\varepsilon_2}\, {}_{\varepsilon_2}\xi_2 . ({}_{\varepsilon_1}\xi_1 . f) \tag{5.10.2}$$

holds. The real superspace $\Gamma_{\mathcal{R}}(\mathcal{T}\mathcal{M})$ of vector fields on $\mathcal{M}$, equpped with the operation $[\ ,\ ]$, is a real Lie superalgebra.

6. Immersions, Submersions, Subsupermanifolds, etc.

6.1. Definitions. We will call every morphism of supermanifolds of the form $\mathcal{M} \approx \mathcal{M}\times P \xrightarrow{\mathrm{Id}\times x} \mathcal{M}\times\mathcal{M}'$ (resp. of the form $\mathcal{M}\times\mathcal{M}' \xrightarrow{\pi} \mathcal{M}$) a standard imbedding (resp. a standard projection).

A morphism $f: \mathcal{M} \to \mathcal{M}'$ of supermanifolds will be called an immersion (resp. a submersion, resp. a local isomorphism) if there exists a family of pullbacks

$$\begin{tikzcd} \mathcal{U}_\alpha \arrow[r, "f_\alpha"] \arrow[d, "\mathrm{Id}"'] & \mathcal{U}'_\alpha \arrow[d, "i'_\alpha"] \\ \mathcal{U}_\alpha \arrow[r, "i_\alpha"] & \mathcal{M} \arrow[r, "f"] & \mathcal{M}' \end{tikzcd} \qquad (\alpha \in A) \tag{6.1.1}$$

such that $\{\mathcal{U}_\alpha \xrightarrow{i_\alpha} \mathcal{M}\}$ is an open covering of $\mathcal{M}$, every i' is an open morphism and every f is a standard imbedding (resp. standard projection, resp. an isomorphism); morphism f will be called an imbedding if there exists a family of pullbacks

$$\begin{array}{ccc} f^{-1}(\mathcal{U}_\alpha) & \xrightarrow{f_\alpha} & \mathcal{U}_\alpha \\ \downarrow & & \downarrow i'_\alpha \\ \mathcal{M} & \xrightarrow{f} & \mathcal{M}' \end{array} \qquad (\alpha \in A) \qquad (6.1.2)$$

such that $\{\mathcal{U}'_\alpha \xrightarrow{i'_\alpha} \mathcal{M}'\}$ is an open covering of $\mathcal{M}'$ and every f_α is a standard imbedding. At last, a supermanifold $\mathcal{M}$ will be called a subsupermanifold of a supermanifold $\mathcal{M}'$, if $\mathcal{M}$ is a set-valued subfunctor of the functor $\mathcal{M}'$ and if, besides, the inclusion morphism $\mathcal{M} \subset \mathcal{M}'$ is an imbedding (which includes that it is supersmooth).

6.2. Morphisms criterions modulo manifolds.

Proposition 6.2.1. a) If some morphism f of supermanifolds is an immersion (resp. submersion, resp. local isomorphism, resp. imbedding) then for every Λ the morphism f_Λ of B.manifolds is an immersion (resp. submersion, resp. local isomorphism, resp. imbedding).

b) If some morphism f of supermanifolds is such that the morphism f_{Λ_1} of manifolds is an immersion (resp. submersion, resp. local isomorphism, resp. isomorphism, resp. imbedding) then f is an immersion (resp. submersion, resp. local isomorphism, resp. isomorphism, resp. imbedding).

Corollary 6.2.2. A morphism $f: \mathcal{M} \to \mathcal{M}'$ of supermanifolds is an isomorphism iff the morphism $\overline{\mathcal{K}}{}^\infty_1(f): \overline{\mathcal{K}}{}^\infty_1(\mathcal{M}) \to \overline{\mathcal{K}}{}^\infty_1(\mathcal{M}')$ of vector bundles (see Sect.4.2) is an isomorphism.

6.3. Differential criterions for morphisms. Let $h: \mathcal{N} \to \mathcal{M}$ be a morphism of supermanifolds. An open subsupermanifold $\mathcal{U}$ of $\mathcal{M}$ will be called an open neighbourhood of the morphism h if h lifts to $\mathcal{U}$ along the inclusion morphism $\mathcal{U} \subset \mathcal{M}$. A morphism $f: \mathcal{M} \to \mathcal{M}'$ is said to be an immersion (resp. a submersion, resp. a local isomorphism) in some neighbourhood of the morphism h if there exists an open neighbourhood $\mathcal{U} \subset \mathcal{M}$ of h such that the morphism $f|_U: \mathcal{U} \to \mathcal{M}'$ is an immersion (resp.submersion, resp. a local isomorphism).

Proposition 6.3.1 (Inverse function theorem). A morphism $f: \mathcal{M} \to \mathcal{M}'$ of supermanifolds is a local isomorphism in some neighbourhood of a point $p \xrightarrow{x} \mathcal{M}$

iff the morphism $\mathcal{T}_x\mathcal{M} \xrightarrow{\mathcal{T}_x f} \mathcal{T}_{f\circ x}\mathcal{M}'$ is an isomorphism of $\overline{\mathcal{R}}$-modules.

To formulate the corresponding criterions for immersions and submersions we need a notion of direct morphisms of modules. Let $\mathcal{V}$ be an $\overline{\mathcal{R}}$-module and $\mathcal{V}'$ be some its submodule. The submodule $\mathcal{V}'$ is called direct if there exists an $\overline{\mathcal{R}}$-module $\mathcal{V}''$ and an isomorphism $\mathcal{V}' \oplus \mathcal{V}'' \simeq \mathcal{V}$ of $\overline{\mathcal{R}}$-modules. More generally, a morphism $g: \mathcal{V}' \to \mathcal{V}$ of $\overline{\mathcal{R}}$-modules is called direct if it is isomorphic (as an object of the category of $\overline{\mathcal{R}}$-modules over $\mathcal{V}$) to the inclusion of some direct submodule of $\mathcal{V}$.

Proposition 6.3.2. A morphism $f: \mathcal{M} \to \mathcal{M}'$ of supermanifolds is an immersion in some neighbourhood of a point $p \xrightarrow{x} \mathcal{M}$ iff the morphism $\mathcal{T}_x\mathcal{M} \xrightarrow{\mathcal{T}_x f} \mathcal{T}_{f\circ x}\mathcal{M}'$ is direct; it is a submersion in some neighbourhood of x iff $\mathcal{T}_x f$ is an epimorphism and $\operatorname{Ker} \mathcal{T}_x f$ is a direct submodule of the $\overline{\mathcal{R}}$-module $\mathcal{T}_x\mathcal{M}$.

In conclusion of this section we will formulate some useful criterion permitting one to see whether some smooth subfunctor $\mathcal{N}$ of a supermanifold $\mathcal{M}$ is a subsupermanifold of $\mathcal{M}$ (a functor $\mathcal{N}$ in $\underline{\mathrm{Man}}^{\mathrm{Gr}}$ is called a smooth subfunctor of the functor $\mathcal{M}$ in $\underline{\mathrm{Man}}^{\mathrm{Gr}}$ if for every Λ the manifold $\mathcal{N}(\Lambda)$ is a submanifold of the manifold $\mathcal{M}(\Lambda)$ and the set of inclusions $\{\mathcal{N}(\Lambda) \subset \mathcal{M}(\Lambda)\}_{\Lambda \in |\underline{\mathrm{Gr}}|}$ is a functor morphism).

Theorem 6.3.3. Let $\mathcal{N}$ be a smooth subfunctor of a supermanifold $\mathcal{M}$. If for every point $p \xrightarrow{x} \mathcal{N}$ of $\mathcal{N}$ the functor $\mathcal{T}_x\mathcal{N}$ is a superrepresentable submodule of the $\overline{\mathcal{R}}$-module $\mathcal{T}_x\mathcal{M}$ then there exists on the functor $\mathcal{N}$ the structure of a subsupermanifold of $\mathcal{M}$ (here, of course, the tangent "bundle" $\mathcal{T}\mathcal{N}$ and its "fiber" $\mathcal{T}_x\mathcal{N}$ for a functor $\mathcal{N}$ in $\underline{\mathrm{Man}}^{\mathrm{Gr}}$ are defined pointwise).

7. Lie Supergroups.

7.1. Definition and examples. A group (object) in the category SMan will be called a Lie supergroup.

Example 1. Let $\mathcal{A}$ be an associative $\overline{\mathrm{K}}$-algebra with unity in SMan. Define $\mathcal{A}^*$ as $\mathcal{A}^* := \mathcal{A}|_{\underline{A}^*}$, where $\underline{A}^*$ is the Lie group of invertible elements of the B.algebra $\underline{A}$. Then for every Λ the manifold $\mathcal{A}^*(\Lambda)$ is a Lie group; the Lie group structures on all $\mathcal{A}^*(\Lambda)$ generate the structure of a Lie supergroup on $\mathcal{A}^*$. In particular, if $\mathcal{V}$ is a $\overline{\mathrm{K}}$-module in SMan then $\mathcal{L}_{\overline{\mathrm{K}}}(\mathcal{V};\mathcal{V})$ is an associative $\overline{\mathrm{K}}$-algebra with unity in SMan (see Sect.1.5 and Coroll.4.4.3). The Lie supergroup $\mathcal{L}_{\overline{\mathrm{K}}}(\mathcal{V},\mathcal{V})^*$ will be denoted $\mathcal{GL}_{\mathrm{K}}(\mathcal{V})$.

Let $\mathcal{G}$ be a Lie supergroup and let $\mathcal{H}$ be a subsupermanifold of $\mathcal{G}$ such that for every Λ the manifold $\mathcal{H}(\Lambda)$ is a subgroup of $\mathcal{G}(\Lambda)$ (and, hence, a Lie subgroup of the Lie group $\mathcal{G}(\Lambda)$). The structures of Lie groups on $\mathcal{H}(\Lambda)$ produce, when Λ runs in Gr, the structure of a Lie supergroup on $\mathcal{H}$; the supermanifold $\mathcal{H}$ equipped with this structure of a Lie supergroup is called a Lie subsupergroup of the Lie supergroup $\mathcal{G}$.

A variety of examples of Lie subsupergroups one can obtain considering involutions in associative algebras with unity in the category SMan. Let $\mathcal{A}$ be an associative $\overline{K}$-algebra with unity in SMan. An $\overline{R}$-linear morphism $I: \mathcal{A} \longrightarrow \mathcal{A}$ is called an involution on the algebra $\mathcal{A}$ if $I^2 = \mathrm{Id}_{\mathcal{A}}$ and, besides, if I is an antiautomorphism of the algebra $\mathcal{A}$, i.e. if for every Λ and every $a, b \in \mathcal{A}(\Lambda)$ the identity $I(a \cdot b) = I(b) \cdot I(a)$ holds.

Proposition 7.1.1. Let $I: \mathcal{A} \longrightarrow \mathcal{A}$ be an involution in an associative K-algebra $\mathcal{A}$ with unity in SMan. Define for every Λ the subset $\mathcal{H}_I(\Lambda)$ of $\mathcal{A}^*(\Lambda)$ as the set of all $a \in \mathcal{A}^*(\Lambda)$ such that $I_\Lambda(a) \cdot a = 1$. The family $\{\mathcal{H}_I(\Lambda)\}$ generates a subfunctor $\mathcal{H}_I$ in $\mathcal{A}^*$. The subfunctor $\mathcal{H}_I$ is a subsupermanifold of $\mathcal{A}^*$ and, moreover, a Lie subsupergroup of the Lie supergroup $\mathcal{A}^*$.

Example 2: Hilbert superspaces and unitary Lie supergroups. Let V be a complex Banach superspace and $\langle , \rangle : V \times V \longrightarrow \mathbb{C}$ be an even nondegenerate superhermitean form on V, i.e. $\langle , \rangle$ is $\mathbb{C}$-semilinear on the first argument, $\mathbb{C}$-linear on the second argument and, besides, the identity

$$\overline{\langle x,y \rangle} = \sum_{\varepsilon,\varepsilon' \in \mathbb{Z}_2} (-1)^{\varepsilon\varepsilon'} \langle {}_\varepsilon y, {}_{\varepsilon'} x \rangle \qquad (x,y \in V) \tag{7.1.1}$$

holds. Note that the non-degeneracy of $\langle , \rangle$ implies, together with condition (7.1.1) that V has the topology of a Hilbert space.

Define the map $\dagger : \mathcal{L}_{\mathbb{C}}(V;V) \longrightarrow \mathcal{L}_{\mathbb{C}}(V;V)$ (superhermitean conjugation) as follows:

$$\langle A^\dagger x, y \rangle = \sum_{\varepsilon,\varepsilon' \in \mathbb{Z}_2} (-1)^{\varepsilon\varepsilon'} \langle {}_\varepsilon x, {}_{\varepsilon'} Ay \rangle \qquad (x,y \in V \,;\; A \in \mathcal{L}_{\mathbb{C}}(V;V)) \tag{7.1.2}$$

(we write, respecting traditions, $A^\dagger$ instead of $\dagger(A)$).

The morphism $\overline{\dagger} : \mathcal{L}_{\overline{\mathbb{C}}}(\overline{V};\overline{V}) \longrightarrow \mathcal{L}_{\overline{\mathbb{C}}}(\overline{V};\overline{V})$ is an involution in the algebra $\mathcal{L}_{\overline{\mathbb{C}}}(\overline{V};\overline{V})$. The Lie subsupergroup $\mathcal{H}_\dagger$ of the Lie group $\mathcal{G}\mathcal{L}_{\mathbb{C}}(\overline{V})$ will be called a pseudounitary group of the of the complex B.superspace V, associated with the superhermitean form $\langle , \rangle$; it will be denoted $\mathcal{U}(V, \langle , \rangle)$.

Note that the forms $\langle , \rangle|_{{}_0V}$ and $\langle , \rangle|_{{}_1V}$ are Hermitean forms on ${}_0V$ and ${}_1V$, respectively. If this two forms are definite then the pair $(V, \langle , \rangle)$ (or, simply, V) will be called a Hilbert superspace, whereas the corresponding Lie supergroup $\mathcal{U}(V, \langle , \rangle)$ will be called the unitary supergroup of the Hilbert superspace V.

7.2. Lie theory. Let $\mathcal{G}$ be a Lie supergroup and $p \xrightarrow{e} \mathcal{G}$ be its unity. For every $\Lambda \in |Gr|$ the B.space $(\mathcal{T}_e \mathcal{G})(\Lambda) = T_e \mathcal{G}(\Lambda)$ is at the same time the Lie algebra $L(\mathcal{G}(\Lambda))$ of the Lie group $\mathcal{G}(\Lambda)$ /18/. The structures of Lie algebras on $(\mathcal{T}_e \mathcal{G})(\Lambda)$ generate, when Λ runs in Gr, the structure of a Lie algebra in the category SMan on the fiber $\mathcal{T}_e \mathcal{G}$ of the tangent bundle $\mathcal{T}\mathcal{G}$ of the Lie supergroup $\mathcal{G}$. The $\overline{R}$-module $\mathcal{T}_e \mathcal{G}$ equipped with this structure will be called the Lie algebra (in the category SMan) of the Lie supergroup $\mathcal{G}$ and will be denoted $L(\mathcal{G})$.

In fact, the function L continues, in an evident way, to some functor from the category of Lie supergroups to the category of Lie algebras of the category SMan (Lie functor); composing the Lie functor L with the superization functor S of Sect.4.4 (see (4.4.7)) we will obtain the functor

$$S \circ L : \text{Lie supergroups} \longrightarrow \text{Banach real Lie superalgebras}\,. \tag{7.2.1}$$

The Lie superalgebra $SL(\mathcal{G})$ will be called the Lie superalgebra of a Lie supergroup $\mathcal{G}$.

7.3. Exponential morphism. Define now for every Lie supergroup $\mathcal{G}$ the exponential morphism

$$\exp_{\mathcal{G}} : L(\mathcal{G}) \longrightarrow \mathcal{G} \tag{7.3.1}$$

pointwise: $(\exp_{\mathcal{G}})_\Lambda = \exp_{\mathcal{G}(\Lambda)}$. It is a functor morphism due to functoriality properties of exponential maps in the ordinary Lie theory.

Proposition 7.3.1. The exponential morphism $\exp_{\mathcal{G}}$ is supersmooth; it is a local isomorphism at some open neighbourhood of the point zero $(0: p \longrightarrow L(\mathcal{G}))$ of the Lie algebra $L(\mathcal{G})$.

7.4. The structure of Lie supergroups. Let $\mathcal{G}$ be a Lie supergroup. For every Λ let $N_{\mathcal{G}}(\Lambda)$ be the kernel of the morphism $\mathcal{G}(\Lambda) \xrightarrow{\mathcal{G}(\varepsilon)} \mathcal{G}$ of Lie groups, where $\varepsilon : \Lambda \longrightarrow \mathbb{R}$ is the terminal morphism of superalgebras. It is evident, that the Lie group $\mathcal{G}(\Lambda)$ is a semidirect product

$$\mathcal{G}(\Lambda) \simeq \mathcal{G} \otimes N_{\mathcal{G}}(\Lambda). \qquad (7.4.1)$$

Besides, $N_{\mathcal{G}}(\Lambda)$ is a nilpotent Lie group.

Consider now the canonical $\bar{R}$-module decomposition $L(\mathcal{G}) = {}_{\bar 0}L(\mathcal{G}) \oplus {}_{\bar 1}L(\mathcal{G})$, where ${}_{\bar 0}L(\mathcal{G}) \simeq \overline{L(\mathcal{G})}$ is the "ordinary" Lie algebra and ${}_{\bar 1}L(\mathcal{G})$ is the superpoint corresponding to the odd part of the Lie superalgebra $SL(\mathcal{G})$. Due to Prop.7.3.1 the exponential morphism $\exp_{\mathcal{G}}$ isomorphically maps the superpoint ${}_{\bar 1}L(\mathcal{G})$ onto some superpoint $\mathcal{P}_{\mathcal{G}} \subset \mathcal{G}$ (in fact, $\mathcal{P}_{\mathcal{G}}(\Lambda) \subset N_{\mathcal{G}}(\Lambda)$ for every Λ).

Let for every Λ the map $I_\Lambda : \mathcal{C}^{\circ}_{\infty}(\underline{\mathcal{G}})(\Lambda) \times \mathcal{P}_{\mathcal{G}}(\Lambda) \longrightarrow \mathcal{G}(\Lambda)$ is the restriction of the multiplication in the group $\mathcal{G}(\Lambda)$, i.e. $I_\Lambda(g,x) = g\,x$ (the functor $\mathcal{C}^{\circ}_{\infty}$ of "Grassman analytical continuation" is defined in Sect.4.2).

Proposition 7.4.1. The family $\{I_\Lambda\}_{\Lambda \in |Gr|}$ determines some supersmooth isomorphism

$$I : \mathcal{C}^{\circ}_{\infty}(\underline{\mathcal{G}}) \times \mathcal{P}_{\mathcal{G}} \longrightarrow \mathcal{G} \qquad (7.4.2)$$

of supermanifolds. In particular, every Lie supergroup is a simple supermanifold.

7.5. Inverse Lie theorem modulo manifolds.

Proposition 7.5.1. Let $\mathcal{J}$ be a Lie superalgebra and G be a Lie group such that $L(G) = {}_{\bar 0}\mathcal{J}$. Let, further, there exists a linear smooth action of the Lie group G on the Banach space ${}_{\bar 1}\mathcal{J}$ such that the corresponding infinitesimal action of the Lie algebra ${}_{\bar 0}\mathcal{J}$ on the space ${}_{\bar 1}\mathcal{J}$ coincides with the ajoint action (determined by the Lie bracket in $\mathcal{J}$). Then there exists the only (up to an isomorphism) Lie supergroup $\mathcal{G}$ such that its Lie superalgebra $SL(\mathcal{G})$ coincides with $\mathcal{J}$ and $\underline{\mathcal{G}}$ coincides with G.

7.6. Linear representations of Lie supergroups. Let $\mathcal{V}$ be a $\bar{K}$-module and $\mathcal{G}$ be a Lie supergroup. An action $\rho : \mathcal{G} \times \mathcal{V} \to \mathcal{V}$ of $\mathcal{G}$ on $\mathcal{V}$ is called a $\bar{K}$-linear representation of $\mathcal{G}$ (or a $\mathcal{G}$-module over $\bar{K}$) if ρ is a family of $\bar{K}$-linear morphisms. As a trivial consequence of Coroll.4.4.3 one obtains that the canonical representation of the Lie supergroup $\mathcal{GL}_K(\mathcal{V})$ on $\mathcal{V}$ (i.e. the restriction of the evaluation morphism ev_1) is universal among all linear actions of Lie supergroups on the $\bar{K}$-module $\mathcal{V}$; in particular, $\bar{K}$-linear representations of a Lie supergroup $\mathcal{G}$ in $\mathcal{V}$ are in a bijective correspondence with the set of all morphisms of $\mathcal{G}$ into $\mathcal{GL}_K(\mathcal{V})$.

7.7. Groups in $SMan^{(m)}$. Groups in the category $SMan^{(m)}$ will be called m- Lie supergroups. The following proposition permits one to reduce Lie supergroups and their representations to m -Lie supergroups and their representations if $m \geqslant 3$.

Proposition 7.7.1. Let $m \geqslant 3$. The functor π^{∞}_{m} (defined in Sect.4.2) generates an equivalence of the category of Lie supergroups with the category of m- Lie supergroups; for a given Lie supergroup $\mathcal{G}$ the category of linear representations of $\mathcal{G}$ is equivalent to the category of linear representations of the m- Lie supergroup $\pi^{\infty}_{m}(\mathcal{G})$.

7.8. Factorsupergroups of Lie supergroups. Let $\mathcal{G}$ be a Lie supergroup and $\mathcal{H}$ be some its Lie subsupergroup. Define the functor $\mathcal{G}/\mathcal{H}$ pointwise as follows: $\mathcal{G}/\mathcal{H}(\Lambda) := \mathcal{G}(\Lambda)/\mathcal{H}(\Lambda)$; canonical projections $\mathcal{G}(\Lambda) \xrightarrow{\pi_\Lambda} \mathcal{G}(\Lambda)/\mathcal{H}(\Lambda)$ define the functor morphism $\pi : \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{H}$.

Proposition 7.8.1. There exists the only structure of a supermanifold on the functor $\mathcal{G}/\mathcal{H}$ such that the morphism π is a submersion.

The functor $\mathcal{G}/\mathcal{H}$ equipped with the structure of a supermanifold determined by Prop. will be called a factorsupermanifold of $\mathcal{G}$ over $\mathcal{H}$.

A Lie subsupergroup $\mathcal{H}$ of the Lie supergroup $\mathcal{G}$ is called normal if for every Λ the subgroup $\mathcal{H}(\Lambda)$ of the Lie group $\mathcal{G}(\Lambda)$ is normal.

Proposition 7.8.2. If $\mathcal{H}$ is normal Lie subsupergroup of the Lie supergroup $\mathcal{G}$, then $\mathcal{G}/\mathcal{H}$ is a Lie supergroup w.r.t. multiplication $\mathcal{G}/\mathcal{H} \times \mathcal{G}/\mathcal{H} \longrightarrow \mathcal{G}/\mathcal{H}$ defined pointwise.

This Lie supergroup will be called the factorsupergroup of $\mathcal{G}$ over $\mathcal{H}$.

8. Supergroups of Superdiffeomorphisms.

In this section will be constructed supergroups of superdiffeomorphisms of supermanifolds being the counterparts of groups of diffeomorphisms in a standard theory of B.manifolds. This supergroups exist as group objects in the functor category Set^{Gr}. The latter topos seems to play the same role in the theory of supermanifolds as the topos Set plays in the theory of manifolds: it is the "environement" for various types of objects which arise naturally in the theory of supermanifolds, but not always could "live" within the category of supermanifolds itself (example: orbits of supersmooth actions of Lie supergroups[4]).

8.1. Geometrized Yoneda functor. In this subsection the natural neglecting functor $\mathbf{SMan} \xrightarrow{N} \mathbf{Man}^{\mathbf{Gr}}$ will be interpreted as a "geometrization" of Yoneda functor $\mathbf{SMan} \xrightarrow{H_*} \mathbf{Set}^{\mathbf{SMan}^\circ}$ composed with the functor $\mathbf{Set}^{\mathbf{SMan}^\circ} \longrightarrow \mathbf{Set}^{\mathbf{Spoint}^\circ}$ of restriction to superpoints. Here $\mathbf{Spoint} \subset \mathbf{SMan}$ is the full subcategory of the category $\mathbf{SMan}$ consisting of finite-dimensional superpoints. It is evident from Th.3.3.2 that the category $\mathbf{Spoint}$ is naturally equivalent to the category $\mathbf{Gr}^\circ$ dual to the category of Grassman superalgebras.

Proposition 8.1.1. The functor $\mathbf{SMan} \xrightarrow{H_*} \mathbf{Set}^{\mathbf{SMan}^\circ} \longrightarrow \mathbf{Set}^{\mathbf{Spoint}^\circ} \approx \mathbf{Set}^{\mathbf{Gr}}$ is naturally isomorphic to the neglecting functor N': $\mathbf{SMan} \xrightarrow{N} \mathbf{Man}^{\mathbf{Gr}} \longrightarrow \mathbf{Set}^{\mathbf{Gr}}$.

This proposition gives the desired interpretation. Besides, choosing and fixing some contravariant functor

$$\mathcal{P} : \mathbf{Gr}^\circ \longrightarrow \mathbf{Spoints} \tag{8.1.1}$$

establishing a natural equivalence of categories, one obtains the following important

Corollary 8.1.2. For every supermanifold $\mathcal{M}$ and every Grassman algebra Λ there exists an isomorphism of sets

$$\mathcal{M}(\Lambda) \approx SC^\infty(\mathcal{P}(\Lambda), \mathcal{M}) \tag{8.1.2}$$

natural both on $\mathcal{M}$ and Λ.

8.2. Functors of supermorphisms and of supersections. Let $\mathcal{M}$ and $\mathcal{M}$ be B.supermanifolds. Define the Set-valued functor $\hat{SC}^\infty(\mathcal{M}, \mathcal{M}')$ on the category of Grassman superalgebras as follows:

$$\hat{SC}^\infty(\mathcal{M}, \mathcal{M}')(\Lambda) := SC^\infty(\mathcal{P}(\Lambda) \times \mathcal{M}, \mathcal{M}') . \tag{8.2.1}$$

The functor $\hat{SC}^\infty(\mathcal{M}, \mathcal{M}')$ will be called the functor of supermorphisms of the supermanifold $\mathcal{M}$ into the supermanifold $\mathcal{M}'$. Note that there exists the evident natural isomorphism

$$\underline{\hat{SC}^\infty(\mathcal{M}, \mathcal{M}')} := \hat{SC}^\infty(\mathcal{M}, \mathcal{M}')(\mathbb{R}) \approx SC^\infty(\mathcal{M}, \mathcal{M}') . \tag{8.2.2}$$

Let now $\mathcal{E} \xrightarrow{\pi} \mathcal{M}$ be a s.vector bundle over the base supermanifold $\mathcal{M}$. Define the Set-valued functor $\hat{\Gamma}(\mathcal{E})$ of supersections of the s.vector bundle $\mathcal{E}$ as follows:

$$\hat{\Gamma}(\mathcal{E})(\Lambda) := \Gamma(\pi_{\mathcal{M}}^* \mathcal{E}) , \tag{8.2.3}$$

where $\pi_{\mathcal{M}} : \mathcal{P}(\Lambda) \times \mathcal{M} \longrightarrow \mathcal{M}$ is the canonical projection. We have that

$$\underline{\hat{\Gamma}(\mathcal{E})} := \hat{\Gamma}(\mathcal{E})(\mathbb{R}) \approx \Gamma(\mathcal{E}) . \tag{8.2.4}$$

Note that the composition of sections with the canonical pullback projection $\pi_{\mathcal{M}}^* \mathcal{E} \longrightarrow \mathcal{E}$ gives for every Λ the canonical monomorphism $\hat{\Gamma}(\mathcal{E})(\Lambda) \rightarrow \hat{SC}^\infty(\mathcal{M}, \mathcal{E})(\Lambda)$; the set of all this monomorphisms produces the canonical functor monomorphism

$$\Gamma(\mathcal{E}) \hookrightarrow \hat{SC}^\infty(\mathcal{M}, \mathcal{E}) . \tag{8.2.5}$$

To visualize the functor $\hat{\Gamma}(\mathcal{E})$ consider the case of trivial s.vector bundle $\mathcal{M} \times \mathcal{V} \longrightarrow \mathcal{M}$. In this case, evidently, there exists the natural isomorphism

$$\hat{\Gamma}(\mathcal{M} \times \mathcal{V}) \approx \hat{SC}^\infty(\mathcal{M}, \mathcal{V}) . \tag{8.2.6}$$

Equip now the $\bar{\mathbb{R}}$-module $\mathcal{V}_{\mathbb{R}}$ with the structure of $\bar{\mathbb{R}}$-supermodule setting

$$\sigma(\mathcal{V}_{\mathbb{R}}) = \mathcal{V} ; \quad \tau(\mathcal{V}_{\mathbb{R}}) = \Pi \mathcal{V} . \tag{8.2.7}$$

Then $SC(\mathcal{M}, \mathcal{V}_{\mathbb{R}})$ becomes an $\mathbb{R}$-superspace.

Proposition 8.2.1. There exists. an isomorphism of functors

$$\hat{SC}^\infty(\mathcal{M}, \mathcal{V}) \approx \overline{SC^\infty(\mathcal{M}, \mathcal{V}_{\mathbb{R}})} \tag{8.2.8}$$

natural on $\mathcal{M}$ and $\mathcal{V}$, turning $\hat{SC}^\infty(\mathcal{M}, \mathcal{V})$ into a superrepresentable $\bar{\mathbb{R}}$-module (in $\mathbf{Set}^{\mathbf{Gr}}$).

8.3. Morphisms of composition and of evaluation. In this subsection it will be more convenient to work directly with the category $\mathbf{Spoint}$ instead of equivalent to it category $\mathbf{Gr}^\circ$. The variable $\mathcal{P}$ will run on the set of objects of the category $\mathbf{Spoint}$.

Let $\mathcal{M}$ and $\mathcal{M}'$ be B.supermanifolds. Define the evaluation morphism

$$ev: \hat{SC}^\infty(\mathcal{M}, \mathcal{M}') \times \mathcal{M} \longrightarrow \mathcal{M}' \tag{8.3.1}$$

as follows: for every morphisms $f: \mathcal{P} \times \mathcal{M} \rightarrow \mathcal{M}'$ and $x: \mathcal{P} \longrightarrow \mathcal{M}$ let $ev_{\mathcal{P}}(f,x)$ be the composition

$$ev_{\mathcal{P}}(f,x): \mathcal{P} \xrightarrow{(Id,x)} \mathcal{P} \times \mathcal{M} \xrightarrow{f} \mathcal{M}' \tag{8.3.2}$$

Let now $\mathcal{M}$, $\mathcal{M}'$ and $\mathcal{M}''$ be supermanifolds. Define the functor morphism

$$comp: \hat{SC}^\infty(\mathcal{M}', \mathcal{M}'') \times \hat{SC}^\infty(\mathcal{M}, \mathcal{M}') \longrightarrow \hat{SC}^\infty(\mathcal{M}, \mathcal{M}'') \tag{8.3.3}$$

of composition as follows. If $f: \mathcal{P} \times \mathcal{M} \rightarrow \mathcal{M}'$ and $f': \mathcal{P} \times \mathcal{M}' \rightarrow \mathcal{M}''$ are some morphisms, let $comp_{\mathcal{P}}(f',f)$ be the composition

$$comp_{\mathcal{P}}(f',f): \mathcal{P} \times \mathcal{M} \xrightarrow{(\pi_{\mathcal{P}},f)} \mathcal{P} \times \mathcal{M}' \xrightarrow{f'} \mathcal{M}'' . \tag{8.3.4}$$

Proposition 8.3.1. Morphism comp is an associative composition on the functor $\hat{SC}(\mathcal{M}, \mathcal{M})$. Besides, the point

$$p \xrightarrow{e} \hat{SC}^\infty(\mathcal{M}, \mathcal{M}) \tag{8.3.5}$$

defined as $e_{\mathcal{P}}(p) = \pi_{\mathcal{M}} : \mathcal{P}\times\mathcal{M} \longrightarrow \mathcal{M}$, is the unity of this composition.

8.4. The supergroup of superdiffeomorphisms. Let $\mathcal{M}$ be a B.supermanifold. Define for every Λ the set $\widehat{\mathrm{SDiff}}(\mathcal{M})(\Lambda)$ as the subset of all invertible elements of the semigroup $\widehat{SC}^{\infty}(\mathcal{M},\mathcal{M})(\Lambda)$ (with the composition comp_{Λ} defined in the preceeding subsection).

Proposition 8.4.1. The family $\{\widehat{\mathrm{SDiff}}(\mathcal{M})(\Lambda)\}_{\Lambda\in|\mathrm{Gr}|}$ forms a subfunctor $\widehat{\mathrm{SDiff}}(\mathcal{M})$ in $\widehat{SC}^{\infty}(\mathcal{M},\mathcal{M})$; this subfunctor coincides with the subfunctor $\widehat{SC}^{\infty}(\mathcal{M},\mathcal{M})|_{\widehat{\mathrm{SDiff}}(\mathcal{M})}$, where $\widehat{\mathrm{SDiff}}(\mathcal{M})$ is naturally isomorphic to the set $\mathrm{SDiff}(\mathcal{M})$ of all superdiffeomorphisms (= isomorphisms in SMan) of $\mathcal{M}$. Moreover, the group structures on all $\widehat{\mathrm{SDiff}}(\mathcal{M})(\Lambda)$ produce the structure of a supergroup (= group object in $\mathrm{Set}^{\mathrm{Gr}}$) on the functor $\widehat{\mathrm{SDiff}}(\mathcal{M})$.

The supergroup $\widehat{\mathrm{SDiff}}(\mathcal{M})$ will be called the supergroup of superdiffeomorphisms of the supermanifold $\mathcal{M}$. This supergroup possesses the following universal property.

Proposition 8.4.2. Let $\mathcal{G}\times\mathcal{M} \xrightarrow{\rho} \mathcal{M}$ be an action of a Lie supergroup $\mathcal{G}$ on a supermanifold $\mathcal{M}$. Then there exists the only morphism $\hat{\rho}$ of supergroups (in the category $\mathrm{Set}^{\mathrm{Gr}}$)

$$\hat{\rho} : \mathcal{G} \longrightarrow \widehat{\mathrm{SDiff}}(\mathcal{M}) \qquad (8.4.1)$$

such that the diagramm

$$\begin{tikzcd} & \widehat{\mathrm{SDiff}}(\mathcal{M})\times\mathcal{M} \arrow[dr, "ev"] & \\ \mathcal{G}\times\mathcal{M} \arrow[ur, "\hat{\rho}\times \mathrm{Id}"] \arrow[rr, "\rho"] & & \mathcal{M} \end{tikzcd} \qquad (8.4.2)$$

is commutative. The morphism $\hat{\rho}$ is determined as follows: for every morphism $g: \mathcal{P}(\Lambda) \longrightarrow \mathcal{G}$ the morphism $\hat{\rho}_{\Lambda}(g)$ is the composition

$$\hat{\rho}_{\Lambda}(g): \mathcal{P}(\Lambda)\times\mathcal{M} \xrightarrow{g\times \mathrm{Id}} \mathcal{G}\times\mathcal{M} \xrightarrow{\rho} \mathcal{M} . \qquad (8.4.3)$$

Proposition 8.4.2 permits one, in particular, to define the induced linear action of a Lie supergroup $\mathcal{G}$ on the functor of superfields $\widehat{SC}^{\infty}(\mathcal{M},\mathcal{R})$ of $\mathcal{M}$, if $\mathcal{G}$ acts on $\mathcal{M}$ (use the universal action $\widehat{\mathrm{SDiff}}(\mathcal{M})\times SC^{\infty}(\mathcal{M},\mathcal{R}) \longrightarrow SC^{\infty}(\mathcal{M},\mathcal{R})$ arising from the composition morphism comp). More generally, one can define linear actions of Lie supergroups on functors of supersections of s.vector bundles (when a corresponding supergroup acts on a s.vector bundle).

8.5. Remarks on locally convex supermanifolds. One can define the category of locally convex, or Frèchet, or tame Frèchet supermanifolds, replacing simply the category $\mathrm{Man}^{\mathrm{Gr}}$ by the category of functors on Gr with values in the category of smooth locally convex, resp. Frèchet, resp. tame Frèchet manifolds (the corresponding theory of manifolds based on the notion of weak derivative morphisms is developed in the paper /19/ of R.Hamilton).

Then one can, on the one hand, to generalize the Nash-Moser inverse function theorem to the case of tame Frèchet supermanifolds; on the other hand, one can equip the functors $\widehat{SC}^{\infty}(\mathcal{M},\mathcal{M}')$ and $\hat{\Gamma}(\mathcal{E}\xrightarrow{\pi}\mathcal{M})$ with structures of tame Frèchet supermanifolds in the case when $\mathcal{M}$ is compact.

The details will be considered elsewhere.

9. $\mathbb{Z}_2^k$-Supermanifolds.

In this section we will construct the "iterated" category $S^k\mathrm{Man}$ of $\mathbb{Z}_2^k$-supermanifolds such that algebras (of any polylinear type $\mathcal{T}$) in this category correspond to $\mathbb{Z}_2^k$-graded Banach superalgebras of the corresponding type.

We could construct the category $S^k\mathrm{Man}$ recursively considering $\mathbb{Z}_2^k$-supermanifolds as functors in the functor category $S^{k-1}\mathrm{Man}^{\mathrm{Gr}}$. Instead we will do it more directly, using the functor category $\mathrm{Man}^{\mathrm{Gr}\times\ldots\times\mathrm{Gr}}$.

9.1. $\mathbb{Z}_2^k$- Grassman superalgebras. We will denote $S^k\mathcal{T}_R(\mathcal{D})$ the category of $\mathbb{Z}_2^k$-graded R-superalgebras of a polylinear type $\mathcal{T}$ in a category $\mathcal{D}$ (see Sect.1.10).

Let

$$i_j : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2^k , \qquad \varepsilon \longmapsto (0,\ldots,0,\overset{j}{\mathcal{E}},0,\ldots,0) \qquad (9.1.1)$$

be the canonical injection of $\mathbb{Z}_2$-modules. It generates, for every commutative ring with unity R in a category $\mathcal{D}$ with finite products and for every polylinear type $\mathcal{T}$ of algebraic structure, some functor

$$I_j : S\mathcal{T}_R(\mathcal{D}) \longrightarrow S^k\mathcal{T}_R(\mathcal{D}) \qquad (9.1.2)$$

from the category of R-superalgebras of the type $\mathcal{T}$ in $\mathcal{D}$ to the category of $\mathbb{Z}_2^k$-graded R-superalgebras of the same type $\mathcal{T}$ in $\mathcal{D}$. In particular, I_j sends commutative superalgebras into commutative $\mathbb{Z}_2^k$-graded superalgebras.

For every map $\varphi: \mathbb{Z}_2^k \longrightarrow \mathbb{N}$ denote Λ_φ some free real $\mathbb{Z}_2^k$-graded commutative superalgebra having exactly $\varphi(\varepsilon)$ free generators with parity ε; the superalgebra Λ_φ with φ determined from equalities

$$\varphi(i_j(\bar{1})) = n_j \text{ , else } \varphi(\varepsilon) = 0 \qquad (j=1,\dots,k) \qquad (9.1.3)$$

will be denoted as $\Lambda_{n_1,\dots,n_k}$ and will be called $\mathbb{Z}_2^k$- Grassman superalgebra.

Let A_1 and A_2 be $\mathbb{Z}_2^k$-graded real superalgebras. The tensor product $A_1 \otimes A_2$ of $\mathbb{Z}_2^k$-graded supermodules equipped with a multiplication defined by equalities

$$(a_1 \otimes a_2)\cdot(b_1 \otimes b_2) = \sum_{\varepsilon,\varepsilon' \in \mathbb{Z}_2^k} (-1)^{\varepsilon\varepsilon'} a_1 \cdot_\varepsilon b_1 \otimes_{\varepsilon'} a_2 \cdot b_2 \quad , \qquad (9.1.4)$$

is called the tensor product of $\mathbb{Z}_2^k$-graded superalgebras A_1 and A_2.

One can easily verify that for every $\mathbb{Z}_2^k$- Grassman superalgebra $\Lambda_{n_1,\dots,n_k}$ the identity

$$\Lambda_{n_1,\dots,n_k} \simeq I_1(\Lambda_{n_1}) \otimes \dots \ I_k(\Lambda_{n_k}) \qquad (9.1.5)$$

holds. It is evident, besides, that $\Lambda_{n_1,\dots,n_k}$ as an algebra (neglecting "super" structures) is the ordinary tensor product of Grassman algebras, which coincides in this particular case with the superproduct (9.1.4).

We will denote the full subcategory of the category of $\mathbb{Z}_2^k$-graded superalgebras with unity, consisting of all $\mathbb{Z}_2^k$- Grassman superalgebras as $\underline{Gr}^{\otimes k}$.

One can observe that the functor

$$\underline{Gr}^k \xrightarrow{\ I\ } \underline{Gr}^{\otimes k} \qquad (9.1.6)$$

determined by isomorphisms (9.1.5) is, in fact, an isomorphism of categories, because $\Lambda_{n_1,\dots,n_k}$ are free. For our purposes it will be more convenient to use directly the category $\underline{Gr}^{\otimes k}$ instead of the equivalent category $\underline{Gr}^k$.

9.2. $\mathbb{Z}_2^k$-supermanifolds. Now we can literally repeat definitions and constructions of preceeding sections for the functor category $\underline{Man}^{\underline{Gr}^{\otimes k}}$ in place of the category $\underline{Man}^{\underline{Gr}}$.

First of all if V is an $\mathbb{Z}_2^k$-graded K-module (in Top, Man or Set), one can define the $\bar{K}$-module $\bar{V}$ in the corresponding functor category ($\underline{Top}^{\underline{Gr}^{\otimes k}}$, $\underline{Man}^{\underline{Gr}^{\otimes k}}$ or $\underline{Set}^{\underline{Gr}^{\otimes k}}$) just by Eq.(2.2) (where Λ runs now in $\underline{Gr}^{\otimes k}$); similarly can be defined $\bar{f}$ for an even K-polylinear map f. $\bar{K}$-algebras of some type $\mathcal{T}$ in the corresponding functor category, which are isomorphic to $\bar{V}$ for some $\mathbb{Z}_2^k$-graded superalgebra V of the type $\mathcal{T}$ (in Top, Man or Set) will

be again called superrepresentable.

Define Banach $\mathbb{Z}_2^k$-superregion as an open subfunctor of a superrepresentable $\bar{\mathbb{R}}$-module in $\underline{Top}^{\underline{Gr}^{\otimes k}}$ (or in $\underline{Man}^{\underline{Gr}^{\otimes k}}$). Every Banach $\mathbb{Z}_2^k$-superregion in $\mathcal{V}$ is again of the form $\mathcal{V}|_U$ for some open U in $\underline{\mathcal{V}} := \mathcal{V}(\mathbb{R})$.

The definitions of supersmooth morphisms and of $\mathbb{Z}_2^k$-supermanifolds literally copy the corresponding definitions for the ordinary case (k=1). Denoting the category of Banach $\mathbb{Z}_2^k$-supermanifolds as $S^k\underline{Man}$ one can formulate the following generalization of Coroll.4.4.2:

Theorem 9.2.1. The category $S^k\mathcal{T}_K(\underline{Man})$ of $\mathbb{Z}_2^k$-graded K-superalgebras of any type $\mathcal{T}$ in Man is naturally equivalent to the category $\mathcal{T}_{\bar{K}}(S^k\underline{Man})$ of $\bar{K}$-algebras of the type $\mathcal{T}$ in $S^k\underline{Man}$.

The theory of $\mathbb{Z}_2^k$-supermanifolds could be developed further along the same lines as the theory of "ordinary" supermanifolds (with the possible exception of the theory of integration). Namely, one can define vector bundles in the category $S^k\underline{Man}$, tangent functor $\mathcal{T}$ and Lie functor, as well as the exponential morphism, following literally the corresponding definitions of the case k=1. In particular, the inverse function theorem is valid in the category $S^k\underline{Man}$ as well.

9.3. Example. We will give, in conclusion, an example showing that it is not easy (if at all possible) to reformulate the theory of finite-dimensional $\mathbb{Z}_2^k$-supermanifolds (for $k \geqslant 2$) in terms of spaces with sheaves of $\mathbb{Z}_2^k$-graded commutative superalgebras on them.

Define a $\mathbb{Z}_2^k$-graded commutative superalgebra $\mathcal{R}^{(k)}$ in $S^k\underline{Man}$ ("coordinate ring") as follows: ${}_\varepsilon(\mathcal{R}^{(k)})(\Lambda) = {}_\varepsilon\Lambda$. The $\mathbb{Z}_2^k$-graded $\bar{\mathbb{R}}$-superalgebra $SC^\infty(\mathcal{M}) := S^k\underline{Man}(\mathcal{M}, \mathcal{R}^{(k)})$ will be called the superalgebra of superfields of the $\mathbb{Z}_2^k$-supermanifold $\mathcal{M}$.

Describe the structure of this superalgebra in a simple case when k=2 and the supermanifold $\mathcal{M}$ is a finite-dimensional $\mathbb{Z}_2^2$-superpoint, i.e. $\mathcal{M} = \bar{V}$, where $\dim V = \{n_{ij}\}_{i,j \in \mathbb{Z}_2}$ is such that $n_{\bar{0}\bar{0}} = 0$ and $n_{\bar{1}\bar{0}}, n_{\bar{0}\bar{1}}, n_{\bar{1}\bar{1}} \in \mathbb{N}$.

It follows from the counterpart of Th.3.3.2 (which generalizes to the case of arbitrary k) that in this case there exists an isomorphism

$$SC^{\infty}(\bar{V}) \approx \Lambda_{n_{\bar1\bar0}, n_{\bar0\bar1}} \otimes \mathbb{R}[[x_1, \ldots, x_{n_{\bar1\bar1}}]] \qquad (9.3.1)$$

of $\mathbb{Z}_2^2$-graded super-algebras, where $\mathbb{R}[[x_1, \ldots, x_{n_{\bar1\bar1}}]]$, considered as an algebra, is simply the algebra of formal power series in variables $x_1, \ldots, x_{n_{\bar1\bar1}}$.

Note that if $\mathcal{U} = U \times \mathcal{P}$ is a finite-dimensional $\mathbb{Z}_2^2$-superregion, such that U is an "ordinary" region (i.e. dimensions of U in "directions" $(1,0)$, $(0,1)$ and $(1,1)$ are zero) and $\mathcal{P}$ is a finite-dimensional superpoint, then, generally speaking, $SC^{\infty}(\mathcal{U}) \not\approx SC^{\infty}(U) \otimes SC^{\infty}(\mathcal{P})$.

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