TOPOSES AND HOMOTOPY TOPOSES (VERSION 0.15)

CHARLES REZK

Contents

This document was created while I gave a series of lectures on "higher topos theory" in Fall 2005. At that time the basic references were the papers of Toen-Vezzosi [TV05] and a document of Lurie [Lura]; Lurie's book on higher topos theory was not yet available. The lectures were my attempt to synthesize what was known at the time (or at least, what was known to me). I've revised them in small ways since they were written. (For instance, I replaced the original term "patching" with the word "descent", as suggested by Lurie.) There are still some gaps in the exposition here.

1. Grothendieck topos

What follows is a quick sketch of Grothendieck's theory of toposes. The emphasis may seem strange; I'll ignore applications to geometry or mathematical logic, and treat a topos as a purely category theoretic object. Also, the definition I'll give is a bit different than (but is equivalent to) the usual one; equivalence with the usual definition is shown in §3. The discussion is designed to make the definition of a model topos seem completely obvious.

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1.1. **Presheaves.** Let $\mathbf{C}$ be a small category. The category $\mathrm{PSh}(\mathbf{C})$ of **presheaves of sets on $\mathbf{C}$** is the category of functors $\mathbf{C}^{\mathrm{op}} \to \mathrm{Sets}$.

The **Yoneda functor** $y\colon \mathbf{C} \to \mathrm{PSh}(\mathbf{C})$ is the functor defined on objects by

$$y(C)(D) \overset{\mathrm{def}}{=} \mathbf{C}(D, C).$$

The "Yoneda lemma" says that $\mathbf{C}(C, C') \xrightarrow{\sim} \mathrm{PSh}(\mathbf{C})(y(C), y(C'))$, so that y induces an equivalence between $\mathbf{C}$ and a full subcategory of $\mathrm{PSh}(\mathbf{C})$.

1.2. **Definition of a topos.** Let us define **site** to be pair consisting of a small category $\mathbf{C}$, together with a full subcategory of $\mathrm{PSh}(\mathbf{C})$, usually denoted by $\mathrm{Sh}(\mathbf{C})$, which is closed under isomorphisms (i.e., if $X \to Y \in \mathrm{PSh}(\mathbf{C})$ is an isomorphism, then $X \in \mathrm{Sh}(\mathbf{C})$ iff $Y \in \mathrm{Sh}(\mathbf{C})$), such that the inclusion functor $i\colon \mathrm{Sh}(\mathbf{C}) \to \mathrm{PSh}(\mathbf{C})$ admits a left adjoint $a\colon \mathrm{PSh}(\mathbf{C}) \to \mathrm{Sh}(\mathbf{C})$, with the property:

(T) the functor a commutes with finite limits.

We will refer to $(\mathbf{C}, \mathrm{Sh}(\mathbf{C}))$ as a site, and the category $\mathrm{Sh}(\mathbf{C})$ as a category of **sheaves on the site**.

Note: the notation "$\mathrm{Sh}(\mathbf{C})$" is ambiguous; a given category $\mathbf{C}$ can admit many sites.

A **topos** is a category which is equivalent to some category of sheaves on a site. (This defintion of "site" is different from, but equivalent to, the usual one; see §3 for a comparison.)

Example 1.3. Let X be a topological space, and let $\mathcal{U}_X$ denote the category whose objects are open subsets of X, and whose morphisms are inclusions of subsets. Let $\mathrm{Sh}(\mathcal{U}_X) \subset \mathrm{PSh}(\mathcal{U}_X)$ be the full subcategory consisting of objects F such that for every open set U, and every open cover $\{U_\alpha\}$ of U, the diagram

$$F(U) \to \prod_{\alpha} F(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} F(U_\alpha \cap U_\beta)$$

is an equalizer of sets. The inclusion of $\mathrm{Sh}(\mathcal{U}_X)$ in $\mathrm{PSh}(\mathcal{U}_X)$ admits a left adjoint a, called **sheafification**, which commutes with finite limits, so that $\mathrm{Sh}(\mathcal{U}_X)$ is a topos (see §3). We usually write $\mathrm{Sh}(X)$ for $\mathrm{Sh}(\mathcal{U}_X)$.

1.4. **Presentable categories.** Property (T) is really the distinguishing property of a site. A **presentable category** is a category equivalent to a subcategory $\mathrm{Sh}(\mathbf{C}) \subseteq \mathrm{PSh}(\mathbf{C})$ satisfying all the properties of a site except (possibly) property (T). We will call such a subcategory a **pseudo-site** of $\mathbf{C}$, and we will refer to the objects of $\mathrm{Sh}(\mathbf{C})$ as **pseudo-sheaves**.

A topos is therefore a presentable category; however, there are many presentable categories which are not toposes.

Example 1.5. Let $\mathbf{A}$ denote the category of abelian groups, and let $\mathbf{C} \subset \mathbf{A}$ be a skeleton of the full subcategory of finitely generated free groups. The evident functor $R\colon \mathbf{A} \to \mathrm{PSh}(\mathbf{C})$ sending $A \mapsto (C \mapsto \mathbf{A}(C, A))$ identifies $\mathbf{A}$ with the full subcategory of $\mathrm{PSh}(\mathbf{C})$ consisting of functors F such that $F(C \oplus C') \approx F(C) \times F(C')$. The functor R admits a left adjoint (which sends a presheaf X to abelian group obtained by taking the coend of $X\colon \mathbf{C}^{\mathrm{op}} \to \mathrm{Sets}$ with the inclusion $\mathbf{C} \to \mathbf{A}$), and therefore the inclusion $\mathrm{Sh}(\mathbf{C}) \to \mathrm{PSh}(\mathbf{C})$ admits a left adjoint, whence $\mathbf{A}$ is presentable. However, this left adjoint does not preserve finite limits, and one can show that $\mathbf{A}$ is not a topos.

Lemma 1.6. *Let $i\colon \mathrm{Sh}(\mathbf{C}) \to \mathrm{PSh}(\mathbf{C})$ be an inclusion of a pseudo-site of $\mathbf{C}$. We have the following.*

(1) *For every $F \in \mathrm{Sh}(\mathbf{C})$ the natural map $a(i(F)) \to F$ is an isomorphism.*
(2) *Let F be object of $\mathrm{PSh}(\mathbf{C})$. The following are equivalent:*
  (a) *F is an object of $\mathrm{Sh}(\mathbf{C})$;*
  (b) *for all $f\colon A \to B \in \mathrm{PSh}(\mathbf{C})$ such that $a(f)\colon a(A) \to a(B) \in \mathrm{Sh}(\mathbf{C})$ is iso, $\mathrm{PSh}(\mathbf{C})(B, F) \to \mathrm{PSh}(\mathbf{C})(A, F)$ is iso.*
  (c) *$F \to i(a(F))$ is an isomorphism in $\mathrm{PSh}(\mathbf{C})$;*

Proof. Since i is fully faithful, $\mathrm{Sh}(\mathbf{C})(F, X) \to \mathrm{Sh}(\mathbf{C})(a(i(F)), X) \approx \mathrm{PSh}(\mathbf{C})(i(F), i(X))$ is an isomorphism, whence $a(i(F)) \to F$ is iso. The second part of the lemma is straightforward: prove (a) implies (b) implies (c) implies (a). □

Remark 1.7. Given any set S of maps in $\mathrm{PSh}(\mathbf{C})$, let $\mathrm{PSh}(\mathbf{C})_S$ denote the full subcategory defined by $F \in \mathrm{PSh}(\mathbf{C})_S$ iff $\mathrm{PSh}(\mathbf{C})(f, F)$ iso for all $f \in S$. Then one can show that the inclusion functor $\mathrm{Sh}(\mathbf{C}) \to \mathrm{PSh}(\mathbf{C})$ admits a left adjoint, and thus $\mathrm{PSh}(\mathbf{C})_S$ is a presentable category.

We'll say that a category admits a **small presentation** if it is equivalent to a $\mathrm{PSh}(\mathbf{C})_S$ as above. In practice, the presentable categories one encounters usually have small presentations, and it turns out that any topos has a small presentation (see §3). (Note: in the literature, this notion is usually called **local presentability**.)

Proposition 1.8. *A presentable category has all small limits and colimits.*

Proof. It suffices to consider a full subcategory $i\colon \mathrm{Sh}(\mathbf{C}) \subset \mathrm{PSh}(\mathbf{C})$ which admits a left adjoint. It is clear that $\mathrm{Sh}(\mathbf{C})$ has limits, which coincide with limits in $\mathrm{PSh}(\mathbf{C})$. To see that $\mathrm{Sh}(\mathbf{C})$ has colimits, consider a functor $F\colon \mathbf{J} \to \mathrm{Sh}(\mathbf{C})$ from a small category $\mathbf{J}$. It is straightforward to check that

$$\mathrm{colim}_{\mathbf{J}} F \approx a(\mathrm{colim}_{\mathbf{J}}\, i(F)).$$

□

1.9. **Basic properties of a topos.** The idea is that a topos is a category which has many of the good property of the category of sets.

Proposition 1.10. *Toposes are cartesian closed. That is, for any pair Y, Z of objects in a topos $\mathbf{E}$, there exists a **function object** $Z^Y \in \mathbf{E}$ with the property that there are isomorphisms*

$$\mathbf{E}(X, Z^Y) \approx \mathbf{E}(X \times Y, Z),$$

natural in $X \in \mathbf{E}$.

We can regard (1.10) as saying that the functor $\mathbf{E}^{\mathrm{op}} \to \mathrm{Sets}$ given by $X \mapsto \mathbf{E}(X \times Y, Z)$ is representable. It is a special case of a more general characterization of representable functors from a presentable category.

Proposition 1.11. *Let $\mathbf{E}$ be a presentable category. If $F\colon \mathbf{E}^{\mathrm{op}} \to \mathrm{Sets}$ is a functor which takes small colimits in $\mathbf{E}$ to limits in Sets, then F is representable by an object of $\mathbf{E}$.*

Note that (1.11) actually gives an equivalence of categories between $\mathbf{E}$ and the category of functors $\mathbf{E}^{\mathrm{op}} \to \mathrm{Sets}$ which take colimits to limits.

Corollary 1.12. *Let* $\mathbf{E}$ *be a presentable category, and let* $L\colon \mathbf{E} \to \mathbf{D}$ *be a functor to some category* $\mathbf{D}$*. Then* L *admits a right adjoint if and only if* L *preserves small colimits.*

Proof of (1.11). First suppose that $\mathbf{E} = \mathrm{PSh}(\mathbf{C})$. Given a functor $F\colon \mathrm{PSh}(\mathbf{C})^{\mathrm{op}} \to \mathrm{Sets}$ which takes colimits to limits, define a presheaf X on $\mathbf{C}$ to be the functor taking the value $X(C) = F(y(C))$ for an object $C \in \mathbf{C}$. We claim that X represents the functor F.

The presheaf X is a colimit of some small diagram of representable presheaves; we have $X = \mathrm{colim}_{y/X}\, yU$, where y/X is the slice category and $U\colon y/X \to \mathbf{C}$ the functor which forgets the map to X. Then

$$F(X) \approx F(\mathrm{colim}_{y/X}\, yU) \approx \lim_{y/X} F(yU) \approx \lim_{y/X} \mathrm{PSh}(\mathbf{C})(yU, X),$$

using the hypothesis on F. Tautologically, there is a map $u_c\colon yU(c) \to X$ for each object $c \in y/X$, and these fit together to give a canonical element $u \in \lim_{y/X} \mathrm{PSh}(\mathbf{C})(U, X) \approx F(X)$.

Consider the natural map

$$\phi\colon\ \mathrm{PSh}(\mathbf{C})(Y, X) \to F(Y)$$

defined by $\phi(f) = F(f)(u)$. I claim that ϕ is an isomorphism for all $Y \in \mathrm{PSh}(\mathbf{C})$. In fact, it is tautologically an isomorphism when $Y = y(C)$; since an arbitrary Y is a colimit of representables, and since both F and $\mathrm{PSh}(\mathbf{C})(-, X)$ carry colimits to limits, we conclude that ϕ is an isomorphism.

To prove the proposition, it suffices to prove the statement when $\mathbf{E} = \mathrm{Sh}(\mathbf{C})$ is a pseudo-site. Given a functor $F\colon \mathbf{E}^{\mathrm{op}} \to \mathrm{Sets}$ taking colimits to limits, we can apply the part already proved to get a presheaf $X \in \mathrm{PSh}(\mathbf{C})$ which represents the functor $Fa\colon \mathrm{PSh}(\mathbf{C})^{\mathrm{op}} \to \mathrm{Sets}$. If $f\colon Y \to Z$ is a map of presheaves such that af is iso, then $\phi\colon\ \mathrm{PSh}(\mathbf{C})(f, X) \approx F(af)$ is iso, and so X is a pseudo-sheaf by (1.6). Clearly, if Y is a pseudo-sheaf, then $\mathrm{Sh}(\mathbf{C})(Y, X) \approx \mathrm{PSh}(\mathbf{C})(iY, X) \approx F(aiY) \approx F(Y)$, and so F is represented by the pseudo-sheaf X. □

Proof of (1.12). It is standard that if L is a left adjoint, then it commutes with colimits. Assume then that L is a functor which preserves colimits. To construct a right adjoint $R\colon \mathbf{D} \to \mathbf{E}$, it suffices to produce for each object $Y \in \mathbf{D}$ an object $RY \in \mathbf{E}$ such that

$$\mathbf{E}(X, RY) \approx \mathbf{D}(LX, Y);$$

that is, RY must represent the functor $X \mapsto \mathbf{D}(LX, Y)\colon \mathbf{E}^{\mathrm{op}} \to \mathrm{Sets}$. Since L preserves colimits, it is clear that this functor takes colimits to limits, and thus is representable by (1.11). □

Proof of (1.10). We claim that the functor $X \mapsto X \times Y\colon \mathbf{E} \to \mathbf{E}$ preserves colimits. This is clear in the category of sets, and thus the result follows if $\mathbf{E}$ is a category of presheaves. To show this for general $\mathbf{E}$, it suffices to show it for a site, and the claim is straightforward using property (T). Thus, by (1.12), the functor $X \mapsto X \times Y$ admits a right adjoint, which is the desired functor $Z \mapsto Z^Y$. □

1.13. **Regular epimorphisms and epi/mono factorizations.** In the category of sets, any function $f\colon X \to Y$ can be factored (up to unique isomorphism)

$$X \xrightarrow{p} I \xrightarrow{i} Y$$

where p is an epimorphism and i is a monomorphism; we call I the image of the map f. In particular, if f is both an epimorphism and a monomorphism, then it is an isomorphism.

This property of sets does not hold for arbitrary categories. To repair this, we replace epimorphisms with the notion of regular epimorphisms.

A **regular epimorphism** $p\colon X \to Y$ in a category $\mathbf{D}$ is a map which is a coequalizer; that is, there exists a coequalizer diagram in $\mathbf{D}$ of the form

$$U \rightrightarrows X \xrightarrow{p} Y.$$

We say $p\colon X \to Y$ is an **effective epimorphism** if

$$X \times_Y X \rightrightarrows X \xrightarrow{p} Y$$

is a coequalizer, where the parallel arrows are the two projections. Every effective epimorphism is a regular epimorphism.

Example 1.14. In Sets, the epimorphisms are regular epimorphisms are effective epimorphisms are surjective maps. To see this, note that if $f\colon X \to Y$ is a surjective function of sets, then $X \times_Y X \rightrightarrows X \to Y$ is a coequalizer, where the parallel arrows are projections; the set $X \times_Y X \subseteq X \times X$ is an equivalence relation on X. The same holds in categegories of presheaves.

Example 1.15. We will soon see that any topos, epimorphisms are regular epimorphism are effective epimorphisms. However, when we consider the homotopy theoretic analogues of these ideas, the generalizations of these notions will diverge.

Example 1.16. In an abelian category, all epimorphisms are regular epimorphisms: if $f\colon A \to B$ is an epimorphism, then $0 \to \ker(f) \xrightarrow{i} A \xrightarrow{f} B \to 0$ is exact (by the axioms for abelian category), and so

$$\ker(f) \underset{0}{\overset{i}{\rightrightarrows}} A \xrightarrow{f} B$$

is a coequalizer.

Example 1.17. In the category of commutative rings, the the inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism, but *not* a regular epimorphism. The regular epimorphisms in this category are precisely the surjections.

Proposition 1.18. *Let $\mathbf{D}$ be a category.*

(a) *Suppose given a commutative square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"'] & X \arrow[d, "i"] \\ B \arrow[r, "g"'] \arrow[ur, dotted] & Y \end{tikzcd}$$

in which p is a regular epimorphism and i is a monomorphism. Then there exists a unique dotted arrow making the diagram commute.

(b) *If $h\colon X \to Y \in \mathbf{D}$ is both a monomorphism and a regular epimorphism, then f is an isomorphism.*

(c) *Let $h\colon X \to Y \in \mathbf{D}$ be a map. Up to isomorphism, there is at most one factorization $h = ip$, up to isomorphism, where i is a monomorphism and p is a regular epimorphism.*

Proof. To prove (a), note that since p is a regular epimorphism, there is a coequalizer diagram

$$U \underset{t}{\overset{s}{\rightrightarrows}} A \xrightarrow{p} B$$

We have $ifs = gps = gpt = ift$, and since i is a monomorphism, it follows that $fs = ft$. Thus the dotted arrow exists and is unique.

Parts (b) and (c) are straightforward, using (a). $\square$

Proposition 1.19. *For any map $f\colon X \to Y$ in a topos $\mathbf{E}$, there exists a factorization $f = ip$ where i is a monomorphism and p is a regular epimorphism (which is unique up to unique isomorphism). Furthermore, p is the coequalizer of the pair of projections $X \times_Y X \rightrightarrows X$; in particular, all regular epis are effective epis.*

Proof of (1.19). First, note that this is true in Sets. In this case, the coequalizer of $X \times_Y X \rightrightarrows X$ is precisely the quotient of X by the equivalence relation: $x_1 \sim x_2$ iff $f(x_1) = f(x_2)$.

Next, note that the proposition is true in $\mathrm{PSh}(\mathbf{C})$, since everything is computed objectwise.

Now if $f\colon X \to Y \in \mathrm{Sh}(\mathbf{C})$, there is a diagram

$$i(X) \times_{i(Y)} i(X) \rightrightarrows i(X) \xrightarrow{q} A \xrightarrow{j} i(Y)$$

in $\mathrm{PSh}(\mathbf{C})$, where q is the coequalizer of the pair and j is a monomorphism, and thus a diagram

$$X \times_Y X \rightrightarrows X \xrightarrow{a(q)} a(A) \xrightarrow{a(j)} Y$$

in $\mathrm{Sh}(\mathbf{C})$, using the fact that $ai \approx \mathrm{id}$ and that a preserves finite limits. Clearly $a(q)$ is a regular epimorphism in $\mathrm{Sh}(\mathbf{C})$, since it is obtained as a coequalizer in $\mathrm{Sh}(\mathbf{C})$. Furthermore, $a(j)$ is a monomorphism, since j is a monomorphism and a preserves finite limits. $\square$

Corollary 1.20. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be maps in a topos.*

(a) *If gf is a regular epi then so is g.*
(b) *If f and g are regular epis then so is gf.*

Proof. Suppose gf is a regular epi. Consider an regular epi/mono factorization of g, and use (1.18)(a) to show that the monomorphism in this factorization is an isomorphism. This proves (a).

Suppose f and g are regular epis. Consider a regular epi/mono factorization of gf, and use (1.18)(a) to show that the monomorphism in this factorization is an isomorphism. This proves (b). $\square$

Proposition 1.21. *Let $\mathbf{E}$ be a topos. Consider a pullback square of the form*

$$\begin{tikzcd} U \arrow[r, "q"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ V \arrow[r, "p"'] & Y \end{tikzcd}$$

If f is a monomorphism/regular epimorphism/isomorphism then so is g, and the converse holds if p is a regular epimorphism.

Proof. By (1.18), the result for isomorphisms follows from the other two cases.

For the monomorphism case, is is clear that pullbacks of monos are monos. Thus we need to show that if p is regular epi and g is a mono, then f is mono. This is straightforward to check for sets, and thus for presheaves. Thus it will suffice to prove the result for a site $\mathbf{E} = \mathrm{Sh}(\mathbf{C})$.

Suppose that the pullback square is in $\mathrm{Sh}(\mathbf{C})$. Let $i(p) = dv$ be the epi/mono factorization of $i(p)$ in $\mathrm{PSh}(\mathbf{C})$. Consider the diagram

$$\begin{tikzcd} iU \arrow[r, "u"] \arrow[d, "ig"'] & X' \arrow[r, "b"] \arrow[d, "f'"] & iX \arrow[d, "if"] \\ iV \arrow[r, "v"'] & Y' \arrow[r, "d"'] & iY \end{tikzcd}$$

in $\mathrm{PSh}(\mathbf{C})$, where X' is the pullback of d along if. Since the left-hand square is a pullback, ig is mono and v is regular epi, we have that f' is a monomorphism. We have $p = ai(p) = a(d)a(v)$ with $a(v)$ mono and $a(d)$ regular epi, and thus $a(v)$ is an isomorphism by (1.18)(c). Therefore f is isomorphic to $ai(f')$ which is a monomorphism in $\mathrm{Sh}(\mathbf{C})$.

For the regular epimorphism case, if p and g are regular epi then $pg = fq$ is regular epi, and hence f is a regular epi, by (1.20). Thus it remains to show that if f is regular epi then so is g. This is straightforward for sets, and hence for presheaves, so we have reduced to the case of a site $\mathbf{E} = \mathrm{Sh}(\mathbf{C})$. Consider

$$\begin{tikzcd} iU \arrow[r] \arrow[d, "u"'] & iX \arrow[d, "b"] \\ V' \arrow[r] \arrow[d, "v"'] & Y' \arrow[d, "d"] \\ iV \arrow[r] & iY \end{tikzcd}$$

where $i(f) = db$ is a epi/mono factorization in $\mathrm{PSh}(\mathbf{C})$, and V' is the pullback of the lower square in $\mathrm{PSh}(\mathbf{C})$. Thus the map u is a pullback of the regular epi b, and thus is a regular epi. Since a preserves colimits, $a(u)$ is a regular epi. Note that since $f = a(db) = a(d)a(b)$ is a regular epi in $\mathrm{Sh}(\mathbf{C})$ then so is $a(d)$ by (1.20). Since $a(d)$ is mono it follows that $a(d)$ is iso by (1.18)(b). Since the bottom square is a pullback and a preserves finite limits, $a(v)$ is an isomorphism. Thus $g = a(v)a(u)$ is regular epi. $\square$

As a consequence, we have the following, which says that the regular epimorphisms are the maps which "locally admit a section".

Corollary 1.22. *In a topos, a map f is a regular epimorphism if and only if there is a pullback square of the form*

$$\begin{tikzcd} U \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ V \arrow[r, "p"'] & Y \end{tikzcd}$$

such that p is a regular epimorphism and g admits a section.

Proof. For the only if part, take $p = f$, so that $g\colon X \times_Y X \to X$ is a projection to a factor; the diagonal map is a section of g.

For the if part, first note that if g admits a section $s\colon V \to U$, then $1_V = gs$ and so g is regular epi by (1.20); that f is regular epi follows from (1.21). □

2. The descent properties of a topos

2.1. **Descent.** We say that a category $\mathbf{E}$ with small colimits and finite limits has **weak descent** if the following four properties hold.

(P1a) Let $\{X_i\}_{i\in I}$ be a collection of objects of $\mathbf{E}$ indexed by a set I, and write $X = \coprod_{i\in I} X_i$. Let $f\colon Y \to X$ be a map in $\mathbf{E}$, and let $Y_i = X_i \times_X Y$. Then the natural map $\coprod_{i\in I} Y_i \to Y$ is an isomorphism.

(P1b) Let $\{f_i\colon Y_i \to X_i\}$ be a collection of maps in $\mathbf{E}$ indexed by a set I, and let $f = \coprod_{i\in I} f_i\colon X \to Y$ be the coproduct of these maps. Then the natural maps $Y_i \to X_i \times_X Y$ are isomorphisms.

(P2a) Let $X_1 \leftarrow X_0 \to X_2$ be a diagram in $\mathbf{E}$, with colimit X. Let $f\colon Y \to X$ be a map in $\mathbf{E}$, and let $Y_i = X_i \times_X Y$ for $i = 0, 1, 2$. Then the natural map $\operatorname{colim}(Y_1 \leftarrow Y_0 \to Y_2) \to Y$ is an isomorphism.

(P2b) Let

$$\begin{tikzcd} Y_1 \arrow[d, "f_1"'] & Y_0 \arrow[l] \arrow[r] \arrow[d, "f_0"'] & Y_2 \arrow[d, "f_2"'] \\ X_1 & X_0 \arrow[l] \arrow[r] & X_2 \end{tikzcd}$$

be a commutative diagram in $\mathbf{E}$ such that both squares are pullbacks, and let $f\colon X \to Y$ be the map between the colimits of the rows. Then the natural maps $Y_i \to X_i \times_X Y$ are regular epimorphisms, for $i = 0, 1, 2$.

Note that property (P2b) only requires *regular epimorphisms*, where one might have expected *isomorphisms.*

Proposition 2.2. *A Grothendieck topos* $\mathbf{E}$ *has weak descent.*

Proof. Sets has weak descent; the usual methods apply. □

Statements (P1a) and (P1b) amount to the following: there is an equivalence of categories

$$\mathbf{E}/X \rightleftarrows \prod_{i\in I} \mathbf{E}/X_i,$$

the functors being given by: pulling back along $X_i \to X$, and taking coproduct, respectively. That is, the category of objects over X can be completely recovered from the categories of objects over "pieces" of X, where "pieces" mean summands.

One might hope that (P2a) and (P2b) give a similar equivalence between $\mathbf{E}/X$ and a suitable category of diagrams of shape $Y_1 \leftarrow Y_0 \to Y_2$ in $\mathbf{E}$ which map to the diagram $X_1 \leftarrow X_0 \to X_2$; it is clearly necessary to require that the squares be pullbacks, if we are to use pullbacks along $X_i \to X$ to produce the Y_i's. However, even this doesn't quite work, since (P2b) in the end only gives us regular epis, rather than isomorphisms. We can see how this happens in a simple example.

Example 2.3. Let $\mathbf{E}$ = Sets, and fix a set X and a non-identity automorphism σ of X. Consider the diagram

$$\begin{tikzcd} X \arrow[d] & X \amalg X \arrow[l, "(\mathrm{id},\mathrm{id})"'] \arrow[r, "(\mathrm{id},\sigma)"] \arrow[d] & X \arrow[d] \\ * & * \amalg * \arrow[l] \arrow[r] & * \end{tikzcd}$$

This satisfies the hypotheses of (P2b). The pushout of the bottom row is a one-point set, while the pushout of the top row is the quotient $X/\sim$ obtained by identifying $x \sim \sigma(x)$ for all $x \in X$. The comparison maps at the ends are $X \to X/\sim$, which are surjections but not isomorphisms.

If we think of sets as being discrete spaces, and we take *homotopy* pushouts, then from this diagram we obtain a fiber bundle over the circle with fiber X. If we then pull back along points, we obtain a set isomorphic to X itself, rather than a quotient. The moral is that working in homotopy theory repairs the difficulty with (P2b).

If $f\colon Y \to X$ is a natural transformation of functors $X, Y\colon \mathbf{C} \to \mathbf{E}$, we say f is **equifibered** if for each morphism $c\colon C \to C' \in \mathbf{C}$, the square

$$\begin{tikzcd} Y(C) \arrow[r, "Y(c)"] \arrow[d, "f(C)"'] & Y(C') \arrow[d, "f(C')"] \\ X(C) \arrow[r, "X(c)"'] & X(C') \end{tikzcd}$$

is a pullback. We can state a descent property for colimits of diagrams of any shape.

Proposition 2.4. *Let* $\mathbf{E}$ *be a category with weak descent.*

(1) *Consider a functor* $X\colon \mathbf{C} \to \mathbf{E}$ *from a small category* $\mathbf{C}$, *with* $\bar{X} = \operatorname{colim} X$ *in* $\mathbf{E}$, *and a map* $f\colon \bar{Y} \to \bar{X}$ *in* $\mathbf{E}$. *Define* $Y\colon \mathbf{C} \to \mathbf{E}$ *by*
$$Y(C) \overset{\mathrm{def}}{=} X(C) \times_{\bar{X}} \bar{Y}.$$
Then the evident map $\operatorname{colim} Y \to \bar{Y}$ *is an isomorphism.*

(2) *Let* $f\colon X \to Y$ *be an equifibered natural transformation of functors* $X, Y\colon \mathbf{C} \to \mathbf{E}$ *from a small category* $\mathbf{C}$. *Define* $\bar{X} = \operatorname{colim} X$ *and* $\bar{Y} = \operatorname{colim} Y$, *and let* $\bar{f} = \operatorname{colim} f\colon \bar{X} \to \bar{Y}$. *Then for each object* $C \in \mathbf{C}$, *the evident map* $g\colon Y(C) \to X(C) \times_{\bar{X}} \bar{Y}$ *is a regular epimorphism. If the category* $\mathbf{C}$ *is a groupoid with at most one map between any two objects, then* g *is an isomorphism.*

Proof. A straightforward exercise using the weak descent properties. □

Part (2) of (2.4) can be improved if f is a monomorphism.

Proposition 2.5. *Consider the situation of part (2) of (2.5), and suppose that* $f(C)\colon X(C) \to Y(C)$ *is a monomorphism for each* $C \in \mathbf{C}$. *Then* $\bar{f}$ *is a monomorphism, and the evident maps* $Y(C) \to X(C) \times_{\bar{X}} \bar{Y}$ *are isomorphisms for all* $C \in \mathbf{C}$.

Proof. By (2.4)(2), the map $g\colon Y(C) \to X(C) \times_{\bar{X}} \bar{Y}$ is a regular epimorphism. The composite of g with the projection $X(C) \times_{\bar{X}} \bar{Y} \to X(C)$ is the map $f(C)$, which by hypothesis is a monomorphism. Therefore g must be an isomorphism by a straightforward argument using (1.18)(a).

To show that $h\colon \bar{Y} \to \bar{Y} \times_{\bar{X}} \bar{Y}$ is an isomorphism and thus that $\bar{f}$ is a monomorphism, note that we can use (2.4)(1) to show that the pullback of this map along each $X(C) \to \bar{X}$ gives a map isomorphic to $h_C\colon Y(C) \to Y(C) \times_{X(C)} Y(C)$, which is iso since $f(C)$ is mono. Again, (2.4) shows that we can recover h as a colimit of the maps h_C, and thus h is an isomorphism. □

Let X be an object in some category $\mathbf{D}$. A **subobject** of X is an isomorphism class of monomorphisms of the form $j\colon A \to X \in \mathbf{D}$. Write $\operatorname{Sub}(X)$ for the class of subobjects of X; this is not necessarily a set. If $\mathbf{D}$ has finite limits, there is a natural map $\operatorname{Sub}(f)\colon \operatorname{Sub}(Y) \to \operatorname{Sub}(X)$ for each map $f\colon X \to Y$, defined by taking pullbacks.

Proposition 2.6. *If $\mathbf{E}$ is a topos, then $\operatorname{Sub}(X)$ is a set for each $X \in \mathbf{E}$, and the functor $\operatorname{Sub}\colon \mathbf{E}^{\mathrm{op}} \to \mathrm{Sets}$ is representable by an object $\Omega \in \mathbf{E}$.*

In the category of sets, the subobject classifier is $\Omega = \{0, 1\}$, and given a set X, the function object Ω^X is just the power set of X. Thus, toposes have an analogue of the power set construction.

Proof. One checks that $\operatorname{Sub}(X)$ is a set if $X \in \operatorname{PSh}(\mathbf{C})$, and that for an object X in $\operatorname{Sh}(\mathbf{C})$ the set $\operatorname{Sub}_{\operatorname{Sh}(\mathbf{C})}(X)$ injects into $\operatorname{Sub}_{\operatorname{PSh}(\mathbf{C})}(iX)$, whence the collection of subobjects is always a set.

Using (2.5) it is straightforward to check that $\operatorname{Sub}(\operatorname{colim}_{\mathbf{I}} X) \approx \lim_{\mathbf{I}} \operatorname{Sub}(X(i))$, where $X\colon \mathbf{I} \to \mathbf{E}$ is any functor from a small category $\mathbf{I}$. The result now follows from (1.11). □

2.7. **A Giraud-type theorem.** One would like to have a theorem which characterizes toposes without reference to a site. The following is a version of a theorem of Giraud.

Theorem 2.8. *Let $\mathbf{E}$ be a category which has all small colimits and all finite limits. Then $\mathbf{E}$ is a Grothendieck topos if and only if*

(i) *$\mathbf{E}$ contains a set of objects which generate $\mathbf{E}$, and*
(ii) *$\mathbf{E}$ has weak descent.*

A collection $\mathbf{C}$ of objects of $\mathbf{E}$ is said to **generate** $\mathbf{E}$ if for any pair $f, g\colon X \rightrightarrows Y$ of maps in $\mathbf{E}$ one has $f = g$ if and only if $fc = gc$ for all $C \in \mathbf{C}$ and all $c\colon C \to X$.

A proof is given in §2.17 below.

Corollary 2.9. *Let $\mathbf{E}$ be a topos. For each object X of $\mathbf{E}$, the slice category $\mathbf{E}/X$ is a topos. For each small category $\mathbf{D}$, the functor category $\mathbf{E}^{\mathbf{D}}$ is a topos.*

Proof. It is clear that $\mathbf{E}/X$ has small colimits and finite limits, and that it satisfies the weak descent properties. If $\mathbf{C}$ is a set of objects which generate $\mathbf{E}$, then $\mathbf{C}/X \overset{\text{def}}{=} \{c\colon C \to X, C \in \mathbf{C}\}$ is a set of objects which generate $\mathbf{E}/X$. Thus $\mathbf{E}/X$ satisfies properties (i) and (ii) of (2.8).

The proof for $\mathbf{E}^{\mathbf{D}}$ is similar; in this case, if $\mathbf{C}$ is a set of generators for $\mathbf{E}$, then $\{F_{C,D}\}$ is a set of generators for $\mathbf{E}^{\mathbf{D}}$, where for objects $C \in \mathbf{C}$ and $D \in \mathbf{D}$, we let $F_{C,D} \in \mathbf{E}^{\mathbf{D}}$ be defined by $F_{C,D}(D') = \coprod_{\mathbf{D}(D,D')} C$. □

2.10. **Geometric morphisms.** A **geometric morphism** $f\colon \mathbf{E} \to \mathbf{F}$ of toposes is a functor $f^*\colon \mathbf{F} \to \mathbf{E}$ which preserves small colimits and finite limits. By (1.12), the functor f^* necessarily admits a right adjoint $f_*\colon \mathbf{E} \to \mathbf{F}$.

Example 2.11. For each Grothendieck site $\mathrm{Sh}(\mathbf{C}) \subseteq \mathrm{PSh}(\mathbf{C})$, the sheafification functor defines a geometric morphism $\mathrm{PSh}(\mathbf{C}) \to \mathrm{Sh}(\mathbf{C})$.

Example 2.12. Given a map $f\colon X \to Y$ of objects in a topos $\mathbf{E}$, there is a geometric morphism $f\colon \mathbf{E}/X \to \mathbf{E}/Y$, given by setting $f^*(B \to Y) \overset{\text{def}}{=} (B \times_Y X \to X)$; it is clear that f^* preserves limits, and the weak descent property shows that f^* preserves small colimits.

Suppose $Y = 1$. Then the right adjoint $f_*\colon \mathbf{E}/X \to \mathbf{E}$ is denoted $\underline{\mathrm{sect}}_X$; it associates to a morphism $f\colon A \to X$ the **object of sections** $\underline{\mathrm{sect}}_X(f)$. Note that $\underline{\mathrm{sect}}_X(X \times Y \to X)$ is canonically isomorphic to the function object Y^X of (1.10).

Example 2.13. Given a topos $\mathbf{E}$, there is a geometric morphism $\pi\colon \mathbf{E} \to \mathrm{Sets}$, where $\pi^* S = \coprod_S 1$. The right adjoint is given by $\pi_* X = \mathbf{E}(1, X)$, the "global sections" functor.

Example 2.14. Given a small category $\mathbf{C}$ and an object $C \in \mathbf{C}$, there is a geometric morphism $f_C\colon \mathrm{Sets} \to \mathrm{PSh}(\mathbf{C})$ given by $f_C^*(X) = X(C)$.

Example 2.15. Given a topological space X and a point $x \in X$, there is a geometric morphism $i\colon \mathrm{Sets} \to \mathrm{Sh}(X)$ given by taking stalks at x, i.e., $i^* F = \operatorname{colim}_{U \ni x} F(U)$.

Example 2.16. Given a continuous map $f\colon X \to Y$ of topological spaces, there is a geometric morphism $f\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$, where for $F \in \mathrm{Sh}(Y)$, f^*F is the sheafification of the presheaf $U \mapsto \operatorname{colim}_{V \supseteq f(U)} F(V)$ on X.

2.17. **Proof of Giraud theorem.** Here I give a proof of (2.8). The reader should skip this section. We start by proving a number of facts which are true for categories with weak descent; some of these we have already proved for toposes.

Proposition 2.18. *Let $\mathbf{E}$ be a category with weak descent. Consider a pullback square of the form*

$$\begin{tikzcd} U \arrow[r, "q"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ V \arrow[r, "p"'] & Y \end{tikzcd}$$

in which p is a regular epi. Then q is also a regular epi, and f is an isomorphism if and only if g is an isomorphism.

Proof. Consider

$$\begin{tikzcd} W \times_Y X \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "h"] & U \arrow[r, "q"] \arrow[d, "g"] & X \arrow[d, "f"] \\ W \arrow[r, shift left] \arrow[r, shift right] & V \arrow[r, "p"'] & Y \end{tikzcd}$$

where the bottom row is a coequalizer. The top row is obtained from the bottom row by pulling back along f, and thus is a coequalizer by (2.4)(1). This proves that q is a regular epi. If g is an isomorphism, then so is h since it is obtained from g by pullback, and thus the map f between colimits is an isomorphism. $\square$

Proposition 2.19. *Let* $\mathbf{E}$ *be a category with weak descent. For any map* $f\colon X \to Y$ *in* $\mathbf{E}$*, there exists a factorization* $f = ip$ *where* i *is a monomorphism and* p *is a regular epimorphism, and* p *can be taken to be the coequalizer of the pair of projections* $X \times_Y X \rightrightarrows X$*. In particular, all regular epimorphisms are effective epimorphisms.*

Proof. Let I be the coequalizer of the projections, giving a diagram

$$X \times_Y X \rightrightarrows X \xrightarrow{p} I \xrightarrow{i} Y.$$

Clearly, p is a regular epi, so we need only check that i is mono, i.e., that the pullback of i along itself is iso. But since p is effective epi, it suffices by (2.18) to show that the pullback of i along $f = ip$ is iso. Thus we must show that j is an isomorphism in

$$\begin{tikzcd} X \times_Y X \times_Y X \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & X \times_Y X \arrow[r, "q"] \arrow[d] & X \times_Y I \arrow[r, "j"] \arrow[d] & X \arrow[d, "f"] \\ X \times_Y X \arrow[r, shift left] \arrow[r, shift right] & X \arrow[r, "p"] & I \arrow[r, "i"] & Y. \end{tikzcd}$$

where the top row is obtained from the bottom row by pullback along f. Since pullbacks of coequalizers are coequalizers by (2.4), q must be a coequalizer of the pair of arrows shown. On the other hand, it is clear that jq is part of a split fork, and so is also a coequalizer of the same pair of arrows. Hence j is an isomorphism, as desired. ☐

Corollary 2.20. *Let* $\mathbf{E}$ *be a category with weak descent, and let* $f\colon F \to G$ *be a natural transformation of functors* $\mathbf{J} \to \mathbf{E}$ *with* $\mathbf{J}$ *a small category. If* $f(j)\colon F(j) \to G(j)$ *is a regular epimorphism for each* $j \in \mathbf{J}$*, then* $\operatorname{colim}_{\mathbf{J}} F \to \operatorname{colim}_{\mathbf{J}} G$ *is a regular epimorphism.*

Proof. By (2.19) and the fact that colimits commute with colimits, the diagram $F \times_G F \rightrightarrows F \to G$ is a coequalizer. ☐

Corollary 2.21. *Let* $\mathbf{E}$ *be a category with weak descent, and let* $f\colon X \to Y$ *and* $g\colon Y \to Z$ *be maps in* $\mathbf{E}$.

(a) *If* gf *is a regular epi then so is* g.
(b) *If* f *and* g *are regular epis then so is* gf.

Proof. The proof is identical to that of (1.20). ☐

Proposition 2.22. *Let* $\mathbf{E}$ *be a category with weak descent. Then every epimorphism in* $\mathbf{E}$ *is a regular epimorphism.*

Proof. Using (2.19) and (2.21), it suffices to show that if $f\colon A \to B$ is an epimorphism and a monomorphism, then it is a regular epimorphism. Recall that $f\colon A \to B$ is an epimorphism if and only if $\operatorname{colim}(B \leftarrow A \to B) \to B$ is an isomorphism. Consider

$$\begin{tikzcd} A \arrow[d, "f"'] & A \arrow[l, "1"'] \arrow[r, "f"] \arrow[d, "1"] & B \arrow[d, "1"] \\ B & A \arrow[l, "f"'] \arrow[r, "f"] & B \end{tikzcd}$$

which is equifibered because f is a monomorphism. Taking colimits along rows recovers the identity map $1_B\colon B \to B$, and (P2b) therefore implies that $f\colon A \to B \times_B B \approx B$ is a regular epimorphism, as desired. ☐

Lemma 2.23. *Let $\mathbf{E}$ be a category with weak descent. Let A, X, Y, and B be functors $\mathbf{J} \to \mathbf{E}$ from some small category $\mathbf{J}$, and consider a pullback square*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

of functors in which the natural transformations f and g are equifibered. Then the canonical map $\operatorname{colim}_{\mathbf{J}} A \to (\operatorname{colim}_{\mathbf{J}} X) \times_{\operatorname{colim}_{\mathbf{J}} B} (\operatorname{colim}_{\mathbf{J}} Y)$ *is a regular epimorphism.*

Proof. Write $\bar{B}$, $\bar{X}$, $\bar{Y}$, and $\bar{A}$ for the colimits of B, X, Y, and A respectively. Set $X'(J) = B(J) \times_{\bar{B}} \bar{X}$ and $Y'(J) = B(J) \times_{\bar{B}} \bar{Y}$, and let

$$A'(J) = B(J) \times_{\bar{B}} (\bar{X} \times_{\bar{B}} \bar{Y}) \approx X'(J) \times_{B(J)} Y'(J).$$

Then (2.4)(1) implies that

$$\operatorname{colim}_{\mathbf{J}} X' \approx \bar{X}, \qquad \operatorname{colim}_{\mathbf{J}} Y' \approx \bar{Y}, \qquad \operatorname{colim}_{\mathbf{J}} A' \approx \bar{X} \times_{\bar{B}} \bar{Y}.$$

Since taking colimits preserves regular epimorphisms (2.20), the lemma will be proved once we show that each map $A(J) \to A'(J)$ is a regular epimorphism.

Since f and g are equifibered, (2.4)(2) tells us that each of the maps $X(J) \to X'(J)$ and $Y(J) \to Y'(J)$ are regular epimorphisms. Therefore, the composite map

$$A(J) \approx X(J) \times_{B(J)} Y(J) \to X(J) \times_{B(J)} Y'(J) \to X'(J) \times_{B(J)} Y'(J) \approx A'(J)$$

is regular epi, since regular epis are preserved under pullback by (2.18), and are closed under composition by (2.21)(b). □

We will prove (2.8) by proving two propositions.

Proposition 2.24. *Let $\mathbf{E}$ be a category which satisfies hypotheses (i) and (ii) of* (2.8)*; that is, it has weak descent, and contains a small generating, full subcategory $\mathbf{C}$. There is an adjoint pair*

$$\ell \colon \operatorname{PSh}(\mathbf{C}) \rightleftarrows \mathbf{E} : r$$

where r is defined by $(rX)(C) = \mathbf{E}(C, X)$, and $\ell r \to 1$ is a natural isomorphism, whence $\mathbf{E}$ is equivalent to a full subcategory of $\operatorname{PSh}(\mathbf{C})$*.*

Proposition 2.25. *Let $\mathbf{C}$ be a small category. Let* $\operatorname{Sh}(\mathbf{C})$ *be a full subcategory of* $\operatorname{PSh}(\mathbf{C})$ *which is closed under isomorphisms, and such that the inclusion functor $i\colon \operatorname{Sh}(\mathbf{C}) \to \operatorname{PSh}(\mathbf{C})$ admits a left adjoint $a\colon \operatorname{PSh}(\mathbf{C}) \to \operatorname{Sh}(\mathbf{C})$. Suppose that the yoneda functor $y\colon \mathbf{C} \to \operatorname{PSh}(\mathbf{C})$ factors through the subcategory* $\operatorname{Sh}(\mathbf{C})$*, and that $\mathbf{C} \to \operatorname{Sh}(\mathbf{C})$ is a full and faithful embedding. If* $\operatorname{Sh}(\mathbf{C})$ *has weak descent, then a preserves finite limits.*

Thus (2.24) says that any category $\mathbf{E}$ satisfying the hypotheses of (2.8) is equivalent to a subcategory $\operatorname{Sh}(\mathbf{C})$ of $\operatorname{PSh}(\mathbf{C})$ satisfying the hypotheses of (2.25), which gives the result.

Proof of (2.24). The adjoint pair is a straightforward hom-tensor adjunction: the left adjoint ℓ sends a presheaf $F\colon \mathbf{C}^{\mathrm{op}} \to \mathrm{Sets}$ to its tensor product with the inclusion $\mathbf{C} \to \mathbf{E}$.

Note that $\ell r X$ is given by the coequalizer

$$\coprod_{C'\to C\to X} C' \rightrightarrows \coprod_{C\to X} C \to \ell r X,$$

where the second coproduct is over all maps $C \to X \in \mathbf{E}$ with $C \in \mathbf{C}$, and the first coproduct is over all sequences of maps $C' \to C \to X \in \mathbf{E}$ with $C, C' \in \mathbf{C}$. We need to show that the evident map $f\colon \ell r X \to X$ is an isomorphism. Let $U = \coprod_{C\to X} C$. Since $\mathbf{C}$ generates $\mathbf{E}$, the map $U \to X$ is an epimorphism, and hence is an effective epimorphism by (2.22), and hence is the coequalizer of the pair of projections $U \times_X U \rightrightarrows U$ by (2.19). Consider

$$\begin{tikzcd}
 & \coprod_{C'\to C\to X} C' \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \coprod_{C\to X} C \arrow[r, "g"] \arrow[d, equal] & \ell r X \arrow[d, "f"] \\
\coprod D \arrow[r, "p"'] & U \times_X U \arrow[r, shift left] \arrow[r, shift right] & U \arrow[r] & X
\end{tikzcd}$$

where the coproduct on the bottom line is over all commutative squares

$$\begin{tikzcd}
D \arrow[r, "d"] \arrow[d, "d'"'] & C \arrow[d, "c"] \\
C' \arrow[r, "c'"'] & X
\end{tikzcd} \tag{2.26}$$

with D, C, and C' in $\mathbf{C}$. Since, by weak descent, $U \times_X U \approx \coprod C \times_X C'$, the map p is an epimorphism.

The rows are coequalizers, and therefore f is an epimorphism. Thus, we will be done if we can show that f admits a retraction, i.e., if $g\colon U \to \ell r X$ equalizes the pair $U \times_X U \rightrightarrows U$. The map g is determined by a collection of maps $g_c\colon C \to \ell r X$ for each $c\colon C \to X$; the fact that g equalizes the parallel arrows $\coprod_{C'\to C\to X} C' \rightrightarrows C$ means that $g_c c' = g_{cc'}$ for every triangle $C' \xrightarrow{c'} C \xrightarrow{c} X$. Therefore for every commutative square (2.26), we have $g_c d = g_{cd} = g_{c'd'} = g_{c'} d'$. Because p is an epimorphism, g must equalize the pair $U \times_X U \rightrightarrows U$, and we are done. $\square$

Proof of (2.25). We call an object of $\mathrm{PSh}(\mathbf{C})$ a "pseudo-sheaf" if it is contained in $\mathrm{Sh}(\mathbf{C})$; these are presicely the presheaves X such that $X \to iaX$ is an isomorphism. Note that by hypothesis, the representable presheaves $y(C)$ lie in the full subcategory, and so are pseudo-sheaves.

The terminal object $1 \in \mathrm{PSh}(\mathbf{C})$ is automatically a pseudo-sheaf, and thus $1 \approx a1$ is the terminal object in pseudo-sheaves. thus, to prove the result, it suffices to show that a commutes with pullbacks. That is, we need to show: (*) that

$$a(X \times_B Y) \to aX \times_{aB} aY$$

is an isomorphism for all objects X, Y, B in $\mathrm{PSh}(\mathbf{C})$. We prove property (*) by a sequence of reductions.

(a) *Property (*) holds if X, Y, B are pseudo-sheaves.* In this case, the presheaf pullback is already a pseudo-sheaf.

(b) *Property (*) holds if Y and B are pseudo-sheaves.* Since representable presheaves are pseudo-sheaves, every presheaf is a colimit (in the category of presheaves) of pseudo-sheaves. Thus, suppose $X \approx \operatorname{colim}_{\mathbf{J}} U$ for some functor $U \colon \mathbf{J} \to \operatorname{Sh}(\mathbf{C}) \subseteq \operatorname{PSh}(\mathbf{C})$ from a small category $\mathbf{J}$. For each $j \in \mathbf{J}$ we have a sequence of pullback squares

$$\begin{tikzcd} U(j) \times_B Y \arrow[r] \arrow[d] & X \times_B Y \arrow[r] \arrow[d] & Y \arrow[d] \\ U(j) \arrow[r] & X \arrow[r] & B \end{tikzcd}$$

By (2.4)(1) in $\operatorname{PSh}(\mathbf{C})$, $\operatorname{colim}_{\mathbf{J}} U \times_B Y \approx X \times_B Y$. Since $U(j)$ is a pseudo-sheaf, part (a) shows that $aU(j) \times_{aB} aY \approx a(U(j) \times_B Y)$ for each $j \in \mathbf{J}$. Now (2.4)(1) applied to $\operatorname{Sh}(\mathbf{C})$ shows that $\operatorname{colim}_{\mathbf{J}}(aU \times_{aB} aY) \approx (\operatorname{colim}_{\mathbf{J}} aU) \times_{aB} aY$ in $\operatorname{Sh}(\mathbf{C})$; hence, $a(X \times_B Y) \approx \operatorname{colim}_{\mathbf{J}} a(U \times_B Y) \approx \operatorname{colim}_{\mathbf{J}}(aU \times_{aB} aY) \approx aX \times_{aB} aY$, as desired.

(c) *Property (*) holds if B is a pseudo-sheaf.* This is proved exactly as in (b), except that we can drop the hypothesis that Y is a pseudo-sheaf by making use of (b).

(d) *The functor a preserves products.* This follows from (c) and the fact that the terminal object is a pseudo-sheaf.

(e) *The functor a preserves monomorphisms.* Let $X \to Y$ be a monomorphism of presheaves. Write $Y = \operatorname{colim}_{\mathbf{J}} V$, where $V \colon \mathbf{J} \to \operatorname{Sh}(\mathbf{C}) \subseteq \operatorname{PSh}(\mathbf{C})$. Let $U(j) = V(j) \times_Y X$, and let $f \colon U \to V$ denote the evident natural map. By (c) proved above, the maps $af(j) \colon aU(j) \to aV(j)$ give an equifibered natural transformation between functors to pseudo-sheaves, and each map $af(j)$ is a monomorphism. Therefore by (2.5) applied to $\operatorname{Sh}(\mathbf{C})$, the map of colimits $aX \to aY$ is a monomorphism.

(f) *Property (*) holds for general X, Y, and B.* Note that the map $h \colon X \times_B Y \to X \times Y$ is a monomorphism for general reasons. The composite map

$$a(X \times_B Y) \xrightarrow{f} aX \times_{aB} aY \to aX \times aY$$

is isomorphic to ah by (d), and therefore is a monomorphism by (e). It follows that f is a monomorphism. Thus, it will suffice to show that f is regular epi.

Write $B = \operatorname{colim}_{\mathbf{J}} W$, where $W \colon \mathbf{J} \to \operatorname{Sh}(\mathbf{C}) \subseteq \operatorname{PSh}(\mathbf{C})$ is a functor from a small category $\mathbf{J}$. Let $U(j) = W(j) \times_B X$ and $V(j) = W(j) \times_B Y$. Then (2.4)(1) in $\operatorname{PSh}(\mathbf{C})$ implies that $X \approx \operatorname{colim}_{\mathbf{J}} U$ and $Y \approx \operatorname{colim}_{\mathbf{J}} V$, and that $\operatorname{colim}_{\mathbf{J}} U \times_W V \approx X \times_B Y$. Using (c), we see that both $aU \to aW$ and $aV \to aW$ are equifibered transformations; therefore (2.23) applied to $\mathbf{E}$ gives that

$$a(X \times_B Y) \approx a(\operatorname{colim}_{\mathbf{J}} U \times_W V) \approx \operatorname{colim} aU \times_{aW} aV \to aX \times_{aB} aY$$

is a regular epimorphism, as desired.

□

3. Grothendieck topologies

Above, I defined a site to be a small category $\mathbf{C}$, together with a full subcategory $\operatorname{Sh}(\mathbf{C})$ of $\operatorname{PSh}(\mathbf{C})$ satisfying certain properties; this is not the usual definition. In this section, I will show that the notion of site I used is equivalent to the more usual notion of a "Grothendieck site", namely a small category $\mathbf{C}$ equipped with a Grothendieck topology.

3.1. **Definition of a Grothendieck topology.** Recall the yoneda embedding $y\colon \mathbf{C} \to \mathrm{PSh}(\mathbf{C})$ which associates to each object $C \in \mathbf{C}$ the representable functor defined by $y(C)(-) = \mathbf{C}(-, C)$. Given an object $C \in \mathbf{C}$, a **sieve** over C is a subfunctor $S \rightarrowtail y(C)$ of $y(C)$. A sieve can be thought of as a collection $\mathcal{F}_S = \bigcup_{C' \in \mathbf{C}} S(C')$ of morphisms with codomain C, which is closed under composition on the right: for any composable pair of maps $f, g \in \mathbf{C}$, $f \in \mathcal{F}_S$ implies $fg \in \mathcal{F}_S$.

Given a map $f\colon C' \to C \in \mathbf{C}$, and a sieve $s\colon S \rightarrowtail y(C)$ over C, we define sieve $f^{-1}S \rightarrowtail y(C')$ over C' by taking the pullback of s along $y(f)$:

$$\begin{tikzcd} f^{-1}S \arrow[r] \arrow[d, tail] & S \arrow[d, tail] \\ y(C') \arrow[r, "y(f)"'] & y(C) \end{tikzcd}$$

In other words, $\mathcal{F}_{f^{-1}S}$ consists of functions $g\colon C'' \to C'$ such that $fg \in \mathcal{F}_S$.

A **Grothendieck topology** τ on $\mathbf{C}$ is a set of sieves $S \rightarrowtail y(C)$ over objects of $\mathbf{C}$ satisfying the following three properties.

(G1) For each $C \in \mathbf{C}$, the identity map $1\colon y(C) \to y(C)$ is in τ.
(G2) For each $f\colon C' \to C \in \mathbf{C}$, if $S \rightarrowtail y(C) \in \tau$ then $f^{-1}S \rightarrowtail y(C') \in \tau$.
(G3) Let $s\colon S \rightarrowtail y(C)$ be a sieve. If $T \rightarrowtail y(C) \in \tau$, such that for each $C' \in \mathbf{C}$ and $f \in T(C')$ the sieve $f^{-1}S \rightarrowtail y(C')$ is contained in τ, then $s \in \tau$.

The elements of τ are called **covering sieves**; we will write τ_C for the set of covering sieves over a given object $C \in \mathbf{C}$.

Remark 3.2. Note that if $\{C_i \to C\}$ is some collection of morphisms in $\mathbf{C}$ with codomain C, then there is a smallest sieve S over C generated by this set. Say that such a set of morphisms is a **covering family** if it generates a covering sieve. It is possible (and usual) to reformulate the notion of a Grothendieck topology in terms of covering families. It is also possible (and usual) to suppose that the category $\mathbf{C}$ has finite limits; this implies that if $\{C_i \to C\}$ is a family of maps generating a sieve S, and if $f\colon C' \to C$ is a map, then $\{C_i \times_C C' \to C'\}$ is a family of maps generating the sieve $f^{-1}S$.

Example 3.3. Let $\mathcal{U}_X$ be the category of open subsets of a topological space X. A sieve S on an open set $U \in \mathcal{U}_X$ corresponds to what is usually called a **filter**, namely a collection $\mathcal{F}$ of open subsets of U such that $W \subseteq V \in \mathcal{F}$ implies $W \in \mathcal{F}$. If we let τ_U correspond to the collection of filters $\mathcal{F}$ over U such that $U = \bigcup_{V \in \mathcal{F}} V$, then we obtain the "usual" Grothendieck topology on X. In this case, the covering families of an open set U correspond precisely to the **open covers** of U.

3.4. **Grothendieck sites and sheafification.** A **Grothendieck site** consists of a small category $\mathbf{C}$ and a Grothendieck topology τ on $\mathbf{C}$. A **sheaf** is a presheaf $X \in \mathrm{PSh}(\mathbf{C})$ such that for every covering sieve $s\colon S \rightarrowtail y(C) \in \tau$, the evident map

$$\mathrm{PSh}(\mathbf{C})(y(C), X) \to \mathrm{PSh}(\mathbf{C})(S, X)$$

is a bijection. We write $\mathrm{Sh}(\mathbf{C},\tau)$ for the full subcategory of sheaves.

Proposition 3.5. *The category* $\mathrm{Sh}(\mathbf{C},\tau)$ *is a site in the sense of §1.2.*

Proof. I will sketch the proof. For more details, see [MLM94, Ch. III] for example.

It is clear that $\mathrm{Sh}(\mathbf{C},\tau)$ is closed under isomorphism, so it remains to produce a left adjoint $a\colon \mathrm{PSh}(\mathbf{C}) \to \mathrm{Sh}(\mathbf{C},\tau)$ to the inclusion functor $i\colon \mathrm{Sh}(\mathbf{C},\tau) \to \mathrm{PSh}(\mathbf{C})$, and show that a preserves finite limits.

To do this, one defines a functor $X \mapsto X^+\colon \mathrm{PSh}(\mathbf{C}) \to \mathrm{PSh}(\mathbf{C})$ together with a natural transformation $\eta\colon X \to X^+$, satisfying the following properties:

(a) $X \mapsto X^+$ preserves finite limits of presheaves,
(b) for any sheaf F, the map $\mathrm{PSh}(\mathbf{C})(X^+,F) \to \mathrm{PSh}(\mathbf{C})(X,F)$ induced by η is a bijection, and
(c) for any presheaf X, the presheaf $(X^+)^+$ is a sheaf.

Given this, it is easy to see that the functor $X \mapsto (X^+)^+$ lands in sheaves, and defines a left adjoint to inclusion which preserves finite limits.

The functor $X \mapsto X^+$ is constructed as follows. For each $C \in \mathbf{C}$, let τ_C denote the set of covering sieves over C. The set τ_C is actually a directed set, by reverse inclusion; the intersection of two covering sieves is a covering sieve. Set

$$X^+(C) \overset{\text{def}}{=} \operatorname{colim}_{S\in\tau_C} \mathrm{PSh}(\mathbf{C})(S,X).$$

Since the identity map $1\colon y(C) \to y(C)$ is a covering sieve, we get a function

$$\eta(C)\colon X(C) = \mathrm{PSh}(\mathbf{C})(y(C),X) \to X^+(C).$$

One shows, using the properties of the Grothendieck topology, that X^+ is a presheaf and η a map of presheaves.

Property (a) follows from the fact that directed colimits are left exact.

Property (b) can be easily checked directly; it is also an immediate consequence of (3.6) below, which gives another description of X^+ and η.

Call a presheaf X a **separated presheaf** if $\mathrm{PSh}(\mathbf{C})(y(C),X) \to \mathrm{PSh}(\mathbf{C})(S,X)$ is a monomorphism for each covering sieve. Property (c) follows from two observations: the functor $X \mapsto X^+$ takes presheaves to separated presheaves, and takes separated presheaves to sheaves. ☐

Lemma 3.6. *Given a presheaf X, let τ/X denote the category whose objects (f,s) are diagrams*

$$X \xleftarrow{f} S \xrightarrow{s} y(C)$$

in $\mathrm{PSh}(\mathbf{C})$ *with $s \in \tau$, and whose morphisms $(f,s) \to (f',s')$ are commutative diagrams*

$$\begin{tikzcd} X & S \arrow[l, "f"'] \arrow[r, tail, "s"] \arrow[d] & y(C) \arrow[d] \\ & S' \arrow[ul, "f'"] \arrow[r, tail, "s'"'] & y(C') \end{tikzcd}$$

Let $F,G\colon \tau/X \to \mathbf{E}$ denote the functors sending $F(f,s) = S$ and $G(f,s) = y(C)$. Then the evident map $\operatorname{colim}_{\tau/X} F \to \operatorname{colim}_{\tau/X} G$ *is naturally isomorphic to* $\eta\colon X \to X^+$.

Proof. First we note a general fact about colimits of sets. Let $K\colon \mathbf{D} \to \text{Sets}$ be some functor from a small category. Let $\mathcal{P}_K$ be the category whose objects are pairs (D, a), with D and object of $\mathbf{D}$ and $a \in K(D)$; morphisms $(D, a) \to (D', a')$ are maps $f\colon D \to D' \in \mathbf{D}$ such that $K(f)(a) = a'$. Then
$$\operatorname{colim}_{\mathbf{D}} K \approx \pi_0 \operatorname{nerve}(\mathcal{P}_K).$$

Let $\mathbf{D} = \tau/X$, and let $\mathrm{ev}_C\colon \mathrm{PSh}(\mathbf{C}) \to \text{Sets}$ denote the functor of evaluation at the object $C \in \mathbf{C}$. The category $\mathcal{P}_{\mathrm{ev}_C F}$ has as objects tuples $((f, s), u)$, where $X \xleftarrow{f} S \xrightarrow{s} y(D)$ is an object of τ/X and $u \in S(C)$; I will also regard the element u as a map $u\colon y(C) \to S$. Consider the full subcategory $\mathcal{P}^0_{\mathrm{ev}_C F}$ of $\mathcal{P}_{\mathrm{ev}_C F}$ consisting of objects $((f, \mathrm{id}_C), \iota_C)$ where $\mathrm{id}_C\colon y(C) \to y(C)$ is the trivial sieve over C, and $\iota_C \in y(C)(C)$ is the element corresponding to the identity map of C. The inclusion functor $\mathcal{P}^0_{\mathrm{ev}_C F} \to \mathcal{P}_{\mathrm{ev}_C F}$ admits a right adjoint, which sends an object $((f, s), u)$ to $((fu, \mathrm{id}_{y(C)}), \iota_C)$.

$$\begin{tikzcd}
& y(C) \arrow[r, "\mathrm{id}"] \arrow[dl, "fu"'] \arrow[d, "u"'] & y(C) \arrow[d, "su"] & \iota_C \in y(C)(C) \arrow[d, mapsto] \\
X & S \arrow[l, "f"] \arrow[r, "s"'] & y(D) & u \in S(C)
\end{tikzcd}$$

Thus, $\operatorname{nerve} \mathcal{P}_{\mathrm{ev}_C F} \approx \operatorname{nerve} \mathcal{P}^0_{\mathrm{ev}_C F}$. Since the only morphisms of $\mathcal{P}^0_{\mathrm{ev}_C F}$ are identity maps, and the objects correspond to elements of $X(C)$, we see that $\operatorname{colim}_{\tau/X} \mathrm{ev}_C F \approx X(C)$.

The category $\mathcal{P}_{\mathrm{ev}_C G}$ has as objects tuples $((f, s), v)$, where $X \xleftarrow{f} S \xrightarrow{s} y(D)$ is an object of τ/X and $v \in y(D)(C)$; I will also regard v as a map $v\colon y(C) \to y(D)$. Consider the full subcategory $\mathcal{P}^1_{\mathrm{ev}_C G}$ of $\mathcal{P}_{\mathrm{ev}_C G}$ consisting of objects $((f, s), \iota_C)$ where $s\colon S \to y(C)$ is a sieve over C, and $\iota_C \in y(C)(C)$ the element corresponding to the identity map of C. The inclusion functor $\mathcal{P}^1_{\mathrm{ev}_C G} \to \mathcal{P}_{\mathrm{ev}_C G}$ admits a right adjoint, which sends an object $((f, s), v)$ to $((fu, t), \iota_C)$ defined by

$$\begin{tikzcd}
& v^{-1}S \arrow[r, "t"] \arrow[dl, "fu"'] \arrow[d, "u"'] & y(C) \arrow[d, "v"] & \iota_C \in y(C)(C) \arrow[d, mapsto] \\
X & S \arrow[l, "f"] \arrow[r, "s"'] & y(D) & v \in y(D)(C)
\end{tikzcd}$$

where the square is a pullback.

Thus, $\operatorname{nerve} \mathcal{P}_{\mathrm{ev}_C G} \approx \operatorname{nerve} \mathcal{P}^1_{\mathrm{ev}_C G}$. The components of $\operatorname{nerve} \mathcal{P}^1_{\mathrm{ev}_C G}$ are in one-to-one correspondence with $X^+(C)$ (and each component is the nerve of a filtered category and thus contractible). Thus $\operatorname{colim}_{\tau/X} \mathrm{ev}_C G \approx X^+(C)$. $\square$

3.7. **Grothendieck sites are sites.**

Proposition 3.8. *A subcategory* $\mathrm{Sh}(\mathbf{C}) \subseteq \mathrm{PSh}(\mathbf{C})$ *is a site (as defined in §1.2) if and only if it is equal to the category of sheaves on the Grothendieck site* $(\mathbf{C}, \tau)$*, for some Grothendieck topology* τ*.*

I'll give a sketch of the proof here.

Let $\mathbf{C}$ be a small category, $\mathrm{Sh}(\mathbf{C}) \subseteq \mathrm{PSh}(\mathbf{C})$ a site as defined in §1.2, and let $a\colon \mathrm{PSh}(\mathbf{C}) \rightleftarrows \mathrm{Sh}(\mathbf{C}) : i$ be the associated adjoint pair. Let τ denote the set of all sieves $s\colon S \rightarrowtail y(C)$ in $\mathrm{PSh}(\mathbf{C})$ such that $a(s)$ is an isomorphism. It is a straightforward exercise to show that τ is a Grothendieck topology on $\mathbf{C}$.

Let $\mathrm{Sh}(\mathbf{C}, \tau)$ denote the full subcategory of sheaves with respect to the topology τ. Write $a_\tau\colon \mathrm{PSh}(\mathbf{C}) \rightleftarrows \mathrm{Sh}(\mathbf{C}, \tau) : i_\tau$ for the adjoint pair associated to the Grothendieck site $(\mathbf{C}, \tau)$.

We want to show that $\mathrm{Sh}(\mathbf{C}) = \mathrm{Sh}(\mathbf{C}, \tau)$. It is clear from the definitions that $\mathrm{Sh}(\mathbf{C}) \subseteq \mathrm{Sh}(\mathbf{C}, \tau)$. The result will thus follow when we prove for any morphism $f\colon X \to Y \in \mathrm{PSh}(\mathbf{C})$, that

(*) $a(f)$ is an isomorphism implies $a_\tau(f)$ is an isomorphism.

Let $f \in \mathrm{PSh}(\mathbf{C})$ such that $a(f)$ is iso. We can factor $f = ip$ where i is a monomorphism and p is a regular epimorphism (in $\mathrm{PSh}(\mathbf{C})$). Since a preserves limits, $a(i)$ is a monomorphism in $\mathrm{Sh}(\mathbf{C})$. Since $a(f)$ is iso, this implies that $a(i)$ and $a(p)$ are iso. Thus we have reduced the problem to showing (*) in the special cases of (a) f is a monomorphism, and (b) f is a regular epimorphism.

(a) *Proof of (*) for f a monomorphism.* Write the presheaf Y as a colimit $Y = \operatorname{colim}_{\mathbf{J}} Y_\alpha$, where the Y_α are representable presheaves. Let $X_\alpha \overset{\text{def}}{=} Y_\alpha \times_Y X$. Since f is a monomorphism, the maps $f_\alpha\colon X_\alpha \to Y_\alpha$ are sieves. Since a commutes with pullbacks, $a(f_\alpha)$ is an isomorphism. Thus $f_\alpha \in \tau$. Any $s \in \tau$ has the property that $a_\tau(s)$ is an isomorphism, and so in particular $a_\tau(f_\alpha)$ is an isomorphism. Since a_τ preserves colimits and $X \approx \operatorname{colim}_{\mathbf{J}} X_\alpha$, $a_\tau(f)$ is an isomorphism.

(b) *Proof of (*) for f a regular epimorphism.* Since a_τ preserves colimits, $a_\tau(f)$ is a regular epimorphism. To complete the proof, we need to show that $a_\tau(f)$ is a monomorphism as well. Let $g\colon X \to X \times_Y X$ be the diagonal map associated to f; the map g is a monomorphism of presheaves. Since a preserves pullbacks and $a(f)$ is iso, we must have that $a(g)$ is iso. By case (a), it follows that $a_\tau(g)$ is iso, and thus $a_\tau(f)$ is a monomorphism since a_τ preserves finite limits.

4. Model Categories

In this section, I give a brief exposition of results from the theory of model categories which I will need. Most proofs are omitted.

Given a category $\mathbf{M}$ and a subcategory $\mathbf{W}$, which we will call the category of "weak-equivalences" in $\mathbf{M}$, the **category of fractions** is category $\mathrm{Ho}_{\mathbf{W}}\,\mathbf{M}$ and a functor $\gamma\colon \mathbf{M} \to \mathrm{Ho}_{\mathbf{W}}\,\mathbf{M}$ which is initial among functors from $\mathbf{M}$ which send morphisms of $\mathbf{W}$ to isomorphisms. We write $\mathrm{Ho}\,\mathbf{M}$ for $\mathrm{Ho}_{\mathbf{W}}\,\mathbf{M}$ when $\mathbf{W}$ is understood. In particular, any functor $F\colon \mathbf{M} \to \mathbf{D}$ which takes weak equivalences to isomorphisms factors uniquely through a functor $\mathrm{Ho}\,F\colon \mathrm{Ho}\,\mathbf{M} \to \mathbf{D}$.

A **Quillen model category** is a pair $(\mathbf{M}, \mathbf{W})$ as above, together with a "model category structure"; this consists of two additional subcategories of $\mathbf{M}$, the subcategories of **fibrations** and the **cofibrations**, which satisfy a number of properties, which I will not list here; [DS95] is the best introduction to this subject. It may be useful to note that given $(\mathbf{M}, \mathbf{W})$, the class of fibrations is determined by the class of cofibrations, and vice versa, so that to specify a model category structure, only one of these two classes needs to be described.

Quillen gave the original axioms for a model category in [Qui67] and [Qui69] (which are those followed by [DS95]). Since then it has been found useful to require some additional properties; in particular, that a model category be complete and cocomplete, and that it has "functorial factorizations". I will take this formulation (as in [DHKS04], [Hov99], and [Hir03]) as the definition of a model category.

A pair $(\mathbf{M}, \mathbf{W})$ can admit more than one model category structure.

Example 4.1. Let $\mathbf{M} = s\,\mathrm{PSh}(\mathbf{C})$ be the category of **simplicial presheaves**, i.e., the category consisting of functors $\mathbf{C}^{\mathrm{op}} \to \mathbf{C}$, where $\mathbf{C}$ is a small category and $\mathbf{S}$ is the category of simplicial sets. Let $\mathbf{W}$ denote the class of maps $f\colon X \to Y \in \mathbf{M}$ for which $f(C)\colon X(C) \to Y(C)$ is a weak equivalence of simplicial sets for all $C \in \mathbf{C}$.

The pair $(\mathbf{M}, \mathbf{W})$ admits at least two model category structures: the **Bousfield-Kan model structure** (or **projective model structure**), in which f is a fibration if and only if each $f(C)$ is a fibration of simplicial sets, and the **Heller model structure** (or **injective model structure**), in which f is a cofibration if and only if each $f(C)$ is a cofibration of simplicial sets, or what is the same thing, if f is a monomorphism.

4.2. **Derived functors.** Model categories are a machine for constructing derived functors. A **left-derived functor** $LF\colon \mathrm{Ho}\mathbf{M} \to \mathbf{D}$ of $F\colon \mathbf{M} \to \mathbf{D}$ is an initial object in the category of pairs consisting of a functor $F'\colon \mathrm{Ho}\,\mathbf{M} \to \mathbf{D}$ and a natural transformation $F \to F' \circ \gamma$. Likewise a **right-derived functor** $RG\colon \mathrm{Ho}\mathbf{M} \to \mathbf{D}$ of $G\colon \mathbf{M} \to \mathbf{D}$ is a final object in the category of pairs consisting of a functor $G'\colon \mathrm{Ho}\,\mathbf{M} \to \mathbf{D}$ and a natural transformation $G' \circ \gamma \to G$.

Derived functors on model categories are constructing using cofibrant or fibrant replacement; for an object X of a model category these are objects X^{cof} and X^{fib} and together with maps

$$0 \xrightarrow{\mathrm{cof}} X^{\mathrm{cof}} \xrightarrow{\mathbf{W}} X \qquad X \xrightarrow{\mathbf{W}} X^{\mathrm{fib}} \xrightarrow{\mathrm{fib}} 1$$

which are cofibrations, fibrations, or weak equivalences as labelled. The cofibrant and fibrant replacements play the same role that projective and injective resolutions play in homological algebra. Given a functor $F\colon \mathbf{M} \to \mathbf{D}$, a left-derived functor $LF\colon \mathrm{Ho}\,\mathbf{M} \to \mathbf{D}$ can be defined if F takes weak equivalences between cofibrant replacements to isomorphisms, so that $LF(X) \approx F(X^{\mathrm{cof}})$, and a right-derived functor $RF\colon \mathrm{Ho}\,\mathbf{M} \to \mathbf{D}$ can be defined if F takes weak equivalences between fibrant replacements to isomorphisms, so that $RF(X) = F(X^{\mathrm{fib}})$.

4.3. **Quillen pairs.** A **Quillen pair** between model categories is a pair of adjoint functors $F\colon \mathbf{M} \rightleftarrows \mathbf{N} : G$ such that F preserves cofibrations and G preserves fibrations. In this case, there are **total derived functors** $\underline{L}F\colon \mathbf{M} \to \mathbf{N}$ and $\underline{L}G\colon \mathbf{N} \to \mathbf{M}$ defined by

$$\underline{L}F(X) \overset{\mathrm{def}}{=} F(X^{\mathrm{cof}}) \qquad \underline{L}G(Y) \overset{\mathrm{def}}{=} G(Y^{\mathrm{fib}}),$$

where $X \mapsto X^{\mathrm{cof}}$ and $Y \mapsto Y^{\mathrm{cof}}$ are functorial cofibrant (resp. fibrant) replacement functors in $\mathbf{M}$ (resp. $\mathbf{N}$). The total derived functors take weak equivalences to weak equivalences, and thus pass to an adjoint pair of functors

$$\mathrm{Ho}\,\underline{L}F\colon\ \mathrm{Ho}\,\mathbf{M} \rightleftarrows \mathrm{Ho}\,\mathbf{N} : \mathrm{Ho}\,\underline{L}G$$

on homotopy categories. In later sections it will be usually understood that we are talking about total derived functors, and thus I will often drop the symbols $\underline{L}$ and $\underline{R}$.

A **Quillen equivalence** is a Quillen pair which induces equivalences of the corresponding homotopy categories. It is the correct notion of "weak equivalence" of model categories.

4.4. **Constructions of model categories.** A nice feature of model categories, is that categories related to a model category are also model categories.

(a) If $\mathbf{M}$ is a model category, then the opposite category $\mathbf{M}^{\mathrm{op}}$ is also a model category, with the same weak equivalences, and the fibrations and cofibrations switched. If $\mathbf{M}_i$ are model categories for $i = 1, 2$, the product category $\mathbf{M}_1 \times \mathbf{M}_2$ is a model category, in which the classes of fibrations, cofibrations, and weak equivalences in the product are the products of those classes of the $\mathbf{M}_i$.
(b) If X is an object of $\mathbf{M}$, then the slice category $\mathbf{M}/X$ inherits a model category structure; the weak equivalences, fibrations, and cofibrations are the maps $(A \to X) \to (B \to X)$ such that the underlying map $A \to B$ in $\mathbf{M}$ is a weak equivalence, fibration, or cofibration.
(c) Similarly, $X\backslash\mathbf{M}$ inherits a model category structure.
(d) For a small category $\mathbf{I}$, let $\mathbf{M}^{\mathbf{I}}$ denote the category of functors $\mathbf{I} \to \mathbf{M}$. Under suitable hypotheses on $\mathbf{I}$ or $\mathbf{M}$ (for instance, if the nerve of $\mathbf{I}$ is a finite simplicial set, or if $\mathbf{M}$ is cofibrantly generated), the category $\mathbf{M}^{\mathbf{I}}$ admits a model category structure, with weak equivalences the natural transformations $\eta\colon F \to G$ such that $\eta(i)\colon F(i) \to G(i)$ is a weak equivalence in $\mathbf{M}$ for every object $i \in \mathbf{I}$.

Remark 4.5. There is a subtlety involving examples (b) and (c), which I'll illustrate for (b). If $f\colon X \to Y$ is a map in $\mathbf{M}$, there is an induced Quillen pair $F\colon \mathbf{M}/X \rightleftarrows \mathbf{M}/Y : G$, where $F(g\colon U \to X) = (fg\colon U \to Y)$ and $G(h\colon V \to Y) = (V \times_Y X \to X)$. One might expect that if f is a weak equivalence, then (F, G) is a Quillen equivalence, but this is *not* always the case. It is true that if X and Y are fibrant objects, then a weak equivalence f gives rise to a Quillen equivalence of slice categories. In other words, the Quillen equivalence type of $\mathbf{M}/X$ is not necessarily a homotopy invariant of X, *unless* we restrict attention to fibrant object X.

For this reason, when I speak of the slice category $\mathbf{M}/X$, I'll implicitly assume that X is to be replaced by a fibrant object, if it is not already so.

A model category is called **right proper** if arbitrary weak equivalences $X \to Y$ induce Quillen equivalences $\mathbf{M}/X \rightleftarrows \mathbf{M}/Y$. There is a dual notion of **left proper**.

4.6. **Homotopy limits and colimits.** An important example of derived functors are homotopy limits and colimits. Thus, the composite of the colimit functor $\operatorname{colim}\colon \mathbf{M}^{\mathbf{I}} \to \mathbf{M}$ with $\gamma\colon \mathbf{M} \to \operatorname{Ho}\mathbf{M}$ has a left derived functor denoted $L\operatorname{colim}\colon \mathbf{M}^{\mathbf{I}} \to \mathbf{M}$. Under suitable hypotheses on $\mathbf{M}$ (i.e., if it is a model category with functorial factorizations), there is a **homotopy colimit functor** $\operatorname{hocolim}\colon \mathbf{M}^{\mathbf{I}} \to \mathbf{M}$ which is the total left derived functor of colim; also, there is a **homotopy limit functor** $\operatorname{holim}\colon \mathbf{M}^{\mathbf{I}} \to \mathbf{M}$ which is the total right derived functor of lim.

4.7. **Derived mapping space.** The set $\operatorname{Ho}\mathbf{M}(X, Y)$ of homotopy classes of maps between two objects in $\mathbf{M}$ is actually the set of path components of a certain space (simplicial set), which I'll call the **derived mapping space**. I refer the reader to [Hov99] for the construction of these derived mapping spaces, which exist for any model category.

We write $\operatorname{map}_{\mathbf{M}}\colon \mathbf{M}^{\mathrm{op}} \times \mathbf{M} \to \mathbf{S}$ for the derived mapping space construction. It has the following properties:

(a) $\pi_0 \operatorname{map}(X, Y) \approx \operatorname{Ho}\mathbf{M}(X, Y)$.
(b) If $F\colon \mathbf{M} \rightleftarrows \mathbf{N} : G$ is a Quillen pair, then there is a weak equivalence
$$\operatorname{map}_{\mathbf{N}}(\underline{L}FX, Y) \approx \operatorname{map}_{\mathbf{M}}(X, \underline{R}GY).$$
(c) If $F\colon \mathbf{M} \rightleftarrows \mathbf{N} : G$ is a Quillen equivalence, then $\operatorname{map}_{\mathbf{M}}(X, Y) \approx \operatorname{map}_{\mathbf{N}}(FX, FY)$ and $\operatorname{map}_{\mathbf{N}}(X, Y) \approx \operatorname{map}_{\mathbf{M}}(GX, GY)$.
(d) For a functor $F\colon \mathbf{I} \to \mathbf{M}$ from a small category $\mathbf{I}$, we have weak equivalences
$$\operatorname{map}_{\mathbf{M}}(\operatorname{hocolim}_{\mathbf{I}} F, Y) \approx \operatorname{holim}_{\mathbf{I}} \operatorname{map}_{\mathbf{M}}(F, Y)$$
and
$$\operatorname{map}_{\mathbf{M}}(X, \operatorname{holim}_{\mathbf{I}} F) \approx \operatorname{holim}_{\mathbf{I}} \operatorname{map}_{\mathbf{M}}(X, F).$$

4.8. **Simplicially enriched model categories.** Although map is a functor in each of its variables, it is not in general composable; in general, there is no natural composition map $\operatorname{map}(X, Y) \times \operatorname{map}(Y, Z) \to \operatorname{map}(X, Z)$. It would be nice to be able to associate to a model category a category enriched over simplicial sets.

A category $\mathbf{M}$ is **enriched over simplicial sets** if for every pair of objects $X, Y \in \mathbf{M}$ there is a simplicial set $\hom(X, Y)$ (the "function complex"), whose vertices are precisely the morphisms in $\mathbf{M}$, and which has associative composition maps $\hom(X, Y) \times \hom(Y, Z) \to \hom(X, Z)$. We say that an enriched model category $\mathbf{M}$ is a **simplicial model category** if

(a) $\mathbf{M}$ has *enriched* limits and colimits, and
(b) if for every cofibration $i\colon A \to B$ and fibration $p\colon X \to Y$ in $\mathbf{M}$, the induced map
$$\hom(B, Y) \to \hom(A, X) \times_{\hom(A,Y)} \hom(B, Y)$$
of simplicial sets is a Kan fibration, and is a weak equivalence of one of i or p is a weak equivalence.

The main point is that if X and Y are fibrant-and-cofibrant objects, then $\hom(X, Y)$ is weakly equivalent to the derived mapping space $\operatorname{map}(X, Y)$. Thus the full simplicially enriched subcategory of $\mathbf{M}$ consisting of fibrant-and-cofibrant objects gives us what we want.

Not every model category is simplicial; however, the presentable model categories of the next section are always simplicial model categories. This includes (up to Quillen equivalence) all the examples of model categories we are interested in.

5. Universal model categories and presentable model categories

5.1. **Universal model categories.** For a small category $\mathbf{C}$, let $s\operatorname{PSh}(\mathbf{C})$ denote the category of simplicial presheaves on $\mathbf{C}$ equipped with the Bousfield-Kan model category structure (4.1).

In [Dug01b], Dugger proves the following result.

Theorem 5.2. *Let $\mathbf{M}$ be a model category, and $\gamma\colon \mathbf{C} \to \mathbf{M}$ a functor from a small category $\mathbf{C}$; let $y\colon \mathbf{C} \to s\operatorname{PSh}(\mathbf{C})$ denote the yoneda embedding. Then there exists a functor $L\colon s\operatorname{PSh}(\mathbf{C}) \to \mathbf{M}$ and a natural transformation $\eta\colon L \circ y \to \gamma$, such that L is the left adjoint of a Quillen pair $s\operatorname{PSh}(\mathbf{C}) \rightleftarrows \mathbf{M}$, and η gives a weak equivalence when evaluated at any* *object of* $\mathbf{C}$*. Furthermore, the choice of* (L, η) *is essentially unique, in the sense that the category of such data* (L, η) *has contractible nerve.*

In Dugger's terminology, this makes $s\,\mathrm{PSh}(\mathbf{C})$ the "universal model category" on the small category $\mathbf{C}$.

As a consequence, if $F\colon s\,\mathrm{PSh}(\mathbf{C}) \to \mathbf{M}$ is any functor to a model category $\mathbf{M}$ which carries weak equivalences to weak equivalences and preserves homotopy colimits, the above theorem shows that there is a Quillen pair $U\colon s\,\mathrm{PSh}(\mathbf{C}) \rightleftarrows \mathbf{M} : V$ such that F is naturally weakly equivalent to the total derived functor $\underline{L}U$.

5.3. **Localization model categories.** Let $\mathbf{M}$ be a model category, and let S be a class of morphisms of $\mathbf{M}$. We say an object W of $\mathbf{M}$ is S**-local** if, for each $f\colon A \to B \in S$, the induced map $\mathrm{map}(B, W) \to \mathrm{map}(A, W)$ on derived mapping spaces is a weak equivalence. Write $\mathcal{L}_S\mathbf{M}$ for the collection of S-local objects.

We say a class of morphisms S is **saturated** if

(a) all weak equivalences are in S,
(b) if the composite gf exists, and any two of f, g, gf are in S, then so is the third,
(c) if
$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[d, "g"] \\ Y \arrow[r] & Y' \end{tikzcd}$$
is a homotopy pushout square, and $f \in S$, then $g \in S$,
(d) if $f\colon F \to G$ is a natural transformation of functors $\mathbf{I} \to \mathbf{M}$ from a small category $\mathbf{I}$, such that $f(i) \in S$ for all $i \in \mathbf{I}$, then $\mathrm{hocolim}_{\mathbf{I}}\, f \in S$.

Given a class S of morphisms $\mathbf{M}$, its **saturation** $\bar{S}$ is the smallest saturated class containing S. It is a formal consequence of the definitions that if $W \in \mathcal{L}_S\mathbf{M}$ and $f\colon X \to Y \in \bar{S}$, then $\mathrm{map}(Y, W) \to \mathrm{map}(X, W)$ is a weak equivalence.

Given a class of maps S in a model category $\mathbf{M}$, a **localization model category structure** on $\mathbf{M}$ is a new model category structure on $\mathbf{M}$, denoted $\mathbf{M}_S$, such that

(a) the weak equivalences of $\mathbf{M}_S$ are precisely the saturation $\bar{S}$ of S, and
(b) the identity functors $\mathrm{id}\colon \mathbf{M} \rightleftarrows \mathbf{M}_S : \mathrm{id}$ are a Quillen pair.

Under certain hypotheses on $\mathbf{M}$, if S is a *set*, then a localization model category structure always exists (see especially [Hir03]). Localization model categories have the following properties.

(a) Let us write $a\colon \mathbf{M} \rightleftarrows \mathbf{M}_S : i$ for the adjoint Quillen pair described above; both are actually the identity functor on the underlying category. Then the natural transformation $(\mathrm{Ho}\, a)(\mathrm{Ho}\, i) \to 1$ of endofunctors of $\mathrm{Ho}\,\mathbf{M}_S$ is an equivalence.
(b) The right adjoint $\mathrm{Ho}\, i\colon \mathrm{Ho}\,\mathbf{M}_S \to \mathrm{Ho}\,\mathbf{M}$ induces an equivalence between $\mathrm{Ho}\,\mathbf{M}_S$ and the full subcategory $\mathrm{Ho}\,\mathcal{L}_S\mathbf{M}$ of S-local objects of $\mathbf{M}$.
(c) Furthermore, the "embedding" $\mathrm{Ho}\, i$ is **homotopically full and faithful**, in the sense that
$$\mathrm{map}_{\mathbf{M}}((\mathrm{Ho}\, i)X, (\mathrm{Ho}\, i)Y) \approx \mathrm{map}_{\mathbf{M}_S}(X, Y).$$
(d) A morphism $f \in \bar{S}$ if and only if $\mathrm{map}_{\mathbf{M}}(f, W)$ is a weak equivalence for all $W \in \mathcal{L}_S\mathbf{M}$.

5.4. **Small presentation.** Let $\mathbf{C}$ be a small category, and let $s\,\mathrm{PSh}(\mathbf{C})$ be the category of simplicial presheaves on $\mathbf{C}$, equipped with the Bousfield-Kan model category structure.

Given a set S of maps in $s\,\mathrm{PSh}(\mathbf{C})$, there is a localization model category structure, denoted $s\,\mathrm{PSh}(\mathbf{C})_S$. Recall (§5.1) that given any functor $\gamma\colon \mathbf{C} \to \mathbf{M}$ to a model category $\mathbf{M}$, there exists an essentially unique Quillen pair extending γ, with left adjoint $L\colon s\,\mathrm{PSh}(\mathbf{C}) \to \mathbf{M}$. It is straightforward to check that L factors through a Quillen pair with left adjoint $L\colon s\,\mathrm{PSh}(\mathbf{C})_S \to \mathbf{M}$ if and only if the total derived functor $\underline{L}L$ carries elements of S to weak equivalences in $\mathbf{M}$. Thus, it is tempting to regard $s\,\mathrm{PSh}(\mathbf{C})_S$ as being given by "generators" and "relations", where the generators are the small category $\mathbf{C}$, and the relations are the set of maps S.

Dugger [Dug01b] defines a **small presentation** of a model category $\mathbf{M}$ to be a Quillen equivalence of the form

$$s\,\mathrm{PSh}(\mathbf{C})_S \rightleftarrows \mathbf{M}$$

where $\mathbf{C}$ is a small category and S is a set of maps in $s\,\mathrm{PSh}(\mathbf{C})$. Dugger shows that any model category which is Quillen equivalent to one with a small presentation also has a small presentation. He also proves [Dug01a] that a very large class of model categories (the "combinatorial model categories") admit a small presentation.

We will need to consider the following variation on these ideas. Let $\mathbf{C}$ be a small, simplicially enriched category; that is, there are simplicial sets of maps $\hom_{\mathbf{C}}(X,Y)$ for each pair of objects. Let $s\,\mathrm{PSh}(\mathbf{C})$ denote the category of simplicial functors from $\mathbf{C}^{\mathrm{op}}$ to simplicial sets. We will define a **small simplicial presentation** of a model category $\mathbf{M}$ to be a simplicial category $\mathbf{C}$, a set of maps S in $s\,\mathrm{PSh}(\mathbf{C})$, and a Quillen equivalence $s\,\mathrm{PSh}(\mathbf{C})_S \rightleftarrows \mathbf{M}$. It is not hard to show that a model category admits a small presentation if and only if it admits a small simplicial presentation (the main point is that even for a simplicial category $\mathbf{C}$, $s\,\mathrm{PSh}(\mathbf{C})_S$ is a combinatorial model category, so Dugger's theorem applies to show that $s\,\mathrm{PSh}(\mathbf{C})_S$ is presentable.)

5.5. **Left-exact localization.** Let $\mathbf{M}$ be a model category, let S be a set of maps for which we can define a localization model category structure $\mathbf{M}_S$, and let $\bar{S}$ denote the saturation of S. We say that localization with respect to S is **left-exact** if the left-derived functor $\underline{L}a$ of $a\colon \mathbf{M} \to \mathbf{M}_S$ preserves finite homotopy limits. Since such a localization always preserves the terminal object, this condition amounts to the requirement that $\underline{L}a$ preserve homotopy pullbacks. Equivalently, if we write $L = \underline{R}i \circ \underline{L}a$ for the localization functor, localization is left-exact if and only if L preserves finite homotopy limits.

Proposition 5.6. *Let $\mathbf{M}$ be a model category, and $\mathbf{M}_S$ be a localization model category structure on $\mathbf{M}$ with respect to a set S. The localization is left-exact if and only if the saturation $\bar{S}$ is closed under homotopy base change.*

Proof. Let $L\colon \mathbf{M} \to \mathbf{M}$ denote the localization functor, and let $\eta_X\colon X \to LX$ denote the coaugmentation. We know that $\bar{S} = \{\, f \mid L(f) \text{ is a weak equivalence}\,\}$ so that $\eta_X \in \bar{S}$ for all X. Furthermore, if L is left-exact, it is clear that $\bar{S}$ is closed under homotopy base change.

Suppose now that $\bar{S}$ is closed under homotopy base change. Consider a homotopy pullback square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & B \end{tikzcd}$$

in $\mathbf{M}$. To show that L carries this to a homotopy pullback, it will be enough to show that the evident map $P \to LX \times^h_{LB} LY$ is in $\bar{S}$. Let $X' = B \times^h_{LB} LX$ and $Y' = B \times^h_{LB} LY$, let $P' = X' \times^h_B Y'$, and consider the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & P' \times^h_{Y'} Y \arrow[r] \arrow[d] & Y \arrow[d] \arrow[dr, "\bar{S}"] & \\ P' \times^h_{X'} X \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & LY \arrow[dd] \\ X \arrow[r] \arrow[dr, "\bar{S}"'] & X' \arrow[r] \arrow[d] & B \arrow[dr, "\bar{S}"] & \\ & LX \arrow[rr] & & LB \end{tikzcd}$$

in which every quadrilateral is a homotopy pullback square. The labelled maps are in $\bar{S}$. Using the two-of-three property of $\bar{S}$ and the fact that $\bar{S}$ is closed under homotopy base change, we see that $P \to P' \approx LX \times^h_{LB} LY$ is in $\bar{S}$, as desired. $\square$

6. Model topos

6.1. Definition of a model topos. A **model site** is defined to be a pair $(\mathbf{C}, S)$, consisting of a small simplicial category and a set of maps S in $s\,\mathrm{PSh}(\mathbf{C})$, such that the left adjoint $a\colon s\,\mathrm{PSh}(\mathbf{C}) \to s\,\mathrm{PSh}(\mathbf{C})_S$ is **left exact**; that is, localization with respect to S preserves homotopy pullbacks.

A **model topos** is a model category which is Quillen equivalent to $s\,\mathrm{PSh}(\mathbf{C})_S$ for some model site.

Example 6.2. Topological spaces, with its usual model category structure, is model topos; of course, so are simplicial sets. The simplicial presheaf categories $s\,\mathrm{PSh}(\mathbf{C})$ are clearly model toposes. We will soon see that if $\mathbf{E}$ is a model topos, then so are its slice categories $\mathbf{E}/X$.

Example 6.3. The Joyal-Jardine model category of simplicial (pre-)sheaves on a Grothendieck site is a model topos; see [Jar87] and [Jar96].

Remark 6.4. Note that in the definition of model topos we allow $\mathbf{C}$ to be a simplicial category, and not merely a discrete category. This is necessary to ensure that the Giraud theorem (6.9) below holds; see (11.9) for an explanation.

The definition of model topos we have given here is basically one given in [TV05].

6.5. **Descent.** Let $\mathbf{M}$ be a model category, and $\mathbf{I}$ a small category. A natural transformation $f\colon X \to Y$ of functors $X, Y\colon \mathbf{I} \to \mathbf{M}$ will be called **equifibered** if for each map $i \to i' \in \mathbf{I}$, the induced square

$$\begin{array}{ccc} X(i) & \longrightarrow & X(i') \\ \downarrow & & \downarrow \\ Y(i) & \longrightarrow & Y(i') \end{array}$$

is a homotopy pullback.

We will say that $\mathbf{M}$ has **descent** (or sometimes **homotopical patching**, or just **patching**), if the following conditions (P1) and (P2) hold:

(P1) Let $\mathbf{I}$ be a small category, $X\colon \mathbf{I} \to \mathbf{M}$ a functor, and $\bar{X} = \operatorname{hocolim}_{\mathbf{I}} X$. Let $f\colon \bar{Y} \to \bar{X}$ be a map in $\mathbf{M}$. Form a functor $Y\colon \mathbf{I} \to \mathbf{M}$ by $Y(i) \overset{\text{def}}{=} X(i) \times^h_{\bar{X}} \bar{Y}$ for $i \in \mathbf{I}$. Then the evident map $\operatorname{hocolim}_{\mathbf{I}} Y \to \bar{Y}$ is a weak equivalence.

(P2) Let $\mathbf{I}$ be a small category, $f\colon Y \to X$ an equifibered natural transformation. Let $\bar{f}\colon \bar{Y} \to \bar{X}$ be the induced map between homotopy colimits $\bar{Y} = \operatorname{hocolim}_{\mathbf{I}} Y$ and $\bar{X} = \operatorname{hocolim}_{\mathbf{I}} X$. Then for each object $i \in \mathbf{I}$ the natural map $Y(i) \to X(i) \times^h_{\bar{X}} \bar{Y}$ is a weak equivalence.

Roughly, "descent" says that for any functor $X\colon \mathbf{I} \to \mathbf{M}$ with $\bar{X} = \operatorname{hocolim}_{\mathbf{I}} X$, there is an equivalence between the homotopy theory of $\mathbf{M}/\bar{X}$, and the homotopically full subcategory of equifibered objects in $\mathbf{M}^{\mathbf{I}}/X$.

It is a standard fact of homotopy theory that the model category of spaces has descent. To prove this, it is enough to check the axioms for two shapes of diagrams: (1) arbitrary coproducts, and (2) pushouts. Case (1) is easy, while case (2) is well known; I think it was first proved explicitly in [Pup74].

From this, it follows that every simplicial presheaf category has descent. It is clear that any left exact localization of a model category with descent also has descent. Thus we have proved

Proposition 6.6. *A model topos* $\mathbf{E}$ *has descent.*

We will need the following.

Proposition 6.7. *Let* $\mathbf{E}$ *be a model category with descent. If*

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & B \end{array}$$

is a homotopy pullback square of natural transformations $A, X, Y, B\colon \mathbf{J} \to \mathbf{E}$ *such that* f *and* g *are equfibered, then the evident map*

$$\operatorname{hocolim} A \to (\operatorname{hocolim} X) \times^h_{\operatorname{hocolim} B} (\operatorname{hocolim} Y)$$

is a weak equivalence.

Proof. Write $\bar{B}$, $\bar{X}$, $\bar{Y}$, and $\bar{A}$ for the homotopy colimits of B, X, Y, and A respectively. Set $X'(J) = B(J) \times^h_{\bar{B}} \bar{X}$ and $Y'(J) = B(J) \times^h_{\bar{B}} \bar{Y}$, and let

$$A'(J) = B(J) \times^h_{\bar{B}} (\bar{X} \times^h_{\bar{B}} \bar{Y}) \approx X'(J) \times^h_{B(J)} Y'(J).$$

Then (P1) implies that

$$\operatorname{hocolim}_{\mathbf{J}} X' \approx \bar{X}, \qquad \operatorname{colim}_{\mathbf{J}} Y' \approx \bar{Y}, \qquad \operatorname{hocolim}_{\mathbf{J}} A' \approx \bar{X} \times_{\bar{B}} \bar{Y}.$$

The proposition will be proved once we show that each map $A(J) \to A'(J)$ is a weak equivalence.

Since f and g are equifibered, (P2) tells us that each of the maps $X(J) \to X'(J)$ and $Y(J) \to Y'(J)$ are weak equivalences. Therefore, the composite map

$$A(J) \approx X(J) \times_{B(J)} Y(J) \to X(J) \times_{B(J)} Y'(J) \to X'(J) \times_{B(J)} Y'(J) \approx A'(J)$$

is a weak equivalence, as desired. □

6.8. **Giraud theorem.**

Theorem 6.9. *A model category* $\mathbf{E}$ *is a model topos if and only if*

(a) $\mathbf{E}$ *admits a small presentation, and*
(b) $\mathbf{E}$ *has descent.*

This has as an easy consequence

Corollary 6.10. *If* $\mathbf{E}$ *is a model topos, so are the slice categories* $\mathbf{E}/X$.

Proof sketch. If $\mathbf{E}$ is a model topos, it certainly admits a small presentation. As noted above, every model topos has patching.

Now suppose that $\mathbf{E}$ is a model category satisfying (a) and (b). Since $\mathbf{E}$ has a small presentation, we can replace it with a Quillen equivalent one which is a simplicial model category. The first step is to choose a set C of fibrant-and-cofibrant objects of $\mathbf{E}$ with the following property: if we let $\mathbf{C}$ be the full simplicial subcategory of the simplicial enrichment of $\mathbf{E}$ with object set C, then the induced adjoint pair $s\,\mathrm{PSh}(\mathbf{C}) \rightleftarrows \mathbf{E}$ identifies $\mathbf{E}$ with a localization of $s\,\mathrm{PSh}(\mathbf{C})$, i.e., there is a set of maps S is $s\,\mathrm{PSh}(\mathbf{C})$ so that $s\,\mathrm{PSh}(\mathbf{C})_S \rightleftarrows \mathbf{E}$ is a Quillen equivalence. This can be proved by ideas similar to those used by Dugger in [Dug01a] (if it is not proved there already).

Given such a simplicial presentation, the simplicial yoneda embedding $y\colon \mathbf{C} \to s\,\mathrm{PSh}(\mathbf{C})_S$ factors through $\mathbf{E}$, and thus each $y(C)$ is an S-local object of $s\,\mathrm{PSh}(\mathbf{C})$. Thus, without loss of generality, we may assume that $\mathbf{E} = s\,\mathrm{PSh}(\mathbf{C})_S$, and that $y\colon \mathbf{C} \to s\,\mathrm{PSh}(\mathbf{C})_S$ factors through the full subcategory of S-local objects.

Write $\mathbf{P} = s\,\mathrm{PSh}(\mathbf{C})$. I'll write $a\colon \mathbf{P} \rightleftarrows \mathbf{E} : i$ for the adjoint pair, and I will not usually bother to distinguish between a and i and their total derived functors. A "pseudosheaf" will be an S-local object of $\mathbf{P}$. Since every object of $\mathbf{P}$ is a homotopy colimit of a diagram of representable presheaves, we have that every object of $\mathbf{P}$ is a homotopy colimit of pseudosheaves.

I need to show:

(*) For each homotopy pullback square $\begin{tikzcd} P \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$ in $\mathbf{P}$, the induced square $\begin{tikzcd} aP \arrow[r] \arrow[d] & aY \arrow[d] \\ aX \arrow[r] & aY \end{tikzcd}$ is a homotopy pullback in $\mathbf{E}$.

I'll prove this in several steps.

(a) *Property (*) holds if Y, B, and X are pseudosheaves.* The homotopy limit in $\mathbf{P}$ of a diagram a pseudosheaves is a pseudosheaf, and this computes the homotopy limit in $\mathbf{E}$.

(b) *Property (*) holds if Y and B are pseudosheaves.* Every $X \in \mathbf{P}$ is a homotopy colimit of pseudosheaves. Thus, suppose $X \approx \operatorname{hocolim}_{\mathbf{J}} U$ for some functor $U : \mathbf{J} \to s\,\mathrm{PSh}(\mathbf{C})$ from a small category $\mathbf{J}$, such that each object $U(j)$ is a pseudosheaf. For each $j \in \mathbf{J}$ we have a sequence of homotopy pullback squares

$$\begin{tikzcd} U(j) \times_B Y \arrow[r] \arrow[d] & X \times_B Y \arrow[r] \arrow[d] & Y \arrow[d] \\ U(j) \arrow[r] & X \arrow[r] & B \end{tikzcd}$$

By descent in $\mathbf{P}$, $\operatorname{hocolim}_J U \times_B Y \approx X \times_B Y$. Since $U(j)$ is a pseudo-sheaf, part (a) shows that $aU(j) \times^h_{aB} aY \approx a(U(j) \times^h_B Y)$ for each $j \in \mathbf{J}$. Now descent in $\mathbf{E}$ implies that $\operatorname{hocolim}_{\mathbf{J}}(aU \times^h_{aB} aY) \approx (\operatorname{hocolim}_{\mathbf{J}} aU) \times^h_{aB} aY$ in $\mathbf{E}$; hence, $a(X \times^h_B Y) \approx \operatorname{hocolim}_{\mathbf{J}} a(U \times^h_B Y) \approx \operatorname{hocolim}_{\mathbf{J}}(aU \times^h_{aB} aY) \approx aX \times^h_{aB} aY$, as desired.

(c) *Property (*) holds if B is a pseudo-sheaf.* This is proved exactly as in (b), except that we can drop the hypothesis that Y is a pseudo-sheaf by making use of (b).

(d) *Property (*) holds for general X, Y, and B.* Write $B = \operatorname{hocolim}_{\mathbf{J}} W$, where $W : \mathbf{J} \to \mathbf{P}$ lands in pseudosheaves. Let $U(j) = W(j) \times^h_B X$ and $V(j) = W(j) \times^h_B Y$. Then descent in $\mathbf{P}$ implies that $X \approx \operatorname{hocolim}_{\mathbf{J}} U$ and $Y \approx \operatorname{colim}_{\mathbf{J}} V$, and that $\operatorname{hocolim}_{\mathbf{J}} U \times^h_W V \approx X \times^h_B Y$. Since $U \to W$ and $V \to W$ are obtained by homotopy pullback from $X \to B$ and $Y \to B$ respectively, these are both equifibered natural transformations. Using (c), we see that both $aU \to aW$ and $aV \to aW$ are equifibered transformations; therefore (6.7) in $\mathbf{E}$ gives that

$$a(X \times^h_B Y) \approx a(\operatorname{hocolim}_{\mathbf{J}} U \times^h_W V) \approx \operatorname{hocolim} aU \times^h_{aW} aV \to aX \times^h_{aB} aY$$

is a weak equivalence, as desired.

□

Both Toën-Vezzosi [TV05] and Lurie [Lura] give versions of a "Giraud theorem" in which descent is replaced with the following three statements:

(A) Formation of homotopy colimits commutes with homotopy pullback.
(B) Homotopy coproducts are disjoint.
(C) Segal groupoid objects are effective.

These three statements are actually equivalent to descent; I'll explain why descent implies (A), (B), and (C); the converse is proved in [Lurb].

Property (A) is really just a restatement of (P1).

Property (B) says that if $\{X_\alpha\}_{\alpha\in I}$ is a collection of objects of $\mathbf{E}$, then for each $\beta \neq \gamma$ in I the square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X_\gamma \arrow[d] \\ X_\beta \arrow[r] & \coprod^h_\alpha X_\alpha \end{tikzcd}$$

is a homotopy pullback; 0 denotes the initial object. To prove this using descent, let

$$Y_\alpha = \begin{cases} 0 & \text{if } \alpha \neq \gamma, \\ X_\gamma & \text{if } \alpha = \gamma. \end{cases}$$

Then the evident natural transformation $Y \to X$ is (trivially) equifibered, and thus for each $\beta \in I$ the square

$$\begin{tikzcd} Y_\beta \arrow[r] \arrow[d] & \coprod^h_\alpha Y_\alpha \arrow[d] \\ X_\beta \arrow[r] & \coprod^h_\alpha X_\alpha \end{tikzcd}$$

is a homotopy pullback. We have $\coprod^h_\alpha Y_\alpha \approx X_\gamma$, and the result follows.

Let $X\colon \Delta^{\mathrm{op}} \to \mathbf{E}$ be a simplicial object in $\mathbf{E}$. Say that X is a **Segal category** in $\mathbf{E}$ if for each $n > 1$ the map

$$f\colon X_n \to \mathrm{holim}(X_1 \xrightarrow{d_0} X_0 \xleftarrow{d_1} \dots \xrightarrow{d_0} X_0 \xleftarrow{d_1} X_1),$$

corresponding to the collection $f_i\colon [1] \to [n]$ of maps sending $(0 < 1)$ to $(i-1 < i)$, is a weak equivalence. That is, X "looks like" the nerve of a category.

Say that a Segal category object X is a **Segal groupoid** if in addition, the map

$$X_2 \xrightarrow{(d_0,d_1)} \mathrm{holim}(X_1 \xrightarrow{d_0} X_0 \xleftarrow{d_1} X_1)$$

is a weak equivalence. That is, X "looks like" the nerve of a groupoid.

Let $Q = \mathrm{hocolim}_{\Delta^{\mathrm{op}}} X$. There is an augmentation $\epsilon\colon X_0 \to Q$. We say that a Segal groupoid is **effective** if the map

$$X_1 \xrightarrow{(d_1,d_0)} \mathrm{holim}(X_0 \xrightarrow{\epsilon} Q \xleftarrow{\epsilon} X_0)$$

is a weak equivalence; this condition implies that $X_n \approx X_0 \times^h_Q \cdots \times^h_Q X_0$ for all $n \geq 0$. I'll sketch the proof that descent implies that all Segal groupoids are effective.

Consider the functor $j\colon \Delta \to \Delta$ defined by $j([n]) = [n+1]$, such that a map $f\colon [k] \to [n]$ is sent to the function $j(f)\colon [k+1] \to [n+1]$ defined by $j(f)(i) = f(i)$ if $0 \leq i \leq k$, and $j(f)(k+1) = n+1$. The functions $[k] \to [k+1]$ defined by $i \mapsto i$ define a natural transformation $1 \to j$. If $X\colon \Delta^{\mathrm{op}} \to \mathbf{E}$ is a functor, we obtain a map $\eta\colon X \circ j \to X$ of simplicial objects. It is not hard to show that $\eta\colon X \circ j \to X$ is equifibered if and only if X is a Segal groupoid. Furthermore, it is a standard fact that $\operatorname{hocolim}(X \circ j) \approx X_0$ (the augmented simplicial object $X \circ j$ has a contracting homotopy to X_0). Thus, if X is a Segal groupoid, descent implies that

$$\begin{tikzcd}
(X \circ j)_0 \arrow[r] \arrow[d, "\eta"'] & \operatorname{hocolim}_{\Delta^{\mathrm{op}}}(X \circ j) \arrow[d, "\eta"] \\
X_0 \arrow[r] & \operatorname{hocolim}_{\Delta^{\mathrm{op}}} X
\end{tikzcd}$$

is a homotopy pullback. Unwinding the definitons gives that $X_1 \xrightarrow{(d_1,d_0)} \operatorname{holim}(X_0 \xrightarrow{\epsilon} Q \xleftarrow{\epsilon} X_0)$ is a weak equivalence, so that X is an effective Segal groupoid, as desired.

6.11. **Morphisms of model toposes.** Let us define a **geometric morphism** $f\colon \mathbf{E} \to \mathbf{F}$ of model toposes to be a functor $f^*\colon \mathbf{F} \to \mathbf{E}$, which has a total left derived functor $\underline{L}f^*$ which preserves homtopy colimits and finite homotopy limits.

Example 6.12. If $\mathbf{E}$ has a small simplicial presentation by $s\,\mathrm{PSh}(\mathbf{C})_S \rightleftarrows \mathbf{E}$, the "sheafification" on $s\,\mathrm{PSh}(\mathbf{C})$ produces a morphism $\mathbf{E} \to s\,\mathrm{PSh}(\mathbf{C})$ of toposes.

Example 6.13. If $\mathbf{E}$ is a model topos and $f\colon X \to Y$ a map in $\mathbf{E}$, the functor $f^*\colon \mathbf{E}/Y \to \mathbf{E}/X$ defined by pullback along f produces a geometric morphism $\mathbf{E}/X \to \mathbf{E}/Y$; clearly f^* preserves homotopy limits, and f^* preserves homotopy colimits exactly by the descent condition (P1).

If $a\colon s\,\mathrm{PSh}(\mathbf{C})_S \rightleftarrows \mathbf{E} : i$ is a small presentation for $\mathbf{E}$, and $f^*\colon \mathbf{E} \to \mathbf{F}$ is the functor associated to a geometric morphism, Dugger's universal model category formalism [Dug01b] allows one to contstruct an adjoint Quillen pair $s\,\mathrm{PSh}(\mathbf{C})_S \rightleftarrows \mathbf{F}$ with the property that the left adjoint of this pair is connected by a chain of weak equivalences to the composite $f^* \circ a$. If we abuse notation and write $f^*\colon s\,\mathrm{PSh}(\mathbf{C})_S \to \mathbf{F}$ for this composite, then we see that there is a right adjoint $f_*\colon \mathbf{F} \to s\,\mathrm{PSh}(\mathbf{C})$. Thus, up to Quillen equivalence, every geometric morphism $\mathbf{E} \to \mathbf{F}$ of model toposes is part of a Quillen pair $f_*\colon \mathbf{F} \rightleftarrows \mathbf{E} : f^*$.

6.14. **Internal function objects.** Let Y be an object of a model topos $\mathbf{E}$, and let $f\colon \mathbf{E}/Y \to \mathbf{E}$ denote the geometric morphism associated to $f\colon Y \to 1 \in \mathbf{E}$, as in (6.13). We write $\underline{\mathrm{sect}}_Y(B \to Y) \overset{\mathrm{def}}{=} f_*(B \to Y)$, and call it the **object of sections** over Y.

Let $\underline{\mathrm{hom}}(Y, Z) \overset{\mathrm{def}}{=} \underline{\mathrm{sect}}_Y(Y \times Z \to Y)$. We call $\underline{\mathrm{hom}}(Y, Z)$ the **internal function object**.

7. Truncation

7.1. **k-truncated objects and maps.** Let X be a space. Say that X is **k-truncated** ($k \geq -1$), if for every choice of basepoint $x \in X$, we have $\pi_q(X, x) = *$ for all $q > k$. We will also say that X is **-2-truncated** iff it is contractible.

Thus, X is -1-truncated iff X is either empty or contractible; X is 0-truncated if it is weakly equivalent to a discrete space; X is 1-truncated if it is weakly equivalent to a disjoint union of $K(\pi, 1)$s; and so forth.

Let $\mathbf{M}$ be a model category, and $X \in \mathbf{M}$ and object. We say that X is **k-truncated** if for every $Y \in \mathbf{M}$, the derived mapping space $\operatorname{map}(Y, X)$ is k-truncated. (If $\mathbf{M}$ is spaces, this agrees with the original notion.)

We say a morphism $f\colon Y \to X$ in $\mathbf{M}$ is k-truncated if it is k-truncated as an object in $\mathbf{M}/X$.

Example 7.2. Let $f\colon X \to Y$ be a map of spaces. Then f is k-truncated if for every $y \in Y$, the homotopy fiber of f over y is a k-truncated space.

We also have that f is k-truncated iff for each $x \in X$, $\pi_q(X, x) \to \pi_q(Y, y)$ is an isomorphism for $q > k+1$, and a monomorphism for $q = k+1$. Thus, f is -1-truncated iff it induces a weak equivalence between X and a union of some of the path components of Y; f is 0-truncated if it is weakly equivalent to a covering map.

Given a map $f\colon X \to Y$ in $\mathbf{E}$, we write $\Delta(f)\colon X \to X \times^h_Y X$ for the evident diagonal map. A useful criterion is a map to be truncated is the following.

Proposition 7.3. *Let $k > -2$. A map $f\colon X \to Y$ in $\mathbf{M}$ is k-truncated iff $\Delta(f)\colon X \to X \times^h_Y X$ is $(k-1)$-truncated. In particular, f is -1-truncated iff the square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is a homotopy pullback.

Remark 7.4. Sometimes we call a morphism a **homotopy monomorphism** if it is -1-truncated; this is the precise analogue in homotopy theory of the notion of a monomorphism in category theory.

There is an analogous notion of **homotopy epimorphism**, obtained by dualizing the above notion, of which we will have no need. It is an amusing exercise to classify the homotopy epimorphisms in spaces.

Proposition 7.5. *If $\mathbf{M}$ is a model category which admits a small presentation, there is for each $k \geq -2$ a truncation functor $\tau_k\colon \mathbf{M} \to \mathbf{M}$, with a natural transformation $\eta\colon 1 \to \tau_k$, such that $\tau_k X$ is k-truncated for each $X \in \mathbf{M}$, and such that for each k-truncated Y in $\mathbf{M}$, the map $\operatorname{map}(\tau_k X, Y) \to \operatorname{map}(X, Y)$ induced by η is a weak equivalence. Furthermore, $f\colon X \to X'$ is such that $\tau_k f$ is a weak equivalence if and only if $\operatorname{map}(X', Y) \to \operatorname{map}(X, Y)$ is a weak equivalence for all k-truncated Y.*

Proof. Without loss of generality, we can set $\mathbf{M} = s\,\mathrm{PSh}(\mathbf{C})_S$. Let $T = \{g_C\}$ be the set of maps in $\mathbf{M}$, where for an object $C \in \mathbf{C}$ the map $g_C\colon S^{k+1} \times y_C \to D^{k+2} \times y_C$ is the evident inclusion. It is easy to see that thet T-local objects of $\mathbf{M}$ are precisely the k-truncated objects. We can form a localization model category $\mathbf{M}_T = s\,\mathrm{PSh}(\mathbf{C})_{S \cup T}$; the localization functor we obtain this way is exactly τ_k. $\square$

We will sometimes write τ_k^X for the k-truncation functor on $\mathbf{M}/X$; this functor associates to each morphism $f\colon Y \to X$ a factorization $Y \to \tau_k^X f \xrightarrow{g} X$ of f in which g is k-truncated.

Proposition 7.6. *If $f\colon \mathbf{F} \to \mathbf{E}$ is a geometric morphism of model toposes, then $f^*\colon \mathbf{E} \to \mathbf{F}$ commutes with truncation, in the sense that there is a weak equivalence $f^*\tau_k^{\mathbf{E}} \approx \tau_k^{\mathbf{F}} f^*$.*

In particular, given a homotopy pullback square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "a"'] & B \arrow[d, "b"] \\ X \arrow[r] & Y \end{tikzcd}$$

in $\mathbf{E}$, there is a homotopy pullback square

$$\begin{tikzcd} \tau_k^X a \arrow[r] \arrow[d] & \tau_k^Y b \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Proof. It is clear that since f^* is left exact, it takes k-truncated objects to k-truncated objects, by (7.3). Thus in the square

$$\begin{tikzcd} f^*X \arrow[r, "f^*\eta"] \arrow[d, "\eta"'] & f^*\tau_k^{\mathbf{E}} X \arrow[d, "\sim"] \\ \tau_k^{\mathbf{F}} f^* X \arrow[r] & \tau_k^{\mathbf{F}} f^*\tau_k^{\mathbf{E}} X \end{tikzcd}$$

the right hand side is a weak equivalence, and thus we need only show that the bottom side is a weak equivalence, i.e., that for all k-truncated Y in $\mathbf{F}$, $\operatorname{map}_{\mathbf{F}}(f^*\tau_k^{\mathbf{E}} X, Y) \to \operatorname{map}_{\mathbf{F}}(f^*X, Y)$ is a weak equivalence. But f^* is left adjoint to a $f_*\colon \mathbf{F} \to \mathbf{E}$, and so this map is equivalent to $\operatorname{map}_{\mathbf{E}}(\tau_k^{\mathbf{E}} X, f_*Y) \to \operatorname{map}_{\mathbf{E}}(X, f_*Y)$. Since f_* preserves homotopy limits, it also preserves k-truncated objects, and the result follows. □

7.7. **Effective epimorphisms and the Cech complex.** In a model topos, there is a direct construction of -1-truncation, by means of a "Cech complex".

Proposition 7.8. *Let $X \in \mathbf{E}$ be an object in a model topos. Let U be the simplicial object in $\mathbf{E}$ defined by $U_n = X^{n+1}$. Let $Y = \operatorname{hocolim}_{\Delta^{\mathrm{op}}} U$. Then the map $X = U_0 \to Y$ is weakly equivalent to the -1-truncation map $X \to \tau_{-1} X$.*

Proof. First, I need to show that Y is -1-truncated; that is, $Y \to Y \times Y$ is a weak equivalence. It is equivalent to show that the projection $\mathrm{pr}_2\colon Y \times Y \to Y$ is a weak equivalence. We have $Y \times Y \approx Y \times (\operatorname{hocolim} U) \approx \operatorname{hocolim}(Y \times U)$. Thus, we need to show that each $\mathrm{pr}_2\colon Y \times U_n \to U_n$ is a weak equivalence; since $\mathrm{pr}_2\colon Y \times U \to U$ is equifibered, it is enough to show that $\mathrm{pr}_2\colon Y \times X = Y \times U_0 \to U_0 = X$ is a weak equivalence.

We have $Y \times X \approx (\operatorname{hocolim} U) \times X \approx \operatorname{hocolim}(U \times X)$. The projection map $\mathrm{pr}_2\colon U_0 \times X = X \times X \to X$ factors through $U_0 \times X \to Y \times X$, and is an augmentation for the simplicial object $U \times X$ which admits a contracting homotopy. Thus $\operatorname{hocolim}(U \times X) \to Y \times X \to X$ are weak equivalences, as desired.

Next, I need to show that if Z is -1-truncated, then $\operatorname{map}(Y, Z) \to \operatorname{map}(X, Z)$ is a weak equivalence. But

$$\operatorname{map}(Y, Z) \approx \operatorname{map}(\operatorname{hocolim} U, Z) \approx \operatorname{holim} \operatorname{map}(U, Z);$$

since Z is -1-truncated, each space $\mathrm{map}(U_n, Z)$ is either empty or contractible. If $\mathrm{map}(X, Z)$ is non-empty, the result is clear. If $\mathrm{map}(X, Z)$ is non-empty, then each $\mathrm{map}(U_n, Z)$ is non-empty and thus contractible, and the homotopy limit is contractible. $\square$

We say that a map $f\colon X \to Y$ in a model topos is an **effective epi** if $\tau_{-1}^{Y}(f) \approx Y$, or equivalently, if the natural map $\mathrm{hocolim}_{\Delta^{\mathrm{op}}} X_Y^{\bullet+1} \to Y$ is a weak equivalence. (Lurie calls such maps "surjections".) Thus, every effective epi with codomain Y is associated to a Cech complex which resolves Y.

Here is a nifty criterion for a map to be an effective epi, given by Lurie. Given $X \in \mathbf{E}$, let $\mathrm{Sub}(X)$ denote set of weak equivalence classes of -1-truncated objects in $\mathbf{E}/X$. (This is a set; there is a set-worth of weak equivalence classes of -1-truncated objects of $s\,\mathrm{PSh}(\mathbf{C})$.) The set $\mathrm{Sub}(X)$ naturally has the structure of a poset; say $A \leq B$ if $\mathrm{map}_{\mathbf{E}/X}(A, B)$ is non-empty. In fact, $\mathrm{Sub}(X)$ is a lattice; the meet of A and B is the homotopy pullback $A \times_X^h B$, viewed as an object over X.

Lemma 7.9. *A map $f\colon X \to Y$ in a model topos $\mathbf{E}$ is an effective epi iff the map $f^*\colon \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ defined by homotopy pullback is injective.*

Proof. The function f^* preserves meets, so it is injective iff $f^*A = f^*B$ implies $A = B$ for all $A \leq B \in \mathrm{Sub}(Y)$.

If f is effective epi, then $Y \approx \mathrm{hocolim}_{\Delta^{\mathrm{op}}} X_Y^{\bullet+1}$. Furthermore, if $A \to Y$ is -1-truncated, then $f^*A \approx \mathrm{hocolim}_{\Delta^{\mathrm{op}}}(X_Y^{\bullet+1} \times_Y^h A)$. In particular, if $g\colon A \to B$ is a map of -1-truncated objects of $\mathbf{E}/Y$ such that the pullback over X is a weak equivalence, then g is a weak equivalence.

Conversely, suppose f^* is injective. There is a map $g\colon \tau_{-1}^{Y}(X) \to Y$ of -1-truncated objects of $\mathbf{E}/Y$. The square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \tau_{-1}^{Y}(X) \arrow[r, "g"'] & Y \end{tikzcd}$$

is a homotopy pullback, and thus since f^* is injective, we conclude that g was a weak equivalence. $\square$

Proposition 7.10.

(a) *The homotopy pullback of an effective epi is effective epi.*
(b) *If f and g are effective epis and gf is defined, then gf is effective epi.*
(c) *If f and g are maps such that gf is defined, and if gf is effective epi, then g is effective epi.*
(d) *If $f\colon X \to Y$ is effective epi, $g\colon A \to Y$ a map, and $h\colon X \times_Y^h A \to X$ the homotopy pullback of g along f, then h is a weak equivalence/k-truncated/effective epi if and only if g is.*

Proof. Part (a) is immediate from (7.6). Part (b) and (c) follow from (7.9). Part (d) is proved by an argument using the Cech complex of (7.8) and descent. $\square$

8. Connectivity

8.1. **k-connected objects and maps.** An object $X \in \mathbf{E}$ is said to be **k-connected** if $\tau_k X \to 1$ is a weak equivalence. A morphism $f: Y \to X \in \mathbf{E}$ is k-connected if it is k-connected as an object in $\mathbf{E}/X$, i.e., if $\tau_k^X(Y) \to X$ is a weak equivalence.

In particular, every map is -2-connected; a map is -1-connected if and only if it is an effective epimorphism.

Example 8.2. For a space, this notion of k-connectedness coincides with the usual one. A space X is -1-connected if and only if it is non-empty. A space X is k-connected ($k \geq -1$) if and only if it is non-empty, and $\pi_q(X, x) \approx *$ for all $x \in X$ and all $q \leq k$.

Under our formulation, a map $f: Y \to X$ of spaces is k-connected if and only if all of its homotopy fibers are k-connected. This differs by one from the usual topological usage, where a map $f: Y \to X$ is called k-connected if it can be modelled by a CW-pair (L, K) such that the complement $L - K$ is a union of open cells having dimensions greater than k. The topological notion of k-connectivity corresponds to what we are calling $(k-1)$-connectivity.

Proposition 8.3. *A map $f: Y \to X$ is k-connected if and only if for every k-truncated map $g: Z \to X$, the map on derived mapping spaces $\mathrm{map}_{\mathbf{E}/X}(X, Z) \to \mathrm{map}_{\mathbf{E}/X}(Y, Z)$ is a weak equivalence.*

Proof. Without loss of generality, assume $X \approx 1$. Then $\tau_k(Y) \approx 1$ if and only if $\mathrm{map}_{\mathbf{E}}(Y, Z) \approx 1$ for every k-truncated Z in $\mathbf{E}$, by the characterization of truncation. □

Proposition 8.4. *In a model topos, the class of k-connected maps is closed under homotopy base change and homotopy cobase change.*

Proof. The statement about homotopy base change follows from the fact that relative truncation is compatible with homotopy pullback (7.6). The statement about homotopy cobase change is easily derived from (8.3). □

Proposition 8.5. *For a morphism $f: X \to Y$ in a model topos $\mathbf{E}$, the k-truncation augmentation $\eta: X \to \tau_k^Y f$ is k-connected. In particular, the relative k-truncation construction produces a factorization of a map into a k-connected map followed by a k-truncated map.*

Proof. Without loss of generality, we may assume $Y \approx 1$, and so show that $\eta: X \to \tau_k X$ is k-connected.

Given a k-truncated map $g: Z \to \tau_k X$, we must show that $\mathrm{map}_{\mathbf{E}/\tau_k X}(\tau_k X, Z) \to \mathrm{map}_{\mathbf{E}/\tau_k X}(X, Z)$ is a weak equivalence. Consider the commutative square

$$\begin{array}{ccc} \mathrm{map}_{\mathbf{E}}(\tau_k X, Z) & \longrightarrow & \mathrm{map}_{\mathbf{E}}(X, Z) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \mathrm{map}_{\mathbf{E}}(\tau_k X, \tau_k X) & \longrightarrow & \mathrm{map}_{\mathbf{E}}(X, \tau_k X) \end{array}$$

induced by the maps η and g. Since g is a k-truncated map and $\tau_k X$ is a k-truncated object, the composite $Z \to \tau_k X \to 1$ is k-truncated. Thus the horizontal maps in the square are weak equivalences by (7.5). The space $\mathrm{map}_{\mathbf{E}/\tau_k X}(\tau_k X, Z)$ is weakly equivalent to the homotopy fiber of α over the point $\mathrm{id} \in \mathrm{map}_{\mathbf{E}}(\tau_k X, \tau_k X)$. Likewise, $\mathrm{map}_{\mathbf{E}/\tau_k X}(X, Z)$ is weakly equivalent to the homotopy fiber of β over the point $\eta \in \mathrm{map}_{\mathbf{E}}(X, \tau_k X)$. This gives the desired equivalence. $\square$

Lemma 8.6. *Let $f\colon Y \to X$ be an ℓ-truncated map in a model topos $\mathbf{E}$, and let $k > \ell$. Then*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] & \tau_k Y \arrow[d] \\ X \arrow[r] & \tau_k X \end{tikzcd}$$

is a homotopy pullback.

Proof. Identify $\mathbf{E}$ with a simplicial presentation $s\,\mathrm{PSh}(\mathbf{C})_S$. Consider the square

$$\begin{tikzcd} iY \arrow[r] \arrow[d, "i(f)"'] & \tau'_k iY \arrow[d] \\ iX \arrow[r] & \tau'_k iX \end{tikzcd}$$

where τ'_k denotes truncation in the presheaf category $s\,\mathrm{PSh}(\mathbf{C})$. The map $i(f)$ is an ℓ-truncated map of presheaves, and a straightforward argument using homotopy groups shows that the square is a homotopy pullback of presheaves. By (7.6), sheafification is compatible with truncation, and we get the desired result. $\square$

Proposition 8.7. *Let $f\colon Y \to X$ be a map in a model topos $\mathbf{E}$.*

(1) *If f is k-connected, then $\tau_k f$ is a weak equivalence.*
(2) *If $\tau_k f$ is a weak equivalence, then f is $(k-1)$-connected.*

Proof. To prove (1), note that if $Z \in \mathbf{E}$ is a k-truncated object, then $\mathrm{pr}_1\colon X \times Z \to X$ is a k-truncated map. The result follows easily using $\mathrm{map}_{\mathbf{E}}(Y, X) \approx \mathrm{map}_{\mathbf{E}/X}(Y, X \times Z)$.

To prove (2), we need to show that if $g\colon Z \to X$ is a $(k-1)$-truncated map, then $\alpha\colon \mathrm{map}_{\mathbf{E}/X}(X, Z) \to \mathrm{map}_{\mathbf{E}/X}(Y, Z)$ is a weak equivalence. Up to weak equivalence, we can identify the map α with

$$\beta\colon \mathrm{map}_{\mathbf{E}/\tau_k X}(X, \tau_k Z) \to \mathrm{map}_{\mathbf{E}/\tau_k X}(Y, \tau_k Z)$$

since $Z \approx X \times^h_{\tau_k X} \tau_k Z$ by (8.6). Since the objects $\tau_k X$ and $\tau_k Z$ are k-truncated, an argument similar to that in the proof of (8.5) shows that we can replace the domains with their truncations, i.e., the map β is equivalent to

$$\gamma\colon \mathrm{map}_{\mathbf{E}/\tau_k X}(\tau_k X, \tau_k Z) \to \mathrm{map}_{\mathbf{E}/\tau_k X}(\tau_k Y, \tau_k Z).$$

Since $\tau_k f$ is an equivalence, the map γ is a weak equivalence. $\square$

Proposition 8.8. *Let $f\colon X \to Y$, $g\colon Y \to Z$ be maps in a model topos $\mathbf{E}$.*

(1) *If f and g are k-connected, then so is gf.*
(2) *If f and gf are k-connected, then so is g.*
(3) *If g and gf are k-connected, then f is $(k-1)$-connected.*

Proof. Statements (1) and (2) are straightforward using the mapping space characterization of k-connected map. Statement (3) follows from (8.5) applied to the slice category $\mathbf{E}/Z$. $\square$

8.9. Connectivity theorems.

Proposition 8.10. *Let $f\colon A\to B$ be a map in a model topos $\mathbf{E}$. The following are equivalent.*

1. *For all k-truncated maps $g\colon X\to Y$ in $\mathbf{E}$, the induced map*
$$h\colon \mathrm{map}_{\mathbf{E}}(B,X)\to \mathrm{map}_{\mathbf{E}}(A,X)\times^h_{\mathrm{map}_{\mathbf{E}}(A,Y)}\mathrm{map}_{\mathbf{E}}(B,Y)$$
of spaces is a weak equivalence.
2. *f is k-connected.*

Proof. Let $g\colon X\to Y$ be a map. Note that for every map $j\colon B\to Y$, the diagram of spaces

$$\begin{tikzcd}
\mathrm{map}_{\mathbf{E}/Y}(B,X) \arrow[r] \arrow[d, "h_j"'] & \mathrm{map}_{\mathbf{E}}(B,X) \arrow[d, "h"] \\
\mathrm{map}_{\mathbf{E}/Y}(A,X) \arrow[r] \arrow[d] & \mathrm{map}_{\mathbf{E}}(A,X)\times^h_{\mathrm{map}_{\mathbf{E}(A,Y)}}\mathrm{map}_{\mathbf{E}}(B,Y) \arrow[d, "\pi_2"] \\
\{j\} \arrow[r] & \mathrm{map}_{\mathbf{E}}(B,Y)
\end{tikzcd}$$

consists of homotopy pullback squares.

Since h is a weak equivalence if and only if h_j is a weak equivalence for every $j\in\mathrm{map}_{\mathbf{E}}(B,Y)$, the result follows from (8.3). □

Next, we give an internal version of (8.3).

Proposition 8.11. *A map $f\colon Y\to X$ is k-connected if and only if for every k-truncated map $g\colon Z\to X$, the map on internal function objects $h\colon \underline{\mathrm{hom}}_{\mathbf{E}/X}(X,Z)\to\underline{\mathrm{hom}}_{\mathbf{E}/X}(Y,Z)$ is a weak equivalence in $\mathbf{E}/X$.*

Proof. Suppose f is k-connected. To show that h is a weak equivalence, it suffices to show that for all maps $a\colon A\to X$, the induced map
$$p=\mathrm{map}_{\mathbf{E}/X}(A,h)\colon\ \mathrm{map}_{\mathbf{E}/X}(A,\underline{\mathrm{hom}}_{\mathbf{E}/X}(X,Z))\to\mathrm{map}_{\mathbf{E}/X}(A,\underline{\mathrm{hom}}_{\mathbf{E}/X}(Y,Z)$$
is a weak equivalence. The map p is equivalent to the map
$$p'\colon\ \mathrm{map}_{\mathbf{E}/X}(A\times^h_X X,Z)\to\mathrm{map}_{\mathbf{E}/X}(A\times^h_X Y,Z).$$
Since connectedness is preserved under base change, $A\times^h_X X\to A\times^h_X Y$ is k-connected, and thus p' is an equivalence by (8.3), as desired.

Conversely, if h is a weak equivalence, apply $\mathrm{map}_{\mathbf{E}/X}(X,-)$ and use (8.3). □

Suppose given maps $f\colon A\to B$ and $g\colon X\to Y$ in $\mathbf{E}$. Let $u_{f,g}$ denote the map
$$\underline{\mathrm{hom}}(B,X)\to\underline{\mathrm{hom}}(A,X)\times^h_{\underline{\mathrm{hom}}(A,Y)}\underline{\mathrm{hom}}(B,Y).$$

Lemma 8.12. *The map $\Delta(u_{f,g})$ is weakly equivalent to $u_{f,\Delta(g)}$, where $\Delta(h)$ denotes the diagonal of the map h, as in §7.1.*

Proof. This is a straightforward exercise, using the fact that $\underline{\mathrm{hom}}$ commutes with homotopy limits in the second variable. □

Proposition 8.13. *Let $f\colon A \to B$ be a map in a model topos $\mathbf{E}$, and let $k \geq -2$. The following are equivalent.*

1. *For all $m \geq -2$, and all m-truncated maps $g\colon X \to Y$ in $\mathbf{E}$, the induced map $u_{f,g}$ in $\mathbf{E}$ is $(m-k-2)$-truncated if $m \geq k$, or is an equivalence if $m \leq k$.*
2. *For all k-truncated maps $g\colon X \to Y$ in $\mathbf{E}$, the induced map $u_{f,g}$ in $\mathbf{E}$ is a weak equivalence.*
3. *f is k-connected.*

Proof. It is clear that (2) is a special case of (1), using $m = k$.

Te show that (2) implies (3), apply (8.10) to $\mathrm{map}_{\mathbf{E}}(1, u_{f,g})$.

It remains to show that (3) implies (1). Suppose that f is k-connected, and consider an m-truncated map $g\colon X \to Y$. We must then show that $u_{f,g}$ is $(m-k-2)$-truncated.

Suppose that $m \leq k$. Then in particular g is k-truncated, and we must show that $u_{f,g}$ is a weak equivalence. It suffices to show that for each object T in $\mathbf{E}$ that $\mathrm{map}_{\mathbf{E}}(T, u_{f,g})$ is a weak equivalence. A straightforward argument shows that $\mathrm{map}_{\mathbf{E}}(T, u_{f,g})$ is equivalent to the map

$$h\colon \mathrm{map}_{\mathbf{E}}(B \times T, X) \to \mathrm{map}_{\mathbf{E}}(A \times T, X) \times^h_{\mathrm{map}_{\mathbf{E}}(A \times T, Y)} \mathrm{map}_{\mathbf{E}}(B \times T, Y).$$

Since connectivity is preserved by base change, $f \times T\colon A \times T \to B \times T$ is k-connected, and it follows that h is an equivalence by (8.10).

Now suppose g is m-truncated with $m > k$, and that we have already proved the claim for maps of lower truncation. Since g is m-truncated, $\Delta(g)\colon X \to X \times^h_Y X$ is $(m-1)$-truncated, by (7.3). By induction, the map $u_{f,\Delta(g)}$ is $(m-k-3)$-truncated. But by (8.12), $u_{f,\Delta(g)}$ is equivalent to $\Delta(u_{f,g})$. Thus we conclude that $u_{f,g}$ is $(m-k-2)$-truncated using (7.3) (which we can apply since $m-k-2 > -2$), as desired. ☐

Lemma 8.14. *Let $f\colon A \to B$, $g\colon A' \to B'$, $h\colon X \to Y$. Then $u_{f,u_{g,h}} \approx u_{f\Box g,h}$, where $f\Box g$ denotes the evident map*

$$\mathrm{hocolim}(A \times B' \xleftarrow{A \times g} A \times A' \xrightarrow{f \times A'} B \times A') \to B \times B'.$$

Proof. Straightforward. ☐

Proposition 8.15 (Join theorem)**.** *Let*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

be a homototpy pullback square in a model topos $\mathbf{E}$. Let $C \overset{\mathrm{def}}{=} \mathrm{hocolim}(X \leftarrow A \to Y)$, and let $h\colon C \to B$ denote the induced map. If f is m-connected and g is n-connected, then h is $(m+n+2)$-connected.

Proof. Without loss of generality, we may assume that B is the terminal object of $\mathbf{E}$, in which case $A \approx X \times Y$, and C may be thought of as the "join" of X and Y.

By (8.13), to show that h is $(m+n+2)$-connected, it suffices to show that $u_{h,q}$ is an equivalence for all $(m+n+2)$-truncated $q\colon U \to V$. Note that $h \approx f\Box g$, so that we have

$u_{h,q} \approx u_{f\Box g,q} \approx u_{f,u_{g,q}}$. We have that $u_{g,q}$ is $(m+n+2)-n-2 = m$-truncated, whence $u_{h,q} \approx u_{f,u_{g,q}}$ is a weak equivalence, using (8.13) twice. $\square$

Proposition 8.16 (Triad connectivity theorem)**.** *Let*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d] \\ X \arrow[r] & B \end{tikzcd}$$

be a homotopy pushout square in a model topos $\mathbf{E}$*. Let* $D \overset{\text{def}}{=} X \times^h_B Y$*, and let* $h\colon A \to D$ *denote the induced map. If* f *is* m*-connected and* g *is* n*-connected, then* h *is* $(m+n)$*-connected.*

Proof. Without loss of generality, assume that $\mathbf{E}$ is a model site $s\,\mathrm{PSh}(\mathbf{C})_S$. Choose a factorization $A \xrightarrow{f'} X' \xrightarrow{f''} X$ of f in $s\,\mathrm{PSh}(\mathbf{C})$ so that f' is m-connected and f'' is m-truncated as maps of simplicial presheaves. The sheafification functor $s\,\mathrm{PSh}(\mathbf{C}) \to \mathbf{E}$ preserves connectivity, from which we conclude that $f'' \in \bar{S}$. Similarly, choose a factorization $A \xrightarrow{g'} Y' \xrightarrow{g''} Y$ so that g' is n-conneced as a map of simplicial presheaves, and $g'' \in \bar{S}$. Then the homotopy pushout square

$$\begin{tikzcd} A \arrow[r, "g'"'] \arrow[d, "f'"'] & Y' \arrow[d] \\ X' \arrow[r] & B \end{tikzcd}$$

in $s\,\mathrm{PSh}(\mathbf{C})$ is equivalent to the original square in $\mathbf{E}$. Thus, it suffices to show that $A \to D'$ is $(m+n)$-connected as a map of simplicial presheaves, where $D' \approx X' \times^h_B Y'$, the homotopy limit being taken in simplicial presheaves. This in turn follows from the classical triad connectivity theorem in spaces. $\square$

9. The topos of discrete objects and homotopy groups

9.1. **The topos of discrete objects.** Given a model category $\mathbf{M}$, write $\tau_k\mathbf{M}$ for the full subcategory of k-truncated objects in $\mathbf{M}$. Write $\mathrm{Ho}(\tau_k\mathbf{M})$ for the full subcategory of $\mathrm{Ho}\,\mathbf{M}$ spanned by the k-truncated objects.

We call an object of $\mathbf{M}$ **(homotopy) discrete** if it is 0-truncated. If $X, Y \in \tau_0\mathbf{M}$, the derived mapping space $\mathrm{map}(X, Y)$ is weakly equivalent to the set (discrete space) $\mathrm{Ho}\,\mathbf{M}(X, Y)$. Thus, all the "homotopical" infomation about maps between discrete objects is already seen by the homotopy classes of maps. In particular, $\tau_0\mathbf{M}$ is "rigid", in the sense that any functor $\bar{F}\colon \mathbf{I} \to \mathrm{Ho}(\tau_0\mathbf{M})$ can be lifted (in a way which is unique up to weak equivalence) to a functor $F\colon \mathbf{I} \to \tau_0\mathbf{M} \subseteq \mathbf{M}$.

By abuse of notation, we write $\tau_0\mathbf{E}$ for $\mathrm{Ho}(\tau_0\mathbf{E})$.

Proposition 9.2. *If* $\mathbf{E}$ *is a model topos, then* $\tau_0\mathbf{E}$ *is a Grothendieck topos.*

Example 9.3. If $\mathbf{E} = \mathbf{S}/X$ for some space X, then $\tau_0\mathbf{E} \approx \mathrm{PSh}(\mathbf{C})$, where $\mathbf{C}$ is the fundamental groupoid of the space X. Note that this shows that distinct (i.e., non-Quillen equivalent) model toposes can have the same topos of discrete objects.

Proof. We use the Giraud theorem for Grothendieck toposes. Identifying $\mathbf{E}$ with a small presentation $s\,\mathrm{PSh}(\mathbf{C})_S$, the collection of objects $\{\tau_0(y_C)\}_{C\in\mathbf{C}}$ is a set of generators for $\tau_0\mathbf{E}$. Thus, it remains to show that $\tau_0\mathbf{E}$ has weak descent (in the non-homotopy theoretic sense).

First, I claim that $\tau_0\mathbf{E}$ is complete and cocomplete. Given a functor $F\colon \mathbf{I}\to\tau_0\mathbf{E}$, we can lift to $\tilde{F}\colon \mathbf{I}\to\mathbf{E}$. The homotopy limit $\operatorname{holim}_{\mathbf{I}}\tilde{F}$ is discrete, and one checks that it computes the limit of F in $\tau_0\mathbf{E}$. The homotopy colimit $\operatorname{hocolim}_{\mathbf{I}}\tilde{F}$ need not be discrete in general, but one can check that $\tau_0\operatorname{hocolim}_{\mathbf{I}}\tilde{F}$ computes the colimit of F in $\tau_0\mathbf{E}$. In the special case where $\mathbf{I}$ is a discrete category, so that the hocolim is a homotopy coproduct, then the homotopy coproduct of homotopy discrete objects is already homotopy discrete; this is related to the fact that homotopy coproducts are disjoint. Thus, $\coprod^h \tilde{F}(i)$ computes $\coprod F(i)$.

It remains to check the weak descent properties (P1a), (P1b), (P2a), (P2b). This is mostly straightforward given the above remarks and the descent properties of $\mathbf{E}$. I'll prove (P2b), which is more subtle that the others.

We start with an equifibered diagram

$$\begin{tikzcd} Y_1 \arrow[d] & Y_0 \arrow[l] \arrow[r] \arrow[d] & Y_2 \arrow[d] \\ X_1 & X_0 \arrow[l] \arrow[r] & X_2 \end{tikzcd}$$

of discrete objects in $\mathbf{E}$. Let $\bar{Y}\to\bar{X}$ be the map obtained by taking homotopy colimits of the rows. I claim that for each $i=0,1,2$ that the map $Y_i\to X_i\times^h_{\tau_0\bar{X}}\tau_0\bar{Y}$ is an effective epi in $\mathbf{E}$.

Let $P\overset{\text{def}}{=}\bar{X}\times^h_{\tau_0\bar{X}}\tau_0\bar{Y}$ be the homotopy pullback in $\mathbf{E}$. The projection $P\to\tau_0\bar{Y}$ is pulled back from the 0-connected map $\bar{X}\to\tau_0\bar{X}$, and thus is 0-connected by (7.6). Since $\bar{Y}\to\tau_0\bar{Y}$ is also 0-connected, we have that $\bar{Y}\to P$ is (-1)-connected by (8.8)(3). The map $Y_i\to X_i\times^h_{\tau_0\bar{X}}\tau_0\bar{Y}$ is obtained from $\bar{Y}\to P$ via base change along $X_i\to\bar{X}$, and thus is (-1)-connected, as desired.

Finally, note that if $A\to B$ is a (-1)-connected map between discrete objects in $\mathbf{E}$, then it is a regular epimorphism viewed as a map in $\tau_0\mathbf{E}$. $\square$

Remark 9.4. As we will soon see, every topos arises as the 0-truncation of a model topos.

Remark 9.5. In a similar way, we may consider the category $\tau_{-1}\mathbf{E}$ of (-1)-truncated objects in $\mathbf{E}$; this category is itself a full subcategory of $\tau_0\mathbf{E}$. An argument much as above shows that $\tau_{-1}\mathbf{E}$ is an example of a **locale** (in the sense of "pointless" topology), and one can show that every locale arises in this way.

9.6. **Homotopy groups.** Let X be an object in a model topos $\mathbf{E}$. For $k\geq 0$, the ***k*-th homotopy sheaf** of X is an object π_kX in the topos $\tau_0(\mathbf{E}/X)$, defined to be the 0-truncation of the map $X^{S^k}\to X$ in $\mathbf{E}/X$ corresponding to evaluation at some chosen point $x\in S^k$.

The projection map $S^k\to\{x\}$ determines a canonical section of π_kX. For $k\geq 1$, the coproduct map $S^k\to S^k\vee S^k$ determines a map $X^{S^k}\times^h_X X^{S^k}\to X^{S^k}$ over X, which makes π_kX into a group object in $\tau_0(\mathbf{E}/X)$; if $k\geq 2$, it is abelian.

Given a map $f\colon X \to Y$ in a model topos $\mathbf{E}$, the **k-th relative homotopy sheaf** of f is an object $\pi_k f$ in $\tau_0(\mathbf{E}/X)$, defined to be the the k-th homotopy sheaf of the object $(X \to Y) \in \mathbf{E}/Y$. This means that $\pi_k f$ is the 0-truncation of the map $X^{S^k} \times^h_{Y^{S^k}} Y \to X$.

Example 9.7. If X is a space, then the object $\pi_k X$ defined above coincides with the usual notion of k-th homotopy group, viewed as a functor of the fundamental groupoid of X.

If $f\colon X \to Y$ is a map of spaces, then $\pi_k f$ corresponds to the functor which associates to each $x \in X$ the set $\pi_k(\mathrm{hofib}_{f(x)}(f), x)$.

If $f\colon X \to Y$ is a map in $\mathbf{E}$, then for any morphism $g\colon U \to Y$ we have $\pi_k(f^*(g)) \approx f^*(\pi_k(g)) \in \tau_0(\mathbf{E}/X)$. More generally, the formation of homotopy sheaves is compatible with the left adjoint functor of any geometric morphism.

If $f\colon X \to Y$ is a map in $\mathbf{E}$, there is a sequence

$$\cdots f^*\pi_{k+1}(Y) \to \pi_k(f) \to \pi_k(X) \to f^*\pi_k(Y) \to \cdots$$

of pointed objects in $\tau_0(\mathbf{E}/X)$ which is exact in the usual sense.

Proposition 9.8. *Let $f\colon X \to Y$ be a morphism in $\mathbf{E}$. Let $k \geq -1$.*

1. *Suppose f is m-truncated for some $m \geq -1$. Then f is k-truncated if and only if $\pi_q(f) \approx *$ for all $q > k$.*
2. *The map f is k-connected if and only if it is (-1)-connected and $\pi_q(f) \approx *$ for all $q \leq k$.*

Proof. Let $X \in \mathbf{E}$ be an object in a model topos, and let $\eta\colon X \to \tau_k X$. First note that $\pi_q(X) \to \eta^*\pi_q(\tau_k X)$ is an isomorphism for all $q \leq k$; this is clear in a presheaf category, and thus follows for a general topos using the fact that everything is compatible with sheafification.

To prove either part of the proposition, we may assume without loss of generality that $Y \approx 1$.

To prove (1), we must show that if X is m-truncated for some m, then X is k-truncated if and only if $\pi_q(X) \approx *$ for all $q > k$. The key observation is that if X is m-truncated, then $X \to X^{S^m}$ is a (-1)-truncated map, and therefore that X is $(m-1)$-truncated iff $X \to X^{S^m}$ is (-1)-connected iff $X^{S^m} \to X$ is 0-connected iff $\pi_m(X) \approx *$.

To prove (2), we must show that X is k-connected if and only if it is (-1)-connected and $\pi_q(X) \approx *$ for all $q \leq k$. First, suppose X is k-connected. Then $\pi_q(X) \approx \eta^*\pi_q(\tau_k X) \approx \eta^*\pi_q(1) \approx *$ for all $q \leq k$. Conversely, if $\pi_q(X) \approx *$ for all $q \leq k$, we have $* \approx \pi_q(X) \approx \eta^*\pi_q(\tau_k X)$; since η is (-1)-connected, it follows that $\pi_q(\tau_k X) \approx *$ for $q \leq k$. Part (1) then shows that $\tau_k X$ is (-1)-truncated. Since X is (-1)-connected this implies $\tau_k X$ is also (-1)-connected, and thus $\tau_k X \approx 1$, as desired. $\square$

10. t-completion

10.1. **∞-connected maps and t-completion.** We say a map $f\colon X \to Y$ is **∞-connected** if it is k-connected for all k. Equivalently, f is (-1)-connected and $\pi_q(f) \approx *$ for all $q \geq 0$.

Example 10.2. In spaces, or presheaves of spaces, or in the Joyal-Jardine model category of simplicial (pre-)sheaves, the ∞-connected maps are precisely the weak equivalences. This is not true in a general model topos.

Let C_∞ denote the class of ∞-connected maps in a model topos $\mathbf{E}$.

Proposition 10.3. *The class C_∞ is saturated. The class C_∞ is closed under homotopy base change. The class C_∞ is the saturation of a set of maps S.*

The localization model category $\mathbf{E}_S \approx \mathbf{E}_{C_\infty}$ obtained by inverting the ∞-connected maps is a model topos, and the adjoint Quillen pair $\mathbf{E} \rightleftarrows \mathbf{E}_S$ defines a geometric morphism $\mathbf{E}_S \to \mathbf{E}$.

We call the model topos $\mathbf{E}_S$ the **t-completion** of $\mathbf{E}$, and write $t\mathbf{E}$ for it.

Proof. Let $C_{\tau_k} = \{\, f \in \mathbf{E} \mid \tau_k f \text{ is a weak equivalence}\,\}$; this is a saturated class of maps. Thus $C_\infty = \bigcap C_{\tau_k}$ by (8.7), and so is saturated.

Let $C_k = \{\, f \in \mathbf{E} \mid f \text{ is } k\text{-connected}\,\}$. Then $C_\infty = \bigcap C_k$, and we have shown that C_k is closed under homotopy base change.

The proof that C_∞ is the saturation of a set of maps is a cardinality argument; see [Lurb].

To show that $\mathbf{E}_S$ is a model topos, it suffices to show that the localization functor $\mathbf{E} \to \mathbf{E}_S$ preserves finite homotopy limits, and this in turn is a straightforward consequence of the fact that the saturation $\bar{S}$ of S is closed under homotopy base change. $\square$

10.4. **Hypercovers.** Let $\mathbf{M}$ be a model category, and consider the category $\mathbf{M}^{\Delta^{\mathrm{op}}}$ of simplicial objects in $\mathbf{M}$. Let $\Delta_{\leq k}$ denote the full subcategory of Δ consisting of objects $[0], \dots, [k]$; there is a restriction functor $u_k \colon \mathbf{M}^{\Delta^{\mathrm{op}}} \to \mathbf{M}^{\Delta^{\mathrm{op}}_{\leq k}}$. The (derived) right adjoint to u_k is called the **kth coskeleton functor**, and is denoted $\operatorname{cosk}^k \colon \mathbf{M}^{\Delta^{\mathrm{op}}_{\leq k}} \to \mathbf{M}^{\Delta^{\mathrm{op}}}$.

By abuse of notation, we will write $\operatorname{cosk}^n \colon \mathbf{M}^{\Delta^{\mathrm{op}}} \to \mathbf{M}^{\Delta^{\mathrm{op}}}$ for the composite functor $\operatorname{cosk}^n \circ u^*$. The adjunction provides a natural transformation $\eta^n \colon X \to \operatorname{cosk}^n X$. Since $\Delta_{\leq k}$ is a full subcategory of Δ, the map $X_j \to (\operatorname{cosk}^k X)_j$ is a weak equivalence for $j \leq k$. Furthermore, when $j \leq k$ the map $\operatorname{cosk}^j(\eta^k) \colon \operatorname{cosk}^j U \to \operatorname{cosk}^j \operatorname{cosk}^k U$ is a weak equivalence, while if $j \geq k$ the map $\eta^j \colon \operatorname{cosk}^k X \to \operatorname{cosk}^j \operatorname{cosk}^k X$ is a weak equivalence.

Let $\mathbf{E}$ be a model topos. A **hypercover** in $\mathbf{E}$ is a simplicial object $U \in \mathbf{E}^{\Delta^{\mathrm{op}}}$ such that for each $n \geq -1$, the evident map $(\eta^n)_{n+1} \colon U_{n+1} \to (\operatorname{cosk}^n U)_{n+1}$ is (-1)-connected. (Note that for $n = -1$, this means that the map $U_0 \to 1$ should be (-1)-connected.) For an object $Y \in \mathbf{E}$ a **hypercover of Y** is a hypercover in $\mathbf{E}/Y$.

Lemma 10.5. *Let $U \in \mathbf{E}^{\Delta^{\mathrm{op}}}$ be a hypercover of $\mathbf{E}$. Then for all q, the object $\operatorname{cosk}^q U \in \mathbf{E}^{\Delta^{\mathrm{op}}}$ is also a hypercover.*

Proof. We must show that if U is a hypercover, then $(\eta^n)_{n+1} \colon (\operatorname{cosk}^q U)_{n+1} \to (\operatorname{cosk}^n \operatorname{cosk}^q U)_{n+1}$ is (-1)-connected for any n and q. If $n \geq q$, then this map is already a weak equivalence, so suppose $n < q$. Consider the commutative diagram

$$\begin{tikzcd} U_{n+1} \arrow[r, "(\eta^q)_{n+1}"] \arrow[d, "(\eta^n)_{n+1}"'] & (\operatorname{cosk}^q U)_{n+1} \arrow[d, "(\eta^n)_{n+1}"] \\ (\operatorname{cosk}^n U)_{n+1} \arrow[r, "(\operatorname{cosk}^n \eta^q)_{n+1}"'] & (\operatorname{cosk}^n \operatorname{cosk}^q U)_{n+1} \end{tikzcd}$$

The bottom horizontal map is a weak equivalence since $n < q$, and the left-hand horizontal map is (-1)-connected, since U is a hypercover. Therefore the right-hand vertical map is (-1)-connected by (7.10)(c). $\square$

The following is based on a clever argument from Dugger-Hollander-Isaksen [DHI04, Prop. A.4].

Proposition 10.6. *Let $U \in \mathbf{E}^{\Delta^{\mathrm{op}}}$ be a hypercover of $\mathbf{E}$. Then for all n, the map $\operatorname{hocolim}_{\Delta^{\mathrm{op}}} \operatorname{cosk}^n U \to 1$ is a weak equivalence.*

Proof. By (10.5), it is enough to prove the result for hypercovers U such that there exists $n \geq -1$ for which $\eta^n\colon U \to \operatorname{cosk}^n U$ is an equivalence. We will prove this by induction on n.

For $n = -1$, $\operatorname{cosk}^{-1} U \approx 1$ in $\mathbf{E}^{\Delta^{\mathrm{op}}}$, and the result follows.

Now suppose that U is a hypercover such that $U \approx \operatorname{cosk}^{n+1} U$, and let $V = \operatorname{cosk}^n U$. By (10.5) we have that V is also a hypercover, and thus the inductive hypothesis gives that $\operatorname{hocolim}_{\Delta^{\mathrm{op}}} V \approx 1$.

Next, I claim that $(\eta^n)_q\colon U_q \to V_q$ is (-1)-connected for all q. For $q \leq n$ the map is already an isomorphism. Let $q > n$, and consider the commutative square

$$\begin{tikzcd} U_q \arrow[r] \arrow[d, "(\eta^n)_q"'] & \prod U_{n+1} \arrow[d, "\prod(\eta^n)_{n+1}"] \\ (\operatorname{cosk}^n U)_q \arrow[r] & \prod(\operatorname{cosk}^n U)_{n+1} \end{tikzcd}$$

where the products in the right-hand column are taken over the set of non-degenerate $(n+1)$-simplices of the standard q-simplex, and the horizontal maps are induced by the corresponding simplicial operators. Because $U \approx \operatorname{cosk}^{n+1} U$, the square is a homotopy pullback square. (This is an expression of the way that the simplicial set $\operatorname{sk}^{n+1}\Delta^q$ is obtained from $\operatorname{sk}^n \Delta^q$ by attaching some $(n+1)$-simplices along their boundary.) The claim now follows from the fact that (-1)-connected maps are preserved under finite products and homotopy base change.

Now define a double complex $Z\colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathbf{E}$ by

$$Z_{p,q} \overset{\mathrm{def}}{=} \underbrace{U_p \times^h_{V_p} \cdots \times^h_{V_p} U_p}_{q+1 \text{ times}}.$$

Fixing p, the fact that $U_p \to V_p$ are (-1)-connected gives that $\operatorname{hocolim}_{\Delta^{\mathrm{op}}} Z_{p,\bullet} \approx V_p$, so $\operatorname{hocolim}_{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}} Z \approx \operatorname{hocolim}_{\Delta^{\mathrm{op}}} V \approx 1$.

Let $D\colon \Delta^{\mathrm{op}} \to \mathbf{E}$ be the diagonal of Z, defined by $D_k = Z_{k,k}$; thus $\operatorname{hocolim}_{\Delta^{\mathrm{op}}} D \approx \operatorname{hocolim}_{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}} Z \approx 1$. We define maps $U \xrightarrow{f} D \xrightarrow{g} U$ as follows. The map f is given by the diagonal map $f_p\colon U_p \to U_p \times^h_{V_p} \cdots \times^h_{V_p} U_p = D_p$. The map $g\colon D \to U$ is adjoint to the map $h\colon D|_{\Delta^{\mathrm{op}}_{\leq n+1}} \to U|_{\Delta^{\mathrm{op}}_{\leq n+1}}$ defined as follows. For $q \leq n$, $D_q = U_q \times^h_{V_q} \cdots \times^h_{V_q} U_q \approx U_q$ since $U_q = V_q$ in these degrees. For $q = n+1$ we define $h_{n+1}\colon D_{n+1} = U_{n+1} \times^h_{V_{n+1}} \cdots \times^h_{V_{n+1}} U_{n+1} \to U_{n+1}$ by projection to the first factor. One checks that this is indeed well-defined.

We have that $gf = 1_U$, so that U is a retract of D, whence $\operatorname{hocolim}_{\Delta^{\mathrm{op}}} U$ is a retract of $\operatorname{hocolim}_{\Delta^{\mathrm{op}}} D \approx 1$, whence $\operatorname{hocolim}_{\Delta^{\mathrm{op}}} U \approx 1$ as desired. ☐

Lurie proves the following.

Proposition 10.7. *A map $f\colon X \to Y \in \mathbf{E}$ is ∞-connected if and only if it is weakly equivalent to one of the form $\operatorname{hocolim}_{\Delta^{\mathrm{op}}} U \to Y$ for some hypercover U of Y.*

In other words, t-completion amounts to "formally inverting hypercovers".

Proof. It is enough to show that an object $X \in \mathbf{E}$ is ∞-connected if and only if it is weakly equivalent to $\operatorname{hocolim}_{\Delta^{\mathrm{op}}} U$ for some hypercover U in $\mathbf{E}$.

Let X be an ∞-connected object in $\mathbf{E}$, and let U be the constant simplicial object in $\mathbf{E}$, with $U_n = X$. I claim that X is a hypercover in $\mathbf{E}$. Since U is a constant simplicial object, it is straightforward to show that $(\operatorname{cosk}^n U)_{n+1} \approx X^{S^n}$. Thus we need to show that $X \to X^{S^n}$ is an effective epimorphism for all $n \geq -1$. This amounts to the fact that $\pi_n X \approx *$ for all n.

Next, we need to show that for a general hypercover U in $\mathbf{E}$, the realization $X \approx \operatorname{hocolim}_{\Delta^{\mathrm{op}}} U$ is ∞-connected. That is, for all $n \geq 0$ and all $(n-1)$-truncated $Y \in \mathbf{E}$, we must show (8.3) that

$$\operatorname{map}(1, Y) \to \operatorname{map}(X, Y) \approx \operatorname{map}(\operatorname{hocolim}_{\Delta^{\mathrm{op}}} U_\bullet, Y) \approx \operatorname{holim}_{\Delta} \operatorname{map}(U_\bullet, Y)$$

is a weak equivalence of spaces.

It is a standard fact that if $A^\bullet$ is a cosimplicial space such that each space A^q is $(n-1)$-truncated, then $\operatorname{holim}_\Delta A^\bullet$ only "depends" on the spaces $A^0, \dots, A^n$, in the sense that if $B^\bullet$ is another cosimplicial spaces with this property and $f\colon A^\bullet \to B^\bullet$ a map which is a weak equivalence is degrees less than or equal to n, then $\operatorname{holim}_\Delta A^\bullet \approx \operatorname{holim}_\Delta B$. We can apply this observation to the map of cosimplicial spaces $\operatorname{map}((\operatorname{cosk}^n U)_\bullet, Y) \to \operatorname{map}(U_\bullet, Y)$, and thus it is enough to show that

$$\operatorname{map}(1, Y) \to \operatorname{map}(\operatorname{hocolim}_{\Delta^{\mathrm{op}}} \operatorname{cosk}^n U, Y)$$

is a weak equivalence when Y is $(n-1)$-truncated. This is an immediate consequence of (10.6), which says that $\operatorname{hocolim}_{\Delta^{\mathrm{op}}} \operatorname{cosk}^n U \approx 1$. $\square$

11. Construction of model toposes

11.1. **The model topos of sheaves on a site.** Let $(\mathbf{C}, \tau)$ be a Grothendieck site; that is, a small category $\mathbf{C}$ with a Grothendieck topology τ. We can regard the category $\operatorname{PSh}(\mathbf{C})$ of presheaves of sets as a full subcategory of the simplicial presheaf category $s\operatorname{PSh}(\mathbf{C})$, by identifying sets with discrete spaces.

With these identifications, we can regard the Grothendieck topology τ (which is a collection of sieves $S \to y_C \in \operatorname{PSh}(\mathbf{C})$) as consisting of a set of maps in $s\operatorname{PSh}(\mathbf{C})$, by identifying sets with discrete spaces. We define

$$s\operatorname{Sh}(\mathbf{C}, \tau) \overset{\text{def}}{=} s\operatorname{PSh}(\mathbf{C})_\tau,$$

the localization of simplicial presheaves with respect to the set of covering sieves of τ. This localization is discussed in [DHI04, App. A]; the following theorem is due to Lurie [Lura] in the ∞-category context.

Proposition 11.2. *The model category $s\operatorname{Sh}(\mathbf{C}, \tau)$ is a model topos with 0-truncation $\tau_0(s\operatorname{Sh}(\mathbf{C}, \tau)) \approx \operatorname{Sh}(\mathbf{C}, \tau)$.*

I'll call this the **model topos of sheaves on** $(\mathbf{C}, \tau)$; this should not be confused with the actual topos of sheaves of sets on $(\mathbf{C}, \tau)$, nor with the Joyal-Jardine model category of simplicial sheaves on $(\mathbf{C}, \tau)$.

Proof. The result will follow if we can produce a functor $L\colon s\operatorname{PSh}(\mathbf{C}) \to s\operatorname{PSh}(\mathbf{C})$ and natural transformation $X \to LX$ such that

(1) LX is τ-local for all $X \in s\operatorname{PSh}(\mathbf{C})$;

(2) for all τ-local Y, $\mathrm{map}(LX, Y) \to \mathrm{map}(X, Y)$ is a weak equivalence; and
(3) L preserves finite homotopy limits.

Using the notation of §3.4, we define a functor $X \mapsto X^+ \colon s\,\mathrm{PSh}(\mathbf{C}) \to s\,\mathrm{PSh}(\mathbf{C})$ and a natural transformation $\eta\colon X \to X^+$, by

$$X^+(C) \overset{\text{def}}{=} \mathrm{hocolim}_{S\in\tau_C} \mathrm{map}_{s\,\mathrm{PSh}(\mathbf{C})}(S, X).$$

(It requires some care to get a well-defined construction which has the right homotopical properties. Perhaps the easiest solution is to define $X^+(C) = \mathrm{colim}_{S\in\tau_C} \mathrm{hom}_{s\,\mathrm{PSh}(\mathbf{C})}(S, X')$, where hom denotes the simplicial mapping space, and X' is the "Heller fibrant replacement" of X; we say a presheaf is **Heller fibrant** if it has the right-lifting property with respect to the class of maps which are both monomorphisms and weak equivalences.)

We define for each ordinal λ a functor $s_\lambda\colon s\,\mathrm{PSh}(\mathbf{C}) \to s\,\mathrm{PSh}(\mathbf{C})$ and natural transformation $X \to s_\lambda(X)$; for a succesor ordinal, take $s_{\lambda+1}(X) = (s_\lambda(X))^+$, and for a limit ordinal, take $s_\lambda(X) = \mathrm{hocolim}_{\mu<\lambda} s_\mu(X)$.

We claim that

(1) there exists an ordinal κ such that $s_\kappa(X)$ is τ-local for all simplicial presheaves X;
(2) for any τ-local object Y, $\mathrm{map}(X^+, Y) \to \mathrm{map}(X, Y)$ is a weak equivalence;
(3) $X \mapsto X^+$ commutes with finite homotopy limits.

Given this, we can take $L = s_\kappa$, which shows that localization is left exact.

For (1), we need to choose κ so that for each covering sieve $S \rightarrowtail y_C$ and each $k \geq 0$, the functor $\mathrm{PSh}(\mathbf{C})(\partial\Delta^k \times S, -)$ commutes with κ-filtered colimits. (Except for some very trivial cases, κ will need to be infinite.) Lurie shows how to do this.

(2) is more difficult, and I can't find a proof in [Lura]. I think a generalization of the argument sketched in §3.4 will work.

(3) is clear, since directed homotopy colimits of spaces commute with finite homotopy limits. ☐

11.3. **An example of a non-t-complete model topos.** I learned about the following example from [DHI04, App. A].

Let $\mathbf{C}$ denote the small category with objects $\{t_k, u_k, v_k\}_{k\geq 0}$ and with the following shape

$$\begin{tikzcd}
& u_0 \arrow[dl] & & u_1 \arrow[dl] & & u_2 \arrow[dl] & & {} \arrow[dl] & \\
t_0 & & t_1 \arrow[ul] \arrow[dl] & & t_2 \arrow[ul] \arrow[dl] & & t_3 \arrow[ul] \arrow[dl] & & \cdots \\
& v_0 \arrow[ul] & & v_1 \arrow[ul] & & v_2 \arrow[ul] & & {} \arrow[ul] &
\end{tikzcd}$$

so that all diagrams commute in $\mathbf{C}$; in particular, there is at most one morphism between any two objects of $\mathbf{C}$. We define a topology τ on $\mathbf{C}$ so that:

(1) the only covering sieves for u_k and v_k are the trivial ones;
(2) the only non-trivial covering sieve for t_k is the one generated by the pair of maps $\{u_k \to t_k, v_k \to t_k\}$.

Let $\mathbf{E} = s\,\mathrm{PSh}(\mathbf{C}, \tau)$. Thus, a simplicial presheaf $F \in s\,\mathrm{PSh}(\mathbf{C})$ is a sheaf iff for each $k \geq 0$, $F(t_{k+1}) \to F(u_k) \times^h_{F(t_k)} F(v_k)$ is a weak equivalence.

It is straightforward to show that the "plus construction" has the property that for a presheaf X, we have

$$X^+(t_{k+1}) \approx X(u_k) \times^h_{X(t_k)} X(v_k), \quad X^+(u_k) \approx X(u_k), \quad X^+(v_k) \approx X(v_k).$$

Furthermore, sheafification is given by iterating the plus construction a countable number of times, since the only non-trivial covering sieve is isomorphic to $S = y_{u_k} \cup^h_{y_{t_{k+1}}} y_{v_k} \to y_{t_k}$, and $s\,\mathrm{PSh}(\mathbf{C})(S, -)$ commutes with countable directed colimits. That is, s_ω computes sheafification.

Let $X \in s\,\mathrm{PSh}(\mathbf{C})$ be the object defined by

$$\begin{tikzcd}
 & D^0_+ \arrow[dr] & & D^1_+ \arrow[dr] & & D^2_+ \arrow[dr] & & {} & \\
S^0 \arrow[ur] \arrow[dr] & & S^1 \arrow[ur] \arrow[dr] & & S^2 \arrow[ur] \arrow[dr] & & S^3 \arrow[ur] \arrow[dr] & & \cdots \\
 & D^0_- \arrow[ur] & & D^1_- \arrow[ur] & & D^2_- \arrow[ur] & & {} &
\end{tikzcd}$$

I claim that aX is a sheaf which is ∞-connected but not contractible, proving that $\mathbf{E}$ is not t-complete.

It is easy to see from the plus-construction that $(s_\omega X)(t_k) \approx \operatorname{hocolim} \Omega^i S^{k+i}$, which is certainly not contractible. To show that the truncations of LX are contractible, we recall that truncation computes with sheafification, so it is enough to show that $s_\omega(\tau_n X)$ is contractible, where $\tau_n X$ denotes truncation in presheaves. In fact, $\tau_n X$ takes contractible values at all objects u_k and v_k and at all objects t_k with $k > n$, and thus $s_{n+1}(\tau_n X)$ is already a contractible presheaf, whence $s_\omega(\tau_n X)$ is contractible.

11.4. **Examples of t-complete model toposes.** In some cases, the construction $s\,\mathrm{Sh}(\mathbf{C}, \tau)$ leads to a model topos which is t-complete. Given a topological space X, write $s\,\mathrm{Sh}(X)$ for $s\,\mathrm{Sh}(\mathcal{U}_{\mathbf{X}}, \tau)$, where $\mathcal{U}_X$ is the category of open subsets of X and τ is the usual topology.

Theorem 11.5 (Brown-Gersten [BG73])**.** *Let X be a topological space such that $\{$open subsets$\}$ and $\{$irreducible closed subsets$\}$ both satisfy the ascending chain condition. Then $s\,\mathrm{Sh}(X)$ is t-complete.*

Lurie [Lura, Ch. 5] proves other results along these lines.

Given a model topos $\mathbf{E}$, we say a set of objects $\{U_\alpha\}$ **generate $\mathbf{E}$** if a map $f\colon X \to Y \in \mathbf{E}$ is a weak equivalence if and only if $\mathrm{map}(U_\alpha, X) \to \mathrm{map}(U_\alpha, Y)$ is for all elements of the set. (For instance, if $\mathbf{E} = s\,\mathrm{PSh}(\mathbf{C})_S$, the image of the yoneda embedding is a generating set.)

Lemma 11.6. *Suppose that $\{U_\alpha\}$ is a set of generators for a model topos $\mathbf{E}$. Then $\mathbf{E}$ is t-complete if and only if every ∞-connected map $F \to U_\alpha$ in $\mathbf{E}$ admits a section (up to homotopy).*

Proof. It is clear that if $\mathbf{E}$ is t-complete, any ∞-connected map $F \to U_\alpha$ is a weak equivalence and so admits a section.

We need to show that if $f\colon X \to Y$ is ∞-connected, then $\operatorname{map}(U_\alpha, X) \xrightarrow{f^*} \operatorname{map}(U_\alpha, Y)$ is a weak equivalence. Equivalently, for every map $g\colon S^q \to \operatorname{map}(U_\alpha, X)$ such that $f_*(g)$ is homotopic to a constant map, there exists a homotopy of g to a constant map.

If $f\colon X \to Y$ is ∞-connected, then for each $q \geq 0$ the map $f_q\colon X^{D^q} \to X^{S^{q-1}} \times^h_{Y^{S^{q-1}}} Y^{D^q}$ is ∞-connected. (In general, if $f\colon X \to Y$ is k-connected, then $\Delta(f)\colon X \to X \times^h_Y X$ is $(k-1)$-connected, by (8.8); the map f_q is weakly equivalent to $\Delta^q(f)$.) Thus, to give a dotted arrow in the commutative diagram

$$\begin{tikzcd} S^{q-1} \arrow[r] \arrow[d] & \operatorname{map}(U_\alpha, X) \arrow[d] \\ D^q \arrow[r] \arrow[ur, dotted] & \operatorname{map}(U_\alpha, Y) \end{tikzcd}$$

is the same as giving a lift in

$$\begin{tikzcd} & X^{D^q} \arrow[d] \\ U_\alpha \arrow[r] \arrow[ur, dotted] & X^{S^{q-1}} \times^h_{Y^{S^{q-1}}} Y^{D^q} \end{tikzcd}$$

which exists by hypothesis. $\square$

Proof of (11.5). Note that since the space X is Noetherian, every closed subset of X is a finite union of closed irreducible subsets.

The yoneda embedding $\mathcal{U}_X \to \mathbf{E} = s\,\mathrm{Sh}(X)$ is subcanonical, so we may as well identify an open set U with its representable functor. In particular, the (-1)-truncated objects of $\mathbf{E}$ correspond exactly to open subsets of X. Furthermore, if U and V are open sets, then

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & U \cup V \end{tikzcd}$$

is a homotopy pushout square in $\mathbf{E}$.

In particular, if F is a fibrant object in $s\,\mathrm{Sh}(X)$, then $F(U)$ is weakly equivalent to the space of lifts

$$\begin{tikzcd} & F \arrow[d] \\ U \arrow[r] \arrow[ur, dotted] & 1 \end{tikzcd}$$

(When referring to $F(U)$, I'll implicitly assume that F is fibrant.)

Note that we can explicitly compute $\tau_{-1}F$ for any F; it is the sheaf associated to the union of open sets V for which $F(V) \neq \varnothing$. In particular, if $F \to U$ is an ∞-connected map to an open set U, so that $\tau_{-1}F \approx U$, then it admits sections "locally in U", i.e., the collection of open sets V such that $F(V) \neq \varnothing$ form a cover of U.

In general, we will say that F is **∞-connected over** Y if Y is an open set of X such that $F \times Y \to Y$ is ∞-connected. Thus, by (11.6), we need to show that if F is ∞-connected over Y then $F(Y)$ is non-empty.

Let $\mathcal{D}$ denote the class of all triples (Y, U, F), where $U \subseteq Y \subseteq X$ are open subsets, $F \in \mathbf{E}$ is ∞-connected over Y, and U is maximal among open subsets of Y with respect to the property that $F(U) \neq \varnothing$. Note that since X is Noetherian, then given Y and F there always exists a $U \subseteq Y$ such that $(Y, U, F) \in \mathcal{D}$.

To prove the claim, we need to show that $(Y, U, F) \in \mathcal{D}$ implies $Y = U$. If $Y \neq U$, then the closure $\overline{Y - U}$ of $Y - U$ in X is a finite union of irreducible components $C_1, \dots, C_r$ with $r > 0$, and for each $i = 1, \dots, r$, $C_i \cap Y \neq \varnothing$ while $C_i \cap U = \varnothing$.

Say that a closed irreducible $C \subseteq X$ is *bad* if there exists $(Y, U, F) \in \mathcal{D}$ such that $C \cap Y \neq \varnothing$ and $C \cap U = \varnothing$. The components C_i of $\overline{Y - U}$ described above are clearly bad; thus, we will have proved $Y = U$ if we can show there are no bad sets.

Suppose there are bad sets, and let C be a maximal bad subset (which exists since closed irreducibles have the ascending chain condition); we'll derive a contradiction.

Let $(Y, U, F) \in \mathcal{D}$ such that $C \cap Y \neq \varnothing$ and $C \cap U = \varnothing$. Since F has sections locally, there is an open $V \subseteq Y$ which touches C and such that $F(V) \neq \varnothing$. Choose elements $\alpha \in F(U)$ and $\beta \in F(V)$, and define G by the homotopy pullback

$$\begin{tikzcd} G \arrow[r] \arrow[d] & V \arrow[d, "\beta"] \\ U \arrow[r, "\alpha"'] & F \end{tikzcd}$$

Note that G is ∞-connected over $U \cap V$, since each of U, V, and F are. Let $W \subseteq U \cap V$ be maximal among open subsets such that $G(W) \neq \varnothing$. Let D be the irreducible component of $X - W$ which contains C. Then $D \cap (U \cap V) \neq \varnothing$ (since D touches U and V, and D is irreducible), while $D \cap W = \varnothing$. Therefore D is bad, since $(U \cap V, W, G) \in \mathcal{D}$, and thus $D = C$ since C is maximally bad. Thus C is one of the irreducible components of $X - W$.

Let B be the union of the irreducible components of $X - W$ other than C, and let $V' = V - B$. We have that $(U \cap V') \cap (X - W) = U \cap V' \cap (C \cup B) = (U \cap V' \cap C) \cup (U \cap V' \cap B) = \varnothing$, so $U \cap V' \subseteq W$. Therefore, $G(U \cap V')$ is nonempty.

Using any element $\gamma \in G(U \cap V')$, we can construct a section of F over $U \cup V'$, extending the sections $\alpha|_{U \cap V'}$ and $\beta|_{U \cap V'}$. But by hypothesis U is maximal among open subsets of Y over which F is non-empty, so we must have $V' \subseteq U$, and so $V' \cap C \subseteq U \cap C = \varnothing$.

But if $V' \cap C = \varnothing$, then $V \cap C \subseteq B$. But since C is irreducible and touches V, $C = \overline{V \cap C} \subseteq B$, which contradicts the hypothesis that C is one of the components of $X - W = C \cup B$. $\square$

11.7. **Simplicial Grothendieck sites, and the classification of t-complete model toposes.** Let $\mathbf{C}$ be a small category enriched over simplicial sets. I'll write $\hom_{\mathbf{C}}(C_1, C_2)$ for the simplicial set of maps from C_1 to C_2 in $\mathbf{C}$. Let $\pi_0\mathbf{C}$ denote the discrete category with the same objects as $\mathbf{C}$, and $\pi_0\mathbf{C}(C_1, C_2) = \pi_0(\hom_{\mathbf{C}}(C_1, C_2))$.

A **simplicial Grothendieck site** is a simplicial category $\mathbf{C}$ together with a Grothendieck topology τ_0 on $\pi_0\mathbf{C}$.

For an object $C \in \mathbf{C}$, let $y_C \in s\,\mathrm{PSh}(\mathbf{C})$ denote the evident representable functor $y_C(C') = \hom_{\mathbf{C}}(C', C)$, and let $\pi_0 y_C$ denote the functor $\pi_0 y_C(C') = \pi_0 \hom_{\mathbf{C}}(C', C)$, where this set is thought of as a discrete simplicial set. For a sieve $S \rightarrowtail \pi_0 y_C$ the pullback square

$$\begin{tikzcd} \tilde{S} \arrow[r, tail, "\tilde{s}"] \arrow[d] & y_C \arrow[d] \\ S \arrow[r, tail, "s"'] & \pi_0 y_C \end{tikzcd}$$

is a homotopy pullback square; let τ denote the set of maps $\tilde{s}$ obtained by such a pullback from $s \in \tau_0$. Define

$$s\,\mathrm{Sh}(\mathbf{C},\tau) \overset{\text{def}}{=} s\,\mathrm{PSh}(\mathbf{C})_\tau.$$

A straightforward generalization of the argument given above shows that this is a model topos.

Theorem 11.8 (Toën-Vezzosi [TV05])**.** *Every t-complete model topos is Quillen equivalent to one of the form $t(s\,\mathrm{Sh}(\mathbf{C},\tau))$ for some simplicial site $(\mathbf{C},\tau)$.*

Sketch proof. Let $\mathbf{E} = s\,\mathrm{PSh}(\mathbf{C})_T$ be a left-exact localization of a simplicial presheaf category on a simplicial category $\mathbf{C}$. Let τ_0 be the collection of sieves $s\colon S \to \pi_0 y_C$ in $\pi_0\mathbf{C}$ with the property that their lift $\bar{s}\colon \bar{S} \to y_C$ is in $\bar{T}$, and let τ denote the set of such lifts. It is easy to show that τ_0 is a topology on $\mathbf{C}$, and that $\bar{\tau} \subseteq \bar{T}$. I am going to show that elements of $\bar{T}$ are actually ∞-connected maps in $s\,\mathrm{PSh}(\mathbf{C})_\tau$ and thus become weak equivalences after t-completion. This shows that $\mathbf{E} \approx t(s\,\mathrm{PSh}(\mathbf{C})_\tau)$ if $\mathbf{E}$ is t-complete.

Let $f\colon X \to Y \in \bar{T}$; we want to show that f is k-connected in $s\,\mathrm{PSh}(\mathbf{C})_\tau$ for all k. Let $g\colon \tau_k^Y f \to Y$ be the relative truncation of f in $s\,\mathrm{PSh}(\mathbf{C})$. Then g is weakly equivalent in $s\,\mathrm{PSh}(\mathbf{C})_\tau$ to the relative truncation of f in $s\,\mathrm{PSh}(\mathbf{C})_\tau$, since truncation commutes with sheafification. For the same reason, $g \in \bar{T}$. Thus, we have reduced to showing that k-truncated maps in $\bar{T}$ are in $\bar{\tau}$ for all k.

We prove if $f \in \bar{T}$ is k-truncated, then $f \in \bar{\tau}$, by induction on k. $f\colon X \to Y \in \bar{T}$ is (-1)-truncated, we can write Y as a colimit of representable presheaves, and thus write f as a colimit of (-1)-truncated maps $\tilde{S} \to y_C$ over representables, i.e., by elements of τ. This implies that f is in $\bar{\tau}$.

Now let $k \geq 0$. Suppose $f\colon X \to Y$ is a k-truncated map of presheaves contained in $\bar{T}$. Consider $g\colon X \to X \times^h_Y X$, which is $(k-1)$-truncated and also in $\bar{T}$, and therefore is in $\bar{\tau}$ by the inductive hypothesis. This means that f is (-1)-truncated when viewed as a map in the model category $s\,\mathrm{PSh}(\mathbf{C})_\tau$, and thus there exists a map f' of presheaves which is (-1)-truncated and is weakly equivalent to f in the $s\,\mathrm{PSh}(\mathbf{C})_\tau$ model structure. Since $\bar{\tau} \subseteq \bar{T}$, the maps f and f' are weakly equivalent in the $s\,\mathrm{PSh}(\mathbf{C})_T$ model structure, and so $f' \in \bar{T}$. By the case already proved, $f' \in \bar{\tau}$, and thus $f \in \bar{\tau}$. ☐

Remark 11.9. Note that the above proof shows that for any model site $\mathbf{E} = s\,\mathrm{PSh}(\mathbf{C})_T$, the homotopically full subcategory $\tau_k\mathbf{E}$ of k-truncated objects (for any k) depends only on the simplicial site $(\mathbf{C},\tau)$ where τ is the collection of sieves in $\bar{T}$.

In particular, if $\mathbf{C}$ is a *discrete* category, this means that $\tau_k\mathbf{E}$ is determined by $\tau_0\mathbf{E} \approx \mathrm{Sh}(\mathbf{C},\tau)$. This explains why we must consider *simplicial* categories $\mathbf{C}$ in the definition of model topos: we want to allow model toposes whose k-truncations are different while their 0-truncations are the same. An example of this phenomenon are $\mathbf{S}$ and $\mathbf{S}/X$, where $\mathbf{S}$ is spaces and X is any simply connected space.

Finally, we mention

Theorem 11.10 (Dugger-Hollander-Isaksen [DHI04]). *For a Grothendieck site* $(\mathbf{C},\tau)$*, the* t*-complete model topos* $t(s\,\mathrm{Sh}(\mathbf{C},\tau))$ *is Quillen equivalent to the Joyal-Jardine model category of simplicial presheaves on* $(\mathbf{C},\tau)$*.*

Remark 11.11. This way of listing the results is not especially historically accurate. Dugger-Hollander-Isaksen [DHI04] show that the Joyal-Jardine model category associated to $(\mathbf{C},\tau)$ ([Jar87], [Jar96]) is Quillen equivalent to a localization of $s\,\mathrm{PSh}(\mathbf{C})$ with respect to certain set of basic hypercovers constructed from τ. Toën-Vezzosi [TV05] show that all t-complete model toposes can be obtained from simplicial sites by a generalization of the Dugger-Hollander-Isaksen construction.

Dugger-Hollander-Isaksen [DHI04][App. A] discuss the localization $s\,\mathrm{PSh}(\mathbf{C})_\tau$, and note that localizing further by inverting basic hypercovers actually formally inverts all hypercovers; they never explicitly discuss the notion of model topos. Lurue [Lura] essentially shows that $s\,\mathrm{PSh}(\mathbf{C})_\tau$ is a model topos, and that inverting hypercovers is the same as t-completion (except that he does not work in the language of model categories, but rather with $(\infty,1)$-categories).

Brown-Gersten [BG73] gave (I think) the very first example of a closed model structure on an interesting category of simplicial sheaves; namely, simplicial sheaves over a topological space X as in the hypotheses of (11.5). Their weak equivalences are maps which induce local isomorphisms of homotopy groups, so that their model structure is a special case of the later Joyal-Jardine model structure; however, their proof relies on the fact [BG73, Thm. 1] that in their setting a simplicial sheaf with trivial local homotopy groups which has descent for covers must be contractible, which in our terms in precisely the claim that the model topos of sheaves (in Lurie's sense) is t-complete.

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Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL
E-mail address: rezk@math.uiuc.edu