On the problem of gauge theories in locally covariant QFT

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Based on published and ongoing works with (various subsets of) the following people:
C. Becker, M. Benini, C. Dappiaggi, T.-P. Hack and R. J. Szabo

Outline

1. Locally covariant quantum field theory
2. Differential cohomology
3. Construction and analysis of the $U(1)$ Yang-Mills model
4. Towards new axioms for gauge QFTs?!?

Locally covariant quantum field theory

Axiomatic QFT on curved spacetimes

- On the Minkowski spacetime $\mathbb{R}^4$, a QFT is an assignment of (C^*-)algebras $\mathcal{A}(U)$ to suitable open subsets $U \subseteq \mathbb{R}^4$ satisfying the Haag-Kastler axioms.

- Parts of the Haag-Kastler axioms can be rephrased by demanding that $\mathcal{A} : \mathcal{O}(\mathbb{R}^4) \to \mathsf{Alg}$ is a covariant functor from a suitable category of open subsets in $\mathbb{R}^4$ to the category of (C^*-)algebras.

- This part can be easily generalized to curved spacetimes; replace $\mathcal{O}(\mathbb{R}^4)$ by the category of m-dimensional globally hyperbolic spacetimes Loc^m.

- Hence, a reasonable starting point for axiomatic QFT on curved spacetimes seems to be the functor category $\mathrm{Fun}(\mathsf{Loc}^m, \mathsf{Alg})$.

- The delicate issue is to impose suitable extra axioms characterizing those functors which "deserve being called a QFT". In other words, we would like to have a full subcategory QFT^m of $\mathrm{Fun}(\mathsf{Loc}^m, \mathsf{Alg})$ such that
  1.) QFT^m is 'big' enough to contain the models relevant in physics.
  2.) QFT^m is 'small' enough to allow for model independent studies.

The axioms of locally covariant QFT

- The subcategory LCT^m of locally covariant QFTs is characterized by the Brunetti-Fredenhagen-Verch (BFV) axioms for functors $\mathcal{A} : \mathsf{Loc}^m \to \mathsf{Alg}$:

Locality: For any Loc^m-morphism $f : M \to N$ the Alg-morphism $f_* := \mathcal{A}(f) : \mathcal{A}(M) \to \mathcal{A}(N)$ is monic.

Causality: If $M_1 \xrightarrow{f_1} N \xleftarrow{f_2} M_2$ are Loc^m-morphisms with causally disjoint image, then $f_{1*}\,\mathcal{A}(M_1)$ and $f_{2*}\,\mathcal{A}(M_2)$ commute in $\mathcal{A}(N)$.

Time-slice: If $f : M \to N$ is Cauchy, then $f_* : \mathcal{A}(M) \to \mathcal{A}(N)$ is Alg-isomorphism.

- Model independent results include for example the spin-statistics theorem [Verch; see also Fewster's talk].

- Question: Does LCT^m include physically relevant models?
  - ☺ Klein-Gordon and Dirac are objects in LCT^m (Dirac requires spin manifolds)
  - ☹ Yang-Mills is **not** an object in LCT^m

- **Goals of my talk:**
  1.) Construct explicitly the full $U(1)$ Yang-Mills model, including all bundles.
  2.) Use the results to explain which axioms are violated and why.
  3.) Sketch some preliminary thoughts on different axioms which are motivated by the theory of stacks [see Schreiber's talk for more on stacks].

Differential cohomology

Why differential cohomology?

◇ Our construction of the quantum $U(1)$ Yang-Mills model is via quantization of the classical model $\rightsquigarrow$ need to understand configuration space first!

◇ The field configurations over a manifold M are pairs (P, ∇) consisting of a $U(1)$-bundle P over M and a connection ∇ on P.

Rem: In previous studies, the bundle P has been fixed from the outside (typically to the trivial one). Here I want to take also bundles as parts of the dynamical degrees of freedom $\rightsquigarrow$ much richer structure!

◇ Gauge equivalence: $(P, \nabla) \sim (P', \nabla')$ iff there exists a $U(1)$-bundle isomorphism $\psi : P \to P'$ preserving the connections.

◇ Take gauge orbit space $\widehat{H}^2(M) = \big\{(P, \nabla)\big\}/\sim$ as configuration space.

◇ The role of differential cohomology is to provide an efficient description of the spaces $\widehat{H}^2(M)$, as well as their analogs $\widehat{H}^k(M)$ for all $k \in \mathbb{N}$.

NB: Physically, $k = 1$ describes the σ-model with target $U(1)$, $k = 2$ the $U(1)$ Yang-Mills model and $k > 2$ connections on higher $U(1)$-bundles.

Differential cohomology theories

Def: A differential cohomology theory is a contravariant functor $\widehat{H}^* : \mathsf{Man} \to \mathsf{Ab}^{\mathbb{Z}}$ together with nat. transformations $(\mathrm{curv}, \mathrm{char}, \iota, \kappa)$, such that the following diagram commutes and has exact rows and columns:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \frac{H^{*-1}(M;\mathbb{R})}{H^{*-1}_{\mathrm{free}}(M;\mathbb{Z})} \arrow[r] \arrow[d] & \frac{\Omega^{*-1}(M)}{\Omega^{*-1}_{\mathbb{Z}}(M)} \arrow[r, "\mathrm{d}"] \arrow[d, "\iota"] & \mathrm{d}\Omega^{*-1}(M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^{*-1}(M;U(1)) \arrow[r, "\kappa"] \arrow[d] & \widehat{H}^*(M) \arrow[r, "\mathrm{curv}"] \arrow[d, "\mathrm{char}"] & \Omega^*_{\mathbb{Z}}(M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^*_{\mathrm{tor}}(M;\mathbb{Z}) \arrow[r] \arrow[d] & H^*(M;\mathbb{Z}) \arrow[r] \arrow[d] & H^*_{\mathrm{free}}(M;\mathbb{Z}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Thm: [Simons,Sullivan; Bär,Becker] Differential cohomology theories exist (e.g. Cheeger-Simons theory) and are unique up to a unique natural isomorphism.

Construction and analysis of the $U(1)$ Yang-Mills model

Bird's-eye view on the construction

◇ The following construction is explained in my paper with Becker and Szabo.

1. Take any differential cohomology theory $\widehat{H}^* : \mathsf{Man} \to \mathsf{Ab}^{\mathbb{Z}}$ and restrict it to the category Loc^m and some fixed degree $k \in \mathbb{N}$.
2. Use Lagrangian $\mathcal{L}(h) = \frac{1}{2}\mathrm{curv}(h) \wedge *\mathrm{curv}(h)$ to construct subfunctor $\mathsf{Sol}^k : \mathsf{Loc}^m \to \mathsf{Ab}$ of $\widehat{H}^k$ describing solutions to EL-equations.
(NB: Sol^k fits into a restriction of the comm. diagram of exact sequences.)
3. Realize that Sol^k is a contravariant functor to Abelian Fréchet-Lie groups with Poisson structure coming from Lagrangian $\mathcal{L}$ (Peierls construction).
(NB: The isomorphism type of $\mathsf{Sol}^k(M)$ is $U(1)^n \oplus G \oplus F$, where $n \in \mathbb{N}^0$, G discrete Abelian group and F ∞-dimensional Fréchet space.)
4. Use smooth Pontryagin duality $\mathrm{Hom}^\infty(\,\cdot\,, U(1))$ to get covariant functor $\mathsf{PS}^k : \mathsf{Loc}^m \to \mathsf{PAb}$ valued in presymplectic Abelian groups.
(NB: PS^k describes classical observables given by characters $\mathsf{Sol}^k \to U(1)$.)
5. Compose PS^k with the CCR-functor $\mathsf{PAb} \to \mathsf{Alg}$.

Thm: The construction above provides an object $\mathcal{A}^k := \mathsf{CCR} \circ \mathsf{PS}^k$ in the functor category $\mathsf{Fun}(\mathsf{Loc}^m, \mathsf{Alg})$.

Properties of the $U(1)$ Yang-Mills model

Thm: $\mathcal{A}^k$ satisfies the causality and time-slice axiom, but it violates the locality axiom (unless accidentally $(m,k) = (2,1)$, i.e. σ-model in 2-dimensions).

- **Mathematical reason:** There exists subfunctor $\mathcal{A}^k_{\mathrm{top}}$ of $\mathcal{A}^k$ isomorphic to CCR $\circ\, (H^{m-k}(\,\cdot\,;\mathbb{R})^\star \oplus H^k(\,\cdot\,;\mathbb{Z})^\star)$. For $f : M \to N$ we have $\mathcal{A}(f)$ monic iff the push-forwards of f on the (dual) cohomology groups are monic. $\lightning$

- **Physical reason:** $\mathcal{A}^k_{\mathrm{top}}$ is a topological theory measuring top. invariants of (higher) $U(1)$-bundles and electric fluxes. Field configurations carrying top. charges in general do not extend from smaller to larger spacetimes.

  (The same should hold true for **any** model with bundles or other topological degrees of freedom! In particular, the non-Abelian Yang-Mills model.)

Thm: For $k \geq 2$, $\mathcal{A}^k$ does not satisfy the additivity property, i.e. there exists M and open cover by diamonds $\{\iota_i : D_i \to M\}$ such that $\mathrm{id}_{\mathcal{A}^k(M)} \ncong \bigvee_i \iota_{i\,*}$.

Sketch: Let M be such that $H^{k-1}(M, U(1)) \ni u \neq 0$ and take Weyl-operator $W(\mathrm{w}) \in \mathcal{A}^k(M)$ such that $\mathrm{w}(\kappa(u)) \neq 0$. Notice that for any Weyl-operator $W(\mathrm{w}_i) \in \iota_{i\,*}\mathcal{A}^k(D_i)$ we have $\mathrm{w}_i(\kappa(u)) = 0$, since $H^{k-1}(D_i, U(1)) = 0$. Hence, $W(\mathrm{w}) \notin \bigvee_i \iota_{i\,*}\mathcal{A}^k(D_i)$.

- **Physical interpretation:** Flat fields can not be measured in diamonds.

Towards new axioms for gauge QFTs?!?

Motivation

- In the following, an open cover of an object M in Loc^m will always mean an open cover by diamonds $\{\iota_i : D_i \to M\}$.
- The problems of $\mathcal{A}^k : \mathsf{Loc}^m \to \mathsf{Alg}$ can already be seen at the level of the configuration spaces $\widehat{H}^k : \mathsf{Loc}^m \to \mathsf{Ab}$, morally:
  - $\mathcal{A}^k$ is not additive, for $k \geq 2$ "$\Leftrightarrow$" $\widehat{H}^k$ is not a sheaf, for $k \geq 2$
  - $\mathcal{A}^k$ violates locality property "$\Leftrightarrow$" $\widehat{H}^k$ is not flabby (or c-soft)
- Should we really use $\widehat{H}^k$ for describing the configuration space?
- A modern point of view is the following:
  - Instead of assigning to M the gauge orbit space $\widehat{H}^{(2)}(M) = \big\{(P,\nabla)\big\}/\sim$ we better should assign the groupoid $\mathcal{G}(M)$ of bundle-connection pairs with morphisms given by gauge equivalence.
  - This gives a contravariant pseudo-functor $\mathcal{G} : \mathsf{Loc}^m \to \mathsf{Groupoids}$ to the category of groupoids (prestack).
  - *Fact:* $\mathcal{G}$ is a stack (called $BU(1)_{\mathrm{con}}$), i.e. it satisfies "sheaf-like gluing conditions up to gauge transformation" for all open covers.
- **Goal:** Develop some axioms for theories of "observables on stacks".

A working definition

- Inspired by groupoid cohomology, it seems reasonable to take as "functions on a groupoid" the differential graded algebra of functions on its simplicial set.

  (Due to the van Est map this reduces to the BRST-BV formalism of Fredenhagen-Rejzner when we work *infinitesimally/perturbatively* with the Lie algebroids of action groupoids.)

- This motivates me to study the covariant pseudo-functor category $\mathrm{PsFun}(\mathsf{Loc}^m, \mathsf{dgAlg})$ for axiomatizing theories of observables.

Def: An object $\mathcal{A}$ in $\mathrm{PsFun}(\mathsf{Loc}^m, \mathsf{dgAlg})$ is called a weak QFT if it satisfies the diamond-locality (DL), causality (C), time-slice (T) and additivity (A) axioms:

(DL): For any diamond $\iota : D \to M$ the dgAlg-morphism $\iota_\ast := \mathcal{A}(\iota) : \mathcal{A}(D) \to \mathcal{A}(M)$ is cochain homotopic to a monic dgAlg-morphism.

(C): If $M_1 \xrightarrow{f_1} N \xleftarrow{f_2} M_2$ are Loc^m-morphisms with causally disjoint image, then the graded commutator $[\,\cdot\,,\,\cdot\,]_{\mathrm{gr}} : f_{1\ast}\,\mathcal{A}(M_1) \otimes_{\mathrm{gr}} f_{2\ast}\,\mathcal{A}(M_2) \to \mathcal{A}(N)$ is cochain homotopic to 0.

(T): If $f : M \to N$ is Cauchy, then $f_\ast : \mathcal{A}(M) \to \mathcal{A}(N)$ is cochain homotopic to a dgAlg-isomorphism.

(A): For any object M and any open cover $\{\iota_i : D_i \to M\}$ we have $\mathrm{id}_{\mathcal{A}(M)} \cong \bigvee_i \iota_{i\,\ast}$ up to cochain homotopy.

Remarks and open questions

◇ **Remarks:**

- Additive locally covariant QFTs are included in my axioms by considering algebras as degree-zero differential graded algebras.
- The axioms are designed to describe the *full content* of a gauge theory, i.e. gauge fields, ghosts, etc., and not only the gauge invariant content.
- For gauge invariant observables we have seen explicitly that additivity doesn't hold, while for the full content I expect this property to hold. This is related to the fact that gauge orbits don't satisfy the gluing conditions of a sheaf, while the full configuration groupoids glue in the sense of stacks.
- It seems reasonable to demand the locality axiom only for diamond embeddings $\iota : D \to M$, since the general locality axiom is dual to a flabbiness property for the configuration groupoids and we already know that bundles (even before taking gauge equivalence classes) in general do not extend.

◇ **Open problems:**

1.) Construct the $U(1)$ Yang-Mills model (modeled on the stack $BU(1)_{\mathrm{con}}$) and show that it fits into the new axioms.

2.) Can we prove some nice model independent results (e.g. spin statistics, perturbative renormalization, ...)?