1. Spectra, ring spectra

A_∞-ring spectrum: An A_∞-ring spectrum is a ring spectrum E whose multiplication is associative up to all higher homotopies. Classically this has been described as an action of an A_∞-operad on E; in more modern terms, an A_∞-ring spectrum can be realized by an associative $\mathbb{S}$-algebra, symmetric, or orthogonal ring spectrum.

E_∞-ring spectrum: An E_∞-ring spectrum is a ring spectrum E whose multiplication is associative and commutative up to all higher homotopies. Classically this has been described as an action of an E_∞-operad on E; in more modern terms, an E_∞-ring spectrum can be realized by an associative, commutative $\mathbb{S}$-algebra, symmetric, or orthogonal ring spectrum.

Even periodic cohomology theory; weakly even periodic theory: A multiplicative cohomology E theory is even periodic if $\pi_*(E)$ is concentrated in even degrees and $\pi_2(E)$ contains a unit. It is weakly periodic if $\pi_2(E)$ is an invertible $\pi_0(E)$ module, and $\pi_{2k}(E) \cong \pi_2(E)^{\otimes k}$ for all $k \in \mathbb{Z}$.

$\mathbb{S}$-module; (commutative) $\mathbb{S}$-algebra: The category of $\mathbb{S}$-modules is one of various point-set models of spectra with a strictly symmetric monoidal smash product, and historically the first. An $\mathbb{S}$-module is indexed not by the natural numbers, but by the finite-dimensional sub-vector spaces of a universe, i.e. an infinite-dimensional real inner product space. It has more structure, though: it is also a module over the sphere spectrum $\mathbb{S}$ with respect to a smash product of so-called $\mathbb{L}$-spectra, which in turn are algebras for a monad derived from the linear isometries operad. The category of $\mathbb{S}$-modules very easily generalizes to equivariant spectra, but its definition is arguably more complicated than symmetric or orthogonal spectra.

Symmetric ((commutative) ring) spectrum: Symmetric spectra are one of various point-set models of spectra with a strictly symmetric monoidal smash product. A symmetric spectrum consists of a sequence of spaces X_n with operations of the symmetric group Σ_n and maps $\sigma\colon X_n \wedge S^1 \to X_{n+1}$ such that the composite $X_n \wedge S^m \xrightarrow{\sigma^m} X_{n+m}$ is $\Sigma_n \times \Sigma_n$-equivariant. An advantage of the symmetric spectrum model of homotopy theory is its simplicity, however they are difficult to use in an equivariant context. A symmetric ring spectrum is a monoid with respect to the smash product; a symmetric commutative ring spectrum a commutative monoid.

Orthogonal ((commutative) ring) spectrum: Orthogonal spectra are one of various point-set models of spectra with a strictly symmetric monoidal smash product. It is a middle ground between $\mathbb{S}$-modules and symmetric spectra. Their definition is identical to symmetric spectra, but the symmetric groups Σ_n are replaced by the orthogonal groups $O(n)$. An advantage over symmetric spectra is the possibility to use this setup in an equivariant context. Orthogonal (commutative) ring spectra are definied in an analogous way to symmetric (commutative) ring spectra.

Units, space of, $GL_1(R)$: If R is an A_∞-ring spectrum, the space of units of R is defined to be the subspace $GL_1(R) \subset \Omega^\infty R$ which is the union of those components that represent an invertible element in the ring $\pi_0 R$.

Units, spectrum of, $gl_1(R)$: If R is an E_∞-ring spectrum, the space $GL_1(R)$ of units of R is naturally the zeroth space of a connective spectrum $gl_1(R)$, the spectrum of units of R.

2. Localization

Arithmetic square: Sullivan's arithmetic square is a way to recover a spectrum X from its p-completions and its $\mathbb{Q}$-localization. That is, X is equivalent to the homotopy limit of the diagram $\prod_p L_p X \to L_\mathbb{Q}(\prod_p L_p X) \leftarrow L_\mathbb{Q} X$. There is a similar arithmetic square when X is a space, in which X can be recovered as the homotopy limit of this diagram given the additional condition that X be nilpotent. More generally, the pullback of the arithmetic square is the $H\mathbb{Z}$-localization of a space.

Bousfield localization; E-localization: Bousfield localization of model categories is a homotopy-theoretic analogue of the usual localization of a category $\mathcal{C}$, with respect to a collection of morphisms I. The localization $C[I^{-1}]$ is the universal category receiving a functor from $\mathcal{C}$ and such that the image of the morphisms in I are all isomorphisms. Similarly, the Bousfield localization of a model category $\mathcal{C}$, with respect to a collection of morphisms I, is the universal model category receiving a left Quillen functor from $\mathcal{C}$, and such that image of the morphisms in I are all weak equivalences. For a given spectrum E, the Bousfield localization of spectra or spaces with respect to the collection of E-equivalences (that is, morphisms f such $E_*(f)$ is an isomorphism), is referred to as Bousfield localization with respect to E. The fibrant replacement in the resulting model category $X \to L_E X$ is then called the Bousfield localization of X with respect to E.

Bousfield-Kuhn functor: A functor Φ_n from spaces to spectra that factors $K(n)$-localization as $L_{K(n)} = \Phi_n \circ \Omega^\infty$. The existence of Φ_n implies that if two spectra have equivalent k-connected covers for some $k \in \mathbb{N}$, then their $K(n)$ localizations agree.

E-acyclic: A spectrum (or space) X is E-acyclic if the E-homology of X is zero. By definition, the Bousfield localization $L_E X$ of an E-acyclic spectrum X is contractible.

E-equivalence: A map of spectra (of spaces) $f\colon X \to Y$ is an E-equivalence if it induces an isomorphism in E-homology.

E-local: An E-local spectrum (or space) is a fibrant object of the Bousfield localized model category. A fibrant spectrum F is E-local iff for any E-acyclic spectrum X, the F-cohomology of X is zero. The Bousfield localization $L_E X$ of a spectrum X is initial (up to homotopy) amongst all E-local spectra receiving a map from X. Is is also terminal (up to homotopy) amongst all spectra Y equipped with an E-equivalence $X \to Y$.

E-nilpotent completion: For a ring spectrum E, the E-nilpotent completion of a spectrum X is the totalization of the cosimplicial spectrum $E^\bullet X$. In general, the E-based Adams spectral sequence for X converges to the homotopy of the E-nilpotent completion of X. If X is connective and $E = H\mathbb{F}_p$, then the E-nilpotent completion is the same as the p-completion.

Hasse square: Similar to the arithmetic square. It recovers the p-completion of an $E(2)$-local spectrum (e.g. and elliptic spectrum) from its $K(1)$- and $K(2)$-localizations, as the homotopy limit of the diagram $L_{K(2)}E \to L_{K(1)}L_{K(2)}E \leftarrow L_{K(1)}E$.

p-localization: The localization of spectra (or spaces) at a prime p is a particular case of Bousfield localization (in this case, with respect to the Eilenberg-MacLane spectrum $\mathrm{H}\mathbb{Z}_{(p)}$). If X is a spectrum (or a simply connected space), then $\pi_*(X_{(p)})$ is isomorphic to $\pi_* X \otimes \mathbb{Z}_{(p)}$.

p-completion: The p-completion – also called $\mathbb{Z}/p$-localization – of spectra (or spaces) at a prime p is a particular case of Bousfield localization (in this case, with respect to the mod p Moore spectrum $M(p)$). If X is a spectrum (or a simply connected space) with finitely presented homotopy groups, then $\pi_*(X_p)$ is isomorphic to $\pi_* X \otimes \mathbb{Z}_p$.

3. Orienations

$\widehat{A}$-genus: The $\widehat{A}$ genus is a $\mathbb{Q}$-valued genus of oriented manifolds given by $\int_M \widehat{A}(M)$, where $\widehat{A}(M) = \prod \frac{\sqrt{x_i}/2}{\sinh(\sqrt{x_i}/2)} \in H^*(M, \mathbb{Q})$ and the total Pontryagin class of M is factored as $\sum p_i(M) = \prod(1 + x_i)$ in some algebraic extension of $H^*(M, \mathbb{Q})$. If M is a spin manifold, then the $\widehat{A}$ is an integer. The $\widehat{A}$-genus of a spin manifold M is the image (mod torsoin) of $[M] \in \pi_n MSpin$ under the Atiyah-Bott-Shapiro orientation $MSpin \to ko$.

Complex oriented cohomology theory: Informally, a multiplicative cohomology theory E^* is called complex oriented if it admits a theory of Chern classes. More precisely, a complex orientation of E^* is a class $z \in E^2(\mathbb{C}P^\infty)$ whose restriction to $E^2(S^2) \cong E^0(S^0)$ along the standard inclusion $S^2 \cong \mathbb{C}P^1 \to \mathbb{C}P^\infty$ maps to the unit element. If E is 2-periodic, the orientation is often taken in degree 0 instead of degree 2. The multiplication on the topological group $\mathbb{C}P^\infty$ gives rise to a map

$$E^*[[z]] \cong E^*(\mathbb{C}P^\infty) \to E^*(\mathbb{C}P^\infty)\hat{\otimes}_{E_*}E^*(\mathbb{C}P^\infty) \cong E^*[[x,y]],$$

and the image of z under this map becomes a formal group law. Examples of complex oriented cohomology theories include singular cohomology, complex K-theory, MU, BP, $E(n)$, Morava K-theory, and several versions of elliptic cohomology.

Elliptic genus: Historically, the elliptic genus was introduced by Ochanine as a genus for oriented manifolds, taking values in the ring $\mathbb{Z}[1/2,\delta,\epsilon]$ (the ring of modular forms for $\Gamma_0(2)$), whose associated complex genus has logarithm $\int \frac{dt}{\sqrt{1-2\delta t^2+\epsilon t^4}}$. Landweber, Ravenel, and Stong subsequently showed that, after inverting Δ, the associated complex genus satisfies the criteria of the Landweber exact functor theorem, yielding a cohomology theory which they denoted Ell, and named "Elliptic Cohomology' (also known as $TMF_0(2)$, topological modular forms for $\Gamma_0(2)$). Today, a variety of genera associated with elliptic curves are called elliptic genera, and a variety of cohomology theories similarly associated to elliptic curves are referred to as elliptic (c.f. "Elliptic spectra").

Genus: A genus φ with values in a graded ring R_* is a map of graded rings $\varphi : MG_* \to R_*$, where MG_* denotes the bordism ring of manifolds with a G-structure on their stable normal bundle. In the case of $G = U$, the genus is called a complex genus.

Landweber exact; Landweber exact functor theorem: A p-typical formal group law over a ring R is classified by a map $BP_* \to R$. The formal group law is called Landweber exact if $X \mapsto BP_*(X)\otimes_{BP_*} R$ is a homology theory (the long exact sequence being the crucial point). Landweber's exact functor theorem characterizes Landweber exact formal group laws as those for which the images $(p, v_1, v_2, \dots, v_n)$ form a regular sequence for all $n \in \mathbb{N}_0$.

Orientation of a cohomology theory: A G-orientation of a multiplicative cohomology theory E with respect to a given a topological structure group G over the infinite-dimensional orthogonal group O is a compatible orientation of all G-vector bundles on any space with respect to E. This gives rise to a theory of characteristic classes for G-bundles in E-cohomology. Such an orientation can be described as a map of ring spectra $MG \to E$, where MG denotes the Thom spectrum associated to $BG \to BO$. Important examples of orientations include the SO-orientation and U-orientations of singular cohomology giving rise to Pontryagin resp. Chern classes, the $Spin$-orientation of real K-theory, and the $String$-orientation of TMF. Note that a *complex oriented cohomology theory* is the same thing as a cohomology theory with a U-orientation.

Orientation of a vector bundle for a group G: Given a topological group G with a morphism $G \to O(n)$, a G-orientation on an n-dimensional vector bundle V on a space X is a homotopy lift of the map $X \to BO(n)$ classifying the bundle V to $X \to BG$. Of particular importance are the groups $U(n)$, $SU(n)$, $SU\langle 6\rangle(n)$, $SO(n)$, $Spin(n)$, $String(n)$. Similarly, a stable G-orientation (where $G \to O$ is a group homomorphism to the infinite orthogonal group) is a lift of the map $X \to BO$ classifying the stabilization of V to $X \to BG$.

Orientation of a vector bundle with respect to a cohomology theory: An orientation of a vector bundle V on a space X with respect to a multiplicative cohomology theory E is a class $u \in E^n(X^V)$, the Thom class, whose restriction to any fiber is a unit in $E^n(\mathbb{R}^n) = \pi_0 E$. Multiplication by u yields the Thom isomorphism $E_{*+n}(X^V) \cong E_*(X)$. The latter can be described as a homotopy equivalence $E \wedge X^V \to \Sigma^n E \wedge X$ between the E-homology of the Thom space X^V and the shifted E-homology of X, at the spectrum level.

String group: The string group $String(n)$ is a group model of the 6-connected cover of the orthogonal group $O(n)$. Unlike $Spin(n)$, it is necessarily infinite-dimensional.

σ-orientation: The σ-orientation is the $String$-orientation of tmf. It is a map of E_∞ ring spectra $MString \to tmf$ that realizes the Witten genus at the level of homotopy groups.

Witten genus: The Witten genus of a string manifold M is the image of $[M] \in \pi_n MString$ under the σ-orientation $MString \to tmf$. The q-expansion of the corresponding modular form can be computed as $\prod_{i\geq 0}(1-q^i)^n \cdot \int_M \widehat{A}(M)ch(\bigotimes_{i\geq 1} Sym_{q^i}T_{\mathbb{C}})$, where ch is the Chern character, $T_{\mathbb{C}}$ is the complexified tangent bundle of M, and, given a vector bundle E, the expression $Sym_t E$ stands for $\sum_{i\geq 0} t^i Sym^i E$, a vector bundle valued formal power series. At a physical level of rigor, the Witten genus can be described as the S^1-equivariant index of the Dirac operator on the free loop space of M.

4. Misc. tools in stable homotopy

Adams condition: A technical hypothesis on a homotopy associative ring spectrum E which guarantees the existence of a universal coefficient spectral sequence. The condition is that E is a filtered colimit of finite spectra E_α, such that the E-cohomology of the Spanier-Whitehead dual E^*DE_α is projective over E_*, and for every E-module M, the map

$$M^*DE_\alpha \to \mathrm{Hom}_{E_*}(E_*DE_\alpha, M_*)$$

is an isomorphism.

Adams spectral sequence: The Adams spectral sequence is a spectral sequence which computes the homotopy groups of the p-nilpotent completion of a spectrum X from its cohomology:

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(X;\mathbb{F}_p), \mathbb{F}_p) \Longrightarrow \pi_{t-s}(X_{H\mathbb{F}_p}),$$

where $\mathcal{A}_p$ denotes the mod-p Steenrod algebra. The Adams spectral sequences converges conditionally in the sense of Boardman, implying that it converges strongly whenever the derived E_∞-term vanishes.

Adams-Novikov spectral sequence (generalized): A (generalized) Adams-Novikov spectral sequence is a variation of the classical Adams spectral sequence where mod-p cohomology is replaced by another (co-)homology theory. If E is a flat homotopy commutative ring spectrum which is either A_∞, or satisfies the Adams condition, then (E_*, E_*E) is a Hopf algebroid, and the E-based ANSS takes the form

$$E^2_{s,t} = \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*(X)) \Longrightarrow \pi_{t-s}(X_E),$$

where X_E denotes the E-completion of X and Ext is the derived functor of homomorphisms of E_*E-comodules. The Adams-Novikov spectral sequence can be seen as a Bousfield-Kan spectral sequence of the tower of spectra $\mathrm{Tot}[E^\bullet X]$ whose totalization is the E-completion.

Descent spectral sequence: The descent spectral sequence associated to a sheaf of spectra $\mathcal{F}$ over some space (or Grothendieck site) X computes the homotopy groups of $\mathcal{F}(X)$. Its E_2 page is the sheaf cohomology of X with coefficient in the homotopy sheaves of $\mathcal{F}$. In the case $X = \mathcal{M}_{ell}$ or $\overline{\mathcal{M}}_{ell}$ and $\mathcal{F} = \mathcal{O}^{top}$, the structure sheaf for TMF, this spectral sequence is also called the elliptic spectral sequence.

Dyer-Lashof algebra: (a.k.a. the big Steenrod algebra). At a prime p, the algebra of Dyer-Lashof algebra is the algebra that acts on the homotopy groups of any E_∞-$\mathrm{H}\mathbb{F}_p$-ring spectrum.

Flat ring spectrum: A homotopy commutative ring spectrum E is said to be flat if E_*E is flat over E_*.

Goerss-Hopkins obstruction theory: Given a flat homotopy commutative ring spectrum E which satisfies the Adams condition, a homotopy commutative E-complete ring spectrum A, and a simplicial resolution $\mathcal{O}_\bullet$ of the commutative operad, the Goerss-Hopkins obstruction defines a sequence of obstructions in the Quillen cohomology of simplicial $E_*\mathcal{O}_\bullet$-algebras in E_*E-comodules to refining the homotopy commutative ring structure on E to an E_∞-structure. More generally, it gives a framework to compute the homotopy groups of the moduli space of E_∞-structures in terms of the aforementioned Quillen cohomology.

Homotopy limit, homotopy colimit: The right (resp. left) derived functors of limit (resp. colimit) on the category of diagrams in a model category, with respect to objectwise weak equivalence.

Hopkins-Miller theorem, Goerss-Hopkins-Miller theorem: The original Hopkins-Miller theorem states that Morava E-theory E_n admits an A_∞ structure, and a point-set level action by the Morava stabilizer group. Subsequently the A_∞ obstruction theory utilized by Hopkins and Miller was refined by Goerss and Hopkins to an E_∞ obstruction theory, resulting in an E_∞ version of the Hopkins-Miller theorem commonly referred to as the Goess-Hopkins-Miller theorem.

Hypercover: A hypercover is a generalization of the Čech nerve

$$\coprod U_\alpha \leftleftarrows \coprod U_\alpha \times_X U_\beta \; \substack{\leftarrow\\ \leftarrow\\ \leftarrow} \; \coprod U_\alpha \times_X U_\beta \times_X U_\gamma \cdots$$

of a covering family $\{U_\alpha \to X\}$. It can be defined as a simpicial sheaf all of whose stalks are contractible Kan complexes.

Hyperdescent: A presheaf $\mathcal{F}$ satisfies hyperdescent if for any hypercover $U_\bullet$ of X, the value of $\mathcal{F}$ on X can be recovered as the homotopy limit of the cosimplicial object $\mathcal{F}(U_\bullet)$.

Injective model structure: For a model category $\mathcal{C}$ and a small category I, a model structure of the diagram category $\mathcal{C}^I$ can often (e.g. for combinatorial model categories $\mathcal{C}$) be defined by defining weak equivalences and cofibrations levelwise. This model structure is referred to as the injective model structure.

Jardine model structure: For a model category $\mathcal{C}$ and a Grothendiek site S, the Jardine model structure is a model structure on the category of $\mathcal{C}$-valued presheaves on S. It is the Bousfield localization of the injective model structure, where the weak equivalences are those morphisms that induce isomorphisms on homotopy sheaves. In this model structure, the fibrant objects satisfy hyperdescent.

Morava stabilizer group: The Morava stabilizer group $\mathbb{G}_n$ is the automorphism group of the unique formal group law of height n over $\overline{\mathbb{F}}_p$. It is a pro-finite group, and it is isomorphic to the maximal order in the central division algebra over $\mathbb{Q}_p$ with Hasse-invariant $1/n$.

Projective model structure: For a model category $\mathcal{C}$ and a small category I, a model structure of the diagram category $\mathcal{C}^I$ can often (e.g. for cofibrantly generated $\mathcal{C}$) be defined by defining weak equivalences and fibrations levelwise. This model structure is referred to as the projective model structure.

Quillen cohomology: Quillen cohomology is a generalization André-Quillen cohomology to model categories. If R is an object of a model category $\mathcal{C}$, and M is an abelian group in the overcategory $\mathcal{C}_{/R}$, then the Quillen cohomology of R with coefficients in M is given by the derived maps $\mathbb{R}\operatorname{Hom}_{\mathcal{C}_{/R}}(R, M)$. If $\mathcal{C}$ is the category of simplicial commutative rings, Quillen cohomology reduces to André-Quillen cohomology. If $\mathcal{C}$ is the category of spaces, Quillen cohomology is equivalent to usual singular cohomology (potentially with twisted coefficients).

θ-algebra: A θ-algebra is a $\mathbb{Z}_p$-algebra equipped with operators ψ^k for all $k \in \mathbb{Z}_p^\times$, ψ^p, and θ. The operators ψ^k and ψ^p are ring homomorphisms, and the operations ψ^k give a continuous action of the profinite group $\mathbb{Z}_p^\times$. The operator ψ^p is a lift of Frobenius, and the operator θ satisfies $\psi^p(x) = x^p + p\theta(x)$. The p-adic K-theory of an E_∞ algebra has the structure of a θ-algebra.

Type-n spectrum: A (usually finite) spectrum X is said to be of type n (at some given prime p) if its nth Morava K-theory, $K(n)_*(X)$, is nonzero while all smaller Morava K-theories are trivial. Every finite spectrum is of type n for some $0 \le n < \infty$.

5. Important examples of spectra

bo, *bso*, *bspin*, *bstring* — **connective covers of real K-theory:** The spectrum *bo* (also written *ko*) denotes connective real K-theory, i.e. the (-1)-connected cover of BO. The spectra *bso*, *bspin*, and *bstring* denote the covers of *bo* that are obtained by consecutively killing the next non-zero homotopy groups: π_1, π_2, and π_4.

BP — **Brown–Peterson spectrum:** When localized at a prime p, the complex cobordism spectrum MU splits as a wedge of spectra by the so-called Quillen idempotents. The summand containing the unit is called BP. It is a commutative ring spectrum itself (up to homotopy). The coefficient ring of BP is $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ with $|v_i| = 2p^i - 2$. The ring BP_* classifies *p-typical* formal group laws; over a torsion-free ring, a formal group law is p-typical if its logarithm series is of the form $x + \sum_{i=1}^{\infty} m_i x^{p^i}$. By a theorem of Cartier, any formal group law over a p-local ring is isomorphic to a p-typical one.

$BP\langle n\rangle$ — **n-truncated Brown-Peterson spectrum:** The notation $BP\langle n\rangle$ stands for any BP-module spectrum with $\pi_* BP\langle n\rangle \cong BP_*/(v_{n+1}, v_{n+2}, \dots)$. They are complex oriented and classify p-typical formal group laws of height bounded by n or infinity. The spectrum $BP\langle 1\rangle$ is the Adams summand of connective K-theory.

$E(n)$ — **Johnson-Wilson spectrum:** For a (fixed, not notated) prime p, the ring spectrum $E(n)$ is a Landweber exact spectrum with coefficients $E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n, v_n^{-1}]$ with $|v_i| = 2p^i - 2$. Explicitly, $E(n)$-homology can be defined by $E(n)_*(X) = BP_*(X) \otimes_{BP_*} E(n)_*$ for the map $BP_* \to E(n)_*$ that sends v_i to the class of the same name for $i \le n$ and to 0 for $i > n$. The Lubin-Tate spectrum E_n can be obtained from $E(n)$ by completing and performing a ring extension; those two spectra belong to the same Bousfield class.

E_n **or** $E(k,\Gamma)$ — **Morava E-theory, aka Lubin-Tate spectrum:** For k a field of characteristic p and Γ a 1-dimensional formal group of height n over k, the Morava E-theory spectrum $E(k,\Gamma)$ is an E_∞-ring spectrum such that $\pi_0 E(k,\Gamma)$ is isomorphic to the universal deformation ring $A(k,\Gamma) \cong W(k)[[u_1, \dots, u_{n-1}]]$ constructed by Lubin-Tate. Morava E-theory is complex-orientable, even-periodic, and Landweber exact; its associated formal group is the universal deformation of Γ. In the case where k is the algebraic closure of the field with p elements, the Morava E-theory spectrum $E(\overline{\mathbb{F}}_p, \Gamma)$ is often abbreviated E_n. (cf entry on Universal deformation, and on Witt vectors.) Morava E-theory is closely related to $L_{K(2)}TMF$, and to the restriction of the sheaf $\mathcal{O}^{top}$ to the locus $\mathcal{M}_{ell}^{ss}$ of supersingular elliptic curves.

EO_n — **higher real K-theory:** The higher real K-theory spectra are the homotopy fixed points of the action on the nth Morava E-theory E_n of a maximal finite subgroup of the Morava stabilizer group. This construction is a consequence of the Hopkins-Miller theorem. At the primes 2 and 3, there is an equivalence $L_{K(2)}TMF \simeq EO_2$.

eo_2 — **p-local topological modular forms:** This is an older name for the p-localization of connective topological modular forms, *tmf* $(p = 2, 3)$. Its notation is in analogy with BO and bo (BO being the periodization of the connective real K-theory spectrum bo).

ko**,** ku**,** KO**,** KU**,** K — **K-theory spectra:** The spectra ko and ku are the connective (i.e. (-1)-connected) covers of KO and $KU = K$. The spectrum KU of complex K-theory is 2-periodic with $(KU)_{2n} = \mathbb{Z} \times BU$ and $(KU)_{2n+1} \cong U$, with one structure map $U \to \Omega(\mathbb{Z} \times BU)$ being the standard equivalence and the other $\mathbb{Z} \times BU \to \Omega U$ given by the Bott periodicity theorem. If X is compact, the group $KU^0(X)$ can be geometrically interpreted as the Grothendieck group of complex vector bundles on X. The spectrum KO is the real equivalent of KU. It is 8-periodic with coefficients $KO_* = \mathbb{Z}[\eta, \mu, \sigma, \sigma^{-1}]$ with $|\eta| = 1$, $|\mu| = 4$, $|\sigma| = 8$, and $2\eta = \eta^3 = \mu\eta = 0$ and $\mu^2 = 4\sigma$.

$K(n)$ — **Morava K-theory:** The nth Morava K-theory at a prime p (p is not included in the notation). $K(n)$ is a complex orientable cohomology theory whose associated formal group is the height n Honda formal group. The coefficient ring $K(n)_* \cong \mathbb{F}_p[v_n^{\pm 1}]$ is a Laurent polynomial algebra on a single invertible generator in degree $2(p^n - 1)$. The generator $v_n \in \pi_{2(p^n-1)} K(n)$ is the image of an element with same name in $\pi_{2(p^n-1)} MU$.

MU — **complex bordism:** MU is the spectrum representing complex cobordism, the cobordism theory defined by manifolds with almost complex structures and bordisms between them, with compatible almost complex structures. MU is a Thom spectrum. MU_* is the Lazard ring, which carries the universal formal group law.

MP — periodic complex bordism: $MP = \bigvee_{n\in\mathbb{Z}} \Sigma^{2n} MU$ is the periodic version of MU, in which we add an invertible element of degree 2. This represents periodic complex cobordism.

$MU\langle 6\rangle$: The ring spectrum $MU\langle 6\rangle$ represent cobordism of manifolds with trivializations of the first and second Chern classes. As a spectrum, it can be constructed as the Thom spectrum over $BU\langle 6\rangle$, the 6-connected cover of BU. The spectrum *tmf* of topological modular forms is oriented with respect to $MU\langle 6\rangle$; this orientation corresponds to the unique cubical structure on every elliptic curve.

$MO\langle 8\rangle$ — string cobordism: The ring spectrum $MO\langle 8\rangle$ represents cobordism of string manifolds, which are spin manifolds equipped with a trivialization of $\frac{1}{2}p_1 \in H^4(M, \mathbb{Z})$, the latter being the pullback of the generator $\frac{1}{2}p_1 \in H^4(BSpin) = \mathbb{Z}$. As a spectrum, it can be constructed as the Thom spectrum over $BO\langle 8\rangle$, the 8-connected cover of BO. The spectrum *tmf* of topological modular forms is oriented with respect to $MO\langle 8\rangle$ (this refines the $MU\langle 6\rangle$-orientation of *tmf*), and that orientation is a topological incarnation of the Witten genus.

$X(n)$ — Ravenel spectrum: The spectra $X(n)$ are defined as Thom spectra of $\Omega SU(n) \to \Omega SU \to BU$, where the second map is the Bott isomorphism. They play a role in the proof of the nilpotence theorem and in constructing a complex oriented theory $A = TMF \wedge X(4)$ classifying elliptic curves with a parameter modulo degree 5, or equivalently, Weierstrass parameterized elliptic curves.

6. Commutative algebra

Adic rings: An adic Noetherian ring A is a topological Noetherian ring with a given ideal $I \subset A$, the ideal of definition, such that the map $A \to \varprojlim A/I^n$ is an isomorphism, and the topology on A is the I-adic topology. Adic rings are the local buildings blocks for formal schemes just as rings are the local buildings blocks for schemes. The functor that assings to an adic ring A, with ideal of definition I, the pro-ring $\{A/I^n\}$ embeds the category of adic rings, with continuous rings maps as morphisms, as a full subcategory of pro-rings: $\mathrm{Hom}_{\mathrm{Adic}}(A, B) \cong \varprojlim_m \varinjlim_n \mathrm{Hom}(A/I^n, B/J^m)$.

André-Quillen cohomology: The André-Quillen cohomology of a commutative ring R with coefficients in an R-module M, is defined as a derived functor of derivations of R into M, $\mathbb{R}\,\mathrm{Der}(R, M)$. This can also be expressed in terms of the cotangent complex of R, L_R, in that there is a natural equivalence $\mathbb{R}\,\mathrm{Hom}_R(L_R, M) \simeq \mathbb{R}\,\mathrm{Der}(R, M)$.

Cotangent complex: (Also called André-Quillen homology.) The cotangent complex L_R of a commutative ring R is the left derived functor of Kähler differentials, Ω^1. If the ring R is smooth, then there is an equivalence $L_R \simeq \Omega^1_R$.

Étale morphism: A map of commutative rings $S \to R$ is étale if it is flat and unramified. Equivalently, the map is étale if and only if the relative cotangent complex $L_{R|S}$ is trivial and R is a finitely presented S-algebra. The conditions "flat and unramified" essentially mean that the map is a local isomorphism (perhaps after base-change), and we should think of étale maps as finite covering maps.

Flat morphism; faithfully flat morphism: A map of commutative rings $S \to R$ is flat if R is a flat S-module, i.e., the functor $R \otimes_S (-)$ is exact. Localizations at ideals are flat. The map is faithfully flat if, additionally, the functor is conservative, meaning that $R \otimes_S M$ is zero if and only if M is zero. Adjoining roots of monic polynomials is a faithfully flat operation.

Kähler differentials: For R a commutative k-algebra, the R-module $\Omega^1_{R|k}$ of Kähler differentials (a.k.a. 1-forms) can be presented as the free R-module on symbols da, $a \in R$, subject to the relations that $d(ab) = adb + bda$ and $da = 0$ for $a \in k$. There is a natural isomorphism $\Omega^1_{R|k} \simeq I/I^2$, where I is the kernel of the multiplication map $R \otimes_k R \to R$. $\Omega^1_{R|k}$ has the important property that it corepresents the functor of derivations; there is a universal derivation $R \to \Omega^1_{R|k}$ that induces an isomorphism $\mathrm{Der}_k(R, M) \cong \mathrm{Hom}_R(\Omega^1_{R|k}, M)$ for any R-module M.

Witt vectors: The Witt vector functor associates to a ring R a new ring $W(R)$ which has R as a quotient and acts as a universal deformation in many cases. In particular, the Witt vectors of a finite field k of characteristic p are a complete local ring with residue field k; for instance, $W(\mathbb{F}_p) \cong \mathbb{Z}_p$.

7. Algebraic geometry, sheaves and stacks

Additive formal group $\mathbb{G}_a$: The additive formal group is the affine 1-dimensional formal group scheme $\mathbb{G}_a = \mathrm{Spf}(\mathbb{Z}[[t]])$ with comultiplication given by $t \mapsto t \otimes 1 + 1 \otimes t$. It is the completion at 0 of the additive group scheme denoted by the same symbol. Topologically, the additive formal group (over a field) arises as the formal group associated with singular cohomology with coefficients in that field.

Deligne-Mumford compactification: The Deligne-Mumford compactification of the stack $\mathcal{M}_g$ of smooth curves of genus g is the stack $\overline{\mathcal{M}}_g$ obtained by allowing certain singularities in those curves: those with at most nodal singularities, and finite automorphism group. The latter are known as *stable curves*.

Étale topology: This is the Grothendieck topology on the category of schemes in which a family $\{f_\alpha : X_\alpha \to X\}$ is covering if the maps f_α are étale and if $\coprod_\alpha X_\alpha(k) \to X(k)$ is surjective for every algebraically closed field k.

étale site (small): Given a scheme (or stack) X, the small étale site of X is the full subcategory of schemes (or stacks) over X whose reference map to X is étale, equipped with the étale Grothendieck topology.

Finite morphism: A morphism of stacks (or schemes) $X \to Y$ is finite if there is a étale cover $\mathrm{Spec}\, S \to Y$ such that $\mathrm{Spec}\, S \times_Y X = \mathrm{Spec}\, R$ is an affine scheme with R finitely generated as an S-module. A morphism is finite iff it is representable, affine, and proper.

Finite type: A morphism of stacks (or schemes) $X \to Y$ is of finite type if there is a cover $\mathrm{Spec}\, S \to Y$ and a cover $\mathrm{Spec}\, R \to \mathrm{Spec}\, S \times_Y X$ such that R is finitely generated as an S-algebra.

Formal spectrum Spf**:** The formal spectrum $\mathrm{Spf}\, A$ of an I-adic Noetherian ring A consists of a topological space with a sheaf of topological rings $(\mathrm{Spf}\, A, \mathcal{O})$. The topological space has points given by prime ideals that contain I, with generating opens $U_x \subset \mathrm{Spf}\, A$ the set of prime ideals not containing an element x of A. The value of the sheaf $\mathcal{O}$ on these opens is $\mathcal{O}(U_x) = A[x^{-1}]^\wedge_I$, the completion of the ring of fractions $A[x^{-1}]$ at the ideal $I[x^{-1}]$.

Formal scheme: A formal scheme is topological space with a sheaf of topological rings, that is locally equivalent to $\mathrm{Spf}\, A$ for some adic Noetherian ring A. The category of affine formal schemes is equivalent to the opposite category of adic Noetherian rings. Formal schemes often arise as the completions, or formal neighborhoods, of a subscheme $Y \subset X$ inside an ambient scheme, just as the completion of a Noetherian ring with respect to an ideal has the structure of an adic ring. Formal schemes embed as a full subcategory of ind-schemes by globalizing the functor that assigns to an adic ring the associated pro-ring.

Formal group: A formal group is a group object in the category of formal schemes. An affine formal group being the same as a cogroup in the category of adic rings, it is thus a certain type of topological Hopf algebra. A 1-dimensional (commutative) formal group over a ring R is a (commutative) formal group whose underlying formal schemes is equivalent to $\mathrm{Spf}\, R[[t]]$ – sometimes this last condition only étale locally in R.

Formal group law: A 1-dimensional formal group law over a commutative ring R is a cocommutative cogroup structure on $R[[t]]$ in the category of adic R-algebras. I.e., it has a commutative comultiplication $R[[t]] \to R[[t]] \widehat{\otimes}_R R[[t]] \cong R[[x, y]]$. This comultiplication is determined by the formal power series that is the image of the element t, so formal group laws are often specified by this single formal power series. A 1-dimensional formal group law is equivalent to the data of a formal group G together with a specified isomorphism $G \cong \operatorname{Spf} R[[t]]$, i.e., a choice of coordinate t on G.

Grothendieck site: A category with a Grothendieck topology.

Grothendieck topology: A Grothendieck topology on a category $\mathcal{C}$ – sometimes also called a Grothendieck pretopology – consists of a distinguished class of families of morphisms $\{X_\alpha \to X\}$, called a covering families, subject to the following conditions: 1. base changing a covering family along any map $Y \to X$ should remain a covering family, 2. if $\{X_\alpha \to X\}$ is a covering family and for every α, $\{X_{\beta\alpha} \to X_\alpha\}$ is a covering family, then the family of composites $\{X_{\beta\alpha} \to X\}$ should also be a covering family. A primary example is the étale topology on the category of schemes.

Group scheme: A group scheme is group object in the category of schemes. Algebraic groups, such as GL_n or SL_n, form a particular class of group schemes. Elliptic curves and, more generally, abelian varieties are also group schemes.

Height: For a homomorphism of formal groups defined over a field of characteristic $p > 0$, say $f : G \to G'$, f may be factorized $G \xrightarrow{(-)^{p^n}} G \to G'$ through a p^n power map, i.e. the n-fold iteration of the Frobenius endomorphism of G. The height of the map f is the maximum n for which such a factorization exists, and is ∞ exactly when $f = 0$. The height of a formal group G is defined as the height of the multiplication by p-map $[p] : G \to G$.

Hopf algebroid: A (commutative) Hopf algebroid A is a cogroupoid object in the category of commutative rings, just as a commutative Hopf algebra is a cogroup object in the category of commutative rings. In other words, if A_0 is a commutative ring, the additional structure of a Hopf algebroid on A_0 is a choice of lift of the functor $\operatorname{Hom}(A_0, -) : \text{Rings} \to \text{Sets}$ to the category of groupoids: A_0 corepresents the objects of the groupoid, and the extra structure provided by the lift amounts to having another ring A_1 that corepresents the morphisms, together with the data of various maps between A_0 and A_1. For a ring spectrum E, the pair (E_*, E_*E) frequently defines a Hopf algebroid. Every Hopf algebroid $A = (A_0, A_1)$ has an associated stack, $\mathcal{M}_A$, defined by forcing the groupoid-valued functor to satisfy descent. In the example of $A = (MP_0, MP_0MP)$, where MP is periodic complex bordism, the associated stack $\mathcal{M}_A$ is the moduli stack of formal groups, $\mathcal{M}_{FG}$.

Hopf algebroid comodule: For a Hopf algebroid A, there is a notion of an A-comodule M, which is roughly a left A_1-comodule in the category of A_0-modules. The category of A-comodules is equivalent to the category of quasicoherent sheaves on the associated stack $\mathcal{M}_A$.

Hopf algebroid cohomology: The nth cohomology of a Hopf algebroid $A = (A_0, A_1)$ with coefficients in an A-comodule M is the nth derived functor of the functor that sends M to $\operatorname{Hom}_{(A_0,A_1)}(A_0, M)$.

Moduli stack of formal groups, $\mathcal{M}_{FG}$: The R-valued points of the stack $\mathcal{M}_{FG}$ are the groupoid of formal groups over R and their isomorphisms. The stack $\mathcal{M}_{FG}$ is the stack associated to the Lazard Hopf algebroid $(L, \Gamma) = (MP_0, MP_0MP)$. The invariant differential on a formal group defines a line bundle ω on $\mathcal{M}_{FG}$, and the E_2-term of the Adams-Novikov spectral sequence can be understood as the stack cohomology group $E_2^{s,2t} = H^s(\mathcal{M}_{FG}, \omega^t)$.

Moduli space of formal group laws, $\mathcal{M}_{FGL}$: A formal group law is a formal group along with a choice of coordinate. The moduli space of formal group laws is the scheme $\operatorname{Spec}(L)$, where $L = MP_0$ is the Lazard ring.

Multiplicative formal group $\mathbb{G}_m$: The multiplicative formal group is the affine 1-dimensional formal group scheme $\mathbb{G}_m = \operatorname{Spf}(\mathbb{Z}[[t]])$ with comultiplication given by $t \mapsto t \otimes 1 + 1 \otimes t + t \otimes t$. It is the completion at 1 of the multiplicative group scheme $\operatorname{Spec}(\mathbb{Z}[u, u^{-1}])$. Topologically, the multiplicative formal group arises as the formal group associated with complex K-theory.

p-divisible group: (Also called Barsotti-Tate groups.) An algebraic group G is a p-divisible group of height n if: the multiplication map $p^i : G \to G$ is surjective; the group $G[p^i] : \operatorname{Ker}(G \xrightarrow{p^i} G)$ is commutative, finite, and flat of rank p^{ni}; the natural map $\varinjlim G[p^i] \to G$

is an isomorphism. Every elliptic curve C defines an associated p-divisible group $C[p^\infty] := \varinjlim C[p^i]$, where $C[p^i]$ is the kernel of $p^i : C \to C$. The Serre-Tate theorem relates the deformation theory of the elliptic curve to that of its p-divisible group.

Proper morphism: A morphism of stacks $X \to Y$ is proper if the following two conditions hold. 1. (separated): For any complete discrete valuation ring V and fraction field K and any morphism $f : \operatorname{Spec} V \to Y$ with lifts $g_1, g_2 : \operatorname{Spec} V \to X$ which are isomorphic when restricted to $\operatorname{Spec} K$, then the isomorphism can be extended to an isomorphism between g_1 and g_2. 2. (proper): for any map $\operatorname{Spec} V \to Y$ which lifts over $\operatorname{Spec} K$ to a map to X, there is a finite separable extension K' of K such that the lift extends to all of $\operatorname{Spec} V'$ where V' is the integral closure of V in K'.

Relative-dimension-zero morphism: A representable morphism $X \to Y$ of stacks (or schemes) has relative dimension zero if all of its fibers have Krull dimension 0.

Relative Frobenius: If S is a scheme over $\mathbb{F}_p$ and $X \to S$ a map of schemes, there are compatible absolute Frobenius maps $F\colon S \to S$ and $F\colon X \to X$ obtained locally by the pth power map. The relative Frobenius $F_{S/X}\colon X \to X(1) = X \times_S S$ is the corresponding map into the fiber product, which is taken using the projection $X \to S$ and the absolute Frobenius $S \to S$.

Representable morphism: A morphism of stacks $f : X \to Y$ is representable if for any map $\operatorname{Spec} R \to Y$, the fiber product $\operatorname{Spec} R \times_Y X$ is representable, i.e., it is equivalent to a scheme. It is called representable and affine, if in addition all the schemes $\operatorname{Spec} R \times_Y X$ are affine.

Stack: A stack is a groupoid-valued functor $\mathcal{F}$ on the category of commutative rings that satisfies descent. The descent property is a generalization of the sheaf property. It says that whenever $\operatorname{Spec}(R) \to \operatorname{Spec}(S)$ is an étale cover, the diagram

$$\mathcal{F}(R \otimes_S R \otimes_S R) \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, \mathcal{F}(R \otimes_S R) \leftleftarrows \mathcal{F}(R) \leftarrow \mathcal{F}(S)$$

exhibits $\mathcal{F}(S)$ as a 2-categorical limit. A standard example is the stack BG, that assigns R the groupoid of principal G-bundles over $\operatorname{Spec}(R)$. Another standard example is the stack $\mathcal{M}_{ell}$, than assigns R the groupoid of elliptic curves over R.

Stack, Deligne-Mumford: A Deligne-Mumford stack is a stack that is locally affine in the étale topology. That is, it is a stack X for which there exists an étale cover $\operatorname{Spec} R \to X$ by an affine scheme. One often also imposes a quasicompactness condition. Deligne-Mumford stacks are the most gentle kinds of stacks and almost all notions that make sense for schemes also make sense for Deligne-Mumford stacks.

Universal deformation (of a formal group): Fix a perfect field k of positive characteristic p, e.g. k could be any finite or algebraically closed field, and a formal group Γ over k of finite height $1 \le n < \infty$. For every Artin local Ring R with residue field k, a deformation of Γ to R is a formal group $\tilde{\Gamma}$ over R together with an isomorphism $\tilde{(\Gamma)} \otimes_R k \simeq \Gamma$. Lubin and Tate determine the deformation theory of Γ by showing that there is a complete local ring R^{univ} such that for every R as above, the set of isomorphism calsses of defomations of Γ to R naturally biject with continous ring homomorphisms from R^{univ} to R. Even more, they prove that R^{univ} is noncanonically isomorphis to a power series ring $R^{univ} \simeq W(k)[[u_1, \dots, u_{n-1}]]$ over the ring of Witt vectors of k.

8. Elliptic curves and their moduli

Discriminant: The discriminant of the elliptic curve $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ is given by $\Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6$, where $b_2 = a_1^2 + 4a_2$, $b_4 = 2a_4 + a_1a_3$, $b_6 = a_3^2 + 4a_6$, $b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2$. Over a field, the discriminant vanishes if and only if the elliptic curve is singular.

Elliptic curve: An elliptic curve C over a ring R is a smooth, projective curve of genus one together with a marked point, i.e., a map $\operatorname{Spec} R \to C$. An elliptic curve has a natural group structure, which can be completed to give a formal group over R.

Elliptic curve, generalized: An algebraic geometer would mean a curve locally given by a Weierstrass equation with one of Δ and c_4 not vanishing at any given point of the base; this is the notion origianlly coined by Deligne and Rapoport and these curves assemble into a proper and smooth Deligne-Mumford stack $\overline{\mathcal{M}_{ell}}$ of relative dimension one over $\mathbb{Z}$. In topology, we consider even more generalized curves: all those given locally by Weierstrass-equation. The resulting stack over $\mathbb{Z}$ is not a Deligne-Mumford stack anymore but an Artin-stack, because the additional curve $y^2 = x^3$ admits non-trivial infinitesimal automorphisms. This point is the only one carrying an additive formal group law which makes it of outstanding topological interest since it is the only point which "knows" about singular (mod p) cohomology.

Elliptic curve, ordinary: An ordinary curve is an elliptic curve over a field of characteristic $p > 0$ whose associated formal group law has height 1.

Elliptic curve, supersingular: A supersingular curve is an elliptic curve over a field of characteristic $p > 0$ whose associated formal group law has height 2.

Hasse invariant: For a fixed prime p, the Hasse invariant is a global section $H \in H^0(\overline{\mathcal{M}}_{ell} \otimes_{\mathbb{Z}} \mathbb{F}_p, \omega^{p-1})$, i.e. a mod p modular form of weight $p-1$. It admits a lift to characteristic 0 (as an Eisenstein series) exactly if $p \neq 2, 3$. The vanishing locus of H are the super-singular points (all with multiplicity 1).

Invariant differentials/canonical bundle: The canonical bundle ω over $\mathcal{M}_{ell}$ (or $\overline{\mathcal{M}}_{ell}$, or $\overline{\mathcal{M}}_{ell}^{+}$) is the sheaf of (translation invariant) relative differentials for the universal elliptic curve over $\mathcal{M}_{ell}$. The stalk of ω at an elliptic curve $C \in \mathcal{M}_{ell}$ is the 1-dimensional vector space of Kähler differentials on C. The sections of $\omega^{\otimes 2k}$ over $\overline{\mathcal{M}}_{ell}$ are modular forms of weight k (for odd n, the line bundle $\omega^{\otimes n}$ has no sections).

j-invariant: The j-invariant of the elliptic curve $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ is given by $(b_2^2 - 24b_4)^3/\Delta$, where b_2, b_4, and Δ are as above. Over an algebraically closed field, the j-invariant is a complete isomorphism invariant of the elliptic curve. More geometrically, the j-invariant is a map $\overline{\mathcal{M}_{ell}} \to \mathbb{P}^1_{\mathbb{Z}}$ which exhibits the projective line as the coarse moduli space of the Deligne-Mumford compactification of $\mathcal{M}_{ell}$.

Level structure: A level structure on an elliptic curve C can refer to either: 1. (a $\Gamma(N)$-structure) an isomorphism between $(\mathbb{Z}/N)^2$ and the group $C[N]$ of N-torsion points of C. 2. (a $\Gamma_1(N)$-structure) an injective homomorphism $\mathbb{Z}/N \to C[N]$. 3. (a $\Gamma_0(N)$-structure) a choice of subgroup of $C[N]$ that is isomoprhic to $\mathbb{Z}/N$. Moduli spaces of elliptic curves with level structures provide examples of stacks over $\mathcal{M}_{ell}$ (or $\overline{\mathcal{M}}_{ell}$) on which one can evaluate $\mathcal{O}^{top}$, the structure sheaf for TMF.

Modular form; weight: A modular form of weight k is a section of $\omega^{\otimes 2k}$ over $\overline{\mathcal{M}}_{ell}$. When restricted to a formal neighborhood of the multiplicative curve $\mathbb{G}_m \in \overline{\mathcal{M}}_{ell}$, the canonical bundle ω trivializes, and one can identify a modular form with an element of $\mathbb{Z}[[q]]$.

Moduli stacks of elliptic curves:

- **$\mathcal{M}_{ell}$:** Also denoted $\mathcal{M}_{1,1}$ in the algebraic geometry literature. The moduli stack of smooth elliptic curves.
- **$\mathcal{M}_{ell}^{ord}$:** The substack of the moduli stack of elliptic curves $\mathcal{M}_{ell}$ over $\mathbb{F}_p$ consisting of ordinary elliptic curves, whose associated formal group has height one. The coarse moduli space $\mathcal{M}_{ell}^{ord}$ at a prime p is a disk with punctures corresponding to the number of supersingular elliptic curves at p.
- **$\mathcal{M}_{ell}^{ss}$:** The substack of the moduli stack $\mathcal{M}_{ell}$ over $\mathbb{F}_p$ consisting of supersingular elliptic curves, whose associated formal group has height two. At a prime p, $\mathcal{M}_{ell}^{ss}$ is a disjoint union of stacks of the form $BG = */G$, where G is the group of automorphisms of a supersingular elliptic curve. Thus, the associated coarse moduli space is a disjoint union of points.
- **$\overline{\mathcal{M}}_{ell}$:** The moduli stack of elliptic curves, possibly with nodal singularities. This is the Deligne-Mumford compactification of the moduli stack of smooth elliptic curves $\mathcal{M}_{ell}$.

$\mathcal{M}_{Weier}$**:** (also denoted $\overline{\mathcal{M}}_{ell}^{+}$) The moduli stack of Weierstrass elliptic curves, associated to the Weierstrass Hopf algebroid. Includes curves with both nodal and cuspidal singularities.

Serre-Tate theorem: The Serre-Tate theorem relates the deformation theory of an elliptic curve to that of its associated p-divisible group. As a consequence, if C is a supersingular elliptic curve, then the map $\mathcal{M}_{ell} \to \mathcal{M}_{FG}$ induces an isomorphism of the formal neighborhood of C in $\mathcal{M}_{ell}$ with the formal neighborhood of the associated formal group $\widehat{C}$ in $\mathcal{M}_{FG}$. More generally, there is a map from $\mathcal{M}_{ell}$ to the moduli stack of p-divisible groups, and this induces an isomorphism of a formal neighborhood of any elliptic curve C with the formal neighborhood of the point given by the associated p-divisible group $C[p^\infty]$. Remark that the p-divisible group $C[p^\infty]$ (which governs the deformation theory of C) is formal if and only if C is supersingular.

Weierstrass curve, Weierstrass form: A Weierstrass curve (or a curve in Weierstrass form) is an affine curve with a parametrization of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

or its projective equivalent. A Weierstrass curve defines an elliptic curve if and only if its discriminant, a polynomial of the a_i, is invertible. Over a field, any elliptic curve can be expressed in Weierstrass form by the Riemann-Roch theorem. More generally, this is true Zariski-locally over any ring, i.e. if R is a ring and C/R is an elliptic curve, there exist elements $r_1, \ldots, r_n \in R$ such that $r_1 + \ldots + r_n = 1$ and for every i, the elliptic curve $C \otimes_R R[\frac{1}{r_i}]$ admits a Weiserstrass equation.

9. Spectra of topological modular forms

Elliptic spectrum: A triple (E, C, α) where E is a weakly even periodic ring spectrum, C is an elliptic curve over $\pi_0 E$, and $\alpha : \mathbb{G}_E \xrightarrow{\cong} \widehat{C}$ is an isomorphism between the formal group of E and the formal group of C.

Elliptic spectral sequence: This can refer to any one of the following spectral sequences: $H^q(\mathcal{M}_{ell}, \omega^p) \Rightarrow \pi_{2p-q}(\mathit{TMF})$, $H^q(\overline{\mathcal{M}}_{ell}, \omega^p) \Rightarrow \pi_{2p-q}(\mathit{Tmf})$, and $H^q(\overline{\mathcal{M}}_{ell}^{+}, \omega^p) \Rightarrow \pi_{2p-q}(\mathit{tmf})$. The first two are examples of descent spectral sequences. The last one is the Adams-Novikov spectral sequence for tmf, and it is not a descent spectral sequence.

$\mathcal{O}^{top}$**, the structure sheaf for** TMF**:** is a sheaf of E_∞-ring spectra on the small étale site of $\overline{\mathcal{M}}_{ell}$, i.e., the site whose objects are stacks equipped with an étale map $\overline{\mathcal{M}}_{ell}$, and whose covering families are étale covers (strictly speaking, this is a 2-category, by it is actually equivalent to a 1-category). The corresponding sheaf over the stack $\overline{\mathcal{M}}_{ell}^{+}$ does not seem to exist, but if it existed its value on $\overline{\mathcal{M}}_{ell}^{+}$ would be tmf.

TMF**, periodic topological modular forms:** The spectrum TMF is the global sections of the sheaf $\mathcal{O}^{top}$ of E_∞-ring spectra over $\mathcal{M}_{ell}$. In other words, it is the value of $\mathcal{O}^{top}$ on $\mathcal{M}_{ell}$. Since $\mathcal{M}_{ell}$ is the open substack of $\overline{\mathcal{M}}_{ell}$ where the discriminant is invertible, there is a natural equivalence $\mathit{TMF} \simeq \mathit{Tmf}[\Delta^{-1}]$ (TMF is also equivalent to $\mathit{tmf}[\Delta^{-1}]$). Note that this is a slight abuse of notation: it is better to write $\mathit{Tmf}[(\Delta^{24})^{-1}]$ (and $\mathit{tmf}[(\Delta^{24})^{-1}]$), as only $\Delta^{24} \in \pi_{576}(\mathit{Tmf})$ survives the descent spectral sequence.

Tmf**:** This is the global section spectrum of the sheaf $\mathcal{O}^{top}$ on $\overline{\mathcal{M}}_{ell}$. In positive degrees, its homotopy groups are rationally equivalent to the ring $\mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta)$ of classical modular forms. The negative homotopy groups of Tmf are related to those in positive degree by $\pi_{-n}\mathit{Tmf} \cong \mathrm{Free}(\pi_{n-21}\mathit{Tmf}) \oplus \mathrm{Tors}(\pi_{n-22}\mathit{Tmf})$.

tmf**, connective topological modular forms:** This is the connective cover of the spectrum Tmf of global sections of $\mathcal{O}^{top}$ on $\overline{\mathcal{M}}_{ell}$. Its homotopy groups are rationally equivalent to the ring $\mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta)$ of classical modular forms. Apart from its $\mathbb{Z}$-free part, $\pi_*(\mathit{tmf})$ also contains intricate patters of 2- and 3-torsion, that approximate rather well the $K(2)$-localizations of the sphere spectrum at those primes.

TMF**, localizations of:** The $K(1)$-localization $L_{K(1)}\,TMF$ is the spectrum of sections of $\mathcal{O}^{top}$ over the ordinary locus $\mathcal{M}_{ell}^{ord}$, while the $K(2)$-localization $L_{K(2)}\,TMF$ is the spectrum of sections of $\mathcal{O}^{top}$ over the supersingular locus $\mathcal{M}_{ell}^{ss}$. The latter is a product of various quotients of the Lubin-Tate spectra E_2, indexed by the finite set of isomorphism classes of supersingular elliptic curves. At the primes 2 and 3, there is only one supersingular elliptic curve, and $L_{K(2)}\,TMF \simeq EO_2$.