“Parallel” transport - revisited

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Abstract

Inspired by recent extensions in the smooth setting of parallel transport to representations of $Sing_{smooth}(B)$ on a smooth fire bundle, I revisit the development of a notion of ‘parallel’ transport in the topological setting of fibrations with the homotopy lifting property and then extend it to representations of $Sing(B)$ on such fibrations.

1 Introduction

Inspired by Block-Smith [**?**] and Igusa (arXiv:0912.0249), Abad and Schaetz [**?**] showed that smooth parallel transport (e.g. as in [**?**, **?**]) can be derived from the A_∞ version of de Rham's theorem due to Gugenheim [**?**]; this allows them to extend parallel transport to an A_∞ functor. Many years ago in a journal not readily available [**?**], I showed that in the topological setting of fibrations (satisfying the homotopy lifting property) there was a notion of ‘parallel’ transport not dependent on having a connection. For my purposes, it was sufficient to consider transport along based loops in the base, though the arguments were sufficient to allow for transport along any path in the base. Instead of the property in the smooth case of giving a holonomy homomorphism from based loops (suitably defined), the parallel transport I constructed gave ‘only’ an A_∞-map.

Recent papers have looked at what is known as the ∞-groupoid $\Pi_\infty(B)$ of a space B and its *representations up to homotopy* on a fibre bundle $E \to B$. The ∞-groupoid $\Pi_\infty(B)$ can be represented as a simplicial set: the singular complex $Sing(B)$. Here we set out to extend known results to the fibrations setting, i.e. without any smoothness or connection operator or differential form.

The ∞-groupoid terminology is perhaps misleading; what I am really after is an analog of parallel transport along paths extended to parallel transport over (maps of) simplices, not just 1-simplices.

2 The ‘classical’ topological case

We first recall what are rightly known as Moore paths [**?**] on a topological space X.

Definition 1 *Let $R^+ = [0,\infty)$ be the nonnegative real line. For a space X, let $Moore(X)$ be the subspace of* Moore paths $\subset X^{R^+} \times R^+$ *of pairs (f,r) such that f is constant on $[r,\infty)$. There are two maps*

- $\partial^-, \partial^+ : Moore(X) \to X$,
- $\partial^-(f,r) = f(0)$,
- $\partial^+(f,r) = f(r)$.

Now composition $\circ$ of Moore paths in $Moore(X)$ is given by sending pairs $(f,r),(g,s) \in Moore(X)$ such that $f(r) = g(0)$ to $h \in Moore(X)$ which is constant on $[r+s,\infty)$, $h|[0,r] = f|[0,r]$ and $h(t) = g(t-r)$ for $t \geq r$. The composite is denoted fg. An identity function $\epsilon : X \to Moore(X)$ is given by $\epsilon(x) = (\hat{x}, 0)$ where $\hat{x}$ is the constant map on R^+ with value x.

Composition is continuous and gives, as is well known, a category structure on $Moore(X)$. If we had used the 'ancient' Poincaré paths $I \to X$, we would have had to work with an A_∞-structure on X^I. Indeed, it was working with that standard parameterization which led to A_∞-structures [**?**, **?**].

For a category C, we denote by $C_{(n)}$ the set of n-tuples of composable morphisms. In partcular, we will be concerned with $Moore(B)_{(n)}$. We will write $\mathbf{t}$ for $(t_1, \cdots, t_n)$ and $\hat{t}_i$ for $(t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n)$,

Definition 2 *A* representation up to homotopy *of $Moore(B)$ on a fibration $E \to B$ is an A_∞-morphism (or shm-morphism [**?**]) from $Moore(B)$ to $End_B(E)$; that is, a collection of maps*

$$\theta_n : I^{n-1} \times Moore(B)_{(n)} \times_B E \to E$$

(where $Moore(B)_{(n)} \times_B E$ consists of $n+1$-tuples $(\lambda_1, \ldots, \lambda_n, e)$ where the λ_i are composable paths, constant on $[r_i, \infty)$, and $p(e) = \lambda_1(0)$) such that

$$p(\Theta_n(\mathbf{t}, \lambda_1, \ldots, \lambda_n, e)) = \lambda_n(r_n),$$

$$\theta_n(\mathbf{t}, \lambda_1, \ldots, \lambda_n, -)$$

is a fibre homotopy equivalence and satisfying the usual/standard relations:

- *if $t_i = 0$, then*
$$\theta_n(t_1, \cdots, t_{n-1}, \lambda_1, \ldots, \lambda_n, e))$$
is given by
$$\theta_{n-1}(\hat{t}_i, \cdots, \lambda_i\lambda_{i+1}, \cdots, e))$$
- *and if $t_i = 1$, then by*

$$\theta_i(\cdots, t_{i-1}, \lambda_1, \ldots, \lambda_{i-1}, \theta_{n-i}(t_{i+1}, \cdots, t_{n-1}, \lambda_i, \cdots, \lambda_n, e))$$

Remark 3 That the parameterization is by cubes, as for Sugawara's strongly homotopy multiplicative maps rather than more general polytopes, reflects the fact that $Moore(X)$ and $End_B(E)$ are strictly associative. Strictly speaking, referring to $Moore(B) \to End_B(E)$ as an A_∞-map raises issues about a topology on $End_B(E)$; the adjoint formulas above avoid this difficulty.

Since our construction uses in a crucial way the homotopy lifting property, we first construct maps

$$\Theta_n : I^n \times Moore(B)_{(n)} \times_B E \to E.$$

Further,

- $\Theta_0 : E \to E$ is the identity,
- if $t_i = 0$, then
$$\Theta_n(t_1, \cdots, t_n, \lambda_1, \ldots, \lambda_n, e))$$
is given by
$$\Theta_{n-1}(\hat{t}_i, \cdots, \lambda_i\lambda_{i+1}, \cdots, e))$$
- and if $t_i = r_i$, then
$$\Theta_{i-1}(\cdots, t_{i-1}, \lambda_1, \ldots, \lambda_{i-1}, \theta_{n-i+1}(t_{i+1}, \cdots, t_n, \lambda_i, \cdots, \lambda_n, e))$$

with the convention that the meaningless $t_{-1}, \lambda_0\lambda_1$ are omitted. The desired θ_n are then recovered at $t_1 = 1$:

$$\Theta_n(1, t_2, \cdots, t_n, \lambda_1, \ldots, \lambda_n, e)) = \theta_n(t_2, \cdots, t_n, \lambda_1, \cdots, \lambda_n, e))$$

Henceforth, when we say 'cube', we really mean a *Moore rectangle*. Notice that if Θ_j has been defined for all $j < i$, then the above conditions already define Θ_i on all faces of the Moore rectangle except for the face where $t_1 = 1$. We 'fill in the box' by using the homotopy lifting property after filling in the trivial image box in B (compare horn-filling in the simplicial setting). That trivial box has image just that of the composite path $\lambda_1 \cdots \lambda_n$.

Theorem 4 *(cf. Theorem A in [**?**]) For any fibration $p : E \to B$, there is an A_∞-action $\{\theta_n\}$ of Moore(B) on E such that θ_1 is a fibre homotopy equivalence.*

In Theorem B in [**?**], I proved further:

Theorem 5 *Given an A_∞-action $\{\theta_n\}$ of the Moore loops ΩB on a space F, there is a fibre space $p_\theta : E_\theta \to B$ such that, up to homotopy, the A_∞-action $\{\theta_n\}$ can be recovered by the above procedure. If the A_∞-action $\{\theta_n\}$ was roiginally obtained by this procedure from a fibre space $p : E \to B$, then p_θ is fibre homotopy equivalent to p.*

This construction gave rise to the slightly more general (re)construction below. It can also be generalized to give an ∞-version of the Borel construction/homotopy quotient: $G \to X \to X_G = X//G$.

Has this appeared in the literature already?

3 Upping the ante

Now suppose instead of looking at just composable paths, we look at $Sing(B)$. For a singular k-simplex $\sigma : \Delta^k \to B$, there are several k-tuples of composable paths from vertex 0 to vertex k by restriction to edges, in fact, $k!$ such. Given σ, we denote by F_i the fibre over vertex $i \in \sigma$.

Following e.g. Abad-Schaetz [**?**], we make the following definition of a representation up to homotopy, where we take a singular k-simplex σ to be (the image of) $< 0, 1, \cdots , k >$ with the p-th face $\partial_p\sigma$ being $< 0, \cdots , p-1, p+1, \cdots , k >$. However, we keep much of the notation above rather than switch to theirs. We also write $\lambda\mu$ to mean traversing λ *then* μ, not worrying about (the analog of) homomorphism versus anti-homomorphism.

Remark 6 *In contrast to the smooth bundle case where a connection provides unique path lifting, the fibration case is considerably more subtle since path and homotopy lifting is far from unique.*

Definition 7 *A* representation up to homotopy *of $Sing(B)$ on a fibration $E \to B$ is a collection of maps $\{\theta_k\}_{k\geq 0}$ which assign to any k-simplex $\sigma : \Delta^k \to B$ a map $\theta_k(\sigma) : I^{k-1} \times F_0 \to F_k$ satisfying the relations for any $e \in F_0$:*

θ_0 is the identity on F_0

For any $(t_1, \cdots , t_{k-1})$,

$\theta_k(\sigma)(t_1, \cdots , t_{k-1}, -) : F_0 \to F_k$ is a homotopy equivalence.

For any $1 \leq p \leq k-1$,

if $t_p = 0$,

$$\theta_k(\sigma)(\cdots , t_p = 0, \cdots , e) = \theta_{k-1}(\partial_p\sigma)(\cdots , \hat{t}_p, \cdots , , e)$$

if $t_p = 1$,

$$\theta_k(\sigma)(\cdots , t_p = 1, \cdots , e) =$$

$$\theta_p(< 0, \cdots , p >)(t_1, \cdots , t_{p-1}, \theta_q(< p, \cdots , k >)(t_{p+1}, \cdots , t_k, e)).$$

Remark 8 In definition 4, we worked with Moore paths so that the A_∞- map was between strictly associative spaces. Here instead the compatible 1-simplices compose just as e.g. a pair of 1-simplices and are related to a single 1-simplex only by an intervening 2-simplex. Associativity is trivial; the subtlety is in handling the 2-simplices and higher ones for multiple compositions. The idea of constructing a representation up to homotopy is very much like that of Theorem 1, the major difference being that instead of comparing two different liftings of the composed paths which are necessarily homotopic, we are comparing a lifting e.g. of a path from 0 to 1 to 2 with a lifting of a path from 0 to 2 IF there is a singular 2-simplex $< 012 >$. However, note that $< 02 >$ plays the role of $\lambda_1\lambda_2$ of Moore paths in the above formulas.

Theorem 9 *For any fibration $p : E \to B$, there is a representation up to homotopy of $Sing(B)$ on E.*

Remark 10 The fact that the representation up to homotopy is by fibre homotopy equivalence (as for the action of Moore(B)) is justified by the following: Since a simplex σ is contractible, the pullback $\sigma^* E$ is fibre homotopy trivial over σ. Choose the requisite lifts in $\sigma^* E$ using a trivialization corresponding to the homotopy we want to lift and then map back into E.

The essence of the proof is in essence the same as that for Theorem 4. The desired θ_n will appear as the missing lid on an open box (defined inductively) which is filled in by homotopy liftings Θ_n of a *coherent* set of maps

$$\gamma_k : I^{k-1} \to P\Delta^n$$

where P denotes the set of paths, i.e. $P\Delta^n = Map(I, \Delta^n)$ and $\gamma_1 : I \to \Delta^1$ is the 'identity'. Such maps were first produced by Adams [**?**] in the topolgical context by induction using the contractability of Δ^n. Later specific formulas were introduced by Chen [**?**, **?**] and, most recently, equivalently but more transparently, by Igusa [**?**].

By coherent I mean precisely

$\gamma_1(0)$ is the trivial path, constant at 0.

For any $1 \leq p \leq k-1$,

$$\gamma_k(\cdots, t_p = 0, \cdots) = \gamma_{k-1}(\cdots, \hat{t}_p, \cdots)$$

and

$$\gamma_k(\sigma)(\cdots, t_p = 0, \cdots) =$$
$$\gamma_p(t_1, \cdots, t_{p-1})\gamma_q(t_{p+1}, \cdots, t_{k-1}).$$

Correspondingly, the liftings $\Theta_n : I^n \times E \to E$ form a collection of maps which assign to any k-simplex $\sigma : \Delta^k \to B$ a map $\Theta_k(\sigma) : I^k \times F_0 \to F_k$ satisfying the relations for any $e \in F_0$:

$\Theta_0(0)$ is the identity on F_0

For any $(t_1, \cdots, t_k)$,

$\Theta_k(\sigma)(t_1, \cdots, t_k, -) : F_0 \to F_k$ is a homotopy equivalence.

For any $1 \leq p \leq k-1$,

if $t_p = 0$,

$$\Theta_k(\sigma)(\cdots, t_p = 0, \cdots, e) = \Theta_{k-1}(\partial_p \sigma)(\cdots, \hat{t}_p, \cdots, e)$$

if $t_p = 1$,

$$\Theta_k(\sigma)(\cdots, t_p = 1, \cdots, e) =$$
$$\Theta_p(< 0, \cdots, p >)(t_1, \cdots, t_{p-1}, \theta_q(< p, \cdots, k >)(t_{p+1}, \cdots, t_k, e)).$$

The desired θ_n is again recovered at $t_1 = 1$.

For example, $\gamma_1 : 0 \to P\Delta^1$ is a path which can be lifted as in Theorem 1 to give $\Theta_1 : I \times F_0 \to E$. Then $\gamma_2 : I \to P\Delta^2$ such that 0 maps to the 'identity' path $I \to < 02 >$ while 1 maps to the concatenated path $< 01 >< 12 >$. (Henceforth, we will assume paths have been normalized to length 1 where appropriate.) Now lift the homotopy γ_2 to a homotopy $\Theta_2(< 012 >) : I \times I \times E \to E$ between $\Theta_1(< 02 >)$ and $\Theta_1(< 01 >< 12 >)$. In particular, $\Theta_2(< 012 >) : 1 \times I \times E \to E$ gives the desired homotopy $\theta_2 : I \times F_0 \to F_k$.

The situation becomes slightly more complicated as we increase the dimension of σ. The case Δ^3 is illustrative. The faces $< 023 >$ and $< 013 >$ lift just as $< 012 >$ had via Θ_2, but that lift must then be 'whiskered' by a rectangle over $< 23 >$ which glues onto $\Theta_3(< 012 >$. In a less complicated way $< 123 >$ is lifted so that vertex 1 agrees with the end of the 'whisker' which is the lift of $< 01 >$. Thus the total lift of < 0123 ends with the desired $\theta_3 : I^2 \times F_0 \to F_3$. The needed whiskering (of various dimensions) is prescribed by the $t_p = 1$ relations of Definition 2 to be satisfied.

INSERT GRAPHIC

4 (Re)-construction of fibrations

In [**?**], I showed how to construct a fibration from the data of an strong homotopy action of ΩB on a 'fibre' F. If the action came from a given fibration $F \to E \to B$, the constructed fibration was fibre homotopy equivalent to the given one. For *representations up to homotopy*, a similar result applies using analogous techniques, with some additional subtlety.

First we try to construct a fibration naively. Over each 1-simplex σ of $Sing(B)$, we take $\sigma \times F_0$ and attempt to glue these pieces appropriately. For the one simplices $< 01 >$ and $< 12 >$, we have $\theta_1 : F_0 \to F_1$ which tells us how to glue $< 01 > \times F_0$ to $< 12 > \times F_1$ at vertex 1, but, since $\theta_1 : F_0 \to F_2$ is not the composite of $\theta_1 : F_0 \to F_1$ and $\theta_? : F_1 \to F_2$, we can not simply plug in $< 012 > \times F_0$ over $< 012 >$. However, we can plug in $I^2 \times F_0$ since $\theta_2 : I \times F_0 \to F_2$ will supply the glue over vertex 2.

To describe the plugging, we use special maps $p_n : I^n \to \Delta^n$ where Δ^n is the set

$$\{(t_1, \cdots, t_n) | 1 \geq t_1 \geq t_2 ... \geq 0\}.$$

The maps p_n are iterated convex linear. The basic example is

$$c : (x, y) \mapsto (x \cdot 1 + (1 - x)y, y).$$

Write $t_1 = t,\ t_2 = s,\ t_3 = r$. For $n = 2$, define $p_2 = c : (t, s) \mapsto (t \cdot 1 + (1 - t)s, s)$ and then

$$p_3 : (t, s, r) \mapsto (c(c(t, s), r), c(s, r), r) = (c(t \cdot 1 + (1 - t)s), r), c(s, r), r).$$

INSERT GRAPHIC

Hopefully, the pattern is clear. Now return to the description of the fibration $\bar{p}_2 : E_2 \to \Delta^2$ above. In greater precision,

$$E_2 =< 01 > \times F_0 \cup_1 < 12 > \times F_1 \cup_0 < 02 > \times F_0\ \cup I^2 \times F_0.$$

The attaching maps over the vertices 0 and 1 are obvious as are the projections to the edges of Δ^2. On $I^2 \times F_0$, the attaching maps are obvious except for the face $t_2 = s = 1$ where it is given by $\theta_2 : I \times F_0 \to F_2$, so as to be compatible with the projection $\bar{p}_2 : I^2 \times F_0 \to \Delta^2$.