Chern-Simons Theory and the Categorified Group Ring

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The talk reports on results of the paper "Topological Quantum Field Theories from Compact Lie Groups" by Freed, Hopkins, Lurie and Teleman [FHLT]. Unless otherwise stated, all results are taken from this paper. I also acknowledge enlightening discussions with Chris Douglas and Constantin Teleman. Any mistakes are of course only due to me.

1 Overview

The goal is to make (Quantum) Chern-Simons theory a 0-1-2-3 theory, extended all the way down to the point. That is, we want to define a 3-functor

$$\mathrm{CS}^{\tau} : \mathrm{Bord}^3 \longrightarrow \mathcal{C},$$

where Bord^3 is a 3-category of (at least) oriented cobordisms, $\mathcal{C}$ is some symmetric monoidal 3-category and $\tau \in \mathrm{H}^4(BG, \mathbb{Z})$ is a parameter.

We recall that τ determines a modular tensor category $\mathrm{PER}^{\tau}(LG)$ of positive energy representations of LG (at least for G simple, connected and simply-connected). This category can be feeded into the Reshetikhin-Turaev construction [RT90], and out comes a 1-2-3 theory – Chern-Simons theory at level τ. Its value on the circle is the category $\mathrm{PER}^{\tau}(LG)$ itself. The 3-functor CS^{τ} we want to define is supposed to reproduce this prescription in 1, 2 and 3 dimensions.

There are two main problems in providing the missing value of CS^{τ} on a point.

1. According to the cobordism hypothesis [BD95] recently proved by Lurie [Lur], the value on the point *determines the whole theory*. So we can't just choose something.

2. Chern-Simons theory is anomalous (as a theory for oriented manifolds). So it will in fact turn out that it is *not* a functor as claimed above, but instead a transformation

$$\begin{tikzcd}
\mathrm{Bord}^4 \arrow[rr, bend left, "\mathbf{I}"] \arrow[rr, bend right, "\mathcal{A}^\tau"'] \arrow[rr, phantom, "\Downarrow \mathrm{CS}^\tau"] & & \mathcal{D}
\end{tikzcd}$$

between 4-dimensional theories: the trivial one and one representing the anomaly. Here, $\mathcal{D}$ is some 4-category to be specified.

2 Strategy

In order to find the correct value of CS^τ at the point, we look (simultaneously) at:

- finite groups as toy models (also known as Dijkgraaf-Witten theory), and generalize to compact groups.
- classical Chern-Simons theory (defined on $\mathrm{Bord}^3_{G,\nabla}$ of oriented bordisms equipped with principal G-bundles with connection), and quantize it.
- analogous TQFT's in dimensions $n = 1, 2$, parameterized by $\mathrm{H}^{n+1}(BG, \mathbb{Z})$, and categorify them.

Diagrammatically, this looks as follows:

$$\begin{tikzcd}
 & & \ast \arrow[rr] & & \ast \arrow[rr] & & \circledast & \\
\text{quantum} & \ast \arrow[ur] \arrow[rr] & & \ast \arrow[ur] \arrow[rr, Rightarrow] & & \ast \arrow[ur, Rightarrow] & & \\
 & & \ast \arrow[uu] \arrow[rr] & & \ast \arrow[uu] \arrow[rr] & & \ast \arrow[uu] & \text{torus} \\
\text{classical} & \ast \arrow[uu] \arrow[ur] \arrow[rr] & & \ast \arrow[uu] \arrow[ur] \arrow[rr] & & \ast \arrow[uu] \arrow[ur] & & \text{finite group} \\
 & n = 1 & & n = 2 & & n = 3 & &
\end{tikzcd}$$

Here, the vertex $\circledast$ is our goal. The paper approach this goal from all directions. In this talk we restrict ourselves to describing the two doubled arrows.

The main tool that we use is a geometrical realization of the parameter $\tau \in \mathrm{H}^4(BG,\mathbb{Z})$, in company with geometric realizations of twisted K-classes. These are to be discussed in the next section.

I should also mention that in this talk we will totally neglect manifolds with boundary (i.e. the functorial aspects of extended field theories), and we will treat all (∞,n)-categories as just n-categories.

3 Geometric realizations

The following table shows possible ways to think about classes in $\mathrm{H}^{n+1}(BG,\mathbb{Z})$.

n	group-theoretical meaning	geometrical meaning
1	characters of G	multiplicative S^1-valued maps
2	central extensions of G by S^1	multiplicative S^1-bundles over G
3	string extensions of G by BS^1	multiplicative S^1-gerbes over G

We will use the ones in the column on the right. As a side remark let me mention that in terms of the objects in this column transgression

$$\mathrm{H}^4(BG,\mathbb{Z}) \longrightarrow \mathrm{H}^3(G,\mathbb{Z})$$

is geometrically realized by simply forgetting the multiplicativity of the respective objects.

Let us concentrate on the last row. The precise statement is – for G a compact Lie group – a group isomorphism

$$\mathrm{H}^4(BG,\mathbb{Z}) \cong \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{multiplicative bundle gerbes over } G \end{array} \right\}.$$

A proof can be found in [CJM$^+$05]. Since the bundle gerbes themselves will shortly disappear, we will only explain their multiplicative structures. It is, however, important to note that bundle gerbes over a manifold form a 2-groupoid.

Definition 1. *A multiplicative bundle gerbe over G is a bundle gerbe $\mathcal{G}$ over G together with a 1-isomorphism K over $G \times G$, fibrewise*

$$K_{x,y} : \mathcal{G}_x \otimes \mathcal{G}_y \longrightarrow \mathcal{G}_{xy},$$

and with a 2-isomorphism θ over $G \times G \times G$, fibrewise

$$\begin{tikzcd}
\mathcal{G}_x \otimes \mathcal{G}_y \otimes \mathcal{G}_z \arrow[r, "K_{x,y}\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes K_{x,y}"'] \arrow[dr, phantom, "\overset{\theta_{x,y,z}}{\swarrow}"] & \mathcal{G}_{xy} \otimes \mathcal{G}_z \arrow[d, "K_{xy,z}"] \\
\mathcal{G}_x \otimes \mathcal{G}_{yz} \arrow[r, "K_{x,yz}"'] & \mathcal{G}_{xyz},
\end{tikzcd}$$

and satisfying a "Pentagon axiom" over G^4.

Let me remark that it is a difficult problem to *construct* this structure from a given class $\tau \in \mathrm{H}^4(BG, \mathbb{Z})$; we only know that it exists and is unique up to equivalence of multiplicative gerbes. The paper [FHLT] does give a construction for the torus $G = T$ performing the following steps:

1. The class $\tau \in \mathrm{H}^4(BG, \mathbb{Z})$ is identified with a bilinear form $\Pi \otimes \Pi \longrightarrow \mathbb{Z}$, where Π is the cocharacter lattice, i.e. $\Pi := \mathrm{Hom}(S^1, T)$.
2. One can identify the Lie algebra $\mathfrak{t}$ of T with its universal covering group, and Π with the fundamental group acting on $\mathfrak{t}$ by Deck transformations, so that $T = \mathfrak{t}/\Pi$.
3. The bilinear form is used to lift the action of $\Pi \times \Pi$ on $\mathfrak{t} \times \mathfrak{t}$ to the trivial principal S^1-bundle over $\mathfrak{t}\times\mathfrak{t}$, so that the latter descends to $T\times T$. This will be the S^1-bundle K.

For more general groups, e.g. compact and simple ones, at least constructions for the underlying gerbe $\mathcal{G}$ are known [GR03, Mei02].

Let us return to the general setup of a multiplicative gerbe. Forgetting the multiplicative structure leaves with a bundle gerbe $\mathcal{G}$ over G, representing a twist for the K-theory of G. In fact, every multiplicative bundle gerbe is in particular equivariant under the conjugation action of G on itself (as one expects for transgressed objects). This will be interesting for us because we are going to talk about twisted *equivariant* K-theory later. The general definition is

Definition 2. *Let X be a smooth manifold with G-action, and let $\mathcal{G}$ be a bundle gerbe over X. A <u>G-equivariant structure</u> on $\mathcal{G}$ is a 1-isomorphism L over $G \times X$, fibrewise*

$$L_{g,x} : \mathcal{G}_x \longrightarrow \mathcal{G}_{gx},$$

together with a 2-isomorphism α over $G \times G \times X$ fibrewise

$$\begin{tikzcd}
\mathcal{G}_x \arrow[rr, "L_{h,x}"] \arrow[dr, "L_{gh,x}"'] & \arrow[d, phantom, "\alpha_{g,h,x}"] & \mathcal{G}_{hx} \arrow[dl, "L_{g,hx}"] \\
& \mathcal{G}_{gh,x}, &
\end{tikzcd}$$

and satisfying a coherence condition over $G^3 \times X$.

The canonical equivariant structure (L, α) on a multiplicative gerbe $(\mathcal{G}, K, \theta)$ looks as follows. The 1-isomorphism L is given by the composite

$$\mathcal{G}_x \longrightarrow \mathcal{G}_g^* \otimes \mathcal{G}_g \otimes \mathcal{G}_x \xrightarrow{\mathrm{id}\otimes K_{g,x}} \mathcal{G}_g^* \otimes \mathcal{G}_{gx} \xrightarrow{\mathrm{id}\otimes K^{-1}_{gxg^{-1},g}} \mathcal{G}_g^* \otimes \mathcal{G}_{gxg^{-1}} \otimes \mathcal{G}_g \longrightarrow \mathcal{G}_{gxg^{-1}},$$

where the unlabelled arrows refer to the symmetry/duality on the 2-groupoid of bundle gerbes. The 2-isomorphism α is then defined as a composition of the 2-isomorphism θ, and the condition on α follows from the pentagon axiom for θ.

It is perfectly possible to represent classes in ${}^{\sigma}K^0(X)$ and ${}^{\sigma}K^0_G(X)$ by vector bundles interacting geometrically with bundle gerbes and equivariant bundle gerbes over X, where $\sigma \in \mathrm{H}^3(X, \mathbb{Z})$ is the Dixmier-Douady class of the gerbe. We will not explain this in full generality, since we pass next to trivial gerbes. The twisted (no-equivariant) case is explained in [BCM+02]. Let me also remark that an equivariant structure with respect to the trivial action (such as the conjugation of the torus on itself) can still contain nontrivial information; this is also familiar from twisted equivariant K-theory.

The groups we are interested in are either finite or the 2-torus T. In both cases we have $\mathrm{H}^3(G, \mathbb{Z}) = 0$, so that all bundle gerbes over G are trivializable and thus disappear from the above definitions. So we are left with multiplicative and equivariant structures on the trivial gerbe. For the trivial gerbe $\mathcal{I}$ over some manifold X, there is a canonical equivalence of monoidal groupoids

$$\mathcal{H}om(\mathcal{I}, \mathcal{I}) \cong \mathcal{B}un_{S^1}(X). \tag{1}$$

It is left as an interesting exercise for the audience to transform the above definitions into principal S^1-bundles and bundle isomorphism assuming $\mathcal{G} = \mathcal{I}$ and using (1). In the following we present the result:

1. A multiplicative structure on the trivial gerbe over G consists of a principal S^1-bundle K over $G \times G$ and of a bundle isomorphism θ over G^3 fibrewise given

as

$$\theta_{x,y,z} : K_{xy,z} \otimes K_{x,y} \longrightarrow K_{x,yz} \otimes K_{y,z}$$

and satisfying a pentagon axiom.

2. A G-equivariant structure on the trivial gerbe over X consists of a principal S^1-bundle L over $G \times X$ and of a bundle isomorphism α over $G^2 \times X$ fibrewise given by

$$\alpha_{g,h,x} : L_{g,hx} \otimes L_{h,x} \longrightarrow L_{gh,x}$$

and satisfying a coherence condition.

3. If the G-equivariant structure is determined by a multiplicative structure, we have

$$L_{g,x} = K^*_{gxg^{-1},g} \otimes K_{g,x}, \tag{2}$$

and also α and θ are related to each other.

Summarizing, a class $\tau \in \mathrm{H}^4(BG, \mathbb{Z})$ that transgresses to $0 \in \mathrm{H}^3(G, \mathbb{Z})$ is represented by a trivial multiplicative bundle gerbe given by a pair (K, θ), and further determining the equivariant structure (L, α).

We will later need the definition of a twisted equivariant vector bundle over G, where the equivariant twist is given by a trivial multiplicative bundle gerbe (K, θ) representing a class $\tau \in \mathrm{H}^4(BG, \mathbb{Z})$. Such a bundle is a vector bundle W over G together with an isomorphism ϕ over $G \times G$, fibrewise

$$\phi_{g,x} : L_{g,x} \otimes W_x \longrightarrow W_{gxg^{-1}}. \tag{3}$$

These pairs (W, ϕ) define classes in ${}^\tau K^0_G(G)$.

4 Starting point: 2d TQFT for finite groups

We denote by $\mathcal{A}lg$ the 2-category of complex algebras, bimodules and intertwiners. We recall that a complex Frobenius algebra is a finite-dimensional algebra A with a trace $\mathrm{tr} : A \longrightarrow \mathbb{C}$; the trace is a linear map that vanishes on all commutators.

Theorem 3. *The following 2-categories are equivalent:*

$$\left\{ \begin{array}{c} \textit{0-1-2 TQFTs for} \\ \textit{oriented manifolds} \\ \textit{with values in } \mathcal{A}lg \end{array} \right\} \cong \left\{ \begin{array}{c} \textit{Fully dualizable and} \\ \mathrm{SO}(2)\textit{-homotopy-fixed} \\ \textit{objects in } \mathcal{A}lg \end{array} \right\} \cong \left\{ \begin{array}{c} \textit{Semisimple complex} \\ \textit{Frobenius algebras} \end{array} \right\}$$

Sketch of proof. The equivalence on the left is the oriented version of the cobordism hypothesis proved by Lurie [Lur]. The equivalence on the right is an explicit computation carried out in [FHLT]. Basically, the condition that an algebra is fully dualizable implies that the algebra is finite-dimensional and semisimple. Being a homotopy fixed point is actually additional structure and translates precisely into a trace. □

Motivated by quantization of classical 2-dimensional quantum field theory, we choose the following Frobenius algebra. We assume a finite group G. Recall that τ is geometrically represented by a multiplicative S^1-bundle (K, θ) over G.

1. $A := \mathbb{C}^\tau[G] := \Gamma(G; K_\mathbb{C})$ is the vector space of sections of the line bundle $K_\mathbb{C} := K \times_{S^1} \mathbb{C}$ over G.
2. The multiplication in A is given by convolution, i.e. we put
$$(\gamma_1 \star \gamma_2)(g) := \sum_{h \in G} \theta(\gamma_1(gh^{-1}) \otimes \gamma_2(h)).$$
3. The trace on A is given by
$$\mathrm{tr}(\gamma) := \frac{1}{|G|}\gamma(e) \in K_\mathbb{C}|_e,$$
employing the fact that the multiplicative structure θ on K determines an identification $K_\mathbb{C}|_e \cong \mathbb{C}$.

According to the above theorem, the Frobenius algebra $A := \mathbb{C}^\tau[G]$ defines a 2d TQFT for oriented manifolds. Its assignments are:

$$\begin{aligned} \star &\longmapsto A \\ S^1 &\longmapsto A \otimes_{A \otimes A^{op}} A \cong A \otimes_{A \otimes A^{op}} A^\vee \cong \mathrm{Hom}_{A \otimes A^{op}}(A, A) \cong Z(A) \\ \Sigma &\longmapsto \sum_{[P]} \frac{1}{|\mathrm{Aut}(P)|} \exp\left(2\pi \mathrm{i} \, \langle \zeta_P^* \tau, \Sigma \rangle\right) \end{aligned}$$

In the second line we have used the trace to obtain an isomorphism $A \cong A^\vee$. In the third line we have summed over equivalence classes of principal G-bundles over Σ, we have denoted by $\zeta_P : \Sigma \longrightarrow BG$ the classifying map of such a bundle, and we regard $\tau \in \mathrm{H}^3(BG, \mathbb{Z})$ as an element in $\mathrm{H}^2(BG, S^1)$, using that G is finite.

It may be helpful to notice that the center $Z(A)$ which is assigned to the circle is a commutative Frobenius algebra. This fits nicely together with the classification of (closed) 1-2 TQFTs in two dimensions; these are exactly classified by commutative Frobenius algebras.

5 Categorification: from 2d to 3d for finite groups

The main idea is to replace $\mathbb{C}$ by Vect, the category of complex vector spaces. One can argue that under this motto the 2-category $\mathcal{A}lg$ becomes replaced by a 3-category 2-$\mathcal{A}lg$, where

1. The objects are tensor categories (over $\mathbb{C}$).
2. The morphisms are bimodule categories.
3. The 2-morphisms are intertwining functors, and the 3-morphisms are natural transformations.

Next we need a fully dualizable object in the 3-category 2-$\mathcal{A}lg$. This will be the "categorified twisted group ring", i.e. we categorify

$$\mathbb{C}^\tau[G] \rightsquigarrow \mathrm{Vect}^\tau[G]$$

according to our motto. The objects the tensor category are vector bundles over G, and the monoidal structure is defined by the "categorified convolution"

$$(W \star W)'(g) := \bigoplus_{hh'=g} K_{\mathbb{C}}|_{h,h'} \otimes W_h \otimes W'_{h'},$$

where $K_{\mathbb{C}} := K \times_{S^1} \mathbb{C}$ is the line bundle associated to the principal S^1-bundle K, which is part of the trivial multiplicative gerbe (K, θ) that represents the twist τ. The isomorphism θ contributes the associator for this monoidal structure.

Theorem 4. *The object* $\mathrm{Vect}^\tau[G]$ *in* 2-$\mathcal{A}lg$ *is fully dualizable and a* $\mathrm{SO}(3)$*-homotopy-fixed point.*

First steps of the proof. This is checking that $\mathrm{Vect}^\tau[G]$ is a dualizable object in the homotopy category $\mathrm{h}_1(2\text{-}\mathcal{A}lg)$. This is clear since the opposite tensor category (i.e. the one with the factors in the tensor product switched) provides a dual. The next thing is to check that evaluation and coevaluation of this duality have adjoints in the homotopy 2-category $\mathrm{h}_2(2\text{-}\mathcal{A}lg)$, but we will not do this here. □

Next we shall verify that the fully dualizable object $\mathrm{Vect}^\tau[G]$ defines the "correct" 3d TQFT – Chern-Simons theory for a finite group.

Theorem 5.

(a) The 3d TQFT determined by $\mathrm{Vect}^\tau[G]$ *assigns to the circle* S^1 *the Drinfeld center* $Z(\mathrm{Vect}^\tau[G])$.

(b) The objects in the Drinfeld center are twisted equivariant vector bundles over G.

Proof. Statement (a) is a purely formal argument, analogous to the fact that a 2d TQFT given by a Frobenius algebra A assigns to the circle the center $Z(A)$. The only thing we have to supply is a natural identification between A and $A^\vee$, where now $A := \mathrm{Vect}^\tau[G]$. Indeed, the category A also has a "trace" just like a Frobenius algebra. This trace is the functor

$$\mathrm{tr} : A \longrightarrow \mathbb{C} : W \longmapsto W_e,$$

taking a vector bundle to its fibre at the identity element. Correspondingly, we obtain a "bilinear form"

$$A \otimes A \longrightarrow \mathrm{Vect} : (W, W') \longmapsto \mathrm{tr}(W \star W'),$$

which defines a functor $A \longrightarrow A^\vee$.

Statement (b) can be verified by an explicit calculation. Suppose (W, ϵ) is in the Drinfeld center. That is, W is a vector bundle over G, and ϵ is a natural transformation

$$\epsilon : W \star - \longrightarrow - \star W.$$

We shall "test" the transformation ϵ on a basis of $\mathrm{Vect}^\tau[G]$ given by vector bundles $\mathbb{C}_y$ that have the fibre $\mathbb{C}$ over y and the trivial vector space elsewhere. Then, for any $x \in G$, ϵ is a bundle isomorphism over $G \times G$ and fibrewise

$$\epsilon_{x,y} : K_{xy^{-1},y} \otimes W_{xy^{-1}} \longrightarrow K_{y,y^{-1}x} \otimes W_{y^{-1}x}.$$

Upon replacing x by yx, tensoring with the dual of $K_{xyx^{-1},x}$ and identifying the S^1-bundle L via (2) one obtains an isomorphism

$$\phi_{x,y} : L_{x,y} \otimes W_x \longrightarrow W_{yxy^{-1}}$$

Consulting (3) we see that W is a twisted equivariant vector bundle. Conversely, one can start with a pair (L, ϕ) and produce a "half-braiding" ϵ promoting W to an object in the Drinfeld center. □

The theorem tells us basically that the fully dualizable object $\mathrm{Vect}^\tau[G]$ defines a TQFT which reproduces Chern-Simons theory in dimensions 1, 2 and 3, and thus is exactly what we were looking for.

To see this, we shall argue informally in the following way:

$$S^1 \longmapsto Z(\mathrm{Vect}^\tau[G]) \cong {}^\tau K_G(G) \cong \mathrm{PER}^\tau(LG).$$

The first "$\cong$" is the above theorem, and the second "$\cong$" is the Freed-Hopkins-Teleman theorem. The disclaimer "informally" refers to the fact that we are here mixing up tensor categories with their homotopy algebras, and also to the fact that the Freed-Hopkins-Teleman theorem doesn't hold this way for disconnected groups. Anyway, we see that the value at the circle is that very tensor category which is used in the Reshetikhin-Turaev construction of Chern-Simons theory.

6 From finite groups to the torus

While we can certainly consider the category of vector bundles over T, the definition of the categorified convolution $\star$ fails for the obvious reasons when going from a finite group to T. So we pass to a "discretized version" of Vect^τ by only admitting "finitely many non-trivial fibres".

More concretely, we consider a category $\mathrm{Sky}^\tau[T]$ whose objects are skyscraper sheaves with values in Vect and finite support. For example, we denote by $\mathbb{C}_x$ the skyscraper at x with value $\mathbb{C}$. Any other object of $\mathrm{Sky}^\tau[T]$ can be obtained by a finite direct sum of these sheaves. The convolution product defined by the twist $\tau \in \mathrm{H}^4(BT, \mathbb{Z})$ and an associated geometrical realization (K, θ) is given by

$$\mathbb{C}_x \star \mathbb{C}_y := K_{x,y} \times_{S^1} \mathbb{C}_{xy},$$

and the associator is again provided by the morphism θ. This way, the category $\mathrm{Sky}^\tau[T]$ becomes a tensor category – i.e. an object in the 3-category 2-$\mathcal{A}lg$.

Let me point out – without any attempt to prove these statements – the following two problems related with the object $\mathrm{Sky}^\tau[T]$.

1. $\mathrm{Sky}^\tau[T]$ is not fully dualizable. In particular, it does not define a 3d TQFT.
2. Its "continuous Drinfeld center" $Z(\mathrm{Sky}^\tau[T])$ is

$$Z(\mathrm{Sky}^\tau[T]) \cong \mathrm{Sky}^\tau[\mathfrak{t}] \otimes \mathrm{Sky}^\tau[F], \tag{4}$$

   where F is a certain subset of the cocharacter lattice Π. The paper claims that the tensor factor $\mathrm{Sky}^\tau[F]$ is a modular tensor category and the right choice to assign to the circle. In that sense, the Drinfeld center is "too big".

One of the main advances of the paper [FHLT] is a "correct" interpretation of these issues as the "framing anomaly" of Chern-Simons theory.

The main idea is to regard the unwanted factor $\mathrm{Sky}^\tau[\mathfrak{t}]$ as an object in the 4-category 3-$\mathcal{A}lg$ whose objects are braided tensor categories, whose morphisms are bimodule tensor categories etc. Indeed, a braiding for $\mathrm{Sky}^\tau[\mathfrak{t}]$ can be obtained from a trivialization of the pullback of the bundle K to $\mathfrak{t} \times \mathfrak{t}$; in the construction of K outlined on page 4 such a trivialization is canonically given. As an object in 3-$\mathcal{A}lg$ the braided tensor category $\mathrm{Sky}^\tau[\mathfrak{t}]$ is fully dualizable. As in the 2-dimensional case of Frobenius algebras, making $\mathrm{Sky}^\tau[\mathfrak{t}]$ a homotopy fixed point under $\mathrm{SO}(4)$ is additional structure, which is here a normalization $\lambda \in \mathbb{C}$ of a certain trace. We will keep λ variable and fix it later. Then, $\mathrm{Sky}^\tau[\mathfrak{t}]$ determines a 4-dimensional TQFT for oriented manifolds, denoted $\mathcal{A}^\tau_\lambda$ and called the "anomaly theory".

We can now regard $\mathrm{Sky}^\tau[T]$ as a $(\mathrm{Vect}, \mathrm{Sky}^\tau[\mathfrak{t}])$-bimodule tensor category, with the action of $\mathrm{Sky}^\tau[\mathfrak{t}]$ defined by the tensor product $\star$ on $\mathrm{Sky}^\tau[\mathfrak{t}]$ and the isomorphism (4). In other words, we have a 1-morphism

$$\mathrm{Sky}^\tau[T] : \mathrm{Vect} \longrightarrow \mathrm{Sky}^\tau[\mathfrak{t}] \tag{5}$$

in the 4-category 3-$\mathcal{A}lg$. Recall that $\mathrm{Sky}^\tau[T]$, Vect and $\mathrm{Sky}^\tau[\mathfrak{t}]$ are, respectively, the values of Chern-Simons Theory, the trivial 4d TQFT $\mathbf{I}$, and the anomaly theory $\mathcal{A}^\tau_\lambda$ at the point. This pattern continues to higher dimensions, and so we must see Chern-Simons Theory as a *transformation*

$$\mathrm{CS}^\tau : \mathbf{I} \longrightarrow \mathcal{A}^\tau_\lambda$$

between four-dimensional theories, over the point reducing to (5). The value of CS^τ on a point is now the relative Drinfeld center of $\mathrm{Sky}^\tau[T]$ over $\mathrm{Sky}^\tau[\mathfrak{t}]$, this is exactly the modular tensor category $\mathrm{Sky}^\tau[F$ that we wanted.

In order to get a better feeling for the anomaly theory $\mathcal{A}^\tau_\lambda$ defined by the fully dualizable object $\mathrm{Sky}^\tau[\mathfrak{t}]$ we describe its value on a closed 4-manifold M, which is the complex number

$$\mathcal{A}^\tau_\lambda(M) := \sqrt{\lambda^{-1}|F|}^{\chi(X)} \cdot \mu^{\mathrm{sign}(\tau)\mathrm{sgn}(X)} \in \mathbb{C}.$$

Here, $\chi(X)$ is the Euler characteristic of X, $\mathrm{sgn}(X)$ is the signature of X and $\mathrm{sign}(\tau)$ is a certain sign related to the bilinear form $\Pi \times \Pi \longrightarrow \mathbb{Z}$ defined by τ. Now we can decide to fix $\lambda := |F|$, so that the first factor vanishes.

Under certain circumstances, the anomaly theory vanishes. A first hint what such circumstances are can be obtained by looking at the number $\mathcal{A}^\tau_\lambda(M)$:

- The sign of τ is a multiple of 8.
- X is a spin manifold, so that $\mathrm{sgn}(X)$ is a multiple of eight.

In these cases, one can choose a trivialization

$$T : \mathcal{A}_\lambda^\tau \longrightarrow \mathbf{I}$$

and form the composition

$$\mathbf{I} \xrightarrow{\mathrm{CS}^\tau} \mathcal{A}_\lambda^\tau \xrightarrow{T} \mathbf{I}\ .$$

This turns out to be a fully dualizable object and a SO(3)-homotopy fixed point in the 3-category of ($\mathbf{I}$-$\mathbf{I}$)-bimodule tensor categories, which can be identified with 2-$\mathcal{A}lg$. Then, Chern-Simons Theory is a proper 3d TQFT.

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