A New Look At The Jones Polynomial of a Knot

Edward Witten, IAS

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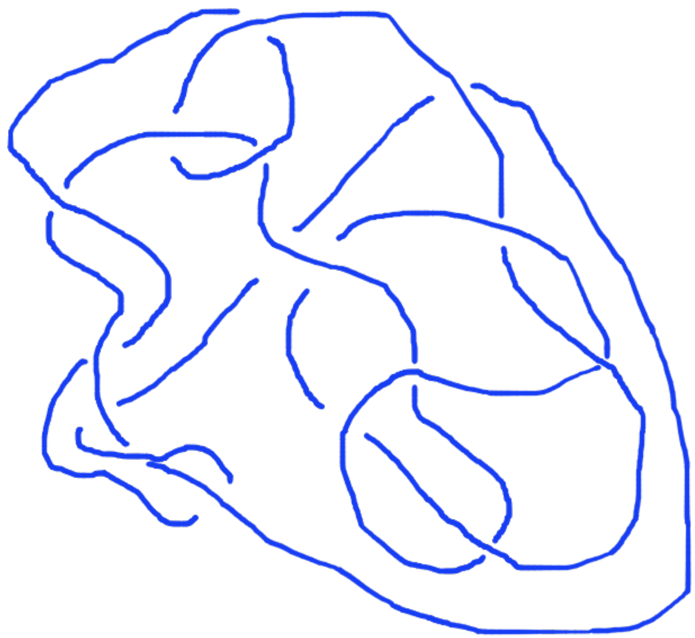

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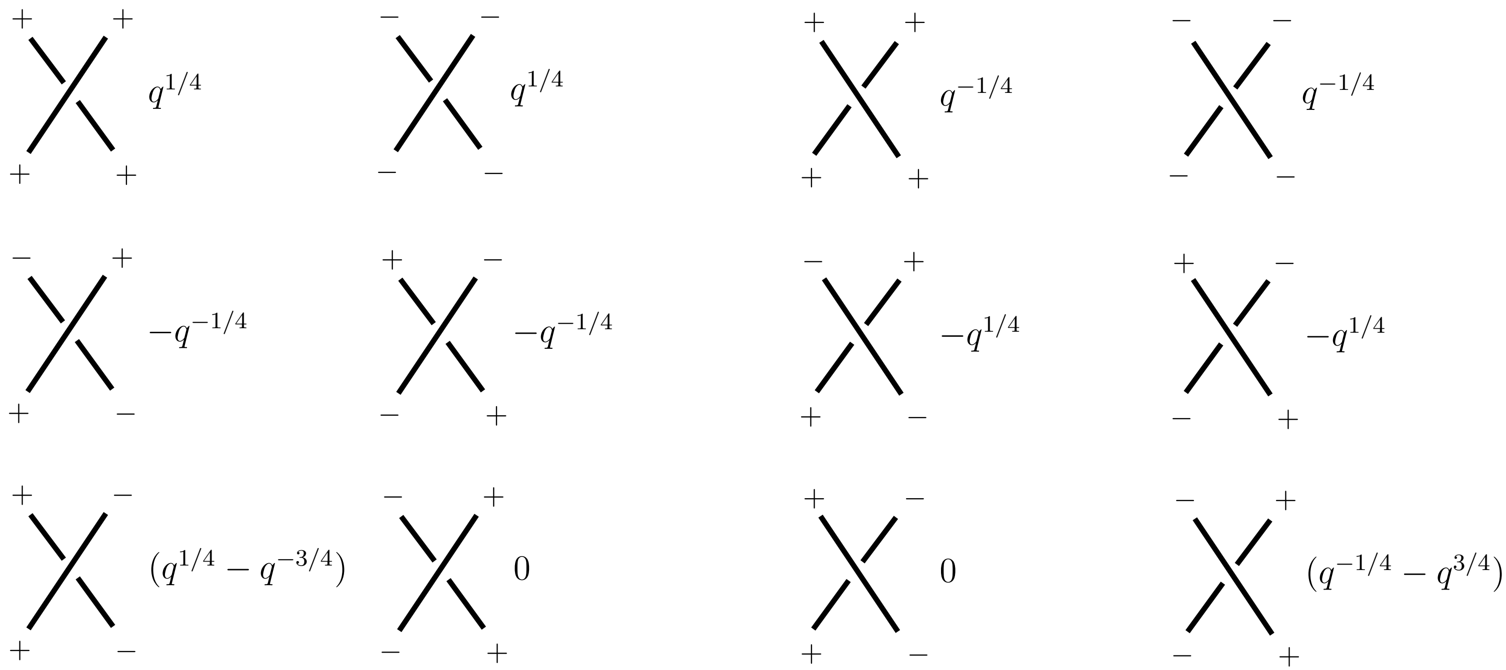

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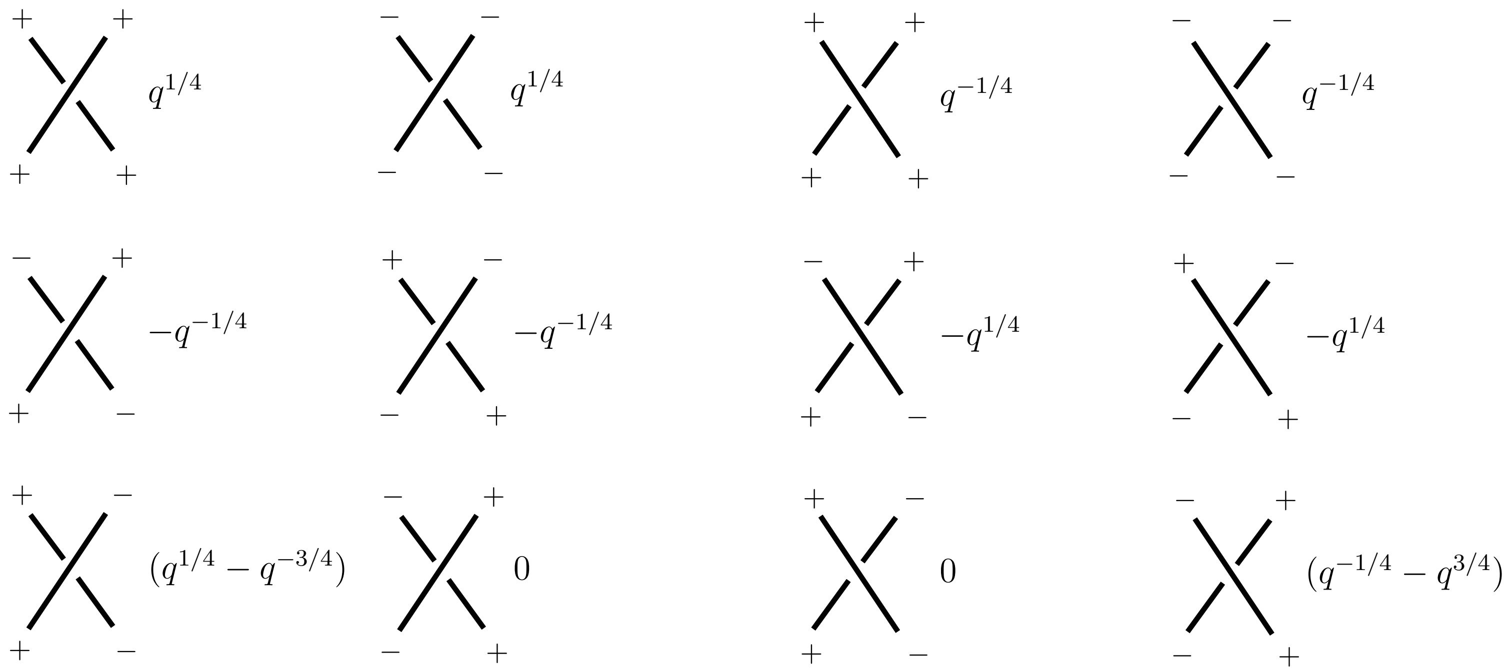
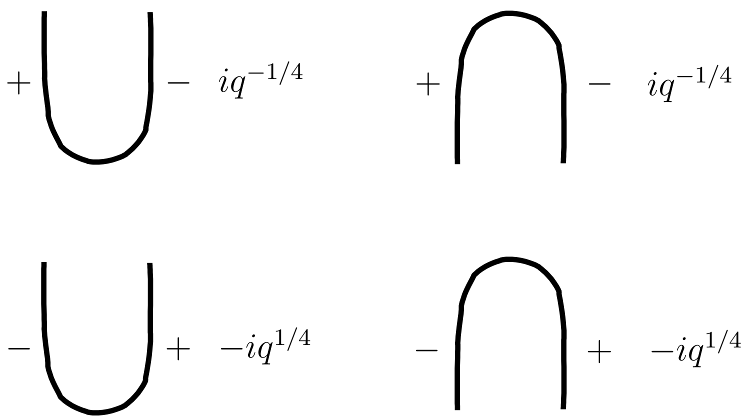


(0 for labelings in which the number of $+$ at the bottom doesn't equal the number at the top.)

and for each creation or annihilation event



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Another relation of the Jones polynomial to two-dimensional mathematical physics was found by A. Tsuchiya and Y. Kanie: they showed that Jones's representations of the braid group (which can be used to give a different definition of the Jones polynomial) were the ones that arise from "conformal blocks" of two-dimensional conformal field theory and the associated Knizhnik-Zamolodchikov equations.

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$$\mathrm{CS}(A) = \frac{1}{4\pi}\int_M \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A\right).$$

(This formula for $\mathrm{CS}(A)$ is a little naive and assumes that the bundle E has been trivialized and the connection A can be regarded as a 1-form valued in the Lie algebra $\mathfrak{g}$ of G.)

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(This formula for $\mathrm{CS}(A)$ is a little naive and assumes that the bundle E has been trivialized and the connection A can be regarded as a 1-form valued in the Lie algebra $\mathfrak{g}$ of G.) All we really need to know for now about $\mathrm{CS}(A)$ is that it is gauge-invariant mod $2\pi\mathbb{Z}$.

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This is a basic construction in quantum field theory, though unfortunately still difficult to understand from a mathematical point of view. k has to be an integer since $\mathrm{CS}(A)$ is only gauge-invariant mod $2\pi\mathbb{Z}$. Formally $Z_k(M)$ is an invariant of an oriented three-manifold; actually, if one follows the logic of what physicists call "renormalization theory," one finds that M must be a "framed" three-manifold (with a simple behavior under change of framing).

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This gives an invariant of the pair (M, K), except that if one looks more closely, one learns that both M and K should be framed.

If we specialize to the case that $M = \mathbb{R}^3$, and we take $G = SU(2)$ and R to be the two-dimensional representation, then $Z_k(M; K, R)$ becomes the Jones polynomial, evaluated at

$$q = \exp(2\pi i/(k+2)).$$

(The analog for an arbitrary simple Lie group G is $q = \exp(2\pi i/(k+h)n_{\mathfrak{g}})$, where $n_{\mathfrak{g}}$ is the ratio of length squared of long and short roots of G.)

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(The analog for an arbitrary simple Lie group G is $q = \exp(2\pi i/(k+h)n_{\mathfrak{g}})$, where $n_{\mathfrak{g}}$ is the ratio of length squared of long and short roots of G.) This is only a discrete set of values of q, but of course these values are enough to determine a Laurent polynomial.

The argument that the invariant obtained from the three-dimensional gauge theory agrees with the Jones polynomial and its usual generalizations involved making contact with the work of Tsuchiya and Kanie, who as I remarked before had interpreted the Jones polynomial in terms of "conformal blocks" of two-dimensional conformal field theory.

The argument that the invariant obtained from the three-dimensional gauge theory agrees with the Jones polynomial and its usual generalizations involved making contact with the work of Tsuchiya and Kanie, who as I remarked before had interpreted the Jones polynomial in terms of "conformal blocks" of two-dimensional conformal field theory. The resulting link between three-dimensional gauge theory and two-dimensional conformal field theory has also been important in condensed matter physics, in studies of the quantum Hall effect and related phenomena.

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25 years ago, it seemed that this was the state of affairs: the gauge theory gives directly a good picture on a general oriented three-manifold M, but if one wants to understand from three-dimensional gauge theory the special things that happen for knots in $\mathbb{R}^3$, one has to proceed by first relating the three-dimensional gauge theory to some other approach (such as that of Tsuchiya and Kanie using two-dimensional conformal field theory).

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The volume conjecture arises if we specialize to the case of knots in $\mathbb{R}^3$, so that k does not have to be an integer. Usually the case $G = SU(2)$ is assumed and we let R be the n-dimensional representation of $SU(2)$. (The corresponding knot invariant is called the colored Jones polynomial.) Then we take $k \to \infty$ through non-integer values, with fixed k/n.

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where k_0 is a fixed complex number (while $n \to \infty$). The large n behavior is now a sum of contributions of *complex* critical points. By a complex critical point, I mean simply a critical point of the analytic continuation of the function $\mathrm{CS}(A)$.

We make this analytic continuation by replacing the gauge group G with its complexification, which I will call $G_{\mathbf{C}}$, replacing the G-bundle $E \to M$ by its complexification, which is a $G_{\mathbf{C}}$-bundle $E_{\mathbf{C}} \to M$, and replacing the connection A on E by a connection $\mathcal{A}$ on $E_{\mathbf{C}}$, which one can think of as a complex-valued connection.

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$$\mathrm{CS}(\mathcal{A}) = \frac{1}{4\pi}\int_M \mathrm{Tr}\left(\mathcal{A}\mathrm{d}\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right).$$

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On a general three-manifold M, a critical point of $\mathrm{CS}(\mathcal{A})$ is simply a complex-valued flat connection, corresponding to a homomorphism $\rho : \pi_1(M) \to G_{\mathbf{C}}$.

In the case of the volume conjecture with $M = \mathbb{R}^3$, the fundamental group is trivial, but we are supposed to also include a holonomy or Wilson loop operator $W_R(K) = \mathrm{Tr}_R \, \mathrm{Hol}_K(A)$ where R is the n-dimensional representation of $SU(2)$.

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Physicists know about various situations (involving "tunneling" problems) in which a path integral is dominated by a complex critical point, but usually this is a complex critical point that makes an exponentially small contribution. What really surprised me about the volume conjecture is that, for many knots, the dominant complex critical point makes an exponentially *large* contribution. In other words, the colored Jones polynomial has oscillatory behavior for $n \to \infty$, $k = k_0 = n$ if k_0 is an integer, but it grows exponentially in this limit as soon as k_0 is not an integer. (Concretely, that is because $k\mathrm{CS}(\mathcal{A})$ has a negative imaginary part, so $\exp(ik\mathrm{CS}(\mathcal{A}))$ grows exponentially for large k.)

This puzzled me for a while, but it turns out that one can find an ordinary integral that does the same thing:

$$I(k, n) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} e^{ik\theta} e^{2in\sin\theta}.$$

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This integral solves Bessel's equation (as a function of n) for any integer k. We want to think of k as an analog of the integer-valued parameter in the Chern-Simons gauge theory that we called by the same name. (The analogy between this toy integral and the problem studied in the volume conjecture is imperfect because in the toy problem, there is no reason for n to be an integer.)

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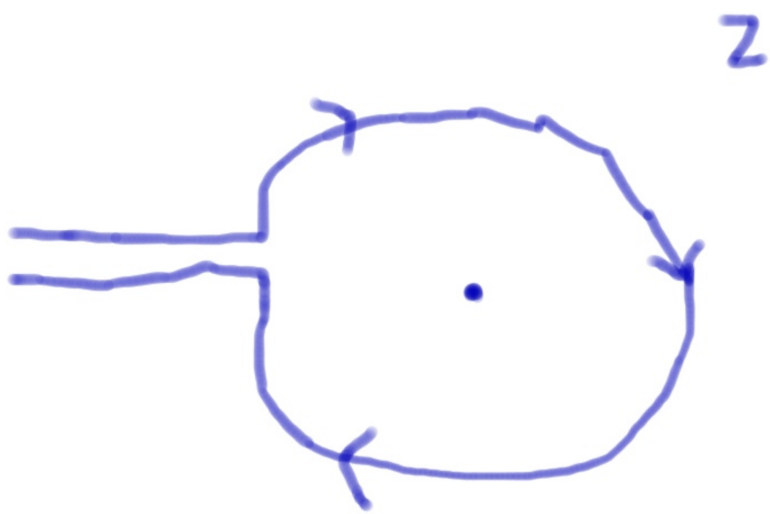
Here the integral is over the unit circle. At this point, k is still an integer. We want to get away from integer values while still obeying Bessel's equation. If $\mathrm{Re}\, n > 0$, this can be done by switching to the following integration contour:



The integral on the new contour converges and it agrees with the integral on the contour if k is an integer, since the extra parts of the contour cancel. But the new contour gives a continuation away from $k \in \mathbb{Z}$, still obeying Bessel's equation.

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We are trying to do an integral of the general form

$$\int_\Gamma \frac{\mathrm{d}z}{2\pi i z} \exp(kF(z))$$

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We note that because of the logarithm, $F(z)$ is multi-valued. To make the analysis properly, we should work on a cover of the punctured z-plane parametrized by $w = \log z$ on which F is single-valued:

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The next step is to find a useful description of all possible cycles on which the desired integral, which now is

$$\int_\Gamma \frac{\mathrm{d}w}{2\pi i} e^{kF(w)},$$

converges.

Morse theory gives an answer to this question. We consider the function $h(w, \overline{w}) = \mathrm{Re}(kF(w))$ as a Morse function.

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The key step is now the following: To every critical point p of h, we can define an integration cycle Γ_p, called a Lefschetz thimble, on which the integral we are trying to do converges.

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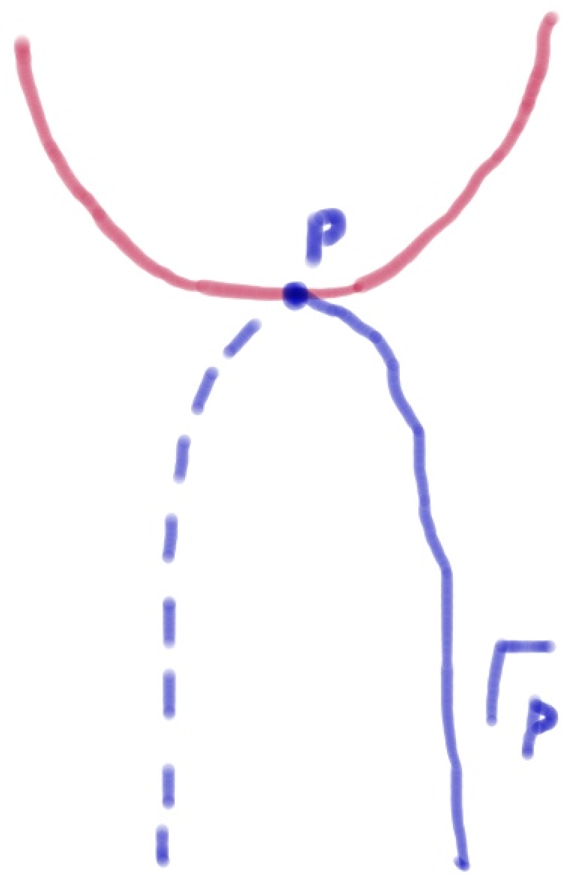
where t is a new "time" coordinate. The Lefschetz thimble Γ_p is defined as the space of all values at $t = 0$ of solutions of the gradient flow equation on the semi-infinite interval $(-\infty, 0]$ that start at p at $t = -\infty$.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converges. Moreover the large k asymptotics of I_p is straightforward:

$$I_p \sim \exp(kF(p)) \cdot (c_0 k^{-1/2} + \dots),$$

because along Γ_p, the real part of the exponent $kF(w)$ has a unique maximum at the point p.

Any other cycle Γ along which the integral converges can be expanded in terms of the Lefschetz thimbles:

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At this stage, I hope it is fairly clear what we should do to understand the analytic continuation to non-integer k of the quantum invariants of knots in $\mathbb{R}^3$, and also to understand the asymptotic behavior of the colored Jones polynomial that is studied in the volume conjecture.

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However, it probably is not clear that this will actually lead to a useful new understanding of the Jones polynomial. That was certainly not clear to me at this stage.

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It turns out that the same miracles happen when we do gradient flow on $\mathcal{U}$, with the Morse function $\mathrm{Re}(ik\mathrm{CS}(\mathcal{A}))$ and the obvious Kahler metric on $\mathcal{U}$

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$$F - \phi\wedge\phi + \star\mathrm{d}_A\phi = 0, \qquad \mathrm{d}_A \star \phi = 0.$$

It turns out that the same miracles happen when we do gradient flow on $\mathcal{U}$, with the Morse function $\mathrm{Re}(ik\mathrm{CS}(\mathcal{A}))$ and the obvious Kahler metric on $\mathcal{U}$

$$|\delta\mathcal{A}|^2 = -\int_M \mathrm{Tr}\,\delta\mathcal{A}\wedge\star\delta\overline{\mathcal{A}}.$$

The gradient flow equation is a gauge-fixed version of an elliptic differential equation that has full four-dimensional symmetry. This equation can be seen as a four-dimensional cousin of N. Hitchin's celebrated two-dimension equation. It is an equation for a pair A,ϕ, where A is a connection on a G-bundle $E\to X$, X being an oriented four-manifold, and $\phi\in\Omega^1(X,\mathrm{ad}(E))$ is a one-form on X valued in $\mathrm{ad}(E)$. The equations (for simplicity I take k real) read

$$F-\phi\wedge\phi+\star\mathrm{d}_A\phi=0,\qquad \mathrm{d}_A\star\phi=0.$$

They are flow equations for the three-dimensional connection $\mathcal{A}=A+i\phi$.

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Something unexpected happens, though it took a while for the implications to sink in. I actually had studied these equations before, in my work with A. Kapustin on gauge theory and geometric Langlands. In a moment we will discuss why this connection is relevant. (These equations – sometimes called the KW equations – have been studied recently in a series of papers by C. Taubes and also by M. Gagliardo and K. Uhlenbeck.)

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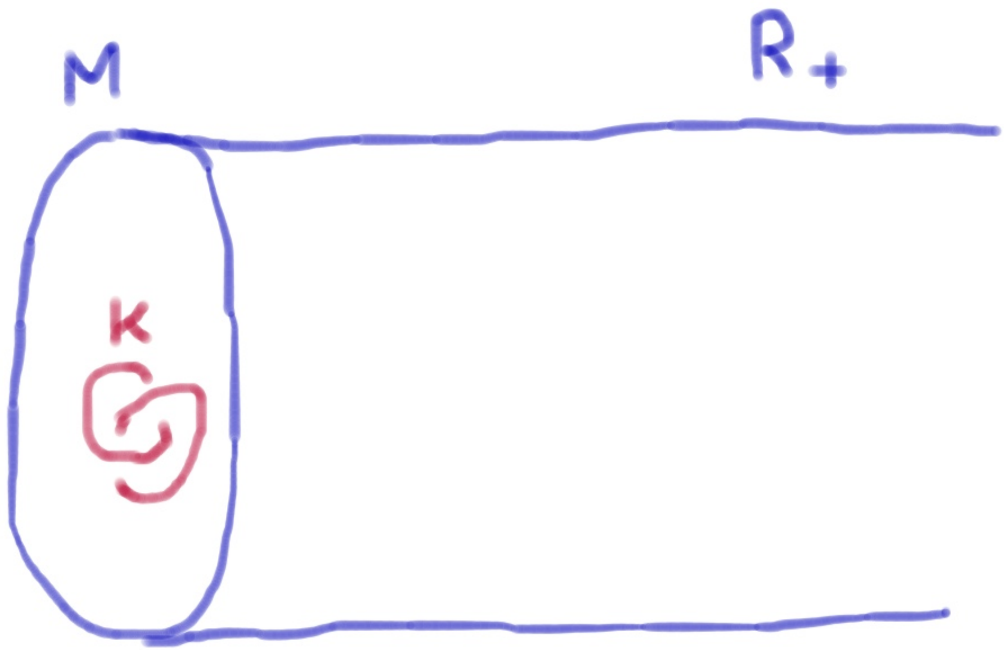

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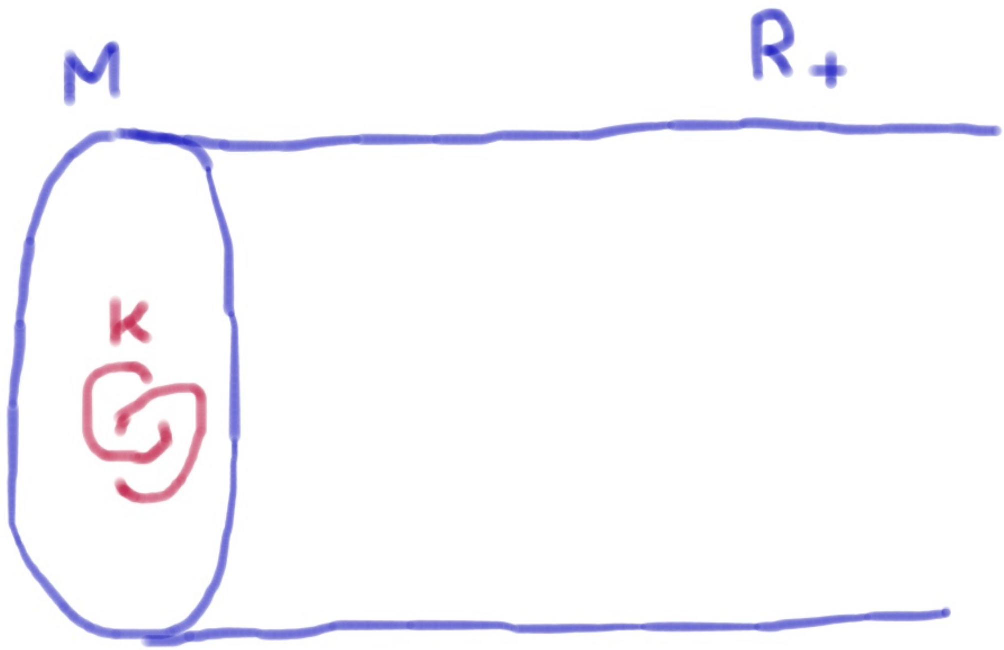

and define the thimble Γ_ρ to consist of all complex connections $\mathcal{A} = A + i\phi$ on $M \times \{0\} \subset M \times \mathbb{R}_+$ that are boundary values of solutions of the KW equations on $M \times \mathbb{R}_+$ that approach $\mathcal{A}_\rho$ at infinity.

For a general M, there are various choices of ρ. To understand the usual quantum knot invariants in this way, we would need to express the integration cycle $U \subset \mathcal{U}$ as a linear combination of the thimbles Γ_ρ. This is technically tricky and also the sort of answer it leads to is not so simple as there will be "Stokes phenomena." (I think it is likely that some things studied in the literature can be understood in this way and I did some very special cases in my paper on "Analytic Continuation Of Chern-Simons Theory.")

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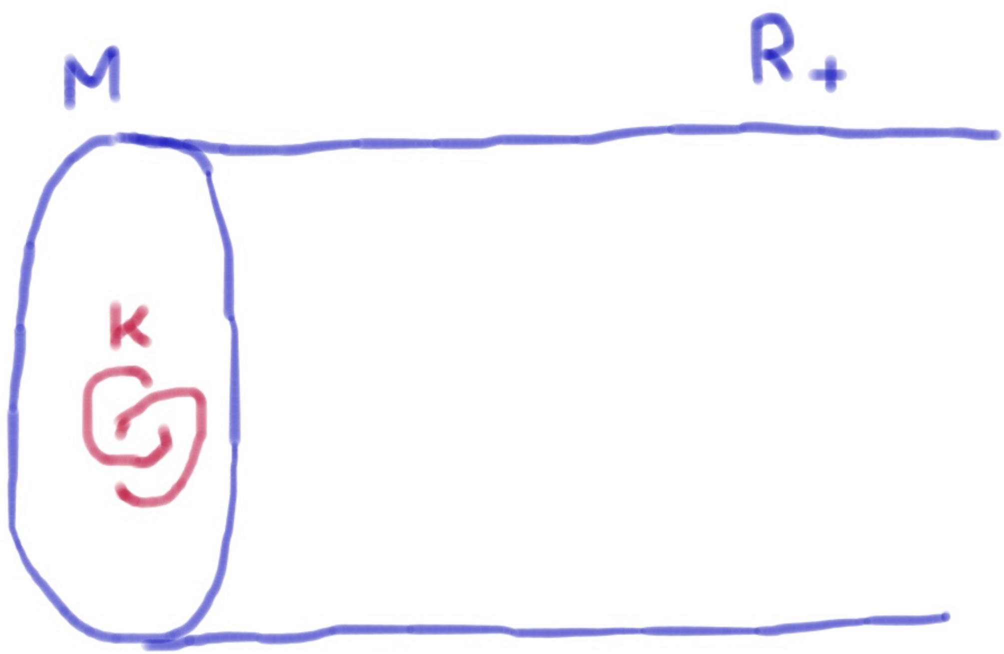
So the Jones polynomial is

$$\int_{\Gamma_0} D\mathcal{A} \exp(ik\mathrm{CS}(\mathcal{A})) \cdot \mathrm{Tr}_R \, \mathrm{Hol}(A, K)$$

where Γ_0 is the space of solutions of the KW equations on $\mathbb{R}^3 \times \mathbb{R}_+$ that vanish on $M \times \{\infty\}$ and $\mathcal{A}$ is the restriction of $A + i\phi$ to $M \times \{0\}$:



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Unfortunately, until quantum field theory is more familiar, this is going to be a hard answer for mathematicians to understand – just like the formula for the Jones polynomial by integration over the space U of real connections – because infinite-dimensional integration is unfamiliar.

Unfortunately, until quantum field theory is more familiar, this is going to be a hard answer for mathematicians to understand – just like the formula for the Jones polynomial by integration over the space U of real connections – because infinite-dimensional integration is unfamiliar. (But the conclusion we just reached could be verified mathematically, in an asymptotic expansion near $k = \infty$ or $q = 1$. Here one would run into an expansion in Feynman diagrams as in the work of Kontsevich and others on knot invariants related to the gauge theory.)

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However, one more step brings us into a world that is accessible mathematically, and also gives a new explanation of why the Jones polynomial is a Laurent polynomial in q. The step in question was also a key step in my work with Kapustin, and, more generally, in much of the work of physicists on the supersymmetric gauge theory in question. This is electric-magnetic duality, the four-dimensional analog of mirror symmetry in two dimensions. $\mathcal{N}=4$ super Yang-Mills theory with gauge group G and "coupling parameter" τ is equivalent to the same theory with gauge group $G^\vee$ – the Langlands or GNO dual of G – and coupling parameter $\tau^\vee=-1/n_{\mathfrak{g}}\tau$ ($n_{\mathfrak{g}}$ is as before the ratio of length squared of long and short roots).

To learn anything about our problem, we need to know what happens to the boundary condition at $\mathbb{R}^3 \times \{0\}$ under electric-magnetic duality. (This question is the analog of asking – in mirror symmetry – what is the object in the Fukaya category that is mirror to a given coherent sheaf.)

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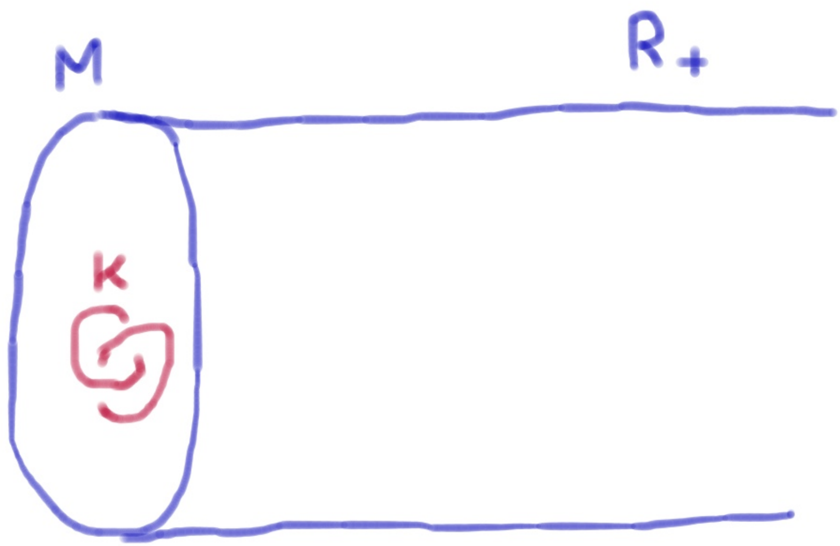
In this situation



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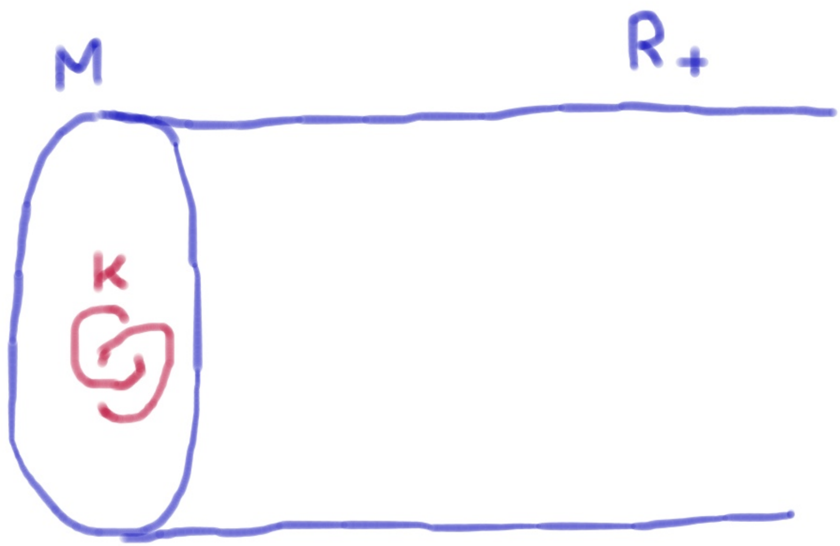

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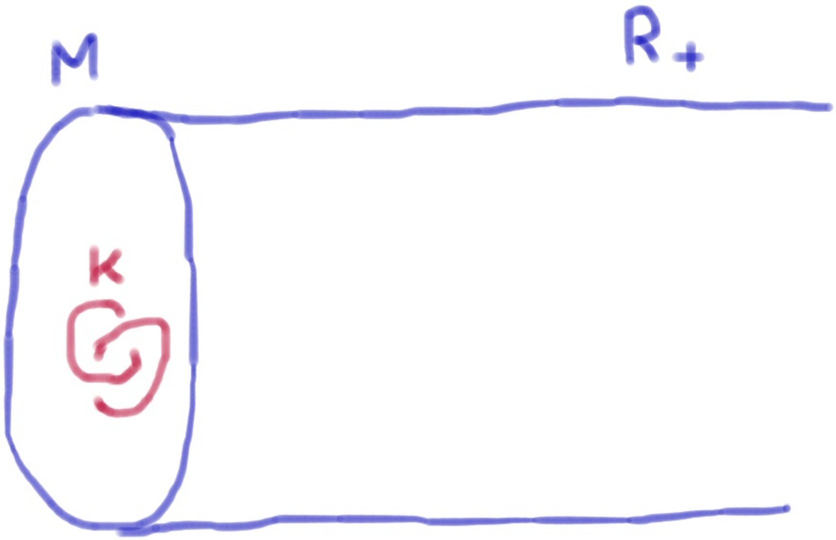

after making the duality transformation, the moduli space of solutions has expected dimension 0 and we just have to "count" (with signs, as in Donaldson theory) the number b_n of solutions for a given value n of the second Chern class. The boundary conditions depend on the knot K and the choice of the representation R by which it is labeled (some details at the workshop tomorrow).

The path integral gives

$$Z_q(K;R) = \sum_n b_n q^n$$

where $q = \exp(2\pi i/n_{\mathfrak{g}} k)$ and b_n is the "number" of solutions for given second Chern class n. This exhibits the Jones polynomial and the related quantum invariants of knots in three dimensions as "Laurent polynomials" in q with integer coefficients.

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