I Homotopy fiber products of homotopy theories in quantum algebra By Julie Bergner (UC Riverside) May 27, 2009

Think of $(\infty, 1)$-categories as models for homotopy theory! General principle: View complete Segal spaces as models for $(\infty, 1)$-categories/homotopy theories. This allows us to generalize model category constructions.

Recall: A model category $\mathcal{M}$ has 3 kinds of specified morphisms, namely weak equivalences, fibrations and cofibrations. Using this structure, we can define $\mathrm{Ho}(\mathcal{M})$.

- $\mathrm{Ho}(\mathcal{M})$ only depends on weak equivalences
- other kinds kinds of maps make $\mathrm{Ho}(\mathcal{M})$ nicely behaved.

Quillen functors (left and right) preserve cofibrations.

II Homotopy fiber products of model categories

Consider $\mathcal{M}_1 \xrightarrow{F_1} \mathcal{M}_1 \xleftarrow{F_2} \mathcal{M}_2$ for F_1 and F_2 left Quillen functors. Define $\mathcal{M} : \mathcal{M}_1 \times^h_{\mathcal{M}_3} \mathcal{M}_2$ to have

- objects (x_1, x_2, x_3, u, v) with x_i an object of $\mathcal{M}_i$ and
$$F_1(x_1) \underset{\sim}{\xrightarrow{u}} x_3 \underset{\sim}{\xleftarrow{v}} F_2(x_2).$$
- morphisms $(f_i : x_i \to y_i)$ such that
$$\begin{tikzcd} F_1(x_1) \arrow[r, "u"] \arrow[d, "f_1"] & x_3 \arrow[d, "f_3"] & F_2(x_2) \arrow[l, "v"'] \arrow[d, "f_2"] \\ F_1(x_1) \arrow[r, "u"] & x_3 & F_2(x_2) \arrow[l, "v"'] \end{tikzcd}$$
commutes.

Note: This carries a (levelwise) homotopy structure, but this is not the correct one. Hope would be to localize this model structure (via left Bousfield localization instead of a right one).

Look at compete Segal Spaces to determine whether this is the "correct" definition!

III Complete Segal spaces

Let W be a simplicial space, so $W : \Delta^{\mathrm{op}} \to \mathrm{sSet}$.

Definition III.1. W is a *Segal space* if $W_n \to W_1 \times_{W_0} ... \times_{W_0} W_1$ (n-times) is a weak equivalence for $n \geq 2$. ∎

Objects of $W : W_{0,0}$, mapping spaces $\mathrm{map}_W(x, y) =$ fiber over (x, y) of $W_1 \xrightarrow{d_1, d_0} W_{0 \times W_0}$, composition, identities, homotopy equivalences, homotopy category...

$$\begin{tikzcd} s_0 : W_0 \arrow[rr] \arrow[dr] & & W_1 \text{ homotopy equivalence} \subseteq W_1 \\ & W_n \arrow[ur] & \end{tikzcd}$$

Definition III.2. W is a *complete Segal space* if $W_0 \to W_h$ is a weak equivalence. ∎

Theorem III.3 (Rezk). *There is a model structure* CSS *on the category of simplicail spaces s.th. the fibrant-cofibrant onjects are complete Segal spacs. The weak equivalences between complete Segal spaces are level-wise.*

Connection with model categories: There is a functor L_C, taking a model category to complete Segal spaces.

$$\mathcal{M} \rightsquigarrow L_C(\mathcal{M}) = \mathrm{nerve}(\mathrm{we}\,\mathcal{M}^{[n]})$$

where objects in $\mathcal{M}^{[n]}$ are sequences of n composable morphisms in $\mathcal{M}$.

Theorem III.4. *$L_C\mathcal{M}$ looks like $\coprod_{\langle\alpha:x\to y\rangle} B\,\mathrm{Aut}^h(\alpha) \Rightarrow \coprod_{\langle x\rangle} B\,\mathrm{Aut}^h(x)$*

Question: Does taking L_c and taking the homotopy fibre product commute (when using only the weak equivalences in order to define $L_C\mathcal{M}$ (for $\mathcal{M} = \mathcal{M}_1 \times^h_{\mathcal{M}_3} \mathcal{M}_2$)?

Theorem III.5. *$L_C(\mathcal{M}_1 \times^h_{\mathcal{M}_3} \mathcal{M}_2)$ is weakly equivalent to $L_C\mathcal{M}_1 \times^h_{L_C\mathcal{M}_3} L_C\mathcal{M}_2$!*

IV Derived Hall algebras

Definition IV.1. Let $\mathcal{A}$ be an abelian category with fin. many iso. classes of objects. Its *Hall algebra* $\mathcal{H}(\mathcal{A})$ is:

- the vector space with basis the isom. classes of objects
- endowed with the multiplication by $A \cdot B = \sum_C g^C_{AB} C$, where C^C_{AB} is the Hall number:

$$g^C_{AB} = \frac{|0 \to A \to B \to C \to 0|}{|\mathrm{Aut}(A)||\mathrm{Aut}(B)|}$$ ∎

Motivation: Let $\mathfrak{g}$ be a Lie algebra of Type A, D, E and let Q be a quiver on its Dynkin diagram. Set $\mathrm{Rep}(Q)$ be the abelian category of $\mathbb{F}_q$-representations $\rightsquigarrow \mathcal{H}(\mathrm{Rep}(Q))$.

$\mathcal{H}(\mathrm{Rep}(Q))$ is closely related to one part of $\mathcal{U}_q(\mathfrak{g})$. **Question:** is there a way to enlarge $\mathcal{H}(\mathrm{Rep}(Q))$ so we can recover all of $\mathcal{U}_q(\mathfrak{g})$?

Conjecture: Want "Hall algebra" associated to $D^b(\mathrm{Rep}(Q))$, which is triangulated, but *not* abelian.

Need: "Derived" Hall algebras for triangulated categories. Toën's construction:

Definition IV.2. Let $\mathcal{M}$ be a model category which is stable (i.e., $\mathrm{Ho}(\mathcal{M})$ is triangulated), having certain finiteness conditions. Then the derived Hall algebra $\mathcal{DH}(\mathcal{M})$ has

- vector space with basis weak equivalence classes of "nice" objects of $\mathcal{M}$
- multiplication: $x \cdot y := \sum_z g^z_{x,y} z$, where $g^y_{x,y}$ is the "derived" Hall number

$$g^z_{x,y} = \frac{|[x,z]_y| \prod_{1>n} |\mathrm{Ext}^{-i}(x,z)|^{(-1)^i}}{|\mathrm{Aut}(x)| \prod_{i>0} |\mathrm{Ext}^{-i}(x,x)|^{(-1)^i}}$$

  where $\mathrm{Ext}^i(x,y) = [x, y[i]]$ ∎

(the upshot of this definition should be that it is somewhat a generalization of the definition of the Hall number from above and that we have an *explicit* formula!)

Connection to homotopy fiber products: They are used to prove that $\mathcal{DH}(\mathcal{M})$ is associative. Moreover, $\mathcal{DH}(\mathcal{M})$ only depends on $\mathrm{Ho}(\mathcal{M})$ and the formula works for any "finitely" triangulated category.

Problem: $D^b(\mathrm{Rep}(Q))$ is *not* finitely triangulated! A remedy could be to generalize the definition of $\mathcal{DH}(\mathcal{M})$ away from model categories Complete Segal spaces look like a promising place to do this!

$\rightsquigarrow$ Want to work in the more general CSS setting.

Theorem IV.3 (work in progress). *Translating Toën's construction into CSS and using homotopy pullbacks gives a derived Hall algebra $\mathcal{DH}(W)$ for any "finitary" stable complete Segal space.*