Матем. Сборник
Том 181 (1990), № 5

Math. USSR Sbornik
Vol. 70(1991), No. 1

# ENHANCED TRIANGULATED CATEGORIES

UDC 512.66

A. I. BONDAL AND M. M. KAPRANOV

ABSTRACT. A solution is given to the problem of describing a triangulated category generated by a finite number of objects. It requires the notion of "enhancement" of a triangulated category, by means of the complexes RHom.

The notion of triangulated category (see [1], [2], and [8]), widely used in homological algebra, is not entirely a satisfactory one, as observed, for example, in [18]. A basic principle of homological algebra is the need always to consider, not just the cohomology of complexes, but the complexes themselves. At the same time, the abelian group $\mathrm{Hom}_{\mathscr{D}}(E, F)$ in a triangulated category $\mathscr{D}$ has the form of the zeroth cohomology group of a certain chain complex (in the categories arising algebraically), or of the zeroth homotopy group of a certain $CW$-spectrum (in the categories arising topologically). This is the situation in the majority of cases encountered in practice. In particular, in the triangulated categories of homological algebra (the homotopy or derived categories) the group $\mathrm{Hom}_{\mathscr{D}}(E, F)$ is the zeroth cohomology group of the naturally arising complex $\mathrm{RHom}(E, F)$. Furthermore, the corresponding exact functors come from functors that preserve the complexes RHom.

An attempt to axiomatize the notion of "triangulated category with RHom complexes" leads to consideration of differential graded ($DG$-) categories. The $DG$-category structure should in a reasonable sense be compatible with the triangulated structure. Firstly, the cohomology of the RHom complexes should be naturally isomorphic to the Ext groups in the triangulated category. Secondly, there are two ways of defining the higher Massey products. The one starts from the $DG$-structure; the other, from the triangulated structure. A natural expectation for a $DG$-structure compatible with a triangulated structure is that both ways should lead to the same result.

In order to satisfy these (and other) requirements, we introduce in the present paper the notion of pretriangulated category (§3). This is a $DG$-category that has functorial cones of closed morphisms (and, more generally, convolutions of the so-called twisted complexes; see §1). The condition for existence of convolutions can be formulated in terms of representability of certain functors. Thus, a pretriangulated category is a $DG$-category with additional properties, not additional structure.

An important example of a pretriangulated category is the category $\text{Pre-Tr}(\mathscr{A})$ of twisted complexes over a $DG$-category $\mathscr{A}$ (see §1). The transition functor from $\mathscr{A}$ to $\text{Pre-Tr}(\mathscr{A})$ is the analogue of that of taking a free object in an algebraic theory.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 18E30.

In fact, a pretriangulated category can be defined as an algebra over a monad (in the sense of Quillen and Mac Lane [3]) constructed by means of the functor Pre-Tr.

A triangulated category, regarded as a mathematical structure, is not a universal algebra in the sense of Kurosh [4], i.e., a set with a given system of operations (in the same way that a topological space is not). In contrast, however, a pretriangulated category is such a universal algebra.

Along with the monad Pre-Tr we can define another one, by means of the one-sided twisted complexes. This is done in §4, where we describe the enhanced triangulated category generated by a finite number of objects.

If a triangulated category is generated by the elements of an ordered exceptional set (see [9] and [17]), the natural question arises as to how the Ext-algebra of the set gets transformed under mutations. Knowing the Ext-groups between elements of the original set is in general, it turns out, insufficient to answer the question. But at the level of $DG$-algebras mutations can be followed through completely; one need only consider a pretriangulated enhancement of the category (see §5).

## §1. $DG$-categories and twisted complexes over them

A *preadditive category* [5] is a category $\mathscr{A}$ in which all the sets $\mathrm{Hom}_{\mathscr{A}}(E, F)$ are provided with abelian group structures and the composition of morphisms is bilinear. A preadditive category with a single object amounts simply to a ring. An *additive category* is a preadditive category that has finite direct sums.

By a *DG-category* is meant a preadditive category $\mathscr{A}$ in which all the abelian groups $\mathrm{Hom}_{\mathscr{A}}(E, F)$ are provided with a $Z$-grading and a differential $d$ of degree $+1$, i.e., with the structure of a complex. Furthermore, it is required that the composition morphisms

$$\mathrm{Hom}_{\mathscr{A}}(E, F) \otimes \mathrm{Hom}_{\mathscr{A}}(F, G) \to \mathrm{Hom}_{\mathscr{A}}(E, G)$$

be morphisms of complexes, and in addition that every $E \in \mathscr{A}$ satisfy the equality $d(\mathrm{id}_E) = 0$. We denote by $\mathrm{Hom}_{\mathscr{A}^k}(E, F)$ the $k$th graded component of the complex $\mathrm{Hom}_{\mathscr{A}}(E, F)$.

Every differential graded ring can be regarded as a $DG$-category with a single object. Another important example of a $DG$-category is constructed as follows. Let $\mathscr{A}$ be an additive category. Consider the category $C(\mathscr{A})$ whose objects are the cochain complexes over $\mathscr{A}$. If $K^{\cdot}$ and $L^{\cdot}$ are such complexes, then

$$\mathrm{Hom}^m_{C(\mathscr{A})}(K^{\cdot}, L^{\cdot}) = \bigoplus_{j-i=m} \mathrm{Hom}_{\mathscr{A}}(K^i, L^j).$$

For

$$f = \{f_i \colon K^i \to L^{i+m}\} \in \mathrm{Hom}^m(K^{\cdot}, L^{\cdot})$$

put

$$df = \{f_{i+1} d_K + (-1)^{m+1} d_L f_i \colon K^i \to L^{i+m+1}\},$$

where $d_K$ and $d_L$ are the differentials in $K^{\cdot}$ and $L^{\cdot}$. This gives $C(\mathscr{A})$ the structure of a $DG$-category. The closed morphisms of degree 0 are the morphisms in the ordinary sense. "Exact" morphisms are those that are homotopic to zero. With any $DG$-category $\mathscr{A}$ is associated a graded category $H(\mathscr{A})$, called the *cohomology category* of $\mathscr{A}$. The objects in $H(\mathscr{A})$ are the same as in $\mathscr{A}$, and the morphisms from $X$ to $Y$ are defined as the cohomology of the complex $\mathrm{Hom}_{\mathscr{A}}(X, Y)$.

Restricting ourselves to the zeroth cohomology of the complexes Hom, we obtain a preadditive category $H^0(\mathscr{A})$. Thus, for example, for an additive category $\mathscr{B}$ the category $H^0C(\mathscr{B}) = \mathrm{Hot}(\mathscr{B})$ is the homotopy category of complexes over $\mathscr{B}$. We also let $\mathscr{A}^{\natural}$ be the graded category obtained from $\mathscr{A}$ by forgetting the differential on morphisms.

By a *DG-functor* between $DG$-categories $\mathscr{A}$ and $\mathscr{B}$ we mean an additive functor $f\colon \mathscr{A} \to \mathscr{B}$ that preserves the grading and differential on morphisms. Such a functor induces a functor $H(f)\colon H(\mathscr{A}) \to H(\mathscr{B})$. We call a $DG$-functor $f$ a *quasi-isomorphism* (*quasi-equivalence*) if $H(f)$ is a category isomorphism (equivalence).

In particular, if $\mathscr{A}$ is the $DG$-category with a single object that corresponds to the $DG$-algebra $A$, and $\mathscr{B} = C(\mathscr{A}b)$, then a $DG$-functor from $\mathscr{A}$ to $\mathscr{B}$ is simply a left $DG$-module over $A$. The category of $DG$-categories (with $DG$-functors as morphisms) will be denoted by $DG$-Cat.

If $\mathscr{A}$ and $\mathscr{A}'$ are two $DG$-categories, then the set of covariant $DG$-functors $\mathscr{A} \to \mathscr{A}'$ is itself the set of objects of a $DG$-category, which we denote by $DG$-$\mathrm{Fun}(\mathscr{A}, \mathscr{A}')$. Namely, let $\varphi$ and $\psi$ be two $DG$-functors. Put $\mathrm{Hom}^k(\varphi, \psi)$ equal to the set of natural transformations $t\colon \varphi^{\natural} \to \psi^{\natural}[k]$ of graded functors from $\mathscr{A}^{\natural}$ to $\mathscr{B}^{\natural}$ (i.e., for each object $E \in \mathrm{Ob}\mathscr{A}$, $t(E) \in \mathrm{Hom}_{\mathscr{B}^k}(\varphi(E), \psi(E))$.

On each $E$, the differential of the transformation $t$ is equal to $dt(E)$. Thus, the closed morphisms of degree 0 are the $DG$-transformations of $DG$-functors. A similar definition gives us the $DG$-category $DG$-$\mathrm{Fun}^0(\mathscr{A}, \mathscr{A}')$ consisting of the contravariant $DG$-functors.

DEFINITION 1 [6]. Let $\mathscr{A}$ be a $DG$-category. A *twisted complex over* $\mathscr{A}$ is a set $\{(E_i)_{i\in \mathbf{Z}}, q_{ij}\colon E_i \to E_j\}$ where the $E_i$ are objects in $\mathscr{A}$, equal to 0 for almost all $i$, and the $q_{ij}$ are morphisms in $\mathscr{A}$ of degree $i-j+1$, satisfying the condition $dq_{ij} + \sum_k q_{kj}q_{ik} = 0$.

EXAMPLE. If $\mathscr{A} = C(\mathscr{B})$ is the category of complexes over an additive category $\mathscr{B}$, then a twisted complex over $\mathscr{A}$ is simply a complex over $\mathscr{B}$ in which each term has an additional grading. This generalizes the notion of twisted complex over $\mathscr{B}$ in the sense of O'Brian, Toledo, and Tong [7].

The twisted complexes over a $DG$-category $\mathscr{A}$ themselves form a $DG$-category, in the following fashion. Let $C = \{E_i, q_{ij}\}$ and $C' = \{E'_i, q'_{ij}\}$ be two twisted complexes. Put

$$\mathrm{Hom}^k(C, C') = \coprod_{l+j-i=k} \mathrm{Hom}^l_{\mathscr{A}}(E_i, E'_j)$$

and, for any $f \in \mathrm{Hom}^l_{\mathscr{A}}(E_i, E'_j)$,

$$df = d_{\mathscr{A}} f + \sum_m (q_{jm} f + (-1)^{l(i-m+1)} f q_{mi}),$$

where $d_{\mathscr{A}}$ is the differential on morphisms of the $DG$-category $\mathscr{A}$.

The closed morphisms of degree 0 between twisted complexes will be called *twisted morphisms.*

Let $\mathscr{A}^{\oplus}$ be the $DG$-category obtained from $\mathscr{A}$ by adjoining finite formal direct sums of objects. (If $\mathscr{A}$ already has direct sums, then $\mathscr{A}^{\oplus}$ is equivalent to $\mathscr{A}$ as a $DG$-category.) The $DG$-category of twisted complexes over $\mathscr{A}^{\oplus}$ will be denoted by $\mathrm{Pre\text{-}Tr}(\mathscr{A})$, and its zeroth cohomology category by $\mathrm{Tr}(\mathscr{A})$.

If $\mathscr{A}'$ is a second $DG$-category, and $F\colon \mathscr{A} \to \mathscr{A}'$ a $DG$-functor, we have the $DG$-functor

$$\mathrm{Pre\text{-}Tr}(F)\colon \mathrm{Pre\text{-}Tr}(\mathscr{A}) \to \mathrm{Pre\text{-}Tr}(\mathscr{A}')$$

and the functor

$$\mathrm{Tr}(F)\colon \mathrm{Tr}(\mathscr{A}) \to \mathrm{Tr}(\mathscr{A}').$$

If $\mathscr{A}$ is a preadditive category with trivial $DG$-structure, then $\mathrm{Tr}(\mathscr{A}) = \mathrm{Hot}(\mathscr{A}^{\oplus})$.

DEFINITION 2. Let $\mathscr{A}$ be a $DG$-category, $C = \{E_i, q_{ij}\}$ and $C' = \{E'_i, q'_{ij}\}$ two objects in $\text{Pre-Tr}(\mathscr{A})$, and $f = \{f_{ij}\colon E_i \to E'_j\}$ a twisted morphism from $C$ to $C'$. By the *cone* of this morphism we mean the object $\mathrm{Cone}\, f = \{E''_i, q''_{ij}\}$ for which

$$E''_i = E_i \oplus E'_{i-1}, \qquad q''_{ij} = \begin{pmatrix} q_{ij} & f_{ij} \\ 0 & q'_{ij} \end{pmatrix}.$$

We have in $\text{Pre-Tr}(\mathscr{A})$ the natural closed morphisms

$$\begin{array}{ccc} C & \xrightarrow{f} & C' \\ & {}^{+1}\nwarrow \quad \swarrow & \\ & \mathrm{Cone}\, f & \end{array}, \tag{1.1}$$

inducing also morphisms in $\mathrm{Tr}(\mathscr{A})$.

By the *distinguished triangles* in $\mathrm{Tr}(\mathscr{A})$ we shall mean the triangles isomorphic to those of the form (1.1).

PROPOSITION 1. *The category* $\mathrm{Tr}(\mathscr{A})$ *with the above-described set of distinguished triangles is a triangulated category.*

Before proving this proposition, we exhibit a construction for another triangulated category also connected with a $DG$-category. This is a generalization of the scheme: algebra $\to$ homotopy category of modules, to the case of $DG$-structures (see [6]).

Let $\mathscr{A}$ be a $DG$-category. We examine the contravariant $DG$-functors from $\mathscr{A}$ to the $DG$-category $C(\mathscr{A}b)$ of complexes of abelian groups. For any $DG$-functor $\varphi\colon \mathscr{A} \to C(\mathscr{A}b)$ we denote by $\overset{+}{\varphi}{}^{\circ}$ the functor from $\mathscr{A}$ to graded abelian groups that is obtained from $\varphi$ by forgetting the differential, and by $H(\varphi)$ the functor, again to graded abelian groups, that is obtained from $\varphi$ by taking cohomology of complexes. The category of 0th cohomology of the $DG$-category $DG\text{-Fun}^0(\mathscr{A}, C(\mathscr{A}b))$ we denote by $\mathrm{Hot}(\mathscr{A})$. If $\mathscr{A}$ has a single object and trivial $DG$-structure, i.e., if it amounts simply to a ring, then $\mathrm{Hot}(\mathscr{A})$ is the homotopy category of complexes of modules over that ring.

Let $t\colon \varphi \to \psi$ be a $DG$-transformation of contravariant $DG$-functors $\mathscr{A} \to C(\mathscr{A}b)$. We define a new $DG$-functor, $\mathrm{Cone}(t)\colon \mathscr{A} \to C(\mathscr{A}b)$, assigning to an object $E \in \mathrm{Ob}\,\mathscr{A}$ the complex $\mathrm{Cone}\{t_E\colon \varphi(E) \to \psi(E)\}$.

We have in $DG\text{-Fun}^0(\mathscr{A}, C(\mathscr{A}b))$

$$\begin{array}{ccc} \varphi & \xrightarrow{t} & \psi \\ & {}^{+1}\nwarrow \quad \swarrow & \\ & \mathrm{Cone}(t) & \end{array}, \tag{1.2}$$

determining also a triangle in $\mathrm{Hot}(\mathscr{A})$.

By the *distinguished triangles* in $\mathrm{Hot}(\mathscr{A})$ we shall mean the triangles isomorphic to those obtained from triangles of the form (1.2).

PROPOSITION 2. *The category* $\mathrm{Hot}(\mathscr{A})$ *with the above-described set of distinguished triangles and componentwise translation functor is a triangulated category.*

PROOF. The verification of Verdier's axioms TR1–TR4 for the case of the ordinary homotopy category of complexes, as carried out, e.g., in [2], applies without change to the present case, which is in essence the case of complexes with a system of operators (generating the $DG$-category).

We now give a definition of representable functor in the present situation. Let $E$ be an object in the $DG$-category $\mathscr{A}$. It determines a contravariant $DG$-functor $h_E\colon \mathscr{A} \to C(\mathscr{A}b)$ that takes $F \in \mathrm{Ob}\,\mathscr{A}$ into the complex $\mathrm{Hom}^{\cdot}_{\mathscr{A}}(F, E)$. The assignment $E \mapsto h_E$ gives a covariant $DG$-functor

$$h\colon \mathscr{A} \to DG\text{-}\mathrm{Fun}^0(\mathscr{A}, C(\mathscr{A}b)).$$

As in the "classical" case (see [18]), one verifies that the functor $h$ is fully strict, i.e., that there exist isomorphisms of complexes

$$\mathrm{Hom}^{\cdot}_{\mathscr{A}}(E, E') \simeq \mathrm{Hom}_{DG\text{-}\mathrm{Fun}^0(\mathscr{A}, C(\mathscr{A}b))}(h_E, h_{E'}). \tag{1.3}$$

A contravariant $DG$-functor $h\colon \mathscr{A} \to C(\mathscr{A}b)$ will be called *representable* if it is isomorphic (as a $DG$-functor) to a functor of the form $h_E$ for some $E \in \mathrm{Ob}\,\mathscr{A}$.

DEFINITION 3. Let $\mathscr{A}$ be a $DG$-category. We define an imbedding of $DG$-categories

$$\alpha\colon \text{Pre-Tr}(\mathscr{A}) \to DG\text{-}\mathrm{Fun}^0(\mathscr{A}, C(\mathscr{A}b)).$$

The imbedding assigns to an object $K = \{E_i, q_{ij}\} \in \mathrm{Ob}\,\text{Pre-Tr}(\mathscr{A})$ the following $DG$-functor $\alpha(K)\colon \mathscr{A} \to C(\mathscr{A}b)$. For each $E \in \mathrm{Ob}\,\mathscr{A}$ the value $\alpha(K)(E)$ is the graded abelian group $\bigoplus \mathrm{Hom}_{\mathscr{A}}(E, E_i)[i]$ provided with the differential $d+Q$, where $Q = \|q_{ij}\|$ and $d$ is the differential in $\bigoplus \mathrm{Hom}_{\mathscr{A}}(E, E_i)[i]$.

PROPOSITION 3. (a) *The functor* $\alpha$ *is an imbedding of* $\text{Pre-Tr}(\mathscr{A})$ *into* $DG\text{-}\mathrm{Fun}^0(\mathscr{A}, C(\mathscr{A}b))$ *as a full DG-subcategory, and it takes the cone of a closed morphism* $f$ *in* $\text{Pre-Tr}(\mathscr{A})$ *into the cone of the morphism* $\alpha(f)$ *in* $DG\text{-}\mathrm{Fun}^0(\mathscr{A}, C(\mathscr{A}b))$.

(b) *The cohomology functor* $H(\alpha)$ *gives an imbedding of* $\mathrm{Tr}(\mathscr{A})$ *into* $\mathrm{Hot}(\mathscr{A})$ *as a full triangulated subcategory.*

The proof consists in verifying the definitions. Part (a) implies Proposition 1, making possible the statement of part (b).

## §2. The monad connected with the functor Pre-Tr

The functor Pre-Tr constructed in the preceding section allows us to define a monad over the category $DG$-Cat. This makes it possible to define a pretriangulated category as an algebra over this monad.

We recall that a monad in a category $\mathscr{B}$ (see [3]) is a functor $C\colon \mathscr{B} \to \mathscr{B}$ together with natural transformations $\mu\colon C \circ C \to C$ and $\eta\colon \mathrm{id}_{\mathscr{B}} \to C$ such that for every $B \in \mathrm{Ob}\,\mathscr{B}$ the composite morphisms

$$C(B) \xrightarrow{\eta_{C(B)}} C(C(B)) \xrightarrow{\mu_B} C(B)$$

and

$$C(B) \xrightarrow{C(\eta_B)} C(C(B)) \xrightarrow{\mu_B} C(B)$$

are the identity, and the diagram

$$\begin{tikzcd} C(C(C(B))) \arrow[r, "\mu_{C(B)}"] \arrow[d, "C(\mu_B)"'] & C(C(B)) \arrow[d, "\mu_B"] \\ C(C(b)) \arrow[r, "\mu_B"] & C(B) \end{tikzcd}$$

commutes.

Let $\mathscr{B} = DG$-Cat be the category of $DG$-categories, and $\mathscr{A} \in \mathrm{Ob}\,\mathscr{B}$ a given $DG$-category.

We construct a $DG$-functor

$$\mathrm{Tot}_{\mathscr{A}}\colon \text{Pre-Tr}(\text{Pre-Tr}(\mathscr{A})) \to \text{Pre-Tr}(\mathscr{A}).$$

Namely, an object in $\text{Pre-Tr}(\text{Pre-Tr}(\mathscr{A}))$ can be regarded as a set $C = \{(C_{ij})_{i,j\in\mathbf{Z}},$ $q_{ij,kl}\colon C_{ij} \to C_{kl}\}$ with appropriate differential conditions on the $q_{ij,kl}$. Put

$$\mathrm{Tot}_{\mathscr{A}}(C) = \{(D_k)_{k\in\mathbf{Z}},\ r_{kl}\colon D_k \to D_l\},$$

where

$$D_k = \bigoplus_{i+j=k} C_{ij}, \qquad r_{kl} = \|q_{ij,mn}\|, \qquad i+j=k,\ m+n=l.$$

We shall call $\mathrm{Tot}_{\mathscr{A}}(C)$ the *convolution* of the twisted complex $C$ over $\text{Pre-Tr}(\mathscr{A})$.

Clearly, the correspondence $\mathscr{A} \mapsto \mathrm{Tot}_{\mathscr{A}}$ extends to a natural transformation

$$\mathrm{Tot}\colon \text{Pre-Tr} \circ \text{Pre-Tr} \to \text{Pre-Tr}$$

on the category $DG$-Cat.

We denote by $\varepsilon_{\mathscr{A}}$ the natural imbedding of $\mathscr{A}$ into $\text{Pre-Tr}(\mathscr{A})$ as a full $DG$-subcategory: $\varepsilon_{\mathscr{A}}(E)$ is the set consisting of just $E$ at the zeroth position. Thus, $\varepsilon$ is a natural transformation $\mathrm{id} \to \text{Pre-Tr}$ on the category $DG$-Cat.

PROPOSITION 1. *The functor*

$$\text{Pre-Tr}\colon DG\text{-Cat} \to DG\text{-Cat}$$

*and the natural transformations*

$$\varepsilon\colon \mathrm{id} \to \text{Pre-Tr}, \qquad \mathrm{Tot}\colon \text{Pre-Tr}(\text{Pre-Tr}) \to \text{Pre-Tr}$$

*define a monad over the category* $DG$-Cat.

The proof consists of a straightforward verification.

PROPOSITION 2. *The functor* $\mathrm{Tot}_{\mathscr{A}}$ *is an equivalence of DG-categories, and the cohomology functor*

$$H^0(\mathrm{Tot}_{\mathscr{A}})\colon \mathrm{Tr}(\text{Pre-Tr}(\mathscr{A})) \to \mathrm{Tr}(\mathscr{A})$$

*is an equivalence of triangulated categories.*

PROOF. If $C$ and $C'$ are any two objects in $(\text{Pre-Tr})^2(\mathscr{A})$, then the complexes

$$\mathrm{Hom}_{(\text{Pre-Tr})^2(\mathscr{A})}(C, C') \quad\text{and}\quad \mathrm{Hom}_{\text{Pre-Tr}(\mathscr{A})}(\mathrm{Tot}\,C, \mathrm{Tot}\,C')$$

are the same. In other words, $\mathrm{Tot}_{\mathscr{A}}$ is an equivalence of $DG$-categories. Furthermore, by Proposition 1, it preserves convolutions of twisted complexes. Therefore $H^0(\mathrm{Tot}_{\mathscr{A}})$ is an equivalence of triangulated categories.

We shall need still another property of the functor

$$\mathrm{Tot}_{\mathscr{A}}\colon (\text{Pre-Tr})^2(\mathscr{A}) \to \text{Pre-Tr}(\mathscr{A}).$$

Namely, let $\mathscr{A}$ be a $DG$-category; $C = \{E_i, q_{ij}\}$ a twisted complex over Pre-Tr$(\mathscr{A})$, i.e., an object in $(\text{Pre-Tr})^2(\mathscr{A})$; and $X$ an object in Pre-Tr$(\mathscr{A})$. We have then the twisted complexes $\mathrm{Hom}(X, C)$ and $\mathrm{Hom}(C, X)$ over the $DG$-category $C(\mathscr{A}b)$. For example,

$$\mathrm{Hom}(C, X) = \{\mathrm{Hom}_{\text{Pre-Tr}(\mathscr{A})}(E_i, X), \tilde{q}_{ij}\},$$

where the $\tilde{q}_{ij}$ are the mappings of the Hom groups induced by the $q_{ij}$. As noted above (the example after Definition 1), a twisted complex over $C(\mathscr{A}b)$ is an ordinary complex of abelian groups (the convolution) with additional grading in the terms. Denote by $\mathrm{THom}(X, C)$ and $\mathrm{THom}(C, X)$ the convolutions of the twisted complexes $\mathrm{Hom}(X, C)$ and $\mathrm{Hom}(C, X)$ over $C(\mathscr{A}b)$, and put $T = \mathrm{Tot}(C) \in \text{Pre-Tr}(\mathscr{A})$.

PROPOSITION 3. *In the above-described situation, there exist natural isomorphisms of complexes*

$$\mathrm{THom}(X, C) \simeq \mathrm{Hom}_{\text{Pre-Tr}(\mathscr{A})}(X, T), \qquad \mathrm{THom}(C, X) \simeq \mathrm{Hom}_{\text{Pre-Tr}(\mathscr{A})}(T, X). \quad \blacksquare$$

## §3. Pretriangulated categories

DEFINITION 1. A $DG$-category $\mathscr{E}$ is called *pretriangulated* if for every twisted complex $K \in \text{Pre-Tr}(\mathscr{E})$ the corresponding contravariant $DG$-functor $\alpha(K)\colon \mathscr{E} \to C(\mathscr{A}b)$ (see §1, Definition 3) is representable.

PROPOSITION 1. *Every pretriangulated category has the structure of an algebra* $(\mathscr{E}, T)$ *over the monad* (Pre-Tr, Tot, $\varepsilon$) (*see* [3]) *in the category* $DG$-Cat, *and the functors*

$$T\colon \text{Pre-Tr}(\mathscr{E}) \to \mathscr{E}, \qquad \varepsilon_{\mathscr{E}}\colon \mathscr{E} \to \text{Pre-Tr}(\mathscr{E})$$

*are quasi-inverse equivalences of* $DG$*-categories.*

PROOF. For $K \in \text{Pre-Tr}(\mathscr{E})$, let $T(K)$ be a representing object for the functor $\alpha(K)$. From (1.3) it follows that the correspondence $K \mapsto T(K)$ extends to a $DG$-functor $T\colon \text{Pre-Tr}(\mathscr{E}) \mapsto \mathscr{E}$ such that $T \circ \varepsilon_{\mathscr{E}} = \mathrm{id}_{\mathscr{E}}$, $T$ and $\varepsilon_{\mathscr{E}}$ are quasi-inverse, and the diagram

$$\begin{tikzcd}
(\text{Pre-Tr})^2(\mathscr{E}) \arrow[r, "\mathrm{Tot}"] \arrow[d, "\text{Pre-Tr}(T)"'] & \text{Pre-Tr}(\mathscr{E}) \arrow[d, "T"] \\
\text{Pre-Tr}(\mathscr{E}) \arrow[r, "T"] & \mathscr{E}
\end{tikzcd} \tag{3.1}$$

commutes. $\blacksquare$

The functor $T$ will be called a *convolution* (of twisted complexes). Its definition depends on the choice of representing objects. Another choice gives a functor $T'$, which is connected with $T$ by a canonical $DG$-functor isomorphism; hence the arbitrariness in the choice is inessential. In what follows we shall assume a fixed choice for the functor $T$.

A pretriangulated category is provided with a translation functor:

$$E[i] = T(\varepsilon(E)[i]).$$

The simplest nontrivial example of a twisted complex over a $DG$-category $\mathscr{E}$ is obtained from a closed morphism $f\colon E \to F$ of degree 0. More precisely, $f$ determines $C_f = \{E_i, q_{ij}\}$, where $E_0 = E$, $E_1 = F$, $E_j = 0$ for $j \neq 0, 1$, $q_{01} = f$, and the remaining $q_{ij}$ are 0. If the category $\mathscr{E}$ is pretriangulated, then the convolution of $C_f$ will be called the *cone* of the closed morphism $f$ and denoted by $\mathrm{Cone}(f) \in \mathrm{Ob}(\mathscr{E})$.

Corresponding to the diagram of closed morphisms

$$\varepsilon(E) \xrightarrow{f} \varepsilon(F) \to C_f \to \varepsilon(E)[1]$$

in the category $\text{Pre-Tr}(\mathscr{E})$ is the diagram of closed morphisms

$$E \xrightarrow{f} F \to \mathrm{Cone}(f) \to E[1]$$

in the category $\mathscr{E}$, and consequently a diagram in the category $H^0(\mathscr{E})$. By the *distinguished triangles in* $H^0(\mathscr{E})$ we mean those diagrams isomorphic to diagrams of this form.

PROPOSITION 2. *Let $\mathscr{E}$ be a pretriangulated category. Then the category $H^0(\mathscr{E})$ with the above-defined translation functor and family of distriguished triangles is triangulated. The functor $H^0(T)$*: $\mathrm{Tr}(\mathscr{E}) \to H^0(\mathscr{E})$ *is an equivalence of triangulated categories.*

PROOF. The verification of Verdier's axioms in the case of the homotopy category of complexes [2] reduces to the construction of certain standard diagrams of closed morphisms of complexes. The construction carries over verbatim to the case of twisted complexes. Consequently, the required standard diagrams can be constructed in the category $\text{Pre-Tr}(\mathscr{E})$. Applying the functor $T$, we obtain corresponding standard diagrams in $\mathscr{E}$, whose existence then establishes the validity of Verdier's axioms in $H^0(\mathscr{E})$. By definition, the functor $T$ is a category equivalence. The fact that it is exact follows from the commutativity of the diagram (3.1).

Given a triangulated category $\mathscr{D}$, by an *enhancement* of $\mathscr{D}$ we shall mean a pretriangulated category $\mathscr{E}$ together with an equivalence $H^0(\mathscr{E}) \to \mathscr{D}$ of triangulated categories. The category $\mathscr{D}$ itself is then said to be *enhanced*.

EXAMPLES. 1. Let $\mathscr{A}$ be an additive category. The homotopy category $\mathrm{Hot}(\mathscr{A})$ is an enhanced triangulated category. The corresponding pretriangulated category is $C(\mathscr{A})$.

2. More generally, for any $DG$-category $\mathscr{A}$, the category $\text{Pre-Tr}(\mathscr{A})$ is pretriangulated.

3. Let $\mathscr{A}$ be an abelian category with enough injectives. The triangulated category $D^b(\mathscr{A})$ is equivalent to the full subcategory in the homotopy category $\mathrm{Hot}^+(\mathscr{A})$ consisting of those complexes all of whose terms are injective and almost all of whose homology groups are zero. If we consider the corresponding full subcategory in $C^+(\mathscr{A})$, we obtain an enhancement of the category $D^b(\mathscr{A})$ ,which we denote by $\text{Pre-}D^b(\mathscr{A})$.

A similar construction can be made in the case of enough projectives.

4. Let $\mathscr{A}$ and $\mathscr{E}$ be two $DG$-categories, with $\mathscr{E}$ pretriangulated. We define on the $DG$-category $DG\text{-Fun}(\mathscr{A}, \mathscr{E})$ a pretriangulated structure, by defining the convolution of twisted complexes of $DG$-functors objectwise. In particular, the category of $DG$-modules over a $DG$-algebra $A$ (morphisms are morphisms that preserve grading and commute with multiplication by elements of $A$, but not necessarily with differentials) is a pretriangulated category. The corresponding enhanced triangulated category is $\mathrm{Hot}(A)$.

REMARK. A construction like that in Example 4 is impossible in the framework of triangulated categories.

DEFINITION 2. Let $\mathscr{E}$ and $\mathscr{E}'$ be two pretriangulated categories.

a) We say that $\mathscr{E}$ is a *full pretriangulated subcategory* of $\mathscr{E}'$ if $\mathscr{E}$ is a full $DG$-subcategory of $\mathscr{E}'$ and the functor $T_{\mathscr{E}'}$ takes Pre-Tr($\mathscr{E}$) into $\mathscr{E}$ and coincides with $T_{\mathscr{E}}$ (remember that the functor $T_{\mathscr{E}}$ is being assumed fixed).

b) By a *pre-exact* (*covariant*) *functor* from $\mathscr{E}$ to $\mathscr{E}'$ we mean a morphism of the corresponding algebras over the monad (Pre-Tr, Tot, $\varepsilon$), i.e., a $DG$-functor $f\colon \mathscr{E} \to \mathscr{E}'$ that commutes with the operation of taking convolutions of twisted complexes. A similar definition applies to contravariant functors.

Clearly, for any pre-exact functor $f$ the functor

$$H^0(f)\colon H^0(\mathscr{E}) \to H^0(\mathscr{E}')$$

is exact. We denote by $\operatorname{Prex}(\mathscr{E}, \mathscr{E}')$ the full $DG$-subcategory of $\operatorname{Prex}^0(\mathscr{E}, \mathscr{E}')$ whose objects are the pre-exact functors. By $\operatorname{Prex}^0(\mathscr{E}, \mathscr{E}')$ we denote the corresponding category of contravariant functors.

PROPOSITION 3. *Let* $\mathscr{E}$ *and* $\mathscr{E}'$ *be two pretriangulated categories. Then* $\operatorname{Prex}(\mathscr{E}, \mathscr{E}')$ *is a full pretriangulated subcategory of* $DG$-$\operatorname{Fun}(\mathscr{E}, \mathscr{E}')$. □

PROPOSITION 4. *Let* $\mathscr{E}$ *be a triangulated category. Then*:

a) *For every* $E \in \operatorname{Ob}\mathscr{E}$, *the* $DG$-*functors*

$$h_E\colon F \to \operatorname{Hom}^{\bullet}_{\mathscr{E}}(F, E), \qquad k^E\colon F \to \operatorname{Hom}^{\bullet}_{\mathscr{E}}(E, F)$$

*from* $\mathscr{E}$ *to* $C(\mathscr{A}b)$ *are pre-exact* (*and, respectively, contravariant and covariant*).

b) *The assignments* $E \mapsto h_E$ *and* $E \mapsto k^E$ *extend to pre-exact functors*

$$h\colon \mathscr{E} \to \operatorname{Prex}^0(\mathscr{E}, C(\mathscr{A}b)), \qquad k\colon \mathscr{E} \to \operatorname{Prex}(\mathscr{E}, C(\mathscr{A}b)),$$

*which are respectively covariant and contravariant.*

PROOF. a) If we regard $E$ as being an object in Pre-Tr($\mathscr{E}$), then the $DG$-functors $h_E$ and $k^E$ from Pre-Tr($\mathscr{E}$) to $C(\mathscr{A}b)$ are pre-exact, by Proposition 3 of §2. Since the convolution functor $T$ is an equivalence of $DG$-categories, $h_E$ and $k^E$ are pre-exact functors on $\mathscr{E}$.

b) The proof is left to the reader.

REMARK. Practically all the known exact functors between triangulated categories that are encountered in homological algebra come from pre-exact functors between the corresponding enhancements. In particular, the "six functorialities" of Grothendieck on sheaves of modules over ringed spaces (the functors $Rf_*, Lf^*, Rf_!, f^!, \otimes^L$, and $\underline{\mathrm{RHom}}$) come from suitable pre-exact functors. These pre-exact functors are obtained by applying the corresponding operations to resolutions. (In the case of infinite homological dimension of a space or of a sheaf of rings, the use of resolutions is validated in [8].)

Let $\{E_i \mid i \in I\}$ be a family of objects in a pretriangulated category $\mathscr{E}$. The smallest strictly full pretriangulated subcategory of $\mathscr{E}$ containing all the $E_i$ will be called their *pretriangulated hull* and denoted by $P_{\mathscr{E}}(\{E_i \mid i \in I\})$.

Thus, $P_{\mathscr{E}}(\{E_i\})$ consists of the convolutions in $\mathscr{E}$ of all possible twisted complexes consisting of direct sums of the $E_i$. Let $\mathscr{A}$ be the full $DG$-subcategory in $\mathscr{E}$ on the objects $E_i$.

PROPOSITION 5. *The composite functor*

$$\text{Pre-Tr}(\mathcal{A}) \to \text{Pre-Tr}(\mathcal{E}) \xrightarrow{T} \mathcal{E}$$

*effects an equivalence between* $\text{Pre-Tr}(\mathcal{A})$ *and* $P_{\mathcal{E}}(\{E_i \mid i \in I\})$ *as pretriangulated categories.*

PROOF. Since $T$ is an equivalence of $DG$-categories, the composite functor effects an equivalence of $\text{Pre-Tr}(\mathcal{A})$ with its image. □

This proposition allows us to identify pretriangulated categories with categories of certain $DG$-modules over suitable $DG$-algebras. Indeed, choosing sufficiently many objects in the category $\mathcal{E}$, we can represent the latter in the form $\text{Pre-Tr}(\mathcal{A})$, which by Proposition 3 of §1 is imbedded as a full $DG$-subcategory in the homotopy category of $DG$-functors $\mathcal{A} \to C(\mathcal{A}b)$ (i.e., explicitly, of $DG$-modules over the $DG$-algebra $\bigoplus_{E,F\in \text{Ob}\mathcal{A}} \text{Hom}_{\mathcal{A}}(E, F))$.

## §4. One-sided twisted complexes

In this section we consider the following problem. Suppose given a set of objects $E_1, \dots, E_n$ in a triangulated category $\mathcal{D}$. Denote by $\langle E_1, \dots, E_n\rangle_{\mathcal{D}}$ the triangulated subcategory of $\mathcal{D}$ generated by the $E_i$, i.e., the smallest strictly full triangulated subcategory of $\mathcal{D}$ that contains the $E_i$. We want to describe this category $\langle E_1, \dots, E_n\rangle_{\mathcal{D}}$. Without the existence of an enhancement for $\mathcal{D}$ the problem is very difficult. For example, if $(E_1, \dots, E_n)$ is a strong exceptional set in a $\mathbf{C}$-linear triangulated category $\mathcal{D}$ (i.e., $\text{Ext}^p_{\mathcal{D}}(E_i, E_j) = 0$ for $p \neq 0$ or $i > j$, and $\text{Hom}_{\mathcal{D}}(E_i, E_i) = \mathbf{C}$; see [9]), there is no obvious way of identifying $\langle E_1, \dots, E_n\rangle_{\mathcal{D}}$ with a derived category of modules over the algebra

$$A = \bigoplus \text{Hom}_{\mathcal{D}}(E_i, E_j).$$

We shall assume, therefore, that the category $\mathcal{D}$ is enhanced: $\mathcal{D} \sim H^0(\mathcal{E})$ for some pretriangulated category $\mathcal{E}$. At the end of the preceding section we described the pretriangulated hull $P(E_1, \dots, E_n)$ of the objects $E_i$, namely,

$$P(E_1, \dots, E_n) \sim \text{Pre-Tr}(\mathcal{A}),$$

where $\mathcal{A} \subset \mathcal{E}$ is the full $DG$-subcategory on the objects $E_i$. The cohomology category $H^0P(\mathcal{E}_1, \dots, \mathcal{E}_n)$ is triangulated, but possibly not generated as a triangulated category by the objects $E_i$. For example, the convolution of the twisted complex

$$E_0 \xleftarrow{q_{00}} E_0 \underset{q_{10}}{\overset{q_{01}}{\rightleftarrows}} E_1 \xrightarrow{q_{11}} E_1$$

cannot be obtained, in general, by means of the operation of taking the cones of closed morphisms from the objects $E_0$ and $E_1$. To surmount this difficulty, we make the following definition.

DEFINITION 1. A twisted complex $C = \{E_i, q_{ij}\}$ over a $DG$-category $\mathcal{A}$ is called one-sided if $q_{ij} = 0$ for $i \geq j$.

EXAMPLE. If $\mathcal{A} = C(\mathcal{B})$ is the category of complexes over an additive category $\mathcal{B}$, then a one-sided twisted complex over $\mathcal{A}$ is precisely a twisted complex over $\mathcal{B}$ in the sense of O'Brian, Toledo, and Tong [7].

The one-sided twisted complexes allow us to define another monad on the category $DG$-Cat.

Let $\mathscr{A}$ be a $DG$-category. Denote by $\widetilde{\mathscr{A}}$ the $DG$-category obtained from $\mathscr{A}$ by adjoining formal translates of objects. Its objects are the symbols $E[i]$, where $E \in \operatorname{Ob}\mathscr{A}$ and $i \in \mathbf{Z}$. If $E, F \in \operatorname{Ob}\mathscr{A}$, then

$$\operatorname{Hom}^{\bullet}_{\widetilde{\mathscr{A}}}(E[i], F[j]) = \operatorname{Hom}^{\bullet}_{\mathscr{A}}(E, F)[j-i].$$

Denote by $\text{Pre-Tr}^{+}(\mathscr{A})$ the full $DG$-subcategory of $\text{Pre-Tr}(\widetilde{\mathscr{A}})$ whose objects are the one-sided twisted complexes. If $C = \{E_i, q_{ij}\}$ is a twisted complex over $\widetilde{\mathscr{A}}$ and $n = \{n_i, i \in \mathbf{Z}\}$ is a set of integers, we define a new twisted complex $C\{n\} = \{E'_i, q'_{ij}\}$, where

$$E'_i = \bigoplus_{j+n_j=i} E_j[n_j], \qquad q'_{ij} = \|\tilde{q}_{kl}\| \qquad (k+n_k = i,\ l+n_l = j),$$

and $\tilde{q}_{kl}$ is the image of $q_{kl} \in \operatorname{Hom}^{k-l+1}(E_k, E_l)$ under identification of the latter group with

$$\operatorname{Hom}^{i-j+1}(E_k[n_k],\ E_l[n_l]).$$

The twisted complex $C\{n\}$ will be described as being *associated with $C$ by means of $n$*. Passage to the associated complexes defines on $\text{Pre-Tr}(\widetilde{\mathscr{A}})$ an action of the free abelian group $\mathbf{Z}^{\infty}$ by means of auto-equivalences of the pretriangulated category. Furthermore, $C\{n\}$ is canonically isomorphic to $C$. By abuse of language, we shall identify $C$ with the associated complexes $C\{n\}$.

In addition, there are defined on the twisted complexes over $\widetilde{\mathscr{A}}$ two commuting translation functors, whose combined action we shall denote by $C \mapsto C[m, n]$. If $C = \{E_i, q_{ij}\}$, then

$$C[m, n] = \{(E_{i-m}[n])_{i\in\mathbf{Z}},\ q_{i-m,j-m}\}.$$

Now let $U = \{V_i, r_{ij}\}$ be a twisted complex over $\text{Pre-Tr}(\widetilde{\mathscr{A}})$, and $n = \{n_i,\ i \in \mathbf{Z}\}$ a set of integers, as above. We define an associated complex $U\{\{n\}\}$, for which the $i$th term is $V_i[-n_i, n_i]$ and the mappings are the images of the $r_{ij}$ under the identifications

$$\operatorname{Hom}^{i-j+1}_{\text{Pre-Tr}(\widetilde{\mathscr{A}})}(V_i, V_j) \simeq \operatorname{Hom}^{i-j+1}_{\text{Pre-Tr}(\widetilde{\mathscr{A}})}(V_i[-n_i, n_i],\ V_j[-n_j, n_j]).$$

The convolution functor

$$\operatorname{Tot}\colon (\text{Pre-Tr})^2(\widetilde{A}) \to \text{Pre-Tr}(\widetilde{\mathscr{A}})$$

takes associated complexes into associated:

$$\operatorname{Tot}(U\{\{n\}\}) = (\operatorname{Tot} U)\{\{n\}\}.$$

LEMMA 1. *Let $U = \{V_i, r_{ij}\}$ be a one-sided twisted complex over* $\text{Pre-Tr}^{+}(\mathscr{A}) \subset \text{Pre-Tr}(\widetilde{\mathscr{A}})$. *Then if the numbers $n_i$, $i \in \mathbf{Z}$, increase sufficiently fast, the twisted complex* $\operatorname{Tot}(U\{\{n\}\})$ *is one-sided.*

PROOF. It suffices to take the $n_i$ so that $n_i - n_{i-1}$ is larger than $M_i - m_{i-1}$ where $m_i$ and $M_i$ are the minimal and maximal index of the nonzero terms of the twisted complex $V_i$. $\square$

DEFINITION 2. We define the functor

$$\operatorname{Tot}^{+}_{\mathscr{A}}\colon (\text{Pre-Tr}^{+})^2(\mathscr{A}) \to \text{Pre-Tr}^{+}(\mathscr{A}),$$

by putting, for any one-sided twisted complex $U$ over $\text{Pre-Tr}^+(\mathscr{A})$,

$$\operatorname{Tot}^+_{\mathscr{A}}(U) = \operatorname{Tot}_{\mathscr{A}}(U\{\{n\}\})$$

for a sufficiently rapidly increasing sequence $n = (n_i)$.

Different choices of the sequence $n$ give canonically isomorphic results.

We denote by $\varepsilon^+_{\mathscr{A}}$ the canonical imbedding

$$\mathscr{A} \subsetneq \text{Pre-Tr}^+(\mathscr{A}).$$

PROPOSITION 1. *The natural transformations*

$$\operatorname{Tot}^+\colon (\text{Pre-Tr}^+)^2 \to \text{Pre-Tr}^+, \qquad \varepsilon^+\colon \mathrm{id} \to \text{Pre-Tr}^+$$

*on the category DG-*Cat *determine a monad.* □

We denote by $\mathrm{Tr}^+(\mathscr{A})$ the zeroth cohomology category of the *DG*-category $\text{Pre-Tr}^+(\mathscr{A})$.

PROPOSITION 2. *The category* $\mathrm{Tr}^+(\mathscr{A})$ *is a full triangulated subcategory of the category* $\mathrm{Tr}(\widetilde{\mathscr{A}})$. *As a triangulated category, it is generated by the objects of* $\mathscr{A}$.

PROOF. The cone of a closed morphism in $\text{Pre-Tr}^+(\mathscr{A})$ is obtained by convolution (i.e., by the operation $\operatorname{Tot}^+_{\mathscr{A}}$) from the corresponding one-sided complex. This proves that $\mathrm{Tr}^+(\mathscr{A})$ is triangulated. We must show that it is generated by $\mathrm{Ob}\,\mathscr{A}$. Let $C = \{E_i, q_{ij}, i < j\}$ be a one-sided twisted complex over $\widetilde{\mathscr{A}}$, with just the objects $E_1, \dots, E_n$ different from zero. Then

$$q_{n-1,n}\colon E_{n-1} \to E_n$$

is a closed morphism; the morphisms $q_{n-2,n-1}$ and $q_{n-2,n}$ define a morphism from $E_{n-2}$ to the cone of $q_{n-1,n}$ (in the category $\text{Pre-Tr}^+(\mathscr{A})$); etc. Continuing in this fashion, we can represent $C$ as an iterated cone (in $\text{Pre-Tr}^+(\mathscr{A})$) of closed morphisms, starting from the objects $E_i$. Passing to homology, we conclude that $\mathrm{Tr}^+(\mathscr{A})$ is generated by $\mathrm{Ob}\,\mathscr{A}$.

THEOREM 1. *Let* $\mathscr{E}$ *be a pretriangulated category,* $E_1, \dots, E_n$ *objects in* $\mathscr{E}$, *and* $\mathscr{A} \subset \mathscr{E}$ *the full DG-subcategory on the objects* $E_i$. *Then* $\langle E_1, \dots, E_n\rangle_{H^0\mathscr{E}}$ *is equivalent to* $\mathrm{Tr}^+(\mathscr{A})$ *as a triangulated category.*

PROOF. Consider the functor

$$\Phi\colon \text{Pre-Tr}^+(\mathscr{A}) \to \mathscr{E}$$

that takes a twisted complex over $\widetilde{\mathscr{A}}$ into its convolution in $\mathscr{E}$ (with the formal translates of objects in $\mathscr{A}$ replaced by their actual translates in $\mathscr{E}$). The functor $\Phi$ defines isomorphisms on the complexes Hom. Consequently, $H^0(\Phi)$ is an equivalence of the category $\mathrm{Tr}^+(\mathscr{A})$ with its essential image. Since $\mathrm{Tr}^+(\mathscr{A})$ is generated by $\mathscr{A}$ as a triangulated category, its essential image under $\Phi$ coincides with $\langle E_1, \dots, E_n\rangle$.

It follows from this theorem that every enhanced triangulated category can be represented as a full triangulated subcategory in the homotopy category of *DG*-modules over a suitable *DG*-algebra. Assertions of this sort (for the case of algebras with trivial *DG*-structure) have already been considered in [9] and [10].

REMARK. We could have taken as the basis of our exposition the monad $(\text{Pre-Tr}^+, \text{Tot}^+, \varepsilon^+)$ and defined "+-pretriangulated" categories as algebras over this monad. From the point of view of the notion of triangulated category this would even seem to be more natural, in view of Theorem 1. But all triangulated categories that actually arise admit a richer structure: the convolution of arbitrary, not just one-sided, twisted complexes. It is therefore this structure that we axiomatize.

## §5. Concluding remarks

**A. Massey products.** If

$$C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \xrightarrow{d^n} C^{n+1}$$

is a sequence of closed morphisms in a $DG$-category $\mathscr{A}$, and $\delta_i = \deg(d_i)$, then there is defined a set

$$\langle\langle d^n, \dots, d^0\rangle\rangle \subset \operatorname{Hom}^{\delta_0+\cdots+\delta_n+1-n}_{H^{\cdot}(\mathscr{A})}(C^0, C^{n+1}),$$

called the set of values of the multivalued Massey product of the morphisms $d^i$ (see, for example, [11]). It is known that this set is nonvoid if and only if the sets of values of all the partial Massey operations

$$\langle\langle d^j, \dots, d^i\rangle\rangle, \qquad |i-j| < n,$$

contain 0; in particular, if the products of adjacent morphisms are coboundaries, i.e., if the $(C^i, d^i)$ define a complex over $H^{\cdot}(\mathscr{A})$.

On the other hand, the definition of Toda brackets in topology [12], [13] carries over easily to the case of arbitrary triangulated categories, giving for a sequence of morphisms

$$C^0 \xrightarrow{d^0} C^1 \to \cdots \xrightarrow{d^n} C^{n+1}$$

(say, of degree 0) in a triangulated category $\mathscr{D}$ a set

$$\langle d^n, \dots, d^0\rangle \subset \operatorname{Ext}t^{1-n}_{\mathscr{D}}(C^0, C^{n+1}).$$

It can be verified that for an enhanced triangulated category the Massey products coincide with the Toda brackets. For the homotopy category of complexes, this is proved in [16].

**B. $K$-theory.** The question of defining the higher Quillen $K$-functors [14] of an abelian category $\mathscr{A}$ for the triangulated category $D^b(\mathscr{A})$ is open. Some progress in this regard was made by Khinich and Shekhtman in [15], where the $K_i\mathscr{A}$ are defined in terms of the category $C^b(\mathscr{A})$ of finite complexes over $\mathscr{A}$ and their quasi-isomorphisms. The Khinich-Shekhtman method can be used to define, for any pretriangulated category $\mathscr{C}$, a pseudosimplicial category $U_{\cdot}(\mathscr{C})$, whose realization can be called the space of the algebraic $K$-theory of $\mathscr{C}$ (and its homotopy groups, the $K$-functors of $\mathscr{C}$). Applying this construction to the natural enhancement of $D^b(\mathscr{A})$ obtained by means of injective resolutions, we get the Quillen $K$-functor for $\mathscr{A}$.

**C. Mutations of exceptional sets.**

DEFINITION 1. a) An *exceptional object* in a $\mathbf{C}$-linear pretriangulated category $\mathscr{C}$ is an object $E$ for which

$$H^0(\operatorname{Hom}_{\mathscr{C}}(E, E)) = \mathbf{C}$$

and

$$H^i(\mathrm{Hom}_{\mathscr{E}}(E, E)) = 0 \quad \text{for } i \neq 0.$$

b) An *exceptional set* is an ordered set of exceptional objects $(E_0, \dots, E_n)$ satisfying the condition

$$\mathrm{Hom}_{\mathscr{E}}(E_i, E_j) \text{ is an acyclic complex for } i > j.$$

For the definition of an exceptional set in a triangulated category, see [9].

An exceptional set of two objects is called an *exceptional pair*. For an exceptional pair $(E, F)$ we define *right* and *left mutations*. These are the pairs $(LF, E)$ and $(F, RE)$, where $LF$ are $RE$ are defined as the following convolutions of complexes:

$$LF = T(\underbrace{\mathrm{Hom}(E, F) \otimes E}_{0} \xrightarrow{l} \underset{1}{F}),$$

$$RE = T(\underset{-1}{E} \xrightarrow{r} \underbrace{\mathrm{Hom}(E, F)^* \otimes E}_{0}).$$

The notation needs explaining. $T$ is the convolution functor; the indices under the objects specify the grading of the complex of objects of $\mathscr{E}$; if $V$ is a $\mathbf{C}$-vector space, $V \otimes E$ is the direct sum of $\dim V$ copies of the object $E$; and $\mathrm{Hom}(E, F) \otimes E$ is a complex of objects of $\mathscr{E}$, since $\mathrm{Hom}(E, F)$ is a complex of $\mathbf{C}$-vector spaces. In the complex $\mathrm{Hom}(E, F)^*$ the grading changes sign under conjugation; and $l$ and $r$ are the canonical morphisms.

A *mutation* of an exceptional set is a (right or left) mutation of any pair of adjoining elements of the set. The result of the mutation is again an exceptional set.

Suppose the pretriangulated category $\mathscr{E}$ is generated by an exceptional set $\sigma = (E_0, \dots, E_n)$. Then, as a triangulated category, $H^0(\mathscr{E})$ is also generated by this set. By Theorem 1 of §4, $H^0(\mathscr{E})$ is determined by the $DG$-category $\mathscr{A}_\sigma$—the full subcategory of $\mathscr{E}$ on the objects $E_i$ in $\sigma$. A mutated set generates the same category $H^0(\mathscr{E})$ (see [9]). It is therefore natural to ask how the category $\mathscr{A}_\sigma$ changes under mutations of the set. For this we must determine the Hom-complexes between elements of the mutated set $\tilde{\sigma}$, which is easily done by using the formulas for mutations and the pre-exactness of representable functors (Proposition 3 of §3). Passing to the homology of the $DG$-category $\mathscr{A}_\sigma$, we obtain the Ext-groups between the elements of the set $\tilde{\sigma}$. Computing them by starting from the Ext-groups of the original set is in general not possible.

If we regard the algebras $\mathscr{A}_\sigma$ as $DG$-algebras up to quasi-isomorphism, then the mutations of such an algebra form an action of the Artin braid group. This follows from the corresponding result for triangulated categories [17], [9].

Steklov Mathematical Institute
Academy of Sciences of the USSR
Moscow

Received 21/FEB/89

## BIBLIOGRAPHY

1. J. L. Verdier, *Catégories dérivées. Quelques résultats* (*état* 0), Séminaire de Géometrie Algébrique du Bois-Marie (SGA $4\frac{1}{2}$), Lecture Notes in Math., vol. 569, Springer-Verlag, 1977, pp. 262–311.

2. Robin Hartshorne, *Residues and duality*, Lecture Notes in Math., vol. 20, Springer-Verlag, 1966.

3. J. P. May, *The geometry of iterated loop spaces*, Lecture Notes in Math., vol. 271, Springer-Verlag, 1972.

4. A. G. Kurosh, *Lectures on general algebra*, 2nd ed., "Nauka", Moscow, 1973; English transls. of 1st ed., Chelsea, New York, 1963, and Pergamon Press, 1965.

5. Saunders Mac Lane, *Homology*, Springer-Verlag, Berlin, and Academic Press, New York, 1963.

6. M. M. Kapranov, *On the derived categories of coherent sheaves on some homogeneous spaces*, Invent. Math. **92** (1988), 479–508.

7. Nigel R. O'Brian, Domingo Toledo, and Yue Lin L. Tong, *The trace map and characteristic classes for coherent sheaves*, Amer. J. Math. **103** (1981), 225–252.

8. N. Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math. **65** (1988), 121–154.

9. A. I. Bondal, *Representation of associative algebras and coherent sheaves*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), 25–44; English transl. in Math. USSR Izv. **34** (1990).

10. Dagmar Baer, *Tilting sheaves in representation theory of algebras*, Manuscripta Math. **60** (1988), 323–347.

11. David Kraines, *Massey higher products*, Trans. Amer. Math. Soc. **124** (1966), 431–449.

12. George W. Whitehead, *Recent advances in homotopy theory*, Conf. Board Math. Sci. Regional Conf. Ser. Math., vol. 5, Amer. Math. Soc., Providence, R. I., 1970.

13. Joel M. Cohen, *The decomposition of stable homotopy*, Ann. of Math. (2) **87** (1968), 305–320.

14. Daniel Quillen, *Higher algebraic $K$-theory*. I, *Algebraic $K$-theory*. I (Proc. Conf., Seattle, Wash., 1972), Lecture Notes in Math., vol. 341, Springer-Verlag, 1973, pp. 85–147.

15. V. A. Hinich [Khinich] and V. V. Schechtman [Shekhtman], *Geometry of a category of complexes and algebraic $K$-theory*, Duke Math. J. **52** (1985), 399–430.

16. M. M. Kapranov, *Derived categories of coherent sheaves on homogeneous spaces*, Candidate's Dissertation, Steklov Math. Inst. Acad. Sci. USSR, Moscow, 1988. (Russian)

17. A. I. Bondal and M. M. Kapranov, *Representable functors, Serre functors, and mutations*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), 1183–1205; English transl. in Math. USSR Izv. **35** (1990).

18. S. I. Gel′fand and Yu. I. Manin, *Methods of homological algebra*, Vol. I, "Nauka", Moscow, 1988; English transl., Springer-Verlag (to appear).

Translated by J. A. ZILBER