I Boundaries and domain walls in 2-d conformal field theories and topological orders By Liang Kong (Caltech) May 29, 2009

Main Theme:

- bulk-boundary duality
- Domain wall $\supset$ duality

I.1 Rational BCFT (open-closed CFT)

CFT: (Segal) ($\phi(x)$: a quantum field associated with a space time insertion, not good in our situation) $\rightsquigarrow$ (Vafa, Huang):

- objects: $[m]$
- $\mathrm{Hom}([m],[n])$ = space of conformal equivalence classes of Riem. surfaces with punctures (m "incoming" and n "outgoing"), togeter with parametrisations of the neighbourhoods (cf. Figure 1)
- sewing operations (is not always well-defined) $\rightsquigarrow$ defines a category RS^p

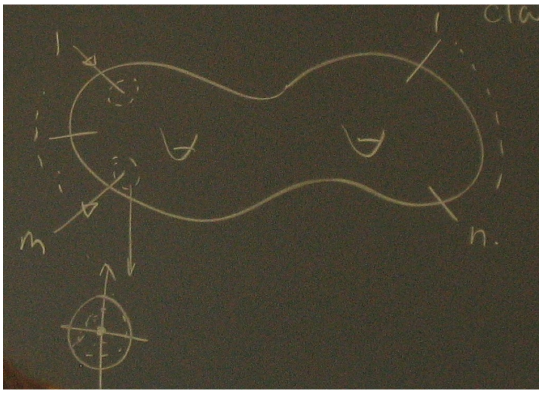

Figure 1: Morphism in RS^p

$A = \oplus_n A_n$ graded vector space $\rightsquigarrow$ $\overline{A} = \prod_n A_n$, $\mathrm{Hom}(A,B)_{\mathrm{GrVect}} := \mathrm{Hom}(A,\overline{B})_C$, $A \xrightarrow{f} \overline{B}$, $B \xrightarrow{g}\rightarrow \overline{C}$ $g \circ f := \sum_n g \circ p_n \circ f$ for $p_n : \overline{B} \to B_n$ $\rightsquigarrow$ defines a category GrVect

CFT: a projective monoidal functor

$$\mathrm{RS}^p \xrightarrow{\mathcal{F}} \mathrm{GrVect}$$

Theorem I.1 (Huang).

$$(\mathcal{F}(1), \{\mathcal{F}(S^2 \text{ with } n \text{ incoming and } 1 \text{ outgoing punctures}) : \mathcal{F}(1)^{\otimes n} \to \overline{\mathcal{F}(1)}\}_{n=0}^{\infty}),$$

such that $\mathcal{F}$ "is holomorphic" and satisfies some natural conditions is a vertex operator algebra (VOA).

Definition I.2. (via Huang's theorem) V is rational if C_V-(cat. of V-modules) is a modular tensor category (MTC). ∎

Theorem I.3. *A BCFT over V gives rise to a triple $(A_{op}|A_{cl}, L)$, where*

1. *A_{cl}: commutative symmetric Frobenius algebra in $\mathcal{C}_V \boxtimes \overline{\mathcal{C}}_V$*
2. *A_{op}: symmetric Frobenius algebra in $\mathcal{C}_V$*
3. *$L : A_{cl} \to \otimes^V A_{op}$ an algebra map*

such that a couple of axioms are satisfied:

- *(cf. Figure 2)*
- *modular invariance*
- *Cardy condition*

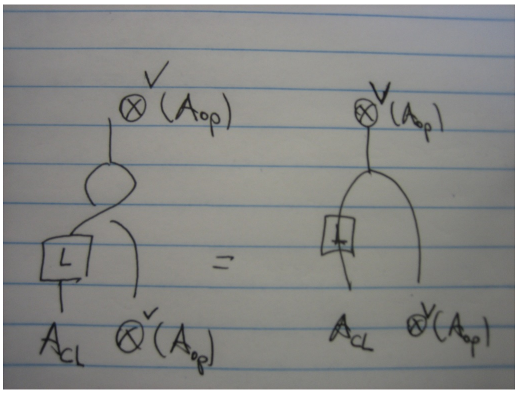

Figure 2: Condition in Theorem I.3

conjecturally, the converse is also true, namely that one can build a BCFT from the given data.

Definition I.4. If A is a special symmetric Frobenius algebra in $\mathcal{C}_V$, then

$$Z(A) = \mathrm{im}(P) \overset{e}{\underset{r}{\rightleftarrows}} \otimes^V(A).$$ ■

Theorem I.5 (with I. Runkel). *$(A|Z(A), e)$ is a BCFT over V.*

Theorem I.6 (with I. Runkel).
1. *If A is simple s.s. Frob. alg., then $A_{cl} = Z(A)$ is unique*
2. *$A_1 \simeq_{mor} A_2$ ifff $Z(A_1) \simeq Z(A_2)$ (boundary duality)*
3. *$\mathrm{Pic}(A) \simeq \mathrm{Aut}(Z(A))$ (deeper fact)*

I.2 topological order

(origin: condensed matter physics, Quantum Hall effect, anyon system)

1. Kitaev's topic code model
2. Levin-Wen Model

Kitaev's model:
Hil $= \otimes_l H_l$, $H_l = \mathbb{C}^2$, $H = -\sum_v Av - \sum_p Bp$ (missed some drawing and motivation) $\rightsquigarrow$ ground state A^q is unique (properties: $Av|0>=|0>$ and $Bp|0>=|0>$)

Excitations: (Superselection sectors)

- $A|e>=-|e>$ (electric charge)
- $Bp|m>=-|m>$ (magnetic vortice)
- $\sigma_1^z|0>=e\otimes e \sim |0>=|$
- $e\otimes m=\epsilon$

$\{1,e,m,\epsilon\} \rightsquigarrow Z(\mathrm{Rep}_{Z_2})$ (Drinfeld center). from this one gets

$$\left.\begin{array}{l} Av=\sigma_1^x\sigma_2^x\sigma_3^x \\ 1\mapsto 1 \\ e\mapsto e \\ m\mapsto 1 \\ \epsilon\mapsto e \end{array}\right\} \rightsquigarrow Z(\mathrm{Rep}_{\mathbb{Z}_2}) \to \mathrm{Rep}_{\mathbb{Z}_2} = \mathrm{Fun}_{\mathrm{Rep}_{\mathbb{Z}_2}}(\mathrm{Rep}_{\mathbb{Z}_2},\mathrm{Rep}_{\mathbb{Z}_2})$$

$$\left.\begin{array}{l} Bp=\sigma_1^x|m> \\ m\mapsto m \\ e\mapsto 1 \\ \epsilon\mapsto m \\ 1\mapsto 1 \end{array}\right\} \rightsquigarrow Z(\mathrm{Rep}_{\mathbb{Z}_2}) \to \mathrm{Rep}_{\mathbb{Z}_2} = \text{...(as above)}$$

...the important example comes from a domain wall: $H=???$, $B_1=\sigma_1^x\sigma_2^z\sigma_3^z\sigma_4^z$, $B_2=\sigma_4^x\sigma_1^z\sigma_5^z\sigma_6^z$ then... (cf. Figure 3)

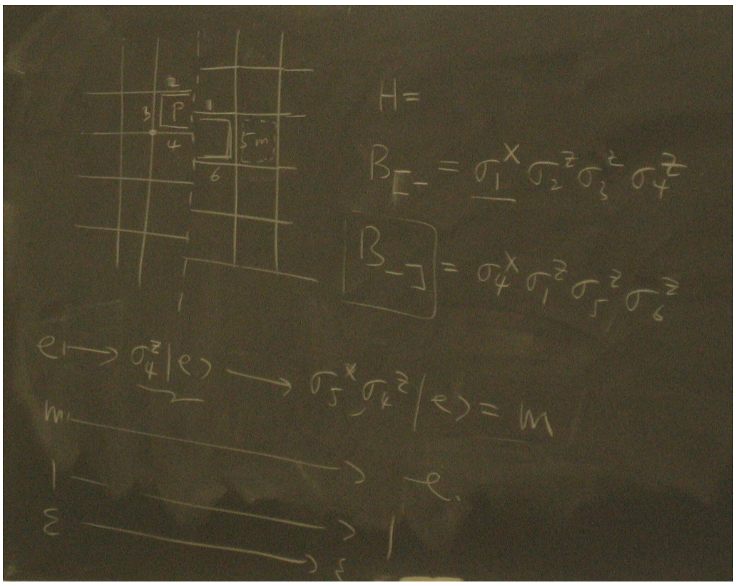

Figure 3: Main example