Multispans

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Abstract

A definition of *multi-(co)spans* inspired by Baas' *hyperstructures* and generalizing ordinary (co)spans and Grandis' higher cubical cospans [2]. Some examples.

Idea: multi-cospans in S formalize *extended cobordisms* in S, generalizing [4]; multi-spans in $\mathcal{C}$ formalize higher linear maps via *groupoidification* [1], morphisms from one to the other respecting multispan composition should capture the idea of *extended QFTs*.

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1 Multispans

We define a multi-cospan as the image of a poset in a category $\mathcal{C}$ with finite colimits (and dually for multi-spans). We think of such a diagram in $\mathcal{C}$ as a hierarchical cell complex where morphisms describe inclusion of boundary pieces.

Definition 1.1 (finite posets) *A finite poset is a finite category D with $D(a,b)$ either empty or the singleton set, for all $a, b \in \mathrm{Obj}(D)$, i.e. a finite category enriched over $(\{\emptyset, \mathrm{pt}\}, \times)$. Write* Posets *for the full sub-category of* $\mathsf{Categories}$ *on finite posets.*

Simple posets of relevance for the following are

- the terminal poset $\{\top\}$;
- the interval $\mathcal{I} = \{a \to \top\}$;
- the abstract cospan $\wedge :=$ $$\left\{\begin{tikzcd} & \top & \\ a_1 \arrow[ur] & & a_2 \arrow[ul] \end{tikzcd}\right\}$$
- the cartesian product of the abstract cospan with itself $\wedge^2 =$ $$\left\{\begin{tikzcd} a_1 \arrow[r] \arrow[d] & b_1 \arrow[d] & a_2 \arrow[l] \arrow[d] \\ b_2 \arrow[r] & \top & b_3 \arrow[l] \\ a_3 \arrow[r] \arrow[u] & b_4 \arrow[u] & a_4 \arrow[l] \arrow[u] \end{tikzcd}\right\};$$
- the co-cone over two glued copies of the abstract cospan $$\left\{\begin{tikzcd} & & \top & & \\ & b_1 \arrow[ur] & & b_2 \arrow[ul] & \\ a_1 \arrow[ur] & & a_2 \arrow[ul] \arrow[ur] & & a_3 \arrow[ul] \end{tikzcd}\right\}$$

Definition 1.2 (terminal and coterminal object) *A terminal object $\top$ in a poset D is an object such that $\forall a \in \mathrm{Obj}(D) : (a, \top) \simeq \mathrm{pt}$. A co-terminal object b is one for which $\forall a \neq b \ \in \mathrm{Obj}(D) : D(a,b) \simeq \emptyset$.*

In the above examples the coterminal objects are the a_i.

Definition 1.3 *Write* $\mathsf{Posets}_{\max}$ *the category whose objects are the posets that have a terminal object and whose morphisms are morphisms of posets that preserve the terminal object.*

There is an obvious forgetful functor $U : \mathsf{Posets}_{\max} \to \mathsf{Posets}$ with a left adjoint $\overline{(\ )} : \mathsf{Posets} \to \mathsf{Posets}_{\max}$ which freely adjoins a terminal object to a given poset.

For instance $$\overline{\left\{\begin{tikzcd} & b_1 & & b_2 & \\ a_1 \arrow[ur] & & a_2 \arrow[ul] \arrow[ur] & & a_3 \arrow[ul] \end{tikzcd}\right\}} = \left\{\begin{tikzcd} & & \top & & \\ & b_1 \arrow[ur] & & b_2 \arrow[ul] & \\ a_1 \arrow[ur] & & a_2 \arrow[ul] \arrow[ur] & & a_3 \arrow[ul] \end{tikzcd}\right\}$$

Definition 1.4 (multi-cospan) *A multi-cospan in a category $\mathcal{C}$ with finite colimits is a finite poset with terminal object, $D \in \mathsf{Posets}_{\max}$, and a functor $K : D \to \mathcal{C}$. The category of multi-cospans in $\mathcal{C}$ is*

$$\mathsf{MultiCospans}(\mathcal{C}) := (\mathsf{Posets}_{\max} \downarrow_{\mathsf{Categories}} \mathcal{C})^{\mathrm{op}}$$

whose morphisms $(f, \phi) : K \to K'$ are triangles

$$\begin{tikzcd} D \arrow[dr, "K"'] \arrow[rr, phantom, shift right=3ex, "\overset{\phi}{\Longrightarrow}"] & & D' \arrow[ll, "f"'] \arrow[dl, "K'"] \\ & \mathcal{C} & \end{tikzcd}$$

of natural transformations ϕ, composition is the obvious pasting of these triangles in Categories

Remark. For K a multi-cospan we think of the object $K(\top) \in \mathcal{C}$, for $\top$ the terminal object of D, as a single top-dimensional cell in a hierarchical complex, in that we think of any morphism $K(a \to \top)$ in $\mathcal{C}$ for every $a \in D$ as embedding a boundary piece $K(a)$ labeled by a into the total space (but there is no requirement that $K(a \to \top)$ be a monomorphism) and think of each morphism $K(b \to a)$ for $b \to a$ in D as describing a boundary piece $K(b)$ of a boundary piece $K(a)$.

1.1 Composition

The idea is that multi-cospans are composed by first gluing them along a common sub-multi-cospan, then forming the colimit cocone over that, and finally picking a sub-multi-cospan in that, containing the tip of the colimit cocone.

Definition 1.5 (multi-cospan closure) *For $D \in$* Posets *and $K : D \to \mathcal{C}$ a functor the multi-cospan closure of K (or rather one of the canonically isomorphic choices) is the unique multi-cospan*

$$\bar{K} : \bar{D} \to \mathcal{C}$$

such that

$$\begin{tikzcd} D \arrow[r, hook] \arrow[rr, bend right, "K"'] & \bar{D} \arrow[r, "\bar{K}"] & \mathcal{C} \end{tikzcd}$$

and

$$\begin{tikzcd} \{\top\} \arrow[r, hook] \arrow[rr, bend right, "\top \mapsto \mathrm{colim}_D K"'] & \bar{D} \arrow[r, "\bar{K}"] & \mathcal{C} \end{tikzcd}$$

For instance for $D = \left\{ \begin{tikzcd}[column sep=small, row sep=small] & b_1 & & b_2 & \\ a_1 \arrow[ur] & & a_2 \arrow[ul] \arrow[ur] & & a_3 \arrow[ul] \end{tikzcd} \right\}$ we have for any $K : D \to \mathcal{C}$

$$\bar{K} : \left\{ \begin{tikzcd}[column sep=small, row sep=small] & & \top & & \\ & b_1 \arrow[ur] & & b_2 \arrow[ul] & \\ a_1 \arrow[ur] & & a_2 \arrow[ul] \arrow[ur] & & a_3 \arrow[ul] \end{tikzcd} \right\} \mapsto \left\{ \begin{tikzcd}[column sep=small, row sep=small] & & \mathrm{colim}_D K & & \\ & K(b_1) \arrow[ur] & & K(b_2) \arrow[ul] & \\ K(a_1) \arrow[ur] & & K(a_2) \arrow[ul] \arrow[ur] & & K(a_3) \arrow[ul] \end{tikzcd} \right\}$$

Definition 1.6 (multi-cospan composition) *For $K_1 : D_1 \to \mathcal{C}$ and $K_2 : D_2 \to \mathcal{C}$ two multi-cospans in $\mathcal{C}$, and given a diagram $U(D_1) \hookleftarrow D_{\mathrm{glue}} \hookrightarrow U(D_2)$ in* Posets *of sub-poset inclusions respecting co-terminal objects, such that*

$$\begin{tikzcd} D_{\mathrm{glue}} \arrow[r] \arrow[d] & D_1 \arrow[d, "K_1"] \\ D_2 \arrow[r, "K_2"] & \mathcal{C} \end{tikzcd}$$

and given a morphism in Posets$_{\max}$ $D_{\rm comp} \hookrightarrow \overline{D_1 \sqcup_{\rm glue} D_2}$ *we say that composition of* K_1 *and* K_2 *along* $D_{\rm glue}$ *to* $D_{\rm comp}$ *is the multi-cospan (or rather any one of the canonically isomorphic choices)*

$$K_{\rm comp} : \; D_{\rm comp} \hookrightarrow \overline{D_1 \sqcup_{\rm glue} D_2} \xrightarrow{\overline{K_1 \sqcup_{\rm glue} K_2}} \mathcal{C}$$

Example: ordinary spans and cospans. We obtain ordinary cospans and their composition by taking all multi-cospan domains to be $D = \left\{ \begin{tikzcd} & \top & \\ a_1 \arrow[ur] & & a_2 \arrow[ul] \end{tikzcd} \right\}$ and $D_{\rm glue} = \{\bullet\}$, so that $D \sqcup_{\rm glue} D = \left\{ \begin{tikzcd} & b_1 & & b_2 & \\ a_1 \arrow[ur] & & a_2 \arrow[ul] \arrow[ur] & & a_3 \arrow[ul] \end{tikzcd} \right\}$ and $\overline{D \sqcup_{\rm glue} D} = \left\{ \begin{tikzcd} & & \top & & \\ & b_1 \arrow[ur] & & b_2 \arrow[ul] & \\ a_1 \arrow[ur] & & a_2 \arrow[ul] \arrow[ur] & & a_3 \arrow[ul] \end{tikzcd} \right\}$ and finally taking $D_{\rm comp} = D$ with $D_{\rm comp} \hookrightarrow \overline{D \sqcup_{\rm glue} D}$ given by $a_1 \mapsto a_1$ and $a_2 \mapsto a_3$.

Dually, we get ordinary multispans in $\mathcal{C}^{\rm op}$. But already in this case we get a little more flexibility spans for handling cospans. For definitenss, consider cospans in$\mathcal{C} =$ Sets$^{\rm op}$.

Composition of the multispan $\bar{K}_1$ which is the closure of

$$K_1 := \left\{ \begin{tikzcd} \Psi_{\rm in} \arrow[d] & \Psi_{\rm out} \arrow[d] \\ X & Y \end{tikzcd} \right\}$$

with

$$K_2 := \left\{ \begin{tikzcd} & R \arrow[d] & \\ & X \times Y \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd} \right\}$$

along the preimages of X and Y *to* $\top$ is the tip of

$$\begin{tikzcd} & \langle \Psi_{\rm in} | R | \Psi_{\rm out} \rangle \arrow[d] \arrow[ddl] \arrow[ddr] & \\ & R \arrow[d] & \\ \Psi_{\rm in} \arrow[d] & X \times Y \arrow[dl] \arrow[dr] & \Psi_{\rm out} \arrow[d] \\ X & & Y \end{tikzcd}$$

and describes the contraction of the matrix K with the vectors $\Psi_{\rm in}$ and $\Psi_{\rm out}$.

Example: more general multi-cospans in Sets$^{\rm op}$. Next, consider two matrices R_1 and R_2 given by the multispan $\bar{K}_1$ which is the closure of

$$K_1 := \left\{ \begin{tikzcd} & R_1 \arrow[d] & & R_2 \arrow[d] & \\ & X \times Y \arrow[dl] \arrow[d] & & Y \times Z \arrow[d] \arrow[dr] & \\ X & Y & & Y & Z \end{tikzcd} \right\}$$

and the consider the multispan

$$K_2 := \left\{ \begin{tikzcd}
X \arrow[ddd, dash, "="] & & & X\times Z \arrow[lll, two heads] \arrow[rrr, two heads] & & & Z \arrow[ddd, dash, "="'] \\
& & & X\times Y\times Z \arrow[u] \arrow[ulll, two heads] \arrow[urrr, two heads] \arrow[dll] \arrow[drr] & & & \\
& X\times Y \arrow[dl] \arrow[drr] & & & & Y\times Z \arrow[dll] \arrow[dr] & \\
X & & & Y & & & Z
\end{tikzcd} \right\}.$$

Then composition *along* the lower zig-zag *to* the resulting top zig-zag yields the matrix product

$$K_{\mathrm{comp}} = \left\{ \begin{tikzcd}
& R_1\cdot R_2 \arrow[d] & \\
& X\times Z \arrow[dl] \arrow[dr] & \\
X & & Z
\end{tikzcd} \right\}$$

These represent $\mathbb{N}$-valued matrices. Somewhat more interestingly, let k be some field and

$$K_1 := \left\{ \begin{tikzcd}
& R_1 \arrow[d] & R_2 \arrow[d] & \\
& (X\times Y)\times k \arrow[dl] \arrow[d] & (Y\times Z)\times k \arrow[d] \arrow[dr] & \\
X & Y & Y & Z
\end{tikzcd} \right\}$$

two k-valued matrices to be composed with the multispan

$$K_2 := \left\{ \begin{tikzcd}
& & (X\times Z)\times k \arrow[dll, two heads] \arrow[drr, two heads] & & \\
X \arrow[ddd, dash, "="] & & & & Z \arrow[ddd, dash, "="'] \\
& & (X\times Y\times Z)\times k\times k \arrow[uu, "(-)\cdot(-)"'] \arrow[ull, two heads] \arrow[urr, two heads] \arrow[dl] \arrow[dr] & & \\
& (X\times Y)\times k \arrow[dl] \arrow[dr] & & (Y\times Z)\times k \arrow[dl] \arrow[dr] & \\
X & & Y & & Z
\end{tikzcd} \right\}.$$

Then the result is (after k-valued cardinality) the product of k-valued matrices.

Example: Grandis' cubical cospans. The cubical multi-cospans in [2] are reproduced by restricting the domain posets to be of the form $\wedge^n$, $n \in \mathbb{N}$.

1.2 Trace and co-trace

Definition 1.7 (co-trace) *If for $K : D \to \mathcal{C}$ a multi-cospan in which for a collection $\{a_i \in \mathrm{Obj}(D)\}_i$ of coterminal objects we have $K(a_i) \simeq X$ for all i and for X a given object of $\mathcal{C}$ the co-trace of K over $\{a_i\}$ is the composition of K with the co-span*

$$K_X : \left\{ \begin{tikzcd}
& & & \top & & \\
a_1 \arrow[urrr, two heads] & a_2 \arrow[urr, two heads] & \cdots & & a_i \arrow[ul] & \cdots
\end{tikzcd} \right\} \overset{\mathrm{const}_X}{\longrightarrow} \mathcal{C}$$

along the obvious identifications of the a_i to the diagram D with the a_i removed.

For spans the co-trace is called the trace.

Examples. Let $\mathcal{C} = \mathsf{Categories}$ and $\mathcal{I} := \{a \to b\}$ be the 1-globe (the (directed) interval) regarded as a co-span

$$\begin{tikzcd}
& \mathcal{I} & \\
\mathrm{pt} \arrow[ur] & & \mathrm{pt} \arrow[ul]
\end{tikzcd}$$

then the co-trace of $\mathcal{I}$ over pt is the the colimit over

$$\begin{tikzcd}
& \mathcal{I} & \\
\mathrm{pt} \arrow[ur] \arrow[dr] & & \mathrm{pt} \arrow[ul] \arrow[dl] \\
& \mathrm{pt} &
\end{tikzcd}$$

which is $\mathbf{B}\mathbb{N}$ (the (directed) circle).

Dually, let $\mathcal{C} = \mathsf{Sets}^{\mathrm{op}}$ and consider spans of finite sets again, with

$$\begin{tikzcd}
& R \arrow[dl] \arrow[dr] & \\
X & & X
\end{tikzcd}$$

an $|X| \times |X|$-matrix, then the trace of this is the limit over

$$\begin{tikzcd}
& R \arrow[dl] \arrow[dr] & \\
X & & X \\
& X \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

which is $\sqcup_{x \in X} Rx, x$, as expected.

2 Extended QFT

Definition 2.1 (extended QFT) *For $S^{\mathrm{op}}, \mathcal{V}$ categories with finite limits an extended S-QFT with coefficients in $\mathcal{V}$ is a functor*

$$Z : \mathsf{MultiSpans}(S^{\mathrm{op}}) \to \mathsf{MultiSpans}(\mathcal{V})$$

which respects composition of multispans.

For instance for $S = \mathrm{Top}$ this is supposed to be an *extended topological QFT*.

Definition 2.2 (σ-model QFT) *If $\mathcal{V}$ is closed monoidal and given a $\mathcal{V}$-enrichment of $[S^{\mathrm{op}}, \mathcal{V}]$, for*

$$B : S \to [S^{\mathrm{op}}, \mathcal{V}]$$

a functor respecting finite colimits and $(P_X \to X) \in [S^{\mathrm{op}}, \mathcal{V}]$, the S-QFT

$$[B(-), P_X] : \mathsf{MultiSpans}(S^{\mathrm{op}}) \to \mathsf{MultiSpans}(\mathcal{V})$$

is a σ-model with target space X and background field P_X.

Consider $S = \mathrm{Top}$, $\mathcal{V} = \mathrm{Cat}$ and $\mathcal{C} = [S^{\mathrm{op}}, \mathcal{V}]$. As noticed in theorem 4.5 in [3], Brown's homotopy van Kampen theorem ensures that the fundamental groupoid assignment $\Pi_1 : \mathrm{Top} \to \mathsf{Categories}$ extends to a functor $\mathsf{MultiSpans}(\mathrm{Top}^{\mathrm{op}}) \to \mathsf{MultiSpans}(\mathsf{Categories}^{\mathrm{op}})$ which respects composition of multispans.

Hence for any $C \in [S^{\mathrm{op}}, \mathcal{V}]$ any category-valued presheaf

$$[\Pi_1(-), C] : \mathsf{MultiSpans}(\mathrm{Top}^{\mathrm{op}}) \to \mathsf{MultiSpans}(\mathsf{Categories}^{\mathrm{op}})$$

respects composition of multispans.

Examples. Let $S = \mathrm{Top}$, $\mathcal{V} = \mathsf{Groupoids}$, G a finite group, then the σ-model

$$[\Pi_1(-), \mathbf{B}G] : \mathsf{MultiSpans}(\mathrm{Top}^{\mathrm{op}}) \to \mathsf{MultiSpans}(\mathsf{Groupoids})$$

is essentially the untwisted Dijkgraaf-Witten model.

References

[1] J. Baez, *Higher dimensional algebra VII: Groupoidification*, [`http://math.ucr.edu/home/baez/hda7.pdf`]

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[4] Marco Grandis, *Cospans in Algebraic Topology, III: Cubical cospans and higher cobordisms*