

# Multispans

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## Abstract

A definition of *multi-(co)spans* inspired by Baas' *hyperstructures* and generalizing ordinary (co)spans and Grandis' higher cubical cospans [2]. Some examples.

Idea: multi-cospans in  $S$  formalize *extended cobordisms* in  $S$ , generalizing [4]; multi-spans in  $\mathcal{C}$  formalize higher linear maps via *groupoidification* [1], morphisms from one to the other respecting multispans. Composition should capture the idea of *extended QFTs*.

## Contents

<b>1</b>	<b>Multispans</b>	<b>2</b>
1.1	Composition . . . . .	3
1.2	Trace and co-trace . . . . .	5
<b>2</b>	<b>Extended QFT</b>	<b>6</b>

# 1 Multispans

We define a multi-cospan as the image of a poset in a category  $\mathcal{C}$  with finite colimits (and dually for multi-spans). We think of such a diagram in  $\mathcal{C}$  as a hierarchical cell complex where morphisms describe inclusion of boundary pieces.

**Definition 1.1 (finite posets)** A finite poset is a finite category  $D$  with  $D(a, b)$  either empty or the singleton set, for all  $a, b \in \text{Obj}(D)$ , i.e. a finite category enriched over  $(\{\emptyset, \text{pt}\}, \times)$ . Write  $\text{Posets}$  for the full sub-category of  $\text{Categories}$  on finite posets.

Simple posets of relevance for the following are

- the terminal poset  $\{\top\}$ ;

- the interval  $\mathcal{I} = \{a \rightarrow \top\}$ ;

- the abstract cospan  $\wedge := \left\{ \begin{array}{c} \top \\ \nearrow \quad \nwarrow \\ a_1 \quad \quad a_2 \end{array} \right\}$

- the cartesian product of the abstract cospan with itself  $\wedge^2 = \left\{ \begin{array}{ccccc} a_1 & \rightarrow & b_1 & \leftarrow & a_2 \\ \downarrow & & \downarrow & & \downarrow \\ b_2 & \rightarrow & \top & \leftarrow & b_3 \\ \uparrow & & \uparrow & & \uparrow \\ a_3 & \rightarrow & b_4 & \leftarrow & a_4 \end{array} \right\}$ ;

- the co-cone over two glued copies of the abstract cospan  $\left\{ \begin{array}{ccccc} & & \top & & \\ & \nearrow & & \nwarrow & \\ & b_1 & & b_2 & \\ \nearrow & & \nwarrow & & \nearrow \\ a_1 & & a_2 & & a_3 \end{array} \right\}$

**Definition 1.2 (terminal and coterminal object)** A terminal object  $\top$  in a poset  $D$  is an object such that  $\forall a \in \text{Obj}(D) : (a, \top) \simeq \text{pt}$ . A coterminal object  $b$  is one for which  $\forall a \neq b \in \text{Obj}(D) : D(a, b) \simeq \emptyset$ .

In the above examples the coterminal objects are the  $a_i$ .

**Definition 1.3** Write  $\text{Posets}_{\max}$  the category whose objects are the posets that have a terminal object and whose morphisms are morphisms of posets that preserve the terminal object.

There is an obvious forgetful functor  $U : \text{Posets}_{\max} \rightarrow \text{Posets}$  with a left adjoint  $(\bar{\phantom{x}}) : \text{Posets} \rightarrow \text{Posets}_{\max}$  which freely adjoins a terminal object to a given poset.

For instance  $\left\{ \overline{\begin{array}{ccccc} & & b_1 & & b_2 \\ & \nearrow & & \nwarrow & \\ & a_1 & & a_2 & \\ \nearrow & & \nwarrow & & \nearrow \\ a_1 & & a_2 & & a_3 \end{array}} \right\} = \left\{ \begin{array}{ccccc} & & \top & & \\ & \nearrow & & \nwarrow & \\ & b_1 & & b_2 & \\ \nearrow & & \nwarrow & & \nearrow \\ a_1 & & a_2 & & a_3 \end{array} \right\}$

**Definition 1.4 (multi-cospan)** A multi-cospan in a category  $\mathcal{C}$  with finite colimits is a finite poset with terminal object,  $D \in \text{Posets}_{\max}$ , and a functor  $\overline{K} : D \rightarrow \mathcal{C}$ . The category of multi-cospans in  $\mathcal{C}$  is

$$\text{MultiCospans}(\mathcal{C}) := (\text{Posets}_{\max} \downarrow_{\text{Categories}} \mathcal{C})^{\text{op}}$$

whose morphisms  $(f, \phi) : K \rightarrow K'$  are triangles

$$\begin{array}{ccc} D & \xleftarrow{f} & D' \\ & \searrow K & \swarrow K' \\ & \mathbb{C} & \end{array}$$

of natural transformations  $\phi$ , composition is the obvious pasting of these triangles in Categories

**Remark.** For  $K$  a multi-cospan we think of the object  $K(\top) \in \mathcal{C}$ , for  $\top$  the terminal object of  $D$ , as a single top-dimensional cell in a hierarchical complex, in that we think of any morphism  $K(a \rightarrow \top)$  in  $\mathcal{C}$  for every  $a \in D$  as embedding a boundary piece  $K(a)$  labeled by  $a$  into the total space (but there is no requirement that  $K(a \rightarrow \top)$  be a monomorphism) and think of each morphism  $K(b \rightarrow a)$  for  $b \rightarrow a$  in  $D$  as describing a boundary piece  $K(b)$  of a boundary piece  $K(a)$ .

## 1.1 Composition

The idea is that multi-cospans are composed by first gluing them along a common sub-multi-cospan, then forming the colimit cocone over that, and finally picking a sub-multi-cospan in that, containing the tip of the colimit cocone.

**Definition 1.5 (multi-cospan closure)** For  $D \in \text{Posets}$  and  $K : D \rightarrow \mathcal{C}$  a functor the multi-cospan closure of  $K$  (or rather one of the canonically isomorphic choices) is the unique multi-cospan

$$\bar{K} : \bar{D} \rightarrow \mathcal{C}$$

such that

$$\begin{array}{ccc} D & \xrightarrow{\quad} & \bar{D} \xrightarrow{\bar{K}} \mathcal{C} \\ & \searrow K & \nearrow \\ & \mathcal{C} & \end{array}$$

and

$$\begin{array}{ccc} \{\top\} & \xrightarrow{\quad} & \bar{D} \xrightarrow{\bar{K}} \mathcal{C} \\ & \searrow \top \mapsto \text{colim}_D K & \nearrow \\ & \mathcal{C} & \end{array}$$

For instance for  $D = \left\{ \begin{array}{c} b_1 \quad b_2 \\ \nearrow \quad \nwarrow \\ a_1 \quad a_2 \quad a_3 \end{array} \right\}$  we have for any  $K : D \rightarrow \mathcal{C}$

$$\bar{K} : \left\{ \begin{array}{c} \top \\ \nearrow \quad \nwarrow \\ b_1 \quad b_2 \\ \nearrow \quad \nwarrow \\ a_1 \quad a_2 \quad a_3 \end{array} \right\} \mapsto \left\{ \begin{array}{c} \text{colim}_D K \\ \nearrow \quad \nwarrow \\ K(b_1) \quad K(b_2) \\ \nearrow \quad \nwarrow \\ K(a_1) \quad K(a_2) \quad K(a_3) \end{array} \right\}$$

**Definition 1.6 (multi-cospan composition)** For  $K_1 : D_1 \rightarrow \mathcal{C}$  and  $K_2 : D_2 \rightarrow \mathcal{C}$  two multi-cospans in  $\mathcal{C}$ , and given a diagram  $U(D_1) \xleftarrow{\quad} D_{\text{glue}} \xrightarrow{\quad} U(D_2)$  in Posets of sub-poset inclusions respecting co-terminal objects, such that

$$\begin{array}{ccc} D_{\text{glue}} & \longrightarrow & D_1 \\ \downarrow & & \downarrow K_1 \\ D_2 & \xrightarrow{K_2} & \mathcal{C} \end{array}$$

and given a morphism in  $\text{Posets}_{\max}$   $D_{\text{comp}} \hookrightarrow \overline{D_1 \sqcup_{\text{glue}} D_2}$  we say that *composition of  $K_1$  and  $K_2$  along  $D_{\text{glue}}$  to  $D_{\text{comp}}$  is the multi-cospan (or rather any one of the canonically isomorphic choices)*

$$K_{\text{comp}} : D_{\text{comp}} \hookrightarrow \overline{D_1 \sqcup_{\text{glue}} D_2} \xrightarrow{\overline{K_1 \sqcup_{\text{glue}} K_2}} \mathcal{C}$$

**Example: ordinary spans and cospans.**

We obtain ordinary cospans and their composition by taking all multi-cospan domains to be  $D = \left\{ \begin{array}{c} \top \\ \nearrow \quad \searrow \\ a_1 \quad a_2 \end{array} \right\}$  and  $D_{\text{glue}} = \{\bullet\}$ , so that  $D \sqcup_{\text{glue}} D = \left\{ \begin{array}{c} b_1 \quad b_2 \\ \nearrow \quad \searrow \\ a_1 \quad a_2 \quad a_3 \end{array} \right\}$

and  $\overline{D \sqcup_{\text{glue}} D} = \left\{ \begin{array}{c} \top \\ \nearrow \quad \searrow \\ b_1 \quad b_2 \\ \nearrow \quad \searrow \\ a_1 \quad a_2 \quad a_3 \end{array} \right\}$  and finally taking  $D_{\text{comp}} = D$  with  $D_{\text{comp}} \hookrightarrow \overline{D \sqcup_{\text{glue}} D}$  given

by  $a_1 \mapsto a_1$  and  $a_2 \mapsto a_3$ .

Dually, we get ordinary multispan in  $\mathcal{C}^{\text{op}}$ . But already in this case we get a little more flexibility spans for handling cospans. For definiteness, consider cospans in  $\mathcal{C} = \text{Sets}^{\text{op}}$ .

Composition of the multispan  $\bar{K}_1$  which is the closure of

$$K_1 := \left\{ \begin{array}{cc} \Psi_{\text{in}} & \Psi_{\text{out}} \\ \downarrow & \downarrow \\ X & Y \end{array} \right\}$$

with

$$K_2 := \left\{ \begin{array}{c} R \\ \downarrow \\ X \times Y \\ \swarrow \quad \searrow \\ X \quad Y \end{array} \right\}$$

along the preimages of  $X$  and  $Y$  to  $\top$  is the tip of

$$\begin{array}{ccc} & \langle \Psi_{\text{in}} | R | \Psi_{\text{out}} \rangle & \\ & \downarrow R & \\ \Psi_{\text{in}} & X \times Y & \Psi_{\text{out}} \\ \downarrow & \swarrow \quad \searrow & \downarrow \\ X & & Y \end{array}$$

and describes the contraction of the matrix  $K$  with the vectors  $\Psi_{\text{in}}$  and  $\Psi_{\text{out}}$ .

**Example: more general multi-cospans in  $\text{Sets}^{\text{op}}$ .**

Next, consider two matrices  $R_1$  and  $R_2$  given by the multispan  $\bar{K}_1$  which is the closure of

$$K_1 := \left\{ \begin{array}{cc} R_1 & R_2 \\ \downarrow & \downarrow \\ X \times Y & Y \times Z \\ \swarrow \quad \searrow & \downarrow \\ X \quad Y & Y \quad Z \end{array} \right\}$$

and then consider the multispan

$$K_2 := \left\{ \begin{array}{c} X \longleftarrow X \times Z \longrightarrow Z \\ \uparrow \qquad \uparrow \\ X \times Y \times Z \\ \swarrow \qquad \searrow \\ X \times Y \qquad Y \times Z \\ \swarrow \qquad \searrow \\ X \qquad Y \qquad Z \end{array} \right\}.$$

Then composition *along* the lower zig-zag to the resulting top zig-zag yields the matrix product

$$K_{\text{comp}} = \left\{ \begin{array}{c} R_1 \cdot R_2 \\ \downarrow \\ X \times Z \\ \swarrow \qquad \searrow \\ X \qquad Z \end{array} \right\}$$

These represent  $\mathbb{N}$ -valued matrices. Somewhat more interestingly, let  $k$  be some field and

$$K_1 := \left\{ \begin{array}{ccc} & R_1 & R_2 \\ & \downarrow & \downarrow \\ & (X \times Y) \times k & (Y \times Z) \times k \\ \swarrow & \downarrow & \searrow \\ X & Y & Z \end{array} \right\}$$

two  $k$ -valued matrices to be composed with the multispan

$$K_2 := \left\{ \begin{array}{c} X \longleftarrow (X \times Z) \times k \longrightarrow Z \\ \uparrow \qquad \uparrow \\ (X \times Y \times Z) \times k \times k \\ \swarrow \qquad \searrow \\ (X \times Y) \times k \qquad (Y \times Z) \times k \\ \swarrow \qquad \searrow \\ X \qquad Y \qquad Z \end{array} \right\}.$$

Then the result is (after  $k$ -valued cardinality) the product of  $k$ -valued matrices.

**Example: Grandis' cubical cospans.** The cubical multi-cospans in [2] are reproduced by restricting the domain posets to be of the form  $\wedge^n$ ,  $n \in \mathbb{N}$ .

## 1.2 Trace and co-trace

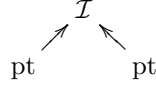
**Definition 1.7 (co-trace)** If for  $K : D \rightarrow \mathcal{C}$  a multi-cospan in which for a collection  $\{a_i \in \text{Obj}(D)\}_i$  of coterminal objects we have  $K(a_i) \simeq X$  for all  $i$  and for  $X$  a given object of  $\mathcal{C}$  the co-trace of  $K$  over  $\{a_i\}$  is the composition of  $K$  with the co-span

$$K_X : \left\{ \begin{array}{c} \top \\ \swarrow \quad \searrow \\ a_1 \quad a_2 \quad \dots \quad a_i \quad \dots \end{array} \right\} \xrightarrow{\text{const}_X} \mathcal{C}$$

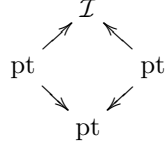
along the obvious identifications of the  $a_i$  to the diagram  $D$  with the  $a_i$  removed.

For spans the co-trace is called the trace.

**Examples.** Let  $\mathcal{C} = \text{Categories}$  and  $\mathcal{I} := \{a \rightarrow b\}$  be the 1-globe (the (directed) interval) regarded as a co-span

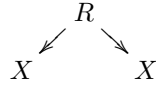


then the co-trace of  $\mathcal{I}$  over  $\text{pt}$  is the the colimit over

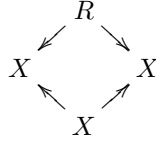


which is  $\mathbf{BN}$  (the (directed) circle).

Dually, let  $\mathcal{C} = \text{Sets}^{\text{op}}$  and consider spans of finite sets again, with



an  $|X| \times |X|$ -matrix, then the trace of this is the limit over



which is  $\sqcup_{x \in X} Rx, x$ , as expected.

## 2 Extended QFT

**Definition 2.1 (extended QFT)** For  $S^{\text{op}}, \mathcal{V}$  categories with finite limits an extended  $S$ -QFT with coefficients in  $\mathcal{V}$  is a functor

$$Z : \text{MultiSpans}(S^{\text{op}}) \rightarrow \text{MultiSpans}(\mathcal{V})$$

which respects composition of multispans.

For instance for  $S = \text{Top}$  this is supposed to be an *extended topological QFT*.

**Definition 2.2 ( $\sigma$ -model QFT)** If  $\mathcal{V}$  is closed monoidal and given a  $\mathcal{V}$ -enrichment of  $[S^{\text{op}}, \mathcal{V}]$ , for

$$B : S \rightarrow [S^{\text{op}}, \mathcal{V}]$$

a functor respecting finite colimits and  $(P_X \rightarrow X) \in [S^{\text{op}}, \mathcal{V}]$ , the  $S$ -QFT

$$[B(-), P_X] : \text{MultiSpans}(S^{\text{op}}) \rightarrow \text{MultiSpans}(\mathcal{V})$$

is a  $\sigma$ -model with target space  $X$  and background field  $P_X$ .

Consider  $S = \text{Top}$ ,  $\mathcal{V} = \text{Cat}$  and  $\mathcal{C} = [S^{\text{op}}, \mathcal{V}]$ . As noticed in theorem 4.5 in [3], Brown's homotopy van Kampen theorem ensures that the fundamental groupoid assignment  $\Pi_1 : \text{Top} \rightarrow \text{Categories}$  extends to a functor  $\text{MultiSpans}(\text{Top}^{\text{op}}) \rightarrow \text{MultiSpans}(\text{Categories}^{\text{op}})$  which respects composition of multispans.

Hence for any  $C \in [S^{\text{op}}, \mathcal{V}]$  any category-valued presheaf

$$[\Pi_1(-), C] : \text{MultiSpans}(\text{Top}^{\text{op}}) \rightarrow \text{MultiSpans}(\text{Categories}^{\text{op}})$$

respects composition of multispans.

**Examples.** Let  $S = \text{Top}$ ,  $\mathcal{V} = \text{Groupoids}$ ,  $G$  a finite group, then the  $\sigma$ -model

$$[\Pi_1(-), \mathbf{B}G] : \text{MultiSpans}(\text{Top}^{\text{op}}) \rightarrow \text{MultiSpans}(\text{Groupoids})$$

is essentially the untwisted Dijkgraaf-Witten model.

## References

- [1] J. Baez, *Higher dimensional algebra VII: Groupoidification*, [<http://math.ucr.edu/home/baez/hda7.pdf>]
- [2] Marco Grandis, *Cospans in Algebraic Topology, I: Higher cospans and weak cubical categories*, TAC, Vol. 18, No. 12, 2007
- [3] Marco Grandis, *Cospans in Algebraic Topology, II: Collared cospans, cohomotopy and TQFT*
- [4] Marco Grandis, *Cospans in Algebraic Topology, III: Cubical cospans and higher cobordisms*