Pasting schemes for the monoidal biclosed structure on ω-**Cat**[*]

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Abstract

Using the theory of pasting presentations, developed in [5], I give a detailed description of the tensor product on ω-categories, which extends Gray's tensor product on 2-categories and which is closely related to Brown-Higgins's tensor product on ω-groupoids.

Immediate consequences are a general and uniform definition of higher dimensional lax natural transformations, and a nice and transparent description of the corresponding internal homs. Further consequences will be in the development of a theory for weak n-categories, since both tensor products and lax structures are crucial in this.

Contents

[*]This paper is also Chapter 3 of the author's PhD thesis.

1 Introduction

In 1945, Eilenberg and Mac Lane invented categories [15]. In 1967, Bénabou invented bicategories [4], and he proved that every bicategory is biequivalent to a 2-category. In 1993, Gordon, Power and Street introduced tricategories, and they proved that every tricategory is triequivalent *not* to a 3-category, but to a **Gray**-*category* [20]. In algebraic topology, some weakening is needed in dimension 3 as well: homotopy 2-types are classified by 2-groupoids [31, 16, 29], but homotopy 3-types are classified not by 3-groupoids, but by **Gray**-*groupoids* [25, 24, 6].

In order to find out what happens in higher dimensions, and to avoid doing dimension 4, then dimension 5, etc., I propose to work ω-dimensionally from the start. I think this is the best way to get grip on the complex notion of *weak n-category.*

So the first step to take is to start with something ω-dimensional which is strict and known, ω-categories [35, 23], and to analyse the structure which in the case of 2-categories leads to the concept of **Gray**-category. This structure, which makes ω-**Cat**, the category of small ω-categories, into a monoidal biclosed category, already has interesting complications, consequences and applications. A complication is, that the machinery of pasting presentations of [5] is needed. A consequence is, that it gives higher homotopies [10] in terms of ω-categories. An application is that "ω-categories might serve as a model for concurrency in computing, and tensor products would be important in this theory" [2, 32].

This paper is connected to previous work of Brown and Higgins [10], Gray [21] and Al-Agl and Steiner [2]. Throughout, I keep close track of the relation between the results there and here. For one thing, the terminology is different: what I call an ω-category is termed ∞-category in [2], and what is called ω-groupoid in [10] I call a cubical ω-groupoid, reserving the name ω-groupoid for what [7, 8] call an ∞-groupoid. For the ω-categories of Street [35] I agree with Verity's suggestion to call these ω^+-categories. Another source is [5], to which the reader is referred for preliminaries on ω-categories, pasting schemes and pasting presentations.

The central idea of this paper is that the tensor product of cubes induces a monoidal biclosed structure on ω-**Cat**. I sketch how this follows formally from results of Day [11, 12], the main point being that ω-**Cat** is *monoidal* monadic over the category **Cub** of cubical sets. Implicitly, Brown and Higgins [10] give the same motivation for the existence of a tensor product of cubical ω-groupoids. There are two disadvantages to the formal approach, though: it doesn't give explicit formulae, and using cubes conflicts with globes representing elements of ω-categories [35]. Therefore, the actual approach uses Johnson's theory of pasting schemes [23], thereby making precise the "appropriate composites of faces" of [2]. Concretely, I give a pasting scheme for the tensor product of two globes as ω-categories, which is used as a basic ingredient in the definition of a pasting presentation for the tensor product. This gives the desired explicit formulae, without the need to ever write out composites as in [34]. It also gives formulae for higher dimensional lax natural transformations and for the internal homs.

This paper is organized as follows. In sections 2 and 3 cubes, cubical sets and the adjunction between cubical sets and ω-categories are treated. Section 4 explains why the tensor product of cubes induces a monoidal biclosed structure on ω-**Cat**. Sections 5 and 6 describe pasting schemes for tensor of and with globes, which are used for the pasting presentation of the tensor product in section 7. In section 8 some properties of the tensor product are checked. Sections 9 to 11 deal with the internal homs, and with higher dimensional lax natural transformations. The final section is on ω-**Cat** as an enriched category.

Some of the ideas here were announced at the Conference on Pure Mathematics of the University of Wales, 24-26 may 1993, Gregynog, UK.

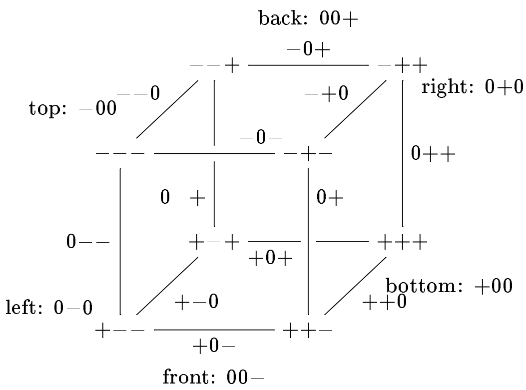
2 Cubes and cubical sets

A simplicial set is usually defined as a collection of cells together with boundary operations and degeneracy operations satisfying some relations [26]. A more categorical definition is that a simplicial set is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Sets}$, where $\mathbf{\Delta}$ is the category of finite ordered sets and order preserving maps between them [13, 19]. For cubical sets, there are *two* analoga of the first description, one without and one with so-called *connections* [9, 36]. Perhaps for this reason there seems to be no categorical description of cubical sets available. I intend to fill part of this gap, by defining a category $\mathbf{\Gamma}$ which is to cubical sets, without connections, what $\mathbf{\Delta}$ is to simplicial sets. In fact, I *define* a cubical set as a functor $\mathbf{\Gamma} \to \mathbf{Sets}$, and I show that this definition coincides with the usual one [9]. Analogous to the simplicial case the objects of $\mathbf{\Gamma}$ are called the *standard cubes*.

2.1 Cubes combinatorially

Aichison has given an extensive account on cubes [1], from which I will use the following combinatorial definition of the n-dimensional cube.

Let $\underline{n}$ be the ordered set $\{1, \ldots, n\}$ and $\Lambda = \{-, 0, +\}$. Then $\Lambda_n = \{x : \underline{n} \to \Lambda\}$ can be thought of as the n-dimensional cube. For example, the three-dimensional cube can be labeled with elements of Λ_3 as in



,

the interior of the cube being labeled by 000.

Some terminology: if x and y are elements of Λ_n then x is a *subcube* of y if for every $l \in \underline{n}$, $x(l) \le y(l)$ in the partial order $- < 0 > +$ on Λ. The *dimension* of $x \in \Lambda_n$ is $\# x^{-1}(0)$.

2.2 A model category for cubes

I will make the set $\{\underline{n} | n \in \omega\}$ into a category $\mathbf{\Gamma}$ by defining morphisms mirroring the behaviour of faces and degeneracies.

Definition 2.1 A *morphism* $f : \underline{n} \to \underline{m}$ is a function $f^* : \underline{m} \to \underline{n} \cup \{+, -\}$ such that $f^*(k) \le f^*(k') \in \underline{n}$ implies $k \le k'$ and $f^*(k) = f^*(k') \in \underline{n}$ implies $k = k'$. $\diamond$

This may look a little bit awkward, but something more complicated than "order preserving" is expected because cubes have opposite faces instead of faces opposite to a vertex. And this notion *is* relevant to cubes since a morphism $f : \underline{n} \to \underline{m}$ induces a function $f : \Lambda_n \to \Lambda_m$ by

$$\begin{aligned} f(x)(k) &= x(f^*(k)) \quad \text{if } f^*(k) \in \underline{n} \\ &= f^*(k) \qquad \text{otherwise.} \end{aligned}$$

This function behaves well:

Lemma 2.2 *If $f : \underline{n} \to \underline{m}$ is a morphism then the induced function $f : \Lambda_n \to \Lambda_m$ sends subcubes to subcubes.*

Proof. Let x and y be subcubes of Λ_n, and let $k \in \underline{m}$. Then $f(x)(k) = x(f^*(k)) \le y(f^*(k)) = f(y)(k)$ if $f^*(k) \in \underline{n}$, otherwise $f(x)(k) = f^*(k) = f(y)(k)$. Thus $f(x)$ is a subcube of $f(y)$. $\square$

Composition of morphisms: let $f : \underline{n} \to \underline{m}$ and $g : \underline{m} \to \underline{r}$. Define

$$\begin{array}{rcll} (g \circ f)^*(p) & = & f^*(g^*(p)) & \text{if } g^*(p) \in \underline{m} \\ & = & g^*(p) & \text{otherwise.} \end{array}$$

With this definition, $g \circ f$ is a morphism $\underline{n} \to \underline{p}$: if $(g \circ f)^*(p) \leq (g \circ f)^*(p') \in \underline{n}$ then $g^*(p)$ and $g^*(p')$ are in $\underline{m}$ and $g^*(p) \leq g^*(p')$, so $p \leq p'$. Similarly for $(g \circ f)^*(p) = (g \circ f)^*(p') \in \underline{n}$. The identity on $\underline{n}$, denoted $\mathrm{id}_{\underline{n}}$, is given by

$$(\mathrm{id}_{\underline{n}})^*(l) = l.$$

Proposition 2.3 *$\mathbf{\Gamma}$ is a category.*

Proof. Composition of morphisms in $\mathbf{\Gamma}$ is associative:

$$\begin{array}{rcll} (h \circ g \circ f)^*(q) & = & f^*\left(g^*(h^*(q))\right) & \text{if } h^*(q) \in \underline{r} \text{ and } g^*(h^*(q)) \in \underline{m} \\ & = & g^*(h^*(q)) & \text{if } h^*(q) \in \underline{r} \text{ and not } g^*(h^*(q)) \in \underline{m} \\ & = & h^*(q) & \text{otherwise} \end{array}$$

for both ways of putting in brackets. And it is easy to see that the identity behaves as an identity should. □

2.3 Generating the model category for cubes

To relate the category $\mathbf{\Gamma}$ to the usual notion of cubical sets, I will show that every morphism is a composite of face and degeneracy morphisms.

Definition 2.4 A morphism $f : \underline{n} \to \underline{m}$ is *surjective* if for all $k \in \underline{m}$, $f^*(k) \in \underline{n}$. It is *injective* if for all $l \in \underline{n}$ there exists $k \in \underline{m}$ with $f^*(k) = l$. ◇

An example of a surjective morphism is $\varepsilon_i : \underline{n} \to \underline{n-1}$, for $1 \leq i \leq n$, which is defined by

$$\begin{array}{rcll} (\varepsilon_i)^*(l) & = & l & \text{if } l < i \\ & = & l+1 & \text{if } l \geq i. \end{array}$$

An example of an injective morphism is $\partial_i^\alpha : \underline{n-1} \to \underline{n}$, for $1 \leq i \leq n$ and $\alpha = \pm$, which is defined by

$$\begin{array}{rcll} (\partial_i^\alpha)^*(l) & = & l & \text{if } l < i \\ & = & \alpha & \text{if } l = i \\ & = & l-1 & \text{if } l > i. \end{array}$$

Note that there are the following relations between the ε_i and the ∂_i^α:

(i) $\partial_j^\alpha \partial_i^\beta = \partial_i^\beta \partial_{j-1}^\alpha$ for all $i < j \leq n$ and $\alpha, \beta = \pm$,

(ii) $\varepsilon_j \varepsilon_i = \varepsilon_i \varepsilon_{j+1}$ for all $i \leq j \leq n$,

(iii)
$$\begin{aligned} \varepsilon_j \partial_i^\alpha &= \partial_i^\alpha \varepsilon_{j-1} && \text{if } i < j && \text{for all } i,\ j \leq n \text{ and } \alpha = \pm \\ &= \partial_{i-1}^\alpha \varepsilon_j && \text{if } i > j \\ &= \mathrm{id}_{K_n} && \text{if } i = j, \end{aligned}$$

These relations are checked easily by immediate calculation.

Proposition 2.5 *Every morphism can be factored as a surjection followed by an inclusion.*

Proof. Given $f : \underline{n} \to \underline{m}$, define $r = \#\{k \in \underline{m} | f^*(k) \in \underline{n}\}$. Define morphisms $g : \underline{n} \to \underline{r}$ and $h : \underline{r} \to \underline{m}$ by

$$\begin{aligned} g^*(p) &= f^*(k) && \text{for } k \text{ the } p\text{-th element of } \underline{m} \text{ for which } f^*(k) \in \underline{n} \\ h^*(k) &= p && \text{if } k \text{ is the } p\text{-th element of } \underline{m} \text{ for which } f^*(k) \in \underline{n} \\ &= f^*(k) && \text{otherwise.} \end{aligned}$$

g is a morphism because if $f^*(k) = g^*(p) \leq g^*(p) = f^*(k')$ then $k \leq k'$ and so $p \leq p'$, and likewise for $g^*(p) = g^*(p)$, and h is a morphism because if $p = h^*(k) \leq h^*(k') = p'$ then $k \leq k'$ and $h^*(k) \leq h^*(k')$ likewise, all these cases when k the p-th element of $\underline{m}$ for which $f^*(k) \in \underline{n}$. g is surjective because for $p \in \underline{r}$, $g^*(p) \in \underline{n}$, and h is injective because for $p \in \underline{r}$ there exists $k \in \underline{m}$ with $h^*(k) = p$, namely the p-th element of $\underline{m}$. Their composite is given by $(h \circ g)^*(k) = g^*(h^*(k)) = g^*(p) = f^*(k)$ if k is the p-th element of $\underline{m}$ for which $f^*(k) \in \underline{n}$ and $(h \circ g)^*(k) = h^*(k) = f^*(k)$ otherwise, so indeed $f = h \circ g$. $\square$

Proposition 2.6 *Every surjection is composite of ε_i's. Every injection is composite of ∂_i^α's.*

Proof. Suppose $f : \underline{n} \to \underline{m}$ is a surjection. Then $n \geq m$, and if $n = m$ then f is the identity. So assume $n > m$, and let i be the first element of $\underline{n}$ which is not $f^*(k)$ for any $k \in \underline{m}$. Define $h : \underline{n-1} \to \underline{m}$ by $h^*(k) = f^*(k)$ if $f^*(k) < i$, and $h^*(k) = f^*(k) - 1$ if $f^*(k) > i$. h is a morphism because of the condition on i, it is surjective by definition, and $(h \circ \varepsilon_i)^*(k) = \varepsilon_i^*(h^*(k)) = f^*(k)$, which shows that $f = h \circ \varepsilon_i$. Induction on the difference of n and m finishes the proof of the first statement.

Suppose $f : \underline{n} \to \underline{m}$ is an injection. Then $n \leq m$, and if $n = m$ then f is the identity. So assume $n < m$, and let i be the first element of $\underline{m}$ for which

$f^*(i) \notin \underline{n}$, say $f^*(i) = \alpha$. Define $g : \underline{n} \to \underline{m-1}$ by $g^*(p) = f^*(p)$ if $p < i$ and $g^*(p) = f^*(p+1)$ if $p \geq i$. Then g is an injective morphism, and $f = \partial_i^\alpha \circ g$, induction finishing the proof of the second statement. $\square$

Thus, $\mathbf{\Gamma}$ is the category generated by the ε_i and the ∂_i^α subject to the relations given above.

2.4 Cubical sets

Definition 2.7 A *cubical set* is a functor $\mathbf{\Gamma}^{\mathrm{op}} \to \mathbf{Sets}$. A *cubical map* is a natural transformation of such functors. $\diamond$

Proposition 2.8 *A cubical set K is a family of sets K_n $(n \geq 0)$, together with face maps $\partial_i^\alpha : K_n \to K_{n-1}$ and degeneracy maps $\varepsilon_i : K_{n-1} \to K_n$, for every $1 \leq i \leq n$ and $\alpha = \pm$, such that*

(i) $\partial_i^\alpha \partial_j^\beta = \partial_{j-1}^\beta \partial_i^\alpha$ *for all $i < j \leq n$ and α, $\beta = \pm$,*

(ii) $\varepsilon_i \varepsilon_j = \varepsilon_{j+1} \varepsilon_i$ *for all $i \leq j \leq n$,*

(iii)
$$\begin{aligned} \partial_i^\alpha \varepsilon_j &= \varepsilon_{j-1} \partial_i^\alpha && \text{if } i < j \quad \text{for all } i,\ j \leq n \text{ and } \alpha = \pm. \\ &= \varepsilon_j \partial_{i-1}^\alpha && \text{if } i > j \\ &= \mathrm{id}_{K_n} && \text{if } i = j \end{aligned}$$

A cubical map $f : K \to L$ is a family of functions $f_n : K_n \to L_n$ commuting with the face and degeneracy maps.

Proof. Because of propositions 2.5 and 2.6, and the relations between the ε_i and the ∂_i^α in $\mathbf{\Gamma}$, which are dual to the ones above. $\square$

The category of cubical sets will be denoted by $\mathbf{Sets}^{\mathbf{\Gamma}^{\mathrm{op}}}$ or by $\mathbf{Cub}$, depending on which viewpoint is taken.

As an example, consider the representative cubical sets, i.e., the standard n-cubes as cubical set. Define a cubical set $\mathcal{I}^n$ by $\mathcal{I}^n(\underline{m}) = \mathbf{\Gamma}(\underline{m}, \underline{n})$. Note that if $m > n$ then all elements of $(\mathcal{I}^n)_m$ are degenerate. $\mathcal{I}^n$ is related to Λ_n: if $A : \underline{m} \to \Lambda$ has $A(k) = 0$ for all $k \in \underline{m}$, then $(f : \underline{m} \to \underline{n}) \in \mathcal{I}^n$ corresponds to $A \circ f^* \in \Lambda_n$.

2.5 Duality

There are three forms of duality of cubical sets that will be of importance in the sequel. The first one is the transposition functor T considered in [10]. For cubical set X, $T(X)$ has the same elements as X in each dimension but has its face and degeneracy operators numbered in reverse order. The second one consists simply of reversing the signs in the exponents of the ∂_i^α, and the third is just the combination of these two.

3 The ω-categorization of cubical sets and the cubical nerve of ω-categories

The standard n-simplex can be given the structure of an n-category: it is Street's n-th oriental [35]. This functor $\mathbf{\Delta} \to \omega$-$\mathbf{Cat}$ induces, by general categorical arguments, two adjoint functors between simplicial sets and ω-categories: ω-categorization and simplicial nerve.

Analogous to a description of the orientals in terms of pasting schemes [22] I define a functor from cubes to ω-categories, and I describe the induced functors between cubical sets and ω-categories. The ω-categorization of a cubical set is given by a pasting presentation, and the cubical nerve of an ω-category is expressed using realizations of pasting schemes.

3.1 Cubical complexes

To describe the orientals, Johnson [22] uses the notion of *simplicial complex.*

Definition 3.1 [Cubical analogue of Johnson's simplicial complexes] A *cubical complex* is a finite or countably infinite set K together with a collection $\mathcal{K}$ of maps $K \to \Lambda$ such that if $B \in \mathcal{K}$ and $B' : K \to \Lambda$ satisfies $B'(k) \leq B(k)$ in the partial order $- < 0 > +$ on Λ for all $k \in K$, then $B' \in \mathcal{K}$. A cubical complex is *oriented* if K is linearly ordered. $\diamond$

An oriented cubical complex generates a cubical set whose non degenerate elements are the same as the elements of the complex.

An example of an oriented cubical complex is $(\omega, \Lambda_\omega^{\mathrm{f}})$, where $\Lambda_\omega^{\mathrm{f}}$ consists of the finite dimensional maps $\omega \to \Lambda$. This cubical complex, and sometimes also the cubical set generated by it, will be called the ω-*cube.* The standard n-cubes can also be seen as cubical sets generated by oriented cubical complexes.

3.2 Pasting schemes for the ω-cube and for the n-cubes

In the simplicial case, a particular simplicial complex is made into a pasting scheme by taking odd faces in the beginning and even faces in the end of a cell.

In the cubical case, I will do the same, but for a different way of expressing and positioning odd and even faces. This will be done such that the direction of the cells is the same as in the oriented cubes of [1].

Let x be an i-dimensional element of the ω-cube, and let $B \in \Lambda_i$. Define $r_B(x) : \omega \to \Lambda$ by

$$\begin{aligned} r_B(x)(k) &= x(k) \quad \text{if } x(k) \neq 0 \\ &= B(l) \quad \text{if } k \text{ is the } l\text{-th element of } x^{-1}(0). \end{aligned}$$

Write $b_B(x)$ for $r_B(x)$ if for all $l \in \underline{i}$, $B(l) \neq 0$ implies $B(l) = (-)^l$, and $e_B(x)$ if for all $l \in \underline{n}$, $B(l) \neq 0$ implies $B(l) = (-)^{l+1}$.

Consider the graded set $\Lambda^{\mathrm{f}}_\omega$. Define relations E and B on $\Lambda^{\mathrm{f}}_\omega$ by $(x, y) \in \mathsf{E}^i_j$ for $x \in (\Lambda^{\mathrm{f}}_\omega)_i$ and $y \in (\Lambda^{\mathrm{f}}_\omega)_j$ if and only if there exists $B : \underline{i} \to \Lambda$ such that $y = e_B(x)$, and $(x, y) \in \mathsf{B}^i_j$ if and only if there exists $B : \underline{i} \to \Lambda$ such that $y = b_B(x)$.

$\Lambda^{\mathrm{f}}_\omega$ is a loop-free pasting scheme, since it is the same pasting scheme as considered by Kapranov-Voevodsky [27].

Taking n instead of ω in the above makes Λ_n into a well-formed loop-free pasting scheme, because it can be viewed as a well-formed subpasting scheme of $\Lambda^{\mathrm{f}}_\omega$. I will need domains and codomains of its cells.

Lemma 3.2 *For $(m+1)$-dimensional $x \in \Lambda_n$, $s_m(\mathsf{R}(x)) = \bigcup\{\mathsf{R}(r_B(x)) | B \in \Lambda_{m+1},\ \dim(B) = m,\ r_B(x) = b_B(x)\}$, and dually.*

Proof. According to [23, Proposition 7] $s_m(\mathsf{R}(x)) = \mathsf{R}(\mathsf{B}_m(x))$, which is exactly the right hand set above. $\square$

In figure 1, the pasting scheme Λ_4.

3.3 Cubes and ω-categories

The morphisms ε_i and ∂^α_i in $\mathbf{\Gamma}$ induce ω-functors $\varepsilon_i : \mathcal{P}(\Lambda_n) \to \mathcal{P}(\Lambda_{n-1})$ and $\partial^\alpha_i : \mathcal{P}(\Lambda_{n-1}) \to \mathcal{P}(\Lambda_n)$ respectively, where $\mathcal{P}(A)$ denotes the ω-category of components of the pasting scheme A [23], as follows.

Define a realization (Λ_n, f_j) of Λ_n in $\mathcal{P}(\Lambda_{n-1})$ by $f_j(x) = \mathsf{R}(x \circ (\varepsilon_i)^*)$. Because $\mathcal{P}(\Lambda_{n-1})$ is considered one sorted this could also mean an identity on this, to get its dimension right! It is an identity exactly when $x(i) = 0$, because ε_i erases the i-th entry. Assume (Λ_n, f_j) is m-appropriate, I will show it is $(m+1)$-appropriate. In case $x(i) = 0$, $s_m(f_{m+1}(x)) = s_m\left(\mathrm{id}_{\mathsf{R}(x \circ (\varepsilon_i)^*)}\right) = \mathsf{R}(x \circ (\varepsilon_i)^*)$,

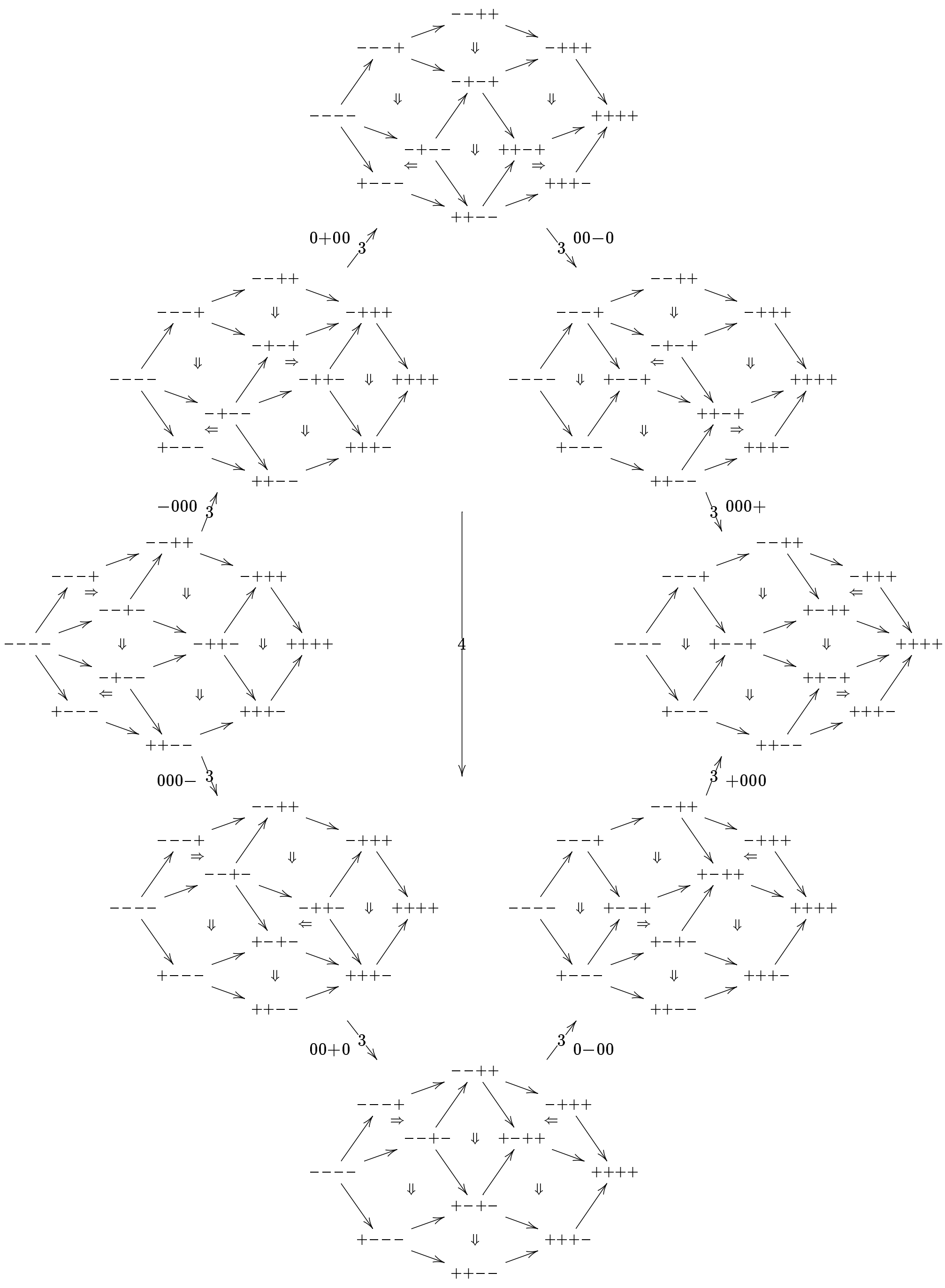


Figure 1: The four dimensional cube as a pasting scheme

and

$$\begin{aligned}
& f\,(s_m(\mathsf{R}(x))) = \\
&\quad = \bigcup\{\mathsf{R}(r_B(x))|\ldots\} && \text{by lemma 3.2} \\
&\quad = \bigcup\{f_m(r_B(x))|\ldots\} && \text{because } f \text{ is the } m\text{-extension of } (\Lambda_n, f_j) \\
&\quad = \bigcup\{\mathsf{R}(r_B(x)\circ(\varepsilon_i)^*)|\ldots\}.
\end{aligned}$$

All these are identities except when $r_B(x)(i) \neq 0$, in which case it is $\mathsf{R}(x\circ(\varepsilon_i)^*)$. So as elements of $\mathcal{P}(\Lambda_{n-1})$, $s_m(f_{m+1}(x))$ and $f\,(s_m(\mathsf{R}(x)))$ are equal. In case $x(i) = \pm$ one sees that for all B, $r_B(x\circ(\varepsilon_i)^*) = r_B(x)\circ(\varepsilon_i)^*$. This, together with lemma 3.2 and that f is the m-extension of (Λ_n, f_j), proves in a similar way that in this case $s_m(f_{m+1}(x)) = f\,(s_m(\mathsf{R}(x)))$ as well. Thus indeed (Λ_n, f_j) is appropriate.

Analogously, one defines an appropriate realization (Λ_{n-1}, g_j) of Λ_{n-1} in $\mathcal{P}(\Lambda_n)$ by $g_j(x) = \mathsf{R}(x\circ(\partial_i^\alpha)^*)$.

These induced ω-functors ε_i and ∂_i^α satisfy the same identities as in $\mathbf{\Gamma}$ because of the identities there and because the induced ω-functors extend (Λ_n, f_j) and (Λ_{n-1}, g_j). Thus, there is a functor $\mathcal{Q} : \mathbf{\Gamma} \to \omega\text{-}\mathbf{Cat}$, defined on objects by $\mathcal{Q}(\underline{n}) = \mathcal{P}(\Lambda_n)$. $\mathcal{Q}(\underline{n})$ could be termed the n-th q-bical oriental, or even the n-th qriental.

3.4 ω-categorization

The functor $\mathcal{Q}$ induces a functor $\Pi_{\mathbf{\Gamma}}$ from cubical sets to ω-categories, which is the left Kan-extension of $\mathcal{Q}$ along the Yoneda embedding $\mathbf{\Gamma} \to \mathbf{Sets}^{\mathbf{\Gamma}^{\text{op}}}$ [30]. It can be given by

$$\Pi_{\mathbf{\Gamma}}(X) = \int^{\underline{n}} X_n \cdot \mathcal{Q}(\underline{n}),$$

where the coend is in ω-**Cat**. A more explicit description of $\Pi_{\mathbf{\Gamma}}(X)$ is by the pasting presentation $(\underline{G}_X, \underline{R}_X)$.

For X a cubical set, generators in $\underline{G}_X$ in dimension n will be $(\Lambda_n, \mathfrak{L}_x)$ for $x \in X_n$, where the elements of Λ_n will be labeled by the corresponding faces of x, i.e., $B \in \Lambda_n$ will be labeled by $X(B)(x)$, the restriction of x along B considered as a map $\Lambda_{\dim(B)} \to \Lambda_n$. Note that it is not required that x is non degenerate! These are indeed generators because $\mathrm{dom}(\Lambda_n)$ is a generated pasting, the cells all being labeled by generators.

Relations in $\underline{R}_X$ in dimension n will come from degeneracies. I want to say that a degenerate cube is equivalent to an identity, but in this I have to use lower dimensional relations. So the approach will be inductively.

Consider the labeled pasting scheme $(\Lambda_{n+1}, \mathfrak{L}_x^{\varepsilon_i})$, where $\mathfrak{L}_x^{\varepsilon_i}$ is equal to $\mathfrak{L}_{\varepsilon_i(x)}$ except that the top-dimensional cell is labeled with the *formal* identity id_x instead of with $\varepsilon_i(x)$. Consider also the labeled pasting scheme $(\Lambda_{n+1}, \mathfrak{L}_x^i)$, where $\mathfrak{L}_x^i$ differs from $\mathfrak{L}_{\varepsilon_i(x)}$ in that cells labeled $\varepsilon_i(x)|_B$ with $B(i) = 0$ in the latter, are labeled $\mathrm{id}_{x|_{\partial_i^+(B)}}$ in the former. It would make no difference to take $\mathrm{id}_{x|_{\partial_i^-(B)}}$ because of the cubical identities. So for example $(\Lambda_3, \mathfrak{L}_{00}^1)$ is

$$\begin{tikzcd}
 & -+_b \arrow[rr] \arrow[dr, phantom, "\Downarrow 00_b"] & & ++_b \arrow[dr, "\mathrm{id}_{++}"] & \\
--_b \arrow[ur] \arrow[rr] \arrow[dr, "\mathrm{id}_{--}"'] & & +-_b \arrow[ur] \arrow[dr, "\mathrm{id}_{+-}"] \arrow[rr, phantom, "\Downarrow \mathrm{id}_{+0}"] & & ++_e \\
 & --_e \arrow[rr] \arrow[ur, phantom, "\Downarrow \mathrm{id}_{0-}"] & & +-_e \arrow[ur] &
\end{tikzcd}
\xrightarrow[3]{\mathrm{id}_{00}}
\begin{tikzcd}
 & -+_b \arrow[rr] \arrow[dr, "\mathrm{id}_{-+}"] & & ++_b \arrow[dr, "\mathrm{id}_{++}"] \arrow[dl, phantom, "\Downarrow \mathrm{id}_{0+}"] & \\
--_b \arrow[ur] \arrow[dr, "\mathrm{id}_{--}"'] \arrow[rr, phantom, "\Downarrow \mathrm{id}_{-0}"] & & -+_e \arrow[rr] \arrow[dr, phantom, "\Downarrow 00_e"] & & ++_e \\
 & --_e \arrow[ur] \arrow[rr] & & +-_e \arrow[ur] &
\end{tikzcd},$$

where the e and b subscripts distinguish different cells with equal labels.

Lemma 3.3 *Suppose that for $n' \leq n$ $(\Lambda_{n'}, \mathfrak{L}_x^{\varepsilon_i})$ is a generated pasting, that it is defined to be related to $(\Lambda_{n'}, \mathfrak{L}_{\varepsilon_i(x)})$, and that with these relations in all dimensions up to n' both are equivalent to $(\Lambda_{n'}, \mathfrak{L}_x^i)$. Then $(\Lambda_{n+1}, \mathfrak{L}_x^{\varepsilon_i})$ is a generated pasting, and if it is defined to be related to $(\Lambda_{n+1}, \mathfrak{L}_{\varepsilon_i(x)})$ then both are equivalent to $(\Lambda_{n+1}, \mathfrak{L}_x^i)$.*

Proof. To prove that $(\Lambda_{n+1}, \mathfrak{L}_x^{\varepsilon_i})$ is a generated pasting, I need to show that its domain is equivalent to $(\Lambda_n, \mathfrak{L}_x)$. For example,

$$\mathrm{dom}(\Lambda_3, \mathfrak{L}_{\varepsilon_1(00)}) = \begin{tikzcd}
 & -+_b \arrow[rr] \arrow[dr, phantom, "\Downarrow 00_b"] & & ++_b \arrow[dr, "\varepsilon_1(++)"] & \\
--_b \arrow[ur] \arrow[rr] \arrow[dr, "\varepsilon_1(--)"'] & & +-_b \arrow[ur] \arrow[dr, "\varepsilon_1(+-)"] \arrow[rr, phantom, "\Downarrow \varepsilon_1(+0)"] & & ++_e \\
 & --_e \arrow[rr] \arrow[ur, phantom, "\Downarrow \varepsilon_1(0-)"] & & +-_e \arrow[ur] &
\end{tikzcd}$$

and

$$(\Lambda_2, \mathfrak{L}_{00}) = \begin{tikzcd}
 & -+ \arrow[dr] & \\
-- \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\Downarrow 00"] & & ++ \\
 & +- \arrow[ur] &
\end{tikzcd}$$

should be equivalent. By adding identities to Λ_n in the right places, first high dimensional ones, then lower dimensional ones to make the higher dimensional ones in the form of cubes, $(\Lambda_n, \mathfrak{L}_x)$ can be seen to be equivalent to a generated pasting with pasting scheme $\mathrm{dom}(\Lambda_{n+1})$. Then identities can be replaced by degeneracies, first low dimensional ones, then higher dimensional ones, because then the relevant subpasting schemes are correctly labeled. The result is exactly $\mathrm{dom}(\Lambda_{n+1}, \mathfrak{L}_x^{\varepsilon_i})$ because the position of the inserted identities is such that they end up in the same position as their corresponding degeneracies. Details, such as full replaceability and full insertability at each stage of this process, are taken for granted.

It is possible to define the relation as claimed since the pasting schemes are equal and the labelings coincide in lower dimensions.

The equivalences hold since again identities can be replaced by degeneracies from low dimensions up. □

So relations can be defined by requiring $\mathfrak{L}_{\varepsilon_i(x)}$ to be related to $(\Lambda_{n+1}, \mathfrak{L}_x^{\varepsilon_i})$ for every $x \in X$.

Proposition 3.4 *The pasting presentation* $(\underline{G}_X, \underline{R}_X)$ *is a pasting presentation for* $\Pi_{\mathbf{\Gamma}}(X)$.

Proof. $\boldsymbol{\omega}(\underline{G}_X, \underline{R}_X)$ and $\Pi_{\mathbf{\Gamma}}(X)$ satisfy the same universal property, as can be seen from the coend description. □

For the representative cubical sets $\mathcal{I}^n$, I will make no notational distinction between the cubical set and its ω-categorization, as usual.

3.5 Cubical nerve

The cubical nerve of an ω-category $\mathbb{C}$ is given by

$$\mathbf{\Gamma}^{\mathrm{op}} \xrightarrow{\mathcal{Q}} \omega\text{-}\mathbf{Cat}^{\mathrm{op}} \xrightarrow{\omega\text{-}\mathbf{Cat}(-,\mathbb{C})} \mathbf{Set}.$$

It is functorial in $\mathbb{C}$, and this functor $N_{\mathbf{\Gamma}}$ is right adjoint to $\Pi_{\mathbf{\Gamma}}$. The existence of such a right adjoint follows from Freyd's adjoint functor theorem [30] since the standard cubes form a generating set of objects of $\mathbf{Sets}^{\mathbf{\Gamma}^{\mathrm{op}}}$. Using the description of $\mathcal{Q}(\underline{n})$ in terms of a pasting scheme,

$$\begin{aligned} N_{\mathbf{\Gamma}}(\mathbb{C})_{\ltimes} &= \omega\text{-}\mathbf{Cat}(\mathcal{Q}(\underline{n}), \mathbb{C}) \\ &= \{(\Lambda_n, f_j) | f_j \text{ is an appropriate realization of } \Lambda_n \text{ in } \mathbb{C}\}. \end{aligned}$$

The cubical operations on $N_{\mathbf{\Gamma}}(\mathbb{C})$ are induced by the ω-functors $\varepsilon_i : \mathcal{P}(\Lambda_n) \to \mathcal{P}(\Lambda_{n-1})$ and $\partial_i^\alpha : \mathcal{P}(\Lambda_{n-1}) \to \mathcal{P}(\Lambda_n)$. More concretely, for (Λ_{n-1}, f_j) an appropriate realization of Λ_{n-1} in $\mathbb{C}$, $(\Lambda_n, \varepsilon_i(f)_j)$ has $\varepsilon_i(f)_j(x) = f_{j'}(x \circ (\varepsilon_i)^*)$, which defines an appropriate realization of Λ_n in $\mathbb{C}$ because of the formula for ε_i as an ω-functor $\mathcal{P}(\Lambda_n) \to \mathcal{P}(\Lambda_{n-1})$, and for (Λ_n, f_j) an appropriate realization of Λ_n in $\mathbb{C}$, $(\Lambda_{n-1}, \partial_i^\alpha(f)_j)$ has $\partial_i^\alpha(f)_j(x) = f_{j'}(x \circ (\partial_i^\alpha)^*)$, which defines an appropriate realization of Λ_{n-1} in $\mathbb{C}$ because of the formula for ∂_i^α as an ω-functor $\mathcal{P}(\Lambda_{n-1}) \to \mathcal{P}(\Lambda_n)$. This also implies that $\partial_i^\alpha(f)_j = f_j|_{\partial_i^\alpha}$, so a face of an element of the nerve is the corresponding face of the composable diagram. For a related approach, which also gives a description of a category of cubical sets with structure making the adjunction an equivalence of categories, see [36].

4 Existence and uniqueness of a monoidal biclosed structure

That the tensor product of cubical sets induces a monoidal biclosed structure on ω-**Cat** was already remarked in [2], and is analogous to the case of crossed complexes [10], which makes use of cubical ω-groupoids as an intermediate stage. As noted there, this works since "ω-**Gpd** is an equationally defined category of many sorted algebras in which the domains of the operations are defined by finite limit diagrams. General theorems on such algebraic theories (see [17, 18, 28, 3]) imply that ω-**Gpd** is complete and cocomplete and that it is monadic over the category **Cub** of cubical sets", and because presentations can be used. Although it is the essence, this is not the whole story. Using methods of Day [11, 12], I show that the monoidal biclosed structure on cubical sets [10] is in fact the extension of a tensor product on $\mathbf{\Gamma}$, and I sketch how pasting presentations can be used to transfer this extension to ω-categories, the main point being that the monad for ω-categories is *monoidal*. Details are omitted in this last step since in sections 5 to 11 I will give a completely independent proof of the existence of a monoidal biclosed structure on ω-**Cat** satisfying $\mathcal{I}^p \otimes \mathcal{I}^q \cong \mathcal{I}^{p+q}$, by describing it explicitly. The uniqueness of such a structure gives that my description is indeed of *the* monoidal biclosed structure on ω-**Cat** induced by the tensor product of cubes.

4.1 Monoidal structure on $\mathbf{\Gamma}$

Addition of natural numbers gives $\mathbf{\Gamma}$ the structure of a strict monoidal category:

let $\underline{m} \otimes \underline{n} = \underline{m+n}$ and let $I = \underline{0} = \varnothing$. To make $\otimes$ into a functor $\mathbf{\Gamma} \to \mathbf{\Gamma}$, define $f \otimes g : \underline{m} \otimes \underline{n} \to \underline{m'} \otimes \underline{n'}$, for $f : \underline{m} \to \underline{m'}$ and $g : \underline{n} \to \underline{n'}$, by

$$\begin{array}{rcll} (f \otimes g)^*(p) & = & f^*(p) & \text{if } p \leq m \\ & = & g^*(p-m)+n & \text{if } p > m \text{ and } g^*(p-m) \in \underline{n} \\ & = & g^*(p-m) & \text{otherwise.} \end{array}$$

It is easily checked that $f \otimes g$ is indeed a morphism in $\mathbf{\Gamma}$, and that $\mathrm{id}_{\underline{n}} \otimes \mathrm{id}_{\underline{m}} = \mathrm{id}_{\underline{n} \otimes \underline{n}}$.

Define two morphisms $\pi_p^{\mathfrak{l}} : \underline{p+q} \to \underline{p}$ and $\pi_q^{\mathfrak{r}} : \underline{p+q} \to \underline{q}$ in Γ by

$$\begin{array}{rcl} \pi_p^{\mathfrak{l}} & = & \varepsilon_{p+1} \circ \ldots \circ \varepsilon_{p+q} \\ \pi_q^{\mathfrak{r}} & = & \underbrace{\varepsilon_1 \circ \ldots \circ \varepsilon_1}_{p}, \end{array}$$

where the ε_i denote morphisms in Γ. These will be used later.

4.2 Induced monoidal biclosed structure on cubical sets

Because of [11], the above monoidal structure on $\mathbf{\Gamma}$ induces a biclosed monoidal structure on the functor category $\mathbf{Sets}^{\mathbf{\Gamma}^{\mathrm{op}}} = \mathbf{Cub}$.

For cubical sets X and Y, their tensor product is

$$X \otimes Y = \int^{\underline{m}, \underline{n}} (X(\underline{m}) \times Y(\underline{n})) \cdot \mathcal{I}^{\underline{m} \otimes \underline{n}}.$$

The unit for the tensor product is $\Pi_{\Gamma}(I) = \mathcal{I}^0$. The internal homs can be described by

$$\mathrm{Hom}^{\mathfrak{r}}(X, Y) = \int_{\underline{n}} \mathbf{Sets}(X(\underline{n}), Y(\underline{n} \otimes -))$$

and

$$\mathrm{Hom}^{\mathfrak{l}}(X, Y) = \int_{\underline{n}} \mathbf{Sets}(X(\underline{n}), Y(- \otimes \underline{n})).$$

Writing out the coend for the tensor product in elementary terms, this gives the same description as in [10]: if K and L are cubical sets, then $(K \otimes L)_n = (\coprod_{p+q=n} K_p \times L_q)/\!\sim$ where $\sim$ is the equivalence relation generated by $(\varepsilon_{r+1}(x), y) \sim (x, \varepsilon_1(y))$ for $x \in K_r$, $y \in L_{n-r-1}$. The equivalence class of (x, y) will be denoted by $x \otimes y$. Define face and degeneracy maps by

$$\begin{array}{rcll} \partial_i^\alpha(x \otimes y) & = & \partial_i^\alpha(x) \otimes y & \text{if } 1 \leq i \leq p \\ & = & x \otimes \partial_{i-p}^\alpha(y) & \text{if } p < i \leq n \\ \varepsilon_i(x \otimes y) & = & \varepsilon_i(x) \otimes y & \text{if } 1 \leq i \leq p+1 \\ & = & x \otimes \varepsilon_{i-p}(y) & \text{if } p+1 \leq i \leq n. \end{array}$$

In particular, $\varepsilon_{p+1}(x) \otimes y = x \otimes \varepsilon_1(y)$ for all $x \in K_p$. $K \otimes L$ is a cubical set.

The description of the internal hom in [10] can be obtained by writing out the end for the *left* internal hom, which fits nicely with the use of the left path complex there.

4.3 Existence of a monoidal biclosed structure on ω-categories

The category of ω-categories is monadic over cubical sets, and the corresponding monad M is the endofunctor induced by the adjunction $\Pi_{\mathbf{\Gamma}} \dashv N_{\mathbf{\Gamma}}$, so

$$M(X)(\underline{r}) = \omega\text{-}\mathbf{Cat}(\mathcal{Q}(\underline{r}), \int^{n} X_n \cdot \mathcal{Q}(\underline{n})),$$

with multiplication induced by the counit of the adjunction. The point is that this monad is *monoidal*, i.e., there are cubical maps $\widetilde{M} : M(X) \otimes M(Y) \to M(X \otimes Y)$ and $M^0 : M(\mathcal{I}^0) \to \mathcal{I}^0$ with respect to which the multiplication and the unit of M are monoidal natural transformations [12]. To describe these maps, however, the structure of ω-categories is essential, and as such it is necessary to make use of pasting presentations, analogous to the use of presentations of cubical ω-groupoids in [10], *and* analogous to the case of modules referred to there.

To give a cubical map $M(X) \otimes M(Y) \to M(X \otimes Y)$ amounts to give an ω-functor $\Pi_{\mathbf{\Gamma}}(M(X) \otimes M(Y)) \to \Pi_{\mathbf{\Gamma}}(X \otimes Y)$. This in turn corresponds to a respectable family of realizations $(\underline{G}_{M(X)\otimes M(Y)}, \varphi_i)$ in $\Pi_{\mathbf{\Gamma}}(X \otimes Y)$ of the pasting presentation $(\underline{G}_{M(X)\otimes M(Y)}, \underline{R}_{M(X)\otimes M(Y)})$ described in section 3. To define such a family of realizations, consider a generator $c = (\Lambda_n, f_i) \otimes (\Lambda_m, f'_j)$ of $(\underline{G}_{M(X)\otimes M(Y)}, \underline{R}_{M(X)\otimes M(Y)})$, so $f_i(z) \in \Pi_{\mathbf{\Gamma}}(X)$ for every $z \in \Lambda_n$, say represented by $(A_{f,z}, \mathfrak{L}_{f,z})$, and similarly for $f'_j(z') \in \Pi_{\mathbf{\Gamma}}(Y)$. To describe $\varphi_i(c)$, take for every $a_{f,z} \in A_{f,z}$ and $a_{f',z'} \in A_{f',z'}$ labeled by $x_{f,z} \in X_p$ and $y_{f',z'} \in Y_q$ a generator $(\Lambda_{p+q}, \mathfrak{L}_{x_{f,z}\otimes y_{f',z'}})$ in $\Pi_{\mathbf{\Gamma}}(X \otimes Y)$. For *fixed* $a_{f,z}$, the cells $x_{f,z} \otimes y_{f',z'}$ can be composed using any way of composing A_{f',Λ_m} to determine the order and the directions of composition. The resulting composites can then be composed using any way of composing A_{f,Λ_n}. The resulting composite is $\varphi_i(c)$. Details, such as what to do with identities, the exact way of composing, that this is independent of the chosen order of composition, and that this family of realizations is respectable, are taken for granted.

Proposition 4.1 *The monoidal biclosed structure on cubical sets induces a monoidal biclosed structure on ω-***Cat***.*

Proof. Well, to prove this I "only" need to go through [12, §4], and check all the requirements of the propositions there. But all of them are immediate from either the monoidal biclosed structure on **Cub** or from completeness of **Cub** and cocompleteness of ω-**Cat**. Note that associativity of the tensor product and its coherence also follow. □

4.4 Uniqueness

It is even possible to speak of *the* monoidal biclosed structure on ω-**Cat** induced by the tensor product of cubes:

Proposition 4.2 *The functor $\otimes : \omega\text{-}\mathbf{Cat} \times \omega\text{-}\mathbf{Cat} \to \omega\text{-}\mathbf{Cat}$ is the unique one, up to isomorphism, for which $\mathbb{C} \otimes -$ and $- \otimes \mathbb{C}$ have right adjoints for every ω-category $\mathbb{C}$ and which satisfies $\mathcal{I}^p \otimes \mathcal{I}^q \cong \mathcal{I}^{p+q}$ for every p, q.*

Proof. That it satisfies these properties is because it is part if a monoidal biclosed structure induced by the tensor product on $\mathbf{\Gamma}$, and that it is the unique such is immediate from the cubes being a generating set of objects for ω-**Cat**, see also [2]. □

5 Globe tensor globe

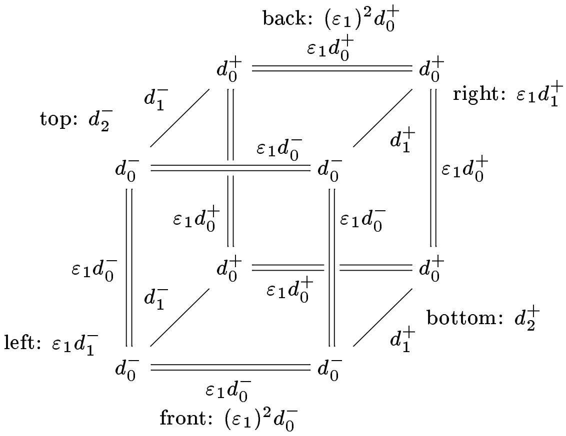
Because the tensor product of ω-categories is induced by the tensor product of cubical sets, and because globes represent elements of ω-categories [35], it is clear that essential information is contained in the ω-categorization of the tensor product of two globes as cubical sets. I show that this ω-categorization is the ω-category of components of a pasting scheme T. This pasting scheme T, or rather, its cells, will be used for the generators of a pasting presentation for the tensor product of ω-categories in section 7.

5.1 n-globes as cubical sets

Consider the cubical set G which has as non degenerate elements in dimension n d_n^+ and d_n^-, with face maps defined by

$$\partial_i^\alpha(d_n^\beta) = (\varepsilon_1)^{i-1}(d_{n-i}^\alpha).$$

So the only non degenerate $(n-1)$-dimensional faces are $\partial_1^\alpha(d_n^\beta) = d_{n-1}^\alpha$. For example, the faces of d_3^α look like



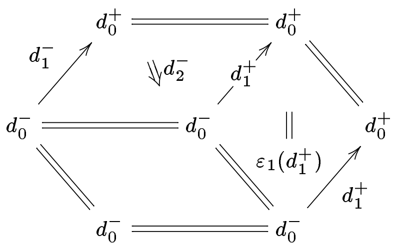
,

the interior of the cube being labeled by d_3^α.

Recall that $\Pi_{\mathbf{\Gamma}}(G)$ has a pasting presentation $(\underline{G}_G, \underline{R}_G)$ which has as generators labeled pasting schemes $(\Lambda_n, \mathfrak{L}_x)$ for $x \in G_n$, which is a degeneracy of some $d_{n'}^\alpha$, and that its relations are that $\mathfrak{L}_{\varepsilon_i(x)}$ is related to $(\Lambda_{n+1}, \mathfrak{L}_x^{\varepsilon_i})$ for every $x \in G_n$.

Lemma 5.1 *In $(\underline{G}_G, \underline{R}_G)$, the generated pastings* $\mathrm{dom}(\Lambda_{m+1}, \mathfrak{L}_{d_{m+1}^\alpha})$ *and* $(\Lambda_m, \mathfrak{L}_{d_m^-})$ *are equivalent. Also,* $\mathrm{dom}(\Lambda_{m+1}, \mathfrak{L}_{\varepsilon_{i_1}\ldots\varepsilon_{i_{m+1-m'}} d_{m'}^\alpha})$ *is equivalent to* $(\Lambda_m, \mathfrak{L}_{\partial_1^- \varepsilon_{i_1}\ldots\varepsilon_{i_{m+1-m'}} d_{m'}^\alpha})$.

Proof. For example,

$\mathrm{dom}(\Lambda_3, \mathfrak{L}_{d_3^-}) =$ 

and

$$(\Lambda_2, \mathfrak{L}_{d_2^-}) = \begin{tikzcd}[column sep=small]
 & d_0^+ \arrow[dr, equal] & \\
d_0^- \arrow[ur, "d_1^-"] \arrow[dr, equal] & \Downarrow d_2^- & d_0^+ \\
 & d_0^- \arrow[ur, "d_1^+"'] &
\end{tikzcd}$$

need to be compared. In the general cases, the difference between the two is a bunch of degeneracies, which can be added by first adding identities and then replacing these by these degeneracies, compare the proof of lemma 3.3! $\square$

I now claim that the ω-categorization of G is free. To this end, define a pasting scheme 2_ω which is the obvious extension of 2_n, i.e., it consists of two cells in every dimension, ending and beginning in the two different cells of one dimension less. For example, $\mathsf{R}_{2_\omega}(d_3^\alpha)$ looks like

$$\begin{tikzcd}
d_0^- \arrow[rr, bend left=50, "d_1^-"] \arrow[rr, bend right=50, "d_1^+"'] \arrow[rr, phantom, "d_2^- \quad d_3^\alpha \quad d_2^+"] & & d_0^+
\end{tikzcd} .$$

Proposition 5.2 $\Pi_{\mathbf{\Gamma}}(G) \cong \mathcal{P}(2_\omega)$.

Proof. I will show that both ω-categories satisfy the same universal property. Thus, that respectable families of realizations of $(\underline{G}_G, \underline{R}_G)$ in $\mathbb{C}$ correspond to appropriate realizations of 2_ω in $\mathbb{C}$.

Let $(\underline{G}_G, \varphi_j)$ be a respectable family of realizations in $\mathbb{C}$. Define a realization $(2_\omega, f_j)$ in $\mathbb{C}$ by

$$f_j(d_j^\alpha) = \varphi_j(\Lambda_j, \mathfrak{L}_{d_j^\alpha}).$$

Suppose it is m-appropriate. Then

$$\begin{array}{rcll}
s_m(f_{m+1}(d_{m+1}^\alpha)) & = & s_m(\varphi_{m+1}(\Lambda_{m+1}, \mathfrak{L}_{d_{m+1}^\alpha})) & \\
 & = & \varphi\left(\overline{\mathrm{dom}(\Lambda_{m+1}, \mathfrak{L}_{d_{m+1}^\alpha})}\right) & \text{by respectability of } (\underline{G}_G, \varphi_j) \\
 & = & \varphi\left(\overline{\Lambda_m, \mathfrak{L}_{d_m^-}}\right) & \text{by lemma 5.1} \\
 & = & \varphi_m(\Lambda_m, \mathfrak{L}_{d_m^-}) & \text{because } \varphi \text{ extends } (\underline{G}_G, \varphi_j) \\
 & = & f_m(d_m^-) & \\
 & = & f(\mathsf{R}(d_m^-)) & \text{because } f\ m\text{-extends } (2_\omega, f_j) \\
 & = & f\left(s_m(\mathsf{R}(d_{m+1}^\alpha))\right) ,&
\end{array}$$

which proves that $(2_\omega, f_j)$ is $(m+1)$-appropriate.

Let $(2_\omega, f_j)$ be an appropriate realization in $\mathbb{C}$. Define a family of realizations $(\underline{G}_G, \varphi_j)$ in $\mathbb{C}$ by

$$\varphi_j(\Lambda_j, \mathfrak{L}_{\varepsilon_{i_1}\ldots\varepsilon_{i_{j-j'}} d^\alpha_{j'}}) = f_{j'}(d^\alpha_{j'}),$$

where $\mathbb{C}$ is considered one-sorted. This family respects relations since if $j' < j$ then $\varphi_j(\Lambda_j, \mathfrak{L}_{\varepsilon_{i_1}\ldots\varepsilon_{i_{j-j'}} d^\alpha_{j'}})$ is indeed the correct identity. Now suppose it respects m-labels. Then if $m' < m+1$

$$\begin{array}{lll} s_m(\varphi_{m+1}(\Lambda_{m+1}, \mathfrak{L}_{\ldots})) = & & \\ \quad = & s_m(f_{m'}(d^\alpha_{m'})) & \\ \quad = & f_{m'}(d^\alpha_{m'}) & \\ \quad = & \varphi_m(\Lambda_m, \mathfrak{L}_{\partial_1^-\ldots}) & \\ \quad = & \varphi\left(\overline{\Lambda_m, \mathfrak{L}_{\partial_1^-\ldots}}\right) & \text{because } \varphi \ m\text{-extends } (\underline{G}_G, \varphi_j) \\ \quad = & \varphi\left(\overline{\mathrm{dom}(\Lambda_{m+1}, \mathfrak{L}_{\ldots})}\right) & \text{by lemma 5.1,} \end{array}$$

and if $m' = m+1$ then

$$\begin{array}{lll} s_m(\varphi_{m+1}(\Lambda_{m+1}, \mathfrak{L}_{\ldots})) = & & \\ \quad = & s_m(f_{m'}(d^\alpha_{m'})) & \\ \quad = & f(\mathsf{R}(d^-_m)) & \text{by } m\text{-appropriateness of } (2_\omega, f_j) \\ \quad = & f_m(d^-_m) & \text{because } f \ m\text{-extends } (2_\omega, f_j) \\ \quad = & \varphi_m(\Lambda_m, \mathfrak{L}_{d^-_m}) & \\ \quad = & \varphi\left(\overline{\Lambda_m, \mathfrak{L}_{d^-_m}}\right) & \text{because } \varphi \ m\text{-extends } (\underline{G}_G, \varphi_j) \\ \quad = & \varphi\left(\overline{\mathrm{dom}(\Lambda_{m+1}, \mathfrak{L}_{\ldots})}\right) & \text{by lemma 5.1.} \end{array}$$

Thus $(\underline{G}_G, \varphi_j)$ respects $(m+1)$-labels.

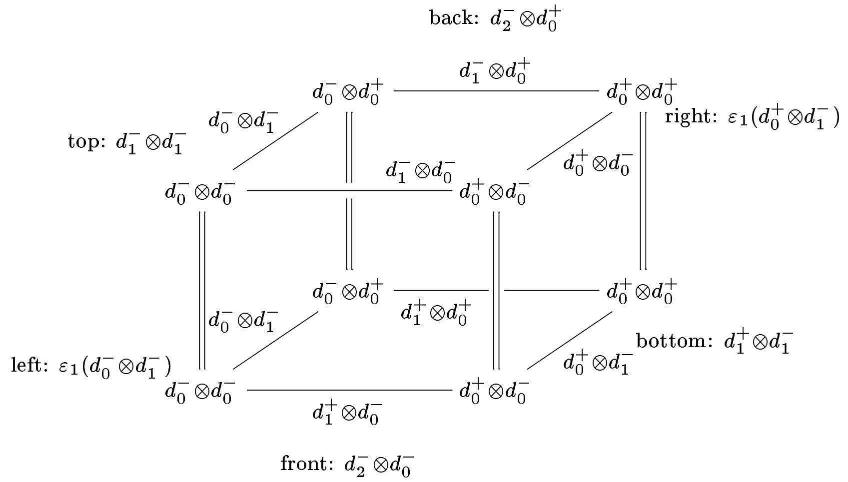
It is immediate that the above gives a bijection between respectable families of realizations of $(\underline{G}_G, \underline{R}_G)$ in $\mathbb{C}$ and appropriate realizations of 2_ω in $\mathbb{C}$. □

It would also be possible to take other faces degenerate in the definition of G, but the above definition is chosen because it it gives rise to formulae similar to ones familiar from homological algebra later on.

A notational convention for later use: in the pasting scheme 2_n the top-dimensional cell d_n can also be denoted by d^-_n and by d^+_n. This convention will avoid unneccessary splitting up in cases where one formula is clearer.

5.2 $2_\omega \otimes 2_\omega$

Recall that $\Pi_{\mathbf{\Gamma}}(G \otimes G)$ has a pasting presentation $(\underline{G}_{G\otimes G}, \underline{R}_{G\otimes G})$. It has as generators labeled pasting schemes $(\Lambda_{p+q}, \mathfrak{L}_{\varepsilon_{i_1}\ldots\varepsilon_{i_{p-p'}} d^\alpha_{p'} \otimes \varepsilon_{i'_1}\ldots\varepsilon_{i'_{q-q'}} d^\alpha_{q'}})$, where the ε_i denote maps in G. To describe the labeling, a face x of Λ_{p+q} can be considered as a morphism $\underline{r} \to \underline{p+q}$ in Γ, and $\mathfrak{L}_{\varepsilon_{i_1}\ldots\varepsilon_{i_{p-p'}} d^\alpha_{p'} \otimes \varepsilon_{i'_1}\ldots\varepsilon_{i'_{q-q'}} d^\alpha_{q'}}$ labels x by $(\pi^{\mathfrak{l}}_p \circ x)^*(\varepsilon_{i_1} \ldots \varepsilon_{i_{p-p'}} d^\alpha_{p'}) \otimes (\pi^{\mathfrak{r}} \circ x)^*(\varepsilon_{i'_1} \ldots \varepsilon_{i'_{q-q'}} d^\alpha_{q'})$. Thus, for example, $(\Lambda_3, \mathfrak{L}_{d_2^- \otimes d_1^-})$ looks like



The relations make that degeneracies are equivalent to identities.

I claim that $\Pi_{\mathbf{\Gamma}}(G \otimes G)$ is free. To this end, define a graded set T where $T_n = \{d^\alpha_p \otimes d^\beta_q \mid \alpha, \beta = \pm, p + q = n\}$. Define relations E and B on T by $(d^\alpha_p \otimes d^\beta_q, y) \in \mathsf{E}^i_j$ for $i > j$ if and only if one of the following:

1. $y = d^\alpha_p \otimes d^{(-)^p}_{q-1}$,

2. $y = d^+_{p-1} \otimes d^\beta_q$,

3. $y = d^+_{p-1} \otimes d^{(-)^p}_{q-1}$.

So E^i_j is empty for $j < i - 2$. B^i_j differs from E^i_j in having $-$ instead of $+$ and $(-)^{p+1}$ instead of $(-)^p$. These relations can be viewed as a modified or generalized version of the Leibniz rule.

In figure 2, a low-dimensional part of T.

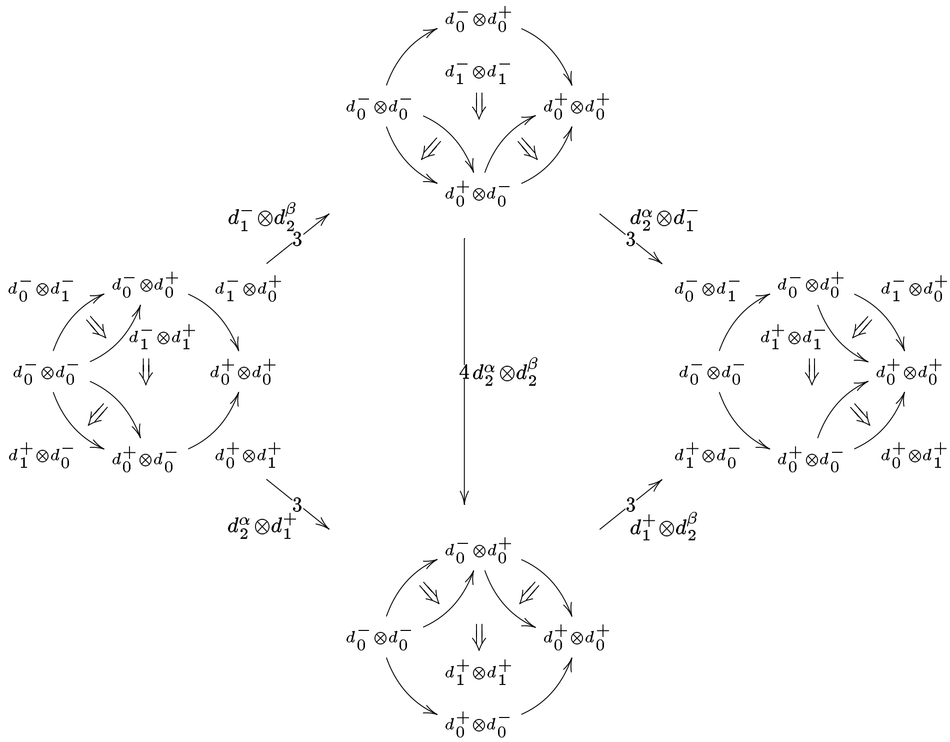


Figure 2: The faces of $d_2^\alpha \otimes d_2^\beta$

Proposition 5.3 *T is a pasting scheme.*

Proof. Pasting axiom (i) is trivial, (ii) can be forced to hold, and (iii) is immediate.

The only non trivial case of pasting axiom (iv) is when $j = i-2$. But the only possibility for either side to hold, with $w = d_p^\alpha \otimes d_q^\beta$, is when $x = d_{p-1}^+ \otimes d_{q-1}^{(-)^p}$, $u = d_p^\alpha \otimes d_{q-1}^{(-)^p}$ and $v = d_{p-1}^+ \otimes d_q^\beta$.

For pasting axiom (v) the case $j = i-1$ is trivial. If $j = i-2$ there are four possibilities, with $w = d_p^\alpha \otimes d_q^\beta$:

- $x = d_{p-1}^+ \otimes d_{q-1}^{(-)^p}$: x is already at the end of w,
- $x = d_p^\alpha \otimes d_{q-2}^{(-)^p}$: take $v = d_p^\alpha \otimes d_{q-1}^{(-)^{p+1}}$,

- $x = d^+_{p-1} \otimes d^{(-)^{p-1}}_{q-1}$: idem,
- $x = d^+_{p-2} \otimes d^\beta_q$: take $v = d^-_{p-1} \otimes d^\beta_q$.

If $j = i - 3$ there are two possibilities:

- $x = d^+_{p-2} \otimes d^{(-)^{p+1}}_{q-1}$: take $v = d^-_{p-1} \otimes d^\beta_q$,
- $x = d^+_{p-1} \otimes d^{(-)^p}_{q-2}$: take $v = d^\alpha_p \otimes d^{(-)^{p+1}}_{q-1}$.

For $j < i - 3$ the relation E^{i-1}_j is empty so the condition is void. $\square$

I will now analyse the situation $a \triangleleft b$ in T for $a = d^\alpha_p \otimes d^\beta_q$, $p + q = i$. Table 1 gives the possibilities for a_1 and a_2.

$a = a_0$	$\mathsf{E}_{i-1}(a_0) \cap \mathsf{B}_{i-1}(a_1)$	a_1	$\mathsf{E}_{i-1}(a_1) \cap \mathsf{B}_{i-1}(a_2)$	a_2
			$d^+_{p-2} \otimes d^{\beta_1}_{q+1}$	$d^-_{p-2} \otimes d^{\beta_2}_{q+2}$ if $\beta_1 = (-)^{p-1}$
	$d^+_{p-1} \otimes d^\beta_q$	$d^+_{p-1} \otimes d^{\beta_1}_{q+1}$ if $\beta = (-)^p$		
			$d^+_{p-1} \otimes d^{(-)^{p-1}}_q$	$-$
$d^\alpha_p \otimes d^\beta_q$				
			$d^+_p \otimes d^{(-)^p}_{q-1}$	$-$
	$d^\alpha_p \otimes d^{(-)^p}_{q-1}$	$d^{\alpha_1}_{p+1} \otimes d^{(-)^p}_{q-1}$ if $\alpha = -$		
			$d^{\alpha_1}_{p+1} \otimes d^{(-)^{p+1}}_{q-2}$	$d^{\alpha_2}_{p+2} \otimes d^{(-)^{p+1}}_{q-2}$ if $\alpha_1 = -$,

and $\mathsf{B}(a) = \{d^\alpha_p \otimes d^\beta_q,\ d^-_{p-1} \otimes d^\beta_q,\ d^\alpha_p \otimes d^{(-)^{p+1}}_{q-1},\ d^-_{p-1} \otimes d^{(-)^{p+1}}_{q-1}\}$.

Table 1: $a \triangleleft a_1 \triangleleft a_2 \ldots$ in T

Lemma 5.4 *The pasting scheme T has no direct loops.*

Proof. Since elements in $\mathsf{E}_{i-2}(a_1)$ always have an index $p-2$ or $q-2$ it follows from the table above that $\mathsf{E}(a_1) \cap \mathsf{B}(a) = \varnothing$. Continuing the table it follows that a_i is always of the form $d^{\alpha_i}_{p\pm i} \otimes d^{\beta_i}_{q\mp i}$ from which follows that for all $i \geq 2$ also

$\mathsf{E}(a_i) \cap \mathsf{B}(a) = \varnothing$. So T has no direct loops since obviously $\mathsf{B}(a) \cap \mathsf{E}(a) = \{a\}$. $\square$

Before well-formedness of $\mathsf{R}(d^\alpha_p \otimes d^\beta_q)$, its m-sources and m-targets need to be considered. I will show that they satisfy a generalized form of the Leibniz rule.

Lemma 5.5 *For $m \le n = p + q$,*

$$\begin{aligned} s_m(\mathsf{R}(d^\alpha_p \otimes d^\beta_q)) = \mathsf{R}(\{d^{\alpha'}_{p'} \otimes d^{\beta'}_{q'} | & p' + q' = m,\ 0 \le p' \le p,\ 0 \le q' \le q, \\ & \textit{if } p' \ne p \textit{ then } \alpha' = -, \\ & \textit{if } p' = p \textit{ then } \alpha' = \alpha, \\ & \textit{if } q' \ne q \textit{ then } \beta' = (-)^{p'+1}, \\ & \textit{if } q' = q \textit{ then } \beta' = \beta\}), \end{aligned}$$

and dually.

Proof. Downward induction on m. If $m = n$ then $s_m(\mathsf{R}(d^\alpha_p \otimes d^\beta_q)) = \mathsf{R}(d^\alpha_p \otimes d^\beta_q)$ which agrees with the formula above. Now suppose this formula is proven for $m+1$, then I have to show that $s_m(\mathsf{R}(d^\alpha_p \otimes d^\beta_q)) = \operatorname{dom} s_{m+1}(\mathsf{R}(d^\alpha_p \otimes d^\beta_q)) = \operatorname{dom} \mathsf{R}(\{d^{\alpha_2}_{p_2} \otimes d^{\beta_2}_{q_2} | p_2 + q_2 = m+1,\ \ldots\}) = \mathsf{R}(\{d^{\alpha_2}_{p_2} \otimes d^{\beta_2}_{q_2} | p_2 + q_2 = m+1,\ \ldots\}) - \mathsf{E}(\mathsf{R}(\{d^{\alpha_2}_{p_2} \otimes d^{\beta_2}_{q_2} | p_2 + q_2 = m+1,\ \ldots\}))$ is equal to $\mathsf{R}(\{d^{\alpha'}_{p'} \otimes d^{\beta'}_{q'} | p' + q' = m,\ \ldots\})$.

That the latter is contained in the former falls apart in two: that $\mathsf{R}(d^{\alpha'}_{p'} \otimes d^{\beta'}_{q'})$ is contained in $\mathsf{R}(\{d^{\alpha_2}_{p_2} \otimes d^{\beta_2}_{q_2} | p_2 + q_2 = m+1,\ \ldots\})$, and that if $d^{\alpha_3}_{p_3} \otimes d^{\beta_3}_{q_3}$ is in $\mathsf{E}(\mathsf{R}(\{d^{\alpha_2}_{p_2} \otimes d^{\beta_2}_{q_2} | p_2 + q_2 = m+1,\ \ldots\}))$ then it is not in $\mathsf{R}(\{d^{\alpha'}_{p'} \otimes d^{\beta'}_{q'} | p' + q' = m,\ \ldots\})$. For the first part it suffices that the $d^{\alpha'}_{p'} \otimes d^{\beta'}_{q'}$ are in the former. Distinguish three cases, namely

- $p' = p$ and $q' < q$: then $\beta' = (-)^{p'+1}$. If $q' = q - 1$ then take $d^{\alpha'}_{p'} \otimes d^\beta_q$, if $q' < q - 1$ then take $d^{\alpha'}_{p'} \otimes d^{\beta'}_{q'+1}$ for $d^{\alpha_2}_{p_2} \otimes d^{\beta_2}_{q_2}$, which in of the correct form, and which satisfies $d^{\alpha_2}_{p_2} \otimes d^{\beta_2}_{q_2} \mathsf{B}^{m+1}_m d^{\alpha'}_{p'} \otimes d^{\beta'}_{q'}$.

- $q' = q$ and $p' < p$: then $\alpha' = -$. If $p' = p - 1$ then take $d^\alpha_p \otimes d^\beta_q$, if $p' < p - 1$ then take $d^{\alpha'}_{p'+1} \otimes d^{\beta'}_{q'}$ for $d^{\alpha_2}_{p_2} \otimes d^{\beta_2}_{q_2}$.

- $p' < p$ and $q' < q$: then $\alpha' = -$ and $\beta' = (-)^{p'+1}$. Do the same as in the first case. Note that it is not possible to do the same as in the second case because then β' would not come out right.

For the second part, there are three possibilities for $d_{p_3}^{\alpha_3} \otimes d_{q_3}^{\beta_3}$, namely $d_{p_2}^{\alpha_2} \otimes d_{q_2-1}^{(-)^{p_2}}$, $d_{p_2-1}^{+} \otimes d_{q_2}^{\beta_2}$, and $d_{p_2-1}^{+} \otimes d_{q_2-1}^{(-)^{p_2}}$. The first one of these three is indeed not one of the $d_{p'}^{\alpha'} \otimes d_{q'}^{\beta'}$ since $\beta' \neq (-)^{p'+1}$, the second one not because $\alpha' \neq -$, and for the third one the requirements on either α' or β' are contradictory as well.

To show that the former is contained in the latter, I have to show that for every $d_{p_3}^{\alpha_3} \otimes d_{q_3}^{\beta_3} \in \mathsf{R}(\{d_{p_2}^{\alpha_2} \otimes d_{q_2}^{\beta_2} | p_2 + q_2 = m+1, \ \ldots\})$, it is either in $\mathsf{R}(\{d_{p'}^{\alpha'} \otimes d_{q'}^{\beta'} | p' + q' = m, \ \ldots\})$, or in $\mathsf{E}(\mathsf{R}(\{d_{p_2}^{\alpha_2} \otimes d_{q_2}^{\beta_2} | p_2 + q_2 = m+1, \ \ldots\}))$. Three cases:

- if $q_3 \leq q_2 - 2$ then one of the reasons for $d_{p_3}^{\alpha_3} \otimes d_{q_3}^{\beta_3}$ to be in $\mathsf{R}(d_{p_2}^{\alpha_2} \otimes d_{q_2}^{\beta_2})$ must be via $d_{p_2}^{\alpha_2} \otimes d_{q_2-1}^{(-)^{p_2+1}}$, which can be taken for $d_{p'}^{\alpha'} \otimes d_{q'}^{\beta'}$ since it is of the correct form. This also works if $q_3 = q_2 - 1$ and $\beta_3 = (-)^{p_2+1}$.
- if $p_3 \leq p_2 - 2$ then via $d_{p_2-1}^{-} \otimes d_{q_2}^{\beta_2}$, which also works if $p_3 = p_2 - 1$ and $\alpha_3 = -$.
- all other cases, namely $d_{p_2}^{\alpha_2} \otimes d_{q_2}^{\beta_2}$, $d_{p_2}^{\alpha_2} \otimes d_{q_2-1}^{(-)^{p_2}}$, $d_{p_2-1}^{+} \otimes d_{q_2}^{\beta_2}$ or $d_{p_2-1}^{-} \otimes d_{q_2-1}^{(-)^{p_2}}$, are in $\mathsf{E}(d_{p_2}^{\alpha_2} \otimes d_{q_2}^{\beta_2})$. □

Lemma 5.6 *For every $d_p^\alpha \otimes d_q^\beta \in T$, $\mathsf{R}(d_p^\alpha \otimes d_q^\beta)$ is well formed.*

Proof. Since $s_m(\mathsf{R}(d_p^\alpha \otimes d_q^\beta))$ and $t_m(\mathsf{R}(d_p^\alpha \otimes d_q^\beta))$ are both R of something, they are subpasting schemes of $\mathsf{R}(d_p^\alpha \otimes d_q^\beta)$. They are also compatible: for $\mathsf{B}_{m-1}(d_{p'}^{\alpha'} \otimes d_{q'}^{\beta'})$ and $\mathsf{B}_{m-1}(d_{p'_1}^{\alpha'_1} \otimes d_{q'_1}^{\beta'_1})$ to have something in common one needs p' and p'_1, and q' and q'_1 at most one apart from each other. This leaves only the consecutive pairs to check, and then the conditions on α', α'_1, β' and β'_1 give that $\mathsf{B}_{m-1}(d_{p'}^{\alpha'} \otimes d_{q'}^{\beta'}) \cap \mathsf{B}_{m-1}(d_{p'_1}^{\alpha'_1} \otimes d_{q'_1}^{\beta'_1}) = \varnothing$. Noting that $s_0(\mathsf{R}(d_p^\alpha \otimes d_q^\beta))$ is always a singleton, namely $\{d_0^- \otimes d_0^-\}$, finishes the proof. □

Proposition 5.7 *The pasting scheme T is loop free.*

Proof. Conditions (i) and (ii) of loop-freeness are lemmas 5.4 and 5.6 respectively.

For condition (iv), consider again table 1, and suppose $a = u \in s_j(\mathsf{R}(x))$, $b = u' \in s_j(\mathsf{R}(x))$, for some $x \in T$. I will show that then also $a_1 \in s_j(\mathsf{R}(x))$, and then induction will do the rest. So $x = d_{p_2}^{\alpha_2} \otimes d_{q_2}^{\beta_2}$, say, and $j = p+q$. Then $\alpha = -$ or $\beta = (-)^p$ or both. In all these cases, for the sequence $a, a_1, \ldots$ to continue til at least a_2 one needs $\beta_1 = (-)^{p-1}$ or $\alpha_1 = -$. In case $\beta_1 = (-)^{p-1}$, a_i will never be in $s_j(\mathsf{R}(x))$ which contradicts $b \in s_j(\mathsf{R}(x))$, and in case $\alpha_1 = -$ indeed $a_1 \in s_j(\mathsf{R}(x))$. Thus T is loop free. $\square$

Thus T is a loop-free pasting scheme. Furthermore, there are also pasting schemes $2_p \otimes 2_q$ defined in the obvious way, which are well formed and loop free either by direct calculation or by viewing them as $\mathsf{R}(x)$ for some $x \in T$.

Now back to the pasting presentation $(\underline{G}_{G\otimes G}, \underline{R}_{G\otimes G})$. Define a labeled pasting scheme $(2_p \otimes 2_q, \mathfrak{L}_{d_p \otimes d_q})$, where $d_{p'}^{\alpha'} \otimes d_{q'}^{\beta'}$ gets labeled by $(\Lambda_{p'+q'}, \mathfrak{L}_{d_{p'}^{\alpha'} \otimes d_{q'}^{\beta'}})$.

Lemma 5.8 *In* $(\underline{G}_{G\otimes G}, \underline{R}_{G\otimes G})$, $\mathrm{dom}(2_p \otimes 2_q, \mathfrak{L}_{d_p\otimes d_q})$ *is a generated pasting which is equivalent to* $\mathrm{dom}(\Lambda_{p+q}, \mathfrak{L}_{d_p \otimes d_q})$. *Also, if* $p' < p$ *or* $q' < q$ *then* $\mathrm{dom}(\Lambda_{p+q}, \mathfrak{L}_{\varepsilon_{i_1}\ldots\varepsilon_{i_{p-p'}} d_{p'}^{\alpha} \otimes \varepsilon_{i'_1}\ldots\varepsilon_{i'_{q-q'}} d_{q'}^{\beta}})$ *is equivalent to* $(\Lambda_{p+q-1}, \mathfrak{L}_{\partial_j(\varepsilon_{i_1}\ldots\varepsilon_{i_{p-p'}} d_{p'}^{\alpha} \otimes \varepsilon_{i'_1}\ldots\varepsilon_{i'_{q-q'}} d_{q'}^{\beta})})$ *for some* j.

Proof. The proof will be by induction on $p+q$. So to show that $\mathrm{dom}(2_p \otimes 2_q, \mathfrak{L}_{d_p \otimes d_q})$ is a generated pasting, take a cell labeled by $(\Lambda_{p'+q'}, \mathfrak{L}_{d_{p'}^{\alpha'} \otimes d_{q'}^{\beta'}})$. By the induction hypothesis, its domain is indeed equivalent to the domain of its label. To show that $(\mathrm{dom}(2_p \otimes 2_q), \mathfrak{L}_{d_p \otimes d_q})$ is equivalent to $\mathrm{dom}(\Lambda_{p+q}, \mathfrak{L}_{d_p \otimes d_q})$, observe that their difference is some identities, which can be inserted and replaced by degeneracies as before.

$$\mathrm{dom}(\Lambda_3, \mathfrak{L}_{d_2^- \otimes d_1^-}) = \begin{tikzcd} & d_0^- \otimes d_0^+ \arrow[rr, "d_1^- \otimes d_0^+"] \arrow[dr, phantom, "\Downarrow d_1^- \otimes d_1^-"] & & d_0^+ \otimes d_0^+ \arrow[dr, equal] & \\ d_0^- \otimes d_0^- \arrow[ur, "d_0^- \otimes d_1^-"] \arrow[rr, "d_1^- \otimes d_0^-"] \arrow[dr, equal] & & d_0^+ \otimes d_0^- \arrow[ur] \arrow[dr, equal] \arrow[dl, phantom, "\Downarrow d_2^- \otimes d_0^-"] & & d_0^+ \otimes d_0^+ \\ & d_0^- \otimes d_0^- \arrow[rr, "d_1^+ \otimes d_0^-"'] & & d_0^+ \otimes d_0^- \arrow[ur, "d_0^+ \otimes d_1^-"'] & \end{tikzcd}$$

and

$$\mathrm{dom}(d_2^- \otimes d_1^-) = \begin{tikzcd} & d_0^+ \otimes d_0^+ \arrow[dr, "d_1^- \otimes d_0^+"] \arrow[dd, phantom, "\Downarrow d_1^- \otimes d_1^-"] & \\ d_0^- \otimes d_0^- \arrow[ur, "d_0^- \otimes d_1^-"] \arrow[dr, bend right, "d_1^+ \otimes d_0^-"'] \arrow[dr, bend left] \arrow[dr, phantom, "\Swarrow"] & & d_0^+ \otimes d_0^+ \\ & d_0^+ \otimes d_0^- \arrow[ur, "d_0^+ \otimes d_1^-"'] & \end{tikzcd} .$$

For the last statement, the difference is a number of degeneracies, which can be dealt with as in lemma 3.3. □

Proposition 5.9 $\Pi_{\mathbf{\Gamma}}(G \otimes G) \cong \mathcal{P}(T)$.

Proof. This proof will follow the lines of the proof of proposition 5.2 closely, I will only do the second part in some detail.

So given an appropriate realization $(2_\omega \otimes 2_\omega, f_i)$ in $\mathbb{C}$, define a family of realizations $(\underline{G}_{G\otimes G}, \varphi_i)$ in $\mathbb{C}$ by

$$\varphi_{p+q}(\Lambda_{p+q}, \mathfrak{L}_{\varepsilon_{i_1}\ldots\varepsilon_{i_{p-p'}} d^\alpha_{p'} \otimes \varepsilon_{i'_1}\ldots\varepsilon_{i'_{q-q'}} d^\beta_{q'}}) = f_{p'+q'}(d^\alpha_{p'} \otimes d^\beta_{q'}).$$

To show it respects $(m+1)$-labels if it respects m-labels, if $p'+q' < m+1$ then

$$\begin{aligned}
&s_m(\varphi_{m+1}(\Lambda_{m+1}, \mathfrak{L}_{\ldots\otimes\ldots})) = \\
&\quad = \ s_m(f_{p'+q'}(d^\alpha_{p'} \otimes d^\beta_{q'})) \\
&\quad = \ f_{p'+q'}(d^\alpha_{p'} \otimes d^\beta_{q'}) \\
&\quad = \ \varphi_m(\Lambda_m, \mathfrak{L}_{\partial^-_j(\ldots\otimes\ldots)}) && \text{for some } j \\
&\quad = \ \varphi\left(\overline{\Lambda_m, \mathfrak{L}_{\partial^-_j(\ldots\otimes\ldots)}}\right) && \text{because } \varphi \ m\text{-extends } (\underline{G}_{G\otimes G}, \varphi_i) \\
&\quad = \ \varphi\left(\overline{\mathrm{dom}(\Lambda_{m+1}, \mathfrak{L}_{\ldots\otimes\ldots})}\right) && \text{by lemma 5.8,}
\end{aligned}$$

and if $m' = m+1$ then

$$\begin{aligned}
&s_m(\varphi_{m+1}(\Lambda_{m+1}, \mathfrak{L}_{\ldots\otimes\ldots})) = \\
&\quad = \ s_m(f_{m'}(d^\alpha_{p'} \otimes d^\beta_{q'})) \\
&\quad = \ f(\mathrm{dom}(d^\alpha_{p'} \otimes d^\beta_{q'})) && \text{by } m\text{-appropriateness of } (2_\omega \otimes 2_\omega, f_j) \\
&\quad = \ \psi^{\mathrm{dom}(d^\alpha_{p'} \otimes d^\beta_{q'})}(\mathrm{dom}(d^\alpha_{p'} \otimes d^\beta_{q'})) && \text{because both are the composite of the } \textit{same} \text{ appropriate realization} \\
&\quad = \ \varphi\left(\overline{\mathrm{dom}(d^\alpha_{p'} \otimes d^\beta_{q'}, \mathfrak{L}_{d^\alpha_{p'} \otimes d^\beta_{q'}})}\right) && \text{by the formula for } \varphi \text{ in section 10 of [5]} \\
&\quad = \ \varphi\left(\overline{\mathrm{dom}(\Lambda_{m+1}, \mathfrak{L}_{d^\alpha_{p'} \otimes d^\beta_{q'}})}\right) && \text{by lemma 5.8.}
\end{aligned}$$

□

Because $\Pi_{\mathbf{\Gamma}}(G) \cong \mathcal{P}(2_\omega)$ is the "generic" ω-category, this proposition suggests that $\mathcal{P}(T)$ is the generic tensor product of ω-categories, a viewpoint that will prove to be fruitful.

6 Pasting scheme tensor globe

Given a p-dimensional pasting scheme A, I describe a well-formed loop-free pasting scheme $A \otimes 2_q$, which could be termed its *right q-th path pasting scheme.* This pasting scheme will be used in the relations of a pasting presentation for the tensor product of ω-categories in section 7.

For a well-formed loop-free pasting scheme A, the i-cells of the graded set $A \otimes 2_q$ are expressions $a \otimes d^{\beta}_{q'}$, where $a \in A_{p'}$, $\beta = \pm$, $p' + q' = i$ and $q' \leq q$. If $q' = q$ then both d^{+}_{q} and d^{-}_{q} are considered synonymous to $d_q \in 2_q$, as before. The relations E^i_j and B^i_j on $A \otimes 2_q$ are such that $(a \otimes d^{\beta}_{q'}, y) \in \mathsf{E}^i_j$ if and only if one of the following:

1. $y = a_2 \otimes d^{\beta}_{q'}, \quad a\mathsf{E}^{p'}_{p_2}a_2,$

2. $y = a_2 \otimes d^{(-)^{p'}}_{q'-1}, \quad a\mathsf{E}^{p'}_{p_2}a_2.$

B^i_j is defined dually, i.e., it has $\mathsf{B}^{p'}_{p_2}$ instead of $\mathsf{E}^{p'}_{p_2}$ and $(-)^{p'+1}$ instead of $(-)^{p'}$.

Proposition 6.1 *If A is a well-formed loop-free pasting scheme, then $A \otimes 2_q$ is a pasting scheme.*

Proof. The proof of this will be analogous to the proof of proposition 5.3, only somewhat more involved. The first three pasting axioms are easy.

For the "$\Rightarrow$" part of pasting axiom (iv), with $w = a \otimes d^{\beta}_{q'}$, there are two possibilities for x. If $x = a_2 \otimes d^{\beta}_{q'}$ where $a\mathsf{E}^{p'}_{p_2}a_2$ then by pasting axiom (iv) in A applied to $a\mathsf{E}^{p'}_{p_2}a_2$ there are b and b' of dimension $p'-1$ which make that $b \otimes d^{\beta}_{q'}$ and $b' \otimes d^{\beta}_{q'}$ can be taken for u and v. If $x = a_2 \otimes d^{(-)^{p'}}_{q'-1}$ where $a\mathsf{E}^{p'}_{p_2}a_2$ then by pasting axiom (iv) in A applied to $a\mathsf{E}^{p'}_{p_2}a_2$ there is a b', and then $a \otimes d^{(-)^{p'}}_{q'-1}$ and $b' \otimes d^{\beta}_{q'}$ can be taken for u and v respectively. Notice that although this situation looks asymmetric it is not, since the dual situation utilizes $b \otimes d^{\beta}_{q'}$ and $a \otimes d^{(-)^{p'+1}}_{q'-1}$.

For the "$\Leftarrow$" part of pasting axiom (iv), with $w = a \otimes d^{\beta}_{q'}$, distinguish the following possibilities for u and v:

- $u = a_2 \otimes d^{\beta}_{q'}$ where $a\mathsf{E}^{p'}_{p'-1}a_2$ and $v = a_3 \otimes d^{\beta}_{q'}$ where $a\mathsf{E}^{p'}_{p'-1}a_3$. Then x must be $a_4 \otimes d^{\beta}_{q'}$ with $a_2\mathsf{E}^{p'-1}_{p_4}a_4$ and $a_3\mathsf{B}^{p'-1}_{p_4}a_4$, and application of pasting axiom (iv) in A gives that $w\mathsf{E}^{i}_{j}x$,

- $u = a_2 \otimes d^{\beta}_{q'}$ where $a\mathsf{E}^{p'}_{p'-1}a_2$ and $v = a \otimes d^{(-)^{p'}}_{q'-1}$. Then there is no x since if the dimensions in 2_q agree then the exponents are different. Compare this with proof of proposition 5.3, where this possibility was also excluded.

- $u = a \otimes d^{(-)^{p'}}_{q'-1}$ and $v = a_3 \otimes d^{\beta}_{q'}$ where $a\mathsf{E}^{p'}_{p'-1}a_3$. Then x must be $a_4 \otimes d^{(-)^{p'}}_{q'-1}$ with $a\mathsf{E}^{p'}_{p_4}a_4$ (and $a_3\mathsf{B}^{p'-1}_{p_4}a_4$), and so $w\mathsf{E}^{i}_{j}x$.

For pasting axiom (v), with $w = a \otimes d^{\beta}_{q'}$, there are four possibilities:

- $x = a_2 \otimes d^{(-)^{p'}}_{q'-1}$ with $a\mathsf{E}^{p'}_{p_2}a_2$: x is already at the end of w,

- $x = a_2 \otimes d^{(-)^{p'}}_{q'-2}$ with $a\mathsf{E}^{p'}_{p_2}a_2$: take $v = a \otimes d^{(-)^{p'+1}}_{q'-1}$,

- $x = a_3 \otimes d^{(-)^{p'}}_{q'-1}$ with $a'\mathsf{E}^{p'-1}_{p_3}a_3$: if $a\mathsf{E}^{p'}_{p_3}a_3$ then take $v = a \otimes d^{(-)^{p'+1}}_{q'-1}$, otherwise there exists, by pasting axiom (v) in A, an a_4 such that $a\mathsf{B}^{p'}_{p'-1}a_4\mathsf{E}^{p'-1}_{p_3}a_3$, then take $v = a_4 \otimes d^{(-)^{p'+1}}_{q'-1}$,

- $x = a_3 \otimes d^{\beta}_{q'}$ with $a'\mathsf{E}^{p'-1}_{p_3}a_3$: if $a\mathsf{E}^{p'}_{p_3}a_3$ then x is already at the end of w, otherwise there exists an a_4 such that $a\mathsf{B}^{p'}_{p'-1}a_4\mathsf{E}^{p'-1}_{p_3}a_3$, then take $v = a_4 \otimes d^{\beta}_{q'}$. □

I will now analyse the situation $b \lhd b'$ in $A \otimes 2_q$ for $b = a \otimes d^{\beta}_{q'}$, $\dim(a) + q' = i$. Table 2 gives the possibilities for a_1 and a_2.

Lemma 6.2 *If A is a well-formed loop-free pasting scheme, then the pasting scheme $A \otimes 2_q$ has no direct loops.*

Proof. Consider table 2. The key to this proof are the dimensions and the exponents in 2_q. For if in b_i this dimension is different from q' then in later b_i's it must be even further away from q'. Also compare the proof of lemma 5.4! So the only relevant dimensions are $q' + 1$, q' and $q' - 1$.

$b = b_0$	$\mathsf{E}_{i-1}(b_0) \cap \mathsf{B}_{i-1}(b_1)$	b_1	$\mathsf{E}_{i-1}(b_1) \cap \mathsf{B}_{i-1}(b_2)$	b_2
				$a_7 \otimes d^{\beta}_{q'}$ with $a_7 \mathsf{B}^{p'}_{p'-1} a_4$
			$a_4 \otimes d^{\beta}_{q'}$ with $a_2 \mathsf{E}^{p'}_{p'-1} a_4$	
				$a_4 \otimes d^{\beta_1}_{q'+1}$ if $\beta = (-)^{p'}$
		$a_2 \otimes d^{\beta}_{q'}$ with $a_2 \mathsf{B}^{p'}_{p'-1} a'$		
			$a_2 \otimes d^{(-)^{p'}}_{q'-1}$	$a_8 \otimes d^{(-)^{p'}}_{q'-1}$ with $a_8 \mathsf{B}^{p'+1}_{p'} a_2$
	$a' \otimes d^{\beta}_{q'}$ with $a \mathsf{E}^{p'}_{p'-1} a'$			
				$a_9 \otimes d^{\beta_1}_{q'+1}$ with $a_9 \mathsf{B}^{p'-1}_{p'-2} a_5$
			$a_5 \otimes d^{\beta_1}_{q'+1}$ with $a' \mathsf{E}^{p'-1}_{p'-2} a_5$	
				$a_5 \otimes d^{\beta_2}_{q'+2}$ if $\beta_1 = (-)^{p'-1}$
		$a' \otimes d^{\beta_1}_{q'+1}$ if $\beta = (-)^{p'}$		
			$a' \otimes d^{(-)^{p-1}}_{q'}$	$a_{10} \otimes d^{(-)^{p'-1}}_{q'}$ with $a_{10} \mathsf{B}^{p'}_{p'-1} a'$
$a \otimes d^{\beta}_{q'}$				
			$a_6 \otimes d^{(-)^{p'}}_{q'-1}$ with $a_3 \mathsf{E}^{p'+1}_{p'} a_6$	$a_{11} \otimes d^{(-)^{p'}}_{q'-1}$ with $a_{11} \mathsf{B}^{p'+1}_{p'} a_6$
	$a \otimes d^{(-)^{p'}}_{q'-1}$	$a_3 \otimes d^{(-)^{p'}}_{q'-1}$ with $a_3 \mathsf{B}^{p'+1}_{p'} a$		
			$a_3 \otimes d^{(-)^{p'+1}}_{q'-2}$	$a_{12} \otimes d^{(-)^{p'+1}}_{q'-2}$ with $a_{12} \mathsf{B}^{p'+2}_{p'+1} a_3$,

and $\mathsf{B}(b) = \{\tilde{a} \otimes d^{\beta}_{q'} | a \mathsf{B}^{p'}_{\tilde{p}} \tilde{a}\} \cup \{\tilde{a} \otimes d^{(-)^{p'+1}}_{q'-1} | a \mathsf{B}^{p'}_{\tilde{p}} \tilde{a}\}$.

Table 2: $b \lhd b_1 \lhd b_2 \ldots$ in $A \otimes 2_q$

- b_i of the form $a_4 \otimes d_{q'+1}^{\beta_1}$: then an element in $\mathsf{B}(b) \cap \mathsf{E}(b_i)$ needs $d_{q'}^{(-)^{p'-1}}$ which is impossible since at the same time $\beta = (-)^{p'}$,

- b_i of the form $a_8 \otimes d_{q'-1}^{(-)^{p'}}$: again, the exponent needs to be $(-)^{p'}$ and $(-)^{p'+1}$ at the same time,

- b_i of the form $a_7 \otimes d_{q'}^{\beta}$: then $a \triangleleft_A a_7$ and $\mathsf{B}(b) \cap \mathsf{E}(b_i) \neq \varnothing$ gives a direct loop in A,

- b_i of the form $a_{10} \otimes d_{q'}^{(-)^{p'-1}}$: either the exponent is wrong, or it reduces to a direct loop in A as well.

So $A \otimes 2_q$ has no direct loops since $\mathsf{B}(b) \cap \mathsf{E}(b) = \{b\}$ because in A $\mathsf{B}(a) \cap \mathsf{E}(a) = \{a\}$. □

I will show that the m-sources and m-targets of $A \otimes 2_q$ also satisfy a generalized form of the Leibnitz rule.

Lemma 6.3 *For A a p-dimensional well-formed loop-free pasting scheme and $m \leq n = p + q$,*

$$s_m(A \otimes 2_q) = \mathsf{R}(\{a \otimes d_{q'}^{\beta'} | a \in s_{p'}(A),\ p' + q' = m,\ 0 \leq q' \leq q, \\ \textit{if } q' \neq q \textit{ then } \beta' = (-)^{p'+1}\})$$

and dually.

Proof. Along the lines of the proof of lemma 5.5. If $m = n$ then the formula above gives $\mathsf{R}(\{a \otimes d_q^{\beta'} | a \in A\})$, which is indeed equal to $A \otimes 2_q$.

For the first part of "$\supseteq$", distinguish two cases:

- $q' < q$: then $\beta' = (-)^{p'+1}$, and take $a \otimes d_q$ or $a \otimes d_{q'+1}^{\beta'}$ for $a_2 \otimes d_{q_2}^{\beta_2}$,

- $q' = q$ and hence $p' < p$: then $a \in s_{p'}(A)$. If $a \in s_{p'+1}(A)$ then take $a \otimes d_q$, otherwise a has an incoming cell a' of dimension $p' + 1$ *which can be chosen in* $s_{p'+1}(A)$ by lemma 4.2 of [5], and take $a' \otimes d_q$ for $a_2 \otimes d_{q_2}^{\beta_2}$.

For the second part of "$\supseteq$" there are two possibilities, namely $a_3 \otimes d_{q_2}^{\beta_2}$ or $a_3 \otimes d_{q_2-1}^{(-)^{p_2}}$, with $a_2 \mathsf{E}_{p_3}^{p_2} a_3$, where in the latter case a_3 can be equal to a_2. If in the first case there is an $a' \otimes d_{q'}^{\beta'}$ then $a_3 \in \mathsf{R}(a')$ and $q_2 \leq q'$ which implies $a_3 \in s_{p'}(A)$ by well-formedness of A and $p_2 > p'$ by the conditions $p' + q' = m$

and $p_2 + q_2 = m + 1$. But a_3 has an incoming p_2-cell a_2, contradiction. And if in the second case there is an $a' \otimes d_{q'}^{\beta'}$ then $a_3 \in \mathsf{R}(a')$ and $q_2 - 1 \leq q'$ so $p_2 \geq p'$. $p_2 > p'$ leads to contradiction as in the first case, and $p_2 = p'$ implies $q' = q_2 - 1 \neq q$ so β' needs to be equal to $(-)^{p'+1}$, which is not the case.

For "$\subseteq$", there are two cases:

- if $q_3 \leq q_2 - 2$ or $q_3 = q_2 - 1$ and $\beta_3 = (-)^{p_2+1}$ then $a_3 \otimes d_{q3}^{\beta_3} \in \mathsf{R}(a_2 \otimes d_{q_2-1}^{(-)^{p_2+1}})$,
- if $q_3 = q_2 - 1$ and $\beta_3 = (-)^{p_2}$ or $q_3 = q_2$ then $a_3 \in \mathsf{R}(a_2)$ which, by well-formedness of A implies $a_3 \in s_{p_2}(A)$. If a_3 has an incoming p_2-dimensional cell $a_2' \in s_{p_2}(A)$ then $a_3 \otimes d_{q3}^{\beta_3} \in \mathsf{E}(a_2' \otimes d_{q2}^{\beta_2})$, otherwise $a_3 \in s_{p_2-1}(A)$ and $a_3 \otimes d_{q3}^{\beta_3} \in \mathsf{R}(a_3 \otimes d_{q2}^{\beta_2})$. □

Proposition 6.4 *For a well-formed loop-free pasting scheme A, the pasting scheme $A \otimes 2_q$ is well formed.*

Proof. Since $s_m(A \otimes 2_q)$ and $t_m(A \otimes 2_q))$ are both R of something, they are subpasting schemes of $A \otimes 2_q)$.

Compatibility: if $\mathsf{B}_{m-1}(a \otimes d_{q'}^{\beta'}) \cap \mathsf{B}_{m-1}(a' \otimes d_{q'}^{\beta'}) \neq \varnothing$ then $\mathsf{B}_{m-1}(a) \cap \mathsf{B}_{m-1}(a') \neq \varnothing$ contradicting compatibility of $s_{p'}(A)$, and if $\mathsf{B}_{m-1}(a \otimes d_{q'}^{\beta'}) \cap \mathsf{B}_{m-1}(a' \otimes d_{q'+1}^{\beta_2}) \neq \varnothing$ then β' needs to be $(-)^{p'+1}$ and $(-)^{p'}$ at the same time. And $s_0(A \otimes 2_q)$ is the singleton $\{a \otimes d_0^- | a \in s_0(A)\}$. □

Lemma 6.5 $\mathsf{R}_{a \otimes 2_q}(a \otimes d_{q'}^{\beta'}) \cong \mathsf{R}_A(a) \otimes 2_{q'}$.

Proof. Immediate. □

It follows that

$$s_m(\mathsf{R}(a \otimes d_{q'}^{\beta'})) = \mathsf{R}(\{a_2 \otimes d_{q2}^{\beta_2} | a_2 \in s_{p_2}(\mathsf{R}(a)),\ p_2 + q_2 = m,\ 0 \leq q' \leq q, \\ \text{if } q_2 \neq q' \text{ then } \beta_2 = (-)^{p_2+1}, \\ \text{if } q_2 = q' \text{ then } \beta_2 = \beta'\}).$$

Lemma 6.6 *For all $a \otimes d_{q'}^{\beta'} \in A \otimes 2_q$, the subpasting scheme $\mathsf{R}(a \otimes d_{q'}^{\beta'})$ is well formed.*

Proof. Combine the above two lemmas with proposition 6.4. $\square$

Before loop-freeness, I need to relate well-formed subpasting schemes of $A \otimes 2_q$ to well-formed subpasting schemes of A. Thus suppose Y' is a j-dimensional well-formed subpasting scheme of $A \otimes 2_q$ containing $a \otimes d^{\beta}_{q'}$. Define a subgraded set Y of A by $Y = \{a' \in A | a' \otimes d^{\beta'}_{q'} \in Y'$ for some $\beta'\}$. It is p'-dimensional because Y' is j-dimensional and $a \in Y$. It is a subpasting scheme of A because Y' is of $A \otimes 2_q$. Before showing it is well formed, I will calculate

$$\begin{aligned}
\mathrm{dom}(Y) &= \{a' \in A | a' \otimes d^{\beta'}_{q'} \in Y' \text{ for some } \beta', \\
&\qquad a' \text{ having no incoming } p'\text{-cell in } Y\} \\
&= \{a' \in A | a' \otimes d^{\beta'}_{q'} \in Y' \text{ for some } \beta', \\
&\qquad \text{there is no } a_2 \otimes d^{\beta_2}_{q'} \in Y' \text{ with } a_2\ p'\text{-} \\
&\qquad \text{dimensional and incoming in } a\} \\
&= \{a' \in A | a' \otimes d^{\beta'}_{q'} \in \mathrm{dom}(Y')\},
\end{aligned}$$

where the last equality is because Y' is a subpasting scheme of $A \otimes 2_q$. So it suffices to show compatibility of Y for every Y'. So suppose $\mathsf{B}_{p'-1}(a') \cap \mathsf{B}_{p'-1}(a_2) \neq \varnothing$ in Y, a' being in Y because $a' \otimes d^{\beta'}_{q'} \in Y'$ and a_2 because $a_2 \otimes d^{\beta_2}_{q'} \in Y'$. But these elements contradict *strong* compatibility of Y' [23, Proposition 10].

Proposition 6.7 *For a well-formed loop-free pasting scheme A, the pasting scheme $A \otimes 2_q$ is loop free.*

Proof. Conditions (i) and (ii) of loop-freeness are lemmas 6.2 and 6.6 respectively.

For condition (iv), consider again table 2, and suppose $b = u \in s_j(\mathsf{R}(x))$, $b' = u' \in s_j(\mathsf{R}(x))$, for some $x = a_2 \otimes d^{\beta_2}_{q2} \in A \otimes 2_q$. I will show that $b_1 \in s_j(\mathsf{R}(x))$. There are three possibilities in the sequence $b \lhd b'$ for q':

- q' going up: the condition on β forces $q' = q_2$. To get below q_2 again this goes via an $a_{10} \otimes d^{p'-1}_{q'}$ which is not in $s_j(\mathsf{R}(x))$, nor is anything further on since the exponent is the wrong one all the time,
- q' going down: look at $a_3 \otimes d^{(-)^{p'}}_{q'-1}$, if it is not in $s_j(\mathsf{R}(x))$ then $a_3 \notin s_{p'+1}(A)$. But then there exists $a'_3 \in s_{p'+1}(A)$ with $a'_3 \mathsf{B}^{p'+1}_{p'} a$ by 4.2 of [5], so $a'_3 \otimes d^{(-)^{p'}}_{q'-1}$

$\in s_j(\mathsf{R}(x)) \subseteq Y$, so if also $a_3 \otimes d_{q'-1}^{(-)^{p'}} \in Y$ then Y is not compatible. Thus $a_3 \in s_{p'+1}(A)$ and $a_3 \otimes d_{q'-1}^{(-)^{p'}} \in s_j(\mathsf{R}(x))$,

- q' doesn't change: take $x' = a_2 \in A$ and the well-formed subpasting scheme of A corresponding to Y, as constructed just before this proposition. Then condition (iv) in A applied to $a \lhd_A a_7 \lhd_A \cdots$ in this situation gives that $a_2 \in s_{p'}(\mathsf{R}(x'))$. And now β needs to be equal to $(-)^{p'+1}$, otherwise Y would't be compatible. So $a_2 \otimes d_{q'}^{\beta} \in s_j(\mathsf{R}(x))$. □

I will also need:

Lemma 6.8 *If A is a round pasting scheme then $A \otimes 2_q$ is round as well.*

Proof. Lemma 6.3 gives that $s_{n-1}(A \otimes 2_q) = \mathsf{R}(\{a \otimes d_{q'}^{\beta'} | a \in s_{p'}(A),\ p'+q' = n-1,\ 0 \leq q' \leq q,\ \text{if } q' \neq q \text{ then } \beta' = (-)^{p'+1}\})$ and $t_{n-1}(A \otimes 2_q) = \mathsf{R}(\{a \otimes d_{q'}^{\beta'} | a \in t_{p'}(A),\ p'+q' = n-1,\ 0 \leq q' \leq q,\ \text{if } q' \neq q \text{ then } \beta' = (-)^{p'}\})$. Suppose $a_2 \otimes d_{q_2}^{\beta_2}$ in their intersection, say in $\mathsf{R}(a_3 \otimes d_{q_3}^{\beta_3})$ and in $\mathsf{R}(a_4 \otimes d_{q_4}^{\beta_4})$. Four cases:

- $a_3 \in s_p(A)$, $q_3 = q-1$, $a_4 \in t_p(A)$, $q_4 = q-1$: either $a_3 \otimes d_{q-2}^{\beta_2}$ or $a_4 \otimes d_{q-2}^{\beta_2}$ must be an intermediate stage, so $a_2 \otimes d_{q_2}^{\beta_2} \in s_{p-2}(A \otimes 2_q) \cup t_{p-2}(A \otimes 2_q)$,
- $a_3 \in s_{p-1}(A)$, $q_3 = q$, $a_4 \in t_{p-1}(A)$, $q_4 = q$: then $a_2 \in s_{p-1}(A) \cap t_{p-1}(A) = s_{p-2}(A) \cup t_{p-2}(A)$. If in $s_{p-2}(A)$ then $a_2 \otimes d_{q_2}^{\beta_2} \in s_{p-2}(A \otimes 2_q)$ and dually,
- $a_3 \in s_{p-1}(A)$, $q_3 = q$, $a_4 \in t_p(A)$, $q_4 = q-1$ ($a_3 \in s_p(A)$, $q_3 = q-1$, $a_4 \in t_{p-1}(A)$, $q_4 = q$ analogous): if via $a_3 \otimes d_{q-1}^{(-)^p}$ then in $s_{p-2}(A \otimes 2_q)$. But it is always possible to do this because if via $a_3 \otimes d_{q-1}^{(-)^{p-1}}$ then β_4 is not right so $q_2 < q-1$. □

Definition 6.9 A well-formed loop-free pasting scheme is *globular* if all m-sources and m-targets are round. ◇

Lemma 6.10 *If A is globular then $A \otimes 2_q$ is globular.*

Proof. Analogous to the proof of the previous lemma. □

7 A pasting presentation for the tensor product of ω-categories

This section is the central part of this paper. In it, I give a detailed description of the tensor product of two ω-categories $\mathbb{C}$ and $\mathbb{D}$ by giving a pasting presentation $(\underline{G}_{\mathbb{C},\mathbb{D}}, \underline{R}_{\mathbb{C},\mathbb{D}})$ for it. The usefulness of this description is that the universal property of pasting presentations makes it relatively easy to deal with ω-functors going *from* a tensor product. This will be used to prove associativity and coherence of the tensor product, in section 8, and to prove the adjunctions between the tensor product and the internal homs, in section 11. It will also give concrete formulae for categories enriched in this monoidal category ω-**Cat**, an example of which is ω-**Cat** itself, see section 12. Another point is that working with pasting schemes is more conceptual than the approach of [2, 34].

Gray's tensor product of 2-categories [21] is defined using essentially the same approach as here: it is defined by generators and relations, and a description of the generated cells is given. Because of the restriction to dimension 2, the tensor product of 2-categories is defined as a 2-category. It can be obtained from the 4-category it is here by taking connected components in dimension 2, i.e., it has the same 0- and 1-cells, and 2-cells are equivalence classes of 2-cells in the 4-category, the equivalence relation being generated by the requirement that two 2-cells are equivalent if there is a 3-cell in between them. This explains *all* extra conditions on the 2-cells of [21]'s tensor product.

7.1 Generators

A generator in $\underline{G}_{\mathbb{C},\mathbb{D}}$ in dimension n is a labeled pasting scheme $(2_p \otimes 2_q, \mathfrak{L}_{c\otimes d})$ such that $p + q = n$, for some p-dimensional $c \in \mathbb{C}$ and some q-dimensional $d \in \mathbb{D}$, where $d^\alpha_{p'} \otimes d^\beta_{q'}$ is labeled by $(2_{p'} \otimes 2_{q'}, \mathfrak{L}_{c'\otimes d'})$ for $c' = d^\alpha_{p'}(c)$ in $\mathbb{C}$ and $d' = d^\beta_{q'}(d)$ in $\mathbb{D}$. Cells x in the domain or codomain of $2_p \otimes 2_q$ all have $\mathsf{R}(x)$ *equal* to a generator of lower dimension, so these labeled pasting schemes can indeed be taken as generators. Sometimes the generator $(2_p \otimes 2_q, \mathfrak{L}_{c\otimes d})$ will be called $c \otimes d$ for short.

7.2 Relations

To define the relations in $\underline{R}_{\mathbb{C},\mathbb{D}}$ I will make use of labeled pasting schemes $(A \otimes 2_q, \mathfrak{L}_{(A,f_i)\otimes d})$, for some appropriate realization (A, f_i) of A in $\mathbb{C}$ and some $d \in \mathbb{D}$, where $a \otimes d^\beta_{q'}$ is labeled by the generator $f_{p'}(a) \otimes d^\beta_{q'}(d)$. Of course, labeled pasting schemes $(2_p \otimes B, \mathfrak{L}_{c\otimes(B,g_i)})$ will also be used, but since the use of these is completely analogous, I will concentrate on the first ones. One might think that also something like $(A \otimes B, \mathfrak{L}_{(A,f_i)\otimes(B,g_i)})$ could be used, *but this is not the case*, because $A \otimes B$, defined in the same way as $A \otimes 2_q$, can fail to be a well-formed loop-free pasting scheme. If $A \otimes B$ is equal to the product of pasting schemes Johnson and Street have in mind, then this failure has been observed by them as well [22]. Here it can be seen from a table like table 2, since the dimension of the second coordinate can go up and down, making direct loops, or to sequences $y \triangleleft y'$ violating condition (iv) of loop-freeness possible. But it isn't *necessary* to consider $A \otimes B$, as the sequel shows.

Now back to the relations. It is not possible to prove directly that $(A, f_i) \otimes d$ is a generated pasting, because for this relations in lower dimensions will be needed. So, as in section 11 of [5], the approach will be inductive, in fact, this whole section is completely along the lines of the proof of section 11 of [5], only worked out a little bit, but only a little bit, more. Some intermediate results will be derived, which illustrate, in fact, are derived from, the intuition behind the tensor product.

For round pasting scheme A with appropriate realization (A, f_i) in $\mathbb{C}$ and $d \in \mathbb{D}$, define a labeled pasting scheme $((A \otimes 2_q)_t, \mathfrak{L}_{((A,f_i)\otimes d)_t})$, which is labeled as $(A, f_i) \otimes d$ except for the top-dimensional cell, which is labeled by $f(A) \otimes d$, where $f(A)$ denotes the composite of (A, f_i).

For $(p-1)$-dimensional $c \in \mathbb{C}$ and q-dimensional $d \in \mathbb{D}$, define a labeled pasting scheme $(2_p \otimes 2_q, \mathfrak{L}_{\mathrm{id}^{\mathfrak{l}}_{c\otimes d}})$, which is labeled as $\mathrm{id}_c \otimes d$ except for the top-dimensional cell, which is labeled by the *formal* expression $\mathrm{id}_{\overline{c\otimes d}}$.

Assume:

- for every appropriate realization (A, f_i) of a p-dimensional well-formed loop-free pasting scheme A in $\mathbb{C}$ and every q-dimensional $d \in \mathbb{D}$ with $p + q \leq n$, the labeled pasting scheme $(A, f_i) \otimes d$ is a generated pasting,

- for every appropriate realization (A, f_i) of a p-dimensional round pasting scheme A in $\mathbb{C}$ and every q-dimensional $d \in \mathbb{D}$ with $p + q \leq n$, the labeled pasting scheme $((A, f_i) \otimes d)_t$ is a generated pasting,

- for every appropriate realization (A, f_i) of a p-dimensional round pasting scheme A in $\mathbb{C}$ and every q-dimensional $d \in \mathbb{D}$ with $p+q \leq n$, the generated pasting $(A, f_i) \otimes d$ is fully replacable in $(A, f_i) \otimes d$,

- for every $(p-1)$-dimensional $c \in \mathbb{C}$ and q-dimensional $d \in \mathbb{D}$ with $p+q \leq n$, the labeled pasting scheme $\mathrm{id}^{\mathfrak{l}}_{c\otimes d}$ is a generated pasting,

- for every $(p-1)$-dimensional $c \in \mathbb{C}$ and q-dimensional $d \in \mathbb{D}$ with $p+q \leq n$, the generated pasting $\mathrm{id}_c \otimes d$ is fully replacable in $\mathrm{id}_c \otimes d$,
- for every appropriate realization (A, f_i) of a p-dimensional round pasting scheme A in $\mathbb{C}$ and every q-dimensional $d \in \mathbb{D}$ with $p+q \leq n$, there is defined a relation between $(A, f_i) \otimes d$ and $((A, f_i) \otimes d)_t$,
- for every $(p-1)$-dimensional $c \in \mathbb{C}$ and q-dimensional $d \in \mathbb{D}$ with $p+q \leq n$, there is defined a relation between $\mathrm{id}_c \otimes d$ and $\mathrm{id}^{\mathfrak{l}}_{c\otimes d}$,

and the same for $(2_p \otimes B, \mathfrak{L}_{c\otimes(B,g_i)})$, etc.

I will derive some consequences of these assumptions that will be used in the next dimension.

Lemma 7.1 *For every appropriate realization (A, f_i) of a p-dimensional globular pasting scheme A in $\mathbb{C}$ and every q-dimensional $d \in \mathbb{D}$ with $p+q \leq n$, $(A, f_i) \otimes d$ is equivalent to $f(A) \otimes d$.*

Proof. The idea is to replace high dimensional pieces by their composite, so that when continuing this for lower dimensions finally $2_p \otimes 2_q$ is reached. The reason for starting with high dimensions is that this leaves not many higher dimensional cells being able to spoil full replacability. This is implemented as follows.

Define $(A \otimes 2_q)^{n.0.4} = A \otimes 2_q[(A \otimes 2_q)_t / A \otimes 2_q]$, and for $0 \leq j' \leq j < n$, define

$$
\begin{array}{rcll}
(A\otimes 2_q)^{j.j'.1} & = & (A\otimes 2_q)^{j.(j'+1).4}[(s_{j'}(A)\otimes d^{(-)^{j'}}_{j-j'})_t / A\otimes d^{(-)^{j'}}_{j-j'}] & \text{if } j'<j \\
 & = & (A\otimes 2_q)^{(j+1).0.4}[(s_{j'}(A)\otimes d^{(-)^{j'}}_{j-j'})_t / A\otimes d^{(-)^{j'}}_{j-j'}] & \text{if } j'=j \\
(A\otimes 2_q)^{j.j'.2} & = & (A\otimes 2_q)^{j.j'.1}[(s_{j'}(A)\otimes d^{(-)^{j'+1}}_{j-j'})_t / s_{j'}(A)\otimes d^{(-)^{j'+1}}_{j-j'}] & \\
(A\otimes 2_q)^{j.j'.3} & = & (A\otimes 2_q)^{j.j'.2}[(t_{j'}(A)\otimes d^{(-)^{j'}}_{j-j'})_t / t_{j'}(A)\otimes d^{(-)^{j'}}_{j-j'}] & \\
(A\otimes 2_q)^{j.j'.4} & = & (A\otimes 2_q)^{j.j'.3}[(t_{j'}(A)\otimes d^{(-)^{j'+1}}_{j-j'})_t / t_{j'}(A)\otimes d^{(-)^{j'+1}}_{j-j'}] &
\end{array}
$$

whenever this makes sense, i.e., when $0 \leq j' \leq p$ and $0 \leq j-j' \leq q$, otherwise don't replace anything, and if $j' = p$ or $j - j' = q$ then do only two of the four replacements. So with index j all j-dimensional pieces are replaced by their composites, and $(A \otimes 2_q)^{0.0.4} \cong 2_p \otimes 2_q$. The above definitions make sense because the pasting schemes that are to be replaced can indeed be considered as subpasting schemes of the $(A \otimes 2_q)^{j.j'.j''}$'s. The pasting schemes to be replaced are round because A is globular and because of lemma 6.8. For the rest of full replacability, I will now describe the pasting schemes. They consist of cells

$d_{p'}^{\alpha'} \otimes d_{q'}^{\beta'}$ in dimensions greater than j, and of the cells of $A \otimes 2_q$ in dimensions less than j, while in dimension j some pieces have been replaced already. The E and B relations make that the $d_{p'}^{\alpha'} \otimes d_{q'}^{\beta'}$'s relate as in $2_p \otimes 2_q$, the cells of $A \otimes 2_q$ relate as in $a \otimes 2_q$, and their mutual relations are such that the pieces act as low dimensional cells of $2_p \otimes 2_q$. The composites are labeled by the appropriate composites, and the cells are labeled by their old labels. Once I've shown full replacability the labeled pasting schemes above are generated pastings because this is a local property.

Now for replacability. Elementary replacability is immediate from the abobe description of the B and E relations. There are no direct loops of dimension $j+1$ in $(A \otimes 2_q)^{j \cdot j' \cdot j''}$ because if there were one not meeting $(\ldots)_t$ this would be a direct loop in the previous step as well, and if there were one meeting $(\ldots)_t$, any p-dimensional $x' \in \widetilde{X}$ instead of it would make it into a direct loop in the previous step. There are no direct loops of dimension j, which follows from a combination of tables 1 and 2. There are no direct loops in dimensions greater than $j+1$ and less than j since such a loop is also a loop in $2_p \otimes 2_q$ and $A \otimes 2_q$ respectively.

Finally, condition (iv) of loop-freeness is proven as in the proof of proposition 6.7. $\square$

Given an appropriate realization (A, f_i) in $\mathbb{C}$, I need an appropriate realization of a globular pasting scheme having the same composite. Take $(\mathrm{Gl}(A), f_i)$, where the identities are realized by the composite of the subpasting schemes they are identities on (see section 11 of [5]). Having defined this, a generated pasting $(A, f_i) \otimes d$ gives rise to a generated pasting $(\mathrm{Gl}(A), f_i) \otimes d$. Note that in this latter pasting scheme all cells are labeled by *actual* generators, not by *formal* identities!

Lemma 7.2 *For every appropriate realization (A, f_i) of a p-dimensional pasting scheme A in $\mathbb{C}$ and every q-dimensional $d \in \mathbb{D}$ with $p+q \leq n$, $(A, f_i) \otimes d$ is equivalent to $(\mathrm{Gl}(A), f_i) \otimes d$.*

Proof. The idea is to use the globularization procedure for A, as described in section 8 of [5], as basis for the insertions that have to occur. For every step of the globularization procedure for A there will be many steps here, in order to ensure the result is of the correct form. Finally, the labeling of the formal identities will be changed in *actual* labels.

Define $(A \otimes 2_q)^{(-1).0.4} = A \otimes 2_q$, and for $0 \leq m \leq p = \dim(A)$ and $q \geq q' \geq 0$, define

$$
\begin{array}{rcll}
(A\otimes 2_q)^{m.q'.1} & = & (A\otimes 2_q)^{m.(q'+1).4}[\mathrm{id}_{s_m(A)\otimes d^-_{q'}} : W^{m.q'.1}] & \text{if } q'<q \\
& = & (A\otimes 2_q)^{(m-1).0.4}[\mathrm{id}_{s_m(A)\otimes d^-_{q'}} : W^{m.q'.1}] & \text{if } q'=q \\
(A\otimes 2_q)^{m.q'.2} & = & (A\otimes 2_q)^{m.q'.1}[\mathrm{id}_{s_m(A)\otimes d^+_{q'}} : W^{m.q'.2}] & \\
(A\otimes 2_q)^{m.q'.3} & = & (A\otimes 2_q)^{m.q'.2}[\mathrm{id}_{t_m(A)\otimes d^-_{q'}} : W^{m.q'.2}] & \\
(A\otimes 2_q)^{m.q'.4} & = & (A\otimes 2_q)^{m.q'.3}[\mathrm{id}_{t_m(A)\otimes d^+_{q'}} : W^{m.q'.2}], &
\end{array}
$$

where if $q' = q$ do only two of the four replacements, and where the witnessing specifications $W^{m.q'.j}$ are such that the position of the identities is the position they are to have in $\mathrm{Gl}(A)\otimes 2_q$. After having completed the m-th stage the pasting scheme is an m-th globulatization of A, so $(A\otimes 2_q)^{p.0.4} \cong \mathrm{Gl}(A)\otimes 2_q$, *as a pasting scheme*. The pasting schemes on which identities are inserted are round because they have been made so in the previous steps. For elementary replacability, the only relevant $(m+q'+2)$-cells are previously added higher-dimensional identities, which indeed have the required property. And the intermediate results are sufficiently like $\mathrm{Gl}_m(A) \otimes 2_q$ to prove them being well-formed loop-free pasting schemes in the same way.

Now I need to replace the labels on the identities. I will do that along the way, so this amount to a modification of the above process, which has been presented nontheless for reasons of clarity. The idea is to add another bunch of identities, so that if the label on id_X is to be replaced, the subpasting scheme X is completely surrounded by identities. Then X can be replaced by some $2_{p'} \otimes 2_{q'}$, on which an identity can be inserted, which can then be relabeled to some $\mathrm{id}_c \otimes d$. Then the whole thing can be undone because this relabeling doesn't change the pasting scheme but only a label, and finally it is ensured that the identity which remains is the one which has been actually labeled. So it remains to describe when and where these extra identities are inserted. To relabel $\mathrm{id}_{s_m(A)\otimes d^-_{q'}}$, say, other cases are analogous, steps $m.q''.1$ and $m.q''.2$ are repeated for all $q'' \leq q'$. So this comes down to globularizing this piece for a second time, and then the piece in between the identities is the isolated copy of X. Going back, there is an extra identity on X, and now the two identities which have been inserted first can be removed, so that indeed id_X remains. □

Proposition 7.3 *For every appropriate realization (A, f_i) of a p-dimensional well-formed loop-free pasting scheme A in $\mathbb{C}$ and every q-dimensional $d \in \mathbb{D}$ with $p + q \leq n$, $(A, f_i) \otimes d$ is equivalent to $f(A) \otimes d$.*

Proof. Combine lemma 7.1 with lemma 7.2. $\square$

Now I can prove all the assumptions in one dimension higher.

Lemma 7.4 *Under the above assumptions, for every appropriate realization (A, f_i) of a p-dimensional well-formed loop-free pasting scheme A in $\mathbb{C}$ and every q-dimensional $d \in \mathbb{D}$ with $p + q \leq n + 1$, the labeled pasting scheme $(A, f_i) \otimes d$ is a generated pasting.*

Proof. I have to show that for every $a \otimes d^{\beta}_{q'} \in A \otimes 2_q$, $\left(\mathrm{dom}(\mathsf{R}(a \otimes d^{\beta}_{q'})), \mathfrak{L}_{(A,f_i)\otimes d}|_{\mathrm{dom}(\mathsf{R}(a\otimes d^{\beta}_{q'}))}\right)$ is equivalent to $(\mathrm{dom}(2_{p'} \otimes 2_{q'}),$ $\mathfrak{L}_{f_{p'}(a)\otimes d^{\beta}_{q'}(d)}|_{\mathrm{dom}(2_{p'}\otimes 2_{q'})})$. This can be done by modifying the constructions of the previous two lemmas in order to make it work on $\mathrm{dom}(\mathsf{R}(a) \otimes 2_{q'})$, by doing only the insertions and the replacements which take place there. This whole construction then uses only generators, generated pastings and relations up to dimension n. $\square$

Lemma 7.5 *Under the above assumptions, for every appropriate realization (A, f_i) of a p-dimensional round pasting scheme A in $\mathbb{C}$ and every q-dimensional $d \in \mathbb{D}$ with $p+q \leq n+1$, the labeled pasting scheme $((A, f_i) \otimes d)_t$ is a generated pasting.*

Proof. The only thing left to check is the labeling of the top-dimensional cell. So the question is, whether $(\mathrm{dom}(A \otimes 2_q), \mathfrak{L}_{(A,f_i)\otimes d}|_{\mathrm{dom}(A\otimes 2_q)})$ is equivalent to $(\mathrm{dom}(2_p \otimes 2_q), \mathfrak{L}_{f(A)\otimes d}|_{\mathrm{dom}(2_p\otimes 2_q)})$. For this the same modification of constructions of lemmas 7.1 and 7.2 as in the previous lemma works. Note, by the way, that roundness of A is needed to make any sense out of $(A \otimes 2_q)_t$, using lemma 6.8. $\square$

Lemma 7.6 *Under the above assumptions,for every appropriate realization (A, f_i) of a p-dimensional round pasting scheme A in $\mathbb{C}$ and every q-dimensional $d \in \mathbb{D}$ with $p + q \leq n + 1$, the generated pasting $(A, f_i) \otimes d$ is fully replacable in $(A, f_i) \otimes d$.*

Proof. By lemma 5.9 of [5] only roundness needs to be checked, but this holds because of lemma 6.8. $\square$

Lemma 7.7 *Under the above assumptions, for every $(p-1)$-dimensional $c \in \mathbb{C}$ and q-dimensional $d \in \mathbb{D}$ with $p+q \leq n$, the labeled pasting scheme $\mathrm{id}^{\mathfrak{l}}_{c \otimes d}$ is a generated pasting.*

Proof. I need prove that $\mathrm{dom}(\mathrm{id}_c \otimes d)$ is equivalent to $c \otimes d$. As before, this can be done by inserting identities and relabeling them, which in this case is easier because in relabeling there's no need to isolate because the relevant cells are already of the correct form. □

Lemma 7.8 *Under the above assumptions, for every $(p-1)$-dimensional $c \in \mathbb{C}$ and q-dimensional $d \in \mathbb{D}$ with $p+q \leq n$, the generated pasting $\mathrm{id}_c \otimes d$ is fully replacable in $\mathrm{id}_c \otimes d$.*

Proof. $2_p \otimes 2_q$ is round because 2_p is. □

Thus, relations in dimension $n+1$ can now be defined by:

- for every appropriate realization (A, f_i) of a p-dimensional round pasting scheme A in $\mathbb{C}$ and every q-dimensional $d \in \mathbb{D}$ with $p+q \leq n+1$, there is a relation between $(A, f_i) \otimes d$ and $((A, f_i) \otimes d)_t$,
- for every $(p-1)$-dimensional $c \in \mathbb{C}$ and q-dimensional $d \in \mathbb{D}$ with $p+q \leq n+1$, there is a relation between $\mathrm{id}_c \otimes d$ and $\mathrm{id}^{\mathfrak{l}}_{c \otimes d}$,

and the same for $(2_p \otimes B, \mathfrak{L}_{c \otimes (B, g_i)})$.

7.3 ω-functoriality

Given an ω-functor $g : \mathbb{C} \to \mathbb{C}'$, define a family of realizations $(\underline{G}_{\mathbb{C},\mathbb{D}}, (g \otimes \mathbb{D})_i)$ of $(\underline{G}_{\mathbb{C},\mathbb{D}}, \underline{R}_{\mathbb{C},\mathbb{D}})$ in $\mathbb{C}' \otimes \mathbb{D}$ by

$$(g \otimes \mathbb{D})_i (c \otimes d) = g(c) \otimes d$$

where the latter is a generator hence a generated pasting, by lemma 7.3 of [5], in the pasting presentation $(\underline{G}_{\mathbb{C}',\mathbb{D}}, \underline{R}_{\mathbb{C}',\mathbb{D}})$ of $\mathbb{C}' \otimes \mathbb{D}$. This family of realizations respects relations:

$$\begin{array}{rll}
(g \otimes \mathbb{D})((A, f_i) \otimes d) = & & \\
= & (A, g \circ f_i) \otimes d & \\
\text{equivalent to} & (g \circ f)(A) \otimes d & \\
= & g(f(A)) \otimes d & \text{because } g \text{ is an } \omega\text{-functor} \\
= & (g \otimes \mathbb{D})(f(A) \otimes d), &
\end{array}$$

and it respects labels:

$$\begin{array}{rcll} \mathrm{dom}((g\otimes\mathbb{D})_i(c\otimes d)) & = & \mathrm{dom}(g(c)\otimes d) & \\ & = & (g\otimes\mathbb{D})(\mathrm{dom}(c\otimes d)) & \text{because both are compos-} \\ & & & \text{ite of the same appropriate} \\ & & & \text{realization of } \mathrm{dom}(2_p\otimes 2_q). \end{array}$$

Thus this defines an ω-functor $g\otimes\mathbb{D}:\mathbb{C}\otimes\mathbb{D}\to\mathbb{C}'\otimes\mathbb{D}$.

8 Associativity and other properties of the tensor product

Various properties of the tensor product of ω-categories necessary for a monoidal structure are checked. Furthermore, there's a brief discussion on duality, the example of tensoring two standard cubes as ω-categories, and a very brief excursion into knot theory.

8.1 Associativity

For associativity of the tensor product, I need to compare $\mathbb{C}\otimes(\mathbb{D}\otimes\mathbb{E})$ with $(\mathbb{C}\otimes\mathbb{D})\otimes\mathbb{E}$. To this end, define an ω-category $\mathbb{C}\otimes\mathbb{D}\otimes\mathbb{E}$ by the following pasting presentation $(\underline{G}_{\mathbb{C},\mathbb{D},\mathbb{E}},\underline{R}_{\mathbb{C},\mathbb{D},\mathbb{E}})$: generators are labeled pasting schemes $(2_p\otimes 2_q\otimes 2_r,\mathfrak{L}_{c\otimes d\otimes e})$ with obvious labeling, and relations are dimensionwise, as in the pasting presentation for $\mathbb{C}\otimes\mathbb{D}$.

Lemma 8.1 *There is a canonical isomorphism between* $\mathbb{C}\otimes(\mathbb{D}\otimes\mathbb{E})$ *and* $\mathbb{C}\otimes\mathbb{D}\otimes\mathbb{E}$*, and also between* $(\mathbb{C}\otimes\mathbb{D})\otimes\mathbb{E}$ *and* $\mathbb{C}\otimes\mathbb{D}\otimes\mathbb{E}$*.*

Proof. Of course, this canonical isomorphism is the unique one which exists because both ω-categories satisfy the same universal property, i.e., I will show that respectable families of realizations of $(\underline{G}_{\mathbb{C},\mathbb{D}\otimes\mathbb{E}},\underline{R}_{\mathbb{C},\mathbb{D}\otimes\mathbb{E}})$ correspond to respectable families of realizations of $(\underline{G}_{\mathbb{C},\mathbb{D},\mathbb{E}},\underline{R}_{\mathbb{C},\mathbb{D},\mathbb{E}})$. The other case will be similar.

So consider a respectable family of realizations $(\underline{G}_{\mathbb{C},\mathbb{D}\otimes\mathbb{E}},\varphi_i)$ of $(\underline{G}_{\mathbb{C},\mathbb{D}\otimes\mathbb{E}},\underline{R}_{\mathbb{C},\mathbb{D}\otimes\mathbb{E}})$ in $\mathbb{F}$. Define a family of realizations $(\underline{G}_{\mathbb{C},\mathbb{D},\mathbb{E}},\overline{\varphi}_i)$ of $(\underline{G}_{\mathbb{C},\mathbb{D},\mathbb{E}},\underline{R}_{\mathbb{C},\mathbb{D},\mathbb{E}})$ in $\mathbb{F}$ by

$$\overline{\varphi}_{p+q+r}(c\otimes d\otimes e)=\varphi_{p+q+r}(c\otimes(d\otimes e)).$$

To show this family of realizations respects labels, define for every $c \in \mathbb{C}$, $d \in \mathbb{D}$ and $e \in \mathbb{E}$ a generated pasting $(2_p \otimes 2_q \otimes 2_r, \mathfrak{L}_{c\otimes d\otimes e})$ in $(\underline{G}_{\mathbb{C},\mathbb{D}\otimes\mathbb{E}}, \underline{R}_{\mathbb{C},\mathbb{D}\otimes\mathbb{E}})$, where $d_{p'}^{\alpha'} \otimes d_{q'}^{\beta'} \otimes d_{r'}^{\gamma'}$ gets labeled by $d_{p'}^{\alpha'}(c) \otimes (d_{q'}^{\beta'}(d) \otimes d_{r'}^{\gamma'}(e))$. Then:

$$\begin{array}{lll}
\multicolumn{3}{l}{\mathrm{dom}(\overline{\varphi}_{p+q+r}(c \otimes d \otimes e)) =} \\
& = \mathrm{dom}\,(\varphi_{p+q+r}(c \otimes (d \otimes e))) & \\
& = \varphi(\mathrm{dom}(c \otimes (d \otimes e))) & \text{because } (\underline{G}_{\mathbb{C},\mathbb{D}\otimes\mathbb{E}}, \varphi_i) \text{ respects labels} \\
& = \varphi(\mathrm{dom}(2_p \otimes 2_q \otimes 2_r, \mathfrak{L}_{c\otimes d\otimes e})) & \text{because of proposition 7.3} \\
& = \overline{\varphi}(\mathrm{dom}(c \otimes d \otimes e)) & \text{because both are composite of the same} \\
& & \text{appropriate realization.}
\end{array}$$

To show it respects relations, define for every $c \in \mathbb{C}$, (B, g_i) an appropriate realization of a round pasting scheme B in $\mathbb{D}$ and $e \in \mathbb{E}$, a generated pasting $(2_p \otimes B \otimes 2_r, \mathfrak{L}_{c\otimes(B,g_i)\otimes e})$ in $(\underline{G}_{\mathbb{C},\mathbb{D}\otimes\mathbb{E}}, \underline{R}_{\mathbb{C},\mathbb{D}\otimes\mathbb{E}})$, where $d_{p'}^{\alpha'} \otimes b \otimes d_{r'}^{\gamma'}$ gets labeled by $d_{p'}^{\alpha'}(c) \otimes (g_{q'}(b) \otimes d_{r'}^{\gamma'}(e))$. Then:

$$\begin{array}{lll}
\multicolumn{3}{l}{\overline{\varphi}(c \otimes (B, g_i) \otimes e) =} \\
= & \varphi(2_p \otimes B \otimes 2_r, \mathfrak{L}_{c\otimes(B,g_i)\otimes e}) & \text{because both are compos-} \\
& & \text{ite of the same appropriate} \\
& & \text{realization of } 2_p \otimes b \otimes 2_r \\
= & \varphi((2_p \otimes B \otimes 2_r)_t, \mathfrak{L}_{(c\otimes(B,g_i)\otimes e)_t}) & \text{because this is a relation} \\
& & \text{in } \underline{R}_{\mathbb{C},\mathbb{D}\otimes\mathbb{E}} \text{ since } B \otimes 2_r \text{ is} \\
& & \text{round} \\
= & \overline{\varphi}\,((c \otimes (B, g_i) \otimes e)_t)\,, &
\end{array}$$

or, more conceptually,

$$\begin{array}{lll}
\multicolumn{3}{l}{\overline{\varphi}(c \otimes (B, g_i) \otimes e) =} \\
= & \varphi(2_p \otimes B \otimes 2_r, \mathfrak{L}_{c\otimes(B,g_i)\otimes e}) & \\
= & \varphi(2_p \otimes 2_q \otimes 2_r, \mathfrak{L}_{c\otimes g(B)\otimes e}) & \text{by the analog of proposition 7.3} \\
= & \overline{\varphi}_{p+q+r}(c \otimes g(B) \otimes e) & \\
= & \overline{\varphi}\,((c \otimes (B, g_i) \otimes e)_t) & \text{because the latter has only one top-} \\
& & \text{dimensional cell,}
\end{array}$$

and

$$\begin{array}{llll}
\overline{\varphi}(c \otimes \mathrm{id}_d \otimes e) & = & \varphi(c \otimes (\mathrm{id}_d \otimes e)) & \\
& = & \varphi(c \otimes \mathrm{id}^{\mathfrak{l}}_{d\otimes e}) & \text{by a relation in } \underline{R}_{\mathbb{D},\mathbb{E}} \\
& = & \varphi(c \otimes \mathrm{id}_{\overline{d\otimes e}}) & \\
& = & \varphi(\mathrm{id}^{\mathfrak{r}}_{c\otimes(d\otimes e))}) & \text{by a relation in } \underline{R}_{\mathbb{C},\mathbb{D}\otimes\mathbb{E}} \\
& = & \varphi(2_p \otimes 2_{q+1} \otimes 2_r, \mathfrak{L}_{\mathrm{id}^{\mathfrak{m}}_{c\otimes d\otimes e}}) & \\
& = & \overline{\varphi}(\mathrm{id}^{\mathfrak{m}}_{c\otimes d\otimes e}), &
\end{array}$$

and the other relations are done similarly.

Conversely, consider a respectable family of realizations $(\underline{G}_{\mathbb{C},\mathbb{D},\mathbb{E}}, \phi_i)$ of $(\underline{G}_{\mathbb{C},\mathbb{D},\mathbb{E}}, \underline{R}_{\mathbb{C},\mathbb{D},\mathbb{E}})$ in $\mathbb{F}$. To define a family of realizations $(\underline{G}_{\mathbb{C},\mathbb{D}\otimes\mathbb{E}}, \widetilde{\phi}_i)$ of $(\underline{G}_{\mathbb{C},\mathbb{D}\otimes\mathbb{E}}, \underline{R}_{\mathbb{C},\mathbb{D}\otimes\mathbb{E}})$ in $\mathbb{F}$, define for every $c \in \mathbb{C}$ and generated pasting $(B, \mathfrak{L}_B^{\mathbb{D}\otimes\mathbb{E}})$ in $(\underline{G}_{\mathbb{D},\mathbb{E}}, \underline{R}_{\mathbb{D},\mathbb{E}})$ a generated pasting $(2_p \otimes B, \mathfrak{L}_{c\otimes\mathfrak{L}_B^{\mathbb{D}\otimes\mathbb{E}}})$ in $(\underline{G}_{\mathbb{C},\mathbb{D},\mathbb{E}}, \underline{R}_{\mathbb{C},\mathbb{D},\mathbb{E}})$, where $d_{p'}^{\alpha'} \otimes b$ gets labeled by $d_{p'}^{\alpha'}(c) \otimes d \otimes e$ when b is labeled by $d \otimes e$ in $(B, \mathfrak{L}_B^{\mathbb{D}\otimes\mathbb{E}})$, and by an identity when b is labeled by an identity in $(B, \mathfrak{L}_B^{\mathbb{D}\otimes\mathbb{E}})$. Define $(\underline{G}_{\mathbb{C},\mathbb{D}\otimes\mathbb{E}}, \widetilde{\phi}_i)$ by

$$\widetilde{\phi}_i\left(c \otimes \overline{(B, \mathfrak{L}_B^{\mathbb{D}\otimes\mathbb{E}})}\right) = \phi(2_p \otimes B, \mathfrak{L}_{c\otimes\mathfrak{L}_B^{\mathbb{D}\otimes\mathbb{E}}}).$$

This family of realizations is well defined because the relations in $(\underline{G}_{\mathbb{D}\otimes\mathbb{E}}, \underline{R}_{\mathbb{D}\otimes\mathbb{E}})$ are componentwise. To show it respects relations, take for every appropriate realization $(\mathsf{B}, \mathsf{g}_i)$ of a pasting scheme B in $\mathbb{D} \otimes \mathbb{E}$, say $\mathsf{g}_i(\mathsf{b})$ represented by $(B_{\mathsf{b}}, \mathfrak{L}_{B_{\mathsf{b}}}^{\mathbb{D}\otimes\mathbb{E}})$, any representative $(B, \mathfrak{L}_B^{\mathbb{D}\otimes\mathbb{E}})$ of $\mathsf{g}(\mathsf{B})$. Then:

$$\begin{aligned}
\widetilde{\phi}(c \otimes (\mathsf{B}, \mathsf{g}_i)) &= \text{composition of } \phi(2_p \otimes B_{\mathsf{b}}, \mathfrak{L}_{B_{\mathsf{b}}}^{\mathbb{D}\otimes\mathbb{E}})\text{'s via } 2_p \otimes \mathsf{B} \\
&= \phi(2_p \otimes B, \mathfrak{L}_{c\otimes\mathfrak{L}_B^{\mathbb{D}\otimes\mathbb{E}}}) \quad \text{because } \phi \text{ is an } \omega\text{-functor} \\
&= \widetilde{\phi}_i(c \otimes \mathsf{g}(\mathsf{B})) \\
&= \widetilde{\phi}_i\left((c \otimes \mathsf{g}(\mathsf{B}))_t\right)
\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{\phi}_i(c \otimes \mathrm{id}_{\overline{(B,\mathfrak{L}_B^{\mathbb{D}\otimes\mathbb{E}})}}) = \\
&= \phi(2_p \otimes B[\mathrm{id}_B\colon\varnothing], \mathfrak{L}_{c\otimes\mathfrak{L}^{\mathbb{D}\otimes\mathbb{E}}_{B[\mathrm{id}_B:\varnothing]}}) && \text{by definition of identity in } \boldsymbol{\omega}(\underline{G}_{\mathbb{D},\mathbb{E}}, \underline{R}_{\mathbb{D},\mathbb{E}}) \\
&= \mathrm{id}_{\phi(2_p\otimes B,\mathfrak{L}_{c\otimes\mathfrak{L}_B^{\mathbb{D}\otimes\mathbb{E}}})} && \text{because the top-cell of } 2_p \otimes B[\mathrm{id}_B\colon\varnothing] \text{ is labeled by an identity} \\
&= \mathrm{id}_{\widetilde{\phi}\left(c\otimes\overline{(B,\mathfrak{L}_B^{\mathbb{D}\otimes\mathbb{E}})}\right)} \\
&= \widetilde{\phi}(\mathrm{id}_{c\otimes\overline{(B,\mathfrak{L}_B^{\mathbb{D}\otimes\mathbb{E}})}}) \\
&= \widetilde{\phi}(\mathrm{id}^{\mathrm{r}}_{c\otimes(B,\mathfrak{L}_B^{\mathbb{D}\otimes\mathbb{E}})}).
\end{aligned}$$

That it respects labels is left to the reader. □

Lemma 8.2 *The tensor product of ω-categories is associative.*

Proof. Compose the isomorphisms of lemma 8.1 to obtain an isomorphism $\mathbb{C} \otimes (\mathbb{D} \otimes \mathbb{E}) \cong (\mathbb{C} \otimes \mathbb{D}) \otimes \mathbb{E}$. Naturality follows from the unicity in the universal property of the ω-category generated by a pasting presentation. □

8.2 Coherence

Lemma 8.3 *The associativity of the tensor product of ω-categories is coherent.*

Proof. Analogous to the pasting presentation $(\underline{G}_{\mathbb{C},\mathbb{D},\mathbb{E}}, \underline{R}_{\mathbb{C},\mathbb{D},\mathbb{E}})$ used in the proof of associativity, define a pasting presentation $(\underline{G}_{\mathbb{C},\mathbb{D},\mathbb{E},\mathbb{F}}, \underline{R}_{\mathbb{C},\mathbb{D},\mathbb{E},\mathbb{F}})$ for an ω-category $\mathbb{C} \otimes \mathbb{D} \otimes \mathbb{E} \otimes \mathbb{F}$. Then in the diagram

$$
\begin{tikzcd}
 & & (\mathbb{C}\mathbb{D})(\mathbb{E}\mathbb{F}) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & & \\
 & \mathbb{C}\mathbb{D}(\mathbb{E}\mathbb{F}) \arrow[ddr, "\cong"] & & (\mathbb{C}\mathbb{D})\mathbb{E}\mathbb{F} \arrow[ddl, "\cong"'] & \\
\mathbb{C}(\mathbb{D}(\mathbb{E}\mathbb{F})) \arrow[ur, "\cong"] \arrow[dd, "\cong"'] & & & & ((\mathbb{C}\mathbb{D})\mathbb{E})\mathbb{F} \arrow[ul, "\cong"'] \arrow[dd, "\cong"] \\
 & & \mathbb{C}\mathbb{D}\mathbb{E}\mathbb{F} & & \\
\mathbb{C}(\mathbb{D}\mathbb{E}\mathbb{F}) \arrow[urr, "\cong"] & & & & (\mathbb{C}\mathbb{D}\mathbb{E})\mathbb{F} \arrow[ull, "\cong"'] \\
 & \mathbb{C}((\mathbb{D}\mathbb{E})\mathbb{F}) \arrow[ul, "\cong"] \arrow[r, "\cong"'] & \mathbb{C}(\mathbb{D}\mathbb{E})\mathbb{F} \arrow[uu, "\cong"] & (\mathbb{C}(\mathbb{D}\mathbb{E}))\mathbb{F} \arrow[ur, "\cong"'] \arrow[l, "\cong"] &
\end{tikzcd}
,
$$

where the $\otimes$'s have been omitted for reasons of space, all squares commute because of the unicity of the ω-functor induced by a respectable family of realizations. So by Mac Lane's coherence theorem [30] coherence of the associativity isomorphism follows. $\square$

Coherence could also be deduced by means of proposition 4.2, but that wouldn't give explicit descriptions of the associativity isomorphism and no filling of the pentagon.

8.3 Another pasting presentation for $\mathbb{C} \otimes \mathcal{P}(2_q)$

Because 2_q is a free ω-category the pasting presentation $(\underline{G}_{\mathbb{C},\mathcal{P}(2_q)}, \underline{R}_{\mathbb{C},\mathcal{P}(2_q)})$ for $\mathbb{C} \otimes \mathcal{P}(2_q)$ as given above can be simplified, in that it doesn't use all cells of $\mathcal{P}(2_q)$ but only generators. This is possible because the set of generators includes the globes.

So the generators for a pasting presentation $(\underline{G}_{\mathbb{C},2_q}, \underline{R}_{\mathbb{C},2_q})$ are $c \otimes d_{q'}^{\beta'}$ for $c \in \mathbb{C}$ and $q' \leq q$. For every appropriate realization (A, f_i) of a round pasting scheme A in $\mathbb{C}$, there is defined a relation between $(A, f_i) \otimes d_{q'}^{\beta'}$ and $((A, f_i) \otimes d_{q'}^{\beta'})_t$, and for every $c \in \mathbb{C}$, there is defined a relation between $\mathrm{id}_c \otimes d_{q'}^{\beta'}$ and $\mathrm{id}^{\mathfrak{f}}_{c \otimes d_{q'}^{\beta'}}$.

Lemma 8.4 $(\underline{G}_{\mathbb{C},2_q}, \underline{R}_{\mathbb{C},2_q})$ *is a pasting presentation for* $\mathbb{C} \otimes \mathcal{P}(2_q)$.

Proof. The idea is that identities in the ω-category $\mathcal{P}(2_q)$ don't matter because of the relations between formal and actual identities in $(\underline{G}_{\mathbb{C},2_q}, \underline{R}_{\mathbb{C},2_q})$. Details are left to the reader. □

8.4 Unit for $\otimes$

The pasting presentation $(\underline{G}_{\mathbb{C},2_0}, \underline{R}_{\mathbb{C},2_0})$ of $\mathbb{C} \otimes 2_0$ is precisely the standard presentation of $\mathbb{C}$. So $\mathbb{C} \otimes 2_0 \cong \mathbb{C}$ via a *canonical* isomorphism, and because also $2_0 \otimes \mathbb{C} \cong \mathbb{C}$, $2_0 = \mathcal{I}^0$ is the two-sided unit for the tensor product of ω-categories.

Proposition 8.5 *The tensor product* $\otimes$ *and unit* $\mathcal{I}^0$ *give* ω-**Cat** *the structure of a monoidal category.*

Proof. Associativity and coherence of the tensor product have been done already in lemmas 8.2 and 8.3, and the axioms for the unit are easy. □

8.5 Duality

The three different dualities of cubical sets, described in section 2, give rise to three dualities of ω-categories. This can be seen best by considering the cubical set G and its duals, and calculating what are their respective ω-categorizations. Of course, they are all isomorphic to $\mathcal{P}(2_\omega)$, but the non-trivial isomorphisms show what changes.

The transposition duality gives rise to an *even duality*, which will be denoted by ${}^{\mathrm{op}}$, and which interchanges source and target of even-dimensional cells. The combined duality gives rise to an *odd duality*, which will be denoted by ${}^{\mathrm{co}}$, and which interchanges source and target of odd-dimensional cells. There's also ${}^{\mathrm{op\,co}}$ which comes from the second duality of cubical sets and which interchanges source and targets of all cells, and ${}^{\mathrm{co\,op}}$, which is equal to ${}^{\mathrm{op\,co}}$.

Before comparing pasting presentations for $(\mathbb{C}\otimes\mathbb{D})^{\mathrm{op}}$ and $(\mathbb{D}^{\mathrm{op}}\otimes\mathbb{C}^{\mathrm{op}})$, I will compare the pasting schemes $(2_q)^{\mathrm{op}}\otimes(2_p)^{\mathrm{op}}$ and $(2_p\otimes 2_q)^{\mathrm{op}}$. The first one is isomorphic to $2_q\otimes 2_p$, but to see how the cells interact in terms of 2_q and 2_p, which will be important for the comparison of pasting presentations following shortly, it will be considered to consist of symbols $(d_q^\beta)^{\mathrm{op}}\otimes(d_p^\alpha)^{\mathrm{op}}$, with relations $((d_q^\beta)^{\mathrm{op}}\otimes(d_p^\alpha)^{\mathrm{op}}, y)\in\mathsf{E}_j^i$ if and only if one of the following:

1. $y=(d_{q-1}^{(-)^{q+1}})^{\mathrm{op}}\otimes(d_p^\alpha)^{\mathrm{op}}$,
2. $y=(d_q^\beta)^{\mathrm{op}}\otimes(d_{p-1}^{(-)^{q+p+1}})^{\mathrm{op}}$,
3. $y=(d_{q-1}^{(-)^{q+1}})^{\mathrm{op}}\otimes(d_{p-1}^{(-)^{q+p+1}})^{\mathrm{op}}$.

Similarly, $(2_p\otimes 2_q)^{\mathrm{op}}$ consists of symbols $(d_p^\alpha\otimes d_q^\beta)^{\mathrm{op}}$ with relations $((d_p^\alpha\otimes d_q^\beta)^{\mathrm{op}}, y)\in\mathsf{E}_j^i$ for $i>j$ if and only if one of the following:

1. $y=d_p^\alpha\otimes d_{q-1}^{(-)^{p+q+1+p}}$,
2. $y=d_{p-1}^{(-)^{p+q+1}}\otimes d_q^\beta$,
3. $y=d_{p-1}^{(-)^{p+q+1}}\otimes d_{q-1}^{(-)^{p+q+1+p}}$.

And indeed, there is an obvious isomorphism given by $(d_q^\beta)^{\mathrm{op}}\otimes(d_p^\alpha)^{\mathrm{op}}\mapsto(d_p^\alpha\otimes d_q^\beta)^{\mathrm{op}}$ which preserves the relations since $2p$ is even.

The pasting presentation $(\underline{G}_{\mathbb{C},\mathbb{D}},\underline{R}_{\mathbb{C},\mathbb{D}})$ of $\mathbb{C}\otimes\mathbb{D}$ gives rise to a pasting presentation $(\underline{G}^{\mathrm{op}}_{\mathbb{C},\mathbb{D}},\underline{R}^{\mathrm{op}}_{\mathbb{C},\mathbb{D}})$ of $(\mathbb{C}\otimes\mathbb{D})^{\mathrm{op}}$ by taking the ${}^{\mathrm{op}}$-dual of the generating pasting schemes, and by essentially keeping the same relations, only taking into account that they are between dualized generated pastings.

Now to compare $(\underline{G}_{\mathbb{D}^{\mathrm{op}},\mathbb{C}^{\mathrm{op}}},\underline{R}_{\mathbb{D}^{\mathrm{op}},\mathbb{C}^{\mathrm{op}}})$, and $(\underline{G}^{\mathrm{op}}_{\mathbb{C},\mathbb{D}},\underline{R}^{\mathrm{op}}_{\mathbb{C},\mathbb{D}})$, let $(\underline{G}_{\mathbb{D}^{\mathrm{op}},\mathbb{C}^{\mathrm{op}}},\varphi_i)$ be a respectable family of realizations of $(\underline{G}_{\mathbb{D}^{\mathrm{op}},\mathbb{C}^{\mathrm{op}}},\underline{R}_{\mathbb{D}^{\mathrm{op}},\mathbb{C}^{\mathrm{op}}})$ in $\mathbb{E}$. Define a family of realizations $(\underline{G}^{\mathrm{op}}_{\mathbb{C},\mathbb{D}},\overline{\varphi}_i)$ in $\mathbb{E}$ by

$$\overline{\varphi}_i((c\otimes d)^{\mathrm{op}})=\varphi_i(d^{\mathrm{op}}\otimes c^{\mathrm{op}}).$$

This family of realizations is respectable because

$$\begin{aligned}
&\mathrm{dom}\,(\overline{\varphi}_i((c\otimes d)^{\mathrm{op}})) = \\
&\quad = \ \mathrm{dom}(\varphi_i(d^{\mathrm{op}}\otimes c^{\mathrm{op}})) \\
&\quad = \ \varphi(\mathrm{dom}(d^{\mathrm{op}}\otimes c^{\mathrm{op}})) \quad \text{by respectability of } (\underline{G}_{\mathbb{D}^{\mathrm{op}},\mathbb{C}^{\mathrm{op}}},\varphi_i) \\
&\quad = \ \overline{\varphi}\,(\mathrm{dom}((c\otimes d)^{\mathrm{op}}))
\end{aligned}$$

because the labelings on the isomorphic pasting schemes coincide: $\varphi_{i-1}(d^-_{q-1}(d^{\mathrm{op}})\otimes c^{\mathrm{op}}) = \overline{\varphi}_{i-1}\left((c\otimes d^{(-)^{q+1}}_{q-1}(d))^{\mathrm{op}}\right)$ and $\varphi_{i-1}(d^{\mathrm{op}}\otimes d^{(-)^q}_{p-1}(c^{\mathrm{op}})) = \overline{\varphi}_{i-1}\left((d^{(-)^{q+p+1}}_{p-1}(c)\otimes d)^{\mathrm{op}}\right)$. Conversely, by the same formula a respectable family of realizations $(\underline{G}^{\mathrm{op}}_{\mathbb{C},\mathbb{D}},\overline{\varphi}_i)$ in $\mathbb{E}$ gives rise to a family of realizations $(\underline{G}_{\mathbb{D}^{\mathrm{op}},\mathbb{C}^{\mathrm{op}}},\varphi_i)$ in $\mathbb{E}$ which is respectable for the same reasons. Thus $\mathbb{C}^{\mathrm{op}}\otimes\mathbb{D}^{\mathrm{op}}\cong(\mathbb{C}\otimes\mathbb{D})^{\mathrm{op}}$. And also for ${}^{\mathrm{co}}$, which is proven completely analogous.

[21] also considers duality. The even (resp. odd) duality here is the extension of the weak or vertical (resp. strong or horizontal) duality considered there.

8.6 Cube tensor cube

I will show that the tensor product of two cubes is again a cube, as expected.

For a face x of Λ_{p+q}, define a labeled pasting scheme $(\Lambda_{p'+q'},\mathfrak{L}_{(\pi^{\mathfrak{l}}_p\otimes\pi^{\mathfrak{r}}_q)(x)})$, where a face x' of $\Lambda_{p'+q'}$, considered as a morphism $\underline{r}\to\underline{p+q}$, is labeled by $(2_{p'}\otimes 2_{q'},\mathfrak{L}_{\mathsf{R}(\pi^{\mathfrak{l}}_p\circ x\circ x')\otimes\mathsf{R}(\pi^{\mathfrak{r}}_q\circ x\circ x')})$, which is indeed a generator in $\underline{G}_{\mathcal{I}^p,\mathcal{I}^q}$: $\mathsf{R}(\pi^{\mathfrak{l}}_p\circ x\circ x')$ and $\mathsf{R}(\pi^{\mathfrak{r}}_q\circ x\circ x')$ are elements of $\mathcal{I}^p$ and $\mathcal{I}^q$ respectively.

Given well-formed subpasting schemes P and Q of $\mathcal{I}^p$ and $\mathcal{I}^q$ respectively, define a subgraded set of Λ_{p+q} by $P\oplus Q=\{x\in\Lambda_{p+q}|\pi^{\mathfrak{l}}_p\circ x\in P,\ \pi^{\mathfrak{r}}_q\circ x\in Q\}$.

Lemma 8.6 *$P\oplus Q$ is a well-formed subpasting scheme of Λ_{p+q}.*

Proof. For $x'\in\mathsf{R}(x)$ for $x\in P\oplus Q$, $\pi^{\mathfrak{l}}_p\circ x\circ x'\in P$ and $\pi^{\mathfrak{r}}_q\circ x\circ x'\in Q$ because P and Q are subpasting schemes of Λ_p and Λ_q respectively, so $P\oplus Q$ is a subpasting scheme of Λ_{p+q}.

For well-formedness, first observe that $s_m(P\oplus Q)=\bigcup\{d^-_{p'}(P)\oplus d^{(-)^{p'+1}}_{q'}(Q)|\ p'+q'=m\}$, which can be proven by induction on m. That this is a subpasting scheme follows by the same argument as just given, so it remains to show compatibility. So let y and y' of dimension m in $s_m(P\oplus Q)$ be such that $z\in\mathsf{E}_{m-1}(y)\cap\mathsf{E}_{m-1}(y')$. Then for dimension reasons exactly one of the equations $\pi^{\mathfrak{l}}_p\circ y=\pi^{\mathfrak{l}}_p\circ z$ and $\pi^{\mathfrak{r}}_q\circ y=\pi^{\mathfrak{r}}_q\circ z$ holds, and also for y'. So this gives four possibilities, two where the equalities occur on the same side, left say, two where they occur on different sides. In the first case well-formedness of Q gives

$\pi_q^{\mathfrak{r}} \circ y = \pi_q^{\mathfrak{r}} \circ y'$ which implies, because of the equalities on the other side, that $y = y'$. In the second case the exponents give a contradiction, precisely as in the proof of well-formedness in lemma 5.6. $\square$

Define a labeled pasting scheme $(P \oplus Q, \mathfrak{L}_{\pi_p^{\mathfrak{l}} \otimes \pi_q^{\mathfrak{r}}})$, where an element x of $P \oplus Q$ is labeled by $(2_{p'} \otimes 2_{q'}, \mathfrak{L}_{\mathsf{R}(\pi_p^{\mathfrak{l}} \circ x) \otimes \mathsf{R}(\pi_q^{\mathfrak{r}} \circ x)})$. Define also a labeled pasting scheme $(\Lambda_{p'+q'}, \mathfrak{L}_{\mathrm{id}(\pi_p^{\mathfrak{l}}) \otimes \mathrm{id}(\pi_q^{\mathfrak{r}})})$, where a face x of $\Lambda_{p'+q'}$ is labeled by $(2_{p'} \otimes 2_{q'}, \mathfrak{L}_{\mathsf{R}(\pi_p^{\mathfrak{l}} \circ x) \otimes \mathsf{R}(\pi_q^{\mathfrak{r}} \circ x)})$ *or by the appropriate identity.*

Lemma 8.7 *In* $(\underline{G}_{\mathcal{I}^p, \mathcal{I}^q}, \underline{R}_{\mathcal{I}^p, \mathcal{I}^q})$, *for every face* x *of* Λ_{p+q}, $\mathrm{dom}(\Lambda_{p'+q'}, \mathfrak{L}_{(\pi_p^{\mathfrak{l}} \otimes \pi_q^{\mathfrak{r}})(x)})$ *is a generated pasting, which is equivalent to* $\mathrm{dom}(2_{p'} \otimes 2_{q'}, \mathfrak{L}_{\mathsf{R}(\pi_p^{\mathfrak{l}} \circ x) \otimes \mathsf{R}(\pi_q^{\mathfrak{r}} \circ x)})$.

Moreover, for all well-formed subpasting schemes P *and* Q *of* $\mathcal{I}^p$ *and* $\mathcal{I}^q$ *respectively,* $\mathrm{dom}(P \oplus Q, \mathfrak{L}_{(\pi_p^{\mathfrak{l}} \otimes \pi_q^{\mathfrak{r}})})$ *is a generated pasting, which is equivalent to* $\mathrm{dom}(2_{p'} \otimes 2_{q'}, \mathfrak{L}_{P \otimes Q})$. *And* $\mathrm{dom}(\Lambda_{p'+q'}, \mathfrak{L}_{\mathrm{id}(\pi_p^{\mathfrak{l}}) \otimes \mathrm{id}(\pi_q^{\mathfrak{r}})})$ *is a generated pasting equivalent to* $\mathrm{dom}(2_{p'} \otimes 2_{q'}, \mathfrak{L}_{\mathrm{id}(P) \otimes \mathrm{id}(Q)})$.

Proof. Left to the reader as an exercise in manipulating generated pastings. The following pictures indicate what needs to be done in the first case:

$$\mathrm{dom}(\Lambda_{2+1}, \mathfrak{L}_{(\pi_2^{\mathfrak{l}} \otimes \pi_1^{\mathfrak{r}})(000)}) = \begin{tikzcd}
& \bullet \arrow[rr, "-0\otimes+"] \arrow[dr, phantom, "\Downarrow -0\otimes 0"] & & \bullet \arrow[dr, "0+\otimes+"] \arrow[dd, phantom, "\Downarrow 0+\otimes 0"] & \\
\bullet \arrow[ur, "--\otimes 0"] \arrow[rr] \arrow[dr, "0-\otimes-"'] & & \bullet \arrow[ur] \arrow[dr] \arrow[dl, phantom, "\Downarrow 00\otimes-"] & & \bullet \\
& \bullet \arrow[rr, "0+\otimes-"'] & & \bullet \arrow[ur, "++\otimes 0"'] &
\end{tikzcd}$$

and

$$\mathrm{dom}(2_2 \otimes 2_1, \mathfrak{L}_{\mathsf{R}(\pi_2^{\mathfrak{l}} \circ 000) \otimes \mathsf{R}(\pi_1^{\mathfrak{r}} \circ 000)}) = \begin{tikzcd}
& \bullet \arrow[dr, bend left, "s_1(00)\otimes+"] \arrow[d, phantom, "\Downarrow s_1(00)\otimes 0"] & \\
\bullet \arrow[ur, "--\otimes 0"] \arrow[dr, bend right=50, "t_1(00)\otimes-"'] \arrow[dr, bend left] \arrow[dr, phantom, "\Swarrow 00\otimes-"] & {} & \bullet \\
& \bullet \arrow[ur, "++\otimes 0"'] &
\end{tikzcd}$$

need to be equivalent, which can be proven, as before, by composing from high dimensions downwards. $\square$

Proposition 8.8 $\mathcal{I}^p \otimes \mathcal{I}^q \cong \mathcal{I}^{p+q}$.

Proof. The proof will be analogous to the proof of propositions 5.2 and 5.9, but a little more involved because the pasting presentation for $\mathcal{I}^p \otimes \mathcal{I}^q$ has more difficult relations, namely those induced by compositions in $\mathcal{I}^p$ and $\mathcal{I}^q$.

Given a respectable family of realizations $(\underline{G}_{\mathcal{I}^p,\mathcal{I}^q}, \varphi_j)$ in $\mathbb{C}$, define a realization (Λ_{p+q}, f_j) in $\mathbb{C}$ by

$$f_j(x) = \varphi_j(2_{p'} \otimes 2_{q'}, \mathfrak{L}_{\mathsf{R}(\pi_p^{\mathfrak{l}} \circ x) \otimes \mathsf{R}(\pi_q^{\mathfrak{r}} \circ x)})$$

for x a face of Λ_{p+q}. Suppose this realization is m-appropriate. Then

$$\begin{array}{lll} s_m(f_{m+1}(x)) = & & \\ = & s_m\left(\varphi_{m+1}(2_{p'} \otimes 2_{q'}, \mathfrak{L}_{\mathsf{R}(\pi_p^{\mathfrak{l}} \circ x) \otimes \mathsf{R}(\pi_q^{\mathfrak{r}} \circ x)})\right) & \\ = & \varphi\left(\overline{\mathrm{dom}(2_{p'} \otimes 2_{q'}, \mathfrak{L}_{\mathsf{R}(\pi_p^{\mathfrak{l}} \circ x) \otimes \mathsf{R}(\pi_q^{\mathfrak{r}} \circ x)})}\right) & \text{because } (\underline{G}_{\mathcal{I}^p,\mathcal{I}^q}, \varphi_j) \text{ respects labels} \\ = & \varphi\left(\overline{\mathrm{dom}(\Lambda_{p'+q'}, \mathfrak{L}_{(\pi_p^{\mathfrak{l}} \otimes \pi_q^{\mathfrak{r}})(x)})}\right) & \text{by lemma 8.7} \\ = & f\left(s_m(\mathsf{R}(x))\right) & \text{because both are composite of the same appropriate realization of } s_m(\mathsf{R}(x)), \end{array}$$

which proves that it is $(m+1)$-appropriate.

Given an appropriate realization (Λ_{p+q}, f_j) in $\mathbb{C}$, define a family of realizations $(\underline{G}_{\mathcal{I}^p,\mathcal{I}^q}, \varphi_j)$ in $\mathbb{C}$ by

$$\varphi_j(2_{p'} \otimes 2_{q'}, \mathfrak{L}_{\mathrm{id}(P) \otimes \mathrm{id}(Q)}) = f(P \oplus Q)$$

for subpasting schemes P and Q of $\mathcal{I}^p$ and $\mathcal{I}^q$ respectively, where the right hand side is defined because of lemma 8.6.

This family respects relations since the identities are allright, and for composites $\varphi_j(2_{p'} \otimes 2_{q'}, \mathfrak{L}_{(P \cup P') \otimes Q}) = f((P \cup P') \oplus Q)$ and $\varphi((2_p \cup 2_{p'}) \otimes 2_q, \mathfrak{L}_{P \otimes Q, P' \otimes Q}) = f(P \oplus Q)$ composed with $f(P' \oplus Q)$ need to be equal, which is the case since f is an ω-functor and composition in $\mathcal{P}(\Lambda_{p+q})$ is union.

Now suppose the family respects m-labels. Then if there are identities around, then

$$\begin{array}{lll} s_m\left(\varphi_{m+1}(2_{p'} \otimes 2_{q'}, \mathfrak{L}_{\mathrm{id}(P) \otimes \mathrm{id}(Q)})\right) = & & \\ = & s_m(f(P \oplus Q)) & \\ = & f(P \oplus Q) & \\ = & \varphi\left(\overline{\mathrm{dom}(\Lambda_{p'+q'}, \mathfrak{L}_{id(\pi_p^{\mathfrak{l}}) \otimes \mathrm{id}(\pi_q^{\mathfrak{r}})})}\right) & \text{because both are composite of the same appropriate realization of } dom(\Lambda_{p'+q'}) \\ = & \varphi\left(\overline{\mathrm{dom}(2_{p'} \otimes 2_{q'}, \mathfrak{L}_{\mathrm{id}(P) \otimes \mathrm{id}(Q)})}\right) & \text{by lemma 8.7.} \end{array}$$

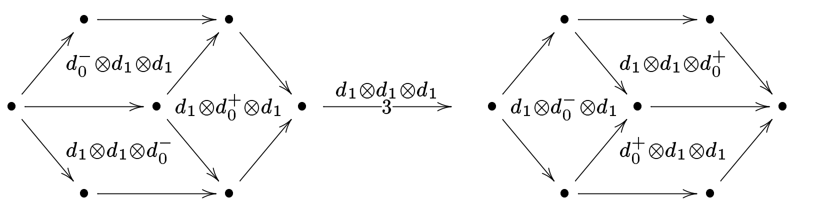
If there are no identities around, then

$$
\begin{aligned}
&s_m(\varphi_{m+1}(2_{p'} \otimes 2_{q'}, \mathfrak{L}_{P\otimes Q})) = \\
&= \ s_m(f(P \oplus Q)) \\
&= \ f(s_m(P \oplus Q)) && \text{because } f \text{ is an } \omega\text{-functor} \\
&= \ \varphi\left(\overline{s_m(P \oplus Q, \mathfrak{L}_{\pi_p^{\mathfrak{l}} \otimes \pi_q^{\mathfrak{r}}})}\right) && \text{because both are composite of the same} \\
& && \text{appropriate realization of } s_m(P \oplus Q) \\
&= \ \varphi\left(\overline{\mathrm{dom}(2_{p'} \otimes 2_{q'}, \mathfrak{L}_{P\otimes Q})}\right) && \text{by lemma 8.7.}
\end{aligned}
$$

Thus it respects $(m+1)$-labels. □

8.7 Triple tensor and Yang-Baxter

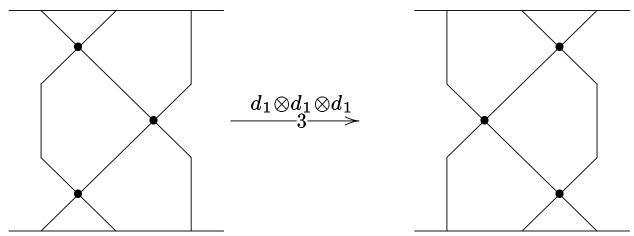
Now look at $2_1 \otimes 2_1 \otimes 2_1$, which is just the 3-dimensional cube, but with an interesting labeling:



.

The two-cells in this pasting scheme can now be seen as Yang-Baxter operators on composites of 1-cells. The domain of $d_1 \otimes d_1 \otimes d_1$ then becomes one side of the Yang-Baxter equation, and the codomain the other. Thus from the ω-categorical viewpoint, Yang-Baxter should be a *cell*, and not an equality.

Another way of seeing the cube above as Yang-Baxter is by taking the planar dual of domain and codomain, resulting in



,

which is one of the Reidemeister moves of knot theory [33].

9 Lax-q-transformations and quasi-ω-functors of more variables

Analogous to the quasi-natural transformations of [21], and to the m-fold homotopies of [10], I introduce the notion of lax-q-transformation. This notion unifies the pseudo-natural transformations, modifications and perturbations of [20], and makes the terminology ready for higher dimensions. It answers a suggestion of [2], that cubes can be used as domains for higher homotopies of ω-categories, negatively: it is the globes that are used as such.

Analogous to the quasi-functors of two and of n variables of [21] and to the bimorphisms of [10], I introduce the notions of quasi-ω-functor of two and of n variables.

All this will be used in the description of the internal homs of ω-categories in section 10.

9.1 Lax-q-transformations

Definition 9.1 A *right lax-q-transformation* $\mathbb{C} \to \mathbb{D}$ is an ω-functor $\mathbb{C} \otimes \mathcal{P}(2_q) \to \mathbb{D}$. A *left lax-q-transformation* $\mathbb{C} \to \mathbb{D}$ is an ω-functor $\mathcal{P}(2_q) \otimes \mathbb{C} \to \mathbb{D}$. $\diamond$

In other words, a right lax-q-transformation is a respectable family of realizations $(\underline{G}_{\mathbb{C},2_q}, \varphi_i)$ of $(\underline{G}_{\mathbb{C},2_q}, \underline{R}_{\mathbb{C},2_q})$ in $\mathbb{D}$. I will describe the data which give rise to this explicitly.

A right lax-q-transformation $(\underline{G}_{\mathbb{C},2_q}, \varphi_i)$ from $\mathbb{C}$ to $\mathbb{D}$ assigns to every p'-dimensional $c \in \mathbb{C}$ and every $d_{q'}^{\beta'} \in 2_q$ with $p' + q' = i$ an i-cell $\varphi_i(c \otimes d_{q'}^{\beta'}) \in \mathbb{D}$, satisfying:

- (respects labels) for every $c \in \mathbb{C}$, the composite of the realization of $\mathrm{dom}(c \otimes d_{q'}^{\beta'})$ induced by φ_i, which is appropriate because φ_i respects m-labels, is equal to $s_m(\varphi_{m+1}(c \otimes d_{q'}^{\beta'}))$, as in section 10 of [5],
- (respects relations) for every appropriate realization (A, f_i) of a round pasting scheme A in $\mathbb{C}$ with composite $f(A)$, the composite of the appropriate (because φ_i respects m-labels) realization of $A \otimes d_{q'}^{\beta'}$ induced by χ_i is equal to $\varphi_i(f(A) \otimes d_{q'}^{\beta'})$, and for every $c \in \mathbb{C}$, $\varphi_{i+1}(\mathrm{id}_c \otimes d_{q'}^{\beta'})$ is equal to $\mathrm{id}_{\varphi_i(c \otimes d_{q'}^{\beta'})}$.

Working out the composite in the first case the condition becomes that $s_m(\varphi_{m+1}(c \otimes d_{q'}^{\beta'}))$ is the composite of $\varphi_m(c \otimes d_{q'-1}^{(-)^{p'+1}})$ and $\varphi_m(s_{p'-1}(c) \otimes d_{q'}^{\beta'})$ with lower dimensional cells, i.e., is the composite of the appropriate realization of a pasting scheme having those two m-cells as highest dimensional cells.

A particular consequence of the second case is, because for *every* A, $(A, f_i) \otimes d_{q'}^{\beta'}$ is equivalent to $f(A) \otimes d_{q'}^{\beta'}$, that the composite of $\varphi_{p'+q'}(c \otimes d_{q'}^{\beta'})$ with $\varphi_{p''+q}(c' \otimes d_{q'}^{\beta'})$ according to the appropriate realization of the pasting scheme $(2_{p'} \circ_m 2_{p''}) \otimes d_{q'}^{\beta'}$ having these as highest dimensional cells is equal to $\varphi_{\max\{p,p'\}+q'}(c' \circ_m c \otimes d_{q'}^{\beta'})$. Because of freeness of $\mathcal{P}(A)$, the previous equality and the condition on the identity also *imply* that φ_i respects relations!

In order to have a short notation for a composite when there is no need to explicitly describe the realization, a composite like the one above will be denoted by $\left\langle (2_{p'} \circ_m 2_{p''}) \otimes d_{q'}^{\beta'}, (\varphi_{p'+q'}(c \otimes d_{q'}^{\beta'}), \varphi_{p''+q'}(c' \otimes d_{q'}^{\beta'}))\right\rangle$. Thus $\langle A, (f_i(a))_{a\in A}\rangle$ would denote $f(A)$ for appropriate realization (A, f_i).

The consequence of the above observation is that:

Lemma 9.2 *A right lax-q-transformation $\mathbb{C} \to \mathbb{D}$ consists of assignments $\varrho_q : C_{p'} \to D_{p'+q}$ and $\varrho_{q'}^{\beta'} : C_{p'} \to D_{p'+q'}$ for every $q' < q$, where ϱ_q can also be denoted by ϱ_q^{β}, such that:*

(i) $\mathrm{dom}(\varrho_{q'}^{\beta'}(c))$ *is the composite of* $\varrho_{q'-1}^{(-)^{p'+1}}(c)$ *and* $\varrho_{q'}^{\beta'}(s_{p'-1}(c))$ *according to* $\mathrm{dom}(2_{p'} \otimes 2_{q'})$*, ie, is equal to* $\left\langle \mathrm{dom}(2_{p'} \otimes 2_{q'}), \left(\varrho_{q'-1}^{(-)^{p'+1}}(c), \varrho_{q'}^{\beta'}(s_{p'-1}(c))\right)\right\rangle$ *and dually,*

(ii) $\varrho_{q'}^{\beta'}(c' \circ_m c)$ *is the composite of* $\varrho_{q'}^{\beta'}(c)$ *with* $\varrho_{q'}^{\beta'}(c')$ *according to* $(2_{p'} \circ_m 2_{p''}) \otimes 2_{q'}$*, i.e., is equal to* $\left\langle (2_{p'} \circ_m 2_{p''}) \otimes 2_{q'}, (\varrho_{q'}^{\beta'}(c), \varrho_{q'}^{\beta'}(c'))\right\rangle$,

(iii) $\varrho_{q'}^{\beta'}(\mathrm{id}_c)$ *is equal to* $\mathrm{id}_{\varrho_{q'}^{\beta'}(c)}$.

Proof. Translate statements about φ_i into statements about $\varrho_{q'}^{\beta'}$. $\square$

In low dimensions this looks as follows. Condition (i) for a right lax-2-

transformation ϱ and a 1-cell c:

$$
\begin{tikzcd}[column sep=small]
 & \varrho_0^+(s_0(c)) \arrow[dr, "\varrho_0^+(c)"] \arrow[dd, phantom, "\Downarrow\varrho_1^+(c)"] & \\
\varrho_0^-(s_0(c)) \arrow[ur, bend left, "\varrho_1^-(s_0(c))"] \arrow[ur, bend right] \arrow[ur, phantom, "\Downarrow\varrho_2(s_0(c))"] \arrow[dr, "\varrho_0^-(c)"'] & & \varrho_0^+(t_0(c)) \\
 & \varrho_0^-(t_0(c)) \arrow[ur, "\varrho_1^+(t_0(c))"'] &
\end{tikzcd}
\xrightarrow[\varrho_2(c)]{3}
\begin{tikzcd}[column sep=small]
 & \varrho_0^+(s_0(c)) \arrow[dr, "\varrho_0^+(c)"] \arrow[dd, phantom, "\varrho_1^-(c)\Downarrow"] & \\
\varrho_0^-(s_0(c)) \arrow[ur, "\varrho_1^-(s_0(c))"] \arrow[dr, "\varrho_0^-(c)"'] & & \varrho_0^+(t_0(c)) \\
 & \varrho_0^-(t_0(c)) \arrow[ur, bend left] \arrow[ur, bend right, "\varrho_1^+(t_0(c))"'] \arrow[ur, phantom, "\Downarrow\varrho_2(t_0(c))"] &
\end{tikzcd}
.
$$

Condition (ii) for a right lax-1-transformation ϱ and 2-cell c and 1-cell c':

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & \bullet \arrow[dr, bend left] \arrow[dd, phantom, "\Downarrow\varrho_1(s_1(c))"] & & \\
\bullet \arrow[ur] \arrow[dr, bend left] \arrow[dr, bend right] \arrow[dr, phantom, "\Downarrow\varrho_0^-(c)"] & & \bullet \arrow[dr] \arrow[dd, phantom, "\Downarrow\varrho_1(c')"] & \\
 & \bullet \arrow[ur] \arrow[dr] & & \bullet \\
 & & \bullet \arrow[ur] &
\end{tikzcd}
\xrightarrow[\varrho_1(c' \circ_0 c)]{3}
\begin{tikzcd}[column sep=small, row sep=small]
 & \bullet \arrow[dr, bend left] \arrow[dr, bend right] \arrow[dr, phantom, "\Downarrow\varrho_0^+(c)"] & & \\
\bullet \arrow[ur] \arrow[dr, bend right] \arrow[dr, phantom, "\varrho_1(t_1(c))\Downarrow"] & & \bullet \arrow[dr] \arrow[dd, phantom, "\Downarrow\varrho_1(c')"] & \\
 & \bullet \arrow[ur] \arrow[dr] & & \bullet \\
 & & \bullet \arrow[ur] &
\end{tikzcd}
.
$$

9.2 Quasi-ω-functors of two variables

Definition 9.3 A *quasi-ω-functor of two variables* $\chi : (\mathbb{C}, \mathbb{D}) \to \mathbb{E}$ consists of a left lax-p-transformation $\chi(c, -) : \mathbb{D} \to \mathbb{E}$ for every p-dimensional $c \in \mathbb{C}$ and a right lax q-transformation $\chi(-, d) : \mathbb{C} \to \mathbb{E}$ for every q-dimensional $d \in \mathbb{D}$, such that

- $\chi(c, -)_p(d) = \chi(-, d)_q(c) \stackrel{\text{def}}{=} \chi(c, d)$,
- $\chi(c, -)_{p'}^{\alpha'} = \chi(d_{p'}^{\alpha'}(c), -)_{p'}$, and
- $\chi(-, d)_{q'}^{\beta'} = \chi(-, d_{q'}^{\beta'}(d))_{q'}$. $\diamond$

Proposition 9.4 *A quasi-ω-functor of two variables $\chi : (\mathbb{C}, \mathbb{D}) \to \mathbb{E}$ corresponds an ω-functor $\mathbb{C} \otimes \mathbb{D} \to \mathbb{E}$.*

Proof. Define a family of realizations $(\underline{G}_{\mathbb{C},\mathbb{D}}, v_i)$ of $(\underline{G}_{\mathbb{C},\mathbb{D}}, \underline{R}_{\mathbb{C},\mathbb{D}})$ in $\mathbb{E}$ by $v_i(c \otimes d) = \chi(c, d)$. Because the relations in $(\underline{G}_{\mathbb{C},\mathbb{D}}, \underline{R}_{\mathbb{C},\mathbb{D}})$ are component-wise, the fact that the $\chi(c,-)$ and the $\chi(-,d)$ are lax transformations implies respectability of this family. Conversely, given $(\underline{G}_{\mathbb{C},\mathbb{D}}, v_i)$, define χ by the same equation, and respectability of the family implies that χ is a quasi-ω-functor of two variables. The reader may check the details of this. □

Thus $\mathbb{C} \otimes \mathbb{D}$ is the universal recipient of quasi-ω-functors of two variables from $(\mathbb{C}, \mathbb{D})$.

The above proposition shows that it is not a coincidence that it is not necessary to look at tensor of pasting schemes in general, but that the theory *forces* that the information can be broken up, so that only pasting schemes $A \otimes 2_q$ and $2_p \otimes B$ need to be considered.

9.3 Quasi-ω-functors of n variables

Contrary to the two-dimensional case [21], it is not possible to define quasi-ω-functors of more variables inductively, in fact because the 3-dimensional cube is not a commutativity condition, but a 3-dimensional cell.

Definition 9.5 A $(p_1, p_2, \ldots, p_{i-1}, -, p_{i+1}, \ldots, p_n)$-lax-transformation $\mathbb{C} \to \mathbb{D}$ is an ω-functor $\mathcal{P}(2_{p_1}) \otimes \mathcal{P}(2_{p_2}) \otimes \ldots \otimes \mathcal{P}(2_{p_{i-1}}) \otimes \mathbb{C} \otimes \mathcal{P}(2_{p_{i+1}}) \otimes \ldots \otimes \mathcal{P}(2_{p_n}) \to \mathbb{D}$. ◇

With this terminology, a left lax-p-transformation is a $(p,-)$-lax-transformation. Note that any tensor product of globes gives a well-formed loop-free pasting scheme by repeated application of the propositions in section 6.

Definition 9.6 A quasi-ω-functor of n variables consists of a multiple lax transformation of the right type for every $(n-1)$-tuple of cells of the respective ω-categories, satisfying obvious compatibility conditions. ◇

Proposition 9.7 *A quasi-ω-functor of n variables $(\mathbb{C}_1, \ldots, \mathbb{C}_n) \to \mathbb{D}$ corresponds to an ω-functor $\mathbb{C}_1 \otimes \ldots \otimes \mathbb{C}_n \to \mathbb{D}$.*

Proof. Straightforward extension of the proof of proposition 9.4. □

9.4 Associativity and coherence revisited

There's a relation between quasi-ω-functors of four variables and coherence of the associativity of the tensor product in section 8, because in the previous proposition it is possible to insert brackets in the tensor product in all possible ways, without affecting its validity. On the one hand, this follows from coherence, see proposition 8.3, on the other hand, it can be proven explicitly analogous to the proof of proposition 9.4, and then coherence follows from this. However, both proofs are equally difficult, since they make use of the universal property of the same pasting presentations.

10 Internal homs

The internal hom ω-categories $\mathrm{Hom}^{\mathfrak{r}}(\mathbb{C},\mathbb{D})$ and $\mathrm{Hom}^{\mathfrak{l}}(\mathbb{C},\mathbb{D})$ extend [21]'s Fun and Fun_u respectively. $\mathrm{Hom}^{\mathfrak{l}}(\mathbb{C},\mathbb{D})$ relates to [10]'s ω-GPD and CRS. I only describe $\mathrm{Hom}^{\mathfrak{r}}(\mathbb{C},\mathbb{D})$ explicitly, the second one is dual in a sense that will be explained in section 11.

As a graded set, $\mathrm{Hom}^{\mathfrak{r}}(\mathbb{C},\mathbb{D})$ consists of right lax transformations, where a right lax-q-transformation is of dimension q. From here on I will omit the prefix since it will invariably be a right one.

10.1 s_m and t_m

The m-sources and m-targets of a lax-q-transformation $\varrho : \mathbb{C} \to \mathbb{D}$ are given by

$$s_m(\varrho)_i(c \otimes d^{\beta'}_{q'}) = t_m(\varrho)_i(c \otimes d^{\beta'}_{q'}) = \varrho_i(c \otimes d^{\beta'}_{q'}) \qquad \text{for } q \leq m,$$

$$s_m(\varrho)_i(c \otimes d^{\beta'}_{q'}) = \begin{cases} \varrho_i(c \otimes d^{-}_{m}) & \text{if } q' = m \\ \varrho_i(c \otimes d^{\beta'}_{q'}) & \text{if } q' < m, \end{cases}$$

and

$$t_m(\varrho)_i(c \otimes d^{\beta'}_{q'}) = \begin{cases} \varrho_i(c \otimes d^{+}_{m}) & \text{if } q' = m \\ \varrho_i(c \otimes d^{\beta'}_{q'}) & \text{if } q' < m \end{cases} \qquad \text{for } q > m,$$

which are right lax-m-transformations.

Or:

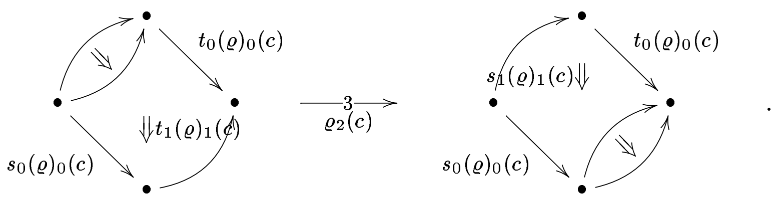
$$(s_m(\varrho))^{\beta'}_{m'}(c) = (t_m(\varrho))^{\beta'}_{m'}(c) = \varrho^{\beta'}_{m'}(c) \qquad \text{for } q \leq m$$

$$(s_m(\varrho))^{\beta'}_{m'}(c) = \begin{cases} \varrho^{-}_{m}(c) & \text{if } m' = m \\ \varrho^{\beta'}_{m'}(c) & \text{if } m' < m, \end{cases}$$

and

$$(t_m(\varrho))^{\beta'}_{m'}(c) = \begin{cases} \varrho^{+}_{m}(c) & \text{if } m' = m \\ \varrho^{\beta'}_{m'}(c) & \text{if } m' < m \end{cases} \qquad \text{for } q > m,$$

which are lax-m-transformations in the terms of lemma 9.2. And the conditions there are obviously satisfied since they are local in some sense. For a lax-2-transformation ϱ and one-dimensional c, this looks as follows:



So it is not necessary that the domain of a lax-q-transformation ϱ in $\mathrm{Hom}^{\mathfrak{r}}(\mathbb{C},\mathbb{D})$ is in the domain of the cells $\varrho_q(c)$!

10.2 Composition

Now suppose ϱ is a lax-q-transformation, σ is a lax-q'-transformation, and that $t_m(\varrho) = s_m(\sigma)$. Their m-composition is given by

$$(\sigma \circ_m \varrho)^{\beta''}_{q''}(c) =$$
$$\begin{cases} \sigma^{\beta''}_{q''}(c) & \text{for } q'' < m, \\ \varrho^-_m(c) \quad \text{if } \beta'' = - & \\ \sigma^+_m(c) \quad \text{if } \beta'' = + & \text{for } q'' = m, \\ \left\langle 2_{p'} \otimes (2_{\min\{q,q''\}} \circ_m 2_{\min\{q',q''\}}), (\varrho^{\beta''}_{\min\{q,q''\}}(c), \sigma^{\beta''}_{\min\{q',q''\}}(c)) \right\rangle & \text{for } q'' > m. \end{cases}$$

The formula for the composition in the cartesian internal hom ω-category $[\mathbb{C},\mathbb{D}]$ in [35] is not correct because it doesn't make $[\mathbb{C},\mathbb{D}]$ into an ω-category, which explains the different format here.

Lemma 10.1 *For ϱ a lax-q-transformation, σ a lax-q'-transformation, and $t_m(\varrho) = s_m(\sigma)$, $\sigma \circ_m \varrho$ is a lax-$(\max\{q,q'\})$-transformation.*

Proof. For condition (i),

$$\mathrm{dom}((\sigma \circ_m \varrho)^{\beta''}_{q''}(c)) =$$

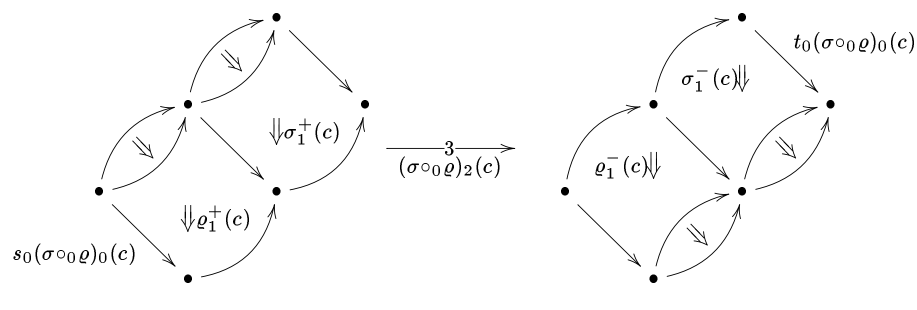
$$
\begin{aligned}
&= \operatorname{dom}\Big(\Big\langle 2_{p'} \otimes (2_{q''} \circ_m 2_{q''}), \big(\varrho^{\beta''}_{q''}(c), \sigma^{\beta''}_{q''}(c)\big)\Big\rangle\Big) \\
&\qquad\qquad\qquad\qquad \text{if } q'' \neq m \text{ and } q'' < \min\{q, q'\} \\
&= \Big\langle \operatorname{dom}(2_{p'} \otimes (2_{q''} \circ_m 2_{q''})), \Big(\varrho^{(-)^{p'+1}}_{q''-1}(c), \varrho^{\beta''}_{q''}(s_{p'-1}(c)), \\
&\qquad \sigma^{(-)^{p'+1}}_{q''-1}(c), \sigma^{\beta''}_{q''}(s_{p'-1}(c)\Big)\Big\rangle \qquad \text{by condition (i) for } \varrho \text{ and } \sigma \\
&= \Big\langle \operatorname{dom}(2_{p'} \otimes 2_{q''}), \Big((\sigma \circ_m \varrho)^{(-)^{p'+1}}_{q''-1}(c), (\sigma \circ_m \varrho)^{\beta''}_{q''}(s_{p'-1}(c))\Big)\Big\rangle \\
&\qquad\qquad\qquad\qquad \text{by the formula for } \sigma \circ_m \varrho,
\end{aligned}
$$

and similar for other q''.

For condition (ii),

$$
\begin{aligned}
(\sigma \circ_m \varrho)^{\beta''}_{q''}(c' \circ_{m'} c) &= \Big\langle 2_{p'} \otimes (2_{q''} \circ_m 2_{q''}), (\varrho^{\beta''}_{q''}(c' \circ_{m'} c), \sigma^{\beta''}_{q''}(c' \circ_{m'} c))\Big\rangle \\
&\qquad\qquad\qquad\qquad \text{say, which needs to be} \\
&= \Big\langle (2_{p''} \circ_{m'} 2_{p'}) \otimes 2_{q''}, (\sigma \circ_m \varrho)^{\beta''}_{q''}(c), (\sigma \circ_m \varrho)^{\beta''}_{q''}(c'))\Big\rangle .
\end{aligned}
$$

Perhaps this could be done by using $(2_{p''} \circ_{m'} 2_{p'}) \otimes (2_{q''} \circ_m 2_{q''})$, provided it is a well-formed loop-free pasting scheme, but I don't want to check this. It is not *necessary*, since the above composites can be obtained as the composite of some appropriate realization in $\mathbb{D}$ of some pasting scheme which can be seen as a generated pasting in the standard presentation of $\mathbb{D}$ and in that respect is equivalent to the two generated pastings described above, which explains their composites being equal. This pasting scheme is $(2_{p''} \circ_{m'} 2_{p'}) \otimes (2_{q''} \circ_m 2_{q''})$ with lots of identities inserted, but that's not the way it is obtained: it is $2_{p'} \otimes (2_{q''} \circ_m 2_{q''})$ with identities inserted and with cells split up, *and* it is $(2_{p''} \circ_{m'} 2_{p'}) \otimes 2_{q''}$ with identities inserted and with cells split up. The highest dimensional cells of this resulting pasting scheme are realized by identities and by the cells $\varrho^{\beta''}_{q''}(c)$, $\varrho^{\beta''}_{q''}(c')$, $\sigma^{\beta''}_{q''}(c)$ and $\sigma^{\beta''}_{q''}(c')$. □



10.3 Identity

For a lax-q-transformation ϱ, define id_ϱ by

$$\begin{array}{lcl}(\mathrm{id}_\varrho)_{q+1}(c) & = & \mathrm{id}_{\varrho_q(c)} \\ (\mathrm{id}_\varrho)^{\beta'}_{q'}(c) & = & \varrho^{\beta'}_{q'}(c) \quad \text{for } q' \leq q\end{array}$$

Lemma 10.2 *For lax-q-transformation ϱ, id_ϱ is a lax-$(q+1)$-transformation.*

Proof. For condition (i),

$$\begin{array}{lcl}\mathrm{dom}((\mathrm{id}_\varrho)_{q+1}(c)) & = & \mathrm{dom}(\mathrm{id}_{\varrho_q(c)}) \\ & = & \varrho_q(c) \\ & = & \left\langle \mathrm{dom}(2_p \otimes 2_{q+1}), \left(\varrho_q(c), \mathrm{id}_{\varrho_q(s_{p'-1}(c))}\right)\right\rangle,\end{array}$$

and for condition (ii) a similar argument as in the previous lemma works. □

10.4 ω-category

Proposition 10.3 $\mathrm{Hom}^{\mathfrak{r}}(\mathbb{C}, \mathbb{D})$, *with the above defined operations, is an ω-category.*

Proof. The elementary properties of s_m and t_m are immediate because they are already incorporated in the data for a lax-q-transformation.

For identity

$$\begin{array}{lcl}(\sigma \circ_m \mathrm{id}_\varrho)_{q+1}(c) & = & \langle 2_{p'} \otimes (2_{q+1} \circ_m 2_{q'}), (\mathrm{id}_{\varrho_q(c)}, \sigma_{q'}(c))\rangle \\ & = & (\mathrm{id}_{\sigma \circ_m \varrho})_{q+1}(c),\end{array}$$

and similarly for other cases.

Associativity and interchange law follow because, for example, $((\tau \circ_m \sigma) \circ_m \varrho)_{\max\{q,q',q''\}}(c)$ and $(\tau \circ_m (\sigma \circ_m \varrho))_{\max\{q,q',q''\}}(c)$ are *both* equal to the composite $\langle 2_{p'} \otimes (2_q \circ_m 2_{q'} \circ_m 2_{q''}), (\varrho_q(c), \sigma_{q'}(c), \tau_{q''}(c))\rangle$, and similarly for interchange. The reader convinces herself or himself of the validity of this statement!

Finally, the other composition axioms are immediate because they are incorporated in the definition of composition. □

10.5 ω-functoriality

Given an ω-functor $g : \mathbb{D} \to \mathbb{D}'$, define an ω-functor $g_* : \mathrm{Hom}^{\mathfrak{r}}(\mathbb{C}, \mathbb{D}) \to \mathrm{Hom}^{\mathfrak{r}}(\mathbb{C}, \mathbb{D}')$ by

$$g_*(\varrho)^{\beta'}_{q'}(c) = g(\varrho^{\beta'}_{q'}(c)).$$

Indeed, it is *completely straightforward* to show that $g_*(\varrho)$ is a lax-q-transformation if ϱ is, and it is only slightly less straightforward to show that g_* is an ω-functor, in both cases making full use of g being an ω-functor, in the sense that g commutes with composites of appropriate realizations.

Given an ω-functor $f : \mathbb{C}' \to \mathbb{C}$, define an ω-functor $f^* : \mathrm{Hom}^{\mathfrak{r}}(\mathbb{C}, \mathbb{D}) \to \mathrm{Hom}^{\mathfrak{r}}(\mathbb{C}', \mathbb{D})$ by

$$f^*(\varrho)^{\beta'}_{q'}(c) = \varrho^{\beta'}_{q'}(f(c)).$$

And indeed, it is completely straightforward to show that f^* is an ω-functor, and it is only slightly less straightforward to show that $f^*(\varrho)$ is a lax-q-transformation if ϱ is.

11 The adjunctions between the tensor product and the internal homs

I prove the adjunctions and mention some consequences, among which dualities relating both internal homs.

11.1 The correspondence

There are two adjunctions to consider: $\mathbb{C} \otimes - \dashv \mathrm{Hom}^{\mathfrak{r}}(\mathbb{C}, -)$ and $- \otimes \mathbb{D} \dashv \mathrm{Hom}^{\mathfrak{l}}(\mathbb{D}, -)$. I will do the first one in some detail, the second one is analogous.

Given an ω-functor $\varphi : \mathbb{C} \otimes \mathbb{D} \to \mathbb{E}$, i.e., a respectable family of realizations $(\underline{G}_{\mathbb{C},\mathbb{D}}, \varphi_j)$ in $\mathbb{E}$, define an ω-functor $\overline{\varphi} : \mathbb{D} \to \mathrm{Hom}^{\mathfrak{r}}(\mathbb{C}, \mathbb{E})$ by

$$\overline{\varphi}(d)^{\beta'}_{q'}(c) = \varphi_{p+q'}(c \otimes d^{\beta'}_{q'}(d)).$$

Indeed, $\overline{\varphi}(d)$ is a lax-q-transformation, for example

$$
\begin{array}{lll}
\multicolumn{3}{l}{\mathrm{dom}(\overline{\varphi}(d)^{\beta'}_{q'}(c)) =} \\
& = & \mathrm{dom}\left(\varphi_{p+q'}(c \otimes d^{\beta'}_{q'}(d))\right) \\
& = & \varphi\left(\overline{\mathrm{dom}(c \otimes d^{\beta'}_{q'}(d))}\right) \qquad\qquad \text{because } (\underline{G}_{\mathbb{C},\mathbb{D}}, \varphi_j) \text{ respects labels} \\
& = & \left\langle \mathrm{dom}\left(2_p \otimes 2_{q'}, \left(\varphi_{p+q'-1}(\mathrm{dom}(c) \otimes d^{\beta'}_{q'}(d)), \varphi_{p+q'-1}(c \otimes d^{(-)^{p+1}}_{q'-1}(d))\right)\right)\right\rangle \\
& = & \left\langle \mathrm{dom}\left(2_p \otimes 2_{q'}, \left(\overline{\varphi}(d)^{\beta'}_{q'}(\mathrm{dom}(c)), \overline{\varphi}(d)^{(-)^{p+1}}_{q'-1}(c)\right)\right)\right\rangle,
\end{array}
$$

so condition (i) of lemma 9.2 holds, and the other conditions are similar. And $\overline{\varphi}$ is an ω-functor, for example

$$
\begin{array}{llll}
\multicolumn{4}{l}{\overline{\varphi}(d' \circ_m d)^{\beta'}_{q'}(c) =} \\
& = & \varphi_{p+q'}(c \otimes d^{\beta'}_{q'}(d' \circ_m d)) & \\
& = & \varphi\left(\overline{c \otimes (d^{\beta'}_{q'}(d') \circ_m d^{\beta'}_{q'}(d))}\right) & \text{if } q' \neq m \\
& = & \varphi\left(\overline{2_p \otimes (2_{q'} \circ_m 2_{q'}), \mathfrak{L}_{c \otimes d^{\beta'}_{q'}(d), c \otimes d^{\beta'}_{q'}(d')}}\right) & \text{because these generated pastings are equivalent} \\
& = & \left\langle 2_p \otimes (2_{q'} \circ_m 2_{q'}), (c \otimes d^{\beta'}_{q'}(d), c \otimes d^{\beta'}_{q'}(d'))\right\rangle & \\
& = & \left\langle 2_p \otimes (2_{q'} \circ_m 2_{q'}), (\overline{\varphi}(d)^{\beta'}_{q'}(c), \overline{\varphi}(d')^{\beta'}_{q'}(c))\right\rangle & \\
& = & (\overline{\varphi}(d') \circ_m \overline{\varphi}(d))^{\beta'}_{q'}(c), &
\end{array}
$$

and the other conditions are easy.

In the other direction, given an ω-functor $\phi : \mathbb{D} \to \mathrm{Hom}^{\mathfrak{r}}(\mathbb{C}, \mathbb{E})$, define an ω-functor $\widetilde{\phi} : \mathbb{C} \otimes \mathbb{D} \to \mathbb{E}$, i.e., a respectable family of realizations $(\underline{G}_{\mathbb{C},\mathbb{D}}, \widetilde{\phi}_j)$ in $\mathbb{E}$ by

$$\widetilde{\phi}_j(c \otimes d) = \phi(d)_q(c).$$

Indeed, $(\underline{G}_{\mathbb{C},\mathbb{D}}, \widetilde{\phi}_j)$ respects labels:

$$
\begin{array}{lll}
s_m(\widetilde{\phi}_{m+1}(c \otimes d)) & = & \mathrm{dom}(\phi(d)_q(c)) \\
& = & \left\langle \mathrm{dom}(2_p \otimes 2_q), \left(\phi(d)_q(\mathrm{dom}(c)), \phi(d)^{(-)^{p+1}}_{q-1}(c)\right)\right\rangle \\
& = & \left\langle \mathrm{dom}(2_p \otimes 2_q), \left(\phi(d)_q(\mathrm{dom}(c)), \phi(d^{(-)^{p+1}}_{q-1})_{q-1}(c)\right)\right\rangle \\
& = & \widetilde{\phi}(\mathrm{dom}(c \otimes d)),
\end{array}
$$

and it respects relations:

$$
\begin{array}{rcll}
\widetilde{\phi}((A,f_i)\otimes d) & = & \left\langle A\otimes 2_q, (\widetilde{\phi}(f_i(a)\otimes d))_{a\in A}\right\rangle & \\
& = & \left\langle A\otimes 2_q, (\phi(d)_q(f_i(a)))_{a\in A}\right\rangle & \\
& = & \phi(d)_q(f(A)) & \text{because } \phi(d), \text{ considered} \\
& & & \text{as a family of realizations,} \\
& & & \text{respects relations} \\
& = & \widetilde{\phi}(f(A)\otimes d), &
\end{array}
$$

and

$$
\begin{array}{rcll}
\widetilde{\phi}(c\otimes(B,g_i)) & = & \left\langle 2_p\otimes B, \left(\widetilde{\phi}(c\otimes g_i(b))\right)_{b\in B}\right\rangle & \\
& = & \left\langle 2_p\otimes B, (\phi(g_i(b))_q(c))_{b\in B}\right\rangle & \\
& = & \phi(g(B))_q(c) & \text{because } \phi \text{ is an } \omega\text{-functor} \\
& = & \widetilde{\phi}(c\otimes g(B)), &
\end{array}
$$

and similarly for the relations with respect to identities.

11.2 Natural

The correspondence is natural in $\mathbb{D}$ because for $g:\mathbb{D}\to\mathbb{D}'$,

$$
\begin{tikzcd}
\mathbb{D} \arrow[d, "g"'] & \omega\text{-}\mathbf{Cat}(\mathbb{C}\otimes\mathbb{D},\mathbb{E}) \arrow[r, shift left, "-"] & \omega\text{-}\mathbf{Cat}(\mathbb{D},\mathrm{Hom}^{\mathrm{r}}(\mathbb{C},\mathbb{E})) \arrow[l, shift left, "\sim"] \\
\mathbb{D}' & \omega\text{-}\mathbf{Cat}(\mathbb{C}\otimes\mathbb{D}',\mathbb{E}) \arrow[u, "-\circ(\mathbb{C}\otimes g)"] \arrow[r, shift left, "-"] & \omega\text{-}\mathbf{Cat}(\mathbb{D}',\mathrm{Hom}^{\mathrm{r}}(\mathbb{C},\mathbb{E})) \arrow[u, "-\circ g"'] \arrow[l, shift left, "\sim"]
\end{tikzcd}
$$

Figure 3: naturality in $\mathbb{D}$

$$
\begin{array}{rcl}
(\widetilde{\phi}\circ(\mathbb{C}\otimes g))(c\otimes d) & = & \widetilde{\phi}(c\otimes g(d)) \\
& = & \phi(g(d))_q(c) \\
& = & ((\phi\circ g)(d))_q(c) \\
& = & \widetilde{(\phi\circ g)}(c\otimes d),
\end{array}
$$

and it is natural in $\mathbb{E}$ because for $h:\mathbb{E}\to\mathbb{E}'$,

$$
\begin{array}{rcl}
(h\circ\widetilde{\phi})(c\otimes d) & = & h(\phi(d)_q(c)) \\
& = & h_*(\phi(d))_q(c) \\
& = & (h_*\circ\phi)(d)_q(c) \\
& = & \widetilde{(h_*\circ\phi)}(c\otimes d).
\end{array}
$$

$$\begin{tikzcd}
\mathbb{E} \arrow[d, "h"'] & \omega\text{-}\mathbf{Cat}(\mathbb{C}\otimes\mathbb{D},\mathbb{E}) \arrow[r, shift left, "\overline{}"] \arrow[d, "h\circ -"'] & \omega\text{-}\mathbf{Cat}(\mathbb{D},\mathrm{Hom}^{\mathfrak{r}}(\mathbb{C},\mathbb{E})) \arrow[l, shift left, "\widetilde{}"] \arrow[d, "h_*\circ -"] \\
\mathbb{E}' & \omega\text{-}\mathbf{Cat}(\mathbb{C}\otimes\mathbb{D},\mathbb{E}') \arrow[r, shift left, "\overline{}"] & \omega\text{-}\mathbf{Cat}(\mathbb{D},\mathrm{Hom}^{\mathfrak{r}}(\mathbb{C},\mathbb{E}')) \arrow[l, shift left, "\widetilde{}"]
\end{tikzcd}$$

Figure 4: naturality in $\mathbb{E}$

Note that because the correspondence is an isomorphism this also makes $\overline{}$ natural in $\mathbb{D}$ and $\mathbb{E}$.

Theorem 11.1 *The internal homs* $\mathrm{Hom}^{\mathfrak{r}}$ *and* $\mathrm{Hom}^{\mathfrak{l}}$ *give the monoidal category* ω-**Cat** *the structure of a monoidal biclosed category. Moreover, this structure coincides with the monoidal biclosed structure of proposition 4.1.*

Proof. The adjunctions have just been proven, and the moreover part is immediate from propositions 4.2 and 8.8. □

11.3 Strength of the adjunctions

One of the consequences of the monoidal biclosed structure is that the natural correspondence $\omega\text{-}\mathbf{Cat}(\mathbb{C}\otimes\mathbb{D},\mathbb{E}) \cong \omega\text{-}\mathbf{Cat}(\mathbb{D},\mathrm{Hom}^{\mathfrak{r}}(\mathbb{C},\mathbb{E}))$ above is in fact the 0-dimensional reflection of a correspondence between *internal* Homs, see e.g. [14]. This can also be seen directly:

$$\begin{aligned}
\mathrm{Hom}^{\mathfrak{r}}(\mathbb{C}\otimes\mathbb{D},\mathbb{E})_q &\cong \omega\text{-}\mathbf{Cat}(\mathbb{C}\otimes\mathbb{D}\otimes 2_q,\mathbb{E}) \\
&\cong \omega\text{-}\mathbf{Cat}(\mathbb{D}\otimes 2_q,\mathrm{Hom}^{\mathfrak{r}}(\mathbb{C},\mathbb{E})) \\
&\cong \mathrm{Hom}^{\mathfrak{r}}(\mathbb{D},\mathrm{Hom}^{\mathfrak{r}}(\mathbb{C},\mathbb{E}))_q .
\end{aligned}$$

It is left to the reader to check this indeed gives an ω-functor between the two.

11.4 Mixed

It is also possible to relate both internal homs:

$$\begin{aligned}
\mathrm{Hom}^{\mathfrak{l}}(\mathbb{C},\mathrm{Hom}^{\mathfrak{r}}(\mathbb{D},\mathbb{E}))_r &\cong \omega\text{-}\mathbf{Cat}(2_r\otimes\mathbb{C},\mathrm{Hom}^{\mathfrak{r}}(\mathbb{D},\mathbb{E})) \\
&\cong \omega\text{-}\mathbf{Cat}(\mathbb{D}\otimes 2_r\otimes\mathbb{C},\mathbb{E}) \\
&\cong \omega\text{-}\mathbf{Cat}(\mathbb{D}\otimes 2_r,\mathrm{Hom}^{\mathfrak{l}}(\mathbb{C},\mathbb{E})) \\
&\cong \mathrm{Hom}^{\mathfrak{r}}(\mathbb{D},\mathrm{Hom}^{\mathfrak{l}}(\mathbb{C},\mathbb{E}))_r .
\end{aligned}$$

11.5 Duality

As a consequence of the adjunction, there are also duals of the internal homs:

$$
\begin{aligned}
&\omega\text{-}\mathbf{Cat}(\mathbb{C}, \mathrm{Hom}^{\mathfrak{r}}(\mathbb{D},\mathbb{E})^{\mathrm{op}}) \cong \\
&\quad\cong\ \omega\text{-}\mathbf{Cat}(\mathbb{C}^{\mathrm{op}}, \mathrm{Hom}^{\mathfrak{r}}(\mathbb{D},\mathbb{E})) \\
&\quad\cong\ \omega\text{-}\mathbf{Cat}(\mathbb{D}\otimes\mathbb{C}^{\mathrm{op}}, \mathbb{E}) && \text{by the adjunction for } \mathrm{Hom}^{\mathfrak{r}} \\
&\quad\cong\ \omega\text{-}\mathbf{Cat}((\mathbb{D}\otimes\mathbb{C}^{\mathrm{op}})^{\mathrm{op}}, \mathbb{E}^{\mathrm{op}}) \\
&\quad\cong\ \omega\text{-}\mathbf{Cat}(\mathbb{C}\otimes\mathbb{D}^{\mathrm{op}}, \mathbb{E}^{\mathrm{op}}) \\
&\quad\cong\ \omega\text{-}\mathbf{Cat}(\mathbb{C}, \mathrm{Hom}^{\mathfrak{l}}(\mathbb{D}^{\mathrm{op}}, \mathbb{E}^{\mathrm{op}})) && \text{by the adjunction for } \mathrm{Hom}^{\mathfrak{l}}.
\end{aligned}
$$

So one internal hom could have been defined in terms of the other by $\mathrm{Hom}^{\mathfrak{l}}(\mathbb{C},\mathbb{D}) = \mathrm{Hom}^{\mathfrak{r}}(\mathbb{C}^{\mathrm{op}},\mathbb{D}^{\mathrm{op}})^{\mathrm{op}}$, *and* by $\mathrm{Hom}^{\mathfrak{r}}(\mathbb{C}^{\mathrm{co}},\mathbb{D}^{\mathrm{co}})^{\mathrm{co}}$. Another consequence is that $\mathrm{Hom}^{\mathfrak{r}}(\mathbb{C}^{\mathrm{op\,co}},\mathbb{D}^{\mathrm{op\,co}}) \cong \mathrm{Hom}^{\mathfrak{r}}(\mathbb{C},\mathbb{D})^{\mathrm{op\,co}}$.

12 ω-Cat is an $(\omega\text{-}\mathbf{Cat})_{\otimes}$-CATegory

As shown in [28], a monoidal closed structure on a category makes this category an enriched category over itself. I describe the resulting structure for one of the internal homs on ω-**Cat**, namely the right one. This structure extends the enrichment which makes 2-**Cat** into a $(2\text{-}\mathbf{Cat})_{\otimes}$-CATegory [21].

There is an ω-functor $\mu : \mathrm{Hom}^{\mathfrak{r}}(\mathbb{C},\mathbb{D}) \otimes \mathrm{Hom}^{\mathfrak{r}}(\mathbb{D},\mathbb{E}) \to \mathrm{Hom}^{\mathfrak{r}}(\mathbb{C},\mathbb{E})$ which can be considered as "horizontal" composition of lax-q-transformations. In fact, $\mu = (\widetilde{\mathrm{id}_{\mathrm{Hom}^{\mathfrak{r}}(\mathbb{D},\mathbb{E})}}) \circ \Big((\widetilde{\mathrm{id}_{\mathrm{Hom}^{\mathfrak{r}}(\mathbb{C},\mathbb{D})}}) \otimes \mathrm{Hom}^{\mathfrak{r}}(\mathbb{D},\mathbb{E})\Big)$, and this ensures that μ is an ω-functor, and that μ is natural in all three variables.

To describe μ, let ϱ be a right lax-q-transformation $\mathbb{C}\to\mathbb{D}$, σ a right lax-r-transformation $\mathbb{D}\to\mathbb{E}$, and c a p-dimensional cell of $\mathbb{C}$. Then

$$\mu(\varrho,\sigma)^{\delta'}_{s'}(c) = \left\langle 2_p \otimes d^{\delta'}_{s'}(2_q\otimes 2_r), \Big(\sigma^{\gamma'}_{r'}(\varrho^{\beta'}_{q'}(c))\Big)_{d^{\beta'}_{q'}\otimes d^{\gamma'}_{r'} \in d^{\delta'}_{s'}(2_q\otimes 2_r)} \right\rangle.$$

In particular, $\mu(\varrho,\sigma)_{q+r}(c) = \sigma_r(\varrho_q(c))$, and the domain of this is a composition of $(\sigma^{(-)^{q+1}}_{r-1}\circ\varrho_q)(c)$ and $(\sigma_r\circ\varrho^{-}_{q-1})(c)$. Other particular instances are when $q=0$ in which case $\mu(\varrho,\sigma) = \varrho^*(\sigma)$, and when $r=0$ in which case $\mu(\varrho,\sigma)=\sigma_*(\varrho)$, see section 10.

The ω-functor $\iota : 2_0 \to \mathrm{Hom}^{\mathfrak{r}}(\mathbb{C},\mathbb{C})$ corresponds, under the adjunction, to the canonical isomorphism $2_0\otimes\mathbb{C}\cong\mathbb{C}$, and as such it is

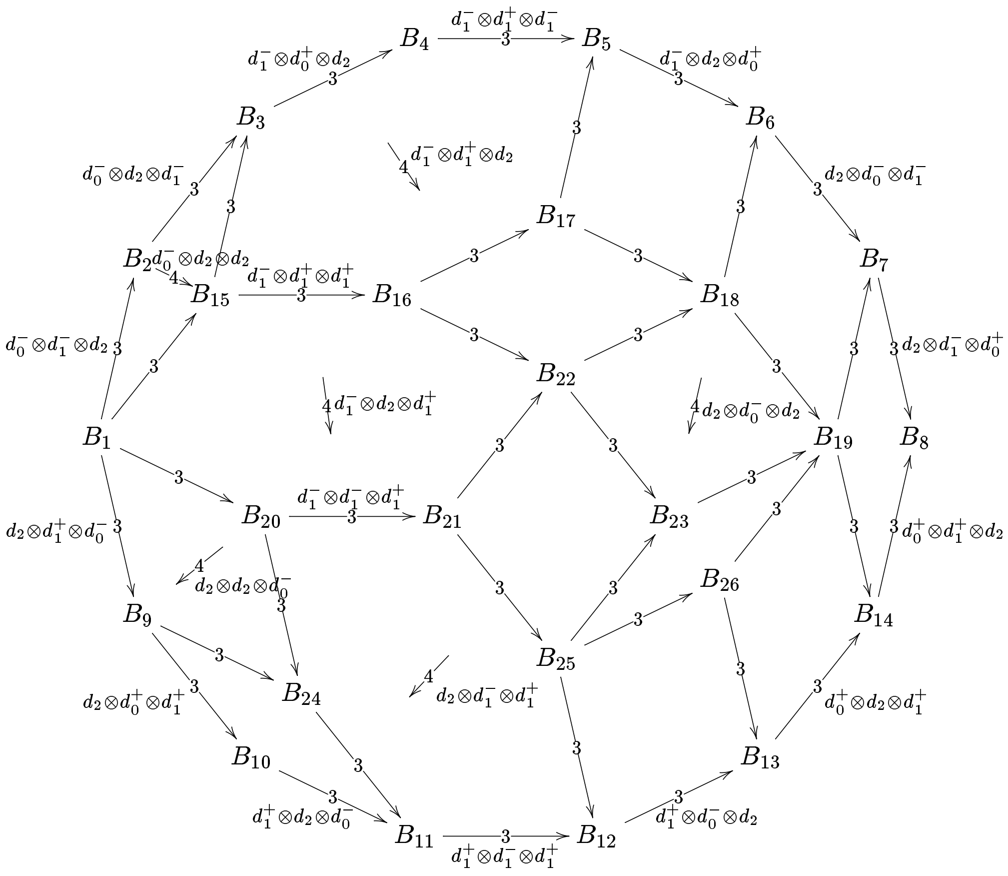
$$\iota(d_0) = (\mathrm{id} : \mathbb{C}\to\mathbb{C}).$$

Finally, a picture of $\mu(\varrho,\sigma)_4(c)$ and all its faces for ϱ a right lax-2-transformation, σ a right lax-2-transformation, and c a 2-dimensional cell of $\mathbb{C}$. As it is a realization of $2_2 \otimes 2_2 \otimes 2_2$ in $\mathbb{E}$, I will give the names of the cells in this pasting scheme, the cell $d_p^\alpha \otimes d_q^\beta \otimes d_r^\gamma$ being realized by $\sigma_r^\gamma\left(\varrho_q^\beta(d_p^\alpha(c))\right)$.

$$
\begin{tikzcd}
 & A_2 \arrow[rr, "d_2 \otimes d_1^- \otimes d_2", "5" description] & {} \arrow[dd, phantom, "\Downarrow 6\, d_2 \otimes d_2 \otimes d_2"] & A_3 \arrow[dr, "d_2 \otimes d_2 \otimes d_1^-", "5" description] & \\
A_1 \arrow[ur, "d_1^- \otimes d_2 \otimes d_2", "5" description] \arrow[dr, "d_2 \otimes d_2 \otimes d_1^+"', "5" description] & & & & A_4 \\
 & A_5 \arrow[rr, "d_2 \otimes d_1^+ \otimes d_2"', "5" description] & {} & A_6 \arrow[ur, "d_1^+ \otimes d_2 \otimes d_2"', "5" description] &
\end{tikzcd}
$$

where the A_i are given by:

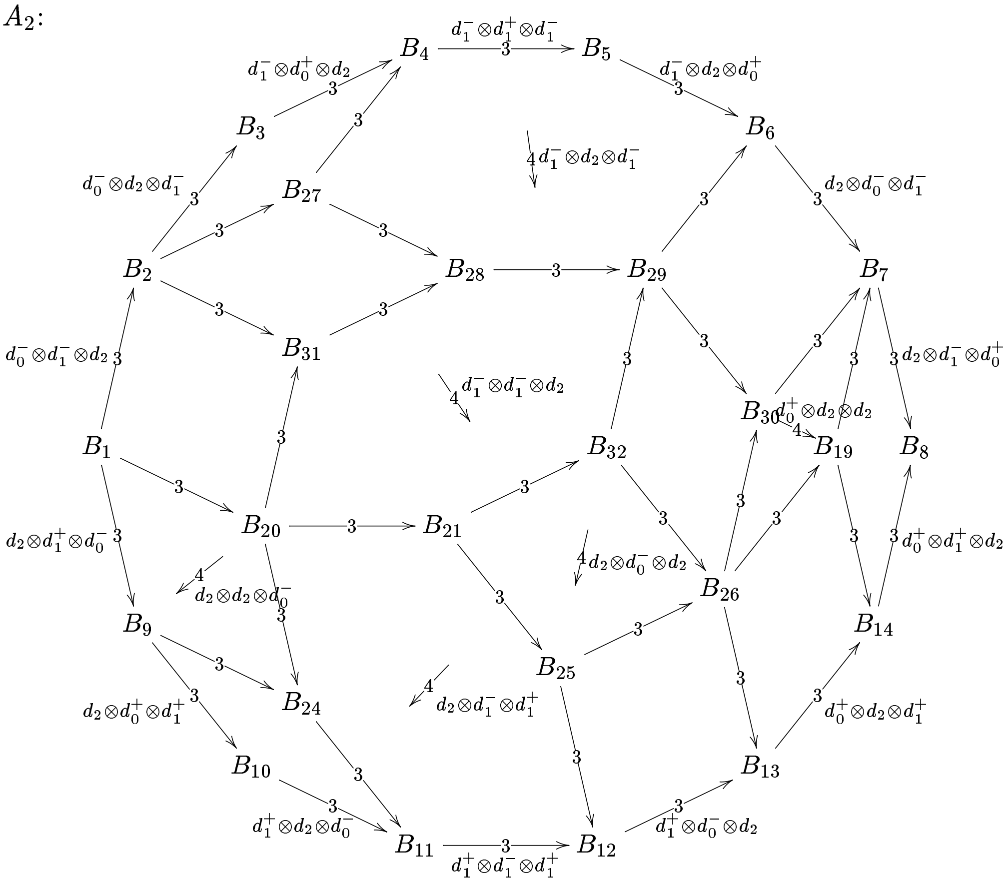
A_1:



in which

$$
\begin{tikzcd}
 & & B_{18} \arrow[dr, "3"] & \\
B_{21} \arrow[r, "3"] & B_{22} \arrow[ur, "3"] \arrow[dr, "3"'] \arrow[rr, phantom, "\Downarrow 4\, d_2\otimes d_0^-\otimes d_2"] & & B_{19} \\
 & & B_{23} \arrow[ur, "3"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & B_{32} \arrow[dr, "3"] & & \\
B_{21} \arrow[ur, "3"] \arrow[dr, "3"'] \arrow[rr, phantom, "\Downarrow 4\, d_2\otimes d_0^-\otimes d_2"] & & B_{26} \arrow[r, "3"] & B_{19} \\
 & B_{25} \arrow[ur, "3"'] & &
\end{tikzcd}
\quad ,
$$

A_2:



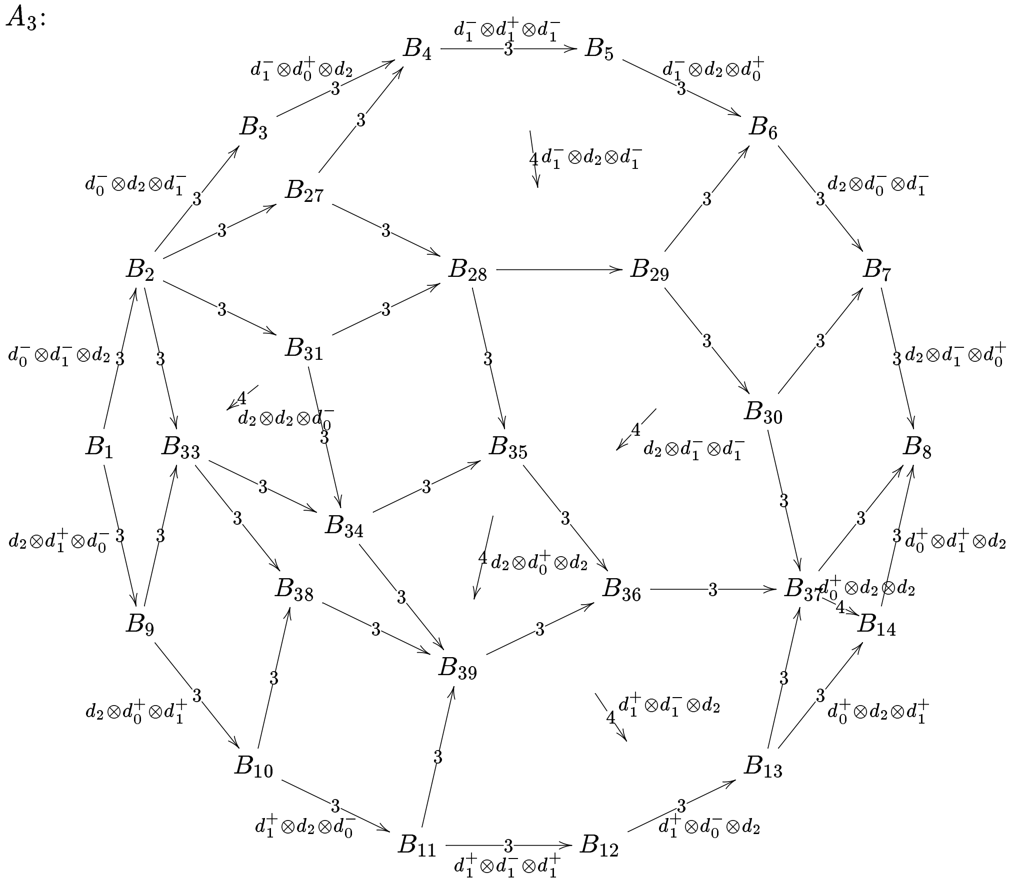
in which

$$
\begin{tikzcd}
 & & B_{30} \arrow[dr, "3"] \arrow[dd, phantom, "\Downarrow 4\; d_0^+\otimes d_2\otimes d_2"] & \\
B_{32} \arrow[r, "3"] & B_{26} \arrow[ur, "3"] \arrow[dr, "3"'] & & B_7 \\
 & & B_{19} \arrow[ur, "3"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & B_{29} \arrow[dr, "3"] \arrow[dd, phantom, "\Downarrow 4\; d_0^+\otimes d_2\otimes d_2"] & & \\
B_{32} \arrow[ur, "3"] \arrow[dr, "3"'] & & B_6 \arrow[r, "3"] & B_7 \\
 & B_{18} \arrow[ur, "3"'] & &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & B_{30} \arrow[dr, "3"] \arrow[dd, phantom, "\Downarrow 4\; d_0^+\otimes d_2\otimes d_2"] & & \\
B_{26} \arrow[ur, "3"] \arrow[dr, "3"'] & & B_7 \arrow[r, "3"] & B_8 \\
 & B_{19} \arrow[ur, "3"'] & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & & B_{33} \arrow[dr, "3"] \arrow[dd, phantom, "\Downarrow 4\; d_0^+\otimes d_2\otimes d_2"] & \\
B_{26} \arrow[r, "3"] & B_{13} \arrow[ur, "3"] \arrow[dr, "3"'] & & B_8 \\
 & & B_{14} \arrow[ur, "3"'] &
\end{tikzcd}
\quad ,
$$

A_3:



in which

$$\begin{tikzcd}
& B_{31} \arrow[dr, "3"] \arrow[dd, phantom, "\Downarrow 4\; d_2\otimes d_2\otimes d_0^-"] & & & & & & B_{28} \arrow[dr, "3"] \arrow[dd, phantom, "\Downarrow 4\; d_2\otimes d_2\otimes d_0^-"] & \\
B_2 \arrow[ur, "3"] \arrow[dr, "3"'] & & B_{34} \arrow[r, "3"] & B_{35} & = & B_2 \arrow[r, "3"] & B_{27} \arrow[ur, "3"] \arrow[dr, "3"'] & & B_{35} \\
& B_{33} \arrow[ur, "3"'] & & & & & & B_{45} \arrow[ur, "3"'] &
\end{tikzcd}$$

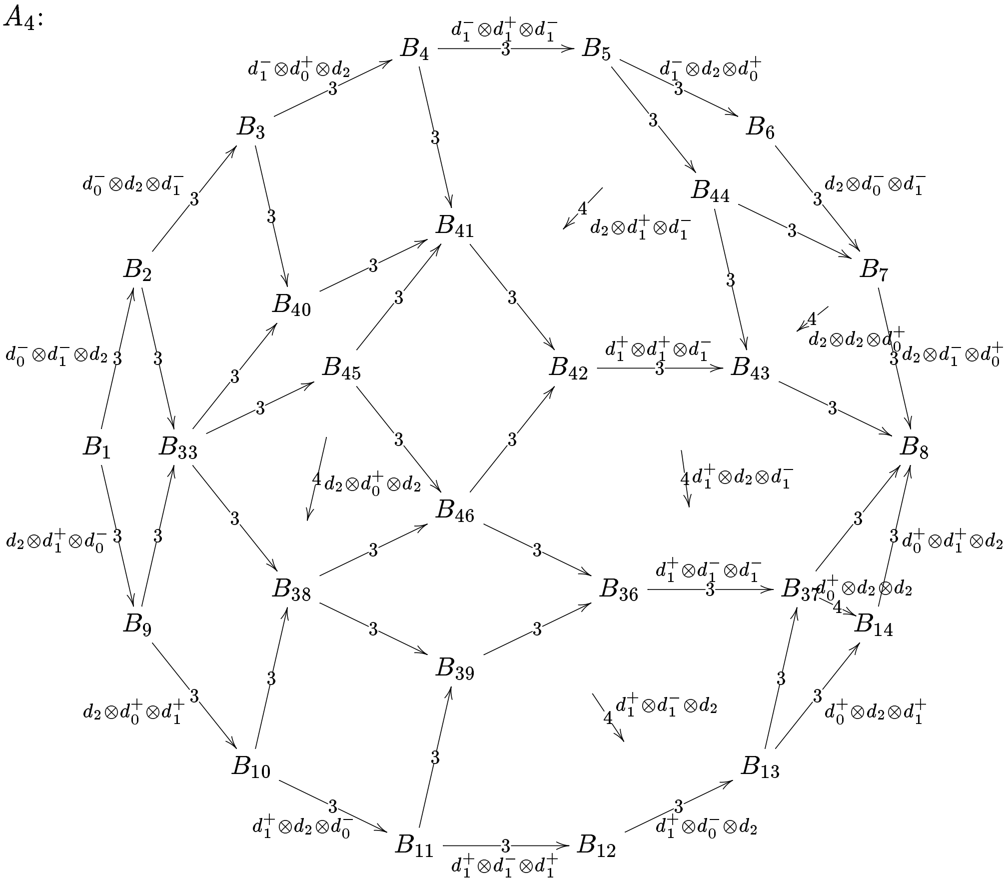
and

$$\begin{tikzcd}
& & B_{31} \arrow[dr, "3"] \arrow[dd, phantom, "\Downarrow 4\; d_2\otimes d_2\otimes d_0^-"] & & & & B_{20} \arrow[dr, "3"] \arrow[dd, phantom, "\Downarrow 4\; d_2\otimes d_2\otimes d_0^-"] & & \\
B_1 \arrow[r, "3"] & B_2 \arrow[ur, "3"] \arrow[dr, "3"'] & & B_{34} & = & B_1 \arrow[ur, "3"] \arrow[dr, "3"'] & & B_{24} \arrow[r, "3"] & B_{34} \\
& & B_{33} \arrow[ur, "3"'] & & & & B_9 \arrow[ur, "3"'] & &
\end{tikzcd}$$
,

and in which

$$
\begin{tikzcd}
 & & B_{35} \arrow[dr, "3"] \arrow[d, phantom, "\Downarrow 4\; d_2\otimes d_0^+\otimes d_2"] & \\
B_{33} \arrow[r, "3"] & B_{34} \arrow[ur, "3"] \arrow[dr, "3"] & & B_{36} \\
 & & B_{39} \arrow[ur, "3"] &
\end{tikzcd}
=
\begin{tikzcd}
 & B_{45} \arrow[dr, "3"] \arrow[d, phantom, "\Downarrow 4\; d_2\otimes d_0^+\otimes d_2"] & & \\
B_{33} \arrow[ur, "3"] \arrow[dr, "3"] & & B_{46} \arrow[r, "3"] & B_{36} \\
 & B_{38} \arrow[ur, "3"] & &
\end{tikzcd}
\quad ,
$$

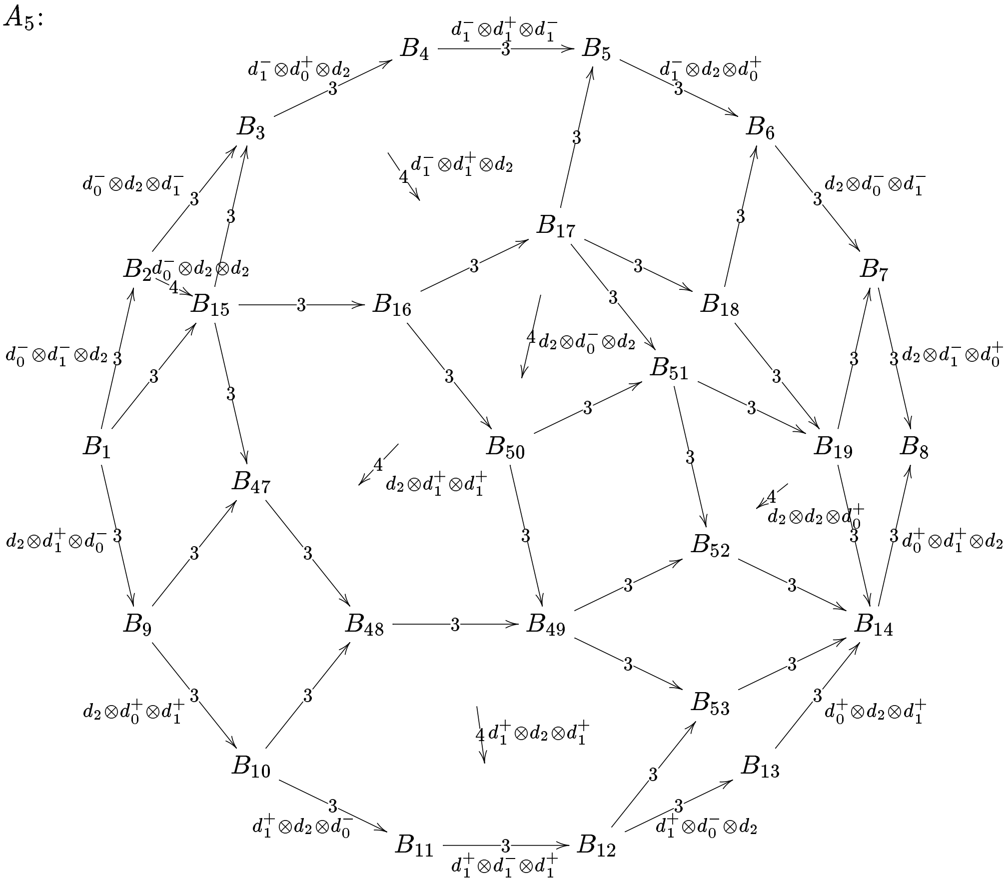
A_4:



in which

$$
\begin{tikzcd}
 & B_{45} \arrow[dr, "3"] \arrow[d, phantom, "\Downarrow 4\; d_2\otimes d_0^+\otimes d_2"] & & \\
B_{33} \arrow[ur, "3"] \arrow[dr, "3"] & & B_{46} \arrow[r, "3"] & B_{42} \\
 & B_{38} \arrow[ur, "3"] & &
\end{tikzcd}
=
\begin{tikzcd}
 & & B_{41} \arrow[dr, "3"] \arrow[d, phantom, "\Downarrow 4\; d_2\otimes d_0^+\otimes d_2"] & \\
B_{33} \arrow[r, "3"] & B_{38} \arrow[ur, "3"] \arrow[dr, "3"] & & B_{42} \\
 & & B_{54} \arrow[ur, "3"] &
\end{tikzcd}
\quad ,
$$

A_5:



in which

$$
\begin{tikzcd}
 & & B_{19} \arrow[dr, "3"] \arrow[dd, phantom, "\Downarrow 4\ d_2\otimes d_2\otimes d_0^+"] & \\
B_{50} \arrow[r, "3"] & B_{51} \arrow[ur, "3"] \arrow[dr, "3"'] & & B_{14} \\
 & & B_{52} \arrow[ur, "3"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & B_{23} \arrow[dr, "3"] \arrow[dd, phantom, "\Downarrow 4\ d_2\otimes d_2\otimes d_0^+"] & & \\
B_{50} \arrow[ur, "3"] \arrow[dr, "3"'] & & B_{53} \arrow[r, "3"] & B_{14} \\
 & B_{49} \arrow[ur, "3"'] & &
\end{tikzcd}
$$

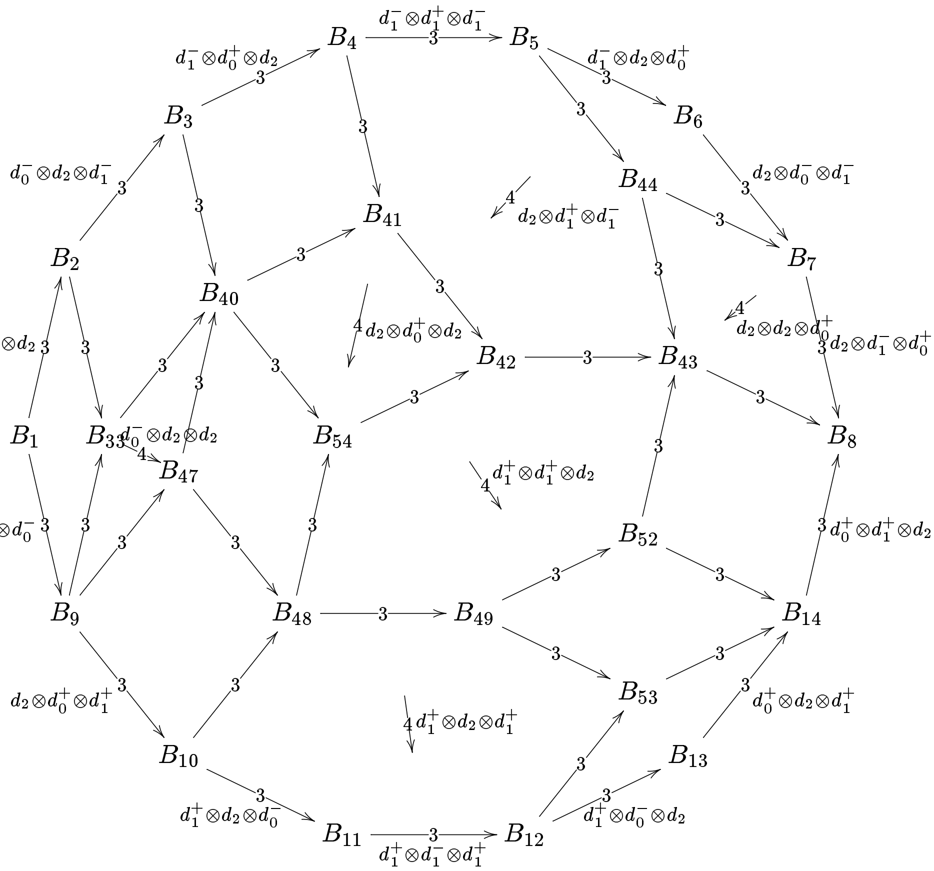
and

$$
\begin{tikzcd}
 & B_{19} \arrow[dr, "3"] \arrow[dd, phantom, "\Downarrow 4\ d_2\otimes d_2\otimes d_0^+"] & & \\
B_{51} \arrow[ur, "3"] \arrow[dr, "3"'] & & B_{14} \arrow[r, "3"] & B_8 \\
 & B_{52} \arrow[ur, "3"'] & &
\end{tikzcd}
=
\begin{tikzcd}
 & & B_7 \arrow[dr, "3"] \arrow[dd, phantom, "\Downarrow 4\ d_2\otimes d_2\otimes d_0^+"] & \\
B_{51} \arrow[r, "3"] & B_{45} \arrow[ur, "3"] \arrow[dr, "3"'] & & B_8 \\
 & & B_{43} \arrow[ur, "3"'] &
\end{tikzcd}
\quad ,
$$

and in which

$$
\begin{tikzcd}
 & B_{17} \arrow[dr, "3"] \arrow[dd, phantom, "\Downarrow 4\ d_0^-\otimes d_2\otimes d_2"] & & & & & & B_{18} \arrow[dr, "3"] \arrow[dd, phantom, "\Downarrow 4\ d_0^-\otimes d_2\otimes d_2"] & \\
B_{16} \arrow[ur, "3"] \arrow[dr, "3"'] & & B_{51} \arrow[r, "3"] & B_{19} & = & B_{16} \arrow[r, "3"] & B_{22} \arrow[ur, "3"] \arrow[dr, "3"'] & & B_{19} \\
 & B_{50} \arrow[ur, "3"'] & & & & & & B_{23} \arrow[ur, "3"'] &
\end{tikzcd}
,
$$

A_6:



in which

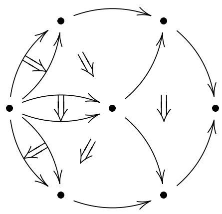
$$
\begin{tikzcd}
 & & B_{33} \arrow[dr, "3"] \arrow[dd, phantom, "\Downarrow 4\ d_0^-\otimes d_2\otimes d_2"] & & & & B_2 \arrow[dr, "3"] \arrow[dd, phantom, "\Downarrow 4\ d_0^-\otimes d_2\otimes d_2"] & & \\
B_1 \arrow[r, "3"] & B_9 \arrow[ur, "3"] \arrow[dr, "3"'] & & B_{40} & = & B_1 \arrow[ur, "3"] \arrow[dr, "3"'] & & B_3 \arrow[r, "3"] & B_{40} \\
 & & B_{47} \arrow[ur, "3"'] & & & & B_{15} \arrow[ur, "3"'] & &
\end{tikzcd}
$$

and

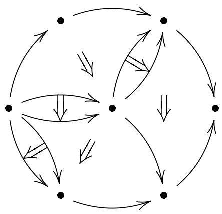
$$
\begin{tikzcd}
 & B_{33} \arrow[dr, "3"] \arrow[dd, phantom, "\Downarrow 4 \; d_0^- \otimes d_2 \otimes d_2"] & & & & & & B_{40} \arrow[dr, "3"] \arrow[dd, phantom, "\Downarrow 4 \; d_0^- \otimes d_2 \otimes d_2"] & \\
B_9 \arrow[ur, "3"] \arrow[dr, "3"'] & & B_{40} \arrow[r, "3"] & B_{54} & = & B_9 \arrow[r, "3"] & B_{10} \arrow[ur, "3"] \arrow[dr, "3"'] & & B_{54} \; , \\
 & B_{47} \arrow[ur, "3"'] & & & & & & B_{48} \arrow[ur, "3"'] &
\end{tikzcd}
$$

and where the B_j are as follows:

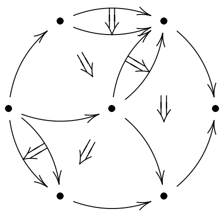
B_1



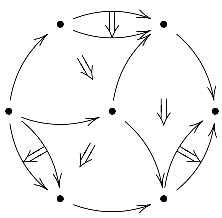
B_2



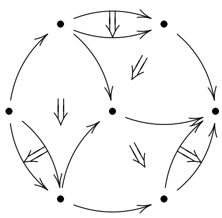
B_3



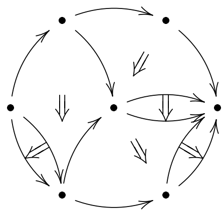
B_4



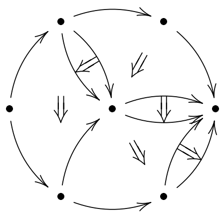
B_5



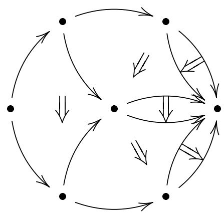
B_6



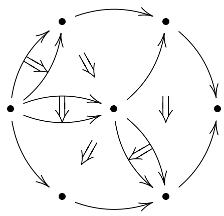
B_7



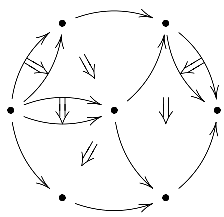
B_8



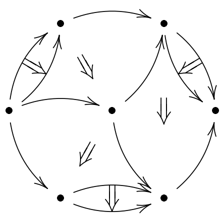
B_9



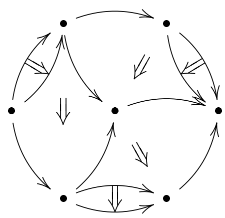
B_{10}



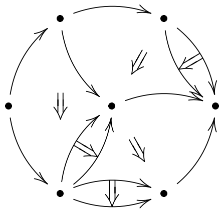
B_{11}



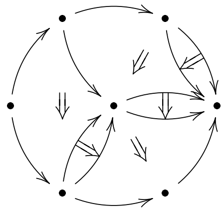
B_{12}



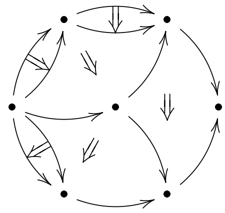
B_{13}



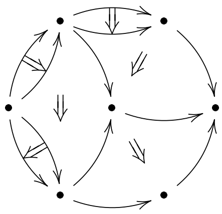
B_{14}



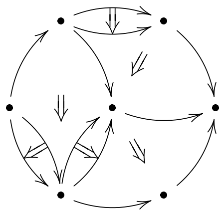
B_{15}



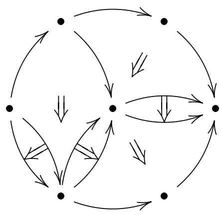
B_{16}



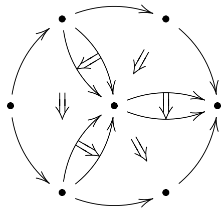
B_{17}



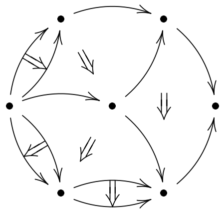
B_{18}



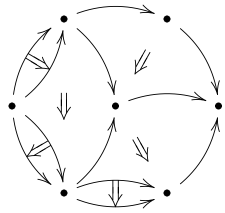
B_{19}



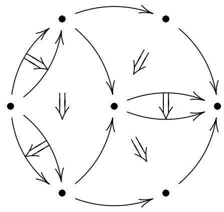
B_{20}



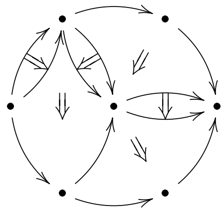
B_{21}



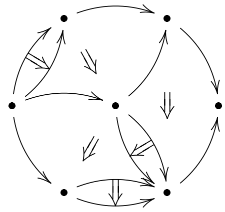
B_{22}



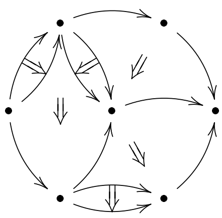
B_{23}



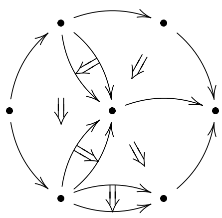
B_{24}



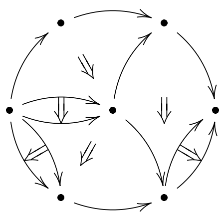
B_{25}



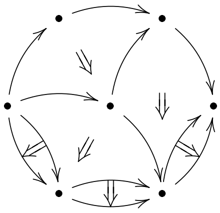
B_{26}



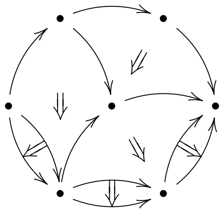
B_{27}



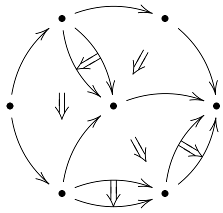
B_{28}



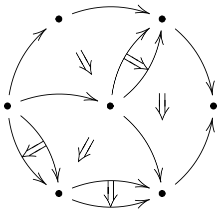
B_{29}



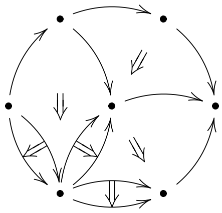
B_{30}



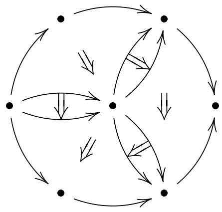
B_{31}



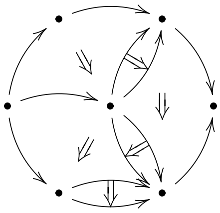
B_{32}



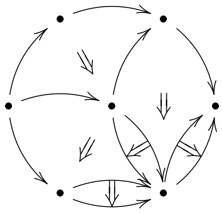
B_{33}



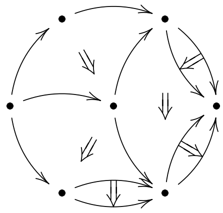
B_{34}



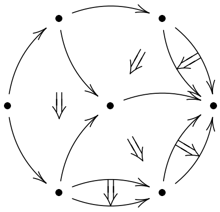
B_{35}



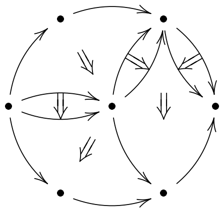
B_{36}



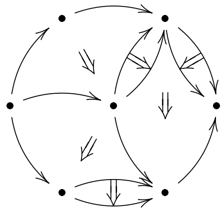
B_{37}



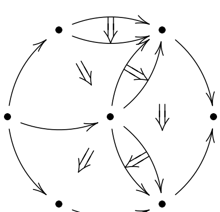
B_{38}



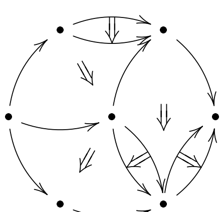
B_{39}



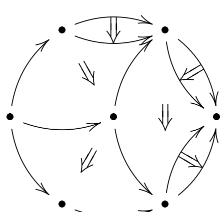
B_{40}



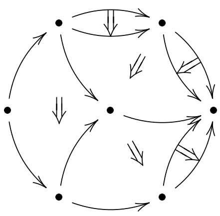
B_{41}



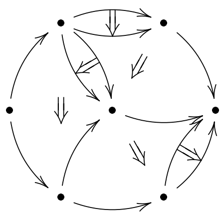
B_{42}



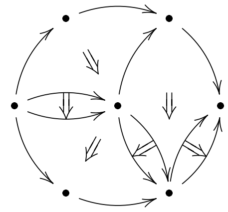
B_{43}



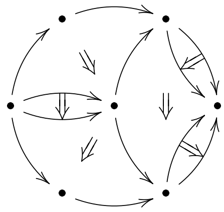
B_{44}



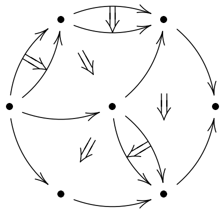
B_{45}



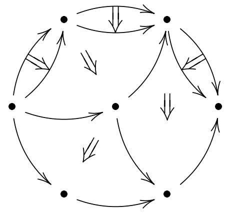
B_{46}



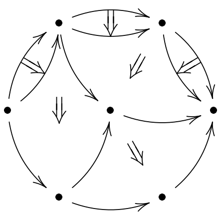
B_{47}



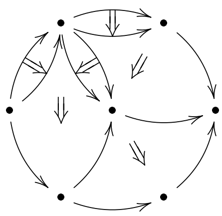
B_{48}



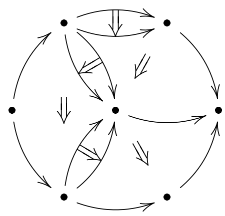
B_{49}



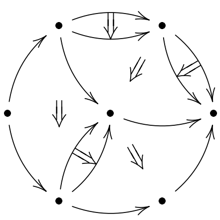
B_{50}



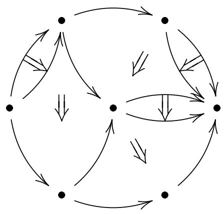
B_{51}



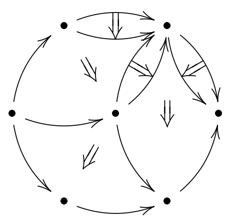
B_{52}



B_{53}



B_{54}



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