Traditionally there are two notions referred to as *compactness* of a space, which are closely related but subtly different.

1. On the one hand a space is called compact if regarded as an object of a certain *site* each of its covering families has a finite subfamily that is still covering.

2. On the other hand, an object in a category with colimits is called compact if the hom-functor out of that object commutes with all filtered colimits.

For instance in the site of topological spaces or of smooth manifolds, equipped with the usual open-cover coverage, the first definition reproduces the the traditional definition of *compact topological space* and of *compact smooth manifold*, respectively. But the notion of compact object in the category of topological spaces in the sense of the second definition is not quite equivalent. For instance the two-element set equipped with the indiscrete topology is compact in the first sense, but not in the second.

The cause of this mismatch, as we will discuss in detail below, becomes clearer once we generalize beyond 1-category theory to ∞-topos theory: in that context it is familiar that locality of morphisms out of an object X into an n-truncated object A (an n-stack) is no longer controled by just the notion of *covers* of X, but by the notion of *hypercover of height* n, which reduces to the ordinary notion of cover for $n = 0$. Accordingly it is clear that the ordinary condition on a compact topological space to admit fintie refinement of any cover is just the first step in a tower of conditions: we may say an object is *compact of height* n if every hypercover of height n over the object is refined by a "finite hypercover" in a suitable sense.

Indeed, the condition on a *compact object* in a 1-category to distribute over filtered colimits turns out to be a compactness condition of *height 1*, which conceptually explains why it is stronger than the existence of finite refinements of covers. This state of affairs in the first two height levels has been known, in different terms, in topos theory, where one distinguishes between a topos being *compact* and being *strongly compact* [MoVe00]:

Definition 0.1. A 1-topos $(\Delta \dashv \Gamma) : \mathcal{X} \underset{\Gamma}{\overset{\Delta}{\leftrightarrows}} \mathrm{Set}$ is called

1. a *compact topos* if the global section functor Γ preserves filtered colimits of subterminal objects (= (-1)-truncated objects);

2. a *strongly compact topos* if Γ preserves all filtered colimits (hence of all 0-truncated objects).

Clearly these are the first two stages in a tower of notions which continues as follows.

Definition 0.2. For $(-1) \leq n \leq \infty$, an ∞-topos $(\Delta \dashv \Gamma) : \mathcal{X} \underset{\Gamma}{\overset{\Delta}{\leftrightarrows}} \infty\mathrm{Grpd}$ is called *compact of height* n if Γ preserves filtered ∞-colimits of n-truncated objects.

Since therefore the traditional terminology concerning "compactness" is not quite consistent across fields, with the category-theoretic "compact object" corresponding, as shown below, to the topos theoretic "strongly compact", we introduce for definiteness the following terminology.

Definition 0.3. For C a subcanonical site, call an object $X \in C \hookrightarrow \mathrm{Sh}(C) \hookrightarrow \mathrm{Sh}_\infty(C)$ *representably compact* if every covering family $\{U_\alpha \to X\}_{i \in I}$ has a finite subfamily $\{U_j \to X\}_{j \in J \subset I}$ which is still covering.

The relation to the traditional notion of compact spaces and compact objects is given by the following

Proposition 0.4. *Let $\mathbf{H}$ be a 1-topos and $X \in \mathbf{H}$ an object. Then*

1. *if X is representably compact, def. 0.3, with respect to the canonical topology, then the slice topos $\mathbf{H}_{/X}$ is a compact topos;*
2. *the slice topos $\mathbf{H}_{/X}$ is strongly compact precisely if X is a compact object.*

Proof. Use that the global section functor Γ on the slice topos is given by

$$\Gamma([E \to X]) = \mathbf{H}(X, E) \times_{\mathbf{H}(X,X)} \{\mathrm{id}_X\}$$

and that colimits in the slice are computed as colimits in $\mathbf{H}$:

$$\varinjlim_i [E_i \to X] \simeq [(\varinjlim_i E_i) \to X] \,.$$

For the first statement, observe that the subterminal objects of $\mathbf{H}_{/X}$ are the monomorphisms in $\mathbf{H}$. Therefore Γ sends all subterminals to the empty set except the terminal object itself, which is sent to the singleton set. Accordingly, if $U_\bullet : I \to \mathbf{H}_{/X}$ is a filtered colimit of subterminals then

- either the $\{U_\alpha\}$ do not cover, hence in particular none of the U_α is X itself, and hence both $\Gamma(\varinjlim_i U_\alpha)$ as well as $\varinjlim_i \Gamma(U_\alpha)$ are the empty set;
- or the $\{U_\alpha\}_{i \in I}$ do cover. Then by assumption on X there is a finite subcover $J \subset I$, and then by assumption that $U_\bullet$ is filtered the cover contains the finite union $\varinjlim_{i \in J} U_\alpha = X$ and hence both $\Gamma(\varinjlim_i U_\alpha)$ as well as $\varinjlim_i \Gamma(U_\alpha)$ are the singleton set.

For the second statement, assume first that X is a compact object. Then using that colimits in a topos are preserved by pullbacks, it follows for all filtered diagrams $[E_\bullet \to X]$ in $\mathbf{H}_{/X}$ that

$$\begin{aligned} \Gamma(\varinjlim_i [E_i \to X]) &\simeq \mathbf{H}(X, \varinjlim_i E_i) \times_{\mathbf{H}(X,X)} \{\mathrm{id}\} \\ &\simeq (\varinjlim_i \mathbf{H}(X, E_i)) \times_{\mathbf{H}(X,X)} \{\mathrm{id}\} \\ &\simeq \varinjlim_i (\mathbf{H}(X, E_i) \times_{\mathbf{H}(X,X)} \{\mathrm{id}\}) \\ &\simeq \varinjlim_i \Gamma[E_i \to X] \end{aligned} ,$$

and hence $\mathbf{H}_{/X}$ is strongly compact.

Conversely, assume that $\mathbf{H}_{/X}$ is strongly compact. Observe that for every object $F \in \mathbf{H}$ we have a natural isomorphism $\mathbf{H}(X, F) \simeq \Gamma([X \times F \to X])$. Using this, we obtain for every filtered diagram $F_\bullet$ in $\mathbf{H}$ that

$$\begin{aligned} \mathbf{H}(X, \varinjlim_i F_i) &\simeq \Gamma([X \times (\varinjlim_i F_i) \to X]) \\ &\simeq \Gamma(\varinjlim_i [X \times F_i \to X]) \\ &\simeq \varinjlim_i \Gamma([X \times F_i \to X]) \\ &\simeq \varinjlim_i \mathbf{H}(X, F_i) \end{aligned}$$

and hence X is a compact object. $\square$

We show now that, while a representably compact object does not distribute over all filtered colimits, it still distributes over some filtered colimits.

Definition 0.5. Call a filtered diagram $A : I \to D$ in a category D *mono-filtered* if for all morphisms $i_1 \to i_2$ in the diagram category I the morphism $A(i_1 \to i_2)$ is a monomorphism in D.

Lemma 0.6. *For C a site and $A : I \to \mathrm{Sh}(C) \hookrightarrow \mathrm{PSh}(C)$ a monofiltered diagram of sheaves, its colimit $\varinjlim_i A_i \in \mathrm{PSh}(C)$ is a separated presheaf.*

Proof. For $\{U_\alpha \to X\}$ any covering family in C with $S(\{U_\alpha\}) \in \mathrm{PSh}(C)$ the corresponding sieve, we need to show that

$$\varinjlim_i A_i(X) \to \mathrm{PSh}_C(S(\{U_\alpha\}), \varinjlim_i A_i)$$

is a monomorphism. An element on the left is represented by a pair $(i \in I, a \in A_i(X))$. Given any other such element, we may assume by filteredness that they are both represented over the same index i. So let (i, a) and (i, a') be two such elements. Under the above function, (i, a) is mapped to the collection $\{i, a|_{U_\alpha}\}_\alpha$ and (i, a') to $\{i, a'|_{U_\alpha}\}_\alpha$. If a is different from a', then these families differ at stage i, hence at least one pair $a|_{U_\alpha}$, $a'|_{U_\alpha}$ is different at stage i. Then by mono-filteredness, this pair differs also at all later stages, hence the corresponding families $\{U_\alpha \to \varinjlim_i A_i\}_\alpha$ differ. $\square$

Proposition 0.7. *For $X \in \mathrm{C} \hookrightarrow \mathrm{Sh}(C)$ a representably compact object, def. 0.3,* $\mathrm{Hom}_{\mathrm{Sh}(C)}(X, -)$ *commutes with all mono-filtered colimits.*

Proof. Let $A : I \to \mathrm{Sh}(C) \hookrightarrow \mathrm{PSh}(C)$ be a mono-filtered diagram of sheaves, regarded as a diagram of presheaves. Write $\varinjlim_i A_i$ for its colimit. So with $L : \mathrm{PSh}(C) \to \mathrm{Sh}(C)$ denoting sheafification, $L \varinjlim_i A_i$ is the colimit of sheaves in question. By the Yoneda lemma and since colimits of presheaves are computed objectwise, it is sufficient to show that for X a representably compact object, the value of the sheafified colimit is the colimit of the values of the sheaves on X

$$(L\varinjlim_i A_i)(X) \simeq (\varinjlim_i A_i)(X) = \varinjlim_i A_i(X)\,.$$

To see this, we evaluate the sheafification by the plus construction. By lemma 0.6, the presheaf $\varinjlim_i A_i$ is already separated, so we obtain its sheafification by applying the plus-construction just *once*.

We observe now that *over a representably compact object* X the single plus-construction acts as the identity on the presheaf $\varinjlim_i A_i$. Namely the single plus-construction over X takes the colimit of the value of the presheaf on sieves

$$S(\{U_\alpha\}) := \varinjlim(\textstyle\coprod_{\alpha,\beta} U_{\alpha,\beta} \rightrightarrows \coprod_\alpha U_\alpha)$$

over the opposite of the category of covers $\{U_\alpha \to X\}$ of X. By the very definition of compactness, the inclusion of (the opposite category of) the category of finite covers of X

into that of all covers is a final functor. Therefore we may compute the plus-construction over X by the colimit over just the collection of finite covers. On a finite cover we have

$$\begin{aligned}
\mathrm{PSh}(S(\{U_\alpha\}), \varinjlim_i A_i) &:= \mathrm{PSh}(\varinjlim(\coprod_{\alpha,\beta} U_{\alpha\beta} \rightrightarrows \coprod_\alpha U_\alpha), \varinjlim_i A_i) \\
&\simeq \varprojlim(\prod_\alpha \varinjlim_i A_i(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \varinjlim_i A_i(U_{\alpha,\beta})) \\
&\simeq \varinjlim_i \varprojlim(\prod_\alpha A_i(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} A_i(U_{\alpha,\beta})) \\
&\simeq \varinjlim_i A_i(X)
\end{aligned} ,$$

where in the second but last step we used that filtered colimits commute with finite limits, and in the last step we used that each A_i is a sheaf.

So in conclusion, for X a representably compact object and $A : I \to \mathrm{Sh}(C)$ a monofiltered diagram, we have found that

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Sh}(C)}(X, L \varinjlim_i A_i) &\simeq (\varinjlim_i A_i)^+(X) \\
&\simeq \varinjlim_i A_i(X) \\
&\simeq \varinjlim_i \mathrm{Hom}_{\mathrm{Sh}(C)}(X, A_i)
\end{aligned}$$

□

Definition 0.8. For C a site, say that an object $X \in C$ is *representably paracompact* if each bounded hypercover over X can be refined by the Čech nerve of an ordinary cover.

The motivating example is

Proposition 0.9. *Over a paracompact topological space, every bounded hypercover is refined by the Čech nerve of an ordinary open cover.*

Proof. Let $Y \to X$ be a bounded hypercover. By lemma 7.2.3.5 in [LuTop] we may find for each $k \in \mathbb{N}$ a refinement of the cover given by Y_0 such that the non-trivial $(k+1)$-fold intersections of this cover factor through Y_{k+1}. Let then $n \in \mathbb{N}$ be a bound for the height of Y and form the intersection of the covers obtained by this lemma for $0 \leq k \leq n$. Then the resulting Čech nerve projection factors through $Y \to X$. □

Proposition 0.10. *We want to show (if true) that if $X \in C \hookrightarrow \mathrm{Sh}_\infty(C) =: \mathbf{H}$ is*

1. *representably paracompact, def. 0.8;*
2. *representably compact, def. 0.3*

then it distributes over sequential ∞-colimits $A_\bullet : I \to \mathrm{Sh}_\infty(C)$ over n-truncated objects for every $n \in \mathbb{N}$.

Proof. Let $A_\bullet : I \to [C^{\mathrm{op}}, \mathrm{sSet}]$ be a presentation of a given sequential diagram in $\mathrm{Sh}_\infty(\mathrm{Mfd})$, such that it is fibrant and cofibrant in $[I, [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}]_{\mathrm{proj}}$. Note for later use that this implies in particular that

- The ordinary colimit $\varinjlim_i A_i \in [C^{\mathrm{op}}, \mathrm{sSet}]$ is a homotopy colimit.
- Every A_i is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ and hence also in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.
- Every morphism $A_i \to A_j$ is (by example **??**) a cofibration in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$, hence in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$, hence in particular in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{inj}}$, hence is over each $U \in C$ a monomorphism.

Observe that $\varinjlim_i A_i$ is still fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$: since the colimit is taken in presheaves, it is computed objectwise, and since it is filtered, we may find the lift against horn inclusions (which are inclusions of degreewise finite simplicial sets) at some stage in the colimit, where it exists by assumption that $A_\bullet$ is projectively fibrant, so that each A_i is projectively fibrant in the local and hence in particular in the global model structure.

Since X, being representable, is cofibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$, it also follows by this reasoning that the diagram

$$\mathbf{H}(X, A_\bullet) : I \to \infty\mathrm{Grpd}$$

is presented by

$$A_\bullet(X) : I \to \mathrm{sSet}\,.$$

Since the functors

$$[I, [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}]_{\mathrm{proj}} \xrightarrow{\mathrm{id}} [I, [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}]_{\mathrm{proj}} \xrightarrow{\mathrm{id}} [I, [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{inj}}]_{\mathrm{proj}} \xrightarrow{\mathrm{id}} [I, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{proj}}$$

all preserve cofibrant objects, it follows that $A_\bullet(X)$ is cofibrant in $[I, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{proj}}$. Therefore also its ordinary colimit presents the corresponding ∞-colimit.

This means that the equivalence which we have to establish can be written in the form

$$\mathbb{R}\mathrm{Hom}(X, \varinjlim_i A_i) \simeq \varinjlim_i A_i(X)\,.$$

If here $\varinjlim_i A_i$ were fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$, then the derived hom on the left would be given by the simplicial mapping space and the equivalence would hold trivially. So the remaining issue is now to deal with the fibrant replacement: the ∞-sheafification of $\varinjlim_i A_i$.

We want to appeal to theorem 7.6 c) in [DuHoIs04] to compute the derived hom into this ∞-stackification by a colimit over hypercovers of the ordinary simplicial homs out of these hypercovers into $\varinjlim_i A_i$ itself. To do so, we now argue that by the assumptions on X, we may in fact replace the hypercovers here with finite Čech covers.

So consider the colimit

$$\varinjlim_{\{U_\alpha \to X\}_{\mathrm{finite}}} [C^{\mathrm{op}}, \mathrm{sSet}](\check{C}(\{U_\alpha\}), \varinjlim_i A_i)$$

over all finite covers of X. Since by representable compactness of X these are cofinal in all covers of X, this is isomorphic to the colimit over all Čech covers

$$\cdots = \varinjlim_{\{U_\alpha \to X\}} [C^{\mathrm{op}}, \mathrm{sSet}](\check{C}(\{U_\alpha\}), \varinjlim_i A_i)\,.$$

Next, by representable paracomopactness of X, the Čech covers in turn are cofinal in all bounded hypercovers $Y \to X$, so that, furthermore, this is isomorphic to the colimit over all bounded hypercovers

$$\cdots = \varinjlim_{Y \to X} [C^{\mathrm{op}}, \mathrm{sSet}](Y, \varinjlim_i A_i)\,.$$

Finally, by the assumption that the A_i are n-truncated, the colimit here may equivalently be taken over all hypercovers.

We now claim that the canonical morphism

$$\lim_{\{U_\alpha \to X\}_{\text{finite}}} [C^{\text{op}}, \text{sSet}](\check{C}(\{U_\alpha\}), \lim_{\longrightarrow_i} A_i) \to \mathbb{R}\text{Hom}(X, \lim_{\longrightarrow_i} A_i)$$

is a weak equivalence. Since the category of covers is filtered, we may first compute homotopy groups and then take the colimit. With the above isomorphisms, the statement is then given by theorem 7.6 c) in [DuHoIs04].

Now to conclude: since maps out of the finite Cech nerves pass through the filtered colimit, we have

$$\begin{aligned} \mathbb{R}\text{Hom}(X, \lim_{\longrightarrow_i} A_i) &\simeq \lim_{\{U_\alpha \to X\}_{\text{finite}}} [C^{\text{op}}, \text{sSet}](\check{C}(\{U_\alpha\}), \lim_{\longrightarrow_i} A_i) \\ &\simeq \lim_{\{U_\alpha \to X\}_{\text{finite}}} \lim_{\longrightarrow_i} [C^{\text{op}}, \text{sSet}](\check{C}(\{U_\alpha\}), A_i) \\ &\simeq \lim_{\longrightarrow_i} \lim_{\{U_\alpha \to X\}_{\text{finite}}} [C^{\text{op}}, \text{sSet}](\check{C}(\{U_\alpha\}), A_i) \\ &\simeq \lim_{\longrightarrow_i} A_i(X) \end{aligned} .$$

Here in the last step we used that each single A_i is fibrant in $[C^{\text{op}}, \text{sSet}]_{\text{proj,loc}}$, so that for each $i \in I$

$$[C^{\text{op}}, \text{sSet}](X, A_i) \to [C^{\text{op}}, \text{sSet}](\check{C}(\{U_\alpha\}), A_i)$$

is a weak equivalence. Moreover, the diagram $[C^{\text{op}}, \text{sSet}](\check{C}(\{U_\alpha\}), A_\bullet)$ in sSet is still projectively cofibrant, by example **??**, since all morphisms are cofibrations in $\text{sSet}_{\text{Quillen}}$, and so the colimit in the second but last line is still a homotopy colimit and thus preserves these weak equivalences. □

References

[DuHoIs04] D. Dugger, S. Hollander, D. Isaksen. *Hypercovers and simplicial presheaves*, Math. Proc. Cambridge Philos. Soc., 136(1):9–51, 2004.

[LuTop] J. Lurie, *Higher topos theory*

[MoVe00] I. Moerdijk, J. Vermeulen, *Relative compactness conditions for toposes*, AMS (2000),
`http://igitur-archive.library.uu.nl/math/2001-0702-142944/1039.pdf`

□