A higher Chern-Weil derivation of AKSZ σ-models

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Abstract

Chern-Weil theory provides for each invariant polynomial on a Lie algebra $\mathfrak{g}$ a map from $\mathfrak{g}$-connections to differential cocycles whose volume holonomy is the the corresponding Chern-Simons theory action functional. We observe that in the context of higher Chern-Weil theory in smooth ∞-groupoids [FSS10, Sch10] this statement generalizes from Lie algebras to L_∞-algebras and further to L_∞-algebroids. It turns out that the symplectic form on a symplectic higher Lie algebroid (for instance a Poisson Lie algebroid or a Courant Lie 2-algebroid) is ∞-Lie-theoretically an invariant polynomial. We show that the higher Chern-Simons action functional associated to this by higher Chern-Weil theory is the action functional of the *AKSZ σ-model* whose target space is the given L_∞-algebroid (for instance the Poisson σ-model or the Courant-σ-model).

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1 Introduction.

The class of topological field theories known as *AKSZ σ-models*[AKSZ95] contains in dimension 3 ordinary Chern-Simons theory (see [Fre] for a comprehensive review) as well as its Lie algebroid generalization (the *Courant σ-model* [Royt06]), and in dimension 2 the Poisson σ-model (see [CaFe00] for a review). It is therefore clear that the AKSZ construction is *some* sort of generalized Chern-Simons theory. Here we demonstrate that this statement is true also in a useful precise sense.

Our discussion proceeds from the observation that the standard Chern-Simons action functional has a systematic origin in Chern-Weil theory (see for instance [GHV] for a classical textbook treatment and [HoSi05] for the refinement to differential cohomology that we need here):

The refined Chern-Weil homomorphism assigns to any invariant polynomial $\langle - \rangle : \mathfrak{g}^{\otimes n} \to \mathbb{R}$ on a Lie algebra $\mathfrak{g}$ of compact type a map that sends $\mathfrak{g}$-connections ∇ on a smooth manifold X to cocycles $[\hat{\mathbf{p}}_{\langle - \rangle}(\nabla)] \in H^{n+1}_{\mathrm{diff}}(X)$ in *ordinary differential cohomology*. These differential cocycles refine the *curvature characteristic class* $[\langle F_\nabla \rangle] \in H^{n+1}_{dR}(X)$ in de Rham cohomology to a fully fledged *line n-bundle with connection*, also known as a *bundle $(n-1)$-gerbe with connection*. And just as an ordinary line bundle (a "line 1-bundle") with connection assigns holonomy to curves, so a line n-bundle with connection assigns holonomy $\mathrm{hol}_{\hat{\mathbf{p}}}(\Sigma)$ to n-dimensional trajectories $\Sigma \to X$. For the special case where $\langle - \rangle$ is the Killing form polynomial and $X = \Sigma$ with $\dim \Sigma = 3$ one finds that this volume holonomy map $\nabla \mapsto \mathrm{hol}_{\hat{\mathbf{p}}_{\langle - \rangle}(\nabla)}(\Sigma)$ is precisely the standard Chern-Simons action functional. Similarly, for $\langle - \rangle$ any higher invariant polynomial this holonomy action functional has as Lagrangian the corresponding higher Chern-Simons form. In summary, this means that Chern-Simons-type action functionals on Lie algebra-valued connections are the images of the refined Chern-Weil homomorphism.

In previous work [Sch10, FSS10] a generalization of the Chern-Weil homomorphism to *higher* ("derived") differential geometry has been established. In this context smooth manifolds are generalized first to orbifolds, then to general Lie groupoids, to Lie 2-groupoids and finally to smooth ∞-groupoids (smooth ∞-stacks), while Lie algebras are generalized to Lie 2-algebras etc., up to L_∞-algebras and more generally to Lie n-algebroids and finally to L_∞-algebroids.

In this context one has for $\mathfrak{a}$ any L_∞-algebroid a natural notion of $\mathfrak{a}$-valued ∞-connections on $\exp(\mathfrak{a})$-principal smooth ∞-bundles (where $\exp(\mathfrak{a})$ is a smooth ∞-groupoid obtained by Lie integration from $\mathfrak{a}$). By analyzing the abstractly defined higher Chern-Weil homomorphism in this context one finds a direct higher analog of the above situation: there is a notion of invariant polynomials $\langle - \rangle$ on an L_∞-algebroid $\mathfrak{a}$ and these induce maps from $\mathfrak{a}$-valued ∞-connections to line n-bundles with connections as before [SSS09, FSS10]. The corresponding class of action functionals we call ∞-*Chern-Simons theory* [FRS].

This construction drastically simplifies when one restricts attention to *trivial* ∞-bundles with (nontrivial) $\mathfrak{a}$-connections. Over a smooth manifold Σ these are simply given by dg-algebra homomorphisms

$$A : \mathrm{W}(\mathfrak{a}) \to \Omega^\bullet(\Sigma)\,,$$

where $\mathrm{W}(\mathfrak{a})$ is the *Weil algebra* of the L_∞-algebroid $\mathfrak{a}$ [SSS09], and $\Omega^\bullet(\Sigma)$ is the de Rham algebra of Σ (which is indeed the Weil algebra of Σ thought of as an L_∞-algebroid concentrated in degree 0). Then for $\langle - \rangle \in \mathrm{W}(\mathfrak{a})$ an invariant polynomial, the corresponding ∞-Chern-Weil homomorphism is presented by a choice of "Chern-Simons element" $\mathrm{cs} \in \mathrm{W}(\mathfrak{a})$, which exhibits the *transgression* of $\langle - \rangle$ to an L_∞-cocycle (the higher analog of a cocycle in Lie algebra cohomology): the dg-morphism A naturally maps the Chern-Simons element cs of A to a differential form $\mathrm{cs}(A) \in \Omega^\bullet(\Sigma)$ and its integral is the corresponding ∞-Chern-Simons action functional $S_{\langle - \rangle}$

$$S_{\langle - \rangle} : A \mapsto \mathrm{hol}_{\hat{\mathbf{p}}_{\langle - \rangle}(A)}(\Sigma) = \int_\Sigma \mathrm{cs}_{\langle - \rangle}(A)\,.$$

Even though trivial ∞-bundles with $\mathfrak{a}$-connections are a very particular subcase of the general ∞-Chern-Weil theory, they are rich enough to contain AKSZ theory. Namely, here we show that a symplectic dg-manifold of grade n – which is the geometrical datum of the target space defining an AKSZ σ-model – is naturally equivalently an L_∞-algebroid $\mathfrak{P}$ endowed with a quadratic and non-degenerate invariant polynomial ω of grade n. Moreover, under this identification the canonical Hamiltonian π on the symplectic target dg-manifold is identified as an L_∞-cocycle on $\mathfrak{P}$. Finally, the invariant polynomial ω is naturally in transgression with the cocycle π via a Chern-Simons element cs_ω that turns out to be the Lagrangian of the AKSZ σ-model:

$$\int_\Sigma L_{\mathrm{AKSZ}}(A) = \int_\Sigma \mathrm{cs}_\omega(A)\,.$$

(An explicit description of L_{AKSZ} is given below in def. 2.13)

In summary this means that we find the following dictionary of concepts:

Chern-Weil theory		**AKSZ theory**
cocycle	π	Hamiltonian
transgression element	cs	Lagrangian
invariant polynomial	ω	symplectic structure

More precisely, we (explain and then) prove here the following theorem:

Theorem 1.1. *For $(\mathfrak{P}, \omega)$ an L_∞-algebroid with a quadratic non-degenerate invariant polynomial, the corresponding ∞-Chern-Weil homomorphism*

$$\nabla \mapsto \mathrm{hol}_{\hat{\mathbf{p}}_\omega(\nabla)}(\Sigma)$$

sends $\mathfrak{P}$-valued ∞-connections ∇ to their corresponding exponentiated AKSZ action:

$$\mathrm{hol}_{\hat{\mathbf{p}}_\omega(\nabla)}(\Sigma) = \int_\Sigma L_{\mathrm{AKSZ}}(\nabla)\,.$$

The local differential form data involved in this statement is at the focus of attention in this article here and contained in proposition 4.2 below. We indicate the global aspects of the construction in 5. The more abstract higher Chern-Weil theoretic interpretation of AKSZ σ-models implies various further constructions and generalizations. We close in 6 by giving an outlook on these.

2 A reminder of AKSZ theory

The first half of the seminal article [AKSZ95] presented some key observations on, what from a modern perspective would be called, symplectic *derived geometry* [Lu09] in its variant of symplectic *dg-geometry* [ToVe05]. In its second half, it describes the role of such symplectic dg-geometry in quantum field theory in general, and σ-model theory in particular.

In this section we briefly review some basics in order to establish the context for our discussion.

2.1 Symplectic dg-geometry

In *higher differential geometry*, smooth manifolds are generalized first to orbifolds – which are special Lie groupoids – then to higher Lie groupoids: smooth ∞-groupoids [Sch10]. Moreover, in *derived differential geometry*, the function algebras are generalized to *smooth ∞-algebras* [Sp08] [Ste01]. All of these ingredients have *presentations* in terms of compound structures in ordinary differential geometry. There is a bit of theory involved in exactly how these presentations model the general abstract theory, but the main statement that we want to discuss here can be described already in a rather simple-minded setup.

Therefore, here we shall be content with the following simple definitions of what might be called *affine smooth graded manifolds* and *affine smooth dg-manifolds*. Despite their simplicity these definitions capture in a precise sense all the relevant structure: namely the *local* smooth structure. Globalizations of these definitions can be obtained, if desired, by general abstract constructions. We give some outlook on this in section 6.

Definition 2.1. The category of *affine smooth $\mathbb{N}$-graded manifolds* – here called *smooth graded manifolds* for short – is the full subcategory

$$\mathrm{SmoothGrMfd} \subset \mathrm{GrAlg}_{\mathbb{R}}^{\mathrm{op}}$$

of the opposite category of $\mathbb{N}$-graded-commutative $\mathbb{R}$-algebras on those isomorphic to Grassmann algebras of the form

$$\wedge^\bullet_{C^\infty(X_0)}\Gamma(V^*)\,,$$

where X_0 is an ordinary smooth manifold, $V \to X_0$ is an $\mathbb{N}$-graded smooth vector bundle over X_0 degreewise of finite rank, and $\Gamma(V^*)$ is the graded $C^\infty(X)$-module of smooth sections of the dual bundle.

For a smooth graded manifold $X \in \mathrm{SmoothGrMfd}$, we write $C^\infty(X) \in \mathrm{cdgAlg}_{\mathbb{R}}$ for its corresponding dg-algebra *of functions*.

Remarks.

- The full subcategory of these objects is equivalent to that of all objects isomorphic to one of this form. We may therefore use both points of view interchangeably.

- Much of the theory works just as well when V is allowed to be $\mathbb{Z}$-graded. This is the case that genuinely corresponds to *derived* (instead of just higher) differential geometry. An important class of examples for this case are BV-BRST complexes which motivate much of the literature. For the purpose of this short note, we shall be content with the $\mathbb{N}$-graded case.

- For an $\mathbb{N}$-graded $C^\infty(X_0)$-module $\Gamma(V^*)$ we have

  $$\wedge^\bullet_{C^\infty}\Gamma(V^*) = C^\infty(X_0) \oplus \Gamma(V_0^*) \oplus \big(\Gamma(V_0^*) \wedge_{C^\infty(X_0)} \Gamma(V_0^*) \oplus \Gamma(V_1^*)\big) \oplus \cdots ,$$

  with the leftmost summand in degree 0, the next one in degree 1, and so on.

- There is a canonical functor

  $$\mathrm{SmoothMfd} \hookrightarrow \mathrm{SmthGrMfd}$$

  which identifies an ordinary smooth manifold X with the smooth graded manifold whose function algebra is the ordinary algebra of smooth functions $C^\infty(X_0) := C^\infty(X)$ regarded as a graded algebra concentrated in degree 0. This functor is full and faithful and hence exhibits a full subcategory.

All the standard notions of differental geometry apply to differential graded geometry. For instance for $X \in \mathrm{SmoothGrMfd}$, there is the graded vector space $\Gamma(TX)$ of vector fields on X, where a vector field is identified with a graded *derivation* $v : C^\infty(X) \to C^\infty(X)$. This is naturally a graded (super) Lie algebra with super Lie bracket the graded commutator of derivations. Notice that for $v \in \Gamma(TX)$ of odd degree we have $[v, v] = v \circ v + v \circ v = 2v^2 : C^\infty(X) \to C^\infty(X)$.

Definition 2.2. The category of (affine, $\mathbb{N}$-graded) *smooth differential-graded manifolds* is the full subcategory

$$\mathrm{SmoothDgMfd} \subset \mathrm{cdgAlg}_{\mathbb{R}}^{\mathrm{op}}$$

of the opposite of differential graded-commutative $\mathbb{R}$-algebras on those objects whose underlying graded algebra comes from SmoothGrMfd.

This is equivalently the category whose objects are pairs (X, v) consisting of a smooth graded manifold $X \in \mathrm{SmoothGrMfd}$ and a grade 1 vector field $v \in \Gamma(TX)$, such that $[v, v] = 0$, and whose morphisms $(X_1, v_1) \to (X_2, v_2)$ are morphisms $f : X_1 \to X_2$ such that $v_1 \circ f^* = f^* \circ v_2$.

Remark 2.3. The dg-algebras appearing here are special in that their degree-0 algebra is naturally not just an $\mathbb{R}$-algebra, but a *smooth algebra* (a "C^∞-ring", see [Ste01] for review and discussion). In a more theoretical account than we want to present here, we would use the corresponding more general notion of *smooth dg-algebras*. For our present purposes, this will only briefly play a role in def. 3.3 below.

Definition 2.4. The *de Rham complex functor*

$$\Omega^\bullet(-) : \mathrm{SmoothGrMfd} \to \mathrm{cdgAlg}_{\mathbb{R}}^{\mathrm{op}}$$

sends a dg-manifold X with $C^\infty(X) \simeq \wedge^\bullet_{C^\infty(X_0)} \Gamma(V^*)$ to the Grassmann algebra over $C^\infty(X_0)$ on the graded $C^\infty(X_0)$-module

$$\Gamma(T^*X) \oplus \Gamma(V^*) \oplus \Gamma(V^*[-1]) \,,$$

where $\Gamma(T^*X)$ denotes the ordinary smooth 1-form fields on X_0 and where $V^*[-1]$ is V^* with the grades *increased* by one. This is equipped with the differential $\mathbf{d}$ defined on generators as follows:

- $\mathbf{d}|_{C^\infty(X_0)} = d_{\mathrm{dR}}$ is the ordinary de Rham differential with values in $\Gamma(T^*X)$;
- $\mathbf{d}|_{\Gamma(V^*)} \to \Gamma(V^*[-1])$ is the degree-shift isomorphism
- and $\mathbf{d}$ vanishes on all remaining generators.

Definition 2.5. Observe that $\Omega^\bullet(-)$ evidently factors through the defining inclusion $\mathrm{SmoothDgMfd} \hookrightarrow \mathrm{cdgAlg}_{\mathbb{R}}$. Write

$$\mathfrak{T}(-) : \mathrm{SmoothGrMfd} \to \mathrm{SmoothDgMfd}$$

for this factorization.

The dg-space $\mathfrak{T}X$ is often called the *shifted tangent bundle* of X and denoted $T[1]X$.

Observation 2.6. *For Σ an ordinary smooth manifold and for X a graded manifold corresponding to a vector bundle $V \to X_0$, there is a natural bijection*

$$\mathrm{SmoothGrMfd}(\mathfrak{T}\Sigma, X) \simeq \Omega^\bullet(\Sigma, V)$$

where on the right we have the set of V-valued smooth differential forms on Σ: tuples consisting of a smooth function $\phi_0 : \Sigma \to X_0$, and for each $n > 1$ an ordinary differential n-form $\phi_n \in \Omega^n(\Sigma, \phi_0^* V_{n-1})$ with values in the pullback bundle of V_{n-1} along ϕ_0.

The standard Cartan calculus of differential geometry generalizes directly to graded smooth manifolds. For instance, given a vector field $v \in \Gamma(TX)$ on $X \in \mathrm{SmoothGrMfd}$, there is the *contraction derivation*

$$\iota_v : \Omega^\bullet(X) \to \Omega^\bullet(X)$$

on the de Rham complex of X, and hence the *Lie derivative*

$$\mathcal{L}_v := [\iota_v, \mathbf{d}] : \Omega^\bullet(X) \to \Omega^\bullet(X)\,.$$

Definition 2.7. For $X \in \mathrm{SmoothGrMfd}$ the *Euler vector field* $\epsilon \in \Gamma(TX)$ is defined over any coordinate patch $U \to X$ to be given by the formula

$$\epsilon|_U := \sum_a \deg(x^a) x^a \frac{\partial}{\partial x^a}\,,$$

where $\{x^a\}$ is a basis of generators and $\deg(x^a)$ the degree of a generator. The *grade* of a homogeneous element α in $\Omega^\bullet(X)$ is the unique natural number $n \in \mathbb{N}$ with

$$\mathcal{L}_\epsilon \alpha = n\alpha\,.$$

Remarks.

- This implies that for x^i an element of grade n on U, the 1-form $\mathbf{d}x^i$ is also of grade n. This is why we speak of *grade* (as in "graded manifold") instead of *degree* here.

- Since coordinate transformations on a graded manifold are grading-preserving, the Euler vector field is indeed well-defined. Note that the degree-0 coordinates do not appear in the Euler vector field.

The existence of ϵ implies the following useful statement (amplified in [Royt99]), which is a trivial variant of what in grade 0 would be the standard Poincaré lemma.

Observation 2.8. *On a graded manifold, every closed differential form ω of positive grade n is exact: the form*

$$\lambda := \frac{1}{n}\iota_\epsilon \omega$$

satisfies

$$\mathbf{d}\lambda = \omega\,.$$

Definition 2.9. A *symplectic dg-manifold* of grade $n \in \mathbb{N}$ is a dg-manifold (X, v) equipped with 2-form $\omega \in \Omega^2(X)$ which is

- non-degenerate;

- closed;

as usual for symplectic forms, and in addition

- of grade n;
- v-invariant: $\mathcal{L}_v\omega = 0$.

In a local chart U with coordinates $\{x^a\}$ we may find functions $\{\omega_{ab} \in C^\infty(U)\}$ such that

$$\omega|_U = \frac{1}{2}\mathbf{d}x^a\,\omega_{ab} \wedge \mathbf{d}x^b\,,$$

where summation of repeated indices is implied. We say that U is a *Darboux chart* for (X, ω) if the ω_{ab} are constant.

Observation 2.10. *The function algebra of a symplectic dg-manifold (X, ω) of grade n is naturally equipped with a Poisson bracket*

$$\{-,-\} : C^\infty(X) \otimes C^\infty(X) \to C^\infty(X)$$

which decreases grade by n. On a local coordinate patch $\{x^a\}$ this is given by

$$\{f, g\} = \frac{f\reflectbox{∂}}{x^a\reflectbox{∂}}\omega^{ab}\frac{\partial g}{\partial x^b}\,,$$

where $\{\omega^{ab}\}$ is the inverse matrix to $\{\omega_{ab}\}$, and where the graded differentiation in the left factor is to be taken from the right, as indicated.

Definition 2.11. For $\pi \in C^\infty(X)$ and $v \in \Gamma(TX)$, we say that π *is a Hamiltonian for* v, or equivalently, that v *is the Hamiltonian vector field of* π if

$$\mathbf{d}\pi = \iota_v\omega\,.$$

Note that the convention $(-1)^{n+1}\mathbf{d}\pi = \iota_v\omega$ is also frequently used for defining Hamiltonians in the context of graded geometry.

Remark 2.12. In a local coordinate chart $\{x^a\}$ the defining equation $\mathbf{d}\pi = \iota_v\omega$ becomes

$$\mathbf{d}x^a\frac{\partial\pi}{\partial x^a} = \omega_{ab}v^a \wedge \mathbf{d}x^b = \omega_{ab}\mathbf{d}x^a \wedge v^b\,,$$

implying that

$$\omega_{ab}v^b = \frac{\partial\pi}{\partial x^a}\,.$$

2.2 AKSZ σ-Models

We now consider, in definition 2.13 below, for any symplectic dg-manifold (X, ω) a functional S_{AKSZ} on spaces of maps $\mathfrak{T}\Sigma \to X$ of smooth graded manifolds. While only this precise definition is referred to in the remainder of the article, we begin by indicating informally the original motivation of S_{AKSZ}. The reader uncomfortable with these somewhat vague considerations can take note of def. 2.13 and then skip to the next section.

Generally, a *σ-model field theory* is, roughly, one

1. whose fields over a space Σ are maps $\phi : \Sigma \to X$ to some space X;
2. whose action functional is, apart from a kinetic term, the transgression of some kind of cocycle on X to the mapping space $\mathrm{Map}(\Sigma, X)$.

Here the terms "space", "maps" and "cocycles" are to be made precise in a suitable context. One says that Σ is the *worldvolume*, X is the *target space* and the cocycle is the *background gauge field*.

For instance, an ordinary charged particle (such as an electron) is described by a σ-model where $\Sigma = (0, t) \subset \mathbb{R}$ is the abstract *worldline*, where X is a (pseudo-)Riemannian smooth manifold (for instance our spacetime), and where the background cocycle is a line bundle with connection on X (a degree-2 cocycle in ordinary differential cohomology of X, representing a background *electromagnetic field*). Up to a kinetic term, the action functional is the holonomy of the connection over a given curve $\phi : \Sigma \to X$. A textbook discussion of these standard kinds of σ-models is, for instance, in [DM99].

The σ-models which we consider here are *higher* generalizations of this example, where the background gauge field is a cocycle of higher degree (a higher bundle with connection) and where the worldvolume is accordingly higher dimensional. In addition, X is allowed to be not just a manifold, but an approximation to a *higher orbifold* (a smooth ∞-groupoid).

More precisely, here we take the category of spaces to be SmoothDgMfd from def. 2.2. We take target space to be a symplectic dg-manifold (X, ω) and the worldvolume to be the shifted tangent bundle $\mathfrak{T}\Sigma$ of a compact smooth manifold Σ. Following [AKSZ95], one may imagine that we can form a smooth $\mathbb{Z}$-graded mapping space $\mathrm{Maps}(\mathfrak{T}\Sigma, X)$ of smooth graded manifolds. On this space the canonical vector fields v_Σ and v_X naturally have commuting actions from the left and from the right, respectively, so that their sum $v_\Sigma + v_X$ equips $\mathrm{Maps}(\mathfrak{T}\Sigma, X)$ itself with the structure of a differential graded smooth manifold.

Next we take the "cocycle" on X (to be made precise in the next section) to be the Hamiltonian π (def. 2.11) of v_X with respect to the symplectic structure ω, according to def. 2.9. One wants to assume that there is a kind of Riemannian structure on $\mathfrak{T}\Sigma$ that allows to form the transgression

$$\int_{\mathfrak{T}\Sigma} \mathrm{ev}^* \omega := p_! \mathrm{ev}^* \omega$$

by pull-push through the canonical correspondence

$$\mathrm{Maps}(\mathfrak{T}\Sigma, X) \xleftarrow{p} \mathrm{Maps}(\mathfrak{T}\Sigma, X) \times \mathfrak{T}\Sigma \xrightarrow{\mathrm{ev}} X \ .$$

When one succeeds in making this precise, one expects to find that $\int_{\mathfrak{T}\Sigma} \mathrm{ev}^* \omega$ is in turn a symplectic structure on the mapping space.

This implies that the vector field $v_\Sigma + v_X$ on mapping space has a Hamiltonian

$$\mathbf{S} \in C^\infty(\mathrm{Maps}(\mathfrak{T}\Sigma, X)) \,, \quad \text{s.t.} \;\; \mathbf{dS} = \iota_{v_\Sigma + v_x} \int_{\mathfrak{T}\Sigma} \mathrm{ev}^* \omega \,.$$

The grade-0 component

$$S_{\mathrm{AKSZ}} := \mathbf{S}|_{\mathrm{Maps}(\mathfrak{T}\Sigma, X)_0}$$

constitutes a functional on the space of morphisms of graded manifolds $\phi : \mathfrak{T}\Sigma \to X$. This is the *AKSZ action functional* defining the AKSZ σ-model with target space X and background field/cocycle ω.

In [AKSZ95], this procedure is indicated only somewhat vaguely. The focus of attention there is on a discussion, from this perspective, of the action functionals of the 2-dimensional σ-models called the *A-model* and the *B-model*. In [Royt06] a more detailed discussion of the general construction is given, including an explicit formula for $\mathbf{S}$, and hence for S_{AKSZ}. That formula is the following:

Definition 2.13. For (X, ω) a symplectic dg-manifold of grade n with global Darboux coordinates $\{x^a\}$, Σ a smooth compact manifold of dimension $(n+1)$ and $k \in \mathbb{R}$, the *AKSZ action functional*

$$S_{\mathrm{AKSZ}} : \mathrm{SmoothGrMfd}(\mathfrak{T}\Sigma, X) \to \mathbb{R}$$

is

$$S_{\mathrm{AKSZ}} : \phi \mapsto \int_{\mathfrak{T}\Sigma} \left(\frac{1}{2} \omega_{ab} \phi^a \wedge d_{\mathrm{dR}} \phi^b - \phi^* \pi \right) ,$$

where π is the Hamiltonian for v_X with respect to ω and where on the right we are interpreting fields as forms on Σ according to prop. 2.6.

This formula hence defines an infinite class of σ-models depending on the target space structure (X, ω). (One can also consider arbitrary relative factors between the first and the second term, but below we shall find that the above choice is singled out). In [AKSZ95], it was already noticed that ordinary Chern-Simons theory is a special case of this for ω of grade 2, as is the Poisson σ-model for ω of grade 1 (and hence, as shown there, also the A-model and the B-model). The main example in [Royt06] spells out the general case for ω of grade 2, which is called the *Courant σ-model* there. (We review and re-derive all these examples in detail in 4.1 below.)

One nice aspect of this construction is that it follows immediately that the full Hamiltonian $\mathbf{S}$ on the mapping space satisfies $\{\mathbf{S}, \mathbf{S}\} = 0$. Moreover, using the standard formula for the internal hom of chain complexes, one finds that the cohomology of $(\mathrm{Maps}(\mathfrak{T}\Sigma, X), v_\Sigma + v_X)$ in degree 0 is the space of functions on those fields that satisfy the Euler-Lagrange equations of S_{AKSZ}. Taken together, these facts imply that $\mathbf{S}$ is a solution of the "master equation" of a BV-BRST complex for the quantum field theory defined by S_{AKSZ}. This is a crucial ingredient for the quantization of the model, and this is what the AKSZ construction is mostly used for in the literature (for instance [CaFe00]).

Here we want to focus on another nice aspect of the AKSZ-construction: it hints at a deeper reason for *why* the σ-models of this type are special. It is indeed one of the very few proposals for what a general abstract mechanism might be that picks out among the vast space of all possible local action functionals those that seem to be of relevance "in nature".

We now proceed to show that the class of action functionals S_{AKSZ} are precisely those that higher Chern-Weil theory canonically associates to target data (X, ω). Since higher Chern-Weil theory in turn is canonically given on very general abstract grounds [Sch10], this in a sense amounts to a derivation of S_{AKSZ} from "first principles", and it shows that a wealth of very general theory applies to these systems. More on this will be discussed elsewhere, some indication are in 5. Here we shall focus on a concrete computation exhibiting S_{AKSZ} as the image of the higher Chern-Weil homomorphism.

3 Chern-Weil theory on L_∞-algebroids

We now discuss the ∞-Lie theoretic concepts in terms of which we shall re-express the AKSZ σ-model below in 4.

3.1 General L_∞-algebroids

We survey some basics of ∞-Lie theory that we need later on. The explicit L_∞-algebraic constructions are from [SSS09], a more encompassing discussion is in [Sch10].

The following definition essentially repeats def. 2.2 with different terminology. While this may look like a redundancy, it is useful to instead regard it as the beginning of a useful dictionary between higher Lie theory and dg-geometry. The examples to follow will illustrate this.

Definition 3.1. The category of *L_∞-algebroids* is equivalent to that of smooth dg-manifolds from def. 2.2:

$$L_\infty\mathrm{Algd} \simeq \mathrm{SmoothDgMfd} \hookrightarrow \mathrm{cdgAlg}_{\mathbb{R}}^{\mathrm{op}}\,.$$

For $\mathfrak{a} \in L_\infty\mathrm{Algd}$ we write $\mathrm{CE}(\mathfrak{a}) \in \mathrm{cdgAlg}_{\mathbb{R}}$ for the corresponding dg-algebra and call it the *Chevalley-Eilenberg algebra* of $\mathfrak{a}$.

If the graded algebra underlying $\mathrm{CE}(\mathfrak{a})$ has generators of grade at most n, we say that $\mathfrak{a}$ is a *Lie n-algebroid*.

Examples.

- Any (degreewise finite-dimensional) L_∞-algebra (and so, in particular, any Lie algebra) $\mathfrak{g}$ can be seen as a (canonically pointed) L_∞-algebroid $b\mathfrak{g}$ over the point: for $D : \vee^\bullet\mathfrak{g} \to \vee^\bullet\mathfrak{g}$ the nilpotent derivation on the free graded coalgebra over $\mathfrak{g}$ which defines the k-ary brackts on $\mathfrak{g}$ by

  $$[-,-,\cdots,-]_k := D|_{\vee^k\mathfrak{g}} : \vee^k\mathfrak{g} \to \mathfrak{g}\,,$$

  we have

  $$\mathrm{CE}(b\mathfrak{g}) := (\wedge^\bullet\mathfrak{g}^*, d := D^*)\,.$$

One directly finds that L_∞-algebroid morphims $b\mathfrak{g} \to b\mathfrak{h}$ are precisely L_∞-algebra morphism $\mathfrak{g} \to \mathfrak{h}$. This means that there is a full and faithful inclusion

$$b : L_\infty \mathrm{Alg} \hookrightarrow L_\infty \mathrm{Algd}$$

of the traditional category of L_∞-algebras into that of L_∞-algebroids.

We refer to $b\mathfrak{g}$ as the *delooping* of $\mathfrak{g}$. This notation is the infinitesimal analog of the notation $\mathbf{B}G$ for the one-object Lie groupoid corresponding to a Lie group G: the loop space object $\Omega\mathbf{B}G$ is equivalent to G, hence the name "delooping" given to $\mathbf{B}G$.

For $\mathfrak{g}$ a Lie algebra, the algebra $\mathrm{CE}(b\mathfrak{g})$ is the ordinary Chevalley-Eilenberg algebra of $\mathfrak{g}$.

- For $n \in \mathbb{N}$ the delooping of the *line Lie n-algebra* is the L_∞-algebroid $b^n\mathbb{R}$ defined by the fact that $\mathrm{CE}(b^n\mathbb{R})$ is generated over $\mathbb{R}$ from a single generator in degree n with vanishing differential.
- For X a smooth manifold, the *tangent Lie algebroid* $\mathfrak{a} = \mathfrak{T}X$ is defined by $\mathrm{CE}(\mathfrak{T}X) = (\Omega^\bullet(X), d_{\mathrm{dR}})$;
- For $(X, \{-,-\})$ a Poisson manifold, the corresponding *Poisson Lie algebroid* $\mathfrak{P}(X)$ is defined by

$$\mathrm{CE}(\mathfrak{P}(X)) = (\wedge^\bullet_{C^\infty(X)}\Gamma(TX), \{\pi, -\}),$$

where $\pi \in \wedge^2_{C^\infty(X)}\Gamma(TX)$ is the Poisson tensor and the bracket means the canonical extension to the tangent bundle: the Schouten bracket.

Remark 3.2. For $\mathfrak{a}$ an L_∞-algebroid and $\{x^i\}$ local coordinates on the corresponding graded manifold, the vector field v corresponding to the Chevellay-Eilenberg differential $d_{\mathrm{CE}(\mathfrak{a})}$ is

$$v|_U = v^i \frac{\partial}{\partial x^i},$$

with $v^i := d_{\mathrm{CE}(\mathfrak{a})} x^i$.

Definition 3.3. For $\mathfrak{a}$ an L_∞-algebroid, its *Weil algebra* is that representative of the free *smooth dg-algebra*, remark 2.3, on the underlying word-length-1 complex of $\mathfrak{a}$ that makes the canonical projection of complexes

$$i^* : W(\mathfrak{a}) \to \mathrm{CE}(\mathfrak{a})$$

into a dg-algebra homomorphism.

Proposition 3.4. *Explicitly, the Weil algebra* $\mathrm{W}(\mathfrak{a})$ *has*

- *as underlying graded algebra the de Rham complex* $\Omega^\bullet(\mathfrak{a})$ *from def. 2.4, applied to the corresponding graded manifold; i.e., the differential graded manifold corresponding to* $\mathrm{W}(\mathfrak{a})$ *is the tangent Lie ∞-algebroid* $\mathfrak{T}\mathfrak{a}$. *This can be equivalently written as*

$$\mathrm{W}(\mathfrak{a}) = \mathrm{CE}(\mathfrak{T}\mathfrak{a}).$$

- *as differential the sum*
$$d_{\mathrm{W}(\mathfrak{a})} = \mathbf{d} + \mathcal{L}_v \,,$$
where $\mathbf{d}$ *is the differential from def. 2.4 and where* $\mathcal{L}_v$ *is the Lie derivative along the vector field* v *corresponding to the Chevalley-Eilenberg differential.*

Remark. Therefore the Weil algebra $\mathrm{W}(\mathfrak{a})$ is a *twisted* de Rham complex on the graded smooth manifold corresponding to $\mathfrak{a}$, where the twist is dictated by the characterizing morphism i^* from def. 3.3. In the abstract theory indicated below in 5 this makes $\mathrm{W}(\mathfrak{a})$ part of the construction of a certain homotopical resolution of the Lie integration of $\mathfrak{a}$. This is the deeper reason for the role played by the Weil in higher Lie theory. But for the present purpose the above explicit definition is sufficient.
Examples.

- For $\mathfrak{g}$ a Lie algebra the definition of $\mathrm{W}(b\mathfrak{g})$ reduces the ordinary definition of the Weil algebra of the Lie algebra $\mathfrak{g}$.
- For $\mathfrak{a} = \Sigma$ an ordinary smooth manifold, $\mathrm{W}(\Sigma) = \Omega^\bullet(\Sigma)$.
- For G a Lie group with Lie algebra $\mathfrak{g}$ acting on a manifold Σ, write $\Sigma//\mathfrak{g}$ for the corresponding action Lie algebroid. Then $\mathrm{W}(\Sigma//\mathfrak{g})$ is the Cartan-Weil model for G-equivariant de Rham cohomology on Σ.
- For $\mathfrak{a} = b^n\mathbb{R}$ the delooping of the line Lie n-algebra, we have that $\mathrm{W}(b^n\mathbb{R})$ is the free dg-algebra on a single generator c in degree n: this is the graded algebra on two generators c and γ, with c in degree n and γ in degree $n+1$, equipped with a differential defined by $d_{\mathrm{W}(b^n\mathbb{R})} : c \mapsto \gamma$.

3.2 Cocycles, invariant polynomials and Chern-Simons elements

The key technical notion for our main theorem is that of Chern-Simons elements witnessing trangression between invariant polynomials and L_∞-algebroid cocycles, which is def. 3.7 below. We show in Section 5 how these notions are related to the ∞-Chern-Weil homomorphism for ∞-bundles with connections.

Definition 3.5. Let $\mathfrak{a}$ be an L_∞-algebroid. A *cocycle* on $\mathfrak{a}$ is an element $\mu \in \mathrm{CE}(\mathfrak{a})$ which is closed.

Definition 3.6. An *invariant polynomial* on $\mathfrak{a}$ is an element $\langle - \rangle$ in $\mathrm{W}(\mathfrak{a})$ which is

1. closed: $d_{\mathrm{W}(\mathfrak{a})}\langle - \rangle = 0$;
2. horizontal: an element of the subalgebra generated by the shifted elements in the Weil algebra.

Definition 3.7. For $\langle - \rangle \in \mathrm{W}(\mathfrak{a})$ an invariant polynomial on an L_∞-algebroid $\mathfrak{a}$, we say a cocycle $\mu \in \mathrm{CE}(\mathfrak{a})$ is *in transgression* with $\langle - \rangle$ if there exists an element cs in $\mathrm{W}(\mathfrak{a})$ such that

1. $d_{\mathrm{W}(\mathfrak{a})}\mathrm{cs} = \langle - \rangle$;
2. $i^*\mathrm{cs} = \mu$.

We say that cs is a *transgression element* or *Chern-Simons element* witnessing this transgression.

As we noticed above, if we look at an ordinary smooth manifold Σ as an L_∞-algebroid, then the Weil algebra of Σ is the de Rham algebra $\Omega^\bullet(\Sigma)$. This motivates the following definition.

Definition 3.8. For $\mathfrak{a}$ an L_∞-algebroid and Σ a smooth manifold, we say a morphism

$$A : \mathrm{W}(\mathfrak{a}) \to \Omega^\bullet(\Sigma)$$

is a degree 1 $\mathfrak{a}$*-valued differential form* on Σ.

Remark 3.9. The name "degree 1 $\mathfrak{a}$-valued differential forms" given to dgca morphisms $\mathrm{W}(\mathfrak{a}) \to \Omega^\bullet(\Sigma)$ has the following origin: if $\mathfrak{g}$ is a Lie algebra, then the Weil algebra $\mathrm{W}(b\mathfrak{g})$ is the free differential graded commutative algebra generated by a shifted copy $\mathfrak{g}^*[-1]$ of the linear dual of $\mathfrak{g}$. Hence a dgca morphism $\mathrm{W}(b\mathfrak{g}) \to \Omega^\bullet(\Sigma)$ is precisely the datum of a morphism of graded vector spaces $\mathfrak{g}^*[-1] \to \Omega^\bullet(\Sigma)$, i.e., an element of $\Omega^1(\Sigma; \mathfrak{g})$.

We say that an $\mathfrak{a}$-valued differential form A is *flat* if the morphism $A : \mathrm{W}(\mathfrak{a}) \to \Omega^\bullet(\Sigma)$ factors through $i^* : \mathrm{W}(\mathfrak{a}) \to \mathrm{CE}(\mathfrak{a})$. The *curvature* of A is the induced morphism of graded vector spaces given by the composite

$$\Omega^\bullet(\Sigma) \xleftarrow{A} \mathrm{W}(\mathfrak{a}) \longleftarrow \wedge^1 V[1] : F_A\,,$$

where the morphism on the right is the inclusion of the linear subspace of the shifted generators into the Weil algebra. A is flat precisely if $F_A = 0$.

Remark 3.10. For $\{x^a\}$ a coordinate chart of $\mathfrak{a}$ and

$$A^a := A(x^a) \in \Omega^{\deg(x^a)}(\Sigma)$$

the differential form assigned to the generator x^a by the $\mathfrak{a}$-valued form A, we have the curvature components

$$F_A^a = A(\mathbf{d}x^a) \in \Omega^{\deg(x^a)+1}(\Sigma)\,.$$

Since $d_{\mathrm{W}} = d_{\mathrm{CE}} + \mathbf{d}$, this can be equivalently written as

$$F_A^a = A(d_{\mathrm{W}}x^a - d_{\mathrm{CE}}x^a)\,,$$

so the *curvature* of A precisely measures the "lack of flatness" of A. Also notice that, since A is required to be a dg-algebra homomorphism, we have

$$A(d_{\mathrm{W}(\mathfrak{a})}x^a) = d_{\mathrm{dR}}A^a\,,$$

so that

$$A(d_{\mathrm{CE}(\mathfrak{a})}x^a) = d_{\mathrm{dR}}A^a - F^a_A\,.$$

Assume now A is a degree 1 $\mathfrak{a}$-valued diffrential form on the smooth manifold Σ, and that cs is a Chern-Simons element transgressing an invariant polynomial $\langle - \rangle$ of $\mathfrak{a}$ to some cocycle μ. We can then consider the image $A(\mathrm{cs})$ of the Chern-Simons element cs in $\Omega^\bullet(\Sigma)$. Equivalently, we can look at cs as a map from degree 1 $\mathfrak{a}$-valued differential forms on Σ to ordinary (real valued) differential forms on Σ.

Definition 3.11. In the notations above, we write

$$\Omega^\bullet(\Sigma) \xleftarrow{A} \mathrm{W}(\mathfrak{a}) \xleftarrow{\mathrm{cs}} \mathrm{W}(b^{n+1}\mathbb{R}) \;\; : \mathrm{cs}(A)$$

for the differential form associated by the Chern-Simons element cs to the degree 1 $\mathfrak{a}$-valued differential form A, and call this the *Chern-Simons differential form* associated with A.

Similarly, for $\langle - \rangle$ an invariant polynomial on $\mathfrak{a}$, we write $\langle F_A \rangle$ for the evaluation

$$\Omega^\bullet_{\mathrm{closed}}(\Sigma) \xleftarrow{A} \mathrm{W}(\mathfrak{a}) \xleftarrow{\langle - \rangle} \mathrm{inv}(b^{n+1}\mathbb{R}) \;\; : \langle F_A \rangle$$

3.3 Symplectic Lie n-algebroids

We now consider L_∞-algebroids that are equipped with certain natural extra structure (symplectic structure) and show how this extra structure canonically induces an invariant polynomial and hence by observation 5.15 a σ-model field theory. In the next section we demonstrate that the field theories arising this way are precisely the AKSZ σ-models.

Definition 3.12. A *symplectic Lie n-algebroid* $(\mathfrak{P}, \omega)$ is a Lie n-algebroid $\mathfrak{P}$ equipped with a quadratic non-degenerate invariant polynomial $\omega \in W(\mathfrak{P})$ of degree $n+2$.

This means that

- on each chart $U \to X$ of the base manifold X of $\mathfrak{P}$, there is a basis $\{x^a\}$ for $\mathrm{CE}(\mathfrak{a}|_U)$ such that

  $$\omega = \frac{1}{2}\mathbf{d}x^a\,\omega_{ab} \wedge \mathbf{d}x^b$$

  with $\{\omega_{ab} \in \mathbb{R} \hookrightarrow C^\infty(X)\}$ and $\deg(x^a) + \deg(x^b) = n$;
- the coefficient matrix $\{\omega_{ab}\}$ has an inverse;

- we have

$$d_{\mathrm{W}(\mathfrak{P})}\omega = d_{\mathrm{CE}(\mathfrak{P})}\omega + \mathbf{d}\omega = 0\,.$$

The following observation essentially goes back to [Sev01] and [Royt99].

Proposition 3.13. *There is a full and faithful embedding of symplectic dg-manifolds of grade n into symplectic Lie n-algebroids.*

Proof. The dg-manifold itself is identified with an L_∞-algebroid by def. 3.1. For $\omega \in \Omega^2(X)$ a symplectic form, the conditions $\mathbf{d}\omega = 0$ and $\mathcal{L}_v\omega = 0$ imply $(\mathbf{d} + \mathcal{L}v)\omega = 0$ and hence that under the identification $\Omega^\bullet(X) \simeq \mathrm{W}(\mathfrak{a})$ this is an invariant polynomial on $\mathfrak{a}$.

It remains to observe that the L_∞-algebroid $\mathfrak{a}$ is in fact a Lie n-algebroid. This is implied by the fact that ω is of grade n and non-degenerate: the former condition implies that it has no components in elements of grade $> n$ and the latter then implies that all such elements vanish. □

The following characterization may be taken as a definition of Poisson Lie algebroids and Courant Lie 2-algebroids.

Proposition 3.14. *Symplectic Lie n-algebroids are equivalently:*

- *for $n = 0$: ordinary symplectic manifolds;*
- *for $n = 1$: Poisson Lie algebroids;*
- *for $n = 2$: Courant Lie 2-algebroids.*

See [Royt99, Sev01] for more discussion.

Proposition 3.15. *Let $(\mathfrak{P}, \omega)$ be a symplectic Lie n-algebroid for positive n in the image of the embedding of proposition 3.13. Then it carries the canonical L_∞-algebroid cocycle*

$$\pi := \frac{1}{n+1}\iota_\epsilon\iota_v\omega \in \mathrm{CE}(\mathfrak{P})$$

which moreover is the Hamiltonian, according to definition 2.11, of $d_{\mathrm{CE}(\mathfrak{P})}$.

Proof. Since $\mathbf{d}\omega = \mathcal{L}_v\omega = 0$, we have

$$\begin{aligned}
\mathbf{d}\iota_\epsilon\iota_v\omega &= \mathbf{d}\iota_v\iota_\epsilon\omega \\
&= (\iota_v\mathbf{d} - \mathcal{L}_v)\iota_\epsilon\omega \\
&= \iota_v\mathcal{L}_\epsilon\omega - [\mathcal{L}_v, \iota_\epsilon]\omega \\
&= n\iota_v\omega - \iota_{[v,\epsilon]}\omega \\
&= (n+1)\iota_v\omega,
\end{aligned}$$

where Cartan's formula $[\mathcal{L}_v, \iota_\epsilon] = \iota_{[v,\epsilon]}$ and the identity $[v, \epsilon] = -[\epsilon, v] = -v$ have been used. Therefore $\pi := \frac{1}{n+1}\iota_\epsilon\iota_v\omega$ satisfies the defining equation $\mathbf{d}\pi = \iota_v\omega$ from definition 2.11. □

Remark 3.16. On a local chart with coordinates $\{x^a\}$ we have

$$\pi\big|_U = \frac{1}{n+1}\omega_{ab}\ \deg(x^a)x^a\ \wedge v^b\,.$$

Our central observation now is the following.

Proposition 3.17. *The cocycle $\frac{1}{n}\pi$ from prop. 3.15 is in transgression with the invariant polynomial ω. A Chern-Simons element witnessing the transgression according to def. 3.7 is*

$$\mathrm{cs} = \frac{1}{n}\left(\iota_\epsilon\omega + \pi\right)\,.$$

Proof. It is clear that $i^*\mathrm{cs} = \frac{1}{n}\pi$. So it remains to check that $d_{\mathrm{W}(\mathfrak{P})}\mathrm{cs} = \omega$. As in the proof of proposition 3.15, we use $\mathbf{d}\omega = \mathcal{L}_v\omega = 0$ and Cartan's identity $[\mathcal{L}_v, \iota_\epsilon] = \iota_{[v,\epsilon]} = -\iota_v$. By these, the first summand in $d_{\mathrm{W}(\mathfrak{P})}(\iota_\epsilon\omega + \pi)$ is

$$\begin{aligned} d_{\mathrm{W}(\mathfrak{P})}\iota_\epsilon\omega &= (\mathbf{d} + \mathcal{L}_v)\iota_\epsilon\omega \\ &= [\mathbf{d} + \mathcal{L}_v, \iota_\epsilon]\omega \\ &= n\omega - \iota_v\omega \\ &= n\omega - \mathbf{d}\pi \end{aligned}\,.$$

The second summand is simply

$$d_{\mathrm{W}(\mathfrak{P})}\pi = \mathbf{d}\pi$$

since π is a cocycle. □

Remark 3.18. In a coordinate patch $\{x^a\}$ the Chern-Simons element is

$$\mathrm{cs}\big|_U = \frac{1}{n}\left(\omega_{ab}\deg(x^a)x^a\ \wedge \mathbf{d}x^b + \pi\right)\,.$$

In this formula one can substitute $\mathbf{d} = d_{\mathrm{W}} - d_{\mathrm{CE}}$, and this kind of substitution will be crucial for the proof our main statement in proposition 4.2 below. Since $d_{\mathrm{CE}}x^i = v^i$ and using remark 3.16 we find

$$\sum_a \omega_{ab}\deg(x^a)x^a \wedge d_{\mathrm{CE}}x^b = (n+1)\pi\,,$$

and hence

$$\mathrm{cs}\big|_U = \frac{1}{n}\left(\deg(x^a)\,\omega_{ab}x^a \wedge d_{\mathrm{W}(\mathfrak{P})}x^b - n\pi\right)\,.$$

In the following section we show that this transgression element cs *is* the AKSZ-Lagrangian.

4 The AKSZ action as a Chern-Simons functional

We now show how an L_∞-algebroid $\mathfrak{a}$ endowed with a triple $(\pi, \mathrm{cs}, \omega)$ consisting of a Chern-Simons element transgressing an invariant polynomial ω to a cocycle π defines an AKSZ-type σ-model action. The starting point is to take as target space the tangent Lie ∞-algebroid $\mathfrak{T}\mathfrak{a}$, i.e., to consider as *space of fields* of the theory the space of maps $\mathrm{Maps}(\mathfrak{T}\Sigma, \mathfrak{T}\mathfrak{a})$ from the worldsheet Σ to $\mathfrak{T}\mathfrak{a}$. Dually, this is the space of morphisms of dgcas from $\mathrm{W}(\mathfrak{a})$ to $\Omega^\bullet(\Sigma)$, i.e., the space of degree 1 $\mathfrak{a}$-valued differential forms on Σ from definition 3.8.

Remark 4.1. As we noticed in the introduction, in the context of the AKSZ σ-model a degree 1 $\mathfrak{a}$-valued differential form on Σ should be thought of as the datum of a (notrivial) $\mathfrak{a}$-valued connection on a trivial principal ∞-bundle on Σ. We will come back to this point of view in Section 5.

Now that we have defined the space of fields, we have to define the action. We have seen in definition 3.11 that a degree 1 $\mathfrak{a}$-valued differential form A on Σ maps the Chern-Simons element $\mathrm{cs} \in \mathrm{W}(\mathfrak{a})$ to a differential form $\mathrm{cs}(A)$ on Σ. Integrating this differential form on Σ will therefore give an AKSZ-type action which, as we will see in Section 5, is naturally interpreted as an higher Chern-Simons action functional:

$$\mathrm{Maps}(\mathfrak{T}\Sigma, \mathfrak{T}\mathfrak{a}) \to \mathbb{R}$$
$$A \mapsto \int_\Sigma \mathrm{cs}(A).$$

Theorem 1.1 then reduces to showing that, when $\{\mathfrak{a}, (\pi, \mathrm{cs}, \omega)\}$ is the set of L_∞-algebroid data arising from a symplectic Lie n-algebroid $(\mathfrak{P}, \omega)$, the AKSZ-type action dscribed above is precisely the AKSZ action for $(\mathfrak{P}, \omega)$. More precisely, this is stated as follows.

Proposition 4.2. *For $(\mathfrak{P}, \omega)$ a symplectic Lie n-algebroid coming by proposition 3.13 from a symplectic dg-manifold of positive grade n with global Darboux chart, the action functional induced by the canonical Chern-Simons element*

$$\mathrm{cs} \in \mathrm{W}(\mathfrak{P})$$

from proposition 3.17 is the AKSZ action from definition 2.13:

$$\int_\Sigma \mathrm{cs} = \int_\Sigma L_{\mathrm{AKSZ}}\,.$$

In fact the two Lagrangians differ at most by an exact term

$$\mathrm{cs} \sim L_{\mathrm{AKSZ}}\,.$$

Proof. We have seen in remark 3.18 that in Darboux coordinates $\{x^a\}$ where

$$\omega = \frac{1}{2}\omega_{ab}\mathbf{d}x^a \wedge \mathbf{d}x^b$$

the Chern-Simons element from proposition 3.17 is given by

$$\mathrm{cs} = \frac{1}{n}\left(\deg(x^a)\,\omega_{ab}x^a \wedge d_{\mathrm{W}(\mathfrak{P})}x^b - n\pi\right) .$$

This means that for Σ an $(n+1)$-dimensional manifold and

$$\Omega^\bullet(\Sigma) \leftarrow \mathrm{W}(\mathfrak{P}) : \phi$$

a (degree 1) $\mathfrak{P}$-valued differential form on Σ we have

$$\int_\Sigma \mathrm{cs}(\phi) = \frac{1}{n}\int_\Sigma \left(\sum_{a,b}\deg(x^a)\,\omega_{ab}\phi^a \wedge d_{\mathrm{dR}}\phi^b - n\pi(\phi)\right) \quad ,$$

where we used $\phi(d_{\mathrm{W}(\mathfrak{P})}x^b) = d_{\mathrm{dR}}\phi^b$, as in remark 3.10. Here the asymmetry in the coefficients of the first term is only apparent. Using integration by parts on a closed Σ we have

$$\begin{aligned}
\int_\Sigma \sum_{a,b}\deg(x^a)\,\omega_{ab}\phi^a \wedge d_{\mathrm{dR}}\phi^b &= \int_\Sigma \sum_{a,b}(-1)^{1+\deg(x^a)}\deg(x^a)\,\omega_{ab}(d_{\mathrm{dR}}\phi^a)\wedge\phi^b \\
&= \int_\Sigma \sum_{a,b}(-1)^{(1+\deg(x^a))(1+\deg(x^b))}\deg(x^a)\,\omega_{ab}\phi^b \wedge (d_{\mathrm{dR}}\phi^a)\,, \\
&= \int_\Sigma \sum_{a,b}\deg(x^b)\,\omega_{ab}\phi^a \wedge (d_{\mathrm{dR}}\phi^b)
\end{aligned}$$

where in the last step we switched the indices on ω and used that $\omega_{ab} = (-1)^{(1+\deg(x^a))(1+\deg(x^b))}\omega_{ba}$. Therefore

$$\begin{aligned}
\int_\Sigma \sum_{a,b}\deg(x^a)\,\omega_{ab}\phi^a \wedge d_{\mathrm{dR}}\phi^b &= \frac{1}{2}\int_\Sigma \sum_{a,b}\deg(x^a)\,\omega_{ab}\phi^a \wedge d_{\mathrm{dR}}\phi^b + \frac{1}{2}\int_\Sigma \sum_{a,b}\deg(x^b)\,\omega_{ab}\phi^a \wedge d_{\mathrm{dR}}\phi^b \\
&= \frac{n}{2}\int_\Sigma \omega_{ab}\phi^a \wedge d_{\mathrm{dR}}\phi^b\,.
\end{aligned}$$

Using this in the above expression for the action yields

$$\int_\Sigma \mathrm{cs}(\phi) = \int_\Sigma \left(\frac{1}{2}\omega_{ab}\phi^a \wedge d_{\mathrm{dR}}\phi^b - \pi(\phi)\right) ,$$

which is the formula for the action functional from definition 2.13. □

4.1 Examples

We unwind the general statement of proposition 4.2 and its ingredients in the central examples of interest, from proposition 3.14: the ordinary Chern-Simons action functional, the Poisson σ-model Lagrangian, and the Courant

σ-model Lagrangian. (The ordinary Chern-Simons model is the special case of the Courant σ-model for $\mathfrak{P}$ having as base manifold the point. But since it is the archetype of all models considered here, it deserves its own discussion.)

By the very content of proposition 4.2 there are no surprises here and the following essentially amounts to a review of the standard formulas for these examples. But it may be helpful to see our general ∞-Lie theoretic derivation of these formulas spelled out in concrete cases, if only to carefully track the various signs and prefactors.

4.1.1 Ordinary Chern-Simons theory

Let $\mathfrak{P} = b\mathfrak{g}$ be a semisimple Lie algebra regarded as an L_∞-algebroid with base space the point and let $\omega := \langle -, - \rangle \in \mathrm{W}(b\mathfrak{g})$ be its Killing form invariant polynomial. Then $(b\mathfrak{g}, \langle -, - \rangle)$ is a symplectic Lie 2-algebroid.

For $\{t^a\}$ a dual basis for $\mathfrak{g}$, being generators of grade 1 in $\mathrm{W}(\mathfrak{g})$ we have

$$d_{\mathrm{W}} t^a = -\frac{1}{2} C^a{}_{bc} t^a \wedge t^b + \mathbf{d} t^a$$

where $C^a{}_{bc} := t^a([t_b, t_c])$ and

$$\omega = \frac{1}{2} P_{ab} \mathbf{d} t^a \wedge \mathbf{d} t^b \,,$$

where $P_{ab} := \langle t_a, t_b \rangle$. The Hamiltonian cocycle π from prop. 3.15 is

$$\begin{aligned}
\pi &= \frac{1}{2+1} \iota_v \iota_\epsilon \omega \\
&= \frac{1}{3} \iota_v P_{ab} t^a \wedge \mathbf{d} t^b \\
&= -\frac{1}{6} P_{ab} C^b_{cd} t^a \wedge t^c \wedge t^d \\
&=: -\frac{1}{6} C_{abc} t^a \wedge t^b \wedge t^c .
\end{aligned}$$

Therefore the Chern-Simons element from prop. 3.17 is found to be

$$\begin{aligned}
\mathrm{cs} &= \frac{1}{2} \left(P_{ab} t^a \wedge \mathbf{d} t^b - \frac{1}{6} C_{abc} t^a \wedge t^b \wedge t^c \right) \\
&= \frac{1}{2} \left(P_{ab} t^a \wedge d_{\mathrm{W}} t^b + \frac{1}{3} C_{abc} t^a \wedge t^b \wedge t^c \right) .
\end{aligned}$$

This is indeed, up to an overall factor 1/2, the familiar standard choice of Chern-Simons element on a Lie algebra. To see this more explicitly, notice that evaluated on a $\mathfrak{g}$-valued connection form

$$\Omega^\bullet(\Sigma) \leftarrow \mathrm{W}(b\mathfrak{g}) : A$$

this is

$$2\mathrm{cs}(A) = \langle A \wedge F_A \rangle - \frac{1}{6} \langle A \wedge [A, A] \rangle = \langle A \wedge d_{dR} A \rangle + \frac{1}{3} \langle A \wedge [A, A] \rangle \,.$$

If $\mathfrak{g}$ is a matrix Lie algebra then the Killing form is proportional to the trace of the matrix product: $\langle t_a, t_b \rangle = \mathrm{tr}(t_a t_b)$. In this case we have

$$\begin{aligned}\langle A \wedge [A, A] \rangle &= A^a \wedge A^b \wedge A^c \,\mathrm{tr}(t_a(t_b t_c - t_c t_b)) \\ &= 2A^a \wedge A^b \wedge A^c \,\mathrm{tr}(t_a t_b t_c) \\ &= 2\,\mathrm{tr}(A \wedge A \wedge A)\end{aligned}$$

and hence

$$2\mathrm{cs}(A) = \mathrm{tr}\left(A \wedge F_A - \frac{1}{3} A \wedge A \wedge A\right) = \mathrm{tr}\left(A \wedge d_{dR} A + \frac{2}{3} A \wedge A \wedge A\right) .$$

4.1.2 Poisson σ-model

Let $(M, \{-,-\})$ be a Poisson manifold and let $\mathfrak{P}$ be the corresponding Poisson Lie algebroid. This is a symplectic Lie 1-algebroid. Over a chart for the shifted cotangent bundle $T^*[-1]X$ with coordinates $\{x^i\}$ of degree 0 and $\{\partial_i\}$ of degree 1, respectively, we have

$$d_{\mathrm{W}} x^i = -\pi^{ij} \partial_j + \mathbf{d} x^i ;$$

where $\pi^{ij} := \{x^i, x^j\}$ and

$$\omega = \mathbf{d} x^i \wedge \mathbf{d} \partial_i .$$

The Hamiltonian cocycle from prop. 3.15 is

$$\pi = \frac{1}{2} \iota_v \iota_\epsilon \omega = -\frac{1}{2} \pi^{ij} \partial_i \wedge \partial_j$$

and the Chern-Simons element from prop. 3.17 is

$$\begin{aligned}\mathrm{cs} &= \iota_\epsilon \omega + \pi \\ &= \partial_i \wedge \mathbf{d} x^i - \frac{1}{2} \pi^{ij} \partial_i \wedge \partial_j\end{aligned} .$$

In terms of d_{W} instead of $\mathbf{d}$ this is

$$\begin{aligned}\mathrm{cs} &= \partial_i \wedge d_{\mathrm{W}} x^i - \pi \\ &= \partial_i \wedge d_{\mathrm{W}} x^i + \frac{1}{2} \pi^{ij} \partial_i \partial_j .\end{aligned}$$

So for Σ a 2-manifold and

$$\Omega^\bullet(\Sigma) \leftarrow \mathrm{W}(\mathfrak{P}) : (X, \eta)$$

a Poisson-Lie algebroid valued differential form on Σ – which in components is a function $X : \Sigma \to M$ and a 1-form $\eta \in \Omega^1(\Sigma, X^* T^* M)$ – the corresponding AKSZ action is

$$\int_\Sigma \mathrm{cs}(X, \eta) = \int_\Sigma \eta \wedge d_{\mathrm{dR}} X + \frac{1}{2} \pi^{ij}(X) \eta_i \wedge \eta_j .$$

This is the Lagrangian of the Poisson σ-model [CaFe00].

4.1.3 Courant σ-model

A Courant algebroid is a symplectic Lie 2-algebroid. By the previous example this is a higher analog of a Poisson manifold. Expressed in components in the language of ordinary differential geometry, a Courant algebroid is a vector bundle E over a manifold M_0, equipped with: a non-degenerate bilinear form $\langle\cdot,\cdot\rangle$ on the fibers, a bilinear bracket $[\cdot,\cdot]$ on sections $\Gamma(E)$, and a bundle map (called the anchor) $\rho\colon E \to TM$, satisfying several compatibility conditions. The bracket $[\cdot,\cdot]$ may be required to be skew-symmetric (Def. 2.3.2 in [Royt99]), in which case it gives rise to a Lie 2-algebra structure, or, alternatively, it may be required to satisfy a Jacobi-like identity (Def. 2.6.1 in [Royt99]), in which case it gives a Leibniz algebra structure.

It was shown in [Royt99] that Courant algebroids $E \to M_0$ in this component form are in 1-1 correspondance with (non-negatively graded) grade 2 symplectic dg-manifolds (M, v). Via this correspondance, M is obtained as a particular symplectic submanifold of $T^*[2]E[1]$ equipped with its canonical symplectic structure.

Let (M, v) be a Courant algebroid as above. In Darboux coordinates, the symplectic structure is

$$\omega = \mathbf{d}p_i \wedge \mathbf{d}q^i + \frac{1}{2}g_{ab}\mathbf{d}\xi^a \wedge \mathbf{d}\xi^b,$$

with

$$\deg q^i = 0, \ \ \deg \xi^a = 1, \ \ \deg p_i = 2,$$

and g_{ab} are constants. The Chevalley-Eilenberg differential corresponds to the vector field:

$$v = P^i_a\xi^a\frac{\partial}{\partial q^i} + g^{ab}\big(P^i_b p_i - \frac{1}{2}T_{bcd}\xi^c\xi^d\big)\frac{\partial}{\partial \xi^a} + \left(-\frac{\partial P^j_a}{\partial q^i}\xi^a p_j + \frac{1}{6}\frac{\partial T_{abc}}{\partial q^i}\xi^a\xi^b\xi^c\right)\frac{\partial}{\partial p_i}.$$

Here $P^i_a = P^i_a(q)$ and $T_{abc} = T_{abc}(q)$ are particular degree zero functions encoding the Courant algebroid structure. Hence, the differential on the Weil algebra is:

$$\begin{aligned} d_W q^i &= P^i_a\xi^a + \mathbf{d}q^i \\ d_W\xi^a &= g^{ab}\big(P^i_b p_i - \frac{1}{2}T_{bcd}\xi^c\xi^d\big) + \mathbf{d}\xi^a \\ d_W p_i &= -\frac{\partial P^j_a}{\partial q^i}\xi^a p_j + \frac{1}{6}\frac{\partial T_{abc}}{\partial q^i}\xi^a\xi^b\xi^c + \mathbf{d}p_i. \end{aligned}$$

Following remark. 3.16, we construct the corresponding Hamiltonian cocycle from prop. 3.15:

$$
\begin{aligned}
\pi &= \frac{1}{n+1}\omega_{ab}\deg(x^a)x^a \wedge v^b \\
&= \frac{1}{3}\big(2p_i \wedge v(q^i) + g_{ab}\xi^a \wedge v(\xi^b)\big) \\
&= \frac{1}{3}\big(2p_i P^i_a \xi^a + \xi^a P^i_a p_i - \frac{1}{2}T_{abc}\xi^a\xi^b\xi^c\big) \\
&= P^i_a \xi^a p_i - \frac{1}{6}T_{abc}\xi^a\xi^b\xi^c .
\end{aligned}
$$

The Chern-Simons element from prop. 3.17 is:

$$
\begin{aligned}
\mathrm{cs} &= \frac{1}{2}\left(\sum_{ab}\deg(x^a)\,\omega_{ab}x^a \wedge d_W x^b - 2\pi\right) \\
&= p_i d_W q^i + \frac{1}{2}g_{ab}\xi^a d_W \xi^b - \pi \\
&= p_i d_W q^i + \frac{1}{2}g_{ab}\xi^a d_W \xi^b - P^i_a \xi^a p_i + \frac{1}{6}T_{abc}\xi^a\xi^b\xi^c .
\end{aligned}
$$

So for a map

$$
\Omega^\bullet(\Sigma) \leftarrow \mathrm{W}(\mathfrak{P}) : (X, A, F)
$$

where Σ is a closed 3-manifold, we have

$$
\int_\Sigma \mathrm{cs}(X,A,F) = \int_\Sigma F_i \wedge d_{\mathrm{dR}}X^i + \frac{1}{2}g_{ab}A^a d_{\mathrm{dR}}A^b - P^i_a A^a F_i + \frac{1}{6}T_{abc}A^aA^bA^c .
$$

This is the AKSZ action for the Courant algebroid σ-model from [Royt99].

5 Higher Chern-Simons field theory

We indicate now the roots of the construction of the higher Chern-Simons action functionals discussed above in a more encompassing general theory. We refer the reader to [Sch10, FSS10] for details on this section.

The first step in this identification involves the *Lie integration* of an L_∞-algebroid $\mathfrak{a}$ to a *smooth ∞-groupoid* $\exp(\mathfrak{a})$ in analogy to how a Lie algebra integrates to a Lie group. This in turn involves two aspects: the notion of a *bare ∞-groupoid* on the one hand, and its *smooth structure* on the other.

Bare ∞-groupoid are presented by *Kan complexes*: simplicial sets such that for all adjacent k-cells there exists a composite k-cell, and such that every k-cell has an inverse, up to $(k+1)$-cells, under this composition. For instance for G any ordinary groupoid there is such a Kan complex whose 0-cells are the objects of the groupoid, and whose k-cells are the sequences of composable k-tuples of morphisms of the groupoid.

These bare ∞-groupoids are equipped with *geometric structure* by providing a rule for what the *geometric families* of k-cells in the ∞-groupoid are supposed to be. In this sense a smooth structure on an ∞-groupoid A is given by declaring for each Cartesian space $U = \mathbb{R}^n$ a set $A_k(U)$ of smooth U-parameterized families of k-morphisms in A, for $k, n \in \mathbb{N}$. Collecting this data for all k and all U produces a functor

$$A : \mathrm{CartSp}^{\mathrm{op}} \to \mathrm{sSet}$$
$$U \mapsto ([k] \mapsto A_k(U))$$

from the opposite of the category of Cartesian spaces to the category of simplicial sets – a *simplicial presheaf* – and this functor encodes the structure of a smooth ∞-groupoid.

For instance if $A = (A_1 \rightrightarrows A_0)$ is an ordinary Lie groupoid, with a smooth manifold of objects A_0 and a smooth manifold of morphisms A_1, this assignment is given in the two lowest degrees by sending U to the set of smooth functions from U to the spaces of objects and morphisms: $A : U \mapsto C^\infty(U, A_k)$.

Definition 5.1. A *smooth ∞-groupoid* is a simplicial presheaf on the category of Cartesian spaces and smooth functions between them,

$$A\colon \mathrm{CartSp}^{\mathrm{op}} \to \mathrm{sSet}\,,$$

such that for each $U \in \mathrm{CartSp}$, the simplicial set $A(U)$ is a Kan complex.

A morphism $f : A_1 \to A_2$ of smooth ∞-groupoids is a morphism of the underlying simplicial presheaves (a natural transformation of functors). A morphism is an *equivalence of smooth ∞-groupoids* if it is stalkwise a weak homotopy equivalence of Kan complexes.

Remark 5.2. Here the category of Cartesian spaces is just the simplest of many possible choices. It can be varied at will, corresponding to which kind of geometric structure the ∞-groupoids are to be equipped with. For instance we can equivalently take instead the full category of smooth manifolds, without changing the notion of smooth ∞-groupoid, up to equivalence. We could also take richer categories, such as that of smooth dg-manifolds. For *non-positively*-graded dg-manifolds we would speak of *derived smooth ∞-groupoids* in this case. These are necessary for discussion of the Lie integration of the full AKSZ BV-action, as opposed to just the grade-0 functional that we concentrate on here.

Remark 5.3. It turns out that under the above notion of equivalence, *every* simplicial presheaf is equivalent to one that is objectwise a Kan complex. In a more abstract discussion than we want to get at here, we would more naturally say that : *the ∞-category of smooth ∞-groupoids is the simplicial localization* $L_W[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$ *at the stalkwise weak equivalences* ([Sch10]).

When regarded as simplicial presheaves on smooth test spaces, smooth ∞-groupoids have a canonical construction from L_∞-algebroids by what is a parameterized version of the classical *Sullivan construction* in rational homotopy theory: the original construction [Sul77] sends – in our ∞-Lie theoretic language – an L_∞-algebroid $\mathfrak{a}$ to the simplicial set

$$\exp(\mathfrak{a})(*) : [k] \mapsto \mathrm{Hom}_{\mathrm{cdgAlg}_{\mathbb{R}}}(\mathrm{CE}(\mathfrak{a}), \Omega^\bullet(\Delta^k)) ,$$

whose k-cells are the flat $\mathfrak{a}$-valued differential forms on the k-simplex (recall definition 3.8). It was noticed in [Hin97, Get09] (for the special case of L_∞-algebras) that this construction deserves to be understood as forming the discrete ∞-groupoid that underlies the Lie integration of $\mathfrak{a}$. In [Henr08] the object $\exp(\mathfrak{a})$, still for the case that $\mathfrak{a} = b\mathfrak{g}$ comes from an L_∞-algebra, is observed to be naturally equipped with a Banach manifold structure. Moreover, a detailed discussion is given showing that the *truncations* $\tau_n \exp(\mathfrak{a})$ (the decategorification of the ∞-groupoid to an n-groupoid) corresponds to the Lie integration to an n-group. For instance for $\mathfrak{a} = b\mathfrak{g}$ coming from an ordinary Lie algebra, $\tau_1 \exp(b\mathfrak{g})$ is $\mathbf{B}G$: the one-object groupoid corresponding to the classical simply connected Lie group integrating $\mathfrak{g}$. A detailed discussion of the smooth structure of $\tau_1 \exp(\mathfrak{a})$ for the case that $\mathfrak{a}$ is a Lie 1-algebroid was given in [CrFe03]. There it is found that a certain cohomological obstruction has to vanish in order that this is a genuine Lie groupoid coming from a simplicial smooth manifold. In [TsZh06] it was pointed out that however for Lie 1-algebroids $\mathfrak{a}$ the 2-truncation $\tau_2 \exp(\mathfrak{a})$ is always a simplicial manifold.

In [FSS10] it was observed that without any assumption on $\mathfrak{a}$ and the truncation degree, the construction always naturally – and usefully – extends to smooth structure as encoded by presheaves on Cartesian test-spaces, simply by declaring the U-parameterized families of k-cells in $\exp(\mathfrak{a})$ to be given by U-parameterized families of flat $\mathfrak{a}$-valued connections:

Definition 5.4. For $\mathfrak{a}$ an L_∞-algebroid, the functor

$$\exp(\mathfrak{a}) : \mathrm{CartSp}^{\mathrm{op}} \to \mathrm{sSet}$$

to the category of simplicial sets is defined by setting, for $U \in \mathrm{CartSp}$ and $k \in \mathbb{N}$,

$$\exp(\mathfrak{a}) : (U, [k]) \mapsto \left\{ \Omega^\bullet_{\mathrm{vert,si}}(U \times \Delta^k) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{a}) \right\} ,$$

where Δ^k is the standard realization of the k-simplex as a smooth manifold with boundary and corners, and where $\Omega^\bullet_{\mathrm{vert,si}}(U \times \Delta^k)$ is the dg-algebra of *vertical* differential forms on $U \times \Delta^k \to U$, that have *sitting instants* towards the boundary faces of the simplex (see [FSS10] for details).
We say that this *simplicial presheaf* presents the *universal Lie integration* of $\mathfrak{a}$. This can be understood as saying that the Lie integration of $\mathfrak{a}$ always exists as a *diffeological* ∞-groupoid [BaHo09].

Remark 5.5. The simplicial presheaf $\exp(\mathfrak{a})$ can naturally be thought of as the presheaf of U-points of the simplicial set $[k] \mapsto \mathrm{Hom}_{\mathrm{cdgAlg}_{\mathbb{R}}}(\mathrm{CE}(\mathfrak{a}), \Omega^\bullet(\Delta^k))$ described above. Indeed, the dg-algebra of vertical differential forms on $U \times \Delta^k$ is naturally isomorphic to $\mathrm{CE}(U \times \mathfrak{T}\Delta^k)$. Also note that this is in turn isomorphic to the (completed) tensor product $\mathrm{CE}(U) \hat{\otimes} \mathrm{CE}(\mathfrak{T}\Delta^k) = C^\infty(U) \hat{\otimes} \Omega^\bullet(\Delta^k)$.

Indeed, the simplicial presheaf given in definition 5.4 is a Lie ∞-groupoid in the sense of definition 5.1.

Proposition 5.6. *For $\mathfrak{a}$ an L_∞-algebroid, the simplicial presheaf* $\exp(\mathfrak{a})$ *is a Lie ∞-groupoid (is objectwise a Kan complex).*

Proof. Since the differential forms in the above definition are required to have sitting instants, they can be smoothly pulled back along the standard continuous retract projections $\Delta^n \to \Lambda^n_i$ of the n-horns, because these are smooth away from the boundary. This provides horn fillers in the standard way. See also the proof of Proposition 4.2.10 in [FSS10].

Remark 5.7. While it can be useful in specific computations to know that $\exp(\mathfrak{a})$ is degreewise a smooth manifold, if indeed it is, no general concept in smooth higher geometry requires this assumption. On the other hand, one can show [Sch10], that every smooth ∞-groupoid is equivalent to a simplicial presheaf that is degreewise a disjoint union of smooth manifolds, even to one that is degreewise a disjoint union of Cartesian spaces.

Remark 5.8. A category with weak equivalences, such as that of smooth ∞-groupoids, is canonically equipped with a *derived hom-functor*, which to smooth ∞-groupoids X and $\exp(\mathfrak{a})$ assigns an ∞-groupoid $\mathbf{RHom}(X, \exp(\mathfrak{a}))$. One finds that the objects of this ∞-groupoid are Čech cocycles for *principal ∞-bundles $P \to X$* that are modeled on $\mathfrak{a}$ in higher analogy of how an ordinary smooth principal bundle is "modeled on" the Lie algebra of its structure group. The 1-morphisms in $\mathbf{RHom}(X, \exp(\mathfrak{a}))$ are the gauge transformations of these principal ∞-bundles, and so on [NSS, Sch10].

Note that in the definition of $\exp(\mathfrak{a})$ only the Chevalley-Eilenberg algebra of $\mathfrak{a}$ is relevant. The Weil algebra $\mathrm{W}(\mathfrak{a})$ is then introduced in order to describe a differential refinement of $\exp(\mathfrak{a})$.

Definition 5.9. For $\mathfrak{a}$ an L_∞-algebroid write $\exp(\mathfrak{a})_{\mathrm{diff}}$ for the simplicial presheaf given by

$$\exp(\mathfrak{a})_{\mathrm{diff}} : (U, [k]) \mapsto \left\{ \begin{array}{ccc} \Omega^\bullet_{\mathrm{vert,si}}(U \times \Delta^k)_{\mathrm{vert}} & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(\mathfrak{a}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(U \times \Delta^k) & \xleftarrow{(A, F_A)} & \mathrm{W}(\mathfrak{a}) \end{array} \right\},$$

where on the right we have the set of horizontal dg-algebra homomorphims that make the square commute, as indicated.

Notice that by definition 3.8 the bottom horizontal morphisms on the right are $\mathfrak{a}$-valued differential forms on $U \times \Delta^k$.

Proposition 5.10. *The canonical projection morphism*

$$\exp(\mathfrak{a})_{\mathrm{diff}} \to \exp(\mathfrak{a})$$

to the Lie integration from definition 5.4 is an equivalence of smooth ∞-groupoids.

Remark 5.11. It is via this property that the Weil algebra serves in ∞-Chern-Weil theory as part of a *resolution* of $\exp(\mathfrak{a})$ from which curvature characteristics are built.

Let now $\langle - \rangle$ be an invariant polynomial on $\mathfrak{a}$ (definition 3.6). Evaluating $\langle - \rangle$ on the curvature F_A of an $\mathfrak{a}$-connection A gives a closed differential form $\langle F_A \rangle$ on $U \times \Delta^k$, according to definition 3.11. This differential form, however, will in general not descend to the base space U. This naturally leads to considering the following definition, which picks the universal subobject of $\exp(\mathfrak{a})_{\mathrm{diff}}$ that makes all *curvature characteristic forms* $\langle F_A \rangle$ descend to base space.

Definition 5.12. Define the simplicial presheaf

$$\exp(\mathfrak{a})_{\mathrm{conn}} : (U, [k]) \mapsto \left\{ \begin{tikzcd} \Omega^\bullet_{\mathrm{vert,si}}(U \times \Delta^k) & \mathrm{CE}(\mathfrak{a}) \arrow[l, "A_{\mathrm{vert}}"'] \\ \Omega^\bullet(U \times \Delta^k) \arrow[u] & \mathrm{W}(\mathfrak{a}) \arrow[l, "A"'] \arrow[u] \\ \Omega^\bullet(U)_{\mathrm{closed}} \arrow[u] & \mathrm{inv}(\mathfrak{a}) \arrow[l, "\langle F_A \rangle"'] \arrow[u] \end{tikzcd} \right\},$$

where on the right we have the set of $\mathfrak{a}$-valued forms A on $U \times \Delta^k$ that make the diagram commute as indicated.

This has the following interpretation (first considered in [SSS09]):

1. The commutativity of the top diagram say that the $\mathfrak{a}$-valued differential form A on $U \times \Delta^k$ is *vertically flat* with respect to the trivial simplex bundle $U \times \Delta \to U$. This is an analogue of the verticality condition for an ordinary *Ehresmann connection*.

2. The commutativity of the lower diagram says that all curvature forms F_A *transform covariantly* along the simplices in such a way as to make all the curvature characteristic form $\langle F_A \rangle$ descent to base space. This is the analogue of the horizonatlity condition on an ordinary Ehresmann connection.

One finds therefore that for X a smooth manifold, an element in $\mathbf{RHom}(X, \exp(\mathfrak{a})_{\mathrm{conn}})$ is

1. a choice of good open cover $\{U_i \to X\}$;

2. on each patch U_i differential form data A_i with values in $\mathfrak{a}$;

3. on each double intersection a choice of 1-parameter gauge transformation between the corresponding differential form data;

4. on each triple intersection a choice of 2-parameter gauge-of-gauge transformation;

5. and so on.

Such a *differential Čech cocycle* is essentially what defines an ∞-connection on a principal ∞-bundle. This is discussed in detail in [FSS10, Sch10].

Remark 5.13. Since for the discussion of the simple case of AKSZ σ-models we can assume that the underlying ∞-bundle is *trivial*, only a single 0-simplex

$$\begin{tikzcd} C^\infty(\Sigma) & \mathrm{CE}(\mathfrak{a}) \arrow[l, "A_{\mathrm{vert}}"'] \\ \Omega^\bullet(\Sigma) \arrow[u] & \mathrm{W}(\mathfrak{a}) \arrow[l, "A"'] \arrow[u] \\ \Omega^\bullet(\Sigma)_{\mathrm{closed}} \arrow[u] & \mathrm{inv}(\mathfrak{a}) \arrow[l, "\langle F_A \rangle"'] \arrow[u] \end{tikzcd}$$

is involved in the description of the AKSZ σ-model.

With these concepts in hand, we can now explain how the datum of a triple $(\mu, \mathrm{cs}, \langle - \rangle)$ consisting of a Chern-Simons element witnessing the transgression between an invariant polynomial and a cocyle (definition 3.7) serves to present a *differential characteristic class* in terms of a morphism of smooth ∞-groupoids. To see this, recall that the line delooping L_∞-algebroid $b^{n+1}\mathbb{R}$ of the Lie line $(n+1)$-algebra is defined by the fact that $\mathrm{CE}(b^{n+1}\mathbb{R})$ is generated over $\mathbb{R}$ from a single generator in degree $n+1$ with vanishing differential. As an immediate consequence, an $(n+1)$-cocylce μ on an L_∞-algebroid $\mathfrak{a}$ is the same thing as a dg-algebra morphism

$$\mu : \mathrm{CE}(b^{n+1}\mathbb{R}) \to \mathrm{CE}(\mathfrak{a}).$$

Similarly, a triple $(\mu, \mathrm{cs}, \langle - \rangle)$ is naturally identified with a commutative diagram of dg-algebras:

$$\begin{tikzcd} \mathrm{CE}(\mathfrak{a}) & \mathrm{CE}(b^{n+1}\mathbb{R}) \arrow[l, "\mu"'] \\ \mathrm{W}(\mathfrak{a}) \arrow[u] & \mathrm{W}(b^{n+1}\mathbb{R}) \arrow[l, "\mathrm{cs}"'] \arrow[u] \\ \mathrm{inv}(\mathfrak{a}) \arrow[u] & \mathrm{inv}(b^{n+1}\mathbb{R}) \arrow[l, "\langle - \rangle"'] \arrow[u] \end{tikzcd} .$$

Pasting this diagram to the one above defining $\exp(\mathfrak{a})_{\mathrm{conn}}$ leads to the following observation, discussed in [FSS10].

Proposition 5.14. *Every triple* $(\mu, \mathrm{cs}, \langle - \rangle)$ *induces a morphism*

$$\exp(\mathfrak{a})_{\mathrm{conn}} \to \exp(b^{n+1}\mathbb{R})_{\mathrm{conn}} .$$

This morphism is in fact the presentation of the ∞-Chern-Weil homomorphism induced by the invariant polynomial $\langle - \rangle$.

This means that for

$$(\nabla : X \to \exp(\mathfrak{a})_{\mathrm{conn}}) \in \mathbf{RHom}(X, \exp(\mathfrak{a})_{\mathrm{conn}})$$

an $\mathfrak{a}$-valued ∞-connection, the composite

$$X \xrightarrow{\nabla} \exp(\mathfrak{a})_{\mathrm{conn}} \xrightarrow{\exp(\mathrm{cs})} \exp(b^{n+1}\mathbb{R})_{\mathrm{conn}}$$

is a representative of the curvature $(n+1)$-form on X that the ∞-Chern-Weil homomorphism induced by $\langle - \rangle$ assigns to ∇.

We can now formalize the observation mentioned in the introduction, that the Chern-Weil homomorphism plays the role of action functional for σ-model quantum field theories. Indeed, in view of the above constructions, the AKSZ σ-model Lagrangian corresponds to forming the following commutative diagram:

$$\begin{tikzcd}
C^\infty(\Sigma) & \mathrm{CE}(\mathfrak{P}) \arrow[l, "\phi_{\mathrm{vert}}"'] & \mathrm{CE}(b^{n+1}\mathbb{R}) \arrow[l, "\frac{1}{n}\pi"'] & : \frac{1}{n}\pi(\phi_{\mathrm{vert}}) \ . \\
\Omega^\bullet(\Sigma) \arrow[u] & \mathrm{W}(\mathfrak{P}) \arrow[l, "\phi"'] \arrow[u] & \mathrm{W}(b^{n+1}\mathbb{R}) \arrow[l, "\mathrm{cs}"'] \arrow[u] & : \mathrm{cs}(\phi) \\
\Omega^\bullet(\Sigma)_{\mathrm{closed}} \arrow[u] & \mathrm{inv}(\mathfrak{P}) \arrow[l, "F_\phi"'] \arrow[u] & \mathrm{inv}(b^{n+1}\mathbb{R}) \arrow[l, "\omega"'] \arrow[u] & : \omega(F_\phi)
\end{tikzcd}$$

In other words, under the identification of the AKSZ action functional as an instance of the ∞-Chern-Weil homomorphism we indeed translate concepts as shown in the table in the introduction: the symplectic form is the invariant polynomial that induces the Chern-Weil homomorphism, the Hamiltonian is the cocycle that it transgresses to, and the Chern-Simons element that witnesses the transgression is the Lagrangian.

This suggests the following general definition of a higher Chern-Simons field theory.

Definition 5.15. Let Σ be an n-dimensional compact smooth manifold and $\mathfrak{a}$ an L_∞-algebroid equipped with an invariant polynomial $\langle - \rangle$. Let cs be a Chern-Simons element witnessing its transgression to a cocycle μ. Then way may say

- A morphism $\phi : \Sigma \to \exp(\mathfrak{a})_{\mathrm{conn}}$ is a *field configuration* on Σ with values in $\mathfrak{a}$.
- The assignment
$$\phi \mapsto \mathrm{cs}(\phi) \in \Omega^{n+1}(\Sigma)$$
is the *Lagrangian* defined by cs;

- The assignment
$$\phi \mapsto \int_\Sigma \mathrm{cs}(\phi) \in \Omega^{n+1}(\Sigma)$$
is the *action functional* defined by cs.

The collection of these notions we call the *higher Chern-Simons field theory* defined by cs.

6 Generalizations

The identification of the AKSZ action functionals as a special case of the general abstract Chern-Weil homomorphism allows to tranfer various insights about the general theory and about its other special cases to AKSZ theory. We close this article by briefly indicating a few. More detailed discussion shall be given elsewhere.

6.1 Symplectic n-groupoids and nontrivial topology

The smooth ∞-groupoid $\exp(\mathfrak{P})$ that Lie integrates a symplectic Lie n-algebroid (as discussed in 5) is the "higher universal" Lie integration of $\mathfrak{P}$. One finds (see [Sch10] for the discussion in the case of smooth ∞-groupoids, following the discussion of Banach-∞-groupoids in [Henr08]) that its geometric realization in topological spaces is ∞-*connected* (hence: contractible) in analogy to how the classical universal Lie integration of a Lie algebra is 1-connected (hence: simply connected).

As we have shown here, this is sufficient for the traditional description of AKSZ σ-models. But more generally one will be interested in the universal integration to the smooth n-groupoid $P := \tau_n \exp(\mathfrak{P})$ obtained as the n-*truncation* (where one retains only equivalence classes of n-cells in $\exp(\mathfrak{P})$).

For instance for $\mathfrak{P} = b\mathfrak{g}$ the delooping of a semisimple Lie algebra $\mathfrak{g}$ (the case of the Courant Lie 2-algebroid over the point) we have that $\tau_1 \exp(b\mathfrak{g}) \simeq \mathbf{B}G$ is the one-object Lie groupoid obtained from the simply-connected Lie group that integrates $\mathfrak{g}$, while the untruncated $\exp(b\mathfrak{g})$ is some higher extension of $\mathbf{B}G$ by ever higher abelian ∞-groups.

This truncation, however, also affects the coefficient object of the ∞-Chern-Weil homomorphism (discussed in detail in [FSS10]): notably the untruncated AKSZ action functional

$$\exp(\mathrm{cs}_\omega) : \exp(\mathfrak{P})_{\mathrm{conn}} \to \exp(b^{n+1}\mathbb{R})_{\mathrm{conn}} \simeq \mathbf{B}^{n+1}\mathbb{R}_{\mathrm{conn}}$$

descends to the truncation only up to a quotient by the group $K \subset \mathbb{R}$ of *periods* of the hamiltonian cocycle π:

$$\exp(\mathrm{cs}_\omega) : \tau_n \exp(\mathfrak{P})_{\mathrm{conn}} \to \mathbf{B}^{n+1}\mathbb{R}/K_{\mathrm{conn}}\,.$$

Typically we have $K \simeq \mathbb{Z}$ and hence $\mathbb{R}/K \simeq U(1)$. This way the properly truncated AKSZ action functional indeed takes values in circle $n+1$-bundles. This becomes a further *quantization condition* for the field configurations, discussed in the next item.

6.2 ∞-Connections on nontrivial $\mathfrak{a}$-principal ∞-bundles: AKSZ instantons

As we have shown in this article, the fields of AKSZ σ-model theories may be understood as ∞-connections on *trivial* $\exp(\mathfrak{P})$-principal ∞-bundles or, by the previous paragraph, on trivial $\tau_n \exp(\mathfrak{P})$-principal ∞-bundles.

The general theory of [FSS10][Sch10] provides also a description of ∞-connections on non-trivial such ∞-bundles and there is no reason to restrict attention to the trivial ones. Such fields with non-trivial underlying principal ∞-bundles correspond to what in the analog situation of Yang-Mills theory are called *instanton* field configurations. These are of importance in a comprehensive discussion of the quantum theory.

This issue plays only a minor role in low dimensions. For instance the reason that the fields of Chern-Simons theory are and can be taken to be connections on trivial G-principal bundles on Σ is that for simply connected Lie groups G the classifying space BG has its first non-trivial homotopy group in degree 4, so that all G-principal bundles on a 3-dimensional Σ are necessarily trivializable.

But by the same argument there are inevitably AKSZ instanton contributions from fields that are connections on non-trivial ∞-bundles as soon as we pass to 4-dimensional AKSZ models and beyond.

6.3 Twisted AKSZ-structures and higher extensions of symplectic L_∞-algebroids

For any differential characteristic class

$$\hat{\mathbf{c}} : A_{\mathrm{conn}} \to \mathbf{B}^{n+1}\mathbb{R}/K_{\mathrm{conn}}$$

such as obtained from Lie integration of a Chern-Simons element:

$$\exp(\mathrm{cs}) : \tau_n \exp(\mathfrak{a})_{\mathrm{conn}} \to \mathbf{B}^{n+1}\mathbb{R}/K$$

it is of interst to study the *homotopy fibers* that this induces on cocycle ∞-groupoids over a given base space X:

in [SSS09c][FSS10] the ∞-groupoid $\hat{\mathbf{c}}\mathrm{Struc}(X)$ of *twisted* $\hat{\mathbf{c}}$-structures is introduced as the homotopy pullback

$$\begin{tikzcd} \hat{\mathbf{c}}\mathrm{Struc}_{\mathrm{tw}}(X) \arrow[r, "\mathrm{tw}"] \arrow[d] & H^n_{\mathrm{diff}}(X, K) \arrow[d] \\ \mathbf{RHom}(X, A_{\mathrm{conn}}) \arrow[r, "\hat{\mathbf{c}}"] & \mathbf{RHom}(X, \mathbf{B}^{n+1}K_{\mathrm{conn}}) \end{tikzcd} ,$$

where the right vertical morphisms – unique up to equivalence – picks one cocycle representative in each cohomology class. The morphism tw sends a given twisted differential cocycle to its *twist*. The fibers over the trivial twist are precisely the $\widehat{A}$-principal ∞-bundles with connection, where $\widehat{A}$ is the extension of A classified by $\mathbf{c}$, wich is characterized by the fact that it sits in a fiber sequence

$$\mathbf{B}^n K \to \widehat{A} \to A\,.$$

In [FSS10] this is discussed in detail for the case that $\mathbf{c} = \frac{1}{2}\mathbf{p}_1$ is a smooth refinement of the first fractional Pontryagin class and for the case $\mathbf{c} = \frac{1}{6}\mathbf{p}_2$ of the fractional second Pontryagin class. In these cases the extension $\hat{A}$ is the delooping, respectively, of the smooth *string 2-group* and of the smooth *fivebrane 6-group*. The corresponding *twisted differential string-structures* and *twisted differential fivebrane structures* are shown there (following [SSS09c]) to encode the Green-Schwarz mechanism in heterotic string theory and dual heterotic string theory, respectively.

By our discussion here, all these constructions have their natural analogs for AKSZ σ-models, too. In particular, every symplectic Lie n-algebroid $(\mathfrak{P}, \omega)$ with Hamiltonian $\pi \in \mathrm{CE}(\mathfrak{P})$ has a canonical ("string-like") extension

$$b^n\mathbb{R} \to \widehat{\mathfrak{P}} \to \mathfrak{P}$$

classified by π regarded as an L_∞-cocycle $\pi : \mathfrak{P} \to b^{n+1}\mathbb{R}$.

This extension is easy to describe: the Chevalley-Eilenberg algebra $\mathrm{CE}(\widehat{\mathfrak{P}})$ is that of $\mathfrak{P}$ with a single generator b in degree n adjoined, and the differential extended to this generator by the formula

$$d_{\mathrm{CE}(\widehat{\mathfrak{P}})} : b \mapsto \pi\,.$$

A *twisted differential* $\exp(\mathfrak{P})$*-structure* is accordingly an $\exp(\mathfrak{P})$-∞-connection ϕ (an AKSZ σ-model field) equipped with an equivalence of its curvature characteristic $\omega(\phi)$ to a presribed "twisting class". When the twisting class is trivial, then these are equivalently $\exp(\widehat{\mathfrak{P}})$-principal ∞-connections.

Notice that these considerations are relevant only over a base space of dimension at least $(n + 2)$. Compare this again to the familiar case of Chern-Simons theory: in Chern-Simons theory itself the G-principal bundle may always be taken to be trivial, since the base space Σ is taken to be 3-dimensional. But all the constructions of Chern-Simons theory make sense also over arbitrary X. Generally, the Chern-Weil homomorphism assigns a *Chern-Simons circle 3-bundle* to every suitabe G-principal bundle on X, and its volume holonomy is the corresponding Chern-Simons functional for this situation. Analogously one can consider AKSZ theory over higher dimensional base spaces.

6.4 Relation to higher dimensional supergravity

AKSZ theory is not the only class of field theories where it was noticed that field configurations have an interpretation in terms of morphisms of dg-algebras. Almost two decades earlier originates the observation that (higher dimensional) *supergravity* (see for instance [DM99] for standard itroductions) has a rather beautiful description in such terms. A detailed exposition of this dg-algebraic approach to supergravity is in the textbook [CaDAFr91].

The authors there speak of "free differential algebras" ("FDAs") where they would mean what in the mathematical literature are called "quasi-free dg-algebras" or "semi-free dg-algebras" – those whose underlying graded algebra is free, as in our definition 3.1 of Chevalley-Eilenberg algebras. Moreover, what we here observe are morphisms out of the Weil algebra (def. 3.8) they call "soft group manifolds". But apart from these purely terminological differences one finds that the observations that drive the developments there are precisely the following, here reformulated in our ∞-Lie theoretic language (see section 4 of [Sch10]):

First of all it is a standard fact that in "first order formulation" the field configurations of gravity in $d+1$ dimensions are naturally presented by $\mathfrak{iso}(d,1)$-vaued connection forms, where $\mathfrak{iso}(d,1) = \mathbb{R}^{d+1} \ltimes \mathfrak{so}(d,1)$ is the Poincaré-Lie algebra. This perspective is inevitable in the context of supergravity, where the first-order formulation is required by the coupling to fermions. There is an evident super-algebra generalization of the Poincaré Lie algebra to the *super-Poincaré*-Lie algebra $\mathfrak{siso}(d,1)$ and a field configuration of supergravity is an $\mathfrak{siso}(d,1)$-valued connection. Or rather, such a connection encodes the *graviton* field and its superpartner field, the *gravitino* field, but not yet the higher bosonic form fields generically present in higher supergravity theories. The first central observation of [DAFr82] is (in our words) that these naturally appear after passage to L_∞*-extensions* of $\mathfrak{siso}(d,1)$.

To put this statement into our context, notice that there is a fairly straightforward super-geometric extension of general abstract higher Chern-Weil theory in which L_∞-algebroids are generalized to *super* L_∞-algebroids as Lie algebras are generalized to super Lie algebras (see section 3.5 of [Sch10]).

It turns out that super-Lie algebra cohomology of $\mathfrak{siso}(d,1)$ contains a certain finite number of *exceptional* cocycles $\mu : \mathfrak{siso}(d,1) \to b^{n+1}\mathbb{R}$ for certain values of d, whose existence is naturally understood from the existence of the four normed division algebras ([Hu11]). In particular, for $d = 10$ there is a 4-cocycle $\mu_4 : \mathfrak{siso}(10,1) \to b^4\mathbb{R}$. The Lie 3-algebra extension that it classifies

$$b^3\mathbb{R} \to \mathfrak{sugra}_{11} \to \mathfrak{siso}(10,1)$$

has been called the *supergravity Lie 3-algebra* in [SSS09]. It turns out ([DAFr82]) that this carries, in turn, a 7-cocycle $\mu_7 : \mathfrak{sugra}_{11} \to b^7\mathbb{R}$. The original observation of [DAFr82] was (not in these words, though, but easily translated into ∞-Lie theoretic terms using our discussion here) that 11-dimensional supergravity, including its higher form field degrees of freedom, is naturally understood as a theory of ∞-connections with values in the corresponding *supergravity Lie 6-algebra*

$$b^7\mathbb{R} \to \widehat{\mathfrak{sugra}}_{11} \to \mathfrak{sugra}_{11}$$

and that the construction of its action functional is governed by the higher Lie theory of this object.

While 11-dimensional supergravity is not entirely a higher Chern-Simons-theory, it crucially does involve Chern-Simons terms in its action functionals. Indeed, one can see that one of the characterizing conditons on a supergravity action functional – the one called the *cosmo-cocycle condition* in [CaDAFr91] – is the defining condition on a Chern-Simons element in our def. 3.7, but solved only up to first order in the curvature terms. It may be noteworthy in this context that there are various speculations (see [BaTrZa96] for discussion and review) that higher dimensional supergravity should be thought of as a limiting theory of a genuine higher Chern-Simons theory.

This shows that there is a close conceptual relation between AKSZ σ-models, higher Chern-Simons theories and higher dimensional supergravity, mediated by abstract higher Chern-Weil theory. The various implications of this observation shall be explored elsewhere.

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