Higher geometric pre-quantization on moduli ∞-stacks

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on joint work with Domenico Fiorenza and Chris L. Rogers
based on joint work with Hisham Sati and Jim Stasheff

1 Motivation

Two formalized methods of quantization:

algebra	Isbell duality $\longleftrightarrow$	**geometry**
algebraic quantization (deformation quantization)		*geometric quantization*

best discussed in the context of:

higher algebra	**higher geometry**
∞-cosheaves	∞-sheaves = ∞-stacks
	e.g. moduli stack of G-connections: $\mathbf{B}G_{\mathrm{conn}}$
considered elsewhere (e.g. Costello)	considered here

for instance for 3d G-Chern-Simons in top dimensions:

dimension		**moduli stack description [Sch]**
$k = 3$	action functional (0-bundle)	$\exp(iS(-)) : [\Sigma_3, \mathbf{B}G_{\mathrm{conn}}] \to U(1)$
$k = 2$	prequantum circle 1-bundle	$[\Sigma_2, \mathbf{B}G_{\mathrm{flat}}] \to \mathbf{B}U(1)_{\mathrm{conn}}$

Question: *What is geometric pre-quantization in lower dimensions k?* **Answer** for 3d CS:

dim.		**prequantum $(3-k)$-bundle**	
$k = 0$	differential fractional first Pontryagin	$\mathbf{c}_{\mathrm{conn}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3U(1)_{\mathrm{conn}}$	[FiScSt]
$k = 1$	WZW background B-field	$[S^1, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{[S^1, \mathbf{c}_{\mathrm{conn}}]} [S^1, \mathbf{B}^3U(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{S^1}(-))} \mathbf{B}^2U(1)_{\mathrm{conn}}$	implicit in [CJMSW]
$k = 2$	off-shell CS prequantum bundle	$[\Sigma_2, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{[\Sigma_2, \mathbf{c}_{\mathrm{conn}}]} [\Sigma_2, \mathbf{B}^3U(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_2}(-))} \mathbf{B}U(1)_{\mathrm{conn}}$	[Sch]
$k = 3$	action functional	$[\Sigma_3, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{[\Sigma_3, \mathbf{c}_{\mathrm{conn}}]} [\Sigma_3, \mathbf{B}^3U(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_3}(-))} U(1)$	[FiScSt]

Higher geometric quantization expected to complete pattern:

symplectic manifold	$\xrightarrow{\text{geometric quantization}}$	quantum mechanics
n-plectic ∞-stack	$\xrightarrow{\text{higher geometric quantization}}$	extended QFT

Already done [Rogers][1]: *n-plectic geometry* and *2-geometric prequantization* over smooth manifolds.

Goal now: formulate generally in *cohesive homotopy type theory*[2], thus generalize from manifolds to smooth moduli ∞-stacks; and apply to ∞-*Chern-Simons theory*[3].

[1] http://ncatlab.org/nlab/show/n-plectic+geometry
[2] http://ncatlab.org/nlab/show/cohesive+homotopy+type+theory
[3] http://ncatlab.org/schreiber/show/infinity-Chern-Simons+theory

2 General theory

∞-geometric quantization [PQ]	**cohesive homotopy type theory** [Sch]	**twisted hyper-sheaf cohomology** [NSS]
n-plectic cohesive ∞-groupoid	$\omega : X \to \Omega^{n+1}_{\mathrm{cl}}(-,\mathbb{G})$ (e.g. $\mathbb{G} = U(1)$ or $= \mathbb{C}^\times$)	twisting cocycle in de Rham cohomology
symplectomorphisms	$\mathbf{Aut}_{/\Omega^{n+1}_{\mathrm{cl}}(-,\mathbb{G})}(\omega) = \left\{ \begin{array}{ccc} X & \xrightarrow{\simeq} & X \\ & {\scriptstyle \omega}\searrow \quad \swarrow{\scriptstyle \omega} & \\ & \Omega^{n+1}_{\mathrm{cl}}(-,\mathbb{G}) & \end{array} \right\}$	twist automorphism ∞-group
Hamiltonian G-action	$\mu : \mathbf{B}G \to \mathbf{B}\mathbf{Aut}_{/\mathbf{B}^n\mathbb{G}_{\mathrm{conn}}}(\mathbf{c}_{\mathrm{conn}})$	G-∞-action on the twisting cocycle
gauge reduction	$\mathbf{c}_{\mathrm{conn}}//G \;:\; X//G \to \mathbf{B}^n\mathbb{G}_{\mathrm{conn}}$	G-∞-quotient of the twisting cocycle
prequantum circle n-bundle	$\begin{array}{ccc} & & \mathbf{B}^n\mathbb{G}_{\mathrm{conn}} \\ & {\scriptstyle \mathbf{c}_{\mathrm{conn}}}\nearrow & \downarrow{\scriptstyle \mathrm{curv}} \\ X & \xrightarrow{\omega} & \Omega^{n+1}_{\mathrm{cl}}(-,\mathbb{G}) \end{array}$	twisting cocycle in differential cohomology
Planck's constant $\hbar$	$\frac{1}{\hbar}\mathbf{c}_{\mathrm{conn}} : X \to \mathbf{B}^n\mathbb{G}_{\mathrm{conn}}$	divisibility of twist class
quantomorphism ∞-group ⊃ Heisenberg ∞-group	$\mathbf{Aut}_{/\mathbf{B}^n\mathbb{G}_{\mathrm{conn}}}(\mathbf{c}_{\mathrm{conn}}) = \left\{ \begin{array}{ccc} X & \xrightarrow{\simeq} & X \\ & {\scriptstyle \mathbf{c}_{\mathrm{conn}}}\searrow \; \swArrow_{\simeq} \; \swarrow{\scriptstyle \mathbf{c}_{\mathrm{conn}}} & \\ & \mathbf{B}^n\mathbb{G}_{\mathrm{conn}} & \end{array} \right\}$	twist automorphism ∞-group
Hamiltonian observables with Poisson L_∞-bracket	$\mathrm{Lie}(\mathbf{Aut}_{/\mathbf{B}^n\mathbb{G}_{\mathrm{conn}}})(\mathbf{c}_{\mathrm{conn}})$	infinitesimal twist automorphisms
Hamiltonian symplectomorphisms	∞-image of $\mathbf{Aut}_{/\mathbf{B}^n\mathbb{G}_{\mathrm{conn}}}(\mathbf{c}_{\mathrm{conn}}) \to \mathbf{Aut}_{/\Omega^{n+1}_{\mathrm{cl}}(-,\mathbb{G})}(\omega)$	twists in de Rham cohomology that lift to differential cohomology
∞-representation of cohesive n-group $\mathbf{B}^{n-1}\mathbb{G}$	$\begin{array}{ccc} V_n & \longrightarrow & V_n//\mathbf{B}^{n-1}\mathbb{G} \\ & & \downarrow{\scriptstyle \mathbf{p}} \\ & & \mathbf{B}^n\mathbb{G} \end{array}$	local coefficient ∞-bundle
prequantum space of states	$\mathbf{\Gamma}_X(E) := [\mathbf{c},\mathbf{p}]_{/\mathbf{B}^n\mathbb{G}_{\mathrm{conn}}} = \left\{ \begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ & {\scriptstyle \mathbf{c}}\searrow \; \swArrow_{\simeq} \; \swarrow{\scriptstyle \mathbf{p}} & \\ & \mathbf{B}^n\mathbb{G} & \end{array} \right\}$	cocycles in [$\mathbf{c}$]-twisted cohomology
prequantum operator	$\widehat{(-)} : \mathbf{\Gamma}_X(E) \times \mathbf{Aut}_{/\mathbf{B}^n\mathbb{G}_{\mathrm{conn}}}(\mathbf{c}_{\mathrm{conn}}) \to \mathbf{\Gamma}_X(E)$	∞-action of twist automorphisms on twisted cocycles
trace to higher dimension	composition with: $\begin{array}{ccc} [S^1, V_n//\mathbf{B}^{n-1}\mathbb{G}_{\mathrm{conn}}] & \xrightarrow{\mathrm{tr}\,\mathrm{hol}_{S^1}} & V_{n-1}//\mathbf{B}^{n-2}\mathbb{G}_{\mathrm{conn}} \\ \downarrow{\scriptstyle \mathbf{p}^{V_n}_{\mathrm{conn}}} & & \downarrow{\scriptstyle \mathbf{p}^{V_{n-1}}_{\mathrm{conn}}} \\ \mathbf{B}^n\mathbb{G}_{\mathrm{conn}} & \xrightarrow{\exp(2\pi i \int_{S^1}(-))} & \mathbf{B}^{n-1}\mathbb{G}_{\mathrm{conn}} \end{array}$	fiber integration in (nonabelian) differential cohomology

3 Example: 1- and 2-geometric pre-quantization

	base ring	local coefficient bundle	n-plectic ∞-groupoid
Choose:	$\mathbb{C}$	$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C}//U(1) \\ & & \downarrow^{\mathbf{p}^{\mathbb{C}}} \\ & & \mathbf{B}U(1) \end{array}$	ordinary symplectic manifold $\omega : X \to \Omega^2_{\mathrm{cl}}(-)$

Proposition. For X an ordinary smooth manifold, $\omega : X \to \Omega^2_{\mathrm{cl}}(-)$ an ordinary symplectic form, the items in the above table 2 reproduce those of traditional geometric prequantization.

	base ∞-ring	local coefficient ∞-bundle	n-plectic ∞-groupoid
Choose:	KU	$\begin{array}{ccccc} \mathbf{B}U(n) & \longrightarrow & (\mathbf{B}U(n))//\mathbf{B}U(1) & \simeq & \mathbf{B}PU(n) \\ & & \downarrow^{\mathbf{p}^{\mathbf{B}U}} & \swarrow_{\mathbf{dd}_n} & \\ & & \mathbf{B}^2U(1) & & \end{array}$	2-plectic manifold $\omega : X \to \Omega^3_{\mathrm{cl}}(-)$

Proposition.

- 2-states are $[\mathbf{c}_{\mathrm{conn}}]$-twisted rank-$n$ bundles with connection
  = D-branes in B-field $\mathbf{c}_{\mathrm{conn}}$ which are anomaly-free (if X is Spin^c)
  (cocycles in $[c]$-twisted K-theory)
- $\mathrm{Lie}(\mathbf{Aut}_{/\mathbf{B}^2U(1)_{\mathrm{conn}}}(\mathbf{c}_{\mathrm{conn}})) \simeq$ the Poisson bracket Lie 2-algebra of [Rogers] (for X a manifold);
- tracing to higher dimension is given by traced $U(n)$-holonomy of twisted bundles with connection:

2-geometric on X	$\xrightarrow{\textbf{trace}}$	**1-geometric on loop space $[S^1, X]$**
$\left\{ \begin{array}{c} \text{prequantum 2-states} \\ \simeq [\mathbf{c}]\text{-twisted } U(n)\text{-bundles} \\ \text{on } X \end{array} \right\}$	$\longrightarrow$	$\left\{ \begin{array}{c} \text{prequantum states} \\ \simeq \text{sections of line bundle } \exp(2\pi i \int_{S^1}[S^1, \mathbf{c}_{\mathrm{conn}}]) \\ \text{on loop space } [S^1, X] \end{array} \right\}$
$\begin{array}{ccc} X & \xrightarrow{\sigma} & (\mathbf{B}U(n))//(\mathbf{B}U(1))_{\mathrm{conn}} \\ & \searrow_{\mathbf{c}_{\mathrm{conn}}} & \downarrow^{\mathbf{p}^{\mathbf{B}U}_{\mathrm{conn}}} \\ & & \mathbf{B}^2U(1)_{\mathrm{conn}} \end{array}$	$\longmapsto$	$\begin{array}{ccccc} [S^1, X] & \xrightarrow{[S^1,\sigma]} & [S^1, (\mathbf{B}U(n))//(\mathbf{B}U(1))_{\mathrm{conn}}] & \xrightarrow{\mathrm{tr\,hol}_{S^1}} & \mathbb{C}//U(1)_{\mathrm{conn}} \\ & \searrow_{[S^1,\mathbf{c}_{\mathrm{conn}}]} & \downarrow^{\mathbf{p}^{\mathbf{B}U}_{\mathrm{conn}}} & & \downarrow^{\mathbf{p}^{\mathbb{C}}_{\mathrm{conn}}} \\ & & [S^1, \mathbf{B}^2U(1)_{\mathrm{conn}}] & \xrightarrow{\exp(2\pi i \int_{S^1}(-))} & \mathbf{B}U(1)_{\mathrm{conn}} \end{array}$
$\left\{ \begin{array}{c} \text{2-Hamiltonian } G\text{-actions} \\ \text{on } X \end{array} \right\}$	$\longrightarrow$	$\left\{ \begin{array}{c} \text{Hamiltonian } G\text{-actions} \\ \text{on loop space } [S^1, X] \end{array} \right\}$
$\begin{array}{ccc} X & \xrightarrow{\simeq} & X \\ & \swArrow & \\ \searrow_{\mathbf{c}_{\mathrm{conn}}} & & \swarrow_{\mathbf{c}_{\mathrm{conn}}} \\ & \mathbf{B}^2U(1)_{\mathrm{conn}} & \end{array}$	$\longmapsto$	$\begin{array}{ccc} X & \xrightarrow{\simeq} & X \\ & \swArrow & \\ \searrow_{\mathbf{c}_{\mathrm{conn}}} & & \swarrow_{\mathbf{c}_{\mathrm{conn}}} \\ & \mathbf{B}^2U(1)_{\mathrm{conn}} & \\ & \downarrow^{\exp(2\pi i \int_{S^1}(-))} & \\ & \mathbf{B}U(1)_{\mathrm{conn}} & \end{array}$

Proposition. For X interpreted as a D-brane and $\mathbf{c}_{\mathrm{conn}}$ as the B-field, this is the Freed-Witten anomaly cancellation mechanism for the type II superstring.

4 Example: Prequantum ∞-Chern-Simons theory

For G a smooth ∞-group and $X = \mathbf{B}G_{\mathrm{conn}}$ the moduli ∞-stack of G-∞-connections [FiScSt][Sch], a prequantum n-bundle $\mathbf{c}_{\mathrm{conn}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^n U(1)_{\mathrm{conn}}$ is a smooth and differential refinement of a universal characteric class $[c] \in H^{n+1}(BG, \mathbb{Z})$. The induced QFT is a higher extended Chern-Simons theory.[3]

Some examples:

model	prequantum n-bundle modulus	prequantum n-bundle total space	quantomorph.
$U(1)$-Chern-Simons in dimension $(4n+3)$	cup product on moduli ∞-stacks of circle n-connections [FiSaSc2] $\mathbf{B}^{2n+1}U(1)_{\mathrm{conn}} \xrightarrow{(-)\cup(-)} \mathbf{B}^{4n+3}U(1)_{\mathrm{conn}}$	moduli 2-stack of differential T-duality structures [Sch]	$\mathbb{Z}_2$
Spin-Chern-Simons at $\hbar = 2$.	differential first fractional Pontryagin class [FiScSt] $\mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}_1} \mathbf{B}^3 U(1)_{\mathrm{conn}}$	smooth moduli 2-stack of String-2-connections $\mathbf{B}\mathrm{String}_{\mathrm{conn}'}$ [SaScSt][FiSaSc]	diagonal transformations in Spin $\times$ Spin theory
String-Chern-Simons at $\hbar = 6$.	differential second fractional Pontryagin class [FiScSt] $\mathbf{B}\mathrm{String}_{\mathrm{conn}} \xrightarrow{\frac{1}{6}\hat{\mathbf{p}}_2} \mathbf{B}^7 U(1)_{\mathrm{conn}}$	smooth moduli 6-stack of Fivebrane-6-connections $\mathbf{B}\mathrm{Fivebrane}_{\mathrm{conn}'}$ [SaScSt]	

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