Higher prequantum string geometry

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1 Motivation

Topological string geometry is controled by the fiber sequence

classifying space for String-principal bundles	$BString \longrightarrow BSpin$	classifying space of Spin-principal bundles	$\in$ Top
	$\downarrow \tfrac{1}{2}p_1$	$\downarrow$ first fractional Pontryagin class	$\simeq$
	$B^3U(1)$	classifying space for degree-4 integral cohomology	∞Grpd

Smooth string geometry is controled by the fiber sequence [S1]

moduli 2-stack of smooth String-principal 2-bundles	$\mathbf{B}String \longrightarrow \mathbf{B}Spin$	moduli stack of smooth Spin-principal bundles	$\in \mathbf{H} :=$
	$\downarrow \tfrac{1}{2}\mathbf{p}_1$	$\downarrow$ smooth refinement of $\tfrac{1}{2}p_1$	Smooth∞Grpd $=$
	$\mathbf{B}^3U(1)$	moduli 3-stack of of circle 3-bundles / bundle 2-gerbes	$Sh_\infty(SmthMfd)$

Theorem. [S1] Generally, for G a compact Lie group: $\pi_0\mathbf{H}(\mathbf{B}G, \mathbf{B}^nU(1)) \simeq H^{n+1}(BG, \mathbb{Z})$.

Differential string geometry is controled by the fiber sequence [FSSt]

moduli 2-stack of String-principal 2-form connections	$\mathbf{B}String_{conn} \longrightarrow \mathbf{B}Spin_{conn}$	moduli stack of Spin-principal connections
	$\downarrow (\tfrac{1}{2}\mathbf{p}_1)_{conn}$	$\downarrow$ differential refinement of $\tfrac{1}{2}p_1$
	$\mathbf{B}^3U(1)_{conn}$	moduli 3-stack of smooth 3-form connections

Observation. [FSSc] This is the *prequantum circle 3-bundle* of extended *Chern-Simons theory*:

dim.		**prequantum $(3-k)$-bundle**	
$k=0$	differential fractional first Pontryagin	$\mathbf{c}_{conn} : \mathbf{B}G_{conn} \to \mathbf{B}^3U(1)_{conn}$	[FSSt]
$k=1$	WZW background B-field	$[S^1, \mathbf{B}G_{conn}] \xrightarrow{[S^1,\mathbf{c}_{conn}]} [S^1, \mathbf{B}^3U(1)_{conn}] \xrightarrow{\exp(2\pi i \int_{S^1}(-))} \mathbf{B}^2U(1)_{conn}$	[∞Quant]
$k=2$	off-shell CS prequantum bundle	$[\Sigma_2, \mathbf{B}G_{conn}] \xrightarrow{[\Sigma_2,\mathbf{c}_{conn}]} [\Sigma_2, \mathbf{B}^3U(1)_{conn}] \xrightarrow{\exp(2\pi i \int_{\Sigma_2}(-))} \mathbf{B}U(1)_{conn}$	[S1]
$k=3$	action functional	$[\Sigma_3, \mathbf{B}G_{conn}] \xrightarrow{[\Sigma_3,\mathbf{c}_{conn}]} [\Sigma_3, \mathbf{B}^3U(1)_{conn}] \xrightarrow{\exp(2\pi i \int_{\Sigma_3}(-))} U(1)$	[FSSt]

Goal here: apply higher geometric quantization to these stringy prequantum n-bundles. Indicate how the extended quantum n-states encode central aspects of

$n=2$: Freed-Witten anomaly cancellation for type II superstrings on D-branes;

$n=3$: Green-Schwarz anomaly cancellation for heterotic superstrings on Hořava-Witten boundaries of M-theory: *twisted differential string structures* [SSS2].

2 General theory

2.1 Kinematics: twisted cohomology

Let $\mathbf{H}$ be any ∞-topos, such as our $\mathbf{H} = \mathrm{Sh}_\infty(\mathrm{SmoothManifolds})$.
Definition A *group* in ∞-topos is a $G \in \mathbf{H}$ equipped with a groupal A_∞-algebra structure: coherently homotopy associative product with coherent homotopy inverses.
Fact. (classical + Lurie) There is an equivalence

$$\{\text{ groups in } \mathbf{H} \} \underset{\text{delooping } \mathbf{B}}{\overset{\text{looping } \Omega}{\underset{\longrightarrow}{\overset{\simeq}{\longleftarrow}}}} \left\{ \begin{array}{c} \text{pointed connected} \\ \text{objects in } \mathbf{H} \end{array} \right\}$$

Definition. A G**-principal** ∞**-bundle** over $X \in \mathbf{H}$ is

- a morphism $P \to X$; with an ∞-action $\rho : P \times G \to P$;
- such that $P \to X$ is ∞-quotient $P \to P//G \Leftrightarrow$ principality : $P \times G^n \xrightarrow[\simeq]{(p_1,\rho)} P \times_X \cdots \times_X P$

Theorem. [NSS] There is equivalence of ∞-groupoids $G\mathrm{Bund}(X) \underset{\underset{\to}{\lim}}{\overset{\text{hofib}}{\underset{\longrightarrow}{\overset{\simeq}{\longleftarrow}}}} \mathbf{H}(X, \mathbf{B}G)$,

where
1. hofib sends a cocycle $X \to \mathbf{B}G$ to its homotopy fiber;
2. $\underset{\longrightarrow}{\lim}$ sends an ∞-bundle to the map on ∞-quotients $X \simeq P//G \to *//G \simeq \mathbf{B}G$.

In particular, G-principal ∞-bundles are classified by the intrinsic cohomology of $\mathbf{H}$

$$G\mathrm{Bund}(X)/_\sim \simeq H^1(X, G) := \pi_0 \mathbf{H}(X, \mathbf{B}G)\,.$$

Observation. By the above theorem, every G-∞-action $\rho : V \times G \to G$ has a classifying map $\mathbf{c}$:

$$\begin{tikzcd} V \arrow[r] & V//G \arrow[d, "\mathbf{c}"] \\ & \mathbf{B}G \end{tikzcd}$$

Theorem. [NSS] This is the *universal ρ-associated V-bundle.*

Observation. Sections σ of the associated ∞-bundle are *lifts* of the cocycle through $\mathbf{c}$; and these locally factor through V:

$$\left\{ \begin{tikzcd} P \times_G V \arrow[r] \arrow[d] & V//G \arrow[d, "\mathbf{c}"] \\ X \arrow[u, bend left, dashed, "\sigma"] \arrow[r, "g"] & \mathbf{B}G \end{tikzcd} \right\} \simeq \left\{ \begin{tikzcd} & V//G \arrow[d, "\mathbf{c}"] \\ X \arrow[ur, dashed, "\sigma"] \arrow[r, "g"] & \mathbf{B}G \end{tikzcd} \right\} \begin{tikzcd} & V \arrow[r] & V//G \arrow[d, "\mathbf{c}"] \\ U \arrow[ur, dashed, "\sigma|_U"] \arrow[r, two heads] & X \arrow[r, "g"] & \mathbf{B}G \end{tikzcd}\,.$$

Hence sections σ are equivalently

- cocycles in $[g]$-*twisted cohomology*;
- $\mathbf{c}$-valued cocycles in the *slice ∞-topos*: $\quad \Gamma_X(P \times_G V) \simeq \mathbf{H}_{/\mathbf{B}G}(g, \mathbf{c})$

2.2 Dynamics: differential refinement

Theorem. [S1] $\mathbf{H} = \mathrm{Smooth}\infty\mathrm{Grpd}$ is *cohesive*. In particular there is a derived endo-adjunction $(\mathbf{\Pi} \dashv \flat) =$ (path ∞-groupoid $\dashv$ flat moduli) and for every ∞-group a *Maurer-Cartan form* $\theta_G : G \to \flat_{\mathrm{dR}}\mathbf{B}G := \flat\mathbf{B}G \prod_{\mathbf{B}G} *$. The moduli for twisted cohomology of the local coefficient bundle $\theta_{\mathbf{B}^{n-1}U(1)}$ are the moduli n-stacks $\mathbf{B}^nU(1)_{\mathrm{conn}}$ of *circle n-bundles with connection*: cocycles in differential cohomology:

$$\begin{tikzcd} \mathbf{B}^nU(1)_{\mathrm{conn}} \arrow[r, "F_{(-)}"] \arrow[d] & \Omega^{n+1}_{\mathrm{cl}}(-) \arrow[d] \\ \mathbf{B}^nU(1) \arrow[r, "\theta_{\mathbf{B}^{n-1}U(1)}"] & \flat\mathbf{B}^{n+1}U(1) \end{tikzcd}$$

Theorem. ([FSSc] using [GT]) For $\nabla : X \to \mathbf{B}^nU(1)_{\mathrm{conn}}$ an n-form connection and Σ a d-dim compact smooth manifold with boundary $\partial\Sigma$ we have **extended higher transgression**:

$$\begin{tikzcd} {[\Sigma, X]} \arrow[rr] \arrow[d, "{[\partial\Sigma \hookrightarrow \Sigma, X]}"'] & & \Omega^{n-d+1}_{\mathrm{cl}}(-) \arrow[d] \arrow[dll, phantom, "\Leftarrow\ \exp(2\pi i \int_\Sigma [\Sigma,-])"] \\ {[\partial\Sigma, X]} \arrow[r, "{[\partial\Sigma,\nabla]}"'] \arrow[rr, bend right, "{\exp(2\pi i \int_{\partial\Sigma}[\partial\Sigma,-])}"'] & {[\partial\Sigma, \mathbf{B}^nU(1)_{\mathrm{conn}}]} \arrow[r, "\exp(2\pi i \int_{\partial\Sigma}(-))"'] & \mathbf{B}^{n-d+1}U(1)_{\mathrm{conn}} \end{tikzcd}.$$

For $d = n$ this is *n-volume parallel transport*. For $\partial\Sigma =$ this is *n-dimensional holonomy*.

∞-geometric quantization [∞Quant]	**cohesive homotopy type theory** [S1]	**twisted hyper-sheaf cohomology** [NSS]
n-plectic cohesive ∞-groupoid	$\omega : X \to \Omega^{n+1}_{\mathrm{cl}}(-,\mathbb{G})$ (e.g. $\mathbb{G} = U(1)$ or $= \mathbb{C}^\times$)	twisting cocycle in de Rham cohomology
prequantum circle n-bundle	$\begin{tikzcd} & \mathbf{B}^n\mathbb{G}_{\mathrm{conn}} \arrow[d, "F_{(-)}"] \\ X \arrow[ur, "\mathbf{c}_{\mathrm{conn}}"] \arrow[r, "\omega"] & \Omega^{n+1}_{\mathrm{cl}}(-,\mathbb{G}) \end{tikzcd}$	twisting cocycle in differential cohomology
quantomorphism ∞-group $\supset$ Heisenberg ∞-group	$\mathbf{Aut}_{/\mathbf{B}^n\mathbb{G}_{\mathrm{conn}}}(\mathbf{c}_{\mathrm{conn}}) = \left\{ \begin{tikzcd} X \arrow[rr, "\simeq"] \arrow[dr, "\mathbf{c}_{\mathrm{conn}}"'] & \arrow[d, phantom, "\swarrow\simeq"] & X \arrow[dl, "\mathbf{c}_{\mathrm{conn}}"] \\ & \mathbf{B}^n\mathbb{G}_{\mathrm{conn}} & \end{tikzcd} \right\}$	twist automorphism ∞-group
∞-representation of cohesive n-group $\mathbf{B}^{n-1}\mathbb{G}$	$\begin{tikzcd} V_n \arrow[r] & V_n//\mathbf{B}^{n-1}\mathbb{G} \arrow[d, "\mathbf{p}"] \\ & \mathbf{B}^n\mathbb{G} \end{tikzcd}$	local coefficient ∞-bundle
prequantum space of states	$\mathbf{\Gamma}_X(E) := \left\{ \begin{tikzcd} X \arrow[rr, "\sigma"] \arrow[dr, "\mathbf{c}"'] & \arrow[d, phantom, "\swarrow\simeq"] & V_n//\mathbf{B}^{n-1}\mathbb{G} \arrow[dl, "\mathbf{p}"] \\ & \mathbf{B}^n\mathbb{G} & \end{tikzcd} \right\}$	cocycles in $[\mathbf{c}]$-twisted cohomology
transgression to higher dimension	composition with: $\begin{tikzcd} {[S^1, V_n//\mathbf{B}^{n-1}\mathbb{G}_{\mathrm{conn}}]} \arrow[r, "\mathrm{tr}\,\mathrm{hol}_{S^1}"] \arrow[d, "\mathbf{p}^{V_n}_{\mathrm{conn}}"] & V_{n-1}//\mathbf{B}^{n-2}\mathbb{G}_{\mathrm{conn}} \arrow[d, "\mathbf{p}^{V_{n-1}}_{\mathrm{conn}}"] \\ \mathbf{B}^n\mathbb{G}_{\mathrm{conn}} \arrow[r, "\exp(2\pi i \int_{S^1}(-))"] & \mathbf{B}^{n-1}\mathbb{G}_{\mathrm{conn}} \end{tikzcd}$	fiber integration in (nonabelian) differential cohomology
higher quantum states	$\begin{tikzcd} Q \arrow[r, dashed, "\text{polarized section of prequantum } n\text{-bundle}"] \arrow[d, hook, "\text{isotropic subspace}"'] & V//G_{\mathrm{conn}} \arrow[d, "\text{differential refinement of universal } \rho\text{-associated } V\text{-bundle}"] \\ X \arrow[r, "\text{prequantum } n\text{-bundle}"'] & \mathbf{B}^nU(1)_{\mathrm{conn}} \end{tikzcd}$	relative cohomology

3 Applications

Higher twisted differential structures/twisted ∞-bundles with connection are induced in string theory by quantum anomaly cancellation conditions [SSS2][S2]. We indicate that

• *Freed-Witten-Kapustin anomaly cancellation* for type II strings on D-branes [K][L] is encoded in the *quantum 2-states* of the extended geometric quantization of the string;

• analogously "lifted to M-theory": the *Green-Schwarz anomaly cancellation* for the heterotic string / twisted differential string structures [SSS2] are encoded in *quantumm 3-states* of the extended geometric quantization of the membrane.

3.1 Freed-Witten-Kapustin quantum anomaly

Consider the σ-model of the open bosonic string with target space X carrying a background B-field and containing n coincident D-branes with oriented worldvolume $Q \hookrightarrow X$ carrying, therefore, a rank-n Chan-Paton bundle. Alternatively, consider the analogous σ-model for the type II superstring and assume, for simplicity, that $W_3(Q) = 0$.

For Σ a 2-dimensional worldsheet with boundary $\partial\Sigma$ the gauge-interaction part of the action functional of the σ-model for a configuration $\phi : \Sigma \to X$ with $\phi|_{\partial\Sigma} \subset Q$ is the product of two contributions: the 2-dimensional parallel transport of the B-field over ϕ, and the traced holonomy over $\phi|_{\partial\Sigma}$ of the connection of the Chan-Paton bundle. By the above, the first term is not a function on the mapping stack $[\Sigma, X]$, but a section of the $U(1)$-principal bundle obtained by transgression of the B-field. Its Chern-class is the *anomaly* of that term. Accordingly, the traced boundary holonomy similarly needs to be not a function but a section of the dual of this bundle, such that the product of the two sections is a well-defined function on $[\Sigma, X]$ (the two Chern-classes/anomalies cancel). In [K] it was found that a sufficient condition for this to happen is that the Chan-Paton bundle is a $\nabla_B|_Q$-*twisted bundle with connection* as in twisted K-theory.

Theorem. All this follows by applying the above extended prequantization via the following dictionary.

∞-geometric quantization [∞Quant]	**string theory**
prequantum circle 2-bundle	B-field $\nabla_B : X_{10} \to \mathbf{B}^2U(1)_{\mathrm{conn}}$
isotropic suspace	n-coincident D-branes: $Q \hookrightarrow X_{10}$
universal local coefficient bundle	$\mathbf{B}U(n) \longrightarrow \mathbf{B}PU(n) \xrightarrow{\mathbf{dd}_n} \mathbf{B}^2U(1)$
polarized section	anomaly-free Chan-Paton background gauge field

2-geometric on Q	$\xrightarrow{\textbf{transg.}}$	**1-geometric on loop space $[S^1, Q]$**
$\left\{ \text{prequantum 2-states} \simeq [\nabla_B\vert_Q]\text{-twisted } U(n)\text{-bundles on } Q \right\}$	$\longrightarrow$	$\left\{ \text{prequantum states} \simeq \text{sections of line bundle } \exp(2\pi i \int_{S^1}[S^1, \nabla_B\vert_Q]) \text{ on loop space } [S^1, Q] \right\}$

$$\begin{tikzcd} Q \arrow[r, "\sigma"] \arrow[d, hook] \arrow[dr, "\nabla_B|_Q"] & (\mathbf{B}U(n))//(\mathbf{B}U(1))_{\mathrm{conn}} \arrow[d, "(\mathbf{dd}_n)_{\mathrm{conn}}"] \\ X \arrow[r, "\nabla_B"] & \mathbf{B}^2U(1)_{\mathrm{conn}} \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} {[S^1, Q]} \arrow[r, "{[S^1,\sigma]}"] \arrow[dr, "{[S^1,\nabla_B|_Q]}"'] & {[S^1, (\mathbf{B}U(n))//(\mathbf{B}U(1))_{\mathrm{conn}}]} \arrow[r, "\mathrm{tr\,hol}_{S^1}"] \arrow[d, "(\mathbf{dd}_n)_{\mathrm{conn}}"] & \mathbb{C}//U(1)_{\mathrm{conn}} \arrow[d, "\mathbf{p}^{\mathbb{C}}_{\mathrm{conn}}"] \\ & {[S^1, \mathbf{B}^2U(1)_{\mathrm{conn}}]} \arrow[r, "\exp(2\pi i \int_{S^1}(-))"] & \mathbf{B}U(1)_{\mathrm{conn}} \end{tikzcd}$$

3.2 Green-Schwarz quantum anomaly

Consider now the bosonic part of the 3-dimensional σ-model of the M2-brane on an 11-dimensional spacetime X_{11}.

The background gauge field that this couples to is the *11d supergravity C-field*, whose moduli are described in [FSSb]. Oversimplifying a bit, we here take the moduli to be $\mathbf{B}^3U(1)_{\mathrm{conn}}$, hence the background gauge field is a circle 3-bundle with connection. The canonical linear action of $\mathbf{B}^2U(1)$ should be on tmf and hence the linear prequantum 3-bundle should be a *smooth* tmf-fiber bundle [ABG]. This is currently out of reach, and so for the time being we fall back to canonical *non-linear* representations of $\mathbf{B}^2U(1)$ on $\mathbf{B}\mathrm{String}$ and on $\mathbf{B}\mathrm{String}(E_8)$ given, according to section 2, by $\frac{1}{2}\mathbf{p}_1$ and similarly by $\mathbf{a} : \mathbf{B}\mathrm{String}(E_8) \to \mathbf{B}^3U(1)$.

From [FSSb] we now find this dictionary:

∞-geometric quantization [∞Quant]	**string theory**
prequantum circle 3-bundle	11d supergravity C-field $\sim\ Y_{11} \to \mathbf{B}^3U(1)_{\mathrm{conn}}$ (plus flux quantization corrections, see [FSSb])
isotropic suspace	Hořava-Witten boundary $X_{10} \hookrightarrow Y_{11}$
universal local coefficient bundle	action of $\mathbf{B}^2U(1)$ on $\mathbf{B}\mathrm{String}(\mathrm{Spin}\times E_8)$: $\begin{array}{ccc} \mathbf{B}\mathrm{String}(\mathrm{Spin}\times E_8) & \longrightarrow & \mathbf{B}\mathrm{Spin}\times E_8 \\ & & \downarrow{\scriptstyle \frac{1}{2}\mathbf{p}_1 - 2\mathbf{a}} \\ & & \mathbf{B}^3U(1) \end{array}$
polarized section	anomaly-free heterotic background gauge fields

With this a quantum 3-state / polarized section is a diagram

$$\begin{tikzcd} X_{10} \arrow[r, "\sigma"] \arrow[d] & \mathbf{B}(\mathrm{Spin}\times E_8)_{\mathrm{conn}} \arrow[d, "(\frac{1}{2}\mathbf{p}_1)_{\mathrm{conn}} - \mathbf{a}_{\mathrm{conn}}"] \\ \Omega^3_{\mathrm{cl}}(-) \arrow[r] & \mathbf{B}^3U(1)_{\mathrm{conn}} \end{tikzcd} .$$

This we recognize from [SSS2] as an anomaly-free heterotic background field configuration. The differential form data over a trivializing cover $U \to X$ encoded by this is [SSS1, FSSt]

- gravity – gauge potential $\omega \in \Omega^1(U, \mathbf{so})$ with field strength F_ω;
- Yang-Mills gauge field – $A \in \Omega^1(U, \mathfrak{e}_8)$ with field strenght F_A;
- B-field – $B \in \Omega^2(-)$ with field strength

$$H = \nabla B = dB + \mathbf{CS}(\omega) - \mathrm{CS}(A) + \mathrm{C}\,.$$

$$dH = \langle F_\omega \wedge F_\omega \rangle - \langle F_A \wedge F_A \rangle\,.$$

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