Synthetic quantum field theory

Talks at

Can. Math. Soc. Summer Meeting 2013

Progress in Higher Categories

Halifax June 7, 2013

Higher Algebras and Lie-infinity Homotopy Theory

Luxembourg June 26, 2013

Type Theory, Homotopy Theory and Univalent Foundations

CRM, September 2013

Urs Schreiber

September 24, 2013

I) Introduction and Overview

(continue reading)

II) Some details and examples

(keep reading after the introduction)

Survey summary of the axiomatics

(turn to as need be)

Hilbert's 6th problem

David Hilbert, ICM, Paris 1900:

Mathematical Problem 6:

To treat [...] by means of **axioms**, *those* **physical sciences** *in which mathematics plays an important part*

[...] try first by a **small number of axioms** *to include as large a class as possible of physical phenomena, and then by adjoining new axioms to arrive gradually at the more special theories.*

[...] take account not only of those theories coming near to reality, but also, [...] of all **logically possible theories**.

Partial Solutions to Hilbert's 6th problem – I) traditional

	physics	**maths**
	prequantum physics	*differential geometry*
18xx-19xx	mechanics	symplectic geometry
1910s	gravity	Riemannian geometry
1950s	gauge theory	Chern-Weil theory
2000s	higher gauge theory	differential cohomology
	quantum physics	*noncommutative algebra*
1920s	quantum mechanics	operator algebra
1960s	local observables	co-sheaf theory
1990s-2000s	local field theory	(∞, n)-category theory

(table necessarily incomplete)

Partial Solutions to Hilbert's 6th problem – II) synthetic

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1. Foundation of **mathematics** in topos theory
("ETCS" [Lawvere 65]).

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by **"synthetic"** formulation:

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by **"synthetic"** formulation:

1. $\left\{ \begin{array}{c} \text{Impose properties on} \\ \text{Add axioms to} \end{array} \right\}$ a $\left\{ \begin{array}{c} \text{topos} \\ \text{intuitionistic type theory} \end{array} \right\}$ which ensure that the $\left\{ \begin{array}{c} \text{objects} \\ \text{types} \end{array} \right\}$ have structure of *differential geometric spaces*.

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2. Then formalize physics by $\left\{ \begin{array}{c} \text{universal constructions} \\ \text{natural deduction} \end{array} \right\}$.

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- *Categorical dynamics* [Lawvere 67]
- *Toposes of laws of motion* [Lawvere 97]
- *Outline of synthetic differential geometry* [Lawvere 98]

But

modern fundamental physics

and

modern foundational maths

are both deeper

than what has been considered in these results...

$\longrightarrow$

Modern natural foundations.

Reconsider Hilbert's 6th in view of modern foundations.

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Modern foundations of fundamental physics is:

local Lagrangian boundary-/defect- quantum gauge field theory

(a recent survey is in [Sati-Schreiber 11])

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$$\boxed{\left\{ \begin{array}{c} \text{homotopy type theory,} \\ \infty\text{-topos theory} \end{array} \right\} + \text{internal } (\infty, n)\text{-category theory}}$$

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Claim

In $\left\{ \begin{array}{c} \textit{homotopy type theory} \\ \infty\textit{-topos theory} \end{array} \right\}$*-foundations*

fundamental physics is synthetically axiomatized

1. *naturally – the axioms are simple, elegant and meaningful;*
2. *faithfully – the axioms capture deep nontrivial phenomena* $\rightarrow$

Project

This is an ongoing project involving joint work with

- Domenico Fiorenza
- Hisham Sati
- Michael Shulman
- Joost Nuiten

and others:

Differential cohomology in a cohesive ∞-topos [Schreiber 11].

You can find publications, further details and further exposition at:

http://ncatlab.org/schreiber/show/differential+cohomology+in+a+cohesive+topos

skip to list of contents

Contents

(higher) gauge-

Lagrangian-

local

(bndry-/defect)-

quantum-

field theory

Contents

	physics	maths	
1)	(higher) gauge-	$\left\{ \begin{array}{l} \infty\text{-topos theory,} \\ \text{homotopy type theory} \end{array} \right.$	
	Lagrangian-		field theory
	local		
	(bndry-/defect)-		
	quantum-		

Contents

	physics	**maths**	
1)	(higher) gauge-	∞-topos theory, homotopy type theory	
2)	Lagrangian-	differential cohomology, cohesion modality	field theory
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Selected **examples and applications**:

Contents

	physics	**maths**	
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Selected **examples and applications**:

Ex1 Classical mechanics and its holographic quantization

Contents

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Ex1 Classical mechanics and its holographic quantization

Ex2 topol. ∞-YM $\overset{\text{bdr}}{\to}$ ∞-CS $\overset{\text{dfct}}{\to}$ ∞-WZW $\overset{\text{dfct}}{\to}$ ∞-Wilson surf.

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Ex3 Super p-branes, e.g. M5 ($\overset{\text{event.}}{\rightsquigarrow}$ Khovanov, Langlands, ...)

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Synthetic QFT Axioms

Synthetic QFT Axioms

(QFT 0) Gauge principle. *Spaces of physical fields are higher moduli stacks:*

$$\left\{ \begin{array}{c} \textit{objects} \\ \textit{types} \end{array} \right\} \textit{ of an } \left\{ \begin{array}{c} \infty\textit{-topos} \\ \textit{homotopy type theory} \end{array} \right\} \mathbf{H}.$$

$$\boxed{\mathbf{Fields} \in \mathbf{H}}$$

We discuss this in more detail below in **1)**.

Synthetic QFT Axioms

(QFT 1) Space of phases.

The $\left\{ \begin{array}{c} \infty\text{-topos} \\ \text{homotopy type theory} \end{array} \right\}$ carries two adjoint triples of $\left\{ \begin{array}{c} \text{idempotent } \infty\text{-(co-)monads} \\ \text{higher modalities} \end{array} \right\}$ that equip $\left\{ \begin{array}{c} \text{objects} \\ \text{types} \end{array} \right\}$ with "differential cohesive" geometric structure.

$$\boxed{\begin{array}{ccccc} \Pi & \dashv & \flat & \dashv & \sharp \\ \mathrm{Red} & \dashv & \Pi_{\mathrm{inf}} & \dashv & \flat_{\mathrm{inf}} \end{array}}$$

This is a joint refinement to homotopy theory of Lawvere's "synthetic differential geometry" and "axiomatic cohesion" [Lawvere 07] .

We discuss this in more detail in **2)** below.

Synthetic QFT Axioms

Theorem

Differential cohesion in homotopy theory implies the existence of differential coefficient $\left\{ \begin{array}{c} \textit{objects} \\ \textit{types} \end{array} \right\}$ *modulating cocycles in differential cohomology.*

$$\begin{array}{ccccc} \flat \mathbf{B}\mathbb{G} & \xrightarrow{\text{include}} & \mathbf{B}\mathbb{G}_{\text{conn}} & \xrightarrow{\text{forget}} & \mathbf{B}\mathbb{G} \\ \text{flat} & & \text{principal} & & \text{principal} \\ \infty\text{-connections} & & \infty\text{-connections} & & \infty\text{-bundles} \end{array}$$

Remark

This is absolutely not the case for differential cohesion interpreted non-homotopically.

Whence the title "*Differential cohomology in a cohesive* ∞*-topos*" [Schreiber 11].

Synthetic QFT Axioms

(QFT 2) Local Lagrangians and action functionals.

The $\left\{ \begin{array}{c} \text{slice objects} \\ \text{dependent types} \end{array} \right\}$ over differential coefficients

$$\boxed{\begin{tikzcd} \mathbf{Fields} \arrow[d, "\exp(iS)"] \\ \mathbf{B}^n \mathbb{G}_{\mathrm{conn}} \end{tikzcd}}$$

are the *local action functionals*.

The $\left\{ \begin{array}{c} \text{correspondence spaces} \\ \text{relations} \end{array} \right\}$

$$\boxed{\begin{tikzcd} & \mathbf{Fields} \arrow[dl] \arrow[dr] & \\ \mathbf{Fields}_1 \arrow[dr, "\exp(iS_1)"'] & \arrow[d, phantom, "\swarrow\!\!\swarrow"] & \mathbf{Fields}_2 \arrow[dl, "\exp(iS_2)"] \\ & \mathbf{B}\mathbb{G}_{\mathrm{conn}} & \end{tikzcd}}$$

are the *field trajectories*,
the *quantum observables*,
and the *defect- and boundary conditions*.

We discuss this in more detail below in **3)**.

Synthetic QFT Axioms

(QFT 3) Quantization.

Quantization is the passage to the "motivic" abelianization of these $\left\{ \begin{array}{c} \text{corespondence spaces} \\ \text{relations} \end{array} \right\}$ of $\left\{ \begin{array}{c} \text{slice objects} \\ \text{dependent types} \end{array} \right\}$ over the differential coefficients.

$$
\begin{tikzcd}
\mathrm{Bord}_n^{\mathrm{bdr}} \arrow[r, "\exp(iS)\, d\mu"'] \arrow[rr, bend left, "\int\limits_{\phi \in \mathbf{Fields}} \exp(iS(\phi))\, d\mu(\phi)"] & \mathrm{Corr}_n^{\mathrm{or}}(\mathbf{H}, \mathbf{B}\mathbb{G}) \arrow[r, "\int (-)\, d\mu(-)"'] & E\mathrm{Mod}(\mathbf{H})
\end{tikzcd}
$$

We discuss this in more detail below in **4)**.

This is established in particular for 2-dimensional theories and their holographic 1-d boundary theories (quantum mechanics) by **Ex1** below.

End

of overview.

→ back to project page

→ on to further details

1)
Higher gauge field theory
∞-Topos theory
Homotopy type theory

back to list of contents

From the gauge principle to higher stacks.

Central principle of modern fundamental **physics** –
the gauge principle:

- Field configurations may be different and yet *gauge equivalent*.
- Gauge equivalences may be different and yet *higher gauge equivalent*.
- Collection of fields forms *BRST complex*, where (higher) gauge equivalences appear as (higher) ghost fields.

This means that moduli spaces of fields are

$$\textbf{geometric homotopy types} \quad \simeq \quad \textbf{higher moduli stacks}$$
$$\simeq \textbf{objects of an } \infty\textbf{-topos } \mathbf{H}$$

$\rightarrow$

Higher moduli stacks of gauge fields

- a moduli stack of fields is **Fields** $\in$ **H**
- a **field configuration** on a $\begin{array}{c}\text{spacetime}\\ \text{worldvolume}\end{array}$ Σ is a map $\phi : \Sigma \to$ **Fields**;
- a *gauge transformation* is a homotopy $\kappa : \phi_1 \overset{\simeq}{\to} \phi_2 : \Sigma \to$ **Fields**
- a *higher gauge transformation* is a higher homotopy;
- the *BRST complex of gauge fields* on Σ is the infinitesimal approximation to the mapping stack $[\Sigma, \mathbf{Fields}]$.

Examples:

- for sigma-model field theory: **Fields** $= X$ is target space;
- for gauge field theory: **Fields** $= \mathbf{B}G_{\mathrm{conn}}$ is moduli stack of G-principal connections.
- in general both: σ-model fields and gauge fields are unified, for instance in "tensor multiplet" on super p-brane, Example 3 below

2)
Lagrangian field theory
Differential cohomology
Cohesion modality

back to list of contents

The action principle

For

- $\Sigma_{\mathrm{in}} \longrightarrow \Sigma \longleftarrow \Sigma_{\mathrm{out}}$ a cobordism (a *Feynman diagram*)
- $\mathbf{Fields}(\Sigma_{\mathrm{in}}) \xleftarrow{(-)|_{\Sigma_{\mathrm{in}}}} \mathbf{Fields}(\Sigma) \xrightarrow{(-)|_{\Sigma_{\mathrm{out}}}} \mathbf{Fields}(\Sigma_{\mathrm{out}})$ the space of *trajectories* of fields,

the *action functional* assigns a phase to each trajectory

$$\exp(iS_\Sigma) : \mathbf{Fields}(\Sigma) \to U(1)$$

and this is *Lagrangian* if there is *differential form* data $\mathbf{L} : \mathbf{Fields} \to \flat \mathbf{B}^n U(1)$ such that

$$\begin{tikzcd}
& \mathbf{Fields}(\Sigma) \arrow[dl] \arrow[dr] & \\
\mathbf{Fields}(\Sigma_{\mathrm{in}}) \arrow[dr] & \arrow[phantom, "{\exp(iS_\Sigma) = \exp(i\int_\Sigma \mathbf{L}) \Swarrow}"] & \mathbf{Fields}(\Sigma_{\mathrm{out}}) \arrow[dl] \\
& \flat \mathbf{B} U(1) &
\end{tikzcd}$$

The need for differential cohesion

In order to formalize the action principle on gauge fields we hence need to

1. Characterize those $\left\{ \begin{array}{c} \infty\text{-toposes} \\ \text{homotopy type theories} \end{array} \right\}$ $\mathbf{H}$ whose $\left\{ \begin{array}{c} \text{objects} \\ \text{types} \end{array} \right\}$ may be interpreted as *differential geometric* spaces.
2. Axiomatize differential geometry and differential cohomology in such contexts.

$\rightarrow$ *differential cohesion*

The adjunction system defining differential cohesion

$$\begin{tikzcd} \mathbf{H} \arrow[rr, shift right, "\Gamma"'] & & \infty\mathrm{Grpd} \arrow[ll, shift right, "\mathrm{LConst}"'] \end{tikzcd}$$

Every ∞-stack ∞-topos has an essentially unique global section geometric morphism to the base ∞-topos.

The adjunction system defining differential cohesion

$$\begin{tikzcd} \mathbf{H} \arrow[rr, shift right, "\Gamma"'] & & \infty\mathrm{Grpd} \arrow[ll, hook', shift right, "\mathrm{Disc}"'] \end{tikzcd}$$

Requiring the formation of locally constant ∞-stacks to be a full embedding means that we have a notion of *geometrically discrete objects* in $\mathbf{H}$.

The adjunction system defining differential cohesion

$$\begin{tikzcd} \mathbf{H} \arrow[r, "\Gamma"] & \infty\mathrm{Grpd} \arrow[l, hook', bend right, "\mathrm{Disc}"'] \arrow[l, hook, bend left, "\mathrm{coDisc}"] \end{tikzcd}$$

Requiring the existence of an extra right adjoint means that we also have the inclusion of geometrically co-discrete (indiscrete) objects.

The adjunction system defining differential cohesion

$$\begin{tikzcd}[column sep=huge] \mathbf{H} \arrow[r, "\Gamma" description] & \infty\mathrm{Grpd} \arrow[l, hook', bend right=15, "\mathrm{Disc}" description] \arrow[l, hook, bend left=15, "\mathrm{coDisc}" description] \end{tikzcd}$$

Now Γ has the interpretation of sending a geometric homotopy type to its underlying ∞-groupoid of points, forgetting the geometric structure.

The adjunction system defining differential cohesion

$$
\begin{tikzcd}
\mathbf{H} \arrow[r, shift left=3, "\Pi"] \arrow[r, shift right=1, "\Gamma"] & \infty\mathrm{Grpd} \arrow[l, hook', shift right=1, "\mathrm{Disc}"'] \arrow[l, hook', shift left=3, "\mathrm{coDisc}"']
\end{tikzcd}
$$

The crucial thing now is that for the ∞-topos $\mathbf{H}$ an extra left adjoint Π sends a geometric homotopy type to its *path ∞-groupoid* or *geometric realization*.

The adjunction system defining differential cohesion

$$\begin{tikzcd}[column sep=huge]
\mathbf{H} \arrow[r, shift left=6, "\Pi"] \arrow[r, shift right=2, "\Gamma"] & \infty\mathrm{Grpd} \arrow[l, hook', shift right=2, "\mathrm{Disc}"'] \arrow[l, hook', shift left=6, "\mathrm{coDisc}"]
\end{tikzcd}$$

If we further require that to preserve finite products then this means that the terminal object in $\mathbf{H}$ is geometrically indeed the point.

The adjunction system defining differential cohesion

$$
\begin{tikzcd}
\mathbf{H} \arrow[r, shift left=4.5, "\Pi"] \arrow[r, shift right=1.5, "\Gamma"] & \infty\mathrm{Grpd} \arrow[l, hook', shift right=1.5, "\mathrm{Disc}"'] \arrow[l, hook', shift left=4.5, "\mathrm{coDisc}"]
\end{tikzcd}
$$

If an adjoint quadruple of this form exists on $\mathbf{H}$ we say that $\mathbf{H}$ *is cohesive* or that its objects have the structure of *cohesively geometric homotopy types*.

The adjunction system defining differential cohesion

$$
\begin{tikzcd}
\mathbf{H}_{\mathrm{red}} \arrow[rr, hook, shift right, "i_*"'] & & \mathbf{H} \arrow[ll, shift right, "i^*"'] \arrow[rr, shift left=3, "\Pi"] \arrow[rr, shift right, "\Gamma"'] & & \infty\mathrm{Grpd} \arrow[ll, hook', shift right, "\mathrm{Disc}"'] \arrow[ll, hook, shift left=3, "\mathrm{coDisc}"]
\end{tikzcd}
$$

Consider moreover the inclusion of a cohesive sub-∞-topos $\mathbf{H}_{\mathrm{red}}$.

The adjunction system defining differential cohesion

$$
\begin{tikzcd}[column sep=huge]
\mathbf{H}_{\mathrm{red}}
  \arrow[r, hook, shift left=3, "i_!"]
  \arrow[r, hook, shift right=1, "i_*"']
& \mathbf{H}
  \arrow[l, shift left=1, "i^*"']
  \arrow[r, shift left=3, "\Pi"]
  \arrow[r, shift right=1, "\Gamma"']
& \infty\mathrm{Grpd}
  \arrow[l, hook', shift right=1, "\mathrm{Disc}"']
  \arrow[l, hook', shift left=3, "\mathrm{coDisc}"]
\end{tikzcd}
$$

If this has an extra left adjoint then this means that i^* is a projection map that contracts away from each object a geometric thickening *with no points*.

The adjunction system defining differential cohesion

$$
\begin{tikzcd}[column sep=huge]
\mathbf{H}_{\mathrm{red}}
\arrow[r, hook, shift left=3, "i_!" description]
\arrow[r, leftarrow, shift left=1, "i^*" description]
\arrow[r, hook, shift right=1, "i_*" description]
& \mathbf{H}
\arrow[r, shift left=3, "\Pi" description]
\arrow[r, hookleftarrow, shift left=1, "\mathrm{Disc}" description]
\arrow[r, shift right=1, "\Gamma" description]
\arrow[r, hookleftarrow, shift right=3, "\mathrm{coDisc}" description]
& \infty\mathrm{Grpd}
\end{tikzcd}
$$

This means that objects of $\mathbf{H}$ may have *infinitesimal thickening* ("formal neighbourhoods") and that $\mathbf{H}_{\mathrm{red}}$ is the full sub-∞-topos of the "reduced" objects: that have no infinitesimal thickening.

The adjunction system defining differential cohesion

$$
\begin{tikzcd}[column sep=huge]
\mathbf{H}_{\mathrm{red}} \arrow[r, hook, shift left=4.5, "i_!" description] \arrow[r, <-, shift left=1.5, "i^*" description] \arrow[r, hook, shift right=1.5, "i_*" description] \arrow[r, <-, shift right=4.5, "i^!" description] \arrow[rr, bend left=25, "\times"' near start] & \mathbf{H} \arrow[r, shift left=4.5, "\Pi" description, "\times"' near start] \arrow[r, <-, hook', shift left=1.5, "\mathrm{Disc}" description] \arrow[r, shift right=1.5, "\Gamma" description] \arrow[r, <-, hook', shift right=4.5, "\mathrm{coDisc}" description] & \infty\mathrm{Grpd}
\end{tikzcd}
$$

Finally that $\mathbf{H}_{\mathrm{red}}$ is itself cohesive means that $\Pi|_{\mathbf{H}_{\mathrm{red}}} = \Pi \circ i_!$ also preserves finite products.

From adjunctions to monads and modalities.

Such a system of two quadruple reflections on $\mathbf{H}$ is equivalently a system of two triple $\left\{ \begin{array}{c} \text{idempotent } \infty\text{-(co-)monads on} \\ \text{higher modalities in} \end{array} \right\}$ $\mathbf{H}$.

- $(\Pi \dashv \flat \dashv \sharp)$: $\mathbf{H} \begin{array}{c} \xrightarrow{\Pi} \\ \hookleftarrow_{\mathrm{Disc}} \\ \xrightarrow{\Gamma} \end{array} \infty\mathrm{Grpd} \begin{array}{c} \xhookrightarrow{\mathrm{Disc}} \\ \xleftarrow{\Gamma} \\ \xhookrightarrow{\mathrm{coDisc}} \end{array} \mathbf{H}$

- $(\mathrm{Red} \dashv \Pi_{\mathrm{inf}} \dashv \flat_{\mathrm{inf}})$: $\mathbf{H} \begin{array}{c} \xrightarrow{i^*} \\ \hookleftarrow_{i_*} \\ \xrightarrow{i^!} \end{array} \mathbf{H}_{\mathrm{red}} \begin{array}{c} \xhookrightarrow{i_!} \\ \xleftarrow{i^*} \\ \xhookrightarrow{i_*} \end{array} \mathbf{H}$

The modality system defining differential cohesion.

Π	shape modality	(idemp. ∞-monad)
$\perp$		
$\flat$	flat modality	(idemp. ∞-co-monad)
$\perp$		
$\sharp$	sharp modality	(idemp. ∞-monad)
Red	reduction modality	(idemp. ∞-co-monad)
$\perp$		
Π_{inf}	infinitesimal shape modality	(idemp. ∞-monad)
$\perp$		
$\flat_{\mathrm{inf}}$	infinitesimal flat modality	(idemp. ∞-co-monad)

The modality system defining differential cohesion.

Π	shape modality
$\perp$	
$\flat$	flat modality
$\perp$	
$\sharp$	sharp modality
Red	reduction modality
$\perp$	
Π_{inf}	infinitesimal shape modality
$\perp$	
$\flat_{\mathrm{inf}}$	infinitesimal flat modality

Models for differential cohesion

The following example accommodates most of contemporaty fundamental physics. (See Example 3 below for more.)

Theorem

Let $\mathrm{CartSp}_{\substack{\mathrm{synth} \\ \mathrm{super}}} := \left\{\mathbb{R}^{p|q;k} = \mathbb{R}^p \times \mathbb{R}^{0|q} \times D^k\right\}_{p,q,k\in\mathbb{N}}$ *be the site of Cartesian formal supergeometric smooth manifolds with its standard open cover topology. The ∞-stack ∞-topos over it*

$$\mathrm{SynthDiffSuperSmooth}\infty\mathrm{Grpd} := \mathrm{Sh}_\infty(\mathrm{CartSp}_{\substack{\mathrm{synth} \\ \mathrm{super}}})$$

is differentially cohesive.

Objects are synthetic differential super-geometric smooth ∞-groupoids.

Remark

This is the homotopy-theoretic and super-geometric refinement of the traditional model for synthetic differential geometry known as the "Cahiers topos". [Dubuc 79].

References: Related work on differential cohesion

- The notion of differential cohesive ∞-toposes is a joint refinement to homotopy theory of W. Lawvere's
  - *synthetic differential geometry* [Lawvere 67, Dubuc 79]
  - *cohesion* [Lawvere 07]

  With hindsight one can see that the article *Some thoughts on the future of category theory* [Lawvere 91] is all about cohesion. What is called a "category of Being" there is a cohesive topos.
- Aspects of the infinitesimal modality triple $(\mathrm{Red} \dashv \Pi_{\mathrm{inf}} \dashv \flat_{\mathrm{inf}})$ appear
  - in [Simpson-Teleman 97] for the formulation of de Rham spacks;
  - in [Kontsevich-Rosenberg 04] for the axiomatization of formally étale maps.

3) Local field theory Higher category theory Higher relations

back to list of contents

(...) [Fiorenza-Schreiber 13a] (...)

Observation

$$\mathrm{Bord}_n^{\mathrm{bry}} \longrightarrow \mathrm{Corr}_n(\mathbf{H}, \flat \mathbf{B}^n U(1))$$

$$(\;\vdash\!\!\longrightarrow * \;) \quad \mapsto \quad
\begin{tikzcd}
 & \mathbf{Fields}_{\color{red}\partial} \arrow[dl] \arrow[dr] & \\
 * \arrow[dr, "0"'] \arrow[rr, phantom, "\swArrow"] & & \mathbf{Fields} \arrow[dl, "\exp(iS)"] \\
 & \flat \mathbf{B}^n U(1) &
\end{tikzcd}
.$$

By theorem 4.3.11 in [L09a].

References: Related work on local QFT by correspondences

- An early unfinished note is [Schreiber 08]
- For the special case of discrete higher gauge theory (∞-Dijkgraaf-Witten theory) a sketch of a theory is in section 3 and 8 of [Freed-Hopkins-Lurie-Teleman 09].

4)

Quantum field theory
Motivic cohomology
Linearized relations

back to list of contents

(with J. Nuiten)

Motivic quantization

The last step – quantization of local prequantum field theory to local quantum field theory– is clearly the most interesting but also the most subtle one.

We indicate now:

1. **a)** The outline of a general abstract formulation.
2. **b)** A concrete implementation for 2-dimensional QFT in the model of smooth cohesion.
3. **c)** A class of examples for the 2-dimensional implementation which reproduces traditional quantum theory.

Motivic quantization – Physics heuristics

$$
\begin{tikzcd}
& \mathbf{Fields}_{\mathrm{traj}} \arrow[dl, "i_{\mathrm{in}}"'] \arrow[dr, "i_{\mathrm{out}}"] & \\
\mathbf{Fields}_{\mathrm{in}} & & \mathbf{Fields}_{\mathrm{out}}
\end{tikzcd}
$$

A space of *field trajectories* is a correspondence of space of fields.

Motivic quantization – Physics heuristics

$$\begin{tikzcd} & \mathbf{Fields}_{\mathrm{traj}} \arrow[dl, "i_{\mathrm{in}}"'] \arrow[dr, "i_{\mathrm{out}}"] & \\ \mathbf{Fields}_{\mathrm{in}} & & \mathbf{Fields}_{\mathrm{out}} \end{tikzcd}$$

$$\begin{tikzcd} & \Sigma & \\ \Sigma_{\mathrm{in}} \arrow[ur, hook] & & \Sigma_{\mathrm{out}} \arrow[ul, hook'] \end{tikzcd}$$

For instance given a cobordism Σ and a moduli space of fields **Fields**...

Motivic quantization – Physics heuristics

$$
\begin{tikzcd}
 & \mathbf{Fields}_{\mathrm{traj}} \arrow[ld, "i_{\mathrm{in}}"'] \arrow[rd, "i_{\mathrm{out}}"] & \\
\mathbf{Fields}_{\mathrm{in}} & {[\Sigma, \mathbf{Fields}]} \arrow[ld, "(-)|_{\Sigma_{\mathrm{in}}}"'] \arrow[rd, "(-)|_{\Sigma_{\mathrm{out}}}"] & \mathbf{Fields}_{\mathrm{out}} \\
{[\Sigma_{\mathrm{in}}, \mathbf{Fields}]} & \Sigma & {[\Sigma_{\mathrm{out}}, \mathbf{Fields}]} \\
\Sigma_{\mathrm{in}} \arrow[ur, hook] & & \Sigma_{\mathrm{out}} \arrow[ul, hook']
\end{tikzcd}
$$

... then fields on Σ form trajectories between the fields on Σ_{in} to the fields on Σ_{out}.

Motivic quantization – Physics heuristics

$$\begin{tikzcd}
 & \textbf{Fields}_{\mathrm{traj}} \arrow[dl, "i_{\mathrm{in}}"'] \arrow[dr, "i_{\mathrm{out}}"] & \\
\textbf{Fields}_{\mathrm{in}} & \textbf{Fields}_{\mathrm{bdr}} \arrow[dl] \arrow[dr] & \textbf{Fields}_{\mathrm{out}} \\
* & & \textbf{Fields}
\end{tikzcd}$$

Or $\textbf{Fields}_{\mathrm{in}} = *$ is trivial, and $\textbf{Fields}_{\mathrm{bdr}}$ encodes a boundary condition for a bulk theory of **Fields**.

Motivic quantization – Physics heuristics

$$
\begin{tikzcd}
& \mathbf{Fields}_{\mathrm{traj}} \arrow[dl, "i_{\mathrm{in}}"'] \arrow[dr, "i_{\mathrm{out}}"] \arrow[dd, "\exp(iS_{\mathrm{codim}=0})" description] & \\
\mathbf{Fields}_{\mathrm{in}} & & \mathbf{Fields}_{\mathrm{out}} \\
& U(1) &
\end{tikzcd}
$$

Traditionally, in codimension 0, an *exponentiated action functional* is a function $\exp(iS)$ from trajectories to $U(1)$.

Motivic quantization – Physics heuristics

$$\begin{tikzcd}
 & \mathbf{Fields}_{\mathrm{traj}} \arrow[dl, "i_{\mathrm{in}}"'] \arrow[dr, "i_{\mathrm{out}}"] & \\
\mathbf{Fields}_{\mathrm{in}} \arrow[dr, "0"'] & \arrow[d, phantom, "\Swarrow \exp(iS_{\mathrm{codim}=0})"] & \mathbf{Fields}_{\mathrm{out}} \arrow[dl, "0"] \\
 & \mathbf{B}U(1) &
\end{tikzcd}$$

More naturally this realized a *homotopy* between two trivial maps to $\mathbf{B}U(1)$.

Motivic quantization – Physics heuristics

$$\begin{tikzcd}
 & {[I,X]} \arrow[dl, "(-)_0"'] \arrow[dr, "(-)_1"] & \\
X \arrow[dr, "\chi(\nabla)"'] \arrow[rr, phantom, "\swarrow \exp(i\int_I [I,\nabla])"] & & X \arrow[dl, "\chi(\nabla)"] \\
 & \mathbf{B}U(1) &
\end{tikzcd}$$

But consider the example the particle on X charged under an electromagnetic field ∇. Here the action functional over an wordline with boundaries $\Sigma = I = [0,1]$ is a section of the pullback of the background field to path space $[I,X]$.

Motivic quantization – Physics heuristics

$$
\begin{tikzcd}
& \mathbf{Fields}_{\mathrm{traj}} \arrow[dl, "i_{\mathrm{in}}"'] \arrow[dr, "i_{\mathrm{out}}"] & \\
\mathbf{Fields}_{\mathrm{in}} \arrow[dr, "\exp(iS_{\mathrm{in}})"'] & & \mathbf{Fields}_{\mathrm{out}} \arrow[dl, "\exp(iS_{\mathrm{out}})"] \arrow[ll, phantom, "\exp(iS_{\mathrm{codim}=(n-1)}) \Swarrow"] \\
& \mathbf{B}^n U(1) &
\end{tikzcd}
$$

So in general the local action functional on trajectories in codimension $(n-1)$ is a homotopy in $\mathbf{B}^n U(1)$ between $\exp(iS_{\mathrm{in}})$ and $\exp(iS_{\mathrm{out}})$

Motivic quantization – Physics heuristics

$$\begin{tikzcd}
& \mathbf{Fields}_{\mathrm{traj}} \arrow[dl, "i_{\mathrm{in}}"'] \arrow[dr, "i_{\mathrm{out}}"] & \\
\mathbf{Fields}_{\mathrm{in}} \arrow[dr, "\exp(iS_{\mathrm{in}})"'] \arrow[rr, phantom, "\swarrow\!\!\swarrow\ \exp(iS_{\mathrm{codim}=(n-1)})"] & & \mathbf{Fields}_{\mathrm{out}} \arrow[dl, "\exp(iS_{\mathrm{out}})"] \\
& \mathbf{B}^n U(1) \arrow[d, "\rho"] & \\
& \mathbf{B}\mathrm{GL}_1(E) \arrow[d] & \\
& E\mathrm{Mod} &
\end{tikzcd}$$

A choice of *linear representation* $\rho : \mathbf{B}^n U(1) \to \mathbf{B}\mathrm{GL}_1(E)$ for E a commutative ∞-ring makes this an *integral kernel*.

Motivic quantization – Physics heuristics

$$\begin{tikzcd}
 & \mathbf{Fields}_{\mathrm{traj}} \arrow[dl, "i_{\mathrm{in}}"'] \arrow[dr, "i_{\mathrm{out}}"] \arrow[d, phantom, "\swarrow \exp(iS_{\mathrm{codim}=(n-1)})"] & \\
\mathbf{Fields}_{\mathrm{in}} \arrow[dr, "\exp(iS_{\mathrm{in}})"] \arrow[ddddr, bend right, "L_{\mathrm{in}}"'] & & \mathbf{Fields}_{\mathrm{out}} \arrow[dl, "\exp(iS_{\mathrm{out}})"'] \arrow[ddddl, bend left, "L_{\mathrm{out}}"] \\
 & \mathbf{B}^n U(1) \arrow[d, "\rho"] & \\
 & \mathbf{B}\mathrm{GL}_1(E) \arrow[d] & \\
 & E\mathrm{Mod} &
\end{tikzcd}$$

Now $L := \rho\left(\exp\left(iS\right)\right)$ is the associated higher *prequantum E-line bundle*.

Motivic quantization – Physics heuristics

$$
\begin{tikzcd}
& \mathbf{Fields}_{\mathrm{traj}} \arrow[dl, "i_{\mathrm{in}}"'] \arrow[dr, "i_{\mathrm{out}}"] \arrow[dd, phantom, "\Swarrow\, \exp(iS_{\mathrm{codim}=(n-1)})"] & \\
\mathbf{Fields}_{\mathrm{in}} \arrow[dr, "\exp(iS_{\mathrm{in}})"] \arrow[dddr, bend right, "L_{\mathrm{in}}"'] & & \mathbf{Fields}_{\mathrm{out}} \arrow[dl, "\exp(iS_{\mathrm{out}})"'] \arrow[dddl, bend left, "L_{\mathrm{out}}"] \\
& \mathbf{B}^n U(1) \arrow[d, "\rho"] & \\
& \mathbf{B}\mathrm{GL}_1(E) \arrow[d] & \\
& E\mathrm{Mod} &
\end{tikzcd}
$$

$$
E^{\bullet + L_{\mathrm{in}}}(\mathbf{Fields}_{\mathrm{in}}) \xleftarrow{i_{\mathrm{in}}^*} E^{\bullet + i_{\mathrm{out}}^*}(\mathbf{Fields}_{\mathrm{traj}}) \xrightarrow{i_{\mathrm{out}}^*} E^{\bullet + L_{\mathrm{out}}}(\mathbf{Fields}_{\mathrm{out}})
$$

Sections $E^{\bullet + L}$ of L are *wavefunctions* hence *quantum states*.

Motivic quantization – Physics heuristics

$$E^{\bullet + L_{\mathrm{in}}}(\mathbf{Fields}_{\mathrm{in}}) \xrightarrow{i_{\mathrm{in}}^*} E^{\bullet + i_{\mathrm{out}}^*}(\mathbf{Fields}_{\mathrm{traj}}) \xleftarrow{i_{\mathrm{out}}^*} E^{\bullet + L_{\mathrm{out}}}(\mathbf{Fields}_{\mathrm{out}})$$

Hence the integral kernel induced from a local action functional $\exp(iS)$ on a space of trajectories $\mathbf{Fields}_{\mathrm{traj}}$ with respect to a superposition principle ρ is a co-correspondence of E-linear maps between E-modules of sections $E^{\bullet+L}(\mathbf{Fields})$.

Here $E^{\bullet+L}(-)$ is known to be equivalently the *L-twisted E-cohomology spectrum*. Integration in twisted E-cohomology is *twisted push-forward*.

Motivic quantization – Physics heuristics

$$\begin{tikzcd} E^{\bullet + L_{\mathrm{in}}}(\mathbf{Fields}_{\mathrm{in}}) \arrow[r, "i_{\mathrm{in}}^*"] \arrow[rr, bend right, "i_{\mathrm{out}}^! \circ i_{\mathrm{in}}^*"'] & E^{\bullet + i_{\mathrm{out}}^* }(\mathbf{Fields}_{\mathrm{traj}}) \arrow[r, "i_{\mathrm{out}}^!"] & E^{\bullet + L_{\mathrm{out}}}(\mathbf{Fields}_{\mathrm{out}}) \end{tikzcd}$$

A choice of *orientation* of i in twisted E-cohomology allows to form the *twisted push-forward map* $i^!$ as in [ABG 10].

Result is cocycle in $(L_{\mathrm{in}}, L_{\mathrm{out}})$-*twisted bivariant* E*-cohomology*.

Motivic quantization – Formalization

$$\begin{tikzcd} \mathrm{Bord}_n^{\otimes} \arrow[rr, bend left, "Z"] & & \mathcal{C}^{\otimes} \end{tikzcd}$$

A generic *topological field theory* is a monoidal (∞, n)-functor Z.

Motivic quantization – Formalization

$$\begin{tikzcd} \mathrm{Bord}_n^{\otimes} \arrow[rr, bend left, "Z"] & & \mathcal{C}^{\otimes} \end{tikzcd}$$

Requiring it to arise via quantization from a local prequantum field theory means...

Motivic quantization – Formalization

$$\begin{tikzcd}
\mathrm{Bord}_n^{\otimes} \arrow[rr, bend left, "Z"] \arrow[dr, "\mathbf{Fields}"'] & & \mathcal{C}^{\otimes} \\
& \mathrm{Corr}_n(\mathbf{H})^{\otimes} &
\end{tikzcd}$$

...first to pick a moduli ∞-stack **Fields** of fields...

Motivic quantization – Formalization

$$\begin{tikzcd}
\mathrm{Bord}_n^{\otimes} \arrow[rrr, bend left, "Z"] \arrow[dr, "\exp(iS)"] \arrow[ddr, "\textbf{Fields}"'] & & & \mathcal{C}^{\otimes} \\
& \mathrm{Corr}_n(\mathbf{H}_{/\mathbf{B}\mathrm{GL}_1(E)})^{\otimes} \arrow[d] & & \\
& \mathrm{Corr}_n(\mathbf{H})^{\otimes} & &
\end{tikzcd}$$

...second to pick a local action functional $\exp(iS)$...

Motivic quantization – Formalization

$$\begin{tikzcd}
\mathrm{Bord}_n^{\otimes} \arrow[r, "\exp(iS)d\mu"] \arrow[rr, bend left, "Z"] \arrow[dr, "\exp(iS)"] \arrow[ddr, "\textbf{Fields}"'] & \mathrm{Corr}_n^{\mathrm{or}}(\mathbf{H}_{/\mathbf{B}\mathrm{GL}_1(E)})^{\otimes} \arrow[d] & \mathcal{C}^{\otimes} \\
 & \mathrm{Corr}_n(\mathbf{H}_{/\mathbf{B}\mathrm{GL}_1(E)})^{\otimes} \arrow[d] & \\
 & \mathrm{Corr}_n(\mathbf{H})^{\otimes} &
\end{tikzcd}$$

...third to pick a path integral measure $\exp(iS)d\mu$...

Motivic quantization – Formalization

$$\begin{tikzcd}
\mathrm{Bord}_n^{\otimes} \arrow[r, "\exp(iS)d\mu"] \arrow[dr, "\exp(iS)"] \arrow[ddr, "\textbf{Fields}"'] \arrow[rr, bend left, "Z"] & \mathrm{Corr}_n^{\mathrm{or}}(\mathbf{H}_{/\mathbf{B}\mathrm{GL}_1(E)})^{\otimes} \arrow[r, "\int(-)d(-)"] \arrow[d] & \mathcal{C}^{\otimes} \\
 & \mathrm{Corr}_n(\mathbf{H}_{/\mathbf{B}\mathrm{GL}_1(E)})^{\otimes} \arrow[d] & \\
 & \mathrm{Corr}_n(\mathbf{H})^{\otimes} &
\end{tikzcd}$$

...such that pull-push integration $\int(-)d(-)$ in twisted E-cohomology is well defined.

Motivic quantization – Formalization

$$
\begin{tikzcd}
\mathrm{Bord}_n^{\otimes} \arrow[r, "\exp(iS)d\mu"] \arrow[rr, bend left, "Z = \int_{\phi \in \mathbf{Fields}} \exp(iS(\phi))\, d\mu(\phi)"] \arrow[dr, "\exp(iS)"] \arrow[ddr, "\mathbf{Fields}"'] & \mathrm{Corr}_n^{\mathrm{or}}(\mathbf{H}_{/\mathbf{B}\mathrm{GL}_1(E)})^{\otimes} \arrow[r, "\int(-)d(-)"] \arrow[d] & \mathcal{C}^{\otimes} \\
& \mathrm{Corr}_n(\mathbf{H}_{/\mathbf{B}\mathrm{GL}_1(E)})^{\otimes} \arrow[d] & \\
& \mathrm{Corr}_n(\mathbf{H})^{\otimes} &
\end{tikzcd}
$$

Then the composite $\int\limits_{\phi \in \mathbf{Fields}} \exp(iS(\phi))\, d\mu(\phi)$ is the quantized field theory.

Theorem (Nuiten)

1. *On nice enough correspondences of differentiable stacks, forming twisted Lie groupoid convolution algebras constitutes a functor*

$$\mathrm{Corr}_2^{\mathrm{nice}}(\mathrm{DiffStacks}, \mathbf{B}^2 U(1)) \xrightarrow{\int [D\phi](-) := C^*(-)} \mathrm{KUMod}$$

to KU*-modules...*

2. *...such that postcomposition with a prequantum boundary field theory*

$$\begin{tikzcd} \mathrm{Bord}_2^{\mathrm{bdr}} \arrow[r, "\exp(iS)"'] \arrow[rr, bend left, "\int [D\phi] \exp(iS(\phi))"] & \mathrm{Corr}_2(\mathrm{DiffStacks}, \mathbf{B}^2 U(1)) \arrow[r, "\int [D\phi](-)"'] & \mathrm{KUMod} \end{tikzcd}$$

produces K-theoretic geometric quantization of Poisson manifolds – Example 1 below.

References: motivic quantization

- Joost Nuiten, *Cohomological quantization of local boundary prequantum field theory*, master thesis, Utrecht 2013, http://ncatlab.org/schreiber/show/master+thesis+Nuiten
- Urs Schreiber, *Motivic quantization of prequantum field theory*, talk at GAP XI Higher Geometry and Quantum Field Theory, http://ncatlab.org/schreiber/show/Motivic+quantization+of+local+prequantum+field+theory
- *n*Lab, *motivic quantization*, http://ncatlab.org/nlab/show/motivic+quantization

References: Related work on motivic quantization

- Bott: quantization of Kähler polarized symplectic manifolds is index map of spin^c-Dirac operator twisted by prequantum bundle;
- [Hörmander 71][Weinstein 71]: the natural domain of quantization are Lagrangian correspondences
- [Landsman 03][Landsman 10]: the natural target of quantization is KK-theory;
- [Connes-Consani-Marcolli 05]: KK-theory is motivic cohomology in noncommutative topology;
- [Brodzki-Mathai-Rosenberg-Szabo 09]: quantize D-branes and T-duality correspondences by index in KK-theory;
- [Baez-Dolan 09]: quantize correspondences of finite groupoids to linear maps of finite vector spaces
- [BenZvi-Francis-Nadler 08]: quantize corespondences of perfect ∞-stacks to maps of stable ∞-categories;
- [Freed-Hopkins-Lurie-Teleman 09] [Lurie 12]: quantize correspondences of finite ∞-groupoids to maps of n-vector spaces.

Survey summary of the axiomatics

back to list of contents

on to examples

Examples

<u>Ex1</u> Classical mchanics and its holographic quantization.

<u>Ex2</u> topol. ∞-YM $\overset{\text{bdr}}{\to}$ ∞-CS $\overset{\text{dfct}}{\to}$ ∞-WZW $\overset{\text{dfct}}{\to}$ ∞-Wilson surf.

<u>Ex3</u> Super p-branes, e.g. M5 ($\overset{\text{event.}}{\rightsquigarrow}$ Khovanov, Langlands, ...)

back to list of contents

Example 1

Classical mechanics and its holographic quantization

back to list of contents

As the most basic example of
the synthetic formulation of quantum field theory
we indicate now

1. The formulation of classical mechanics.
2. Its quantization to quantum mechanics.

Newton's laws

Newton's laws say that *mechanics* is described by *differential equations of second order*. Traditionally one writes

$$a = F/m$$

to mean

Acceleration "a" is proportional to prescribed external *forces* "F".

Here if

- q denotes the position of a particle
- of mass m
- at time t

then

$$a := \ddot{q} := \frac{d^2 q}{dt^2} .$$

Phase space

Therefore the *initial value data* for mechanics is specified by

1. *positions* q in space
2. *momenta* p: their first derivatives $p := m\dot{q}$.

And so

1. the *phase of motion* of a mechanical system is coordinatized by positions and momenta (q, p);
2. the *phase space* of a system with k degrees of freedom is locally a Cartesian space of the form

$$\mathbb{R}^{2k} = \mathbb{R}^k_{\text{positions}} \oplus \mathbb{R}^k_{\text{momenta}} \,.$$

Hamilton's equations

Let

$$H : \mathbb{R}^{2k} \longrightarrow \mathbb{R}$$

be a smooth function, to be thought of as sending the phase of motion (q, p) to its *energy*.

Definition

A trajectory $(q, p) : \mathbb{R} \longrightarrow \mathbb{R}^{2k}$ satisfies *Hamilton's equations* if

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}\,.$$

Standard form of Hamiltonian energy

The standard form of the energy is

$$\begin{aligned} H &= H_{\mathrm{kin}} + H_{\mathrm{pot}} \\ &= \tfrac{1}{2m} p^2 + V(q) \end{aligned}$$

for

$$V : \mathbb{R}^n_{\mathrm{position}} \longrightarrow \mathbb{R}$$

a smooth "potential" function.
In this case the first Hamilton equation identifies momentum proportionally with velocity

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{\partial H_{\mathrm{kin}}}{\partial p} = p/m$$

and the second Hamilton equation reproduces Newton's law for a force that is the gradient of the potential:

$$\dot{p} = -\frac{\partial H}{\partial q} = -\frac{\partial H_{\mathrm{pot}}}{\partial q} =: F\,.$$

Hamilton's equations in Symplectic geometry

The traditional modern formulation of Hamilton's equations (e.g. Arnold 89) is in *symplectic geometry*:

Fact

Define the differential 2-form (notational supressing the contraction)

$$\omega := \mathbf{d}p \wedge \mathbf{d}q \quad \in \Omega^2(\mathbb{R}^{2k}) \,,$$

then Hamilton's equations are equivalent to

$$\iota_v \omega = \mathbf{d}H \,.$$

We now explain this by providing a *sheaf semantics* for differential geometry.

Abstract coordinate systems

Definition

Let CartSp be the category whose

- objects are the Cartesian spaces $\mathbb{R}^n$ for $n \in \mathbb{N}$;
- morphisms are smooth functions $\mathbb{R}^{n_1} \longrightarrow \mathbb{R}^{n_2}$.

Think of $\mathbb{R}^n$ as the abstract n-dimensional *coordinate system* and think of a smooth function between Cartesian spaces as a *coordinate transformation* (possibly degenerate).

Gluing of coordinate systems

Definition

An open cover $\{U_i \hookrightarrow \mathbb{R}^n\}$ is *differentially good* if every finite intersection of the patches is diffeomorphic to an $\mathbb{R}^n$.

Remark

There are diffeomorphisms

$$\mathbb{R}^n \simeq D^n$$

smoothly identifying the n-dimensional Cartesian space with the n-dimensional open unit ball.

Smoooth 0-types

Definition

Write

$$\mathrm{Smooth0Types} := \mathrm{Func}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Set})[\{\text{local bijections}\}^{-1}]$$

for the sheaf topos over the site of Cartesian spaces with Grothendieck pre-topology the differentiably good open covers.

We often abbreviate

$$\mathbf{H} := \mathrm{Smooth0Types}\,.$$

An object/type $X \in \mathbf{H}$ is a like a set with *smooth structure* which can be "probed" by mapping it out by smooth coordinate systems.

Cohesion of smooth 0-types

Proposition

The topos $\mathbf{H} = \mathrm{Smooth0Types}$ *is*

- *local,*
- *locally connected,*
- *globally connected,*
- *such that taking connected components preserves products;*

in that there exists a quadruple of adjoint functors

$$(\Pi_0 \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{coDisc}) \;:\; \begin{tikzcd} \mathrm{Smooth0Types} \arrow[r, shift left=3, "\Pi_0"] & \mathrm{Set} \arrow[l, hook', shift left=1, "\mathrm{Disc}"] \arrow[l, shift left=1, phantom] \end{tikzcd} \;\; \begin{tikzcd} \mathrm{Smooth0Types} \arrow[r, shift left=3, "\Pi_0"] \arrow[r, shift right=1, "\Gamma"'] & \mathrm{Set} \arrow[l, hook', shift left=1, "\mathrm{Disc}"'] \arrow[l, hook', shift left=5, "\mathrm{coDisc}"'] \end{tikzcd} \,.$$

This notion is secretly what Lawvere's "Some thoughts on the future of category theory" [Lawvere 91] is about. In [Lawvere 07] such toposes are called *cohesive*.

Concrete objects and diffeological space

Definition

Given a local topos, write $\sharp := \mathrm{coDisc} \circ \Gamma$ for the induced monad. We call this the *sharp modality*.

Definition

An object/type X in a cohesive topos is *concrete* if the unit of the sharp modality is a monomorphism $X \hookrightarrow \sharp X$.

Proposition

The concrete smooth 0-types are equivalently the diffeological spaces.

Diffeological spaces were introduced by Chen for studying differential forms on loop spaces. Iglesias-Zemmour has a textbook that develops all of differentials geometry with smooth manifolds generalized to diffeological spaces.

Smooth manifolds as smooth 0-types

Side remark:

A *smooth manifold* X is a smooth 0-type that admits an *étale* cover of the form

$$\coprod_i \mathbb{R}^n \longrightarrow\!\!\!\!\!\rightarrow X \;\; .$$

This cannot be axiomatized in plain cohesion, but can be axiomatized in *differential cohesion* [Schreiber 11].

But for the moment we should skip over that discussion...

The smooth type of differential 1-forms

Definition

Write

$$\mathbf{\Omega}^1 \in \mathrm{Smooth0Types}$$

for the smooth 0-type which is probed by coordinate systems by the rule

$$\mathbb{R}^n \;\mapsto\; (C^\infty(\mathbb{R}^n))^n =: \left\{ \sum_{i=1}^n \alpha_i \mathbf{d}x^i \mid \alpha_i \in C^\infty(\mathbb{R}^n) \right\}$$

and which sends a change of coordinates $(\mathbb{R}^{n_1} \xrightarrow{f} \mathbb{R}^{n_2})$ to the $C^\infty(\mathbb{R}^{n_2})$-linear map given by

$$\mathbf{d}x_2^j \;\mapsto\; \sum_{i=1}^n \frac{\partial f^j}{\partial x_1^i} \mathbf{d}x_1^i \,.$$

The smooth type of differential 2-forms

Similarly:

Definition

Write

$$\mathbf{\Omega}^2 \in \mathrm{smooth0Types}$$

for the smooth 0-type with coordinate probes being

$$\mathbb{R}^n \mapsto (C^\infty(\mathbb{R}^n))^{\binom{n}{2}} =: \left\{ \sum_{i,j=1}^{n} \omega_{ij} \mathbf{d}x^i \wedge \mathbf{d}x^j \right\} ,$$

where on the right we have formal basis elements subject to the relation

$$\mathbf{d}x^i \wedge \mathbf{d}x^j = -\mathbf{d}x^j \wedge \mathbf{d}x^i ,$$

and where the pullback operation is componentwise as before.

Differentiation

Proposition

There is a morphism of smooth 0-types $\mathbf{d} : \mathbb{R} \longrightarrow \mathbf{\Omega}^1$ *given by sending for each* $n \in \mathbb{N}$

$$f \in \mathbb{R}(\mathbb{R}^n) \overset{\text{Yoneda}}{=} C^\infty(\mathbb{R}^n, \mathbb{R})$$

to

$$\mathbf{d}f := \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \mathbf{d}x^i \,.$$

Proposition

There is also a morphism $\mathbf{d} : \mathbf{\Omega}^1 \longrightarrow \mathbf{\Omega}^2$ *given by*

$$f \mathbf{d}x^i \mapsto \mathbf{d}f \wedge \mathbf{d}x^i := \sum_j \frac{\partial f}{\partial x^j} \mathbf{d}x^j \wedge \mathbf{d}x^i \,.$$

Symplectic form

Example

The *canonical symplectic form* on phase space $\mathbb{R}^{2k} \in \mathrm{Smooth0Types}$ is

$$\omega := \mathbf{d}p \wedge \mathbf{d}q \ : \ \mathbb{R}^{2n} \longrightarrow \mathbf{\Omega}^2 \,.$$

Remark

Therefore a phase space is naturally an object in the slice topos

$$(X, \omega) \in \mathbf{H}_{/\mathbf{\Omega}^2} := \mathrm{Smooth0Types}_{/\mathbf{\Omega}^2} \,.$$

Symplectomorphism

A *symplectomorphism* between phase spaces is an equivalence in the slice $\mathbf{H}_{/\mathbf{\Omega}^2}$, hence a diagram in $\mathbf{H}$ of the form

$$\begin{tikzcd} X_1 \arrow[rr, "f", "\simeq"'] \arrow[dr, "\omega_1"'] & & X_2 \arrow[dl, "\omega_2"] \\ & \mathbf{\Omega}^2 & \end{tikzcd}$$

In traditional language this means that

1. f is a diffeomorphism
2. which respects the symplectic form in that $f^*\omega_2 = \omega_1$.

Pairing with vector fields

For the moment consider this:

A *vector field* on $\mathbb{R}^n$ is a smooth function of the form

$$(v_i)_{i=1}^n : \mathbb{R}^n \longrightarrow \mathbb{R}^n \,.$$

The *pairing* of v with a differential 1-form $\alpha = \sum_i \alpha_i \mathbf{d}x^i$ is the smooth function

$$\iota_v \alpha := \sum_i \alpha_i v^i \;:\; \mathbb{R}^n \longrightarrow \mathbb{R} \,.$$

The pairing of v with a differential 2-form is the differential 1-form defined by

$$\iota_v \left(\mathbf{d}x^i \wedge \mathbf{d}x^j \right) := (\iota_v \mathbf{d}x^i)\mathbf{d}x^j - (\iota_v \mathbf{d}x^j)\mathbf{d}x^i$$

Hamilton's equations in symplectic geometry

Now we understand:

Fact

With

$$\omega = \mathbf{d}p \wedge \mathbf{d}q : X \longrightarrow \mathbf{\Omega}^2$$

the canonical symplectic form, Hamilton's equations are equivalent to

$$\iota_v \omega = \mathbf{d}H \, .$$

This is nice ($\rightarrow$ symplectic geometry)... ...but not as nice as it could be, because "ι_v" is not yet nicely defined internally. For that we need *smooth 1-types*.

Smooth 1-types

Write

$$\mathrm{Smooth1Types} := \mathrm{Func}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Grpd})[\{\text{local equivalences}\}^{-1}]$$

for the groupoid-enriched category obtained from groupoid-valued functors by universally turning local equivalences of groupoids into genuine homotopy equivalences.

This is the $(2,1)$-*topos of smooth 1-types*.

From now on we often abbreviate

$$\mathbf{H} := \mathrm{Smooth1Types}\,.$$

Smooth delooping

Let

$$U(1) := \mathbb{R}/\mathbb{Z} \in \mathrm{Smooth0Types} \hookrightarrow \mathrm{Smooth1Types}$$

be the *smooth circle group*.

Definition

The *smooth delooping* of $U(1)$ is the smooth 1-type

$$\mathbf{B}U(1) \in \mathrm{Smooth1Types}$$

given by

$$\mathbf{B}U(1) : \mathbb{R}^n \mapsto \left(C^\infty(\mathbb{R}^n, U(1)) \rightrightarrows * \right) .$$

Proposition

This is the moduli stack for smooth $U(1)$-principal bundles:

$$\mathbf{H}_{/\mathbf{B}U(1)} \simeq \{\text{Smooth } U(1)\text{-principal bundles}\} .$$

Smooth action groupoid homotopy quotients

Generally:

Definition

For $X \in \mathrm{Smooth0Types}$ and

$$\rho : X \times U(1) \longrightarrow X$$

a smooth group action, then

$$X /\!/ U(1) := \left(X \times U(1) \underset{\rho}{\overset{p_1}{\rightrightarrows}} X \right) \in \mathrm{Smooth1Types}\,.$$

is the *smooth action groupoid* or *smooth quotient stack*.

Example

$$\mathbf{B}U(1) \;\simeq\; * /\!/ U(1)\,.$$

Differential moduli

Example

There is a canonical action

$$\mathbf{\Omega}^1 \times U(1) \longrightarrow \mathbf{\Omega}^1$$

given by

$$(\alpha, f) \mapsto \alpha + \mathbf{d} f \,.$$

Definition

Write

$$\mathbf{B}U(1)_{\mathrm{conn}} := \mathbf{\Omega}^1 // U(1)\,.$$

Proposition

$$\mathbf{H}_{/\mathbf{B}U(1)_{\mathrm{conn}}} \simeq \{U(1)\text{-}\textit{principal connections}\}$$

Universal curvature

Proposition

The morphism $\mathbf{d} : \mathbf{\Omega}^1 \longrightarrow \mathbf{\Omega}^2$ *extends to a morphism*

$$F_{(-)} : \mathbf{B}U(1)_{\mathrm{conn}} \longrightarrow \mathbf{\Omega}^2$$

$$\begin{tikzcd}
X \arrow[dr, "\text{principal connection}"] \arrow[drr, bend left, "\text{curvature}"] \arrow[ddr, bend right, "\text{principal bundle}"'] & & \\
& \mathbf{B}U(1)_{\mathrm{conn}} \arrow[r, "F_{(-)}"] \arrow[d] & \mathbf{\Omega}^2 \\
& \mathbf{B}U(1) &
\end{tikzcd}$$

Pre-quantization

A standard notion in the physics of phase spaces is now formalized as follows:

Definition

Given a phase space $(X, \omega) \in \mathbf{H}_{/\mathbf{\Omega}^2}$ then a *pre-quantization* is a dashed lift in

$$
\begin{tikzcd}
X \arrow[r, dashed, "\nabla"] \arrow[dr, "\omega"'] & \mathbf{B}U(1)_{\mathrm{conn}} \arrow[d, "F_{(-)}"] \\
& \mathbf{\Omega}^2
\end{tikzcd}
$$

hence a lift of (X, ω) through the dependent sum along the universal curvature map

$$
\sum_{F_{(-)}} \; : \; \mathbf{H}_{/\mathbf{B}U(1)_{\mathrm{conn}}} \longrightarrow \mathbf{H}_{/\mathbf{\Omega}^2}
$$

The standard local pre-quantization

Example

For the canonical symplectic form

$$\omega = \mathbf{d}p \wedge \mathbf{d}q \;:\; \mathbb{R}^{2n} \longrightarrow \mathbf{\Omega}^2$$

the standard pre-quantization is $p\mathbf{d}q$:

$$\begin{tikzcd}
\mathbb{R}^{2n} \arrow[rr, dashed, "p\mathbf{d}q"] \arrow[drr, "\nabla" description] \arrow[ddrr, "\omega"'] & & \mathbf{\Omega}^1 \arrow[d] \arrow[dd, bend left, "\mathbf{d}"] \\
& & \mathbf{B}U(1)_{\mathrm{conn}} \arrow[d, "F_{(-)}" description] \\
& & \mathbf{\Omega}^2
\end{tikzcd}$$

Hamilton's equations via Slice automorphism

Theorem ([Fiorenza-Rogers-Schreiber 13a])

Concrete functions

$$\mathbf{B}\mathbb{R} \longrightarrow \mathbf{BAut}_{/\mathbf{B}U(1)_{\mathrm{conn}}}(\nabla) \hookrightarrow \mathbf{H}_{/\mathbf{B}U(1)_{\mathrm{conn}}}$$

are equivalent to $\{H \in C^\infty(\mathbb{R}^{2k})\}$ *and send*

$$t \;\mapsto\; \begin{tikzcd} \mathbb{R}^{2k} \arrow[rr, "\exp(t\{H,-\})"] \arrow[dr, "\nabla"'] & \arrow[d, phantom, "\Swarrow \exp(iS_t)"] & \mathbb{R}^{2k} \arrow[dl, "\nabla"] \\ & \mathbf{B}U(1)_{\mathrm{conn}} & \end{tikzcd}$$

where

- $\exp(t\{H,-\})$ *is Hamilton's flow of (time) length* t;
- $S_t := \int_{0^t} \mathbf{Ldt}$ *is the "action" where* $L := p\frac{\partial H}{\partial p} - H$ *is the "Lagrangian".*

Intermediate conclusion

So once the homotopy type theory has *differential moduli* types such as $\mathbf{B}U(1)_{\mathrm{conn}}$...

...then it serves as a context for solving Hilbert's 6th problem, the axiomatization of physics.

Therefore we need to axiomatize the construction of differential moduli types like $\mathbf{B}U(1)_{\mathrm{conn}}$...

Smooth homotopy types

Definition

Write

$$\begin{aligned}&\mathrm{SmoothHomotopyTypes}\\&:= \mathrm{Func}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet})[\{\text{local weak homotopy equivalences}\}^{-1}]\end{aligned}$$

for the ∞-topos over the site of smooth Cartesian spaces.

From now on we abbreviate

$$\mathbf{H} := \mathrm{SmoothHomotopyTypes}\,.$$

We think of an object/type of $\mathbf{H}$ as a homotopy type equipped with *smooth structure*.

Cohesion of smooth homotopy types

Proposition ([Schreiber 11])

Smooth homotopy types are cohesive in that there exists an adjoint quadruple of ∞-functors

$$(\Pi \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{coDisc}) \;:\; \mathrm{SmoothTypes} \underset{\underset{\mathrm{coDisc}}{\hookleftarrow}}{\overset{\overset{\Pi}{\longrightarrow}}{\underset{\Gamma}{\overset{\mathrm{Disc}}{\hookleftarrow \atop \longrightarrow}}}} \infty Grpd \;.$$

Cohesive homotopy type theory

This means that the homotopy type theory of $\mathrm{SmoothHomotopyTypes}$ is equipped with an adjoint triple of *higher modalities*

$$\underset{\text{shape modality}}{\int} \quad \dashv \quad \underset{\text{flat modality}}{\flat} \quad \dashv \quad \underset{\text{sharp modality}}{\sharp}$$

The $\flat$*-modal types* are *geometrically discrete*

Example

We have

1. $\int \mathbf{B}U(1) \simeq BU(1)$;
2. $\flat \mathbf{B}U(1) \simeq K(U(1)_{\mathrm{disc}}, 1)$.

Maurer-Cartan forms

Definition

For $\mathbb{G} \in \mathrm{Grp}(\mathbf{H})$ a cohesive homotopy type with group structure, define

$$\flat_{\mathrm{dR}}\mathbf{B}\mathbb{G} := \mathrm{hfib}\,(\flat\mathbf{B}\mathbb{G} \longrightarrow \mathbf{B}\mathbb{G}) \quad \in \mathbf{H}$$

Definition

The *Maurer-Cartan form* of $\mathbb{G}$

$$\theta_{\mathbb{G}} : \mathbb{G} \longrightarrow \flat_{\mathrm{dR}}\mathbf{B}\mathbb{G}$$

is

$$\theta_G := \mathrm{hfib}\,(\flat_{\mathrm{dR}}\mathbf{B}\mathbb{G} \longrightarrow \flat\mathbf{B}\mathbb{G}) \; .$$

Differential forms, 0-truncated

Given $\flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1)$ as above, say that a function

$$\mathbf{\Omega}^{n+1}_{\mathrm{cl}} \longrightarrow \flat\mathbf{B}^{n+1}U(1)$$

is a *choice of global curvature n-forms* if

1. $\mathbf{\Omega}^{n}_{\mathrm{cl}}$ is 0-truncated (is an h-set);
2. for every smooth manifold Σ the map

   $$[\Sigma, \mathbf{\Omega}^{n+1}] \longrightarrow [\Sigma, \flat\mathbf{B}^{n+1}U(1)]$$

   is a 1-epimorphism
3. $\mathbf{\Omega}^{n+1}_{\mathrm{cl}}$ is minimal with these properies.

Differential moduli

Definition

For $n \in \mathbb{N}$ write $\mathbf{B}^n U(1)_{\mathrm{conn}} \in \mathrm{SmoothHomotopyTypes}$ for the *Deligne complex* of smooth 0-types

$$[U(1) \stackrel{\mathbf{d}}{\to} \mathbf{\Omega}^1 \stackrel{\mathbf{d}}{\to} \cdots \stackrel{\mathbf{d}}{\to} \mathbf{\Omega}^n] \in \mathrm{Ch}_\bullet(\mathrm{Smooth0Types})$$

regarded as a smooth homotopy type under the *Dold-Kan corespondence*

$$\mathbf{Ch}_{\bullet \geq 0} \xrightarrow{\simeq} \mathrm{sAbGrp} \xrightarrow{\text{forget}} \mathrm{sSet}\,.$$

For $n = 1$ this reproduces the moduli for circle-principal connections from above. For general n these modulate higher-degree analogs of circle-principal connections (cocycles in "ordinary differential cohomology").

Synthetic differential cohomology

With this we finally find that the differential moduli indeed have an axiomatic/synthetic characterization, as follows:

Theorem ([Schreiber 11])

In $\mathrm{SmoothHomotopyTypes}$ *there is a homotopy pullback diagram of the form*

$$\begin{tikzcd}
\mathbf{B}^n U(1)_{\mathrm{conn}} \arrow[rr, "F_{(-)}"] \arrow[d] && \mathbf{\Omega}^{n+1}_{\mathrm{cl}} \arrow[d] \\
\mathbf{B}^n U(1) \arrow[rr, "\theta_{\mathbf{B}^n U(1)}"] && \flat_{\mathrm{dR}} \mathbf{B}^{n+1} U(1)
\end{tikzcd}$$

Outlook

With classical mechanics synthetically formulated in cohesive homotopy type theory this way...

... we can now study what happens as we increase the degree n on the differential moduli types $\mathbf{B}^n U(1)$.

In Schreiber 13 we find that concrete functions of the form

$$\mathbf{B}\mathbb{R}^n \longrightarrow \mathbf{H}_{/\mathbf{B}^n U(1)_{\mathrm{conn}}}$$

encode n-dimensional *classical field theory*
(describing for instance electromagnetism and gravity).

References on classical mechanics via Cohesive homotopy types

A standard textbook in the traditional modern formulation of classical mechanics is

- V. Arnold, *Mathematical methods of classical mechanics*, Graduate Texts in Mathematics (1989)

The technical results of the above synthetic formulation are due to [Fiorenza-Rogers-Schreiber 13a]. An exposition is in

- U.Schreiber, *Classical field theory via Cohesive homotopy types*, `http://www.nlab.org/schreiber/show/Classical+field+theory+via+Cohesive+homotopy+types`

Now we discuss the *qauntization* of classical mechanics to *quantum mechanics*.

Holographic motivic quantization of Poisson manifolds

physics	**mathematics**
mechanical system	symplectic manifold (X, ω)
foliation by mechanical systems	Poisson manifold (X, π)
quantization of mechanical systems	quantization of Poisson manifolds

Observation: each Poisson manifold induces a 2-dimensional local Poisson-Chern-Simons theory whose moduli stack of fields is the "symplectic groupoid" $\mathrm{SymGrp}(X, \pi)$ with local action functional

$$\begin{array}{c} \mathrm{SympGrpd}(X, \pi) \\ \big\downarrow{\scriptstyle \exp(i S_{PCS})} \\ \mathbf{B}^2 U(1)_{\mathrm{conn}^1} \end{array}$$

Holographic motivic quantization of Poisson manifolds

The original Poisson manifold includes into the symplectic groupoid and naturally trivializes $\exp(iS_{PCS})$. So by Observation B it constitutes a canonical boundary condition for the 2-d Poisson-CS theory, exhibited by the correspondence

$$
\begin{tikzcd}
 & X \arrow[dl] \arrow[dr, "i"] & \\
* \arrow[dr] \arrow[rr, phantom, "\Swarrow \xi"] & & \mathrm{SymGrp}(X,\pi) \arrow[dl, "\chi"] \\
 & \mathbf{B}^2 U(1) &
\end{tikzcd}
\quad \simeq \quad
\begin{tikzcd}
 & X \arrow[dl] \arrow[dd, "i^*\chi"] \arrow[dr, "i"] & \\
* \arrow[dr] \arrow[r, phantom, "\Swarrow \xi"] & \, & \mathrm{SymGrp}(X,\pi) \arrow[dl, "\exp(iS_{\mathrm{PCS}})"] \\
 & \mathbf{B}^2 U(1) &
\end{tikzcd}
$$

Holographic motivic quantization of Poisson manifolds

Applying Theorem N, the groupoid convolution functor sends this to the co-correspondence of Hilbert bimodules

$$\mathbb{C} \xrightarrow{\Gamma(\xi)} C^*(X, i^*\chi) \xleftarrow{i^*} C^*(\mathrm{SymGrpd}, \chi) \ .$$

So if i is KK-orientable, then this boundary condition of the 2d PCS theory quantizes to the KK-morphism

$$\mathbb{C} \xrightarrow{\Gamma(\xi)} C^*(X, i^*\chi) \xrightarrow{i!} C^*(\mathrm{SymGrpd}, \chi)$$

hence to the class in twisted equivariant K-theory

$$i_![\xi] \in K(\mathrm{SympGrp}(X, \pi), \chi) \, .$$

The groupoid $\mathrm{SymGrp}(X, \pi)$ is a smooth model for the possibly degenerate space of symplectic leafs of (X, π) and this class may be thought of as the leaf-wise quantization of (X, π).

Holographic motivic quantization of Poisson manifolds

In particular when (X, π) is symplectic we have $\mathrm{SymGrpd}(X, \pi) \simeq *$ and $\xi = \mathbb{L}$ is an ordinary prequantum bundle and i is KK-oriented precisely if X is Spin^c. In this case

$$i_![\xi] = i_![\mathbb{L}] \in K(*) = \mathbb{Z}$$

is the traditional K-theoretic geometric quantization of (X, ω).

Holographic motivic quantization of Poisson manifolds

Similarly, for

$$\chi_B : X \to \mathbf{B}^2 U(1)$$

a B-field, a D-brane $i : Q \to X$ is a boundary condition given by

$$\begin{tikzcd}
 & Q \arrow[dl] \arrow[dr, "i"] \arrow[dd, phantom, "\xi\ \swarrow"] & & & & Q \arrow[dl] \arrow[dr, "i"] \arrow[dd, "i^*\chi"] & \\
* \arrow[dr] & & X \arrow[dl, "\chi"] & \simeq & * \arrow[dr] \arrow[r, phantom, "\xi\ \swarrow"] & {} & X \arrow[dl, "\chi"] \\
 & \mathbf{B}^2 U(1) & & & & \mathbf{B}^2 U(1) &
\end{tikzcd}$$

where now ξ is the <u>*Chan-Paton bundle*</u> on the D-brane.

Holographic motivic quantization of Poisson manifolds

Proceeding as above shows that the quantization of this boundary condition in the 2d QFT which is the topological part of the 2d string σ-model gives the *D-brane charge*

$$i_![\xi] \in K(X, \chi)\,.$$

[Brodzki-Mathai-Rosenberg-Szabo 09]

Holographic motivic quantization of Poisson manifolds

In conclusion:

- The quantization of a Poisson manifold is equivalently its *brane charge* when regarded as a boundary condition of its 2d Poisson-Chern-Simons theory.

Conversely:

- The charge of a D-brane is equivalently the quantization of a particle on the brane charged under the Chan-Paton bundle.

References: Related work on holographic quantization of Poisson manifolds and D-branes

- [Kontsevich 97] + [Cattaneo-Felder 99] realize *perturbative* algebraic deformation quantization of Poisson manifold holographically by perturbative quantization of 2d Poisson σ-model;
- [EH 06] completes Weinstein-Landsman program of geometric quantization of symplectic groupoids by secretly quantizing a prequantum 2-bundle
- [Gukov-Witten 08] realize geometric quantization of symplectic manifold holographically by quantization of 2d A-model
- [Brodzki-Mathai-Rosenberg-Szabo 09] formalize D-brane charge in KK-theory

Example 2

∞-Chern-Simons local prequantum field theory

back to list of contents

(...) [Fiorenza-Schreiber 13a] (...)

Example 3

Super L_∞-extensions and the super p-brane bouquet

based on [Fiorenza-Sati-Schreiber 13b]

back to list of contents

Outline of Example 3

We will indicate the following story:

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i) Motivation: The **localized WZW σ-model**

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a) $\left\{ \begin{array}{l} \underline{\textbf{cohomological quantization}} \\ \quad \text{implies} \\ \mathbb{S}\textbf{-graded geometry} \\ \quad \text{implies} \\ \textbf{supergeometry}; \end{array} \right.$

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c)
- **higher cocycles on super-spacetime**
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- **the super p-brane models in string theory/M-theory**.

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induce
the super p-brane models in string theory/M-theory.

d) $\{$**higher Noether current algebras**

i)

Motivation:

The localized WZW σ-model

back to Example 3 contents

WZW σ-model field theory describing a bosonic string on a simple Lie group G is all controled by the canonical Lie algebra 3-cocycle

$$\langle \theta, [\theta, \theta] \rangle : \mathfrak{g} \longrightarrow \mathbf{B}^2\mathbb{R} \ .$$

This σ-model famously has an affine Lie current algebra of Noether currents. This is the symmetry algebra of the transgression of the theory to loop space $[S^1, G]$:

$$\boxed{\text{affine Lie algebra}} \simeq \boxed{\begin{array}{c}\text{Heisenberg Lie algebra} \\ \text{of prequantum geometry on } [S^1, G]\end{array}} .$$

This is the infinitesimal approximation to:

$$\boxed{\text{Kac-Moody loop group}} \simeq \boxed{\begin{array}{c}\text{Heisenberg Lie group} \\ \text{of prequantum geometry on } [S^1, G]\end{array}}$$

(the geometric loop representation theory of [Pressley-Segal]).

But the WZW is a **local field theory**. It is not defined just on loop space. Its transgression to loop space loses information. Therefore we want to

- "**de-transgress**" or "**localize**" from $[S^1, G]$ to $[*, G] \simeq G$.

In [Fiorenza-Rogers-Schreiber 13a] the following is made precise and proven (we come back to this below):

$$\boxed{\begin{array}{c}\text{String}\\ \text{Lie 2-algebra}\end{array}} \simeq \boxed{\begin{array}{c}\text{Noether current Lie}\\ \text{2-algebra of } L_{\mathrm{WZW}}\end{array}}$$

$$\simeq \boxed{\begin{array}{c}\text{Heisenberg Lie 2-algebra of}\\ \text{prequantum 2-geometry } (G, \langle \theta \wedge [\theta \wedge \theta] \rangle)\end{array}}$$

Here String is homotopy fiber of L_∞-algebras of the WZW curvature:

$$\begin{tikzcd} \mathfrak{string}(\mathfrak{g}) \arrow[r] \arrow[d, "\mathrm{hfib}"'] \arrow[dr, phantom, "\swArrow"] & 0 \arrow[d] \\ \mathfrak{g} \arrow[r, "\langle\theta\wedge[\theta\wedge\theta]\rangle"'] & \mathbf{B}^2\mathbb{R} \end{tikzcd} .$$

Hence an interesting **question** is:

- How does this **generalize** to higher WZW-type field theories?
- What are **examples**?

Hence an interesting **question** is:

- How does this **generalize** to higher WZW-type field theories?
- What are **examples**?

A famous class of field theories of higher WZW type are the Green-Schwarz action functionals for **super-p-brane σ-models**. [Green-Schwarz 84].

These are WZW-type models induced by the exceptional invariant super Lie algebra cocycles on the super translation Lie algebra, hence on super-Minkowski spacetime:

$$\mathbb{R}^{d;N=1} \xrightarrow{\quad \langle \Psi \wedge [E^p \wedge \Psi] \rangle \quad} \mathbf{B}^{p+1}\mathbb{R} \qquad .$$

$\boxed{\text{super-spacetime}}$ $\qquad$ $\boxed{\text{higher gauge background field}}$

The old branescan

These cocycles have been **classified** in the "**old brane scan**" [Achucarro-Evans-Townsend-Wiltshire 87], [Azcćarraga-Townsend 80] [1]:

[1] See [JH 12] for an introduction with an eye towards the L_∞-perspective below, and see [Brandt 13] for a comprehensive classification.

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$d =$	$p = 0$	1	2	3	4	5	6
11			(1) $\mathfrak{m2brane}$				
10		(1) $\mathfrak{string}_{\mathrm{het}}$				(1) $\mathfrak{ns5brane}_{\mathrm{het}}$	
9					(1)		
8				(1)			
7			(1)				
6		(1) $\mathfrak{littlestring}_{\mathrm{het}}$		(1)			
5			(1)				
4		(1)	(1)				
3		(1)					

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7			(1)				
6		(1) $\mathfrak{littlestring}_{\mathrm{het}}$		(1)			
5			(1)				
4		(1)	(1)				
3		(1)					

But the old brane scan is still missing many branes, for instance the M5-brane.

Where are the missing branes? They have been proposed and built by hand [BLNPST 97]...

...but can we **discover them as local higher WZW models**?

[1] See [JH 12] for an introduction with an eye towards the L_∞-perspective below, and see [Brandt 13] for a comprehensive classification.

a)

Supergeometry

back to Example 3 contents

Superalgebra

Definition

Write

$$\begin{aligned}\mathrm{SuperPoints} &:= \mathrm{GrassmannAlgebras}^{\mathrm{op}}_{\mathrm{fin-gen}/\mathbb{R}} \\ &\simeq \left\{\mathbb{R}^{0|q}\right\}_{q\in\mathbb{N}}\end{aligned}$$

for the opposite category of real *Grassmann algebras*, the category of *super-points*.

Write

$$\mathrm{SuperSet} := \mathrm{Sh}(\mathrm{SuperPoints})$$

for the (pre-)sheaf topos over super points.

Example

The canonical line object here is $\underline{\mathbb{R}} \in \mathrm{SuperSet}$, given by

$$\underline{\mathbb{R}} \;:\; \mathbb{R}^{0|q} \mapsto (\wedge^\bullet \mathbb{R}^q)_{\mathrm{even}} \,.$$

Superalgebra

Observation
([Schwarz 84, Voronov 84, Konechny-Schwarz 97])

Algebra over $\underline{\mathbb{R}}$ is superalgebra:

- $\underline{\mathbb{R}}$*-modules* $\underline{V}$ *are super-vector spaces* V*.*

$$\underline{V} : \mathbb{R}^{0|q} \mapsto ((\wedge^\bullet \mathbb{R}^q) \otimes V)_{\mathrm{even}}$$

- *A (commutative)* $\underline{\mathbb{R}}$*-algebra* $\underline{A}$ *is a (super-commutative) super-algebra* A *over* $\mathbb{R}$*.*

From superalgebra to higher supergeometry

	superalgebra	smooth geometry	homotopy theory
modeled on	superpoints $\{\mathbb{R}^{0\|q}\}_{q \in \mathbb{N}}$	Cartesian spaces $\{\mathbb{R}^p\}_{p \in \mathbb{N}}$	simplices $\{\Delta^k\}_{k \in \mathbb{N}}$

$$\boxed{\text{geometry} + \text{homotopy theory} = \infty\text{-topos theory}}$$

Definition

Write

$$\mathbf{H} = \mathrm{SmoothSuper}\infty\mathrm{Grpd} := L_{whe}\mathrm{Sh}(\{(\mathbb{R}^{p|q}, \Delta^k)\}_{p,q,k \in \mathbb{N}})$$

for the **homotopy theory** obtained
from simplicial sheaves on super-Cartesian spaces
by
universally turning
local homotopy equivalences into actual homotopy equivalences.

Differential cohomology in cohesive higher geometry

This supergeometric homotopy theory is differentially cohesive which in particular implies the following. For every **higher super group** G (super group ∞-stack) there is

the coefficient object which modulates higher G-**principal bundles**	$\mathbf{B}G$
the coefficient object which modulates **flat** G-**principal connections**	$\flat\mathbf{B}G$
the coefficient object which modulates **flat** G-**valued differential forms**	$\flat_{\mathrm{dR}}\mathbf{B}G$
the **higher Maurer-Cartan form**	$G \xrightarrow{\theta_G} \flat_{\mathrm{dR}}\mathbf{B}G$
if $G = \mathbb{G}$ is abelian (braided), then a differential coefficient object which modulates $\mathbb{G}$-**principal connections** (with curvature)	$\mathbf{B}\mathbb{G}_{\mathrm{conn}}$

Higher line bundles

In particular the *Dold-Kan correspondence*

$$\mathrm{Ch}_{\bullet \geq 0} \xrightarrow{\simeq} \mathrm{sAb} \xrightarrow{\text{forget}} \mathrm{KanCplx}$$

yields examples:

$$\mathbf{B}^n U(1) := \mathrm{DK}(\underline{U(1)}[n])$$

$\mathbf{H}$ is cohesive $\Rightarrow$ geometric realization

$$| - | : \mathbf{H} \to L_{\mathrm{whe}}\mathrm{Top}$$

Example

$|\mathbf{B}^n \underline{\mathbb{C}}^\times| \simeq K(\mathbb{Z}, n+1)$

Higher circle-principal connections

So for $X \in \mathrm{SmoothSuper}\infty\mathrm{Grpd}$ any higher super-orbispace (super ∞-stack), a map

$$\nabla : X \to \mathbf{B}^n U(1)_{\mathrm{conn}}$$

is equivalently

- a circle n-bundle with n-form connection on X with curvature F_∇ [Fiorenza-Schreiber-Stasheff 10];
- a higher prequantization of the pre-n-plectic form F_∇ [Fiorenza-Rogers-Schreiber 13a];
- a local Lagrangian/action functional for an n-dimensional local prequantum field theory with moduli stack of fields given by X [Fiorenza-Sati-Schreiber 12b, Fiorenza-Schreiber 13a].

b)

Higher cocycles and higher gauged higher WZW models

back to Example 3 contents

Local action functionals

In the last interpretation of ∇, the σ-model induced by ∇ is the local prequantum field theory which to a closed oriented manifold Σ_k assigns the $(n-k)$-bundle with $(n-k)$-connection which is the **transgression** of ∇ to the space $[\Sigma_k, X]$ of fields on Σ [Fiorenza-Sati-Schreiber 12a], [Fiorenza-Sati-Schreiber 12b]:

$$\begin{tikzcd}
{[\Sigma_k, X]} \arrow[r, "{[\Sigma_k,\nabla]}"] \arrow[dr, "{\exp(2\pi i \int_{\Sigma_k} [\Sigma_k,\nabla])}"'] & {[\Sigma_k, \mathbf{B}^n U(1)_{\mathrm{conn}}]} \arrow[d, "{\exp(2\pi i \int_{\Sigma_k}(-))}"] \\
 & \mathbf{B}^{n-k} U(1)_{\mathrm{conn}}
\end{tikzcd} .$$

For for $k = n$ we have $\mathbf{B}^0 U(1)_{\mathrm{conn}} = U(1)$ and so

- in codimension 0 this is the **action functional**;
- in codimension 1 it is the (off-shell) **prequantum bundle**.

The higher gauged higher WZW models

By [Fiorenza-Schreiber-Stasheff 10] we may **Lie integrate** each super Lie $(p+2)$-cocycle such as $\langle \Psi \wedge E^p \wedge \Psi \rangle$ to a map of super ∞-stacks of the form

$$\mathbf{c} : \ \mathbf{B}G \xrightarrow{\exp(\langle \Psi \wedge [E^p \wedge \Psi] \rangle)} \mathbf{B}^{p+2}(\mathbb{R}/\Gamma) \ .$$

The higher gauged higher WZW models

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The corresponding higher WZW model is supposed to have

- underlying prequantum $(p+1)$-bundle the looping

$$\Omega\mathbf{c} : \ G \longrightarrow \mathbf{B}^{p+1}(\mathbb{R}/\Gamma)$$

- curvature $(p+2)$-form $\langle \Psi \wedge E^p \wedge \Psi \rangle$.

The higher gauged higher WZW models

We obtain this $\mathbf{L}_{\mathrm{WZW}}$ by a universal construction using the above cohesion in $\mathrm{SmoothSuper}\infty\mathrm{Grpd}$ as follows:

$$
\begin{tikzcd}
\tilde{G} \arrow[r, "\tilde{\theta}_G"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega_{\mathrm{flat}}(-,\mathfrak{g}) \arrow[rr, "\mathrm{CS}_{\mathbf{c}}"] \arrow[d] & & \Omega^{p+2}_{\mathrm{cl}} \arrow[d] \\
G \arrow[r, "\theta_G"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \flat_{\mathrm{dR}} \mathbf{B} G \arrow[rr, "\flat_{\mathrm{dR}} \mathbf{c}"] \arrow[d] & & \flat_{\mathrm{dR}} \mathbf{B}^{p+2}(\mathbb{R}/\Gamma) \arrow[d] \\
* \arrow[r] & \flat \mathbf{B} G \arrow[rr, "\flat \mathbf{c}"] & & \flat \mathbf{B}^{p+2}(\mathbb{R}/\Gamma)
\end{tikzcd}
$$

rewrite diagram $\rightarrow$

The higher gauged higher WZW models

We obtain this $\mathbf{L}_{\mathrm{WZW}}$ by a universal construction using the above cohesion in $\mathrm{SmoothSuper}\infty\mathrm{Grpd}$ as follows:

$$\begin{tikzcd}
\tilde{G} \arrow[r, dashed, "\boxed{\mathbf{L}_{\mathrm{WZW}}}"] \arrow[d] & \mathbf{B}^{p+1}U(1)_{\mathrm{conn}} \arrow[r, "F_{(-)}"] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \Omega^{p+2}_{\mathrm{cl}} \arrow[d] \\
G \arrow[r, "\Omega\mathbf{c}"] \arrow[d] & \mathbf{B}^{p+1}(\mathbb{R}/\Gamma) \arrow[r, "\theta_{\mathbf{B}^n U(1)}"] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \flat_{\mathrm{dR}}\mathbf{B}^{p+2}(\mathbb{R}/\Gamma) \arrow[d] \\
* \arrow[r] & * \arrow[r] & \flat\mathbf{B}^{p+2}(\mathbb{R}/\Gamma)
\end{tikzcd}$$

rewrite diagram ←

The higher gauged higher WZW models

This means:

- $\mathbf{L}_{\mathrm{WZW}}$ is the Lagrangian of a local σ-model prequantum field theory as above;
- defined on a higher super-orbispace $\tilde{G}$ which is a differential extension of the higher super group G;
- such that its curvature is the original super L_∞-cocycle, regarded as a left-invariant form on the super ∞-group;
- such that its integral class is the above integral lift of this cocycle.

Together this identifies $\mathbf{L}_{\mathrm{WZW}}$ as a higher analog of the "WZW gerbe", an n-connection whose local n-connection form is a WZW potential for the given cocycle.

The higher gauged higher WZW models

Remark

That $\tilde{G}$ is a differential extension of G means that a σ-model on $\tilde{G}$ has fields which are multiplets consisting of maps from the worldvolume to G and of differential forms on the worldvolume. Hence $\tilde{G}$ is the target super orbispace for **tensor multiplets** *on branes (notably the DBI 1-forms on the D-branes and the 2-form multiplet on the M5-brane).*

With a general higher geometric prequantum theory and a general construction of higher WZW terms in hand, we can now

- formulate their higher prequantum geometry;
- formulate and compute their higher Hesenberg/Noether current Lie n-algebras and the corresponding super n-groups.

c)

Higher cocycles on super-spacetime
and
the super p-brane bouquet
of string theory/M-theory.

back to Example 3 contents

Open branes ending on branes

By the rules of prequantum boundary field theory [Fiorenza-Schreiber 13a] a **boundary condition for an open brane** involves a trivialization/gauging-away of its gauge coupling term on the boundary, for instance for the 3d σ-model of the M2-brane:[2]

$$
\begin{tikzcd}
\partial(\Sigma_{2+1}) \arrow[r, dotted, "\phi|_{\partial\Sigma}"] \arrow[d] & \Sigma_{5+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow \phi_{\mathrm{bdr}}"] & * \arrow[d] \\
\Sigma_{2+1} \arrow[r, dotted, "\phi"] & \mathbb{R}^{11;N=1} \arrow[r, "\langle \Psi \wedge [E^2 \wedge \Psi] \rangle"] & \mathbf{B}^3\mathbb{R}
\end{tikzcd}
\quad \boxed{\text{topological boundary condition}} \,.
$$

$\boxed{\text{open brane } \sigma\text{-model field}}$ $\boxed{\text{background field}}$

[2] Here the maps on the left are displayed by dotted arrows because strictly speaking they live in a different category, for ease of exposition. This is resolved after Lie integration, which we suppress here.

Brane intersection law from super L_∞-extensions

By the universal property of the **homotopy pullback** of super L_∞-algebras, this means, that the map $\Sigma_{5+1} \to \mathbb{R}^{11|N=1}$ equivalently factors through the **homotopy fiber** super L_∞-algebras

$$\mathfrak{m2brane} := \mathrm{hfib}(\langle \Psi \wedge E^2 \wedge \Psi \rangle)$$

so that we have a factorization as such:

$$\begin{tikzcd}
\partial\Sigma \arrow[r, dotted, "\phi_\partial"] \arrow[d] & \Sigma_{5+1} \arrow[r, dotted] \arrow[d] & \mathfrak{m2brane} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\swArrow"] & & * \arrow[d] \\
\Sigma_{2+1} \arrow[r, dotted, "\phi"] & \mathbb{R}^{11;N=1} \arrow[r, equal] & \mathbb{R}^{11;N=1} \arrow[rr, "\langle \Psi \wedge [E^2, \Psi] \rangle"'] & & \mathbf{B}^2\mathbb{R} \, .
\end{tikzcd}$$

Brane intersection law from super L_∞-extensions

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$$\begin{tikzcd}
\partial\Sigma \arrow[r, dotted, "\phi_\partial"] \arrow[d] & \Sigma_{5+1} \arrow[r, dotted] \arrow[d] & \mathfrak{m2brane} \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & * \arrow[d] \\
\Sigma_{2+1} \arrow[r, dotted, "\phi"] & \mathbb{R}^{11;N=1} \arrow[r, equal] & \mathbb{R}^{11;N=1} \arrow[r, "\langle \Psi \wedge [E^2, \Psi] \rangle"'] & \mathbf{B}^2\mathbb{R} \, .
\end{tikzcd}$$

Consequently:

- the M5-brane itself is a σ-model not on super-spacetime itself, but on a **higher extension super Lie 3-algebra** $\mathfrak{m2brane}$ of spacetime.

One checks that this reproduces the proposals [BLNPST 97]...

Brane intersection laws from super L_∞-extensions

In summary we find

- a super p-brane on which no other branes may end is induced by a super L_∞-extension

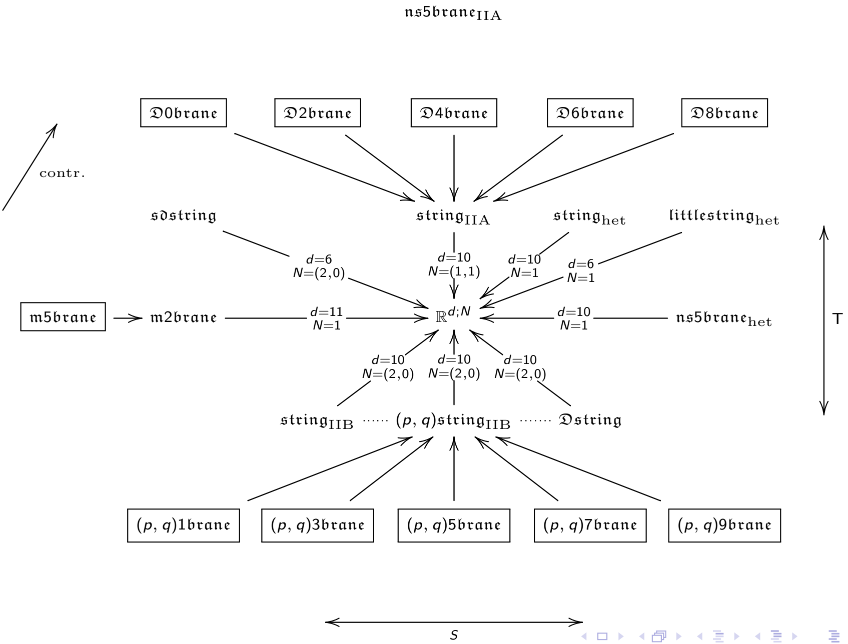
$$\mathbf{B}^p\mathbb{R} \longrightarrow p\mathfrak{brane} \longrightarrow \mathbb{R}^{d;N}$$

  of super-spacetime itself;
- a super p_2-brane on which an open p_1-brane may end is induced by an super L_∞-extension

$$\mathbf{B}^{p_2}\mathbb{R} \longrightarrow p_2\mathfrak{brane} \longrightarrow p_1\mathfrak{brane}$$

Tabulating all these extensions we get the following diagram of super L_∞-extensions... $\rightarrow$

The brane bouquet



Proof of the brane bouquet:
After translation of supergravity theorist's "FDA"-notation to homotopy theory of super-L_∞-algebras as in [Sati-Schreiber-Stasheff 08], [Fiorenza-Rogers-Schreiber 13b] this follows

- with [Azcćarraga-Townsend 80] for the "old" $N = 1$ classification,
- with section 3 of [Auria-Fré 82] for the M2/M5-brane,
- with section 6 of [Chryssomalakos-Azcárraga-Izquierdo-Bueno 99] for type IIA,
- with section 2 of [Sakaguchi 00] for the type IIB branes,
- with section 6 of [Brandt 13] for the self-dual string in $d = 6$, $N = (2, 0)$.

$\square$

Remark

This brane bouquet is reminiscent of the famous cartoon of "M-theory" (figure 4 in [Witten 98]), but the brane bouquet is a theorem in super L_∞-algebra cohomology theory.

d)

Higher Noether current algebras

back to Example 3 contents

skip to conclusions

For gauge-coupling terms in higher prequantum geometry the fully localized version of the prequantum bundle coincides with the local action functional

$$\boxed{\text{fully localized higher prequantum bundle}} \simeq \boxed{\text{local action functional}}$$

namely the connection $(p+1)$-form

$$\mathbf{L}_{\mathrm{WZW}} : G \to \mathbf{B}^n U(1)_{\mathrm{conn}}$$

is the Lagrangian form and hence the transgression to codimension zero [Fiorenza-Sati-Schreiber 12b]is the ∞-**WZW action functional** [Fiorenza-Sati-Schreiber 12b]

$$\begin{tikzcd} : [\Sigma_{\mathrm{p+1}}, G] \arrow[r, "{[\Sigma_{p+1}, \mathbf{L}_{\mathrm{WZW}}]}"] \arrow[dr, "\exp(iS_{\mathrm{WZW}})"'] & {[\Sigma_{p+1}, \mathbf{B}^n U(1)_{\mathrm{conn}}]} \arrow[d, "\exp(2\pi i \int_{\Sigma_{p+1}}(-))"] \\ & U(1) \end{tikzcd} .$$

Using this one can observe that

$$\boxed{\text{higher quantomorphism}} \simeq \boxed{\text{higher Noether current}}$$

Because a higher quantomorphism is [Fiorenza-Rogers-Schreiber 13a] a transformation of the form

$$\begin{tikzcd} \tilde{G} \arrow[rr, "\simeq"] \arrow[dr, "\mathbf{L}_{\mathrm{WZW}}"'] & \arrow[d, phantom, "\overset{\simeq}{\swarrow}\alpha"] & \tilde{G} \arrow[dl, "\mathbf{L}_{\mathrm{WZW}}"] \\ & \mathbf{B}^n U(1)_{\mathrm{conn}} & \end{tikzcd}$$

and infinitesimally and locally this is

$$\mathcal{L}_{\delta\phi} L_{\mathrm{WZW}} = d\alpha \ ,$$

where $\mathcal{L}$ is the Lie derivative, given as $\mathcal{L} = d\iota + \iota d$. Hence

$$\iota_v \langle \theta \wedge \cdots \theta \rangle = d\left(\iota_{\delta\phi} L_{\mathrm{WZW}} - \alpha\right) \ .$$

The term on the left vanishes on shell (here gauge coupling sector only) and so $J_\phi := \iota_{\delta\phi} L_{\mathrm{WZW}} - \alpha$ is a **conserved p-form Noether current**. This gives us the corresponding **super-Lie** $(p+1)$**-group of exponentiated currents**

$$\mathbf{Noeth}(\mathbf{L}_{\mathrm{WZW}}) \simeq \mathbf{Heis}(\mathbf{L}_{\mathrm{WZW}}) \simeq \left\{ \begin{tikzcd} \tilde{G} \arrow[rr, "\simeq"] \arrow[dr, "\mathbf{L}_{\mathrm{WZW}}"'] & \arrow[d, phantom, "\swarrow\alpha"] & \tilde{G} \arrow[dl, "\mathbf{L}_{\mathrm{WZW}}"] \\ & \mathbf{B}^{p+1} U(1)_{\mathrm{conn}} & \end{tikzcd} \right\} .$$

In [Fiorenza-Rogers-Schreiber 13a] is proven that:

Theorem. For each ∞-WZW model $\mathbf{L}_{\mathrm{WZW}}$, there is a homotopy fiber sequence of higher super-groups

$$\mathbf{B}^p U(1) \longrightarrow \mathbf{Noeth}(\mathbf{L}_{\mathrm{WZW}}) \longrightarrow \tilde{G} \ .$$

which differentiates to an extension of the super L_∞-algebra $\mathfrak{g}$ by $\mathbf{B}^p\mathbb{R}$:

$$\mathbf{B}^p\mathbb{R} \longrightarrow \mathfrak{Noether}(\mathbf{L}_{\mathrm{WZW}}) \longrightarrow \mathfrak{g} \ .$$

For the ordinary WZW model this reproduces the $\mathrm{String}(G)$-extension that motivated us back on p. 2.
For the M2/M5 brane system this yields the integrated M-theory super Lie algebra and more...

Conclusion

back to list of contents

Conclusion

Dieses Ergebnis scheint uns fast auf den Hegelschen Standpunkt zu führen, wonach aus blossen Begriffen alle Beschaffenheit der Natur rein logisch deduziert werden kann.

[Hilbert 21]

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back to list of contents

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