

# Super Lie $n$ -algebra extensions, higher WZW models and super $p$ -branes with tensor multiplet fields

Domenico Fiorenza, Hisham Sati, Urs Schreiber

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## Abstract

We formalize higher dimensional and higher gauge WZW-type sigma-model local prequantum field theory, and discuss its rationalized/perturbative description in (super-)Lie  $n$ -algebra homotopy theory (the true home of the “FDA”-language used in the supergravity literature). We show generally how the intersection laws for such higher WZW-type  $\sigma$ -model branes (open brane ending on background brane) are encoded precisely in (super-) $L_\infty$ -extension theory and how the resulting “extended (super-)spacetimes” formalize spacetimes containing  $\sigma$ -model brane condensates. As an application we prove in Lie  $n$ -algebra homotopy theory that the complete super  $p$ -brane spectrum of superstring/M-theory is realized this way, including the pure sigma-model branes (the “old brane scan”) but also the branes with tensor multiplet worldvolume fields, notably the D-branes and the M5-brane. For instance the degree-0 piece of the higher symmetry algebra of 11-dimensional spacetime with an M2-brane condensate turns out to be the “M-theory super Lie algebra”. We also observe that in this formulation there is a simple formal proof of the fact that type IIA spacetime with a D0-brane condensate is the 11-dimensional sugra/M-theory spacetime, and of (prequantum) S-duality for type IIB string theory. Finally we give the non-perturbative description of all this by higher WZW-type  $\sigma$ -models on higher super-orbispaces with higher WZW terms in stacky differential cohomology.

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# 1 Introduction: Traditional WZW and the need for higher WZW

For  $G$  be a simple Lie group, write  $\mathfrak{g}$  for its semisimple Lie algebra. The Killing form invariant polynomial  $\langle -, - \rangle : \text{Sym}^2 \mathfrak{g} \rightarrow \mathbb{R}$  induces the canonical Lie algebra 3-cocycle

$$\mu := \langle -, [-, -] \rangle : \text{Alt}^3(\mathfrak{g}) \rightarrow \mathbb{R}$$

which by left-translation along the group defines the canonical closed and left-invariant 3-form

$$\langle \theta \wedge [\theta \wedge \theta] \rangle \in \Omega_{\text{cl,L}}^3(G),$$

where  $\theta \in \Omega_{\text{flat}}^1(G, \mathfrak{g})$  is the canonical *Maurer-Cartan form* on  $G$ . What is called the *Wess-Zumino-Witten sigma-model* induced by this data (see for instance [28] for a decent review) is the prequantum field theory given by an action functional, which to a smooth map  $\Sigma_2 \rightarrow G$  out of a closed oriented smooth 2-manifold assigns the product of the standard exponentiated kinetic action with an exponentiated “surface holonomy” of a 2-form connection whose curvature 3-form is  $\langle \theta \wedge [\theta \wedge \theta] \rangle$ .

In the special case that  $\phi : \Sigma_2 \rightarrow G$  happens to factor through a contractible open subset  $U$  of  $G$  – notably in the *perturbative expansion* about maps constant on a point – the Poincaré lemma implies that one can find a potential 2-form  $B \in \Omega^2(U)$  with  $dB = \langle \theta \wedge [\theta \wedge \theta] \rangle|_U$  and with this perturbative perspective understood one may take the action functional to be simply of the naive form that is often considered in the literature:

$$\exp(iS_{\text{WZW}}) := \exp\left(i \int_{\Sigma_2} \mathcal{L}_{\text{WZW}}\right) : \phi \mapsto \exp\left(2\pi i \int_{\Sigma_2} \phi^* B\right).$$

There are plenty of hints and some known examples which point to the fact that this construction of the standard WZW model is just one in a large class of examples of higher dimensional boundary local (pre-)quantum field theories [46, 36] which generalize traditional WZW theory in two ways:

1. the cocycle  $\mu$  is allowed to be of arbitrary degree;
2. the Lie algebra  $\mathfrak{g}$  is allowed to be a (super-)Lie  $n$ -algebra for  $n \geq 1$  ( $L_\infty$ -algebra).

One famous class of examples of the first point are the Green-Schwarz type action functionals for the super  $p$ -branes of string/M-theory [1]. These are the higher dimensional analog of the action functional for the superstring that was first given in [29] and then recognized as a super WZW-model in [25], induced from an exceptional 3-cocycle on super-Minkowski spacetime of bosonic dimension 10, regarded a super-translation Lie algebra. These higher dimensional Green-Schwarz type  $\sigma$ -model action functionals are accordingly induced by higher exceptional super-Lie algebra cocycles on super-Minkowski spacetime, regarded as a super-translation Lie algebra. Remarkably, while ordinary Minkowski spacetime is cohomologically fairly uninteresting, super-Minkowski spacetime has a finite number of *exceptional* super-cohomology classes. The higher dimensional WZW models induced by the corresponding higher exceptional cocycles account precisely for the  $\sigma$ -models of those super- $p$ -branes in string/M-theory which are pure  $\sigma$ -models, in that they do not carry (higher) gauge fields (“tensor multiplets”) on their worldvolume, a fact known as “the old brane scan” [1]. This includes, for instance, the heterotic superstring and the M2-brane, but excludes the D-branes and the M5-brane.

However, as we discuss below in section 4, this restriction to pure  $\sigma$ -model branes without “tensor multiplet” fields on their worldvolume is due to the restriction to ordinary super Lie algebras, hence to super Lie  $n$ -algebras for just  $n = 1$ . If one allows genuinely higher WZW models which are given by higher cocycles on Lie  $n$ -algebras for higher  $n$ , then *all* the super  $p$ -branes of string/M-theory are described by higher WZW  $\sigma$ -models. This is an incarnation of the general fact that in higher differential geometry, in the sense of [16, 45], the distinction between  $\sigma$ -models and (higher) gauge theory disappears, as (higher) gauge theories are equivalently  $\sigma$ -models whose target space is a smooth higher moduli *stack*, infinitesimally approximated by a Lie  $n$ -algebra for higher  $n$ .

This general phenomenon is particularly interesting for the M5-brane (see for instance the Introduction of [19] for plenty of pointers to the literature on this). According to the higher Chern-Simons-theoretic formulation of  $\text{AdS}_7/\text{CFT}_6$  in [49], the 6-dimensional  $(2, 0)$ -superconformal worldvolume theory of the M5-brane is related to the 7-dimensional Chern-Simons term in 11-dimensional supergravity compactified on a 4-sphere in direct analogy to the famous relation of 2d WZW theory to the 3d-Chern-Simons theory controlled by the cocycle  $\mu$  (see [28] for a review). In previous work we have discussed the bosonic nonabelian (quantum corrected) component of this 7d Chern-Simons theory as a higher gauge local prequantum field theory [19, 20]; the discussion here provides the fermionic terms and the formalization of the 6d WZW-type theory induced from a (flat) 7-dimensional Chern-Simons theory.

Up to the last section in this paper we discuss general aspects and examples of higher WZW-type sigma-models in the rational/perturbative approximation, where only the curvature  $n$ -form matters while its lift to a genuine cocycle in differential cohomology is ignored. However, in order to define already the traditional WZW action functional in a sensible way on *all* maps to  $G$ , one needs a more global description of the WZW term  $\mathcal{L}_{\text{WZW}}$ . Since [27, 26], this is understood to be a circle 2-connection/bundle gerbe/Deligne 3-cocycle whose curvature 3-form is  $\langle \theta \wedge [\theta \wedge \theta] \rangle$ , hence a *higher prequantization* [17] of the curvature 3-form, which, following [16, 45] we write as a lift of maps of smooth higher stacks

$$\begin{array}{ccc}
 & & \mathbf{B}^2U(1)_{\text{conn}} \\
 & \nearrow \mathcal{L}_{\text{WZW}} & \downarrow H_{(-)} \\
 G & \xrightarrow{\langle \theta \wedge [\theta \wedge \theta] \rangle} & \Omega_{\text{cl}}^3,
 \end{array}$$

where  $\mathbf{B}^2U(1)_{\text{conn}}$  denotes the smooth 2-stack of smooth circle 2-connections. Then for  $\phi : \Sigma_2 \rightarrow G$  a smooth map from a closed oriented 2-manifold to  $G$ , the globally defined value of the action functional is the corresponding *surface holonomy* expressed as the composite

$$\exp(iS_{\text{WZW}}) := \exp\left(2\pi i \int_{\Sigma_2} [(-), \mathcal{L}_{\text{WZW}}]\right) : [\Sigma, G] \xrightarrow{[\Sigma, \mathcal{L}_{\text{WZW}}]} [\Sigma, \mathbf{B}^2U(1)_{\text{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_2})(-)} U(1),$$

of the functorial mapping stack construction followed by a stacky refinement of fiber integration in differential cohomology, as discussed in [21, 22].

Towards the end, in section 5 we demonstrate a general universal construction of such non-perturbative refinements of all the local higher WZW terms considered in the main text. We show how these are in a precise sense boundary local prequantum field theories for flat higher Chern-Simons type local prequantum field theories as explained in [17, 46] (which is in line with the Chern-Simons theoretic holography in [49]). Therefore we know in principle how to quantize them non-perturbatively in generalized cohomology, namely along the lines of [36]. This, however, is to be discussed elsewhere.

## 2 Lie $n$ -algebraic formulation of perturbative higher WZW

We start with the traditional WZW model and show how in this example we may usefully reformulate its rationalized/perturbative aspects in terms of Lie  $n$ -algebraic structures. Then we naturally and seamlessly generalize to a definition of higher WZW-type  $\sigma$ -models.

We recall the notion of  $L_\infty$ -algebra valued differential forms/connections from [44, 16] to establish our notation. All the actual  $L_\infty$ -homotopy theory that we need can be found discussed or referenced in [18]. Just for simplicity of exposition and since it is sufficient for the present discussion, here we take all  $L_\infty$ -algebras to be of finite type, hence degreewise finite dimensional; see [39] for the general discussion in terms of pro-objects.

A (super-)Lie  $n$ -algebra is a (super-) $L_\infty$ -algebra concentrated in the lowest  $n$  degrees. Given a (super-) $L_\infty$ -algebra  $\mathfrak{g}$ , we write  $\text{CE}(\mathfrak{g})$  for its *Chevalley-Eilenberg algebra*; which is a  $(\mathbb{Z} \times \mathbb{Z}_2)$ -graded commutative dg-algebra with the property that the underlying graded super-algebra is the free graded commutative super-algebra on the dual graded super vector space  $\mathfrak{g}[1]^*$ . These are the dg-algebras which in parts of the supergravity literature are referred to as “FDA”s, a term introduced in [34] and then picked up in [3, 4, 13] and followups. Precisely all the (super-)dg-algebras of this *semi-free* form arise as Chevalley-Eilenberg algebras of (super-) $L_\infty$ -algebras this way, and a homomorphism of  $L_\infty$ -algebras  $f : \mathfrak{g} \rightarrow \mathfrak{h}$  is equivalently a homomorphism of dg-algebras of the form  $f^* : \text{CE}(\mathfrak{h}) \rightarrow \text{CE}(\mathfrak{g})$ . See [18] for a review in the context of the higher prequantum geometry of relevance here and for further pointers to the literature on  $L_\infty$ -algebras and their homotopy theory.

**Definition 2.1.** For  $\mathfrak{g}$  a Lie  $n$ -algebra, and  $X$  a smooth manifold, a *flat  $\mathfrak{g}$ -valued differential form* on  $X$  (of total degree 1, with  $\mathfrak{g}$  regarded as cohomologically graded) is equivalently a morphism of dg-algebras  $A^* : \text{CE}(\mathfrak{g}) \rightarrow \Omega_{\text{dR}}^\bullet(X)$  to the de Rham complex. Dually we write this as  $A : X \rightarrow \mathfrak{g}$ . These differential forms naturally pull back along maps of smooth manifolds, and we write  $\Omega_{\text{flat}}^1(-, \mathfrak{g})$  for the sheaf, on smooth manifolds, of flat  $\mathfrak{g}$ -valued differential forms of total degree 1.

Notice that, in general, these forms of total degree 1 involve differential forms of higher degree with coefficients in higher degree elements of the  $L_\infty$ -algebra:

**Example 2.2.** For  $n \in \mathbb{N}$  write  $\mathbb{R}[n]$  for the abelian Lie  $n$ -algebra concentrated on  $\mathbb{R}$  in degree  $-n$ . Its Chevalley Eilenberg algebra is the dg-algebra which is genuinely free on a single generator in degree  $n + 1$ . A flat  $\mathbb{R}[n]$ -valued differential form is equivalently just an ordinary closed differential  $(n + 1)$ -form

$$\Omega_{\text{flat}}^1(-, \mathbb{R}[n]) \simeq \Omega_{\text{cl}}^{n+1}.$$

**Definition 2.3.** A  $(p+2)$ -cocycle  $\mu$  on a Lie  $n$ -algebra  $\mathfrak{g}$  is a degree  $p+2$  closed element in the corresponding Chevalley-Eilenberg algebra  $\mu \in \text{CE}(\mathfrak{g})$ .

**Remark 2.4.** A  $(p+2)$ -cocycle on  $\mathfrak{g}$  is equivalently a map of dg-algebras  $\text{CE}(\mathbb{R}[p+1]) \rightarrow \text{CE}(\mathfrak{g})$  and hence, equivalently, a map of  $L_\infty$ -algebras of the form  $\mu : \mathfrak{g} \rightarrow \mathbb{R}[p+1]$ . So, if  $\{t_a\}$  is a basis for the graded vector space underlying  $\mathfrak{g}$ , then the differential  $d_{\text{CE}}$  is given in components by

$$d_{\text{CE}} t^a = \sum_{i \in \mathbb{N}} C^a_{a_1 \dots a_i} t^{a_1} \wedge \dots \wedge t^{a_i},$$

where  $\{C^a_{a_1 \dots a_i}\}$  are the structure constants of the  $i$ -ary bracket of  $\mathfrak{g}$ . Consequently, a degree  $p+2$  cocycle is a degree  $(p+2)$ -element

$$\mu = \sum_i \mu_{a_1 \dots a_i} t^{a_1} \wedge \dots \wedge t^{a_i}$$

such that  $d_{\text{CE}} \mu = 0$ .

**Example 2.5.** For  $\{t_a\}$  a basis as above and  $\omega \in \Omega_{\text{flat}}^1(X, \mathfrak{g})$  a  $\mathfrak{g}$ -valued 1-form on  $X$ , the pullback of the cocycle is the closed differential  $(p+2)$ -form which in components reads

$$\mu(\omega) = \sum_i \mu_{a_1 \dots a_i} \omega^{a_1} \wedge \dots \wedge \omega^{a_i},$$

where  $\omega^a = \omega(t^a)$ .

**Remark 2.6.** Composition  $\omega \mapsto (X \xrightarrow{\omega} \mathfrak{g} \xrightarrow{\mu} \mathbb{R}[p+1])$  of  $\mathfrak{g}$ -valued differential forms  $\omega$  with an  $L_\infty$ -cocycle  $\mu$  yields a homomorphism of sheaves

$$\Omega_{\text{flat}}^1(-, \mu) : \Omega_{\text{flat}}(-, \mathfrak{g}) \longrightarrow \Omega_{\text{cl}}^{p+2}.$$

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<sup>1</sup>The reader familiar with  $L_\infty$ -algebroids should take this as shorthand for the  $L_\infty$ -algebroid homomorphism from the tangent Lie algebroid of  $X$  to the delooping of the  $L_\infty$ -algebra  $\mathfrak{g}$ .

This is the sheaf incarnation of  $\mu$  regarded as a universal differential form on the space of all flat  $\mathfrak{g}$ -valued differential forms. More on this is below in 5.

**Example 2.7.** By the Yoneda lemma, for  $X$  a smooth manifold, morphisms<sup>2</sup>  $X \rightarrow \Omega_{\text{flat}}^1(-, \mathfrak{g})$  are equivalently just flat  $\mathfrak{g}$ -valued differential forms on  $X$ . Specifically, for  $G$  an ordinary Lie group, its Maurer-Cartan form is equivalently a map

$$\theta : G \longrightarrow \Omega_{\text{flat}}^1(-, \mathfrak{g}) .$$

Therefore, given a field configuration  $\phi : \Sigma_2 \rightarrow G$  of the traditional WZW model, postcomposition with  $\theta$  turns this into

$$\phi^*\theta : \Sigma \xrightarrow{\phi} G \xrightarrow{\theta} \Omega_{\text{flat}}^1(-, \mathfrak{g}) .$$

Here if  $\mathfrak{g}$  is represented as a matrix Lie algebra then this is the popular expression  $\phi^*\theta = \phi^{-1}d\phi$

**Definition 2.8.** Given an  $L_\infty$ -algebra  $\mathfrak{g}$  equipped with a cocycle  $\mu : \mathfrak{g} \rightarrow \mathbb{R}[p+1]$  of degree  $p+2$ , a *perturbative  $\sigma$ -model datum* for  $(\mathfrak{g}, \mu)$  is a triple consisting of

- a space  $X$ ;
- equipped with a flat  $\mathfrak{g}$ -valued differential form  $\theta_{\text{global}} : X \rightarrow \Omega_{\text{flat}}^1(-, \mathfrak{g})$  (a “global Maurer-Cartan form”);
- and equipped with a factorization  $\mathcal{L}_{\text{WZW}}$  through  $d_{\text{dR}}$  of  $\mu(\theta_{\text{global}})$ , as expressed in the following diagram

$$\begin{array}{ccccc} X & \xrightarrow{\theta_{\text{global}}} & \Omega_{\text{flat}}(-, \mathfrak{g}) & \xrightarrow{\mu} & \Omega_{\text{cl}}^{p+2} . \\ & \searrow \mathcal{L}_{\text{WZW}} & & \nearrow d_{\text{dR}} & \\ & & \Omega^{p+1} & & \end{array}$$

The *action functional* associated with this data is the functional

$$S_{\text{WZW}} : [\Sigma, X] \longrightarrow \mathbb{R}$$

given by

$$\phi \mapsto \int_{\Sigma} \mathcal{L}_{\text{WZW}}(\phi) ,$$

where the integrand is the differential form

$$\mathcal{L}_{\text{WZW}}(\phi) : \Sigma \xrightarrow{\phi} X \xrightarrow{\mathcal{L}_{\text{WZW}}} \Omega_{\text{cl}}^{p+1} .$$

**Remark 2.9.** Here  $X$  actually need not be a (super-)manifold but may be a smooth higher (super-) stack, hence what we may suggestively call *higher super orbi-space*. We make this precise below in section 5.

**Remark 2.10.** The notation  $\theta_{\text{global}}$  serves to stress the fact that we are considering globally defined one-forms on  $X$  as opposed to cocycles in hypercohomology, which is where the higher Maurer-Cartan forms on *higher* (super-)Lie groups take values, due to presence of nontrivial higher gauge transformations. See section 5 for more discussion.

<sup>2</sup>of sheaves, by thinking of  $X$  as the sheaf  $C^\infty(-, X)$ .

**Remark 2.11.** The diagram in Def. 2.8 manifestly captures a local description, when  $X$  is a contractible manifold. An immediate global version is captured by the following diagram

$$\begin{array}{ccccc} \Sigma & \xrightarrow{\eta} & X & \xrightarrow{\theta_{\text{global}}} & \Omega_{\text{flat}}(-, \mathfrak{g}) & \xrightarrow{\mu} & \Omega_{\text{cl}}^{p+2}, \\ & & & \searrow^{\mathcal{L}_{\text{WZW}}} & & \nearrow^{F(-)} & \\ & & & & \mathbf{B}^{p+1}U(1)_{\text{conn}} & & \end{array}$$

where  $\mathbf{B}^{p+1}U(1)_{\text{conn}}$  is the stack of  $U(1)$ - $(p+1)$ -bundles with connections, and  $F(-)$  is the curvature morphism; see, for instance, [16]. This globalization is what one sees, for example, in the ordinary WZW model. This, too, we come to below in section 5.

Finally, we notice for discussion in the examples one aspect of the higher symmetries of such perturbative higher WZW models:

**Definition 2.12.** Given a (super-)  $L_\infty$ -algebra  $\mathfrak{g}$ , its *graded Lie algebra of infinitesimal automorphisms* is the Lie algebra whose elements are graded derivations  $v \in \text{Der}(\text{Sym}^\bullet \mathfrak{g}[1]^*)$  on the graded algebra underlying its Chevalley-Eilenberg algebra  $\text{CE}(\mathfrak{g})$ , acting as the corresponding Lie derivatives.

### 3 Boundary conditions and brane intersection laws

In the context of fully extended (i.e. local) boundary topological field theories, one has the following general notion of boundary condition, see [36, 46].

**Definition 3.1.** A *prequantum boundary condition for an open brane* (hence a “background brane” on which the given brane may end) is given by boundary gauge trivializations  $\phi_{\text{bdr}}$  of the Lagrangian restricted to the boundary fields, hence by diagrams of the form

$$\begin{array}{ccc} & \text{Boundary Field} & \\ & \swarrow & \searrow \\ * & & \text{Bulk Fields} \\ & \swarrow^{\phi_{\text{bdr}}} & \searrow^{\text{Lagrangian}} \\ & 0 & \text{Phases,} \end{array}$$

where “Phases” denotes generally the space where the Lagrangian takes values.

Specializing this general principle to our current situation, we have the following

**Definition 3.2.** A *boundary condition* for a rational  $\sigma$ -model datum,  $(X, \mathfrak{g}, \mu)$  of Def. 2.8, is

1. an  $L_\infty$ -algebra  $Q$  and a homomorphism  $Q \rightarrow \mathfrak{g}$ ,

2. equipped with a homotopy  $\phi_{\text{brd}}$  of  $L_\infty$ -algebras morphisms

$$\begin{array}{ccc} & Q & \\ & \swarrow & \searrow \\ * & & \mathfrak{g} \\ & \swarrow^{\phi_{\text{brd}}} & \searrow^{\mu} \\ & 0 & \mathbb{R}[p+1] \end{array}$$

**Remark 3.3** (Background branes). Since  $\mathfrak{g}$  is to be thought of as the *spacetime target* for a  $\sigma$ -model, we are to think of  $Q \rightarrow \mathfrak{g}$  in Def. 3.2 as a *background brane* “inside” spacetime. For instance, as demonstrated below in Section 4, it may be a D-brane in 10-dimensional super-Minkowski space on which the open superstring ends, or it may be the M5-brane in 11-dimensional super-Minkowski spacetime on which the open M2-brane ends. To say then that the  $p$ -brane described by the  $\sigma$ -model may end on this background brane  $Q$  means to consider worldvolume manifolds  $\Sigma_n$  with boundaries  $\partial\Sigma_{p+1} \hookrightarrow \Sigma_{p+1}$  and *boundary field configurations*  $(\phi, \phi|_{\partial})$  making the left square in the following diagram commute:

$$\begin{array}{ccccc} \partial\Sigma_{p+1} & \xrightarrow{\phi|_{\partial\Sigma}} & Q & \xrightarrow{\quad} & * \\ \downarrow & & \downarrow & \swarrow \phi_{\text{bdr}} & \downarrow \\ \Sigma_{p+1} & \xrightarrow{\phi} & \mathfrak{g} & \xrightarrow{\mu} & \mathbb{R}[p+1] . \end{array}$$

The commutativity of the diagram on the left encodes precisely that the boundary of the  $p$ -brane is to sit inside the background brane  $Q$ . But now – by the defining universal property of the homotopy pullback of super  $L_\infty$ -algebras – this means, equivalently, that the background brane embedding map  $Q \rightarrow \mathfrak{g}$  factors through the *homotopy fiber* of the cocycle  $\mu$ . If we denote this homotopy fiber by  $\widehat{\mathfrak{g}}$ , then we have an essentially unique factorization as follows

$$\begin{array}{ccccc} \partial\Sigma_{p+1} & \xrightarrow{\phi|_{\partial\Sigma}} & Q & \dashrightarrow & \widehat{\mathfrak{g}} & \xrightarrow{\quad} & * \\ \downarrow & & \downarrow & & \downarrow & \swarrow \phi_{\text{bdr}}^{\text{univ.}} & \downarrow \\ \Sigma_{p+1} & \xrightarrow{\phi} & \mathfrak{g} & \xlongequal{\quad} & \widehat{\mathfrak{g}} & \xrightarrow{\mu} & \mathbb{R}[p+1] , \end{array}$$

where now  $\widehat{\mathfrak{g}} \rightarrow \mathfrak{g}$  is the *homotopy fiber*  $\widehat{\mathfrak{g}}$  of the cocycle  $\mu$ . Notice that here in homotopy theory *all* diagrams appearing are understood to be filled by homotopies/gauge transformations, but only if we want to label them explicitly do we display them.

The crucial implication to emphasize is that what used to be regarded as a background brane  $Q$  on which the  $\sigma$ -model brane  $\Sigma_n$  may end is itself characterized by a  $\sigma$ -model map  $Q \rightarrow \widehat{\mathfrak{g}}$ , not to the original target space  $\mathfrak{g}$ , but to the *extended target space*  $\widehat{\mathfrak{g}}$ . In the class of examples discussed below in Section 4, this extended target space is precisely the *extended superspace* in the sense of [14].

**Remark 3.4.** The  $L_\infty$ -algebra  $\widehat{\mathfrak{g}} \rightarrow \mathfrak{g}$  is the *extension* of  $\mathfrak{g}$  classified by the cocycle  $\mu$ , in generalization to the traditional extension of Lie algebras classified by 2-cocycles. If  $\mu$  is an  $(n_2 + 1)$ -cocycle on an  $n_1$ -Lie algebra  $\mathfrak{g}$  for  $n_1 \leq n_2$ , then the extended  $L_\infty$ -algebra  $\widehat{\mathfrak{g}}$  is an Lie  $n_2$ -algebra. See [18] for more details on this.

**Proposition 3.5.** *The Chevalley-Eilenberg algebra  $\text{CE}(\widehat{\mathfrak{g}})$  of the extension  $\widehat{\mathfrak{g}}$  of  $\mathfrak{g}$  by a cocycle  $\mu$  admits, up to equivalence, a very simple description; namely, it is the differential graded algebra obtained from  $\text{CE}(\mathfrak{g})$  by adding a single generator  $c_n$  in degree  $n$  subject to the relation*

$$d_{\text{CE}(\widehat{\mathfrak{g}})} c_n = \mu .$$

Here we are viewing  $\mu$  as a degree  $n + 1$  element in  $\text{CE}(\mathfrak{g})$ , and hence also in  $\text{CE}(\widehat{\mathfrak{g}})$ .

Proof. First observe that we have a commuting diagram of (super-)dg-algebras of the form

$$\begin{array}{ccc} \text{CE}(\widehat{\mathfrak{g}}) & \longleftarrow & \text{CE}\left(\left(\mathbb{R} \xrightarrow{\text{id}} \mathbb{R}\right)[n-1]\right) \\ \uparrow & & \uparrow \\ \text{CE}(\mathfrak{g}) & \longleftarrow & \text{CE}(\mathbb{R}[n]) \end{array} .$$

Here the top left dg-algebra is the dg-algebra of the above statement, the top morphism is the one that sends the unique degree- $(n + 1)$ -generator to  $\mu$  and the unique degree- $n$  generator to  $c_n$ , the vertical morphisms are the evident inclusions, and the bottom morphism is the given cocycle. Consider the dual diagram of  $L_\infty$ -algebras

$$\begin{array}{ccc} \widehat{\mathfrak{g}} & \longrightarrow & (\mathbb{R} \xrightarrow{\text{id}} \mathbb{R})[n - 1] \\ \downarrow & & \downarrow \\ \mathfrak{g} & \xrightarrow{\mu} & \mathbb{R}[n]. \end{array}$$

Then observe that the underlying graded vector spaces here form a pullback diagram of linear maps (the linear components of the  $L_\infty$ -morphisms). From this the statement follows directly with the recognition theorem for  $L_\infty$ -homotopy fibers, theorem 3.1.13 in [18].  $\square$

**Remark 3.6.** The construction appearing in Prop. 3.5 is of course well familiar in the “FDA”-technique in the supergravity literature [13], and we recall famous examples below in Section 4. The point to highlight here is that this construction has a universal  $L_\infty$ -homotopy-theoretic meaning, in the way described above.

The crucial consequence of this discussion is the following:

**Remark 3.7.** If the extension  $\widehat{\mathfrak{g}}$  itself carries a cocycle  $\mu_Q : \widehat{\mathfrak{g}} \rightarrow \mathbb{R}[n]$  and we are able to find a local potential/Lagrangian  $\mathcal{L}_{\text{WZW}}$  for the closed  $(n + 1)$ -form  $\mu_Q$  (and we will see in the full description in 5 that this is always the case), then this exhibits the background brane  $Q$  itself as a rational WZW  $\sigma$ -model, now propagating not on the original “target spacetime”  $\mathfrak{g}$  but on the “extended spacetime”  $\widehat{\mathfrak{g}}$ .

**Remark 3.8.** Iterating this process gives rise to a tower of extensions and cocycles

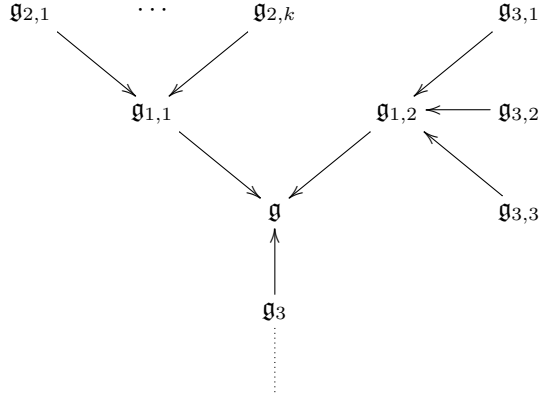
$$\begin{array}{ccc} \dots & & \\ \widehat{\widehat{\mathfrak{g}}} & \xrightarrow{\mu_3} & \mathbb{R}[n_3] \\ \downarrow & & \\ \widehat{\mathfrak{g}} & \xrightarrow{\mu_2} & \mathbb{R}[n_2] \\ \downarrow & & \\ \mathfrak{g} & \xrightarrow{\mu_1} & \mathbb{R}[n_1], \end{array}$$

which is like a Whitehead tower in rational homotopy theory, only that the cocycles in each degree here are not required to be the lowest-degree nontrivial ones. In fact, the actual rational Whitehead tower is an example of this. In the language of Sullivan’s formulation of rational homotopy theory this says that  $\mathfrak{g}_n$  is exhibited by a sequence of cell attachments as a *relative Sullivan algebra* relative to  $\mathfrak{g}$ .

Since this is an important concept for the present purpose, we give it a name:

**Definition 3.9.** Given an  $L_\infty$ -algebra  $\mathfrak{g}$ , the *brane bouquet* of  $\mathfrak{g}$  is the rooted tree consisting of, iteratively, all possible equivalence classes of nontrivial  $\mathbb{R}[\bullet]$  extensions (corresponding to equivalence classes of nontrivial  $\mathbb{R}[\bullet]$ -cocycles) starting with  $\mathfrak{g}$  as the root.





This *brane bouquet* construction in  $L_\infty$ -homotopy theory that we introduced serves to organize and formalize the following two physical heuristics.

**Remark 3.10** (Brane intersection laws). By the discussion above in Remark 3.3, each piece of a brane bouquet of the form

$$\begin{array}{ccc}
 \mathfrak{g}_2 & \xrightarrow{\mu_2} & \mathbb{R}[n_2] \\
 \downarrow & & \\
 \mathfrak{g}_1 & \xrightarrow{\mu_1} & \mathbb{R}[n_1]
 \end{array}$$

may be thought of as encoding a *brane intersection law*, meaning that the WZW  $\sigma$ -model brane corresponding to  $(\mathfrak{g}_1, \mu_1)$  can end on the WZW  $\sigma$ -model brane corresponding to  $(\mathfrak{g}_2, \mu_2)$ . Therefore, the brane bouquet of some  $L_\infty$ -algebra  $\mathfrak{g}$  lists all the possible  $\sigma$ -model branes and all their intersection laws in the “target spacetime”  $\mathfrak{g}$ .

**Remark 3.11** (Brane condensates). To see how to think of the extensions  $\widehat{\mathfrak{g}}$  as “extended spacetimes”, observe that by Prop. 3.5 and Def. 2.1 a  $\sigma$ -model on the extension  $\widehat{\mathfrak{g}}$  of  $\mathfrak{g}$  which is classified by a  $(p+2)$ -cocycle  $\mu$  is equivalently a  $\sigma$ -model on  $\mathfrak{g}$  together with an  $p$ -form higher gauge field on its worldvolume, one whose curvature  $(p+1)$ -form satisfies a twisted Bianchi identity controlled by  $\mu$ . The examples discussed below in Section 4 show that this  $p$ -form field (“tensor field” in the brane literature) is that which is “sourced” by the charged boundaries of the original  $\sigma$ -model branes on  $\mathfrak{g}$ . For instance for superstrings ending on D-branes it is the Chan-Paton gauge field sourced by the endpoints of the open string, and for M2-branes ending on M5-branes it is the latter’s B-field which is sourced by the self-dual strings at the boundary of the M2-brane. In conclusion, this means that we may think of the extension  $\widehat{\mathfrak{g}}$  as being the original spacetime  $\mathfrak{g}$  but *filled with a condensate* of branes whose  $\sigma$ -model is induced by  $\mu$ .

## 4 Example: Super $p$ -branes and their intersection laws

We now discuss higher rational/perturbative WZW models on super-Minkowski spacetime regarded as the super-translation Lie algebra over itself, as well as on the *extended superspaces* which arise as exceptional super Lie  $n$ -algebra extensions of the super-translation Lie algebra. We show then that by the brane intersection laws of Remark 3.10 this reproduces precisely the super  $p$ -brane content of string/M-theory including the  $p$ -branes with tensor multiplet fields, notably including the D-branes and the M5-brane. The discussion is based on the work initiated in [4] and further developed in articles including [14]. The point here is to show that this “FDA”-technology is naturally and usefully reformulated in terms of super- $L_\infty$ -homotopy theory, and that this serves to clarify and illuminate various points that have not been seen, and are indeed hard to see, via the “FDA”-perspective.

We set up some basic notation concerning the super-translation- and the super-Poincaré super Lie algebras, following [4]. For more background see lecture 3 of [23] and appendix B of [38].

Write  $\mathfrak{o}(d-1, 1)$  for the Lie algebra of the Lorentz group in dimension  $d$ . If  $\{\omega_a{}^b\}_{a,b}$  is the canonical basis of Lie algebra elements, then the Chevalley-Eilenberg algebra  $\text{CE}(\mathfrak{o}(d-1, 1))$  is generated from elements  $\{\omega_a{}^b\}_{a,b}$  in degree (1, even) with the differential given by<sup>3</sup>  $d_{\text{CE}} \omega_a{}^b := \omega_a{}^c \wedge \omega_c{}^b$ . Next, write  $\mathfrak{iso}(d-1, 1)$  for the Poincaré Lie algebra. Its Chevalley-Eilenberg algebra in turn is generated from the  $\{\omega_a{}^b\}$  as before together with further generators  $\{e^a\}_a$  in degree (1, even) with differential given by  $d_{\text{CE}} e^a := \omega_a{}^b \wedge e^b$ . Now for  $N$  denoting a real spinor representation of  $\mathfrak{o}(d-1, 1)$ , also called the number of supersymmetries (see for instance part 3 of [23]), write  $\{\Gamma^a\}$  for a representation of the Clifford algebra in this representation and  $\{\Psi_\alpha\}_\alpha$  for the corresponding basis elements of the spinor representation. There is then an essentially unique symmetric  $\text{Spin}(d-1, 1)$ -equivariant bilinear map from two spinors to a vector, traditionally written in components as

$$(\psi_1, \psi_2)^a := \frac{i}{2} \bar{\psi} \Gamma^a \psi.$$

This induces the super Poincaré Lie algebra  $\mathfrak{siso}_N(d-1, 1)$  whose Chevalley-Eilenberg super-dg-algebra is generated from the generators as above together with generators  $\{\Psi^\alpha\}$  in degree (1, odd) with the differential now defined as follows

$$\begin{aligned} d_{\text{CE}} \omega_a{}^b &= \omega_a{}^c \wedge \omega_c{}^b, \\ d_{\text{CE}} e^a &= \omega_a{}^b \wedge e^b + \frac{i}{2} \bar{\psi} \wedge \Gamma^a \psi, \\ d_{\text{CE}} \psi^\alpha &= \frac{1}{4} \omega_a{}^b \wedge \Gamma^a{}_\beta \Psi^\beta. \end{aligned}$$

Here and in the following  $\Gamma^{a_1 \dots a_p}$  denotes the skew-symmetrized product of the Clifford matrices and in the above matrix multiplication is understood whenever the corresponding indices are not displayed. In summary, the degrees are

$$\deg(e^a) = (1, \text{even}), \quad \deg(\omega^a) = (1, \text{even}), \quad \deg(\psi^\alpha) = (1, \text{odd}), \quad \deg(d_{\text{CE}}) = (1, \text{even}).$$

Notice that this means that, for instance,  $e^{a_1} \wedge e^{a_2} = -e^{a_2} \wedge e^{a_1}$  and  $e^a \wedge \psi^\alpha = -\psi^\alpha \wedge e^a$  but  $\psi^{\alpha_1} \wedge \psi^{\alpha_2} = +\psi^{\alpha_2} \wedge \psi^{\alpha_1}$ .

**Example 4.1.** For  $\Sigma$  a supermanifold of dimension  $(d; N)$ , a flat  $\mathfrak{siso}(d-1, 1)$ -valued differential form  $A : \text{CE}(\mathfrak{siso}(d-1, 1)) \rightarrow \Omega_{\text{dR}}^\bullet(\Sigma)$ , according to Def. 2.1 and subject to the constraint that the  $\mathbb{R}^{d; N}$ -component is induced from the tangent space of  $\Sigma$  (this makes it a *Cartan connection*) is

1. a *vielbein* field  $E^a := A(e^a)$ ,
2. with a *Levi-Civita connection*  $\Omega^a{}_\beta := A(\omega^a{}_\beta)$  (graviton),
3. a spinor-valued 1-form field  $\Psi^\alpha := A(\psi^\alpha)$  (gravitino),

subject to the flatness constraints which here say that the torsion of the Levi-Civita connection is the super-torsion  $\tau = \bar{\Psi} \wedge \Gamma^a \Psi \wedge E_a$  and that the Riemann curvature vanishes. This is the gravitational field content (for vanishing field strength here, one can of course also consider non-flat fields; see [44]) of supergravity on  $\Sigma$ , formulated in first order formalism. By passing to  $L_\infty$ -extensions of  $\mathfrak{siso}$  this is the fomulation of supergravity fields which seamlessly generalizes to the higher gauge fields that higher supergravities contain, including their correct higher gauge transformations. This is the perspective on supergravity originating around the article [4] and expanded on in the textbook [13]. Recognizing the “FDA”-language used in this book as secretly being about Lie  $n$ -algebra homotopy theory (the “FDA”s are really Chevalley-Eilenberg algebras super- $L_\infty$ -algebras) allows to uncover some natural and powerful higher gauge theory and geometric homotopy theory [45] hidden in traditional supergravity literature.

The *super translation Lie algebra* corresponding to the above is the quotient

$$\mathbb{R}^{d; N} := \mathfrak{siso}(d-1, 1) / \mathfrak{o}(d-1, 1)$$

---

<sup>3</sup>Here and in all of the following a summation over repeated indices is understood.

whose CE-algebra is as above but with the  $\{\omega^a{}_b\}$  discarded. We may think of the underlying super vector space of  $\mathbb{R}^{d;N}$  as  $N$ -super Minkowski spacetime of dimension  $d$ , i.e. with  $N$  supersymmetries. Regarded as a supermanifold, it has canonical super-coordinates  $\{x^a, \vartheta^\alpha\}$  and the CE-generators  $e^a$  and  $\psi^\alpha$  above may be identified under the general equivalence  $\text{CE}(\mathfrak{g}) \simeq \Omega_{\mathbb{L}}^\bullet(G)$  (for a (super-)Lie group  $G$  with (super-)Lie algebra  $\mathfrak{g}$ ) with the corresponding canonical left-invariant differential forms on this supermanifold:

$$\begin{aligned} e^a &= d_{\text{dR}} x^a + \bar{\vartheta} \Gamma^a d_{\text{dR}} \vartheta, \\ \psi^\alpha &= d_{\text{dR}} \vartheta^\alpha. \end{aligned}$$

This defines a morphism  $\theta : \text{CE}(\mathbb{R}^{d;N}) \rightarrow \Omega^{\bullet|\bullet}(\mathbb{R}^{d;N})$  to super-differential forms on super Minkowski space, and via Def. 2.1 this is the Maurer-Cartan form, Example 2.7, on the supergroup  $\mathbb{R}^{d;N}$  of supertranslations. As such  $\{e^a, \psi^\alpha\}$  is the canonical *super-vielbein* on super-Minkowski spacetime.

Notice that the only non-trivial piece of the above CE-differential remaining on  $\text{CE}(\mathbb{R}^{d;N})$  is

$$d_{\text{CE}(\mathbb{R}^{d;N})} e^a = \bar{\psi} \wedge \Gamma^a \psi.$$

Dually this is the single non-trivial super-Lie bracket on  $\mathbb{R}^{d;N}$ , the one which pairs two spinors to a vector. All the exceptional cocycles considered in the following exclusively are controlled by just this equation and Lorentz invariance.

We next consider various branches of the brane bouquet, Def. 4.14, of these super-spacetimes  $\mathbb{R}^{d;N}$ .

#### 4.1 $N = 1$ $\sigma$ -model super $p$ -branes — The old brane scan

As usual, we write  $N$  for a choice of number of irreducible real (Majorana) representations of  $\text{Spin}(d-1, 1)$ , and  $N = (N_+, N_-)$  if there are two inequivalent chiral minimal representations. For instance, two important cases are

$d = 10$	$d = 11$
$N = (1, 0) = \mathbf{16}$	$N = 1 = \mathbf{32}$

For  $0 \leq p \leq 9$  consider the dual bispinor element

$$\mu_p := e^{a_1} \wedge \cdots \wedge e^{a_p} \wedge (\bar{\psi} \wedge \Gamma^{a_1 \cdots a_p} \psi) \in \text{CE}(\mathbb{R}^{d;N}),$$

where here and in the following the parentheses are just to guide the reader's eye. Observe that the differential of this element is of the form

$$d_{\text{CE}} \mu_p \propto e^{a_1} \wedge \cdots \wedge e^{a_{p-1}} \wedge (\bar{\psi} \Gamma^{a_1 \cdots a_p} \psi) \wedge (\bar{\psi} \wedge \Gamma^{a_p} \psi).$$

This is zero precisely if after skew-symmetrization of the indices, the spinorial expression

$$\bar{\psi} \Gamma^{[a_1 \cdots a_p} \psi \wedge \bar{\psi} \wedge \Gamma^{a_p]} \psi = 0$$

vanishes identically (on all spinor components). The spinorial relations which control this are the *Fierz identities*. If this expression vanishes, then  $\mu_p$  is a  $(p+2)$ -cocycle on  $\mathbb{R}^{d;N=1}$ , Def. 2.3, hence a homomorphism of super Lie  $n$ -algebras of the form

$$\mu_p : \mathbb{R}^{d;N=1} \longrightarrow \mathbb{R}[p+1].$$

If this is the case then, by Def. 2.8, this defines a  $\sigma$ -model  $p$ -brane propagating on  $\mathbb{R}^{d;N=1}$ .

The combinations of  $d$  and  $p$  for which this is the case had originally been worked out in [1]. The interpretation in terms of super-Lie algebra cohomology was clearly laid out in [5]. See [9, 10, 11] for a rigorous treatment and comprehensive classification for all  $N$ . The non-trivial cases (those where  $\mu_p$  is closed but not itself a differential) correspond precisely to the non-empty entries in the following table.

$d \setminus p$	1	2	3	4	5	6	7	8	9
11		(1) <b>m2brane</b>							
10	(1,0) <b>string<sub>het</sub></b>				(1,0) <b>ns5brane<sub>het</sub></b>				
9				(1)					
8			(1)						
7		(1)							
6	(1,0) <b>littfestring</b>		(1,0)						
5		(1)							
4	(1)	(1)							
3	(1)								

This table is known as the “old brane scan” for string/M-theory. Each non-empty entry corresponds to a  $p$ -brane WZW-type  $\sigma$ -model action functional of Green-Schwarz type. For  $(d = 10, p = 1)$  this is the original Green-Schwarz action functional for the superstring [29] and, therefore, we write **string<sub>het</sub>** in the respective entry of the table (similarly there are cocycles for type II strings, discussed in the following sections), which at the same time is to denote the super Lie 2-algebra extension of  $\mathbb{R}^{10, N=1}$  that is classified by  $\mu_p$  in this dimension, according to Remark 3.4:

$$\begin{array}{ccc}
 \mathbf{string}_{\text{het}} & & \\
 \downarrow & & \\
 \mathbb{R}^{10; N=(1,0)} & \xrightarrow{\mu_1} & \mathbb{R}[2] .
 \end{array}$$

This Lie 2-algebra has been highlighted in [30, 6].

Analogously we write **m2brane** for the super Lie 3-algebra extension of  $\mathbb{R}^{11; N=1}$  classified by the nontrivial cocycle  $\mu_2$  in dimension 11 (this was called the *supergravity Lie 3-algebra sugra<sub>11</sub>* in [44])

$$\begin{array}{ccc}
 \mathbf{m2brane} & & \\
 \downarrow & & \\
 \mathbb{R}^{11; N=1} & \xrightarrow{\mu_2} & \mathbb{R}[3] ,
 \end{array}$$

and so on.

While it was a pleasant insight back then that so many of the extended objects of string/M-theory do appear from just super-Lie algebra cohomology this way in the above table, it was perhaps just as curious that not all of them appeared. Later other tabulations of string/M-branes were compiled, based on less mathematically well defined physical principles [15]. These “new brane scans” are what make the above an “old brane scan”. But we will show next that if only we allow ourselves to pass from (super-)Lie algebra theory to (super-) Lie  $n$ -algebra theory, then the old brane scan turns out to be part of a brane bouquet that accurately incorporates all the information of the “new brane scan”, all the branes of the new brane scan, altogether with their intersection laws, with their tensor multiplet field content and its correct higher gauge transformation laws.

## 4.2 Type IIA superstring ending on D-branes and the D0-brane condensate

We consider the branes in type IIA string theory and point out how their  $L_\infty$ -homotopy theoretic formulation serves to provide a formal statement and proof of the folklore relation between type IIA string theory with a D0-brane condensate and M-theory.

Write  $N = (1, 1) = \mathbf{16} + \mathbf{16}'$  for the Dirac representation of  $\text{Spin}(9, 1)$  given by two 16-dimensional real irreducible representations of opposite chirality. We write  $\{\Gamma^a\}_{a=1, \dots, 10}$  for the corresponding representation of the Clifford algebra and  $\Gamma^{11} := \Gamma^1 \Gamma^2 \dots \Gamma^{10}$  for the chirality operator. Finally write  $\mathbb{R}^{10; N=(1,1)}$  for the corresponding super-translation Lie algebra, the super-Minkowski spacetime of type IIA string theory.

**Definition 4.2.** The type IIA 3-cocycle is

$$\mu_{\text{string}_{\text{IIA}}} := \bar{\psi} \wedge \Gamma^a \Gamma^{11} \psi \wedge e^a \quad : \quad \mathbb{R}^{10; N=(1,1)} \longrightarrow \mathbb{R}[2] .$$

The type IIA superstring super Lie 2-algebra is the corresponding super  $L_\infty$ -extension

$$\begin{array}{ccc} \text{string}_{\text{IIA}} & & \\ \downarrow & & \\ \mathbb{R}^{10; N=(1,1)} & \xrightarrow{\mu_{\text{string}_{\text{IIA}}}} & \mathbb{R}[2] . \end{array}$$

Its Chevalley-Eilenberg algebra is that of  $\mathbb{R}^{10; N=(1,1)}$  with one generator  $F$  in degree  $(2, \text{even})$  adjoined and with its differential being

$$d_{\text{CE}} F = \mu_{\text{string}_{\text{IIA}}} = \bar{\psi} \wedge \Gamma^a \Gamma^{11} \psi \wedge e^a .$$

This dg-algebra appears as equation (6.3) in [14]. It can also be deduced from op.cit. that the IIA string Lie 2-algebra of Def. 4.2 carries exceptional cocycles of degrees  $p + 2 \in \{2, 4, 6, 8, 10\}$  of the form

$$\begin{aligned} \mu_{\text{D}p\text{brane}} &:= C \wedge e^F \\ &:= \sum_{k=0}^{(p+2)/2} c_k^p (e^{a_1} \wedge \dots \wedge e^{a_{p-2k}}) \wedge (\bar{\psi} \Gamma^{a_1} \dots \Gamma^{a_{p-2k}} \Gamma^{11} \psi) \underbrace{F \wedge \dots \wedge F}_{k \text{ factors}} , \end{aligned} \quad (1)$$

where  $\{c_k^p \in \mathbb{R}\}$  are some coefficients, and where  $C$  denotes the inhomogeneous element of  $\text{CE}(\mathbb{R}^{10; N=(1,1)})$  defined by the second line. For each  $p \in \{0, 2, 4, 6, 8\}$  there is, up to a global rescaling, a unique choice of the coefficients  $c_k^p$  that make this a cocycle. This is shown on p. 19 of [14].

**Remark 4.3.** Here the identification with physics terminology is as follows

- $F$  is the field strength of the *Chan-Paton gauge field* on the D-brane, a “tensor field” that happens to be a “vector field”;
- $C = \sum_p k^p \bar{\psi} \underbrace{e \wedge \dots \wedge e}_p \psi$  is the *RR-field*.

It is interesting to notice the special nature of the cocycle for the D0-brane:

**Remark 4.4.** According to (1) for  $p = 0$ , the cocycle defining the D0-brane as a higher WZW  $\sigma$ -model is just

$$\mu_{\text{D0brane}} = \bar{\psi} \Gamma^{11} \psi .$$

Since this independent of the generator  $F$ , it restricts to a cocycle on just  $\mathbb{R}^{10; N=(1,1)}$  itself.

Concerning this, we highlight the following fact, which is mathematically elementary but physically noteworthy (see also Section 2.1 of [14]), as it has conceptual consequences for arriving at M-theory starting from type IIA string theory.

**Proposition 4.5.** *The extension of 10-dimensional type IIA super-Minkowski spacetime  $\mathbb{R}^{10;N=(1,1)}$  by the D0-brane cocycle as in Remark 4.4 is the 11-dimensional super-Minkowski spacetime of 11-dimensional supergravity/M-theory:*

$$\begin{array}{ccc} \mathbb{R}^{11;N=1} & & \\ \downarrow & & \\ \mathbb{R}^{10;N=(1,1)} & \xrightarrow{\mu_{\text{D0brane}}} & \mathbb{R}[1] . \end{array}$$

*Proof.* By Prop. 3.5 the Chevalley-Eilenberg algebra of the extension classified by  $\mu_{\text{D0brane}}$  is that of  $\mathbb{R}^{10;N=(1,1)}$  with one new generator  $e^{11}$  in degree  $(1, \text{even})$  adjoined and with its differential defined to be

$$d_{\text{CE}} e^{11} = \mu_{\text{D0brane}} = \bar{\psi} \Gamma^{11} \psi .$$

An elementary basic fact of Spin representation theory says that the  $N = 1$ -representation of the Spin group  $\text{Spin}(10, 1)$  in odd dimensions is the  $N = (1, 1)$ -representation of the even dimensional Spin group  $\text{Spin}(9, 1)$  regarded as a representation of the Clifford algebra  $\{\Gamma^a\}_{a=1}^{10}$  with  $\Gamma^{11}$  adjoined as in Def. 4.2. Using this, the above extended CE-algebra is exactly that of  $\mathbb{R}^{11;N=1}$ .  $\square$

**Remark 4.6.** In view of Remark 3.11 the content of Prop. 4.5 translates to heuristic physics language as: *A condensate of D0-branes turns the 10-dimensional type IIA super-spacetime into the 11-dimensional spacetime of 11d-supergravity/M-theory. Alternatively: The condensation of D0-branes makes an 11th dimension of spacetime appear.*

In this form the statement is along the lines of the standard folklore relation between type IIA string theory and M-theory, which says that type IIA with  $N$  D0-branes in it is M-theory compactified on a circle whose radius scales with  $N$ ; see for instance [8, 37]. See also [32] for similar remarks motivated from phenomena in 2-dimensional boundary conformal field theory. Here in the formalization via higher WZW  $\sigma$ -models a version of this statement becomes a theorem, Prop. 4.5.

**Remark 4.7.** The mechanism of remark 4.6 appears at several places in the brane bouquet. First of all, since by Prop. 1 the D0-brane cocycle is a summand in each type IIA D-brane cocycle, it follows via the above translation from  $L_\infty$ -homotopy theory to physics language that: *Any type IIA D-brane condensate extends 10-dimensional type IIA super-spacetime to 11-dimensional super-spacetime.* If we lift attention again from the special case of D-branes of type IIA string theory to general higher WZW-type  $\sigma$ -models, then this mechanism is seen to generalize: the 10-dimensional super-Minkowski spacetime itself is an extension of the *super-point* by 10-cocycles (one for each dimension):

$$\begin{array}{ccc} \mathbb{R}^{10;N=(1,1)} & & \\ \downarrow & & \\ \mathbb{R}^{0;N=(1,1)} & \xrightarrow{\sum_{a=1}^{10} \overline{(-)} \Gamma^a (-)} & \mathbb{R}[1] . \end{array}$$

Here the cocycle describes 10 different 0-brane  $\sigma$ -models, each propagating on the super-point as their target super-spacetime. Again, by remark 3.11, this mathematical fact is a formalization and proof of what in physics language is the statement that *Spacetime itself emerges from the abstract dynamics of 0-branes.* This is close to another famous folklore statement about string theory. In our context it is a theorem.

### 4.3 Type IIB superstring ending on D-branes and S-duality

We consider the branes in type IIB string theory as examples of higher WZW-type  $\sigma$ -model field theories and observe how their  $L_\infty$ -homotopy theoretic formulation serves to provide a formal statement of the prequantum S-duality equivalence between F-strings and D-strings and their unification as  $(p, q)$ -string bound states.

Write  $N = (2, 0) = \mathbf{16} + \mathbf{16}$  for the direct sum representation of  $\text{Spin}(9, 1)$  given by two 16-dimensional real irreducible representations of the same chirality. We write  $\{\Gamma^a\}_{a=1, \dots, 10}$  for the corresponding representation of the Clifford algebra on one copy of  $\mathbf{16}$  and  $\Gamma^a \otimes \sigma^i$  for the linear maps on their direct sum representation that act as the  $i$ th Pauli matrix on  $\mathbb{C}^2$  with components  $\Gamma^a$ , under the canonical identification  $\mathbf{16} \oplus \mathbf{16} \simeq \mathbf{16} \otimes \mathbb{C}^2$ . Finally write  $\mathbb{R}^{10; N=(2,0)}$  for the corresponding super-translation Lie algebra, the super-Minkowski spacetime of type IIB string theory.

There is a cocycle  $\mu_{\text{string}_{\text{IIB}}} \in \text{CE}(\mathbb{R}^{10; N=(2,0)})$  given by

$$\mu_{\text{string}_{\text{IIB}}} = \bar{\psi} \wedge (\Gamma^a \otimes \sigma^3) \psi \wedge e^a .$$

The corresponding WZW  $\sigma$ -model is the Green-Schwarz formulation of the fundamental type IIB string. Of course we could use in this formula any of the  $\sigma^i$ , but one fixed such choice we are to call the type IIB string. That the other choices are equivalent is the statement of *S-duality*, to which we come in a moment. The corresponding  $L_\infty$ -algebra extension, hence by Remark 3.11 the IIB spacetime “with string condensate” is the homotopy fiber

$$\begin{array}{ccc} \text{string}_{\text{IIB}} & & \\ \downarrow & & \\ \mathbb{R}^{10; N=(2,0)} & \xrightarrow{\mu_{\text{string}_{\text{IIB}}}} & \mathbb{R}[2] . \end{array}$$

As for type IIA, its Chevalley-Eilenberg algebra  $\text{CE}(\text{string}_{\text{IIB}})$  is that of  $\mathbb{R}^{10; N=(2,0)}$  with one generator  $F$  in degree  $(2, \text{even})$  adjoined. The differential of that is now given by

$$\begin{aligned} d_{\text{CE}} F &= \mu_{\text{string}_{\text{IIB}}} \\ &= \bar{\psi} \wedge (\Gamma^a \otimes \sigma^3) \psi \wedge e_a . \end{aligned}$$

Now this Lie 2-algebra itself carries exceptional cocycles of degree  $(p+2)$  for  $p \in \{1, 3, 5, 7, 9\}$  of the form

$$\begin{aligned} \mu_{\text{D}p\text{brane}} &:= C \wedge e^F \\ &:= \sum_{k=0}^{(p+2)/2+1} c_k^p (e^{a_1} \wedge \dots \wedge e^{a_{p-2k}}) \wedge \left( \bar{\psi} \wedge (\Gamma^{a_1} \dots \Gamma^{a_{p-2k}} \otimes \sigma^{1/2}) \psi \right) \underbrace{F \wedge \dots \wedge F}_{k \text{ factors}} , \end{aligned} \quad (2)$$

where on the right the notation  $\sigma^{1/2}$  is to mean that  $\sigma^1$  appears in summands with an odd number of generators “ $e$ ”, and  $\sigma^2$  in the other summands. The corresponding WZW models are those of the type IIB D-branes.

**Remark 4.8.** According to expression (2) the cocycle of the D1-brane is of the form

$$\mu_{\text{D1brane}} = \bar{\psi} \wedge (\Gamma^a \otimes \sigma^1) \wedge e^a ,$$

which is the same form as that of  $\mu_{\text{string}_{\text{IIB}}}$  itself, only that  $\sigma^3$  is replaced by  $\sigma^1$ . In fact since this is the D-brane cocycle which is independent of the new generator  $F$ , it restricts to a cocycle on just  $\mathbb{R}^{10; N=(2,0)}$  itself. So the cocycle for the “F-string” in type IIB is on the same footing as that of the “D-string”. Both differ only by a “rotation” in an internal space.

**Remark 4.9.** There is a circle worth of  $L_\infty$ -automorphisms

$$S(\alpha) : \mathbb{R}^{10;N=(2,0)} \rightarrow \mathbb{R}^{10;N=(2,0)} ,$$

hence a group homomorphism

$$U(1) \rightarrow \text{Aut}(\mathbb{R}^{10;N=(2,0)}) ,$$

given dually on Chevalley-Eilenberg algebras by

$$\begin{aligned} e^a &\mapsto e^a \\ \psi &\mapsto \exp(\alpha\sigma^2)\psi . \end{aligned}$$

This mixes the cocycles for the F-string and for the D-string in that for a quarter rotation it turns one into the other

$$S(\pi/4)^*(\mu_{\text{string}_{\text{IIA}}}) = \mu_{\text{D1brane}} ,$$

and for a rotation by a general angle it produces a corresponding superposition of both. In particular, we can form *bound states* of  $F$ -strings and D1-branes by adding these cocycles

$$\mu_{(p,q)\text{string}} = p\mu_{\text{string}_{\text{IIB}}} + q\mu_{\text{D1brane}} \in \text{CE}(\mathbb{R}^{10;N=(2,0)}) .$$

These define the  $(p,q)$ -string bound states as WZW-type  $\sigma$ -models.

#### 4.4 The M-theory 5-brane and the M-theory super Lie algebra

We discuss here the single M5-brane as a higher WZW-type  $\sigma$ -model, show that it is defined by a 7-cocycle on the M2-brane super Lie-3 algebra and observe that this 7-cocycle is indeed the relevant fermionic 7d Chern-Simons term of 11-dimensional supergravity compactified on  $S^4$ , as required by  $\text{AdS}_7/\text{CFT}_6$  in the Chern-Simons interpretation of [49]. We see that the truncation of the symmetry algebra of this higher 5-brane superalgebra to degree 0 is the ‘‘M-algebra’’.

Write  $N = 1 = \mathbf{32}$  for the irreducible real representation of  $\text{Spin}(10, 1)$ . Write  $\{\Gamma^a\}_{a=1}^{11}$  for the corresponding representation of the Clifford algebra. Finally write  $\mathbb{R}^{11;N=1}$  for the corresponding super-translation Lie algebra. According to the old brane scan in section 4.1, the exceptional Lorentz-invariant cocycle for the M2-brane is

$$\mu_{\text{m2brane}} = \bar{\psi} \wedge \Gamma^{ab}\psi \wedge e^a \wedge e^b .$$

The Green-Schwarz action functional for the M2-brane is the  $\sigma$ -model defined by this cocycle

$$\mathbb{R}^{11;N} \xrightarrow{\mu_{\text{m2brane}}} \mathbb{R}[3] .$$

By the  $L_\infty$ -theoretic brane intersection law of Remark 3.10, for the M2-brane to end on another kind of brane, that other WZW model is to have the extended spacetime  $\mu_{\text{m2brane}}$  (the original spacetime including a condensate of M2s) as its target space. By Prop. 3.5, the Chevalley-Eilenberg algebra of the M2-brane algebra is obtained from that of the super-Poincaré Lie algebra by adding one more generator  $c_3$  with  $\text{deg}(c_3) = (3, \text{even})$  with differential defined by

$$\begin{aligned} d_{\text{CE}} c_3 &:= \mu_{\text{m2brane}} \\ &= \bar{\psi} \wedge \Gamma^{ab}\psi \wedge e^a \wedge e^b . \end{aligned}$$

We can then define an extended spacetime Maurer-Cartan form  $\hat{\theta}$  in  $\Omega_{\text{flat}}^1(\mathbb{R}^{11;N}, \mathbf{m2brane})$ , extending the canonical Maurer-Cartan form  $\theta$  in  $\Omega_{\text{flat}}^1(\mathbb{R}^{11;N}, \mathbb{R}^{d;N})$ , by picking any 3-form  $C_3 \in \Omega^3(\mathbb{R}^{11;N})$  such that  $d_{\text{dR}} C_3 = \bar{\psi} \Gamma^{ab} \wedge \psi \wedge e^a \wedge e^b$ .



Next, for every  $(n + 1)$  cocycle on **m2brane** we get an  $n$ -dimensional WZW model defined on  $\mathbb{R}^{11;N}$  this way. In particular, the next one we meet is the M5-brane cocycle. Indeed, there is the degree-7 cocycle

$$\mu_7 = \bar{\psi}\Gamma^{a_1\cdots a_5}\psi e^{a_1} \wedge \cdots \wedge e^{a_5} + C_3 \wedge \bar{\psi}\Gamma^{ab}\psi \wedge e^a \wedge e^b : \mathbf{m2brane} \longrightarrow \mathbb{R}[6]$$

that was first observed in [4], then rediscovered several times, for instance in [47], in [7] and in [14]. Here we identify it as an  $L_\infty$  7-cocycle on the **m2brane** super Lie 3-algebra. The  $L_\infty$ -extension of **m2brane** associated with the 7-cocycle is a super Lie 6-algebra that we call **m5brane**.

It follows from this, with remark 3.10, that the M2-brane may end on a M5-brane whose WZW term  $\mathcal{L}_{\text{WZW}}$  locally satisfies

$$d\mathcal{L}_{\text{WZW}} = \mu_7 = \bar{\psi}\Gamma^{a_1\cdots a_5}\psi e^{a_1} \wedge \cdots \wedge e^{a_5} + C_3 \wedge \bar{\psi}\Gamma^{ab}\psi \wedge e^a \wedge e^b$$

This is precisely what in [7] is argued to be the action functional of the M5-brane (here displayed in the absence of the bosonic contribution of the C-field). However, in order to get the expected structure of gauge transformations, we need to go further. Namely, while the above local expression for the action functional appears to be correct on the nose, its gauge transformations are not as expected for the M5: for the M5-brane worldvolume theory the 2-form with curvature  $C_3$  is supposed to be a genuine higher 2-form gauge field on the worldvolume, directly analogous to the Neveu-Schwarz B-field of 10-dimensional supergravity spacetime; see [19]. As such, it is to have gauge transformations parameterized by 1-forms. But in the above formulation fields are maps  $\Sigma_6 \rightarrow \mathbb{R}^{11;N}$  into spacetime itself, and as such have no gauge transformations at all. We can fix this by finding a better space  $\hat{X}$ . In fact we should take that to be **m2brane** itself. As indicated above, this is an extension

$$\mathbb{R}[2] \longrightarrow \mathbf{m2brane} \longrightarrow \mathbb{R}^{11;N} ,$$

and, hence, a twisted product of spacetime with  $\mathbb{R}[2]$ , the infinitesimal version of the moduli space of 2-form connections. We see this more precisely below in Section 5.

**Remark 4.10.** By AdS<sub>7</sub>/CFT<sub>6</sub> duality and by [49] the M5-brane is supposed to be the 6-dimensional WZW model which is holographically related to the 7-dimensional Chern-Simons term inside 11-dimensional supergravity compactified on a 4-sphere in analogy to how the traditional 2d WZW model is the holographic dual of ordinary 3d Chern-Simons theory. By our discussion here that 7d Chern-Simons theory ought to be the one given by the 7-cocycle. Indeed, we observe that this 7-cocycle does appear in the compactification according to D'Auria-Fre [4]. Back in that article these authors worked locally and discarded precisely this term as a global derivative, but in fact it is a topological term as befits a Chern-Simons term and may *not* be discarded globally. This connects the discussion here to the holographic AdS<sub>7</sub>/CFT<sub>6</sub>-description of the *single* M5-brane. Now a coincident  $N$ -tuple of M5-branes is supposed to be determined by a semisimple Lie algebra and nonabelian higher gauge field data. Since AdS<sub>7</sub>/CFT<sub>6</sub> is still supposed to apply, we are to consider the *nonabelian* contributions to the 7-dimensional Chern-Simons term in 11d sugra compactified to AdS<sub>7</sub>. These follow from the 11-dimensional anomaly cancellation and charge quantization. Putting this together as discussed in [19, 20] yields the corresponding 7d Chern-Simons theory. Among other terms it is controlled by the canonical 7-cocycle  $\mu_7^{\mathfrak{so}}$  on the semisimple Lie algebra  $\mathfrak{so}$ . Since this extends evidently to a cocycle also on the super Poincaré Lie algebra, we may just add it to the bispinorial cocycle that defines the single M5, to get

$$\mathbb{R}^{11;N=1} \times \mathfrak{so}(10, 1) \xrightarrow{\bar{\psi}e^5\psi + \langle \omega \wedge [\omega \wedge \omega] \wedge [\omega \wedge \omega] \wedge [\omega \wedge \omega] \rangle} \mathbb{R}[6] .$$

By the general theory indicated here this defines a 6-dimensional WZW model. By the discussion in [19, 20] it satisfies all the conditions imposed by holography. It is to be expected that this is part of the description of the nonabelian M5-brane.

Finally it is interesting to consider the symmetries of the M5-brane higher WZW model obtained this way.

**Definition 4.11.** The *polyvector extension* [2] of  $\mathfrak{sl}(10,1)$  – called the *M-theory Lie algebra* [47] – is the super Lie algebra obtained by adjoining to  $\mathfrak{sl}(10,1)$  generators  $\{Q_\alpha, Z^{ab}\}$  that transform as spinors with respect to the existing generators, and whose non-vanishing brackets among themselves are

$$\begin{aligned} [Q_\alpha, Q_\beta] &= i(C\Gamma^a)_{\alpha\beta}P_a + (C\Gamma_{ab})Z^{ab}, \\ [Q_\alpha, Z^{ab}] &= 2i(C\Gamma^{[a})_{\alpha\beta}Q^{b]\beta}. \end{aligned}$$

**Proposition 4.12.** *The degree-0 piece of the graded Lie algebra of infinitesimal automorphisms of m2brane, Def. 2.12, is the “M-theory algebra” polyvector extension of the 11d super Poincaré algebra of Def. 4.11.*

*Proof.* We leave this as an exercise to the reader. Hint: under the identification of FDA-language with ingredients of  $L_\infty$ -homotopy theory as discussed here, one can see that this involves the computations displayed in [12].  $\square$

## 4.5 The complete brane bouquet of string/M-theory

We have discussed various higher super Lie  $n$ -algebras of super-spacetime. Here we now sum up, list all the relevant extensions and fit them into the full brane bouquet. To state the brane bouquet, we first need names for all the branches that it has

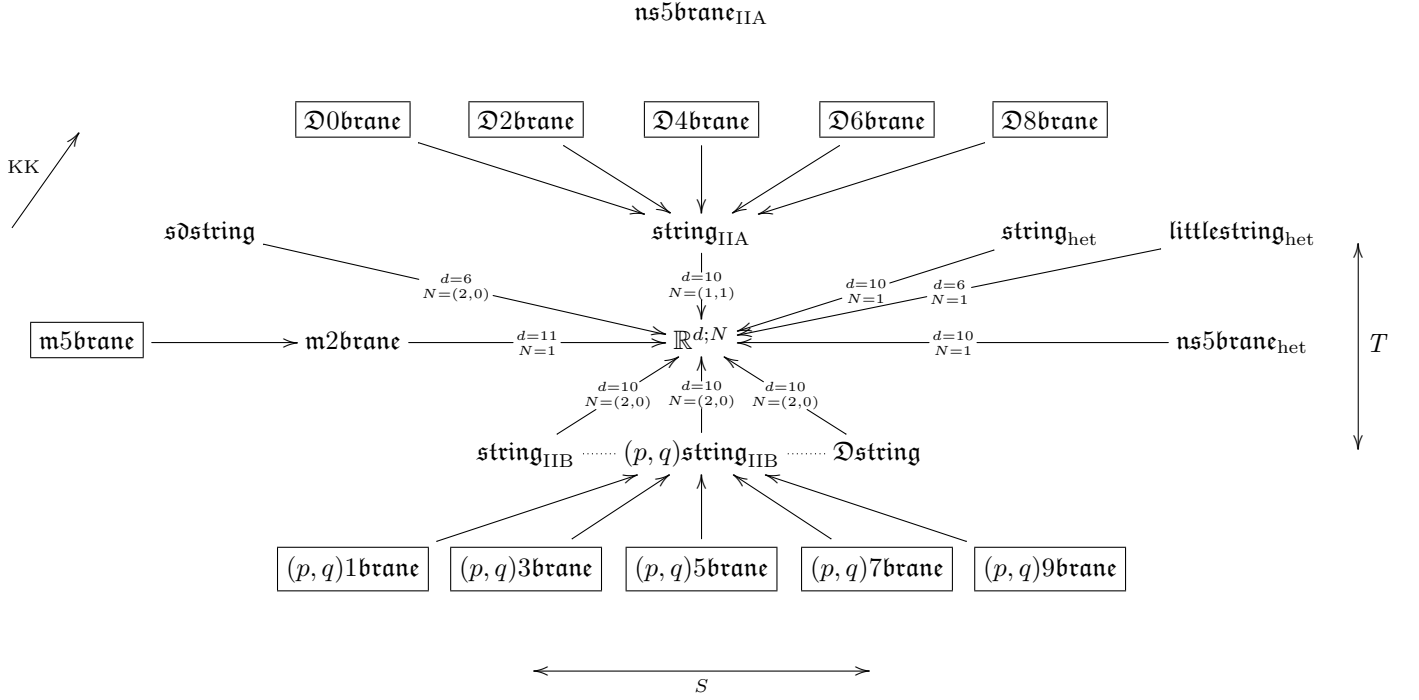
**Definition 4.13.** The *refined brane scan* is the following collection of values of triples  $(d, p, N)$ .

$\underline{D}$	$p=0$	1	2	3	4	5	6	7	8	9
11			(1) m2brane			(1) m5brane				
10	(1,1) D0brane	(1,0) string <sub>het</sub> (1,1) string <sub>IIA</sub> (2,0) string <sub>IIB</sub> (2,0) D1brane	(1,1) D2brane	(2,0) D3brane	(1,1) D4brane	(1,0) ns5brane <sub>het</sub> (1,1) ns5brane <sub>IIA</sub> (2,0) ns5brane <sub>IIB</sub> (2,0) D5brane	(1,1) D6brane	(2,0) D7brane	(1,1) D8brane	(2,0) D9brane
9					(1)					
8				(1)						
7			(1)							
6		(2,0) sDstring		(2,0)						
5			(1)							
4		(1)	(1)							
3		(1)								

The entries of this table denote super- $L_\infty$ -algebras that organize themselves as nodes in the brane bouquet according to the following proposition.

**Proposition 4.14** (The brane bouquet). *There exists a system of higher super-Lie- $n$ -algebra extensions of the super-translation Lie algebra  $\mathbb{R}^{d,N}$  for  $(d = 11, N = 1)$ ,  $(d = 10, N = (1, 1))$ , for  $(d = 10, N = (2, 0))$  and*

for  $(d = 6, N = (2, 0))$ , which is jointly given by the following diagram



where

- An object in this diagram is precisely a super-Lie- $(p + 1)$ -algebra extension of the super translation algebra  $\mathbb{R}^{d;N}$ , with  $(d, p, N)$  as given by the entries of the same name in the refined brane scan, def. 4.13;
- every morphism is a super-Lie  $(p + 1)$ -algebra extension by an exceptional  $\mathbb{R}$ -valued  $\mathfrak{o}(d)$ -invariant super- $L_\infty$ -cocycle of degree  $p + 2$  on the domain of the morphism;
- the unboxed morphisms are hence super Lie  $(p + 1)$ -algebra extensions of  $\mathbb{R}^{d;N}$  by a super Lie algebra  $(p + 2)$ -cocycle, hence are homotopy fibers of the form

$$\begin{array}{ccc}
 p\text{brane} & \longrightarrow & * \\
 \downarrow & \lrcorner & \downarrow \\
 \mathbb{R}^{d;N} & \xrightarrow{\text{some cocycle}} & \mathbb{R}[p + 1] ,
 \end{array}$$

- and the boxed super- $L_\infty$ -algebras are super Lie  $(p + 1)$ -algebra extensions of genuine super- $L_\infty$ -algebras (which are not plain super Lie algebras), again by  $\mathbb{R}$ -cocycles

$$\begin{array}{ccc}
 p_2\text{brane} & \longrightarrow & * \\
 \downarrow & \lrcorner & \downarrow \\
 p_1\text{brane} & \xrightarrow{\text{some cocycle}} & \mathbb{R}[p_2 + 1] .
 \end{array}$$

Proof. Using prop. 3.5 and the dictionary that we have established above between the language used in the physics literature (“FDA”s) and super- $L_\infty$ -algebra homotopy theory, this is a translation of the following results that can be found scattered in the literature (some of which were discussed in the previous sections).

- All  $N = 1$ -extensions of  $\mathbb{R}^{d;N=1}$  are those corresponding to the “old brane scan” [1]. Specifically the cocycle which classifies the super Lie 3-algebra extension  $\mathbf{m2brane} \rightarrow \mathbb{R}^{11;1}$  had been found earlier in the context of supergravity around equation (3.12) of [4]. These authors also explicitly write down the “FDA” that then in [44] was recognized as the Chevalley-Eilenberg algebra of the super Lie 3-algebra  $\mathbf{m2brane}$  (there called the “supergravity Lie 3-algebra”). Later all these cocycles appear in the systematic classification of super Lie algebra cohomology in [9, 10, 11].
- The 7-cocycle classifying the super-Lie-6-algebra extension  $\mathbf{m5brane} \rightarrow \mathbf{m2brane}$  together with that extension itself can be traced back, in FDA-language, to (3.26) in [4]. This is maybe still the only previous reference that makes explicit the Lie 6-algebra extension (as an “FDA”), but the corresponding 7-cocycle itself has later been rediscovered several times, more or less explicitly. For instance it appears as equations (6) and (9) in [7]. A systematic discussion is in section 8 of [14].
- The extension  $\mathbf{string}_{\text{IIA}} \rightarrow \mathbb{R}^{10;N=(1,1)}$  by a super Lie algebra 3-cocycle and the cocycles for the further higher extensions  $\mathfrak{D}(2n)\mathbf{brane} \rightarrow \mathbf{string}_{\text{IIA}}$  can be traced back to section 6 of [14].
- The extension  $\mathbf{string}_{\text{IIB}} \rightarrow \mathbb{R}^{10;N=(2,0)}$  by a super Lie algebra 2-cocycle and the cocycles for the further higher extensions  $\mathfrak{D}(2n+1)\mathbf{brane} \rightarrow \mathbf{string}_{\text{IIA}}$ , as well as the extension  $\mathbf{ns5brane}_{\text{IIB}} \rightarrow \mathfrak{D}\mathbf{string}$  follow from section 2 of [40].

□

**Remark 4.15.** The look of the brane bouquet, Prop. 4.14, is reminiscent of the famous cartoon that displays the conjectured coupling limits of string/M-theory, e.g. figure 4 in [48], or fig. 1 in [37]. Contrary to that cartoon, the brane bouquet is a theorem. Of course that cartoon alludes to more details of the nature of string/M-theory than we are currently discussing here, but all the more should it be worthwhile to have a formalism that makes precise at least the basic structure, so as to be able to proceed from solid foundations.

## 5 Non-perturbative higher WZW models on higher super-orbispaces

In this final section we give a non-perturbative (globalized) refinement of the perturbative higher WZW-models that we discussed so far. These non-perturbative higher WZW models are naturally formulated not just in higher Lie theory as used so far, but in genuine higher differential geometry, which means in higher smooth and supergeometric *stacks*. In the language of physics, stacks may best be thought of as *higher orbispaces*, the generalization of *orbifolds* and more generally of orbispaces (dropping the finiteness condition) to the case where there are not just gauge transformations between points, but also higher gauge transformations between these. The idea of considering  $\sigma$ -models on orbifold target spaces is traditionally familiar, and here we generalize this naturally by allowing these target spaces to be such higher (super-)orbispaces. The reader can find an exposition of the technology relevant for the following in [22], a collection of all the relevant definitions and constructions in [17], and the full technical details in [45].

In higher (super-)differential geometry every (super-)  $L_\infty$ -algebra  $\mathfrak{g}$  has *Lie integrations* to higher smooth (super-)groups  $G$ ; see [16] for details. (For  $\mathfrak{g} = \mathbf{string}_{\text{het}}$  the Lie integration is discussed in [31].) For instance, the abelian  $L_\infty$ -algebra  $\mathbb{R}[n]$  integrates to the *circle  $n+1$ -group*  $\mathbf{B}^n U(1)$ . This is at the same time the higher *moduli stack* for circle  $n$ -bundles (also called  $(n-1)$ -bundle gerbes).

Recall then from the Introduction that a perturbative higher WZW model of dimension  $n$  is all encoded by a morphism of (super-)  $L_\infty$ -algebras of the form

$$\mu : \mathfrak{g} \longrightarrow \mathbb{R}[n] .$$

Therefore, its non-perturbative refinement is to be an  $n$ -form connection on a circle  $n$ -bundle over the higher group  $G$ . The latter is given by a morphism of higher smooth (super-)groups the form

$$\Omega \mathfrak{c} : G \longrightarrow \mathbf{B}^n U(1) .$$

(This is the higher and smooth analog of the canonical morphism  $G \rightarrow K(\mathbb{Z}, 3)$  defining the fundamental class  $[\omega_G] \in H^3(G; \mathbb{Z})$  for a compact, simple and simply connected Lie group  $G$ , in the traditional WZW model.) Equivalently, this is a morphism of the corresponding delooping stacks

$$\mathbf{c} : \mathbf{B}G \longrightarrow \mathbf{B}^{n+1}U(1) .$$

It is shown in [16] that this always and canonically exists, it is just the Lie integration  $\mathbf{c} = \exp(\mu)$  of the original  $L_\infty$ -cocycle. <sup>4</sup>

Now, as indicated in the Introduction, the local Lagrangian for the non-perturbative WZW model is to be an  $n$ -connection on this  $n$ -bundle whose curvature  $n+1$ -form is  $\mu(\theta_{\text{global}})$ , the value of the original cocycle applied to a *globally defined* Maurer-Cartan form on  $G$ . Every higher group (in cohesive geometry [45]) does carry a higher Maurer-Cartan form (see also [17]), given by a canonical map  $\theta_G : G \rightarrow \mathfrak{b}_{\text{dR}}\mathbf{B}G$  with values in the (nonabelian) *de Rham hypercohomology* stack  $\mathfrak{b}_{\text{dR}}\mathbf{B}G$ . Exactly as  $[\omega_G]$  for a Lie group is represented by the closed left-invariant 3-form  $\omega_G = \mu(\theta_G \wedge \theta_G \wedge \theta_G)$ , where  $\theta_G$  is the Maurer-Cartan form of  $G$ , the morphism  $\Omega\mathbf{c}$  has a canonical factorization

$$\begin{array}{ccc} G & \xrightarrow{\theta_G} & \mathfrak{b}_{\text{dR}}\mathbf{B}G & \xrightarrow{\mathfrak{b}_{\text{dR}}\mathbf{c}} & \mathfrak{b}_{\text{dR}}\mathbf{B}^{n+1}U(1) \\ & \searrow & \Omega\mathbf{c} & \searrow & \uparrow \text{curv} \\ & & & & \mathbf{B}^nU(1) , \end{array}$$

where  $\mathfrak{b}_{\text{dR}}\mathbf{B}G$  and  $\mathfrak{b}_{\text{dR}}\mathbf{B}^nU(1)$  are the higher smooth stacks of flat  $G$ -valued and of flat  $\mathbf{B}^nU(1)$ -valued differential forms, respectively,  $\theta_G$  is the Maurer-Cartan form, and  $\text{curv} : \mathbf{B}^nU(1) \rightarrow \mathfrak{b}_{\text{dR}}\mathbf{B}^{n+1}U(1)$  is the canonical curvature morphism (see [16, 17] for details).

There is, however, a fundamental difference between the general case of a higher smooth group and the classical case of a compact Lie group. Namely, the higher Maurer-Cartan form  $\theta_G : \mathbf{B}G \rightarrow \mathfrak{b}_{\text{dR}}\mathbf{B}G$  will not, in general, be represented by a globally defined flat differential form with coefficients in the  $L_\infty$ -algebra  $\mathfrak{g}$ . In other words, we do not have, in general, a factorization

$$\begin{array}{ccc} & & \Omega_{\text{flat}}^1(-; \mathfrak{g}) \\ & \nearrow & \downarrow \\ G & \xrightarrow{\theta_G} & \mathfrak{b}_{\text{dR}}\mathbf{B}G \end{array}$$

as in the case of compact Lie groups. Rather, in general  $\theta_G$  is a genuine hyper-cocycle: a collection of local differential forms on an atlas for  $G$ , with gauge transformations where their domain of definition overlaps and higher gauge transformations on higher intersections. The universal way to force a globally defined curvature form is to consider the smooth stack  $\tilde{G}$  which is the universal solution to the above factorization problem. That is, we consider the (higher) smooth stack  $\tilde{G}$  defined as the following homotopy pullback

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\theta_{\text{global}}} & \Omega_{\text{flat}}^1(-; \mathfrak{g}) \\ \downarrow & \lrcorner & \downarrow \\ G & \xrightarrow{\theta_G} & \mathfrak{b}_{\text{dR}}\mathbf{B}G \end{array}$$

in higher supergeometric smooth stacks. In conclusion then the non-perturbative WZW-model induced by the cocycle  $\mu$  is to be an  $n$ -connection local Lagrangian of the form

$$\mathcal{L}_{\text{WZW}} : \tilde{G} \longrightarrow \mathbf{B}^nU(1)_{\text{conn}} ,$$

<sup>4</sup>Here and in the following we use  $U(1) = \mathbb{R}/\mathbb{Z}$  for brevity, but in general what appears is  $\mathbb{R}/\Gamma$ , for  $\Gamma \hookrightarrow \mathbb{R}$  the discrete subgroup of *periods* of  $\mu$ ; see [16] for details.

satisfying two conditions:

1. its curvature  $(n + 1)$ -form is the evaluation of  $\mu$  on the globally defined Maurer-Cartan form;
2. the underlying  $n$ -bundle is the higher group cocycle  $\Omega \mathfrak{c}$  given by Lie integration of  $\mu$ .

The following proposition now asserts that this indeed exists canonically and is essentially uniquely.

**Proposition 5.1.** *On  $\tilde{G}$  there is an essentially unique factorization of the globally defined invariant form  $\mu(\theta_{\text{global}})$  through an extended WZW action functional  $\mathcal{L}_{\text{WZW}}$*

$$\begin{array}{ccc}
 \tilde{G} & \xrightarrow{\theta_{\text{global}}} & \Omega_{\text{flat}}(-, \mathfrak{g}) \xrightarrow{\mu} \Omega_{\text{cl}}^{n+1} \\
 & \searrow \mathcal{L}_{\text{WZW}} & \nearrow F_{(-)} \\
 & & \mathbf{B}^n U(1)_{\text{conn}} ,
 \end{array}$$

such that the underlying smooth class  $G \rightarrow \mathbf{B}^n U(1)$  is the looping of the exponentiated cocycle  $\mathfrak{c} = \exp(\mu)$ .

*Proof.* One considers the smooth stacks  $\mathfrak{b}\mathbf{B}G$  and  $\mathfrak{b}\mathbf{B}^{n+1}U(1)$  of  $G$ -principal bundles and  $U(1)$ -principal  $(n + 1)$ -bundles with flat connections, respectively, together with the canonical morphisms  $\mathfrak{b}_{\text{dR}}\mathbf{B}G \rightarrow \mathfrak{b}\mathbf{B}G$  and  $\mathfrak{b}_{\text{dR}}\mathbf{B}^{n+1}U(1) \rightarrow \mathfrak{b}\mathbf{B}^{n+1}U(1)$  (again, see [16, 17] for definitions). By naturality of these morphisms one has a homotopy commutative diagram of the form

$$\begin{array}{ccc}
 \mathfrak{b}_{\text{dR}}\mathbf{B}G & \xrightarrow{\mathfrak{b}_{\text{dR}}\mathfrak{c}} & \mathfrak{b}_{\text{dR}}\mathbf{B}^{n+1}U(1) \\
 \downarrow & & \downarrow \\
 \mathfrak{b}\mathbf{B}G & \xrightarrow{\mathfrak{b}\mathfrak{c}} & \mathbf{B}^{n+1}U(1) .
 \end{array}$$

Then, by naturality of the inclusions  $\Omega_{\text{flat}}^1(-; \mathfrak{g}) \rightarrow \mathfrak{b}_{\text{dR}}\mathbf{B}G$  and  $\Omega_{\text{cl}}^{n+1} = \Omega_{\text{flat}}^1(-; \mathbb{R}[n]) \rightarrow \mathfrak{b}_{\text{dR}}\mathbf{B}^{n+1}U(1)$ , one has a homotopy commutative diagram

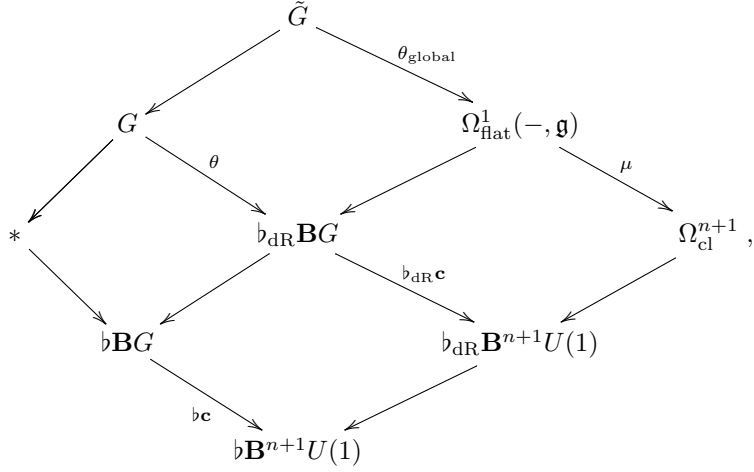
$$\begin{array}{ccc}
 \Omega_{\text{flat}}^1(-; \mathfrak{g}) & \xrightarrow{\mu} & \Omega_{\text{cl}}^{n+1} \\
 \downarrow & & \downarrow \\
 \mathfrak{b}_{\text{dR}}\mathbf{B}G & \xrightarrow{\mathfrak{b}_{\text{dR}}\mathfrak{c}} & \mathfrak{b}_{\text{dR}}\mathbf{B}^{n+1}U(1) .
 \end{array}$$

Finally, since by definition  $\mathfrak{b}_{\text{dR}}\mathbf{B}G$  is the homotopy fiber of the forgetful morphism  $\mathfrak{b}\mathbf{B}G \rightarrow \mathbf{B}G$ , we have a homotopy pullback diagram of the form

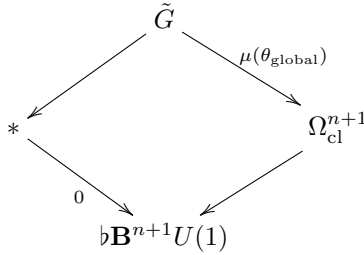
$$\begin{array}{ccc}
 G \simeq \Omega \mathbf{B}G & \longrightarrow & \mathfrak{b}_{\text{dR}}\mathbf{B}G \\
 \downarrow & & \downarrow \\
 * & \longrightarrow & \mathfrak{b}\mathbf{B}G .
 \end{array}$$

Pasting together the above three diagrams and the homotopy commutative diagram defining  $\tilde{G}$  we obtain

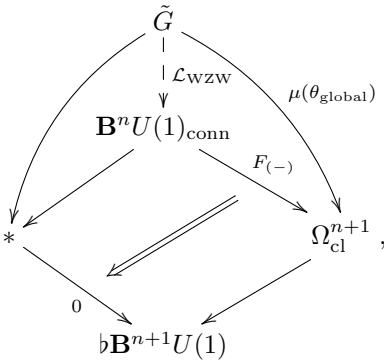
the big homotopy commutative diagram



and hence the homotopy commutative diagram



as the outermost part of the above big diagram. Then, by the universal property of the homotopy pullback, this factors essentially uniquely as



where we have used the fact that the stack  $\mathbf{B}^n U(1)_{\text{conn}}$  of  $U(1)$ - $n$ -bundles with connection is naturally the homotopy fiber of the inclusion  $\Omega_{\text{cl}}^{n+1} \rightarrow \mathfrak{b}\mathbf{B}^{n+1}U(1)$ ; see [16].  $\square$

**Remark 5.2.** The above proposition has been stated having in mind a cocycle with integral periods, so that  $\mathbb{R}/\mathbb{Z} \cong U(1)$ . The generalization to an arbitrary subgroup of periods  $\Gamma \hookrightarrow \mathbb{R}$  is immediate.

**Remark 5.3.** The construction of the full higher WZW term  $\mathcal{L}_{\text{WZW}}$  in Prop. 5.1 turns out to canonically exhibit the higher WZW-type theory as the boundary theory of a higher Chern-Simons-type theory, in the

precise sense of Prop. 3.1. To see this, first recall from [17, 46, 45] that an  $(n + 1)$ -dimensional local Chern-Simons-type prequantum field theory for a cocycle  $\mathbf{c} : \mathbf{B}G \rightarrow \mathbf{B}^{n+1}U(1)$  as above is a map of smooth higher moduli stacks of the form

$$\mathcal{L}_{\text{CS}} : \mathbf{B}G_{\text{conn}} \rightarrow \mathbf{B}^{n+1}U(1)_{\text{conn}}$$

which fits into a homotopy commutative diagram of the form

$$\begin{array}{ccc} \mathfrak{b}\mathbf{B}G & \xrightarrow{\mathfrak{b}\mathbf{c}} & \mathfrak{b}\mathbf{B}^{n+1}U(1) \\ \downarrow & & \downarrow \\ \mathbf{B}G_{\text{conn}} & \xrightarrow{\mathcal{L}_{\text{CS}}} & \mathbf{B}^{n+1}U(1)_{\text{conn}} \\ \downarrow & & \downarrow \\ \mathbf{B}G & \xrightarrow{\mathbf{c}} & \mathbf{B}^{n+1}U(1) . \end{array}$$

This hence is a refinement to differential cohomology that respects both the inclusion of flat higher connections as well as the underlying universal principal  $n$ -bundles. In [16] is given a general construction of such  $\mathcal{L}_{\text{CS}}$  by a stacky/higher version of Chern-Weil theory, which applies whenever the cocycle  $\mu$  is in transgression with an invariant polynomial on the  $L_\infty$ -algebra  $\mathfrak{g}$ . For instance ordinary 3d Chern-Simons theory is induced this way from the transgressive 3-cocycle  $\langle -, [-, -] \rangle$  on a semisimple Lie algebra, and the nonabelian 7d Chern-Simons theory on String 2-connections which appears in quantum corrected 11d supergravity is induced by the corresponding 7-cocycle [19].

Now by pasting this diagram below the diagram

$$\begin{array}{ccccc} \tilde{G} & \xrightarrow{\theta_{\text{global}}} & \Omega_{\text{flat}}^1(-, \mathfrak{g}) & \xrightarrow{\mu} & \Omega_{\text{cl}}^{n+1} \\ \downarrow & & \downarrow & & \downarrow \\ G & \xrightarrow{\theta_G} & \mathfrak{b}_{\text{dR}}\mathbf{B}G & \xrightarrow{\mathfrak{b}_{\text{dR}}\mathbf{c}} & \mathfrak{b}_{\text{dR}}\mathbf{B}^{n+1}U(1) \end{array}$$

appearing in the proof of Prop. 5.1 we obtain the homotopy commutative diagram of smooth higher moduli stacks

$$\begin{array}{ccccc} \tilde{G} & \xrightarrow{\theta_{\text{global}}} & \Omega_{\text{flat}}^1(-, \mathfrak{g}) & \xrightarrow{\mu} & \Omega_{\text{cl}}^{n+1} \\ \downarrow & & \downarrow & & \downarrow \\ G & \xrightarrow{\theta_G} & \mathfrak{b}_{\text{dR}}\mathbf{B}G & \xrightarrow{\mathfrak{b}_{\text{dR}}\mathbf{c}} & \mathfrak{b}_{\text{dR}}\mathbf{B}^{n+1}U(1) \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \mathfrak{b}\mathbf{B}G & \xrightarrow{\mathfrak{b}\mathbf{c}} & \mathfrak{b}\mathbf{B}^{n+1}U(1) \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{B}G_{\text{conn}} & \xrightarrow{\mathcal{L}_{\text{CS}}} & \mathbf{B}^{n+1}U(1)_{\text{conn}} . \end{array}$$



Inside the above diagram one reads the correspondence

$$\begin{array}{ccc}
 & \tilde{G} & \\
 \swarrow & & \searrow^{\theta_{\text{global}}} \\
 * & & \mathbf{B}G_{\text{conn}} , \\
 \searrow & \swarrow^{\mathcal{L}_{\text{WZW}}} & \swarrow^{\mathcal{L}_{\text{CS}}} \\
 & \mathbf{B}^{n+1}U(1)_{\text{conn}} & 
 \end{array}$$

which equivalently expresses the higher WZW term as a cocycle in degree  $n$  differential cohomology twisted by the Chern-Simons term evaluated on the globally defined Maurer-Cartan form. By Prop. 3.1 this precisely exhibits  $\mathcal{L}_{\text{WZW}}$  as a boundary condition for  $\mathcal{L}_{\text{CS}}$ .

This general mathematical statement seems to be well in line with the relation between higher Chern-Simons terms and higher WZW models found in [49]. Notice that with  $\mathcal{L}_{\text{WZW}}$  realized as a boundary theory of  $\mathcal{L}_{\text{CS}}$  this way, any further boundary of  $\mathcal{L}_{\text{WZW}}$ , notably as in Def. 3.2, makes that a *corner* of  $\mathcal{L}_{\text{CS}}$ . In fact, in [46] is shown that  $\mathcal{L}_{\text{CS}}$  itself is already naturally a boundary theory for a topological field theory of yet one dimension more, namely a universal higher topological Yang-Mills theory. Hence we find here a whole cascade of *corner field theories* of arbitrary codimension. For instance from the results above we have the sequence of higher order corner theories that looks like

$$\text{M2-brane} \xleftarrow{\text{ends on}} \text{M5-brane} \xleftarrow{\text{WZW boundary of}} \text{7d CS in 11d SUGRA} \xleftarrow{\text{boundary of}} \text{8d tYM} .$$

Such hierarchies of higher order corner field theories have previously been recognized and amplified in string theory and M-theory [41, 42, 43]. More discussion of the above formalization of these hierarchies in local (multi-tiered) prequantum field theory is in [46]. Closely related considerations have appeared in [24].

To further appreciate the abstract construction of the higher WZW term  $\mathcal{L}_{\text{WZW}}$  in Prop. 5.1, it is helpful to notice the following two basic examples, which are in a way at opposites ends of the space of all examples.

**Example 5.4.** For  $\mathfrak{g}$  an ordinary (super-)Lie algebra and  $G$  an ordinary (super-)Lie group integrating it, we have  $\mathfrak{b}_{\text{dR}}\mathbf{B}G \simeq \Omega_{\text{flat}}^1(-, \mathfrak{g})$  [45]. This implies that in this case  $\tilde{G} \simeq G$ , hence that there is no extra “differential extension”. Now for  $\mu$  a 3-cocycle, the induced  $\mathcal{L}_{\text{WZW}}$  is the traditional WZW term, refined to a Deligne 2-cocycle/bundle gerbe with connection as in [27, 26].

**Example 5.5.** For  $\mathfrak{g} = \mathbb{R}[n]$  we can take the smooth higher group integrating it to be the  $(n+1)$ -group  $G = \mathbf{B}^n U(1)$ . In this case, as shown in [45], the definition of  $\tilde{G}$  is precisely the characterization of the moduli  $n$ -stack of  $U(1)$ - $n$ -bundles with connections, so that

$$\tilde{G} \simeq \mathbf{B}^n U(1)_{\text{conn}}$$

in this case. Then for  $\mu : \mathfrak{g} \rightarrow \mathbb{R}[n]$  the canonical cocycle (the identity), it follows that  $\mathcal{L}_{\text{WZW}}$  is the identity, hence is the canonical  $U(1)$ - $n$ -connection on the moduli  $n$ -stack of all  $U(1)$ - $n$ -connections. This describes the extreme case of a higher WZW-type field theory with *no*  $\sigma$ -model fields and *only* a “tensor field” on its worldvolume, and whose action functional is simply the higher volume holonomy of that higher gauge field.

Generic examples of higher WZW theories are twisted products of the above two basic examples:

**Example 5.6.** Consider  $K$  a higher (super-)group extension of a Lie (super-)group  $G$  of the form

$$\mathbf{B}^n U(1) \longrightarrow K \longrightarrow G .$$

For instance  $G$  may be a translation super-group  $\mathbb{R}^{d;N}$  and  $K$  the Lie integration of one of the extended superspaces such as **m2brane** considered above (spacetime filled with a brane condensate, Remark 3.11). This

means that  $K$  is a *twisted product* of the (super-)Lie group  $G$  and the  $(n + 1)$ -group  $\mathbf{B}^n U(1)$ , which appear in examples 5.4 and 5.5 above. Since the construction of  $\mathcal{L}_{\text{WZW}}$  in the proof of Prop. 5.1 suitably respects products, it follows that the field content of a higher WZW model on the higher smooth (super-)group  $K$  is a tuple consisting of

1. a  $\sigma$ -model field with values in  $G$ ;
2. an  $n$ -form higher gauge field,

both subject to a twisting condition which gives the higher gauge field a twisted Bianchi identity depending on the  $\sigma$ -model fields.

In particular, for the extended spacetime given by an M2-brane condensate in 11-dimensional ( $N = 1$ )-super spacetime, this says that the M5-brane higher WZW model according to Section 4.4 has fields given by a multiplet consisting of embedding fields into spacetime and a 2-form higher gauge field (“tensor field”) on its worldvolume. Notice that the higher gauge transformations of the 2-form field are correctly taken into account by this full (in particular non-perturbative) construction of the WZW term as a higher prequantum bundle.

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