1 Adjoint higher modalities

Given an ∞-topos $\mathbf{H}$, we say that the operation induced by an *idempotent* ∞*-(co)monad* on $\mathbf{H}$

$$\Box, \Diamond : \mathbf{H} \to \mathbf{H}$$

on the internal homotopy type theory is a (higher) *modality* on the type system. Externally, these structures correspond to adjunctions

$$(L \dashv R) : \ \mathbf{H} \underset{R}{\overset{L}{\rightleftarrows}} \mathbf{B}$$

such that L or R is a fully faithful ∞-functor, by $\Box \simeq L \circ R$ and $\Diamond \simeq R \circ L$, or the other way around.

Here we are interested in the case that the comonadic $\Box$ is itself part of an adjunction with the monadic $\Diamond$, as $\Box \dashv \Diamond$ or $\Diamond \dashv \Box$. Such a situation corresponds externally to adjoint triples

$$(f_! \dashv f^* \dashv f_*) : \ \begin{tikzcd} \mathbf{H} \arrow[r, shift left=2, "f_!"] \arrow[r, shift right=2, "f_*"'] & \mathbf{B} \arrow[l, "f^*" description] \end{tikzcd} \qquad \text{or} \qquad (f^* \dashv f_* \dashv f^!) : \ \begin{tikzcd} \mathbf{H} \arrow[r, "f_*" description] & \mathbf{B} \arrow[l, shift right=2, "f^*"'] \arrow[l, shift left=2, "f^!"] \end{tikzcd}$$

such that the middle functor or the two outer functors are fully faithful:

$$(\Diamond \dashv \Box) \simeq (f^* f_! \dashv f^* f_*) \quad \text{or} \quad (\Box \dashv \Diamond) \simeq (f^* f_* \dashv f^! f_*)\,.$$

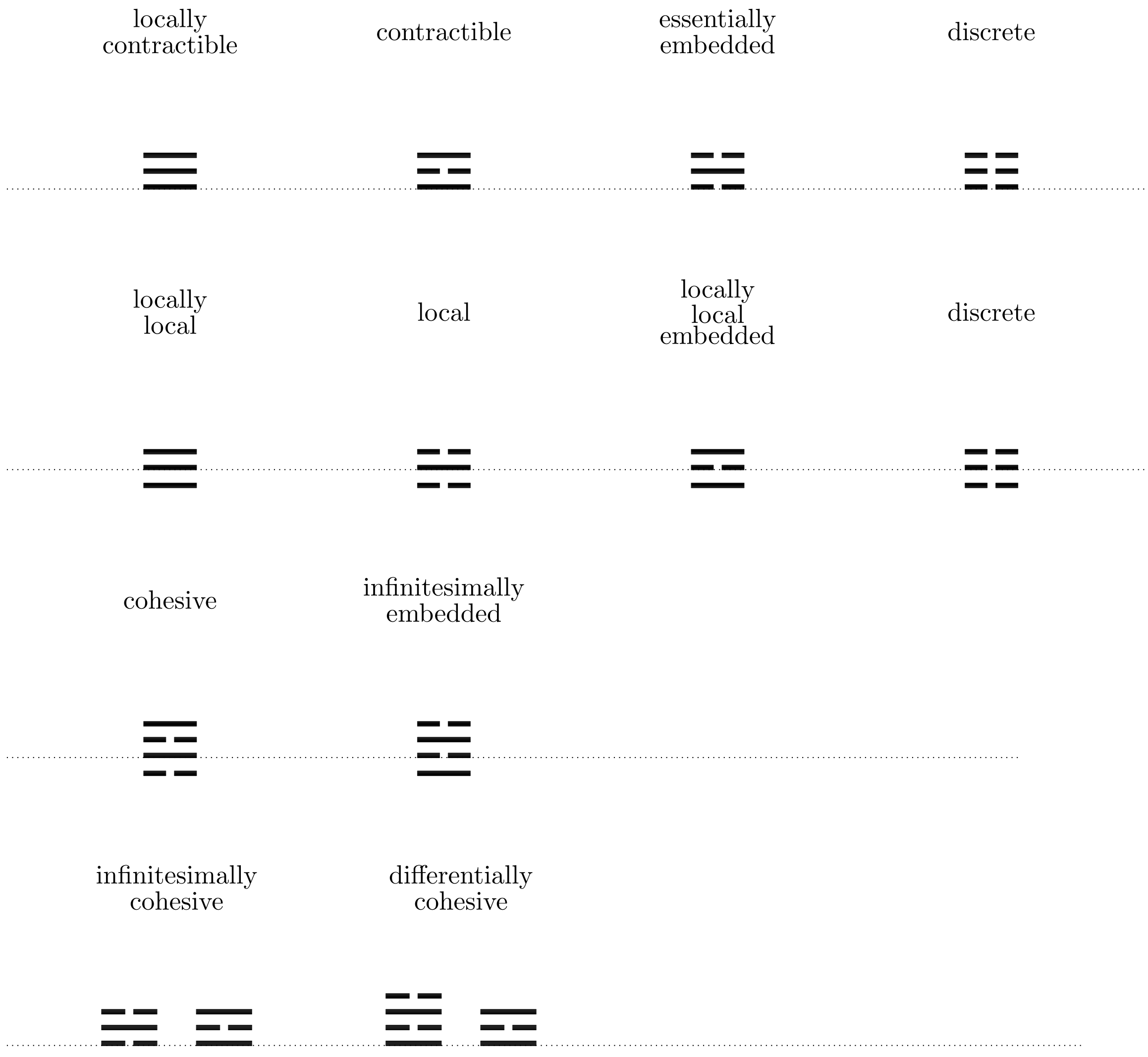
All that matters for the nature of the induced modalities is in which direction these functors go and which of them are fully faithful. Moreover, both direction and fully faithfulness are necessarily alternating through the adjoint triple, so what really matters is only which functor we regard as the direct image, the number of adjoints it has to the left and to the right, and whether it is itself fully faithful or its adjoints are. To bring that basic information out more clearly it may be helpful to introduce the following condensed notation.

Let stand for an adjoint pair where the direct image f_* points from $\mathbf{H}$ to $\mathbf{B}$, (this is the bar on the dotted baseline) and such that it has a single left adjoint f^* (the second bar on top). Accordingly, if there is a further left adjoint $f_!$ then we draw a further bar on top . If there is a further right adjoint $f^!$ then we draw a further bar on the bottom . And so forth: bars on top are left adjoint to bars below them, and the direction is left-to-right for the bar on the base line and for every second bar next to it, while it is right-to-left for every other bar. Finally, we mark the fully faithful functors by breaking the corresponding bar. For instance the notation means that the inverse image is fully faithful, hence is shorthand for an adjunction of the form $\begin{tikzcd} \mathbf{H} \arrow[r, shift right, "f_*"'] & \mathbf{B} \arrow[l, hook', shift right, "f^*"'] \end{tikzcd}$,

and so forth.

The following table lists, in this notation, the possibilities for adjoint higher modalities together with the common (or not so common) name of the corresponding attribute of $\mathbf{H}$ as an ∞-topos over the base $\mathbf{B}$.

2 The possibilities



Remark. Here some of the listed attributes strictly speaking require further conditions: for the first two cases the adjunction is to be *indexed* over the base, in the second row the adjunctions are to be geometric morphisms, hence the functor corresponding to the top bar is required to preserve finite limits. In the last two rows the top bar is required to preserve finite products.

The last line denotes a pair of *composite* adjoint quadruples. For instance the first one in the last line describes a factorization $\mathbf{H} \hookrightarrow \mathbf{H}_{\mathrm{th}} \to \mathbf{B}$ by adjoint quadruples. The second one in the last line is obtained from that, equivalently, by reversing the direction of the first factor.