Cohomological quantization of local prequantum boundary field theory

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Abstract

We discuss how local prequantum field theories with boundaries can be described in terms of n-fold correspondence diagrams in the ∞-topos of smooth stacks equipped with higher circle bundles. This places us in a position where we can linearize the prequantum theory by mapping the higher circle groups into the groups of units of a ring spectrum, and then quantize the theory by a pull-push construction in the associated generalized cohomology theory. In such a way, we can produce quantum propagators along cobordisms and partition functions of boundary theories as maps between certain twisted cohomology spectra. We are particularly interested in the case of 2d boundary field theories, where the pull-push quantization takes values in the twisted K-theory of differentiable stacks.

Many quantization procedures found in the literature fit in this framework. For instance, propagators as maps between spectra have been considered in the context of string topology and in the realm of Chern-Simons theory, transgressed to two dimensions. Examples of partitions functions of boundary theories are provided by the D-brane charges appearing in string theory and the K-theoretic quantization of symplectic manifolds. Here we extend the latter example to produce a K-theoretic quantization of Poisson manifolds, viewed as boundaries of the non-perturbative Poisson sigma-model. This involves geometric quantization of symplectic groupoids as well as the K-theoretic formulation of Kirillov's orbit method. At the end we give an outlook on the 2d string sigma-model on the boundary of the membrane, quantized over tmf-cohomology with partition function the Witten genus.

Contents

1 Introduction

In the past two decades there has been much progress in understanding the mathematical structure underlying quantum field theories (a nice overview is presented in [SS11]). This has resulted in a well-developed mathematical theory of cobordism representations (functorial QFT) and the dual theory of nets of algebras (algebraic QFT). Both formalizations of quantum field theory, in one way or another, emphasize one important aspect of field theories:

- field theories are *local*. This means that a field configuration (or state) on a spacetime Σ can be obtained by gluing configurations on smaller pieces of Σ.

We will restrict our attention to the formalism of functorial quantum field theory (FQFT), in particular in the context of *topological* field theories, which require no extra geometric structure on Σ like a metric. From a mathematical point of view, such topological field theories are very well-behaved objects. Indeed, the definition of a topological FQFT just gives a rigorous meaning to the notion of locality: a topological field theory is an assignment of any kind of data to manifolds Σ which can be obtained by chopping up Σ into pieces and gluing the data over each of these pieces along common boundaries. Since any manifold Σ can be cut into very simple pieces, these functorial topological field theories are very tractable objects from the mathematical perspective.

To construct quantum field theories of actual physical interest, we need to specify the sort of local data we want to use, before we can actually make use of the FQFT formalism. There is currently little known about how to concretely implement physically realistic QFTs in these axioms. In particular, it is unclear what kind of local data we should choose to produce a functorial description of such theories. On the other hand, the last 50 years of fundamental physics have all revolved around quantum field theory. Even though a robust structural description of QFT itself is missing, it has been used to produce stunningly precise descriptions of elementary particles.

The success of these computations can be attributed to a remarkable property that all physical quantum field theories share: although we do not know what QFTs of physical interest will look like, we do have a rough idea of how to produce them.

- Quantum field theories are obtained from geometric data by a process of 'quantization'.

All theories of physical relevance arise in this way. In fact, quantum field theory has not been introduced for its own sake, but rather to account for the small-scale behaviour of the *prequantum field theories* (pQFTs) one encounters in everyday life (like electromagnetism).

Such a prequantum theory is constructed by choosing a (gauge) bundle, whose sections are the field configurations, together with a Lagrangian or an action functional S on the space of sections. If we consider an n-dimensional field theory and Σ^n is a n-dimensional closed manifold, then such an (exponentiated) action functional is given by a smooth function

$$\exp(iS)\colon \mathrm{Fields}(\Sigma^n) \to U(1). \tag{1.1}$$

The locality axiom for field theories tells us that the value of this action functional on a field $\phi \in \mathrm{Fields}(\Sigma^n)$ is determined by action functionals applied to the restrictions of the field ϕ to smaller pieces of the manifold Σ^n.

Locality therefore forces us to also assign some kind of extended action functional to lower dimensional manifolds, so that we can glue action functionals along lower dimensional boundaries. Good targets for such extended action functionals are the higher stacky analogues $\mathbf{B}^kU(1)$ of the circle group $U(1)$. For each closed $(n-k)$-dimensional manifold Σ^{n-k}, an extended (exponentiated) action functional can then be described as a smooth map

$$\exp(iS)\colon \mathrm{Fields}(\Sigma^{n-k}) \to \mathbf{B}^kU(1).$$

When $k=0$, the target is simply the smooth circle group $U(1)$ and we obtain the traditional action functional. If Σ has dimension $n-1$, then the codomain $\mathbf{B}U(1)$ of the extended action functional is the smooth moduli stack of circle bundles. The map $\mathrm{Fields}(\Sigma^{n-1}) \to \mathbf{B}U(1)$ then gives rise to a $U(1)$-bundle over the space $\mathrm{Fields}(\Sigma)$, which is called the prequantum circle bundle.

We can use these higher refinements of the circle group to obtain action functionals by pasting along lower dimensional boundaries. For example, if Σ^n has a boundary consisting of two components $\partial\Sigma_{\mathrm{in}}$ and $\partial\Sigma_{\mathrm{out}}$, then the extended action functional will determine a map of $U(1)$-bundles over $\mathrm{Fields}(\Sigma)$, which we can depict as a *correspondence diagram*

$$\begin{tikzcd} & \mathrm{Fields}(\Sigma) \arrow[dl, "|_{\partial\Sigma_{\mathrm{in}}}"'] \arrow[dr, "|_{\partial\Sigma_{\mathrm{out}}}"] & \\ \mathrm{Fields}(\partial\Sigma_{\mathrm{in}}) \arrow[dr, "\alpha"'] \arrow[rr, phantom, "\Rightarrow"] & & \mathrm{Fields}(\partial\Sigma_{\mathrm{out}}) \arrow[dl, "\beta"] \\ & \mathbf{B}U(1) & \end{tikzcd} \tag{1.2}$$

This diagram describes the possible trajectories of a field from the ingoing boundary to the outgoing boundary, each with its own phase assigned to it. In case both boundary components are empty, then the two $U(1)$-bundles on $\mathrm{Fields}(\Sigma)$ are trivial, so that the map between them is simply the $U(1)$-valued action functional 1.1 we started with.

The same kind of diagrams appear when we add boundaries of all possible dimensions $\leq n$: in that case, all spaces come equipped with maps into $\mathbf{B}^n U(1)$, together with homotopies between them. If there are no boundaries of dimension below k, then all higher homotopies give rise to maps into $\mathbf{B}^{n-k}U(1)$, just like we found a map into $U(1)$ if there are no boundaries at all. Summarizing, a prequantum field theory is a functor that assigns certain (higher) correspondence diagrams to cobordisms, i.e. it gives rise to a functor

$$\mathrm{Bord}_n \longrightarrow \mathrm{Corr}_n\big(\mathbf{H}_{/\mathbf{B}^n U(1)}\big)$$

into a certain category of n-fold correspondences between spaces of fields and extended action functionals on them. We will give a more detailed account of this perspective on pQFTs in section 2.3.

Having chosen a pQFT, one has to quantize it. Of course, since it is unclear what the resulting QFT should look like, there is even less known about this quantization step. In any case, it should morally consist of two steps:

(i) linearize the theory. We want the action functional S to take values in something linear, so that we are able to add the values of S. A result of this linearity is the superposition principle, which says that we can add up quantum states. In particular, different states might add up to zero, which results in interference.

(ii) perform a path integral. We want to integrate the action functional against some measure on the space of field configurations. Over an n-dimensional closed manifold Σ, this just means that we integrate the $U(1)$-valued function S

$$\int_{\phi\in\mathrm{Fields}(\Sigma)} [D\phi] S(\phi)$$

where $[D\phi]$ denotes a path integral measure on $\mathrm{Fields}(\Sigma)$. Over an $(n-1)$-dimensional surface, we have that S determines a $U(1)$-bundle. Picking the associated line bundle L over $\mathrm{Fields}(\Sigma^{n-1})$, the 'path integral' is supposed to be some sort of linear space of sections of L.

The path integral is notorious for tending to be ill-defined. The spaces of field configurations are usually infinite dimensional and almost never admit a natural measure against which to integrate. Only in very simple cases do we have a concrete path integral measure, like the Wiener measure. In general, the path integral is considered more as a heuristic principle, rather than a meaningful object on itself.

On the other hand, problems arise in trying to quantize the space $\mathrm{Fields}(\Sigma^{n-1})$ carrying the line bundle L. We might again run into infinite dimensional spaces of fields, but problems arise even in the nice cases that these spaces are finite dimensional. This happens for instance in 3d Chern-Simons theory, where we obtain a smooth manifold $M = \mathrm{Fields}(\Sigma^2)$ of flat G-bundles over the surface Σ (these form a stack rather than a manifold, but we ignore this for now).

In such finite-dimensional cases, the manifold $M = \mathrm{Fields}(\Sigma^{n-1})$ usually carries a symplectic form and the line bundle L is a prequantum line bundle for this symplectic form. To quantize such a symplectic manifold, one picks a polarization on M - a splitting into position and momentum coördinates. The quantization of M is then the space

$$\Gamma_{\mathrm{pol}}(M, L)$$

of polarized sections of the line bundle L - those sections which depend only on the 'position' coördinates. This process is called the *geometric quantization* of the symplectic manifold M. In the case of 3d Chern-Simons theory, Witten [Wit89] discusses how one can obtain a Kähler polarization on the space of flat G-bundles on a 2d surface Σ from a choice of complex structure on the surface Σ. Geometric quantization then produces a Hilbert space of polarized sections of L for each complex structure on Σ. He shows that these spaces form a vector bundle over the moduli space of complex structures on Σ, and that this vector bundle carries a projectively flat connection, the KZ-connection.

This demonstrates that quantization is a very nontrivial step: even in the nice, classical setup of a finite-dimensional symplectic manifold, it is unclear how the quantization depends on the choice of polarization. For example, in the case of Chern-Simons theory the dependence on the choice of polarization requires delicate analysis. In fact, often one has to adjust the above procedure by hand to produce the 'correct' quantization: there might be too few polarized sections, or one has to choose a 'metaplectic correction' to obtain an inner product on the space of states. In short, while the idea of geometric quantization looks promising from the outset, the choices and modifications needed in practice make it non-systematic and certainly non-functorial.

In this text, we change perspective on the above quantization scheme. Instead of viewing the path integral as a measure-theoretic structure, we consider it from the point of view of cohomology. This follows the viewpoint due to Freed [Fre99], which has been developed in many different contexts [FHT10], [BZFN10], [FHLT10], [CG04], [BMRS08]. Recall that for a compact oriented manifold M, there is a natural integration map

$$H^{n-*}(M) \xrightarrow{\int_M -} H^*(*) = \mathbb{C}$$

in (complex) ordinary cohomology. In terms of differential forms, this map precisely integrates a top degree form on M against a measure; the choice of orientation corresponds to the choice of measure. On the other hand, the above map can be constructed purely in terms of algebraic topology, without any reference to a measure.

More generally, if we have a diagram like 1.2 where the circle bundles are flat, then we can construct a map in twisted cohomology

$$H^{*+\alpha}(\mathrm{Fields}(\partial\Sigma_{\mathrm{in}})) \longrightarrow H^{*+\beta}(\mathrm{Fields}(\partial\Sigma_{\mathrm{out}}))$$

by pulling back along the left map and integrating along the right map. This map depends on a choice of orientation for the right map, but the possible choices can be easily described in terms of cohomology.

As was amplified in [ABG+08], twisted cohomology groups (or rather their refinements to spectra) can be interpreted as the spaces of sections of line bundles. As such, we can interpret them as spaces of higher wave functions, or higher quantum states. As we will see, these higher states assigned to the point do not require a polarization, which is obtained automatically in higher dimensions from the pull-push formalism. The resulting map between these cohomology groups (spectra) can then be interpreted as the quantum propagator from the ingoing to the outgoing boundary.

The main aim of this text is to extend this picture to the situation where the group $U(1)$ is replaced by one of the higher groups $\mathbf{B}^k U(1)$.

(i) To linearize the theory, we have to map $\mathbf{B}^k U(1)$ in the ∞-group of units of a (smooth) E_∞-ring R. In the above situation, we embed $U(1)$ in the ring $\mathbb{C}$ of complex numbers. Such an E_∞-ring R classifies a (smooth) multiplicative generalized cohomology theory, and the map from $\mathbf{B}^k U(1)$ to its group of units produces a *twist* in R-cohomology. This allows us to form the composite functor

$$\mathrm{Bord}_n \longrightarrow \mathrm{Corr}_n\big(\mathbf{H}_{/\mathbf{B}^n U(1)}\big) \longrightarrow \mathrm{Corr}_n(\mathbf{H}_{/\mathbf{B}GL_1(R)})$$

where $GL_1(R)$ is the group of twists for R-cohomology. The result describes the linearized prequantum field theory.

(ii) We then quantize a correspondence diagram like 1.2 (with $\mathbf{B}U(1)$ replaced by $\mathbf{B}^nU(1)$) by a pull-push construction in R-cohomology. As in ordinary cohomology, the pushforward step requires a choice of orientation, but these are usually easier to keep track of than the choices of polarization needed in standard geometric quantization. The result of such a pull-push construction is a linear map between twisted R-cohomology spectra.

Again, these twisted R-cohomology spectra are given by the R-modules of sections of line bundles, whose fiber is the ring R. They can therefore be viewed as the spaces of higher wave functions, or higher quantum states. The pull-push map presents the propagator of these states.

We will only discuss the pull-push quantization of a single such correspondence diagram. However, our general treatment of pull-push quantization points to a general setup for quantizing full pQFTs: a considerable number of results suggest that the pull-push constructions we encounter can be made functorial [CG04], [CS84], [EM09], [FHT10]. Given such a functorial way of quantizing correspondence diagrams, we could then try to quantize the n-fold correspondence diagrams that form the target of our pQFT, since these are just built up from single correspondences. Such a quantization process might also be called 'motivic quantization', since it is obtained by passing from the category of correspondences to some linearized version thereof. In the outlook section 6, we will say a bit more about how one would expect this to come about. For the rest of the text we are mainly interested in quantizing a single correspondence diagram like 1.2.

In fact, such diagrams do not only appear as descriptions of trajectories of prequantum fields. As we will see in section 2.3, a similar kind of diagram describes a *boundary theory* to an n-dimensional topological field theory:

$$\begin{tikzcd}
 & \mathbf{Fields}^{\partial} \arrow[dl] \arrow[dr] & \\
* \arrow[dr] & \arrow[d, phantom, "\Searrow"] & \mathbf{Fields} \arrow[dl] \\
 & \mathbf{B}^nU(1) &
\end{tikzcd} \tag{1.3}$$

Here the spaces involved are not the spaces of fields on a surface, but instead they are classifying spaces of fields. For example, for Chern-Simons theory, where the fields on a manifold Σ are (flat) G-bundles with connection, we have a classifying space (or rather, a stack) $\mathbf{B}G$ of G-bundles (ignoring the differential refinement to the stack $\mathbf{B}G_{\text{conn}}$ of G-bundles with connection).

By applying the pull-push quantization method to such diagrams, we quantize the boundary theory, which produces an *element* in the quantization of the bulk field theory. In the physics literature, this phenomenon is known as the *holographic principle*, which says that the correlators of a boundary QFT can be identified with certain states of the bulk TQFT. We provide several examples of this holographic quantization in section 5.

Since the pushforward requires a choice of orientation, we need extra geometric structure to quantize such a boundary theory. This is not unexpected: often the boundary theories of topological field theories are themselves not topological, but require extra geometric structure on the spaces Σ. The prototypical example is the WZW-model, which is the 2d boundary theory to 3d Chern-Simons theory: this is a conformal field theory, depending on a choice of complex structure on a surface Σ.

Again, this brings us to the problem of *functorially* quantizing correspondence diagrams, which lies outside the scope of this thesis. Instead, we will focus on the pull-push quantization of a single such correspondence diagram: we embed the group $\mathbf{B}^nU(1)$ in the group of units of a higher ring R, pass to twisted R-cohomology and perform the pushforward in twisted R-cohomology.

In this thesis, we try to give a structural account of the general setup of quantization along the above lines, together with the mathematical structures involved. Section 2 discusses prequantum field theories and their boundary theories in terms of correspondence diagrams of smooth stacks equipped with higher circle bundles. The next two sections describe the two steps in the proposed quantization scheme:

section 3 concerns the passage to twisted generalized cohomology and section 4 treats pushforward maps in generalized cohomology, which give rise to quantum propagators. In section 5, we show how various phenomena considered in the literature can be put in the abstract framework of cohomological quantization.

The main body of the text consists of putting into place the available mathematical machinery needed in the quantization of topological field theories. In addition, original results presented in this thesis include the following:

- a systematic discussion of pushforward maps in twisted generalized cohomology in section 4.1.4. This mainly reviews the discussion of pushforward maps in [ABG11], but highlights the fact that pushforward maps are obtained systematically from fiberwise duality in the category of (parametrized) spectra. This idea appears in [ABG11] as well, but we put more emphasis on its central role in the pushforward construction than is maybe made apparent there. This formulation in terms of duality points towards an immediate generalization in the context of *smooth* spectra (see section 3.1.5 and remark 4.1.26).

- as an expected topological approximation to smooth K-theory, we present a functorial description of twisted K-theory for differentiable stacks, by constructing a $(2,1)$-functor assigning to each stack carrying a circle 2-bundle its twisted convolution algebra. This has been done in the 1-categorical, untwisted case by [Lan01] and [AG06]. In section 3.3 we add to this the $(2,1)$-functoriality together with the twists by circle 2-bundles.

- we give a plausible proposal for completing Weinstein's program [WX91] of geometrically quantizing Poisson manifolds using their symplectic groupoids. This connects the K-theoretic quantization of symplectic manifolds with the geometric quantization of symplectic groupoids proposed in [Haw08], by making the latter invariant under Morita equivalence. Our proposal for the quantization of a Poisson manifold makes essential use of the holographic principle, by realizing a Poisson manifold as a boundary to its 2d Poisson sigma model. This is the analogue in geometric quantization of the well-known result by Kontsevich and Cattaneo-Felder identifying the deformation quantization of a Poisson manifold with the perturbative quantization of its Poisson sigma model.

- an extension of the geometric quantization of symplectic manifolds to the case where they carry a Hamiltonian G-action, by pushing forward in equivariant K-theory. The resulting quantization carries a natural action of the group G by quantum operators.

- using the previous two results, we find a natural interpretation of the 'inverse orbit method' of Freed-Hopkins-Teleman [FHT13] as a defect between the Lie-Poisson sigma model and a chosen coadjoint orbit.

- finally, we discuss how various phenomena considered in the literature can be put in the abstract framework of cohomological quantization:
  - the string topology operations [CS99].
  - the quantization of Chern-Simons theory transgressed to two dimensions [FHT10].
  - D-brane charges as the quantization of particle at the boundary of a string [BMRS08].
  - the 'M9-brane charge' as the quantization of a string sitting at the end of a 2-brane [HW96], [FSS12].

We conclude this thesis with an outlook on the issue of *functorially* quantizing correspondence diagrams and n-fold correspondences. Such a functorial pull-push construction would be an important next step in the fully extended geometric quantization of prequantum field theories.

2 Local prequantum field theory

In this section we describe the ingredients that give the input of our quantization process. The main aim is to provide a formalization of local prequantum field theory suitable for our needs, in terms of n-fold correspondences. This description requires the theory of functorial QFTs, which we shortly recall in section 2.1.

As we said in the introduction, prequantum field theories are described in terms of geometric structures. The structures we will be using are (higher) smooth stacks. On the one hand, stacks arise naturally in field theory because the fields involved carry a notion of equivalence between them: they are *gauge fields*. On the other hand, we have seen that locality forces the action functional of a field theory to be extended: we should not only assign numbers to fields over top dimensional manifolds, but we should also assign data to fields over lower dimensional manifolds which allows us to glue action functionals together. This requires a stacky refinement of the circle group $U(1)$ to the smooth stacks $\mathbf{B}^n U(1)$ classifying higher circle bundles.

In any case, stacks are inevitable in our presentation of pQFTs. We will give an overview of the theory of stacks in section 2.2. With an eye towards the twisted K-theory for differentiable stacks developed in section 3, we will pay special attention in section 2.2.3 to the presentation of differentiable stacks in terms of Lie groupoids and bibundles. We add a slight modification to this where we present differentiable stacks carrying a circle 2-bundle. This seems well-known in the literature, but we could not find a reference for it.

Having the language of stacks and FQFT at our disposal, we describe pQFTs in section 2.3. We conclude with a discussion of boundaries and defects in prequantum field theory: under the cobordism hypothesis, these are described by certain correspondence diagrams in the category of stacks. Precisely these kind of correspondence diagrams will be the the diagrams that we quantize in the next sections.

2.1 Functorial field theory

From a physical perspective, a field theory describes a system whose configurations can be written down *locally*, in terms of local coordinates. More precisely, such a configuration - a *field* - is some piece of data on a manifold Σ that can be described locally. This means that any field on Σ can be defined by giving fields on subsets U_i covering Σ, and providing gluing information between them on overlapping regions.

For example, a field can be a section of a fiber bundle. This happens in the theory of general relativity, where the studied field is the spacetime metric. Of course, we can give metrics on open subsets and require them to agree on the intersections. This then gives a globally defined metric on the manifold Σ.

A different kind of field arises in the (prequantum) standard model. There the relevant fields are the gauge fields responsible for the strong and electroweak force. Such gauge field is not the section of a fiber bundle, but rather is itself a principal bundle, together with a connection. Such bundles are also local objects: from bundles on smaller opens U_i, together with chosen equivalences between their restrictions to the intersections $U_i \cap U_j$, we can construct a bundle on the entire manifold Σ.

Although both fields are local, they seem to behave differently. While we can glue sections of a fiber bundle by requiring them to be equal on overlapping regions, we can glue bundles by giving explicit equivalences between them on the overlaps (in fact, we could give a similar description of general relativity). In the next section 2.2 we will describe the geometry that unifies these two types of fields: we can obtain principal bundles as sections of a (trivial) bundle whose fibers are not manifold, but stacks.

The previous examples of fields were of a differential geometric nature, but there are many other possibilities. In particular, in quantum field theory the fields are not expected to be of this geometric kind. The only requirement we put on a field is that it is local, so that it can be obtained by some process of gluing.

These physical fields are not static, but can evolve along certain trajectories. More precisely, suppose we have an cobordism Σ from an ingoing boundary $\partial\Sigma_0$ (which we think of a sitting at time $t = 0$), to an outgoing boundary $\partial\Sigma_1$ at time $t = 1$. Then a field configuration on the boundary $\partial\Sigma_0$ can evolve via the manifold Σ until it reaches a configuration on the outgoing boundary $\partial\Sigma_1$. Meanwhile, the cobordism Σ

might changed the shape of $\partial\Sigma_0$, so that $\partial\Sigma_0 \not\cong \partial\Sigma_1$. In any case our physical fields can move over Σ from Σ_0 to Σ_1, so that we get a rule that tells us how fields can evolve

$$\Sigma \longmapsto \{\text{Fields on } \Sigma_0\} \xrightarrow{\text{Evolve}} \{\text{Fields on } \Sigma_1\}$$

From this perspective, a field theory is supposed to assign certain spaces of field configurations (or 'initial value data') to manifolds of dimension $n-1$, while it assigns 'a rule for evolving fields' to an n-dimensional cobordism between two such manifolds.

Remark 2.1.1. The nature of this rule depends on the type of field theory one considers. For the prequantum theories that we will consider in 2.3, the rule gives all possible trajectories of the field: it assigns to a cobordism Σ all possible field configurations that agree with the given fields on the boundary. On the other hand, in quantum field theory such a rule is really a function, called the propagator: it tells us how an ingoing state propagates to an outgoing state.

Remark 2.1.2. The number n is called the dimension of the field theory. An n-dimensional field theory only has fields on manifolds of dimension $\leq n$. We can always restrict field configurations (by locality), so that we inevitably obtain fields on manifolds of dimension lower than n. We do not consider manifolds of larger dimension.

The locality condition also applies to the ways fields can evolve along Σ: if we know how a field moves along smaller pieces of Σ, then we can determine how the field moves along the whole of Σ. More precisely, if a field moves from Σ_0 to Σ_1 along Σ, and from Σ_1 to Σ_2 along $\tilde{\Sigma}$, then we also know how the field evolves from Σ_0 to Σ_2 along $\Sigma \cup_{\Sigma_1} \tilde{\Sigma}$. We first move over Σ first and then we evolve along $\tilde{\Sigma}$.

Finally, the locality condition tells us that the fields on a disjoint union are independent, as is the evolution along a disjoint union of cobordisms. This means that a space of fields on a coproduct of manifolds looks like a 'product' of spaces of fields on each of the components.

At this point, we have not specified what we mean by such a product. In fact, we did not even say what the 'spaces of field configurations' should look like. Choosing a different meaning of these terms leads to different types of field theories and the first task in specifying a field theory is to pick these definitions wisely. Generally, we have to give maps and products of spaces of fields, so that they are organized into a symmetric monoidal category $\mathbf{C}$ (the symmetry comes from the symmetry of the disjoint union of manifolds).

Once we have fixed our category $\mathbf{C}$, the above considerations essentially give us the definition of an n-dimensional topological field theory (TFT), as it was given by Atiyah [Ati89b].

Definition 2.1.3. An n-dimensional TFT is a monoidal functor $\mathrm{Bord}_{n-1,n} \to \mathbf{C}$ for some symmetric monoidal category $\mathbf{C}$.

This definition makes sense for any choice of category $\mathbf{C}$. Atiyah was mainly interested in fields with a linear structure, so he chose $\mathbf{C} = \mathrm{Vect}$ to be the category of vector spaces, with the tensor product as the monoidal structure. We will see other choices in the next sections.

Remark 2.1.4. Here $\mathrm{Bord}_{n-1,n}$ denotes the category whose objects are closed $n-1$-dimensional smooth manifolds and whose morphisms are diffeomorphism classes of n-dimensional smooth cobordisms. We require manifolds to come with an n-framing of their tangent bundle, i.e. a trivialization of the vector bundle $T\Sigma^k \oplus \mathbb{R}^{n-k}$ for a k-dimensional manifold (with boundary). The restriction to framed manifolds is mainly for simplicity of exposition.

Composition is by gluing along boundaries (this obtains a smooth structure by identifying collar neighbourhoods of the boundary), the unit on Σ^{n-1} is the cylinder $\Sigma^{n-1} \times [0, 1]$ and the monoidal structure is given by the disjoint union, with monoidal unit the empty manifold.

Remark 2.1.5. Since we do not require any geometric structure on our cobordisms Σ, like a metric or a complex structure, such field theories are called topological. However, the bordisms are smooth manifolds up to diffeomorphism, not topological spaces up to homeomorphism. In constrast to topological bordisms, *smooth* cobordisms have the main property that they admit a handle decomposition, which means that they can be constructed from elementary pieces by surgery. Locality expresses the idea that fields can be obtained in a similar way, by gluing along lower dimensional boundaries.

Many physical theories are not topological, but depend on a background metric. Such theories can be defined in a similar way, by replacing cobordisms by cobordisms with extra geometric structure, in this case a metric. However, when coupled to relativity, usually this metric becomes a field on its own and the theory becomes topological (except possibly for a choice of orientation).

Atiyah's definition of a TFT gives the locality of fields in terms of a functoriality condition. To get a field evolving over a manifold M, we just chop it up into pieces along $n-1$-dimensional surfaces and take fields moving over each of the pieces, matching at the boundary. However it does not yet give the full locality condition on the field configurations: given an $n-1$-dimensional surface Σ, there is no rule that tells us how the field configurations on Σ can be built up from data on smaller pieces. We would therefore want a structure that also tells us how field can be glued if we cut $(n-1)$-dimensional surfaces into pieces along $n-2$-dimensional manifolds.

We can now continue chopping up our manifolds along lower-dimensional boundaries, until we end up with zero-dimensional manifolds that cannot be split up anymore. A truly local field theory should allow us to cut and paste like this, all the way until we end up with a point. The object that describes all this cutting an pasting is that of an extended (topological) field theory.

Definition 2.1.6. An n-dimensional extended TFT is a monoidal (∞, n)-functor $\mathrm{Bord}_n \to \mathbf{C}$ to a symmetric monoidal (∞, n)-category $\mathbf{C}$.

The (∞, n)-category Bord_n is roughly the category whose

- objects are framed points (points with a trivialization of the stabilized tangent bundle $T\{*\} \oplus \mathbb{R}^n$).
- 1-morphisms are framed, smooth 1-dimensional cobordisms between such points.
- 2-morphisms are framed, smooth 2-dimensional cobordisms between such 1d cobordisms. Note that these are given by manifolds with corners.
- ...
- n-morphisms are framed smooth n-dimensional cobordisms.
- $n+1$-morphisms are diffeomorphisms of n-dimensional cobordisms.
- $n+2$-morphisms are smooth homotopies of diffeomorphisms.
- ...
- symmetric monoidal structure is the disjoint union of manifolds, with the empty set as the monoidal unit.

A precise interpretation of this definition requires some theory of (∞, n)-categories. In [Lur09b], the presentation of (∞, n)-categories by n-fold complete Segal spaces is used to describe Bord_n.

Remark 2.1.7. Although there is an abstract theory of (∞, n)-categories, it is hard to give specific examples of them and do computations with them. In this text, we therefore try to circumvent the technical details on (∞, n)-categories.

Observe that all one-dimensional cobordisms can be obtained by gluing intervals, which are the unit 1-morphisms in Bord_n. Similarly, all two-dimensional (framed) cobordisms can be obtained by sewing squares, which are the identity 2-morphisms. In fact, since all manifolds involved are smooth, they can essentially be obtained by gluing k-cubes, which are all (higher) identity morphisms. In other words, any morphism can be obtained as a nontrivial composition of identity maps (and disjoint union of identity maps). This suggests the following result, which has been proven by Lurie in [Lur09b]:

Theorem 2.1.8 (Baez-Dolan-Lurie Cobordism Hypothesis)**.** Bord_n *is the free symmetric monoidal (∞, n)-category with all duals generated by a single object $*$. In particular, any monoidal functor*

$$F\colon \mathrm{Bord}_n \to \mathbf{C}$$

into a symmetric monoidal (∞, n)-category is defined by a fully dualizable object

$$F(*) \in \mathbf{C}_{\mathrm{fd}}.$$

Remark 2.1.9. The object $F(*)$ has to be *fully dualizable* to really serve as an object that a TFT can assign to the point. The condition of full dualizability is a higher-categorical analogue of the notion of dualizability, which allows us to take the higher dimensional traces necessary to construct a TFT from a single object.

This result captures the full locality of a TFT: everything can be obtained from a the data assigned to the point by taking (higher) traces and tensor products. By using this, we can reduce the structure of a complete TFT to a single object.

In the rest of this section, we will describe how to obtain such extended TFTs from geometric data. The resulting field theories are called prequantum field theories. Our next task is therefore to describe the geometric objects that arise in such pQFTs: stacks.

2.2 Stacks

At the beginning of the previous section we mentioned how the fields that arise in the standard model are described by principal bundles carrying a connection. Recall that such bundles are local objects; we can glue principal bundles just like we can glue sections of bundles, and we can also glue the connections on such bundles. However, if we want to glue principal G-bundles $P_i \to U_i$ over an open cover $\{U_i\}$ of the manifold Σ, it is wrong to require them to be the same on overlaps $U_i \cap U_j$. Rather, we have to specify equivalences between them

$$\begin{tikzcd} P_i \arrow[rr, "g_{ij}"] \arrow[dr] & & P_j \arrow[dl] \\ & U_i \cap U_j & \end{tikzcd}$$

Then the bundles P_i glue to a bundle $P \to \Sigma$ if these equivalences satisfy the cocycle identity

$$g_{ij} g_{jk} = g_{ik} \qquad g_{ii} = 1 \tag{2.1}$$

on a triple intersection $U_i \cap U_j \cap U_k$. If this is satisfied, we can obtain P as the union of all P_i, where we identify $p \sim g_{ij}(p)$ for any $p \in P_i$. The cocycle identity makes sure we do not identify a point $p \in P_i$ with another point in P_i.

This form of gluing is a consequence of the fact that principal bundles (with connection) do not form a set, but rather a *groupoid*. Different bundles can be equivalent in many inequivalent ways and a bundle can have nontrivial automorphisms. To glue different bundles we have to provide gauge transformations between them, but we should not obtain non-trivial self-equivalences by composing these transformations.

A field theory where the spaces of fields carry such a notion of equivalence is called a *gauge* theory. Although two fields can be equivalent (hence describing physically equivalent configurations), they are not the same: it is generally *wrong* to identify them. This is known as the gauge principle. From a mathematical perspective, the gauge principle simply states that it is generally wrong to identify a groupoid with its set of path-components. Instead, one is only allowed to identify equivalent groupoids.

Note that we can identify the groupoids X and Y with homotopy 1-types, in which case they are equivalent iff there is a weak equivalence between them. The fields in a gauge theory are therefore organized in groupoids, or homotopy 1-types, which themselves form a $(2,1)$-category (the 2-morphisms are natural transformations or homotopies between maps, depending on the perspective).

We see that gauge fields are naturally organized into groupoids, but for our description of pQFTs we need even more. Recall from the introduction that an extended action functional was supposed to assign data to fields on lower dimensional manifolds admitting not only equivalences, but also higher equivalences between equivalences. Once we know what the action functional assigns to fields over the point, we can obtain the value of the action functional over higher dimensional manifolds by taking higher equivalences of its value over the point. We therefore need the notion of an n-groupoid, or a homotopy n-type, to describe the values of extended action functionals. Turning n into ∞, we see that fully generally, the objects we have to consider are ∞-groupoids, or homotopy types.

The groupoids we have described so far are all discrete; we started with sets and added sets of equivalences to those. To do geometry we need to endow these ∞-groupoids with a smooth structure.

The basic idea is that a smooth structure on an ∞-groupoid X allows us to consider smooth maps from test spaces into X. For example, once we know the smooth structure on X, we can consider maps from a manifold M into X.

In case X is a smooth manifold, we can reconstruct its smooth structure from this. Indeed, if we know all smooth maps into X, then we certainly know about all the charts mapping into the manifold X. We can now turn this around and instead *define* a smooth structure on an ∞-groupoid X to be the collection of all smooth maps into X. This realizes X as a presheaf (of ∞-groupoids) on the category of smooth manifolds.

Now suppose that M is a manifold that is covered by opens U_i. Then by definition a smooth map $M \to X$ into a manifold X can be obtained by gluing smooth maps $U_i \to X$ that agree on the overlap. This makes the presheaf

$$X : \mathrm{SmMfd}^{op} \to \mathrm{Set}$$

into a sheaf (of sets) on the category of smooth manifolds. The same kind of gluing condition should hold for the smooth maps into any smooth groupoid X. But as we have seen, when we work with higher groupoids, we should really glue by specifying equivalences on overlapping regions, equivalences between equivalences on triple overlaps, etcetera. A presheaf of ∞-groupoids satisfying such a gluing law is called a stack (or ∞-stack). These stacks describe the spaces with equivalences we are interested in.

2.2.1 The ∞-topos of smooth stacks

Inspired by the previous section, we will now work our way towards the definition of a stack.

Definition 2.2.1. Let SmMfd be the category of (Hausdorff, second countable) smooth manifolds, equipped with the Grothendieck topology consisting of the smooth open covers. Let Shv(SmMfd) be the corresponding sheaf category.

Remark 2.2.2. Recall that the category of sheaves on a site is obtained as a left exact localization of the category of presheaves on that site. In particular, if $\coprod U_i \to M$ is an open cover of a smooth manifold, we can consider the simplicial diagram

$$\cdots \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \coprod_{ijk} U_{ijk} \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \coprod_{ij} U_{ij} \substack{\longrightarrow\\[-1ex]\longrightarrow} \coprod_i U_i \tag{2.2}$$

in the category of presheaves on SmMfd, where the $U_{ij} = U_i \cap U_j$ denote intersections of opens. The colimit $\check{C}(U)$ of this diagram in the category of presheaves is called the Cech nerve of the cover. It comes with a natural map

$$\check{C}(U) \to M$$

We obtain the category of sheaves on SmMfd by formally inverting these morphisms from Cech nerves to manifolds. Since all these morphisms arise from a coverage, the localization of the presheaf category at these morphisms forms a reflexive subcategory of the category of presheaves

$$\begin{tikzcd} \mathrm{Shv}(\mathrm{SmMfd}) \arrow[r, hook, shift right] & \mathrm{PShv}(\mathrm{SmMfd}) \arrow[l, shift right, "L"'] \end{tikzcd}$$

where the left adjoint L preserves finite limits. Explicitly, Shv(SmMfd) consists of those presheaves X that satisfy the descent condition for each cover of a manifold M:

$$\mathrm{Hom}(M, X) \xrightarrow{\simeq} \mathrm{Hom}(\check{C}(U), X).$$

For any manifold M, the smooth maps into M form a sheaf over the category of smooth manifolds, so we obtain

Lemma 2.2.3. *The Yoneda embedding gives rise to a fully faithful embedding*

$$\mathrm{SmMfd} \to \mathrm{Shv}(\mathrm{SmMfd}).$$

We will always think of manifolds as sheaves on the site of smooth manifolds.

Any manifold can be obtained by gluing cartesian spaces, i.e. spaces of the form $U = \mathbb{R}^n$. The value of a sheaf X on a manifold M is therefore determined uniquely by its value on the cartesian spaces. In fact, we have the following result:

Proposition 2.2.4 ([Mil63])**.** *Any open cover of a (paracompact) smooth manifold refines to a good open cover, i.e. an cover by opens $\{U_i\}_{i\in I}$ such that for any finite subset $J \subseteq I$, we have that*

$$\bigcap_J U_j \simeq \begin{cases} \mathbb{R}^n & \textit{for some } n \\ \emptyset & \textit{otherwise} \end{cases}$$

This implies that the cartesian spaces form a dense subsite of SmMfd:

Lemma 2.2.5. *Let* CartSp *be the category of cartesian spaces (spaces diffeomorphic to $\mathbb{R}^n$) and smooth maps between them. Endow* CartSp *with the coverage consisting of good open covers. Then* CartSp *forms a dense subsite and consequently, the restriction*

$$\mathrm{Shv}(\mathrm{SmMfd}) \to \mathrm{Shv}(\mathrm{CartSp})$$

is an equivalence of categories.

Again, the smooth manifolds embed fully faithfully into the category of sheaves on CartSp, and by the above equivalence we have for any smooth manifold M and sheaf X on SmMfd that

$$X(M) \simeq \mathrm{Shv}(\mathrm{CartSp})\big(M, X\big|_{\mathrm{CartSp}}\big)$$

To obtain stacks, all we have to do is repeat this story, replacing sets by ∞-groupoids. The only difference is that ∞-groupoids have a homotopy theory: since ∞-groupoids themselves carry equivalences, maps between two such ∞-groupoids (i.e. ∞-functors) can be equivalent, and in turn two such equivalences can be equivalent. This means that ∞-groupoids are naturally organized into an $(\infty, 1)$-category, where we allow for higher morphisms that are all invertible, up to homotopy.

A well-developed model for such categories is by the quasi-categories of Joyal, of which an extensive account is [Lur09a]. Essentially all of category theory passes on to ∞-categories, as long as we never ask two things to be the same, but rather ask for an (explicit) equivalence between them.

Definition 2.2.6. Let SmMfd be the category of smooth manifolds, equipped with the Grothendieck topology consisting of open covers. We let $\mathbf{H} = \mathrm{Sm}\infty\mathrm{Gpd}$ denote the category of ∞-sheaves of ∞-groupoids on SmMfd. We also call such a sheaf of ∞-groupoids a stack.

As discussed in [Lur09a], the construction of $\mathbf{H}$ mimics the construction of Shv(SmMfd). One first considers the category PShv(SmMfd) of functors $\mathrm{SmMfd} \to \infty\mathrm{Gpd}$. Since any set can be viewed as a discrete ∞-groupoid (without any nontrivial morphisms), we have that a smooth manifold gives rise to such a presheaf of ∞-groupoids.

Given a cover $\{U_i\}$ of a smooth manifold M, we can construct its Cech nerve $\check{C}(U)$ by forming the colimit of the diagram 2.2 in the category PShv(SmMfd) of presheaves with values in ∞-groupoids. Since we are not allow to identify two objects, the Cech nerve in the ∞-category of presheaves looks different from the Cech nerve in the 1-category: in the latter, we essentially just identify two points x_i, x_j if they correspond to the same point x in M. In the ∞-categorical colimit, we formally add an equivalence that witnesses that these two points are equivalent

$$U_i \ni x_i \xrightarrow{x_{ij}} x \in U_j.$$

In order not to get a nontrivial self-equivalence of x_i, we add a two-cell for each triple intersection containing x

$$\begin{tikzcd} x_i \arrow[r, "x_{ij}"] & x_j \arrow[d, "x_{jk}"] \arrow[dl, phantom, "\swarrow x_{ijk}"] \\ & x_k \arrow[ul, "x_{ki}"] \end{tikzcd}$$

We continue like this, adding n-cells for each n-fold intersection.

The category $\mathbf{H}$ of sheaves of ∞-groupoids on SmMfd is now obtained by universally turning the maps $\check{C}(\{U_i\}) \to M$ into equivalences. The procedure is exactly the same as in the set-valued setting: one realizes $\mathbf{H}$ as the subcategory of PShv(SmMfd) on the presheaves X that satisfy descent, in the sense that

$$\mathrm{Hom}(M, X) \to \mathrm{Hom}(\check{C}(\{U_i\}), X)$$

is a weak equivalence of ∞-groupoids. Here Hom is the ∞-groupoid of morphisms in the presheaf ∞-category, i.e. it is the *derived* mapping space. The inclusion of the stacks into the presheaves of groupoids admits a left adjoint that maps a presheaf of groupoids to its associated stack.

Example 2.2.7. Any sheaf on the category of manifolds gives rise to a sheaf of ∞-groupoids on SmMfd. For example, the sheaf of differential forms Ω^k defines a sheaf of ∞-groupoids over SmMfd.

Example 2.2.8. Let G be a Lie group. Then there is a stack $\mathbf{B}G$, the classifying stack of G-bundles, which to each manifold M assigns the groupoid of G-bundles over M. The descent condition precisely says that any G-bundle over M can be obtained from G-bundles over opens U_i covering M, together with equivalences between their restrictions to $U_i \cap U_j$ satisfying the cocycle condition 2.1. Moreover, we can also obtain equivalences between G-bundles by gluing local equivalences.

Example 2.2.9. Combining the above two examples, we can construct a stack $\mathbf{B}G_{\mathrm{conn}}$ of G-bundles with connection.

Example 2.2.10. For any manifold M carrying a smooth action of a Lie group G, we can form the quotient stack $M//G$. This quotient serves as the 'correct' placeholder of the (singular) quotient space. In particular, if the action of G is free and proper, then $M//G$ is equivalent to the quotient manifold M/G.

Example 2.2.11. For each k, there is a stack $\mathbf{B}^k U(1)$ that classifies circle k-bundles. For $k = 0$ this is just the manifold $U(1)$, for $k = 1$ this is the stack of circle bundles and for $k = 2$ this is the stack classifying $U(1)$-gerbes.

The category $\mathbf{H}$ has the same kind of properties as the ordinary category of sheaves on SmMfd:

- $\mathbf{H}$ is an ∞-topos, i.e. it satisfies the higher analogue of the Giraud axioms for a topos (see [Lur09a] for an extensive treatment).
- $\mathbf{H}$ is equivalent to the ∞-topos of sheaves on the site of cartesian spaces, endowed with the coverage of good open covers.
- Being an ∞-topos, $\mathbf{H}$ comes equipped with an adjunction [Lur09a]

$$\begin{tikzcd} \mathbf{H} \arrow[r, shift right, "\Gamma"'] & \infty\mathrm{Gpd} \arrow[l, shift right, "\mathrm{Disc}"'] \end{tikzcd}$$

The right adjoint sends a smooth stack to its underlying groupoid of points. When M is a smooth manifold, $\Gamma(M)$ is simply the set M and for a Lie group G, we have that

$$\Gamma(\mathbf{B}G) \simeq \mathbf{H}(*, \mathbf{B}G) \simeq \left\{ \begin{tikzcd} * \arrow[r, "g"] \arrow[dr, "hg"'] & * \arrow[d, "h"] \\ & * \end{tikzcd} \right\}$$

is the groupoid with one object and the set G as automorphisms. Indeed, there is a unique G-bundle over the point, with G as its automorphism group.

The left adjoint sends $X \in \infty\mathrm{Gpd}$ to X carrying the discrete smooth structure: a morphism $M \to \mathrm{Disc}(X)$ is necessarily constant [Sch12].

- $\mathbf{H}$ is a cohesive ∞-topos (see [Sch12] for an extensive treatment). This means that Γ has a further right adjoint and that Disc has a further left adjoint

$$\begin{tikzcd} \mathbf{H} \arrow[r, shift left=3, "\Pi"] \arrow[r, shift left=1, leftarrow, "\mathrm{Disc}"'] \arrow[r, shift right=1, "\Gamma"] \arrow[r, shift right=3, leftarrow, "\mathrm{CoDisc}"'] & \infty\mathrm{Gpd} \end{tikzcd}$$

The left adjoint Π sends a stack to its homotopy type (also called its geometric realization). For example, for M a smooth manifold, $\Pi(M)$ is the fundamental ∞-groupoid of the topological space underlying M. For G a Lie group, we have that $\Pi(\mathbf{B}G) \simeq BG$ is the classifying space of the topological group underlying G. The final axiom of cohesion says that this functor Π preserves finite products.

This provides the basic abstract structure on the ∞-topos $\mathbf{H}$ of smooth stacks. To do actual computations with stacks, we need a good presentation of them. In the next section we will give the presentation of smooth stacks by a model structure on the category of simplicial presheaves on the site SmMfd of smooth manifold (or the site CartSp of cartesian spaces). In section 2.2.3 we will discuss the class of differentiable stacks, which can be presented by Lie groupoids.

2.2.2 Presentation

The main reason for $(\infty, 1)$-categories to be tractable (in contrast to higher categories) is that they can often be presented by 1-categorical structures. For example, by Hammock localization every category $\mathbf{C}$ together with a class W of weak equivalences in $\mathbf{C}$ gives rise to a presentation of an ∞-category $\mathbf{C}[W]$. A morphism is given by a zig-zag of morphisms and weak equivalences.

In the case of ∞-toposes, even more tools are available to describe them: we can present them by simplicial model categories [Lur09a]. The fundamental example of this is the presentation of ∞Gpd by the standard Quillen model structure on the category of simplicial sets (or the Quillen equivalent structure on topological spaces). Other toposes are obtained by localizing toposes of presheaves of ∞-groupoids, which in turn have an easy presentation in terms of simplicial presheaves:

Proposition 2.2.12 ([Lur09a]). *For any small category $\mathbf{C}$, the ∞-category of presheaves of ∞-groupoids on $\mathbf{C}$ can be presented by the category $[\mathbf{C}^{op}, \mathbf{sSet}]$ of simplicial presheaves on $\mathbf{C}$, equipped with either the projective or the injective model structure.*

In the present case of $\mathbf{H} = \mathrm{Sm}\infty\mathrm{Gpd}$, there are (at least) two equivalent model structures that present $\mathbf{H}$, each with its own benefits. The first option is to view $\mathbf{H}$ as sheaves on the site of all smooth manifolds, in which case the corresponding model structure has many good cofibrant objects, but few easily described fibrant objects. Secondly, we might realize $\mathbf{H}$ as sheaves on the site of cartesian spaces, in which case there are fewer good cofibrant objects but the fibrant objects are much more tractable. We can freely move between the two perspectives.

Proposition 2.2.13 ([Sch12]). *The ∞-topos $\mathbf{H}$ of smooth stacks has the following two presentations:*

- *Let $[\mathrm{SmMfd}^{op}, \mathbf{sSet}]_{proj}$ be the category of simplicial presheaves on SmMfd with the projective model structure. Then $\mathbf{H}$ can be presented by the left Bousfield localization of $[\mathrm{SmMfd}^{op}, \mathbf{sSet}]$ at the maps*

$$\check{C}(U) \to M$$

for any open cover of a manifold M.

- *Let $[\mathrm{CartSp}^{op}, \mathbf{sSet}]_{proj}$ be the category of simplicial presheaves on the category of catesian spaces, with the projective model structure. Then $\mathbf{H}$ can be presented by the left Bousfield localization of $[\mathrm{CartSp}^{op}, \mathbf{sSet}]$ at the maps*

$$\check{C}(U) \to V$$

for any good open cover of a cartesian space V.

The resulting model structures are called the Cech model structures on the categories of simplicial presheaves.

Remark 2.2.14. Recall the definition of the Cech nerve as the ∞-colimit over the diagram 2.2 in the ∞-category of presheaves of ∞-groupoids on SmMfd. Each of the manifolds arising in the diagram can be presented by an ordinary presheaf (seen as a simplicial presheaf concentrated in degree 0). Since all representable presheaves are cofibrant, the homotopy colimit of this diagram is just the diagram 2.2 itself, seen as a single simplicial presheaf.

Remark 2.2.15. The projective model structure on simplicial presheaves has as weak equivalences (resp. fibrations) the objectwise weak equivalences (resp. fibrations). Consequently, the fibrant objects are easily described, but the cofibrant objects are not that tractable. Since representable presheaves are cofibrant, the first presentation of $\mathbf{H}$ has all smooth manifolds as cofibrant objects.

On the other hand, the smooth manifolds are not cofibrant in the second presentation: instead, a cofibrant replacement of a manifold M is obtained by taking a *good* open cover $U = \{U_i\}$ of M and taking the Cech nerve $\check{C}(U)$, presented by the simplicial presheaf 2.2. That this is cofibrant follows from a cofibrancy criterion by Dugger [Dug01].

A simplicial presheaf X is fibrant in the Cech model structure if it is objectwise fibrant and has the property that

$$\mathrm{Hom}(M, X) \to \mathrm{Hom}(\check{C}(U), X)$$

is a weak equivalence of simplicial sets, for any (good) open cover of a manifold M. This is precisely the descent condition on X.

Example 2.2.16. Let G be a Lie group. The classifying stack $\mathbf{B}G$ of G-bundles has two fibrant presentations:

- as a simplicial presheaf on SmMfd, it is presented by the presheaf $\mathbf{B}G$ that sends each manifold M to the groupoid of principal G-bundles over M. This is fibrant by the gluing construction of principal bundles we already saw. For any manifold M, we have that $\mathbf{H}(M, \mathbf{B}G)$ is presented by the *derived* mapping space $R\mathrm{Hom}(M, \mathbf{B}G)$. Since each manifold is cofibrant, we see that in the ∞-topos $\mathbf{H}$, the space of maps from M to $\mathbf{B}G$ is the groupoid of G-bundles over M.

- by restricting to the category of cartesian spaces, we obtain an easier fibrant presentation of $\mathbf{B}G$. Since each G-bundle over $\mathbb{R}^k$ is trivializable, we have that $\mathbf{B}G$ can be presented by the Lie groupoid $*//G$, consisting of a single point and the manifold G as space of morphisms. The Lie groupoid $*//G$ gives rise to the simplicial presheaf

  $$U \mapsto N\Big(* //\mathcal{C}^\infty(U, G)\Big)$$

  that sends each cartesian space U to (the nerve of) the groupoid with a single object (the trivial bundle over U), whose self-equivalences are smooth G-valued functions on U. This is precisely the groupoid of G-bundles over U, and the above presentation is easily seen to be fibrant.

  If M is a smooth manifold, then the mapping space $\mathbf{H}(M, \mathbf{B}G)$ is presented by

  $$R\mathrm{Hom}(M, *//G) \simeq \mathrm{Hom}(\check{C}(U), *//G)$$

  by cofibrantly replacing M by the Cech nerve of a good open cover $\{U_i\}$. Writing out the right hand side, we find that a map $M \to \mathbf{B}G$ is given by transition functions

  $$g_{ij} : U_i \cap U_j \to G$$

  such that $g_{ii} = 1$, and satisfying the cocycle identity $g_{ij}g_{jk} = g_{ik}$ over a triple overlap. This is precisely the cocycle desciption of a principal G-bundle. Similarly, one can compute that morphisms between two such cocycles are precisely the coboundaries describing an equivalence of G-bundles.

We now use the ideas of the previous example to to describe the targets of our extended action functionals. These are the stacks $\mathbf{B}^k U(1)$ classifying circle k-bundles. The idea is that such higher bundles are trivial on cartesian spaces, so that we only see the higher gauge transformations. We obtain bundles over generic manifolds by gluing the locally trivial bundles using cocycles.

Definition 2.2.17. The stack $\mathbf{B}^k U(1) \in \mathbf{H}$ of circle k-bundles can be described by the simplicial presheaf on CartSp given by

$$U \mapsto DK^{-1}\Big[\cdots \longrightarrow 0 \longrightarrow \mathcal{C}^\infty(U, U(1)) \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0\Big].$$

The right hand side is the simplicial abelian group given under the Dold-Kan correspondence (cf. [GJ09] for a textbook treatment) by the chain complex having $\mathcal{C}^\infty(U, U(1))$ in degree k.

Remark 2.2.18. More abstractly, the stacks $\mathbf{B}^k U(1)$ can be decribed inductively as follows: each $\mathbf{B}^k U(1)$ carries the natural structure of a smooth ∞-group (i.e. a group up to coherent homotopy). This is manifest in the above presentation, since it assigns a simplicial abelian group to each cartesian space. For $k = 1$, the group structure can also be given by the tensor product of line bundles.

By the Giraud axioms [Lur09a], it follows that the group $\mathbf{B}^k U(1)$ is equivalent to the loop group of a stack, which is unique up to a contractible space of choices. It is easily checked that the above presentation of $\mathbf{B}^{k+1} U(1)$ provides such a delooping for $\mathbf{B}^k U(1)$. We can deloop the group $U(1)$ as often as we want to, due to the fact that $U(1)$ is an *abelian* group.

The main reason for passing to the site of cartesian spaces is that simplicial presheaves like this are often fibrant:

Lemma 2.2.19. *The above simplicial presheaf presenting* $\mathbf{B}^k U(1)$ *is fibrant.*

Proof. Clearly $\mathbf{B}^k U(1)$ is fibrant in the projective model structure. To show it satisfies descent, suppose V is a cartesian space and $U = \{U_i\}$ is a good open cover of V. Following the procedure in example 2.2.16, we find that a map $\check{C}(U) \to \mathbf{B}^k U(1)$ is given by a Cech cocycle

$$g_{i_0 \dots i_k} \colon U_{i_0} \cap \dots \cap U_{i_k} \to U(1) \qquad\qquad (\delta g)_{i_0 \dots i_{k+1}} = 1$$

with values in the sheaf of smooth $U(1)$-valued functions. A transformation between two such maps is given by a Cech coboundary between the two cocycles, a 2-morphisms in the mapping space correspond to coboundaries between coboundaries, etcetera. But over a cartesian space V, all cohomology groups with values in the sheaf $\mathcal{C}^\infty(-, U(1))$ vanish, except for the zeroth group which is of course $\mathcal{C}^\infty(V, U(1))$.

This implies that any Cech cocycle is trivial (up to equivalence), any automorphism of the trivial Cech cocycle is trivial (up to equivalence), etcetera. When we arive at degree k, we find that a self-equivalence of the trivial $k-1$-morphism is given by a Cech 0-cocycle with values in $\mathcal{C}^\infty(-, U(1))$, i.e. by a smooth $U(1)$-valued function. In other words, we find that the natural map

$$\mathrm{Hom}(V, \mathbf{B}^k U(1)) \to \mathrm{Hom}(\check{C}(U), \mathbf{B}^k U(1))$$

is a weak equivalence. □

Since $\mathbf{B}^k U(1)$ is fibrant, we can apply the procedure of example 2.2.16 to produce a description of circle k-bundles over generic smooth manifolds. Using an open cover $\{U_i\}$ of a smooth manifold M, we find that circle k-bundle are described by Cech k-cocycle with values in the sheaf $\underline{U(1)} = \mathcal{C}^\infty(-, U(1))$, while gauge equivalences are witnessed by Cech coboundaries and higher gauge equivalences are witnessed by coboundaries between coboundaries. In particular, we find that equivalence classes of circle k-bundles are classified by the cohomology group

$$H^k(M, \underline{U(1)}) \simeq H^{k+1}(M, \mathbb{Z}).$$

Furthermore, suppose that we have a circle k-bundle $\alpha \colon M \to \mathbf{B}^k U(1)$ described by a cocycle $\alpha_{i_0 \dots i_k} \colon U_{i_0 \dots i_k} \to U(1)$. Then a *trivialization* of this circle k-bundle is given by a Cech coboundary $\beta_{i_0 \dots i_{k-1}}$ which satisfies the condition

$$\big(\delta\beta\big)_{i_0 \dots i_k} = \alpha_{i_0 \dots i_k}$$

In other words, β almost defines a circle $(k-1)$-bundle, up to a *twist* by α:

Corollary 2.2.20. *A trivialization of a circle k-bundle over M is given by an α-twisted circle $(k-1)$-bundle over M.*

This perspective of trivializations of higher circle bundles as twisted circle bundles of lower degree provides a way to think geometrically about these higher circle bundles.

Remark 2.2.21. We do not have a nice (fibrant) description of $\mathbf{B}^k U(1)$ in terms of simplicial presheaves on SmMfd. But we do know that there exists such a presentation, and that its value on a smooth manifold is a groupoid weakly equivalent to the groupoid given in terms of the above cocycles.

The presentation of stacks by simplicial presheaves is very convenient, in particular when we want to describe higher principal bundles in terms of cocycle data, as we did for circle n-bundles. However, most of the stacks we will consider have a much easier description: they can be presented by Lie groupoids.

2.2.3 Differentiable stacks

We conclude our discussion of stacks with the treatment of a particularly nice accessible of stacks: differentiable stacks. We will give an overview of the well-known presentation of the category of differentiable stacks in terms of Lie groupoids and bibundles beteen them (as reviewed for instance in [Blo08]). Both Lie groupoids and bibundles are described in terms of manifolds, and therefore allow us to apply classical differential geometric constructions to them. This makes them particularly useful in the construction of convolution algebras, as was realized in [MRW87].

In section 3.3.1, we will show how to functorially assign a twisted convolution algebra to a differentiable stack carrying a circle 2-bundle. Such stacks with a circle 2-bundle can be presented by Lie groupoid central extensions, as discussed for instance in [TXLG04]. We extend this description of a single stack with a circle 2-bundle to a presentation of the full (2, 1)-category of stacks with such a 2-bundle.

Any (Hausdorff, second countable) Lie groupoid X presents a smooth stack. Indeed, the Lie groupoid X gives rise to presheaf of groupoids on CartSp given by

$$U \mapsto \mathcal{C}^\infty(U, X)$$

This serves as a presentation of the stack associated to this presheaf of groupoids.

Definition 2.2.22. A differentiable stack is a stack which can be presented in the above way by a Lie groupoid.

The class of differentiable stacks contains many examples of interest: it contains the smooth manifolds, classifying stacks for smooth G-bundles (G a Lie group) and the Cech nerve associated to a (countable) cover of a smooth manifold. Note that a differentiable stack might be presented by two non-isomorphic Lie groupoids (in the category of Lie groupoids and smooth functors). For example, a manifold M almost never admits an equivalence by smooth functors to the Cech groupoid $\check{C}(U)$ for some cover $\{U_i\}$ of M. On the other hand, both present the same stack.

To get a better feeling for this, we give a slightly more abstract definition of a differentiable stack. As a first step, observe the following

Lemma 2.2.23. *Let X be a Lie groupoid and let $X_\bullet$ be the associated simplicial diagram of smooth manifolds (where X_n is the manifold of n composable morphisms). Then we obtain the stack presented by X by viewing $X_\bullet$ as a diagram in $\mathbf{H}$ and taking its homotopy colimit.*

Proof. This follows from the fact that the associated stack functor is a left adjoint, so preserves colimits. Colimits of presheaves of groupoids are computed objectwise, so we compute the homotopy colimit of the simplicial diagram $\mathcal{C}^\infty(U, X_\bullet)$. But the homotopy colimit of a simplicial diagram of sets is simply that simplicial set itself, which in this case is the nerve of the groupoid $\mathcal{C}^\infty(U, X)$. It follows that the homotopy colimit of $X_\bullet$ in $\mathbf{H}$ is the stack presented by the Lie groupoid X. □

In particular, if Y is a differentiable stack, then a presentation of Y by a Lie groupoid X gives rise to a map $X_0 \to Y$, which is called an *atlas* for Y. Different presentations of Y give rise to different atlasses.

In fact, we have that the groupoid X is uniquely defined by the atlas $X_0 \to Y$.

Lemma 2.2.24 ([Lur09a]). *Let X be a Lie groupoid presenting a stack Y and let $X_\bullet$ be the associated simplicial manifold. Then there is an equivalence of simplicial objects in $\mathbf{H}$ between $X_\bullet$ and the simplicial object whose value on $[n] \in \Delta$ is given by the n-fold homotopy pullback*

$$X_0 \times_Y X_0 \times_Y \ldots \times_Y X_0 \tag{2.3}$$

with the natural face and degeneracy maps.

Proof. This follows from the fact that $X_\bullet$ is a groupoid object in $\mathbf{H}$, which means that we can obtain X_n as an iterated pullback of the spaces X_1 over X_0 (see [Lur09a] for more details). Here we use that the Yoneda embedding and the associated stack functor preserve finite limits, so that the pullback in SmMfd agrees with the pullback of manifolds inside $\mathbf{H}$. The result is then the Giraud axiom for an ∞-topos that says that groupoid objects are effective. $\square$

This gives rise to the following abstract definition of a differentiable stack:

Definition 2.2.25. A differentiable stack X is a stack that admits an atlas $X_0 \to X$ from a smooth manifold X_0. An atlas for a stack is a map so that

- the resulting groupoid object 2.3 is a simplicial manifold, where the face maps are surjective submersions.
- the homotopy colimit of that groupoid object $X_\bullet$ is the stack X.

A choice of atlas gives rise to a presentation of the stack X by a Lie groupoid.

Remark 2.2.26. The first defining property of an atlas can be sharpened: an atlas $X_0 \to X$ of a differentiable stack X has the property that for any smooth manifold M and a map $M \to X$, the natural map

$$M \times_X X_0 \to M$$

from the homotopy pullback is a surjective submersion of smooth manifolds.

Remark 2.2.27. The second property states that the map $X_0 \to X$ is an effective epimorphism, which is just a categorical notion (see [Lur09a]). In this case, a map $X_0 \to X$ is an effective epi if it induces an epi of homotopy sheaves $\pi_0(X_0) \to \pi_0(X)$, i.e. the sheaves associated to the presheaf that takes the path components over each test space U (this uses that $\mathbf{H}$ is hypercomplete).

Example 2.2.28. Let M be a smooth manifold, viewed as an object in $\mathbf{H}$. Of course, the simplest atlas for M is the identity map $M \to M$. The corresponding groupoid presenting M is just the discrete groupoid $M \rightrightarrows M$.

On the other hand, a surjective submersion $Y \to M$ is also an atlas. The corresponding groupoid is the groupoid

$$Y \times_M Y \underset{p_2}{\overset{p_1}{\rightrightarrows}} Y$$

If $Y = \coprod U_i$ is the union of the opens of an open cover of M, then the resulting groupoid is the Cech groupoid $\check{C}(U)$.

Example 2.2.29. Let $\mathbf{B}G$ be the classifying stack of G-bundles, for G a Lie group. Then an atlas for $\mathbf{B}G$ is the inclusion $* \to \mathbf{B}G$ that classifies the trivial bundle over the point. The corresponding groupoid object is the Lie groupoid $*//G$, which indeed presents $\mathbf{B}G$ as we have seen in example 2.2.16.

To see that the point inclusion $* \to \mathbf{B}G$ gives the presentation of $\mathbf{B}G$ by $*//G$, note that we have a homotopy pullback diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \mathbf{B}G \end{tikzcd}$$

Indeed, a map from M into the homotopy pullback is given by the the two constant maps $M \to * \to \mathbf{B}G$ *and* a homotopy between these two maps. In other words, a map from M into the homotopy pullback is a gauge equivalence of the trivial G-bundle on M, which is precisely a G-valued function on M.

Remark 2.2.26 has a nice interpretation in this case, which is fundamental for the presentation of differentiable stacks we need later. Suppose $M \to \mathbf{B}G$ is a map into $\mathbf{B}G$, classifying a G-bundle P over the manifold M. A map from a test space U into the homotopy pullback $M \times_{\mathbf{B}G} *$ is given by a map of manifolds $U \xrightarrow{f} M$, together with a trivialization of the pullback bundle ξ. Such a trivialization is obtained by choosing for each point $x \in U$ a point $p \in P_{f(x)}$, where $P_{f(x)}$ is the fiber of P over $f(x) \in M$. In other words, a map from U into $M \times_{\mathbf{B}G} *$ is the same as a map into the total space P of the G-bundle, so we have a homotopy pullback diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & * \arrow[d] \\ M \arrow[r] & \mathbf{B}G \end{tikzcd}$$

The above example easily generalizes to any presentation of a differentiable stack X by a Lie groupoid $X_1 \rightrightarrows X_0$. A map from a manifold into X classifies a (left) groupoid principal bundle over M.

Definition 2.2.30 (cf. [MM03])**.** Let G be a Lie groupoid and P a smooth manifold. A left action of G on P is given by

- a submersion $\tau : P \to G_0$ (the 'target'), so that we can form the pullback $(G_1)_s \times_\tau P$. This space consists of arrows $g \in G_1$ whose source agrees with the 'target' of $p \in P$, so that the expression $g \cdot p$ is going to make sense.
- a map $(G_1)_s \times_\tau P \xrightarrow{\rho} P; \rho(g,p) = g \cdot p$ so that the natural equations hold: the target of $g \cdot p$ is the target of g, $g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p$ whenever this makes sense and $1 \cdot p = p$.

We will also say that G acts on P over the map $\tau : P \to G_0$.

A left action is principal if there is a surjective submersion $P \to M$ onto some manifold, such that the natural map

$$(G_1)_s \times_\tau P \xrightarrow{(p_2,\rho)} P \times_M P; (p,g) \mapsto (p, g \cdot p)$$

is a diffeomorphism. In this case $M := P/G$ is the quotient space, or orbit space.

Remark 2.2.31. If a groupoid G acts from the left on a manifold P, then we can form the *action groupoid* $P//G$. In a picture, it looks like

$$\begin{tikzcd} p \arrow[rr, "g"] \arrow[dr, "hg"'] & & g \cdot p \arrow[dl, "h"] \\ & (hg) \cdot p & \end{tikzcd}$$

for all $p \in P, g, h \in G_1$ for which this makes sense. In low degrees, this Lie groupoid looks like

$$(G_2)_s \times_\tau P \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; (G_1)_s \times_\tau P \; \underset{d_0 = \rho}{\overset{d_1 = p_1}{\rightrightarrows}} \; P$$

where the left three maps send a 2-simplex of the above form to the data of each of its three boundaries.

The following result shows that maps into differentiable stacks give rise to groupoid principal bundles. This is really a fundamental property of ∞-toposes, which holds in greater generality: essentially it is the principle that a map of stacks $Y \to X$ always classifies some kind of datum *over* Y.

Proposition 2.2.32 ([Sch12]). *Let X be a differentiable stack with atlas $X_0 \to X$ and suppose we have a homotopy pullback square*

$$\begin{tikzcd} P \arrow[r, "\tau"] \arrow[d] & X_0 \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

where P is a smooth manifold and Y is any stack. Then the groupoid $X_1 \rightrightarrows X_0$ acts from the left on P over the map τ. The map $P \to Y$ is the atlas for Y corresponding to the presentation of Y by the action groupoid defined above. In particular, Y is a differentiable stack.

Conversely, any manifold P carrying an action of the groupoid $X_1 \rightrightarrows X_0$ gives rise to a pullback diagram of the above form.

Proof. Pulling back along the source and target maps $X_1 \to X_0$, we obtain a cube where each face is a cartesian square

$$\begin{tikzcd} & (X_1)_s \times_\tau P \arrow[rr] \arrow[dl, "p_1"'] \arrow[dd, dotted, "\rho" near end] & & X_1 \arrow[dl, "s"'] \arrow[dd, "t"] \\ P \arrow[rr, "\tau" near end] \arrow[dd] & & X_0 \arrow[dd] & \\ & P \arrow[rr, dotted] \arrow[dl, dotted] & & X_0 \arrow[dl] \\ Y \arrow[rr] & & X & \end{tikzcd}$$

By pasting pullbacks, we see that this diagram commutes (up to homotopy). In particular, there are natural equivalences

$$(X_1)_s \times_\tau P \simeq P_\tau \times_t X_1 \simeq P \times_Y P$$

The first equivalence describes the equivalence between left and right X-actions by sending an arrow to its inverse. By choosing the presentation $(X_1)_s \times_\tau P$, the map ρ in the back gives a *left* X-action on P. The associativity of this action will follow by pulling back along each of the three maps $X_2 \to X_1$. Continuing this, we produce the simplicial diagram describing the action groupoid from the remark. In particular, we obtain the left X-action from the claim.

The second equivalence shows that the arrow space in the action groupoid, $(X_1)_s \times_\tau P$, is equivalent to the pullback $P \times_Y P$. This continues this way, so the space of n composable arrows in the action groupoid is equivalent to the $n+1$-fold homotopy pullback $P \times_Y P \times_Y \ldots \times_Y P$. It follows that the action groupoid presents the stack

$$\operatorname*{colim}_n P^{\times_Y n}$$

Now effective epimorphisms are stable under pullback [Lur09a]. Since the atlas $X_0 \to X$ is an effective epi, it follows that $P \to Y$ is an effective epi as well. By the characterization of an effective epi used in definition 2.2.25, it follows that the above homotopy colimit is Y. This proves the second claim.

Finally, if P is any smooth manifold carrying a left X-action, then we have a diagram of groupoid objects

$$\begin{tikzcd} (X_2)_s \times_\tau P \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & (X_1)_s \times_\tau P \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & P \arrow[r] \arrow[d, "\sigma"] & Y \arrow[d, dotted, "f"] \\ X_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X_1 \arrow[r, shift left] \arrow[r, shift right] & X_0 \arrow[r, "\pi"'] & X \end{tikzcd}$$

which gives rise to a map f of homotopy colimits. To check that the right hand square is homotopy cartesian, we use the Giraud axiom that colimits are preserved by pullback [Lur09a]. A simple computation shows that pulling Y back along the atlas π gives

$$\pi^* Y \simeq \operatorname*{colim}_n \pi^*((X_n)_s \times_\tau P) \simeq \operatorname*{colim}_n (X_{n+1})_s \times_\tau P \simeq P.$$

This shows the converse statement. □

We thus see that morphisms of differentiable stacks give rise to groupoid bundles, once we have chosen atlases. This motivates the following presentation of DiffStack:

Definition 2.2.33. Let LieGpd_{bib} be the 2-category defined as follows:

- objects are Lie groupoids
- for G and H Lie groupoids, a morphism $G \to H$ is given by a *bibundle* from G to H, i.e. a smooth manifold P together with submersions

$$G_0 \xleftarrow{\sigma} P \xrightarrow{\tau} H_0$$

  which are part of the data of a right action of G on P and a left action of H on P. Moreover, these actions are required to commute and the action of H on P is required to be principal, with quotient map $\sigma : P \to G_0$.

- given bibundles $G_0 \leftarrow P \to H_0$ and $H_0 \leftarrow Q \to K_0$, their composition is given by the bibundle

$$\begin{tikzcd}
 & & P \times_H Q \arrow[dl] \arrow[dr] \arrow[ddll, bend right, "\sigma"'] \arrow[ddrr, bend left, "\tau"] & & \\
 & P \arrow[dl, "\sigma_1"'] \arrow[dr, "\tau_1"] & & Q \arrow[dl, "\sigma_2"'] \arrow[dr, "\tau_2"] & \\
G_0 & & H_0 & & K_0
\end{tikzcd}$$

  where $P \times_H Q$ is obtained by forming the pullback $P \times_{H_0} Q$ and dividing our the diagonal H-action (this is well defined since the left H-action is principal).

- given two morphisms $G \xrightarrow{P,Q} H$, a 2-morphism from P to Q is a smooth map intertwining both of the actions. Note that by principality of the H-action, this smooth map has an inverse.

It is well-known that this bicategory indeed presents the bicategory of differentiable stacks:

Proposition 2.2.34 ([Blo08], [Car11])**.** *There is an equivalence of* $(2,1)$*-categories*

$$\mathrm{LieGpd}_{bib} \xrightarrow{\sim} \mathrm{DiffStack}$$

that sends each Lie groupoid to the associated stack.

Proof (sketch). Let G, H be differentiable stacks with atlasses G_0, H_0. Given a map of stacks $G \to H$, we can consider the diagram consisting of two homotopy cartesian squares

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "\sigma"'] \arrow[rr, bend left, "\tau"] & P//G \arrow[r] \arrow[d] & H_0 \arrow[d] \\
G_0 \arrow[r] & G \arrow[r] & H
\end{tikzcd}$$

Then P is a smooth manifold and the map $P \xrightarrow{\sigma} G_0$ is the quotient of a (say left) principal H-action on P over τ. On the other hand, $P \to P//G$ is the quotient of a (say right) G action on P over σ. Moreover, the quotient $P//G$ by the G-action still carries an H-action, due to the right square being homotopy cartesian. In other words, the G- and H-actions on P commute and turn P into a bibundle from the Lie groupoid $G_1 \rightrightarrows G_0$ to the Lie groupoid $H_1 \rightrightarrows H_0$.

Conversely, starting from a bibundle P, we know the left pullback square exists (from the G-action). Since the H-action commuted with the G-action, we see that $P//G \to G$ is a principal H-bundle, whose pullback is the principal H-bundle $P \to G_0$. Hence we can form the right pullback square, which includes the map $G \to H$ we wanted. □

Before discussing circle 2-bundles on differentiable stacks, let us conclude with the definition of a proper map of stacks. We will need this properness assumption to be able to pull back the functions that form the convolution algebras considered in section 3.3.1.

Definition 2.2.35 ([AG06])**.** Let $G \xrightarrow{f} H$ be a morphism of differentiable stacks. We say f is proper if for any two atlases G_0, H_0, the right G-action on $G_0 \times_H H_0$ is a proper action and the map $\tau\colon G_0 \times_H H_0 \to H_0$ has the property that for any compact $K \subseteq H_0$, there is a compact subset of $G_0 \times_H H_0$ whose G-orbit constains $\tau^{-1}(K)$.

By definition, such maps are presented by *proper* bibundles. If M and N are smooth manifolds, then a map between them is proper in the above sense precisely if it is proper in the traditional sense.

Presentation of differentiable stacks with circle 2-bundle

Definition 2.2.36. Let $\mathrm{DiffStack}_{/\mathbf{B}^2U(1)}$ be the full subcategory of the slice ∞-category $\mathbf{H}_{/\mathbf{B}^2U(1)}$ on those morphisms $\alpha\colon X \to \mathbf{B}^2U(1)$ for which X is a differentiable stack.

Remark 2.2.37. $\mathrm{DiffStack}_{/\mathbf{B}^2U(1)}$ is a (2,1)-category: given two maps $\alpha : X \to \mathbf{B}^2U(1)$ and $\beta : Y \to \mathbf{B}^2U(1)$, we can compute the (derived) mapping space between them as the homotopy fiber [Lur09a]

$$\begin{tikzcd}
\mathbf{H}_{/\mathbf{B}^2U(1)}(\alpha,\beta) \arrow[r] \arrow[d] & * \arrow[d, "\alpha"] \\
\mathbf{H}(X,Y) \arrow[r, "\beta_*"'] & \mathbf{H}(X,\mathbf{B}^2U(1))
\end{tikzcd}$$

Since differentiable stacks are 1-truncated (they assign ordinary groupoids to each manifold) and $\mathbf{B}^2U(1)$ is 2-truncated, the long exact sequence of homotopy groups associated to such a fiber sequence shows that $\mathbf{H}_{/\mathbf{B}^2U(1)}(\alpha,\beta)$ is 1-truncated.

Our first aim is to obtain a presentation of a circle 2-bundle over a differentiable stack which only involves data given by Lie groupoids. If the stack X is presented by a Lie groupoid $X_1 \rightrightarrows X_0$, we have seen that we can realize it as the homotopy colimit of the simplicial manifold $X_\bullet$ in $\mathbf{H}$. Now we have that

$$\mathbf{H}(X,\mathbf{B}^2U(1)) \simeq \mathbf{H}(\operatorname{colim} X_\bullet, \mathbf{B}^2U(1)) \simeq \lim \mathbf{H}(X_\bullet, \mathbf{B}^2U(1))$$

is the homotopy limit of a simplicial diagram of ∞-groupoids. This gives us a concrete description of a map $X \xrightarrow{\alpha} \mathbf{B}^2U(1)$: it is given by

- a map $X_0 \xrightarrow{\beta} \mathbf{B}^2U(1)$
- an equivalence ξ between the pullbacks $d_0^*\beta$ and $d_1^*\beta$ over X_1
- a 2-cell ϕ between the equivalences $d_0^*\xi \circ d_2^*\xi$ and $d_1^*\xi$ over X_2
- a 3-cell in between the pullbacks $d_i^*\phi$ over X_3. Since $\mathbf{B}^2U(1)$ is 2-truncated, such a 3-cell is unique up to a contractible space of choices, if it exists. All higher equivalences exist and are unique up to a contractible space of choices.

Moreover, these maps have to satisfy certain unitality constraints. For example, the equivalence ξ should be trivial when restricted to the identity morphisms.

The map $\beta\colon X_0 \to \mathbf{B}^2U(1)$ classifies a circle 2-bundle over X_0, for which we do not have a much better description then the cocycle model discussed in section 2.2.2. If we assume that β is trivial, then the above description of a circle 2-bundle $\alpha : X \to \mathbf{B}^2U(1)$ simplifies drastically: it is given by a circle bundle $p\colon X_1^\alpha \to X_1$, which is trivial over the unit morphisms, together with $U(1)$-equivariant maps between its fibers

$$\phi_{g,h} : p^{-1}(g) \times p^{-1}(h) \to p^{-1}(gh)$$

whenever the composition gh is defined. These maps depend smoothly on g and h, and satisfy $\phi_{1,g}(1,\lambda) = \phi_{g,1}(\lambda,1) = \lambda$ for all $\lambda \in p^{-1}(g)$ (this makes sense since X_1^α is trivial over the unit morphisms). Finally, they have to satisfy the cocycle constraint

$$\phi_{gh,k}\phi_{g,h} = \phi_{g,hk}\phi_{h,k} \colon p^{-1}(g) \times p^{-1}(h) \times p^{-1}(k) \to p^{-1}(ghk)$$

Following [TXLG04], we observe that this has a very simple interpretation in terms of Lie groupoids:

Lemma 2.2.38. *The above data precisely determines a $\mathbf{B}U(1)$-central extension of the Lie groupoid $X = \left[\, X_1 \rightrightarrows X_0 \,\right]$.*

Proof. Let $X_1^\alpha \xrightarrow{p} X_1$ be the given circle bundle. We write elements in X_1^α as pairs (g,λ), with $g \in X_1$ and $\lambda \in p^{-1}(g)$. The maps $d_1 p, d_0 p : X_1^\alpha \to X_0$ give a source and target map (both are surjective submersions). Given a pair (g,λ), (h,μ) such that $d_1(g) = d_0(h)$, we define its composition by $\big(gh, \phi_{g,h}(\lambda,\mu)\big)$. This is associative and unital by the cocycle conditions, so we find a groupoid X^α together with a smooth functor $X^\alpha \to X$.

This map fits in an exact sequence of groupoids

$$\begin{tikzcd}
X_0 \times U(1) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & X_1^\alpha \arrow[r, "p"] \arrow[d, shift left] \arrow[d, shift right] & X_1 \arrow[d, shift left] \arrow[d, shift right] \\
X_0 \arrow[r, equal] & X_0 \arrow[r, equal] & X_0
\end{tikzcd}$$

where the left map is the restriction of X_1^α to the identity arrows in X_1. The left groupoid $X_0 \times \mathbf{B}U(1)$ is central in X^α, so this indeed gives a central extension of groupoids. Conversely, given a diagram as above, the inclusion of $X_0 \times U(1)$ in the space of arrows X_α^1 defines an $U(1)$-action on that space of arrows, respecting the groupoid structure, whose quotient is X_1. This gives the bundle $X_1^\alpha \to X_1$ and the maps $\phi_{g,h}$. □

We combine this observation with the bibundle picture from the previous section to obtain a bicategory of Lie groupoids with a $\mathbf{B}U(1)$-central extension.

Definition 2.2.39. Let $\mathrm{LieGpd}_{/\mathbf{B}^2U(1)}$ be the $(2,1)$-category whose

- objects are $\mathbf{B}U(1)$-central extensions of Lie groupoids $G^\alpha \to G$.
- a morphism $G^\alpha \to H^\beta$ is given by a bibundle P, such that for any $p \in P, g \in G^\alpha, h \in H^\beta$ and $\lambda \in U(1)$ we have:
$$(h \cdot \lambda) \cdot p \cdot g = h \cdot p \cdot (g \cdot \lambda)$$
whenever the composition makes sense.
Stated in terms of central extensions: the central copy of $\mathbf{B}U(1)$ in both G^α and H^β acts *the same* on P.
- a morphism $P \to Q$ is just a smooth map intertwining the actions of G^α and H^β (hence in particular the $U(1)$-action).

Given two morphisms $G \xrightarrow{P} H \xrightarrow{Q} K$, we define their composition as the pullback of the bibundles.

Remark 2.2.40. Given such a bibundle P, the quotient of P by the principal $U(1)$-action gives a bibundle $P/U(1)$ from G to H. The bibundle P describes the whole diagram

$$\begin{tikzcd}
G \arrow[rr] \arrow[dr, "\alpha"'] & \arrow[d, phantom, "\Leftarrow"] & H \arrow[dl, "\beta"] \\
& \mathbf{B}^2U(1) &
\end{tikzcd} \tag{2.4}$$

while the bundle $P/U(1)$ only describes the top morphism.

Theorem 2.2.41. *There is an equivalence of* $(2,1)$*-categories*

$$\mathrm{LieGpd}_{/\mathbf{B}^2U(1)} \xrightarrow{\sim} \mathrm{DiffStack}_{/\mathbf{B}^2U(1)}$$

Proof (sketch). We have already seen that the central extensions $G^\alpha \to G$ of Lie groupoids correspond to circle 2-bundles on G whose restriction to the object space G_0 gives a trivial circle 2-bundle. On the other hand, any differentiable stack can be presented by a Lie groupoid whose space of objects is a disjoint union of cartesian spaces: indeed, if $X_0 \to X$ is an atlas, then for any open cover $\coprod U_i \to X_0$ we have that the resulting map $\coprod U_i \to X$ is an atlas for the stack X. Hence any circle 2-bundle over a stack can be presented by a central extension of some Lie groupoid presenting that stack.

Now suppose G^α, H^β are central extensions of groupoids presenting the maps $G \xrightarrow{\alpha} \mathbf{B}^2U(1)$ and $H \xrightarrow{\beta} \mathbf{B}^2U(1)$. A diagram of the form 2.4 corresponds precisely to a $\mathbf{B}U(1)$-equivariant map from G^α to H^β. Such a map is presented by a bibundle on which the central copy of $\mathbf{B}U(1)$ in both G^α and H^β acts the same. ☐

Finally, a morphism in $\mathrm{DiffStack}_{/\mathbf{B}^2U(1)}$ is called proper if the underlying map of differentiable stacks is proper. Since $U(1)$ is compact, this implies that the bibundle between the central extensions is also a proper bibundle.

2.3 Prequantum field theory via correspondences

The ∞-topos $\mathbf{H}$ of smooth ∞-groupoids provides a suitable context to describe smooth spaces carrying gauge equivalences. We now show how to realize a prequantum field theory using correspondences in $\mathbf{H}$, or rather in a slice thereof. We will first argue in section 2.3.1 how the field content of an n-dimensional prequantum field theory is naturally encoded into a monoidal functor

$$\mathrm{Bord}_n \xrightarrow{\mathrm{Fields}} \mathrm{Corr}_n\big(\mathbf{H}\big)$$

into the (∞,n)-category of n-fold correspondences in $\mathbf{H}$. In section 2.3.2 we add an extended action functional by choosing a lift of this functor to a monoidal functor

$$\mathrm{Bord}_n \xrightarrow{\mathrm{Fields}} \mathrm{Corr}_n\big(\mathbf{H}_{/\mathbf{B}^nU(1)}\big)$$

Such a functor describes a (topological) *prequantum field theory.*

The cobordism hypothesis gives an easy classification of such functors: each local pQFT is classified by a fully dualizable object in the (∞,n)-category of n-fold correspondences. We can use this simple description of pQFTs by a single object to add extra structure to our field theory. In particular, we can add *boundary theories* and *defects* along the lines of section 4.3 in [Lur09b]. This will be done in section 2.3.3

2.3.1 Prequantum fields

Given a manifold Σ, a prequantum field theory gives a stack of fields on Σ, denoted $\mathrm{Fields}(\Sigma)$. Since fields are supposed to be local entities, we can restrict them to subsets of Σ. In particular, when Σ has a boundary $\partial\Sigma = \partial\Sigma_{\mathrm{in}} \coprod \partial\Sigma_{\mathrm{out}}$ decomposed into an ingoing and outgoing boundary, then we have restriction maps

$$\begin{tikzcd} & \mathrm{Fields}(\Sigma) \arrow[dl, "|_{\partial\Sigma_{\mathrm{in}}}"'] \arrow[dr, "|_{\partial\Sigma_{\mathrm{out}}}"] & \\ \mathrm{Fields}(\partial_{\mathrm{in}}\Sigma) & & \mathrm{Fields}(\partial_{\mathrm{out}}\Sigma) \end{tikzcd}$$

A cobordism $\Sigma : \partial_{\mathrm{in}}\Sigma \to \partial_{\mathrm{out}}\Sigma$ thus gives rise to a correspondence, or span, in $\mathbf{H}$. The top space in such a correspondence gives the space of all trajectories from the ingoing to the outgoing boundary: it gives all possible ways for a field to evolve over the manifold Σ.

A composition of cobordisms gives rise to the composition of correspondences, which proceeds by pullback. In other words, we have

$$\begin{tikzcd}[column sep=tiny]
 & & \mathrm{Fields}(\Sigma_1 \coprod_{\partial_{\mathrm{out}}\Sigma^1} \Sigma^2) \arrow[dl] \arrow[dr] & & \\
 & \mathrm{Fields}(\Sigma^1) \arrow[dl] \arrow[dr] & & \mathrm{Fields}(\Sigma^2) \arrow[dl] \arrow[dr] & \\
\mathrm{Fields}(\partial_{\mathrm{in}}\Sigma^1) & & \mathrm{Fields}(\partial_{\mathrm{out}}\Sigma^1) = \mathrm{Fields}(\partial_{\mathrm{in}}\Sigma^2) & & \mathrm{Fields}(\partial_{\mathrm{out}}\Sigma^2)
\end{tikzcd}$$

$$\partial_{\mathrm{in}} u\Sigma^1 \xrightarrow{\Sigma^1} \partial_{\mathrm{out}}\Sigma^1 = \partial_{\mathrm{in}}\Sigma^2 \xrightarrow{\Sigma^2} \partial_{\mathrm{out}}\Sigma^2$$

where the top square is (homotopy) cartesian. Field configurations on $\Sigma = \Sigma^1 \coprod_{\partial_{\mathrm{out}}\Sigma^1} \Sigma^2$ are obtained from fields on Σ^1 and Σ^2 by providing an equivalence between their restrictions to the boundary along which we glue.

Remark 2.3.1. It is not obvious that one wants to impose the condition $\mathrm{Fields}(\Sigma^1 \coprod_{\partial_{\mathrm{out}}\Sigma^1} \Sigma^2) \simeq \mathrm{Fields}(\Sigma^1) \times_{\mathrm{Fields}(\partial\Sigma_{\mathrm{out}})} \mathrm{Fields}(\Sigma^2)$. To glue fields *smoothly*, we should not only give equivalences over the boundary, but also over an open neighbourhood of the boundary. Since the formalism using spans does not capture this, the resulting fields will not depend smoothly on the manifold Σ. On the other hand, the spaces of fields themselves still carry a nontrivial smooth structure.

On the other hand, the use of spans results in finiteness properties of the prequantum field theory that make it much more tractable. For example, we cannot describe the pQFT where $\mathrm{Fields}(\Sigma)$ is the space of all G-bundles with connection over Σ: we cannot glue two such bundles if we have an equivalence over the boundary. Instead, we can describe the pQFT that describes flat G-bundles over Σ, which is a good approximation of the pQFT of all G-bundles. But where the moduli of all G-bundles are usually infinite dimensional, the moduli of flat bundles are finite dimensional stacks (compare thediscussion in [FHT10]).

Now suppose Σ describes a cobordism from Σ_{left} to Σ_{right}, where both $\Sigma_{\mathrm{left}}, \Sigma_{\mathrm{right}}$ have a boundary. A good example is the square, which gives a cobordism from its left side to its right side. Σ now has another two boundary components, which we call Σ_{top} and Σ_{bottom}, which give the cobordism between the boundaries of Σ_{left} and Σ_{right}.

Then Σ_{left} and Σ_{right} gives rise to spans, and the whole cobordism Σ gives rise to a span of spans

$$\begin{tikzcd}
\mathrm{Fields}(\partial_{\mathrm{in}}\Sigma_{\mathrm{left}}) & \mathrm{Fields}(\Sigma_{\mathrm{top}}) \arrow[l] \arrow[r] & \mathrm{Fields}(\partial_{\mathrm{in}}\Sigma_{\mathrm{right}}) \\
\mathrm{Fields}(\Sigma_{\mathrm{left}}) \arrow[u] \arrow[d] & \mathrm{Fields}(\Sigma) \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \mathrm{Fields}(\Sigma_{\mathrm{right}}) \arrow[u] \arrow[d] \\
\mathrm{Fields}(\partial_{\mathrm{out}}\Sigma_{\mathrm{left}}) & \mathrm{Fields}(\Sigma_{\mathrm{bottom}}) \arrow[l] \arrow[r] & \mathrm{Fields}(\partial_{\mathrm{out}}\Sigma_{\mathrm{right}})
\end{tikzcd}$$

We can repeat this until we arive at the point. In the end this means that the fields of an n-dimensional manifold (possibly with corners etc.) form the center of an n-fold correspondence.

The empty manifold carries a unique field, so $\mathrm{Fields}(\emptyset) \simeq *$. If we view $\emptyset$ as an n-dimensional manifold, this means that it gives rise to the n-fold span consisting of points only. Consequently, a *closed* n-dimensional manifold Σ gives rise to an n-fold span in which almost all objects are points, except the tip which is Fields(Σ). For example, a closed 2-dimensional manifold Σ gives rise to a diagram

$$\begin{tikzcd}
* & * \arrow[l] \arrow[r] & * \\
* \arrow[u] \arrow[d] & \mathrm{Fields}(\Sigma) \arrow[l] \arrow[r] \arrow[u] \arrow[d] & * \arrow[u] \arrow[d] \\
* & * \arrow[l] \arrow[r] & *
\end{tikzcd}$$

since there are no boundaries.

Finally, if we have two cobordisms Σ_1 and Σ_2, then of course $\mathrm{Fields}(\Sigma_1 \coprod \Sigma_2) \simeq \mathrm{Fields}(\Sigma_1) \times \mathrm{Fields}(\Sigma_2)$, since the fields on the two components are completely unrelated. The discussion is now summarized in the following two definitions:

Definition 2.3.2. Let Corr_k denote the fullsubcomplex of $[k] \times [k]$ on the pairs (i, j) such that $i+j \leq k$. For example, Corr_1 is given by

$$(0,1) \longleftarrow (0,0) \longrightarrow (1,0)$$

and Corr_2 is given by

$$\begin{tikzcd}
 & & (0,0) \arrow[dl] \arrow[dr] & & \\
 & (0,1) \arrow[dl] \arrow[dr] & & (1,0) \arrow[dl] \arrow[dr] & \\
(0,2) & & (1,1) & & (2,0)
\end{tikzcd}$$

The maps $\partial_i \times \partial_{k-i} : [k-1] \times [k-1] \to [k] \times [k]$ for $i = 0, ...k$ and the similar degeneracies turn this into a cosimplicial object in **sSet**.

For $\mathbf{C}$ an $(\infty, 1)$-category with finite limits, consider the n-fold simplicial space $\infty\mathrm{Cat}(\mathrm{Corr}_\bullet^{\times n}, \mathbf{C})$. This assigns to each $[k_1], ..., [k_n]$ the ∞-groupoid of functors

$$\mathrm{Corr}_{k_1} \times ... \times \mathrm{Corr}_{k_n} \to \mathbf{C}$$

The associated (∞, n)-category (modeled by a complete Segal space) is the (∞, n)-category $\mathrm{Corr}_n(\mathbf{C})$ of n-fold correspondences, or spans, in $\mathbf{C}$.

If the category $\mathbf{C}$ carries a symmetric monoidal structure, then the category of n-fold spans has an induced symmetric monoidal structure:

Proposition 2.3.3 ([Lur09b], remark 3.2.3)**.** *A symmetric monoidal structure on* $\mathbf{C}$ *naturally induces a symmetric monoidal structure on* $\mathrm{Corr}_n(\mathbf{C})$*, where the tensor product of a k-fold span is obtained by applying the tnesor product in catC to the whole diagram. In particular, the tensor product of objects is simply the tensor product in* $\mathbf{C}$*. We will denote by* $\mathrm{Corr}_n(\mathbf{C}^{\otimes})$ *the corresponding symmetric monoidal* (∞, n)*-category.*

Definition 2.3.4. Since $\mathbf{C}$ has finite limits, the cartesian product gives a symmetric monoidal structure on $\mathbf{C}$. In this case we implicitly assume $\mathrm{Corr}_n(\mathbf{C})$ to carry the induced monoidal structure.

Definition 2.3.5. Let Bord_n be the symmetric monoidal (∞, n)-category of framed $\leq n$-dimensional cobordisms. A local prequantum field in n dimensions is a symmetric monoidal (∞, n)-functor

$$\mathrm{Bord}_n \xrightarrow{\mathrm{Fields}} \mathrm{Corr}_n(\mathbf{H})$$

with $\mathbf{H} = \mathrm{Sm}\infty\mathrm{Gpd}$.

The following result follows immediately from the cobordism hypothesis [Lur09b]:

Proposition 2.3.6. *A local prequantum field is uniquely determined by a* moduli stack of fields **Fields** $\in$ **H**.

Proof. By the cobordism hypothesis, Bord_n is the free symmetric monoidal (∞, n)-category with all duals generated by a single object $*$. As such, the value of the monoidal functor Fields is determined by its value on the point

$$\mathbf{Fields} := \mathrm{Fields}(*)$$

which is required to be fully dualizable in $\mathrm{Corr}_n(\mathbf{H})$. The result now follows from the next lemma. ☐

Lemma 2.3.7 ([FSS]). *Let* $\mathbf{C}$ *be an* ∞*-category with finite limits (with the cartesian monoidal structure). Then every object in* $\mathrm{Corr}_n(\mathbf{C})$ *is fully dualizable. In fact, every object has all duals.*

Proof. This is essentially remark 3.2.3 in [Lur09b]. For example, the evaluation and coevaluation on an object X are given by

$$\begin{tikzcd} \mathrm{coev}: & * & X \arrow[l] \arrow[r, "\Delta"] & X \times X \\ \mathrm{ev}: & X \times X & X \arrow[l, "\Delta"'] \arrow[r] & * \end{tikzcd}$$

A quick computation shows that $(1 \times \mathrm{ev})(\mathrm{coev} \times 1)$ is given by the identity correspondence

$$\begin{tikzcd} & & X \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & & \\ & X \times X \arrow[dl, "p_2"'] \arrow[dr, "\Delta\times 1"] & & X \times X \arrow[dl, "1\times\Delta"'] \arrow[dr, "p_1"] & \\ X & & X \times X \times X & & X \end{tikzcd}$$

The other zigzag law follows in a similar way, as do the higher dualizability conditions (see [FSS] for more details). ☐

If we now consider the composite $\mathrm{ev} \circ \mathrm{coev}$, which should correspond to the circle, we see that it is given by the correspondence

$$\begin{tikzcd} & & {[\Pi(S^1), X]} \arrow[dl] \arrow[dr] & & \\ & X \arrow[dl] \arrow[dr, "\Delta"] & & X \arrow[dl, "\Delta"'] \arrow[dr] & \\ * & & X \times X & & * \end{tikzcd}$$

In other words, we see that the span describing the circle gives has as its tip the mapping stack from the homotopy type of the circle (viewed as a geometrically discrete stack) into X. In fact, an inductive argument (see [FSS] for details) shows that this holds for spheres and discs as well. Using a handle decomposition of a cobordism, we then obtain:

Proposition 2.3.8 ([FSS]). *Any local pQFT in the above sense is a topological sigma-model: for any manifold* Σ*, we have that*

$$\mathrm{Fields}(\Sigma) \simeq [\Pi(\Sigma), \mathbf{Fields}]$$

is the stack of maps from the homotopy type of Σ *into the stack* **Fields** *(viewed as the tip of a k-fold correspondence).*

We thus obtain the fields content of our field theory. Next, we need an (exponentiated) action functional $\exp(iS)$ to determine the evolution of the fields.

2.3.2 Extended action functional

An action functional is an assignment of phases to field configurations. One uses the action functional to describe the dynamics of the field configurations. At the classical level, we obtain classical fields as those field configurations where the action functional is stationary. On the other hand, at the quantum level the phases that the action assigns to fields are supposed to add up, together describing a quantum propagator.

We already saw in the introduction that the locality condition forces the action functional to be 'extended': we do not only assign $U(1)$-phases to fields over an n-dimensional manifold, but we also have to assign higher phases to fields on lower dimensional manifolds. These phases are given by maps into the stacky refinements of the circle group, the stacks $\mathbf{B}^k U(1)$ of circle k-bundles. Recall that an n-dimensional cobordism gave rise to a diagram of the form

$$\begin{tikzcd}
 & \mathrm{Fields}(\Sigma) \arrow[dl, "|_{\partial\Sigma_{\mathrm{in}}}"'] \arrow[dr, "|_{\partial\Sigma_{\mathrm{out}}}"] & \\
\mathrm{Fields}(\partial\Sigma_{\mathrm{in}}) \arrow[dr, "\alpha"'] \arrow[rr, phantom, "\Rightarrow"] & & \mathrm{Fields}(\partial\Sigma_{\mathrm{out}}) \arrow[dl, "\beta"] \\
 & \mathbf{B}U(1) &
\end{tikzcd}$$

In case that the boundary of Σ was empty, this gave a $U(1)$-valued function on $\mathrm{Fields}(\Sigma)$, the traditional action functional. The same thing happens in lower degrees: if the boundary $\partial\Sigma_{\mathrm{in}}$ has itself a boundary, then it is given by a span itself. In that case, $\mathrm{Fields}(\partial\Sigma_{\mathrm{in}})$ does not come equipped with a map into $\mathbf{B}U(1)$, but rather fits in a diagram like the above one, where the $\mathbf{B}U(1)$-valued function is replaced by an equivalence between two maps into $\mathbf{B}^2U(1)$.

In turn, these maps into $\mathbf{B}^2U(1)$ should really be interpreted as equivalences between $\mathbf{B}^3U(1)$-valued maps, etcetera. In the end we end up with 0-dimensional manifolds which come equipped with maps to $\mathbf{B}^nU(1)$. The action functionals in higher dimension arise from correspondence diagrams in the slice over $\mathbf{B}^nU(1)$: the higher cells, like the 2-cell in the above diagram, describe the action functionals as higher gauge equivalences of certain maps into $\mathbf{B}^nU(1)$.

Summarizing, we expect a prequantum field together with an action function to be a functor into the category $\mathrm{Corr}_n\big(\mathbf{H}_{\mathbf{B}^nU(1)}\big)$ of n-fold correspondences in the slice ∞-topos over $\mathbf{B}^nU(1)$. Moreover, suppose we have a field on the disjoint union of two cobordisms $\Sigma_1 \coprod \Sigma_2$. Since the two fields on the two components are completely independent, we expect that the action functional is given by the product

$$\exp(iS)\big(\phi_1 \coprod \phi_2\big) = \exp(iS)\big(\phi_1\big) \cdot \exp(iS)\big(\phi_2\big)$$

In other words, the disjoint union of cobordisms $\Sigma_1 \coprod \Sigma_2$ gives rise to a product action functional, given by

$$\mathrm{Fields}\big(\Sigma_1 \textstyle\coprod \Sigma_2\big) = \mathrm{Fields}\big(\Sigma_1\big) \times \mathrm{Fields}\big(\Sigma_2\big) \xrightarrow{\exp(iS)\times\exp(iS)} \mathbf{B}^nU(1) \times \mathbf{B}^nU(1) \xrightarrow{\mu} \mathbf{B}^nU(1)$$

where μ is the multiplication on the higher group $\mathbf{B}^nU(1)$. We thus make use of an additional monoidal structure on the slice topos (such a topos with an extra monoidal structure has also been called a 'quantum topos'):

Proposition 2.3.9 ([FSS])**.** *Let* $\mathbf{C}$ *an* ∞*-category and* E *an* E_∞*-algebra in* $\mathbf{C}$ *(with respect to the cartesian monoidal structure). The the slice* $\mathbf{C}_{/E}$ *has a natural symmetric monoidal structure, given by*

$$\begin{pmatrix} X \\ \downarrow \\ E \end{pmatrix} \otimes \begin{pmatrix} Y \\ \downarrow \\ E \end{pmatrix} = \begin{pmatrix} X \times Y \\ \downarrow \\ E \times E \\ \downarrow{\scriptstyle \mu} \\ E \end{pmatrix}$$

where the latter map is the multiplication in E.

Remark 2.3.10. The E_∞-structure on the object E makes sure that thre resulting monoidal structure is symmetric.

Definition 2.3.11. For E an E_∞-monoid in $\mathbf{C}$, we let $\mathrm{Corr}_n(\mathbf{C}, E)$ denote the (∞, n)-category $\mathrm{Corr}_n(\mathbf{C}_{/E})$ with the symmetric monoidal structure induced by the above one.

Observe that there is a natural monoidal functor

$$\mathrm{Corr}_n(\mathbf{C}, E) \longrightarrow \mathrm{Corr}_n(\mathbf{C})$$

which forgets the map into E. The above discussion then motivates the following definition of a prequantum field theory:

Definition 2.3.12. Let

$$\mathrm{Bord}_n \xrightarrow{\mathrm{Fields}} \mathrm{Corr}_n(\mathbf{H})$$

be a local prequantum field in n dimensions. An action functional on this field is a lift of monoidal functors

$$\begin{tikzcd} & \mathrm{Corr}_n(\mathbf{H}, \mathbf{B}^n U(1)) \arrow[d] \\ \mathrm{Bord}_n \arrow[ur, dotted, "\exp(iS)"] \arrow[r, "\mathrm{Fields}"'] & \mathrm{Corr}_n(\mathbf{H}) \end{tikzcd}$$

An n-dimensional prequantum field theory is a choice of prequantum field and action functional like this.

Proposition 2.3.13. *An n-dimensional prequantum field theory is uniquely determined by a map*

$$\mathbf{Fields} \to \mathbf{B}^n U(1)$$

This follows immediately from the cobordism hypothesis, together with the following result

Lemma 2.3.14. *If E is a groupal E_∞-algebra in $\mathbf{C}$, then every object in $\mathrm{Corr}_n(\mathbf{C}, E)$ is fully dualizable.*

Proof. The same argument as for lemma 2.3.7 applies. In this case, the dual of a map $\alpha \colon X \to E$ is given by the inverse of that map with respect to the group structure on E, which we denote by $-\alpha \colon X \to E$. $\square$

The value of the pQFT on a k-dimensional closed manifold Σ is given by

$$\big[\Pi(\Sigma), \mathbf{Fields}\big] \to \mathbf{B}^{n-k} U(1)$$

where the circle $(n-k)$-bundle over the stack of fields is determined by gluing higher gauge transformations. Essentially this means that we form the map $\big[\Pi(\Sigma), \mathbf{Fields}\big] \to \big[\Pi(\Sigma), \mathbf{B}^n U(1)\big]$ and take higher 'traces' of the circle n-bundle over $\Pi(\Sigma)$.

Remark 2.3.15. Actually, this can be made more precise: the structure of a cohesive ∞-topos on $\mathbf{H}$ allows us to identify a map from a cartesian space $U \to \big[\Pi(\Sigma), \mathbf{B}^n U(1)\big]$ with a circle n-bundle over $U \times \Sigma$, which is flat along Σ. Taking the holonomy of this circle n-bundle along Σ (where the bundle carries a higher flat connection) gives rise to a map $U \to \mathbf{B}^{n-k} U(1)$. This is natural in U, so we obtain a holonomy map

$$\big[\Pi(\Sigma), \mathbf{B}^n U(1)\big] \xrightarrow{\int_\Sigma -} \mathbf{B}^{n-k} U(1).$$

then the composition with the map $\big[\Pi(\Sigma), \mathbf{Fields}\big] \to \big[\Pi(\Sigma), \mathbf{B}^n U(1)\big]$ induces the circle $(n-k)$-bundle on the space of fields on Σ.

This is precisely the kind of structure that appears in physics: the action functional over a manifold Σ is obtained by integrating a Lagrangian over that manifold. Here the process of integration is given by the process of taking holonomy.

We thus see how the description of prequantum field theory in terms of correspondences agrees with the structure of field theories in physics. On the other hand, we see that topological pQFTs have a very simple mathematical description: the cobordism hypothesis tells us that they are determined by a classifying stack of fields carrying an extended action functional. We can use this to give a straightforward description of boundary theories, or defects, due to [FV13][1] using the results from the last section of [Lur09b].

2.3.3 Boundaries and defects

The cobordism hypothesis tells us that the (∞, n)-category of (framed) cobordisms, while complicated-looking from the outset, has a surprisingly simple description: since any smooth manifold has a handle decomposition, we can construct any cobordism from basic building blocks like points, intervals, squares etcetera. Since these n-fold cubes describe the unit n-morphisms on the point in Bord_n, one sees that one can obtain Bord_n as the free symmetric monoidal (∞, n)-category with all duals on a single generator: the point. This is the fundamental insight we gain from the cobordism hypothesis: forming cobordisms is just a free construction.

As was amplified in the last section of [Lur09b], this insight allows us to give simple categorical descriptions of categories of decorated cobordisms as well. For example, we could consider a category of cobordisms where part of the boundary are constrained: fields cannot propagate through them. This means that the cobordisms come equipped with a (codimension 1) *marked* boundary along which we are not allowed to glue (since gluing along a boundary implies that fields can propagate through that boundary).

In this case (ignoring all corners and sigularities), such a cobordism Σ with marked boundaries can be decomposed in a very simple way: around each marked boundary component $\partial\Sigma$ we can pick a collar, i.e. a manifold of the form $\partial\Sigma \times [0, 1]$. One of the boundaries of this collar is the marked boundary $\partial\Sigma$, while the other boundary is that same manifold $\partial\Sigma$, but now unmarked, so that we can glue along it.

We thus see that each cobordism with a marked boundary can be obtained from an ordinary cobordism by gluing such collars along some of its boundary components. Such a collar witnesses that we cannot glue along that specific boundary: we should see marked boundaries a being 'capped off'. This means that the collar to the marked boundary should be seen as a morphism

$$\emptyset \xrightarrow{\partial\Sigma} \partial\Sigma$$

which makes it impossible to glue along the boundary $\partial\Sigma$ once we have added such a collar to it. This motivates the following definition from [Lur09b]

Definition 2.3.16. the (∞, n)-category $\mathrm{Bord}_n^{\mathrm{bdr}}$ of framed cobordisms with a marked codimensino 1 boundary is the free symmetric monoidal (∞, n)-category with all duals on a single object $*$, together with a morphism

$$\emptyset \relbar\joinrel\mapstochar\joinrel\rightarrow *$$

from the monoidal unit.

This single morphism gives rise to all the collars around the marked boundaries we just described.

Now a topological field theory might be constrained on these boundaries. The system can evolve *along* these marked boundaries, but not through them. On the other hand, the fields on the marked boundary might be different from the fields on the bulk: they might be more constrained, but they might also consist of extra fields that can only live on the marked boundary and not on the rest of the manifold. Summarizing, on the marked boundary the system behaves as an $(n-1)$-dimensional TFT itself, except that it interacts with the bulk n-dimensional theory.

It turns out that the following simple definition of a boundary TFT precisely describes this behaviour:

Definition 2.3.17. A boundary TFT is a symmetric monoidal (∞, n)-functor $\mathrm{Bord}_n^{\mathrm{bdr}} \to \mathbf{C}$ into some symmetric monoidal (∞, n)-category.

[1]Many thanks to Domenico Fiorenza and Alessandro Valentino for the invitation to the Max Planck institute in Bonn where these ideas were discussed.

The universal property of Bord_n^∂ tells us that such a boundary TFT is described by a single morphism

$$1 \to C$$

from the monoidal unit into a fully dualizable object C. Moreover, this morphism has to be dualizable itself, i.e. it has to admit all adjoints.

Applying this to the case where $\mathbf{C} = \mathbf{Corr}_n(\mathbf{H})_{/\mathbf{B}^n U(1)}$, we obtain the following result, due to [FV13]:

Proposition 2.3.18. *A boundary for a prequantum field theory is given by a diagram of the form*

$$\begin{tikzcd}
 & \mathbf{Fields}^\partial \arrow[dl] \arrow[dr] & \\
* \arrow[dr] \arrow[rr, phantom, "\Downarrow"] & & \mathbf{Fields} \arrow[dl] \\
 & \mathbf{B}^n U(1) &
\end{tikzcd}$$

As was amplified by [FV13], this precisely captures the kind of behaviour we want our boundary theories to satisfy: over the marked boundary components, the fields are confined to live in $\mathbf{Fields}^\partial$, or there might be extra fields on the extra marked boundary if $\mathbf{Fields}^\partial$ is an extension of $\mathbf{Fields}$. Moreover, the extended action functional is supposed to be trivialized over the boundary: this makes sure that the total action functional (adding the contributions from the boundary and the bulk) is well-defined. As such it appears for instance in the Freed-Witten anomaly cancellation condition, which we discuss in section 5.2.4.

There are many variations of this theme (see [Lur09b]): for example, the boundary theory might not be the boundary of a single bulk theory, but might live on a membrane between two different field theories. This is described by taking a bordism category with two different generators and a morphism between them. Furthermore, we might add boundary theories to boundary theories, which also interact with the bulk theory.

We give one more example of such a situation:

Definition 2.3.19. Let $\mathrm{Bord}_n^{\mathrm{def}}$ be the free symmetric monoidal (∞, n)-category with all duals generated by a two objects $*_1, *_2$ together with the 2-morphism and the three extra arrows that appear in the following diagram

$$\begin{tikzcd}
\emptyset \arrow[r, "\mathrm{id}"] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \emptyset \arrow[d] \\
*_1 \arrow[r] & *_2
\end{tikzcd}$$

This describes two bulk theories together with a boundary, a membrane between them and a *defect* where this membrane meets the boundary. A functor

$$\mathrm{Bord}_n^{\mathrm{def}} \longrightarrow \mathrm{Corr}_n\big(\mathbf{H}_{/\mathbf{B}^n U(1)}\big)$$

is classified by a diagram in the category of n-fold correspondences in $\mathbf{H}_{/\mathbf{B}^n U(1)}$ of the form

$$\begin{tikzcd}
* & * \arrow[l] \arrow[r] & * & \text{Trivial} \\
\mathbf{Fields}_1^\partial \arrow[u] \arrow[d] & \mathbf{Fields}^{def} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \mathbf{Fields}_2^\partial \arrow[u] \arrow[d] & \text{Boundary theory} \\
\mathbf{Fields}_1 & \mathbf{Fields}^{mem} \arrow[l] \arrow[r] & \mathbf{Fields}_2 & \text{Bulk theory} \\
\text{Bulk theory} & \text{Defect} & \text{Bulk theory} &
\end{tikzcd}$$

This whole diagram sits over $\mathbf{B}^n U(1)$. We will encounter such a diagram in relation to the orbit method 5.2.2.

We have seen how prequantum field theories are modeled by (∞, n)-functors into the category of n-fold correspondences in the slice $\mathbf{H}_{/\mathbf{B}^n U(1)}$. Using the cobordism hypothesis, such pQFTs have a very simple description: they are just classified by a single classifying stack of fields, together with an extended action functional modeled by a circle n-bundle on that stack. Analogously, boundary pQFTS and defects are classified by certain correspondence diagrams in $\mathbf{H}_{/\mathbf{B}^n U(1)}$.

In the rest of this text, we describe a first step in the quantization of such theories. In the next section we will show how one can linearize a pQFT to end up with correspondence diagrams in categories of modules. We will then show how to turn such a correspondence diagram into an actual linear map of modules, by a pull-push construction. Since correspondence diagrams arise in two slightly different ways, this process of pull-push quantization produces two slightly different quantum phenomena:

- on the one hand, the value of a pQFT on a cobordism is given by an n-fold correspondence. To quantize the pQFT, we have to consistenly give the pull-push quantization of all correspondence diagrams that appear in this n-fold correspondence.
- on the other hand, we can classify a boundary theory by a single span. Then the pull-push quantization of that span can be interpreted as the quantization of the boundary theory. We might also want to quantize defects like the one described above. But then we again run into n-fold spans, where we need a functoriality condition to form a consistent quantization.

We will only discuss how to quantize a single correspondence diagram and will not deal with the functoriality issues that go into the formation of pushforward maps. Interesting examples already appear in the case where we have only a single correspondence diagram (see section 5). In the outlook section 6 we will sketch how one should proceed to also quantize n-fold spans.

3 Linearization in generalized cohomology

In the previous section we have seen how a topological pQFT

$$\mathrm{Bord}_n \to \mathrm{Corr}_n(\mathbf{H}_{/\mathbf{B}^n U(1)})$$

is classified by a moduli stack of fields with an extended action functional $\mathbf{Fields} \to \mathbf{B}^n U(1)$. The possible trajectories of these fields are described by certain correspondence diagrams in the slice category $\mathbf{H}_{/\mathbf{B}^n U(1)}$.

A codimension 1 boundary to such a field theory is given by a functor

$$\mathrm{Bord}_n^{\partial} \to \mathrm{Corr}_n(\mathbf{H}_{/\mathbf{B}^n U(1)})$$

which is classified by a correspondence diagram of the form

$$\begin{tikzcd}
& \mathbf{Fields}^{\partial} \arrow[dl] \arrow[dr, "f"] & \\
* \arrow[dr] \arrow[rr, phantom, "\Downarrow \xi"] & & \mathbf{Fields} \arrow[dl, "\exp(iS)"] \\
& \mathbf{B}^n U(1) &
\end{tikzcd} \tag{3.1}$$

Recall from the introduction our cohomological approach to the quantization of such diagrams:

(i) we have to linearize the theory by embedding the higher abelian group $\mathbf{B}^{n-1}U(1)$ into the group of units of a higher commutative ring R. For example, $U(1)$ can be embedded into the ring $\mathbb{C}$. The additive structure on R allows us to add up the phases in $\mathbf{B}^{n-1}U(1)$.

(ii) such higher commutative ring R classifies a (multiplicative) generalized cohomology theory. The quantization is now obtained by pulling and pushing in R-cohomology. This will be the content of the next section 4.

In this section, we will show how we can linearize prequantum field theories and their boundary theories using generalized cohomology.

In the case where all objects involved are discrete, this has been studied in [ABG+08]. In this setup we replace the category $\mathbf{H}$ of smooth stacks by the category of ∞-groupoids and the smooth groups $\mathbf{B}^n U(1)$ by the discrete ∞-groups $B^n U(1) = \Pi\big(\mathbf{B}^n U(1)\big)$ that classify (not necessarily smooth) circle n-bundles. As the authors show, a map from the group $B^{n-1}U(1)$ to the group of units of an E_∞-ring R gives rise to an ∞-functor

$$B^n U(1) \to BGL_1(R) \to \mathrm{Pic}(R) \to R\mathrm{Mod}$$

that describes a universal *twist* in R-cohomology. If X is a discrete ∞-groupoid equipped with a morphism $\alpha\colon X \to B^n U(1)$, then the associated map $X \to R\mathrm{Mod}$ classifies a line bundle over X, whose fiber is the ring R. The α-twisted R-cohomology of the ∞-groupoid X is then obtained by taking sections of the dual line bundle over X. We will give an overview of this in section 3.1.

From a physical perspective, the twisted cohomology spectrum can be interpreted as the analogue of the space of wave functions that appears in ordinary quantum mechanics. As such, one might view this as the space of higher quantum states which the TQFT associates to the point. Notice that we take all sections and do not introduce a polarization: instead, the analogue of a polarization appears in higher dimensions from the pull-push quantization of boundary theories (see section 4).

Remark 3.0.20. In principle, there is no need to have the twist given by $B^n U(1)$. Any abelian ∞-group can serve as a twist in the above way, by mapping it to the group of units of an E_∞ ring.

Postcomposition with the functor $B^nU(1) \to BGL_1(R)$ gives rise to a functor

$$\infty\mathrm{Gpd}_{/B^nU(1)U(1)} \to \infty\mathrm{Gpd}_{/BGL_1(R)} \to R\mathrm{Mod}^{op}$$

where the last functor sends each $X \to BGL_1(R)$ to the corresponding twisted cohomology spectrum. In fact, each of these functors is a monoidal functor.

The above functor allows us to linearize our prequantum field theory: a k-dimensional cobordism Σ gives rise to a certain k-fold correspondence diagram in $\infty\mathrm{Gpd}_{/B^nU(1)}$. Then we can apply the above functor to turn this into a k-fold correspondence in the opposite of the 'linear' category of R-modules. The maps of R-modules that arise in such a diagram are induced by pulling back sections and applying gauge equivalences between R-line bundles.

We have a similar situation if we apply the above functor to the correspondence diagram 3.1 describing a boundary theory. We then obtain a cospan of twisted R-cohomology spectra

$$R \xrightarrow{\xi} R^{*+f^*\exp(iS)}\big(\mathrm{Fields}^\partial\big) \xleftarrow{f^*} R^{*+\exp(iS)}\big(\mathrm{Fields}\big)$$

In the next section 4 on quantization, we describe how we can turn such a cospan diagram of R-modules into a single linear map of R-modules by forming the pushforward map along the right map.

However, the above picture only applies when our spaces of fields are discrete stacks, carrying a trivial smooth structure. We could ignore this by replacing each smooth stack X by its homotopy type $\Pi(X)$. In some cases (in particular when X is a smooth manifold) this gives a good approximation to X, at least from the point of view of cohomology, but in some situations this approximation is too coarse. In section 3.1.5, we therefore describe how one should generally extend the discussion in the discrete case to the case where objects carry smooth structure.

Unfortunately, we cannot apply the abstract ideas from 3.1.5 in concrete cases because we lack concrete descriptions of ring spectra carrying a smooth structure: essentially the only examples we have are the rings $\mathbb{R}$ and $\mathbb{C}$. In the second part of this section we present a workable approximation to the abstract picture sketched in section 3.1.5. We do this in the case where $n = 2$. In this case, the ring R to consider is the ring KU classifying complex K-theory, which carries a twist

$$B^2U(1) \to BGL_1(KU).$$

This gives the well-known degree 2 twist in complex K-theory.

Although we do not have a well-established description of a smooth K-theory spectrum, there is a good theory of K-theory for differentiable stacks. For instance, this produces the G-equivariant K-theory for a quotient stack $M//G$ and allows for twists by smooth circle 2-bundles. Many of the abstract ideas from the discrete case can be transfered to this 'smooth' K-theory.

A large part of this section is therefore dedicated to describing twisted K-theory for differentiable stacks. We describe two models for this: the first approach uses spaces of Fredholm operators to present K-theory spectra. This is an extension by [FHT11] of the classical picture of twisted K-theory in the discrete case, as discussed in [AS04]. This description of twisted K-theory is close in spirit to the abstract theory of twisted generalized cohomology.

The second description is based on the fact that the (compactly supported) K-theory of a topological space is entirely encoded in the C^*-algebraic structure of its function algebra. We show in section 3.3.1 how a differentiable stack with a circle 2-bundle gives rise to a C^*-algebra $C^*_\alpha(X)$ of 'twisted functions', whose K-theory gives the twisted (compactly supported) K-theory of that stack.

The two approaches are not unrelated: for large classes of stacks, the two definitions of the twisted K-theory groups agree, as was shown in [TXLG04]. As such, the Fredholm picture of twisted K-theory serves as a bridge between the abstract picture of twisted generalized cohomology from section 3.1 and the description of twisted K-theory in terms of operator algebras.

The algebraic description of twisted K-theory has the advantage that it fits in a more structural framework: it extends to a bivariant form of K-theory, which describes the maps that give rise to maps in K-theory. More precisely, there is an additive category KK whose objects are C^*-algebras and where the sets of maps $\mathbb{C} \to C^*_\alpha(X)$ coincides with the set of twisted K-theory classes of the stack X. This category KK forms the natural home for index theory, and therefore provides a good framework for the kind of pushforwards we want to consider in section 4.

The construction of a twisted function algebra from a differentiable stack gives rise to a monoidal functor

$$C^*(-)\colon \mathrm{DiffStack}^{prop}_{/\mathbf{B}^2U(1)} \to \mathrm{KK}^{op}$$

which describes the smooth analogue of the functor $\infty\mathrm{Gpd}_{/B^nU(1)} \to R\mathrm{Mod}^{op}$. In fact there is a functor $\mathrm{KK} \to \mathrm{ho}\big(KU\mathrm{Mod}\big)$ that assigns to each algebra its K-theory spectrum as an object in the homotopy category of KU-modules [DEKM11], [JS09]. Moreover, this construction of a K-theory spectrum gives a lax monoidal functor. This allows us to pass from operator algebras back to the setting of modules over ring spectra.

In analogy with the discrete setting, we can use the functor $C^*(-)$ to turn a k-fold correspondence diagram in $\mathrm{DiffStack}^{prop}_{/\mathbf{B}^2U(1)}$ into a k-fold cospan diagram in the category KK. By passing to K-theory spectra, we obtain a k-fold cospan diagram in the homotopy category of KU-modules. Again, all maps arising in such a diagram come from pullback maps in K-theory and equivalences of twists in K-theory.

Similarly, the single correspondence describing a boundary to a two-dimensional pQFT gives rise to a diagram of C^*-algebras

$$\mathbb{C} \xrightarrow{\ \xi\ } C^*_{f^*\exp(iS)}\big(\mathbf{Fields}^\partial\big) \xleftarrow{\ f^*\ } C^*_{\exp(iS)}\big(\mathbf{Fields}\big)$$

in the category KK. A single such diagram will be the input of the K-theoretic pull-push quantization we discuss in section 4.

3.1 Linearization in abstract homotopy theory

We discuss how the process of linearization leads us to consider twisted cohomology spectra. In section 3.1.1 we give a quick presentation of the theory of higher groups, rings and their modules. In the next section 3.1.2 we discuss how the embedding of the groups $B^nU(1)$ in the units of a ring spectrum R gives rise to twists in R-cohomology. We will give some examples of such twists in section 3.1.4. These will come back in the examples of our pull-push quantization in section 5.

The first two sections only apply to the case where our groups and rings are discrete. In section 3.1.5 we sketch how one can extend these constructions to the smooth case. While this looks promising, we lack a concrete description of smooth spectra that allows us to produce examples. In the next section we therefore discuss an approach to twisted K-theory of differentiable stacks which is tractable and which is expected to approximate a smooth version of the K-theory spectrum.

3.1.1 Homotopical algebra

The homotopy theoretic analogue of an ordinary abelian group is a spectrum A, i.e. a sequence of pointed ∞-groupoids $\{A_n\}$ together with weak equivalences

$$A_n \to \Omega A_{n+1}$$

where $\Omega A_{n+1} = * \times_{A_{n+1}} *$ is the loop space of A_{n+1} at its basepoint. These spectra naturally form an ∞-category of spectra, denoted Sp. Abstractly, this category can be realized as the full subcategory of the functor category $\mathrm{Fun}(\mathbb{Z}\times\mathbb{Z}, \infty\mathrm{Gpd}_*)$ on those functors such that

(i) $F(i,j) = *$ if $i \neq j$

(ii) for all i, the diagram

$$\begin{tikzcd} F(i,i) \arrow[r] \arrow[d] & F(i,i+1) \simeq * \arrow[d] \\ * \simeq F(i+1,i) \arrow[r] & F(i+1,i+1) \end{tikzcd}$$

is homotopy cartesian.

A functor satisfying only (i) is often called a prespectrum: it is just a sequence of pointed topological spaces $\{A_n\}$ with maps $A_n \to \Omega A_{n+1}$. The inclusion of the spectra in the prespectra has a left adjoint L that assigns to each prespectrum the spectrum given by the familiar formula

$$(LA)_n = \operatorname*{colim}_k \Omega^k A_{n+k}$$

The category of spectra comes equipped with adjunctions

$$\begin{tikzcd} \infty\mathrm{Gpd} \arrow[r, shift left] \arrow[rr, bend left, "\Sigma^\infty_+"] & \infty\mathrm{Gpd}_* \arrow[l, shift left] \arrow[r, shift left, "\Sigma^\infty"] & \mathrm{Sp} \arrow[l, shift left, "\Omega^\infty"] \end{tikzcd}$$

to the ∞-categories of (pointed) ∞-groupoids. The left adjoint Σ^∞ sends a pointed topological space to its suspension spectrum, while the right adjoint Ω^∞ sends a spectrum $A = \{A_n\}$ to the space $\Omega^\infty A := A_0$.

Remark 3.1.1. The space $\Omega^\infty A$ has the natural structure of an infinite loop space

$$\Omega^\infty A \simeq \operatorname*{colim}_n \Omega^n A_n.$$

As such, it carries the structure of a group-like algebra over the E_∞ operad (in the cartesian monoidal structure on ∞Gpd). We call such an object an abelian ∞-group, and let AbGrp_∞ denote the ∞-category of abelian ∞-groups.

It is well-known that AbGrp_∞ is equivalent to the category of (-1)-connective spectra: we have just seen how such spectra give rise to infinite loop spaces. Conversely, any abelian ∞-group G admits an infinite tower of deloopings $B^n G$, which together give a (-1)-connective spectrum whose looping produces G.

Example 3.1.2. The classifying spaces of circle n-bundles $B^n U(1) = \Pi(\mathbf{B}^n U(1))$ are abelian ∞-groups. Indeed, we have that $B^n U(1) \simeq \Omega B^{n+1} U(1)$, just as we saw in remark 2.2.18 for the smooth stacks $\mathbf{B}^n U(1)$.

Having spectra as our analogue of abelian groups, we need a symmetric monoidal structure on them to define rings. This is provided by the smash product of spectra, whose unit is the sphere spectrum $\mathbb{S} = \Sigma^\infty_+ *$. In fact, this monoidal structure is uniquely characterized by the fact that its unit is the sphere spectrum $\mathbb{S}$, while it preserves homotopy colimits in each variable (see [Lur12]).

Remark 3.1.3. There are multiple model-theoretic descriptions areound for the ∞-category of spectra. One choice, which has a monoidal smash product of spectra already at the 1-categorical level, is given by the model category of symmetric spectra, as discussed in [Sch07].

Since the functor $(-) \wedge E$ preserves small colimits, it has a right adjoint $[E, -]$. We call $[E, A]$ the mapping spectrum from E to A. It satisfies

$$\mathrm{Sp}(F \wedge E, A) \simeq \mathrm{Sp}(F, [E, A]).$$

Observe that

$$\Omega^\infty [E, A] \simeq \infty\mathrm{Gpd}\big(*, \Omega^\infty [E, A]\big) \simeq \mathrm{Sp}\big(\Sigma^\infty_+ *, [E, A]\big) \simeq \mathrm{Sp}\big(\mathbb{S} \wedge E, A\big) \simeq \mathrm{Sp}\big(E, A\big)$$

so the infinite looping of the mapping spectrum is the (derived) mapping space.

The symmetric monoidal structure on Sp allows us to consider commutative algebra objects in Sp.

Definition 3.1.4. A commutative ∞-ring is an E_∞ ring spectrum, i.e. an E_∞ algebra with respect to the smash product on Sp. We denote the ∞-category of commutative ∞-rings by CRing_∞.

Example 3.1.5. The sphere spectrum $\mathbb{S}$, being the monoidal unit, carries an E_∞-ring structure. It is the initial ∞-ring, since any other ring spectrum R has to be equipped with a map $\mathbb{S} \to R$ giving the multiplicative unit. This map is the unique (unital) ring map from $\mathbb{S}$ into R.

Example 3.1.6. If R is an ordinary commutative ring, then its Eilenberg-Maclane spectrum $\mathrm{H}R = \{K(R,n)\}$ has the natural structure of an E_∞-ring spectrum.

Having the above theory of higher groups and rings, we can make sense of the group of units of an ∞-ring. To do this we have to make sure that we preserve the ring structure when passing from spectra to spaces. This is guaranteed by the following result:

Proposition 3.1.7 ([ABG+08], 3.27)**.** *The $(\infty-)$functors $\Sigma^\infty_+ \colon \infty\mathrm{Gpd} \rightleftarrows \mathrm{Sp} \colon \Omega^\infty$ preserve E_∞ algebras. More precisely, they induce an adjunction between the ∞-categories of E_∞ algebras, with respect to the cartesian product on the left and the smash product on the right.*

Given an E_∞ ring spectrum R, the associated infinite loop space $\Omega^\infty R$ therefore carries an additional E_∞ algebra structure, corresponding to the multiplication on R. This gives an ordinary ring structure on the set of connected components of $\Omega^\infty R$. We now define the group of units $GL_1(R)$ to be the homotopy pullback

$$\begin{tikzcd} GL_1(R) \arrow[r] \arrow[d] & \Omega^\infty R \arrow[d] \\ \pi_0(\Omega^\infty R)^\times \arrow[r] & \pi_0(\Omega^\infty R) \end{tikzcd}$$

of pointed spaces. Since each space carries the structure of an E_∞ monoid, so does the homotopy pullback $GL_1(R)$. Note that being group-like is a property rather than a structure: we just require that each element has an inverse, up to (not necessarily coherent) homotopy. This construction of the group of units gives the left adjoint in an adjunction of ∞-categories:

Proposition 3.1.8 ([ABG+08], 3.2)**.** *There is an adjunction*

$$\begin{tikzcd} \mathrm{CRing}_\infty \arrow[r, shift right, "GL_1(-)"'] & \mathrm{AbGrp}_\infty \arrow[l, shift right, "\mathbb{S}[-]"'] \end{tikzcd}$$

where the right adjoint sends a group to its suspension spectrum (which is an E_∞ ring spectrum by proposition 3.1.7).

Finally, given an E_∞ ring spectrum R, we can consider left modules over R: these are just spectra carrying a left action from R, in the homotopical sense. These modules are organized in an ∞-category $R\mathrm{Mod}$. The functor that sends each R-module to the underlying spectrum has a left adjoint

$$\begin{tikzcd} R\mathrm{Mod} \arrow[r, shift right] & \mathrm{Sp} \arrow[l, shift right, "R\wedge(-)"'] \end{tikzcd}$$

that sends a spectrum E to $R \wedge E$, where R simply acts on the copy of R by its own ring structure. This also carries a symmetric monoidal structure $\wedge_R$, by taking the smash product over R. This preserves colimits in both of its arguments, so again we obtain a mapping spectrum satisfying

$$R\mathrm{Mod}(L \wedge_R M, N) \simeq R\mathrm{Mod}(L, [M, N]).$$

We have that $M \wedge_R (R \wedge E) \simeq M \wedge E$ for any R-module M and spectrum E.

Example 3.1.9. When $R = \mathbb{S}$, we simply have that $\mathbb{S}\mathrm{Mod} \simeq \mathrm{Sp}$ (since $\mathbb{S}$ is the monoidal unit, it can only act trivially on each spectrum). The monoidal structure on $\mathbb{S}\mathrm{Mod}$ agrees with the smash product of spectra.

Definition 3.1.10. If R is an E_∞ ring spectrum, then the Picard groupoid $\mathrm{Pic}(R)$ is the core of the full subcategory of $R\mathrm{Mod}$ on those R-modules M that are invertible with respect to the monoidal structure (these are sometime called 'lines').

An R-module M is invertible if there is a module $M^\vee$ such that $M \wedge_R M^\vee \simeq R$. Of course R itself is a line, with dual R. Note that invertibility can be tested at the level of the homotopy category.

Remark 3.1.11. The Picard groupoid $\mathrm{Pic}(R)$ carries itself the structure of an abelian ∞-group: indeed, the smash product $(-) \wedge_R (-)$ gives an E_∞ algebra structure on $\mathrm{Pic}(R)$, while each object in $\mathrm{Pic}(R)$ is invertible by definition.

Observe that

$$R\mathrm{Mod}\big(R, R\big) \simeq R\mathrm{Mod}\big(R \wedge \mathbb{S}, R\big) \simeq \mathrm{Sp}(\mathbb{S}, R) \simeq \Omega^\infty R$$

The composition is precisely the multiplication on $\Omega^\infty R$, so the group of automorphisms of the module R is simply $GL_1(R)$. It follows that there is an inclusion of ∞-categories

$$BGL_1(R) \to \mathrm{Pic}(R) \to R\mathrm{Mod}$$

where the first inclusion is a monoidal functor. We will use this inclusion in the next section to define the twisted R-cohomology of an ∞-groupoid X. A map $X \to BGL_1(R)$ takes value on the trivial module R, but 1-cells in X can give rise to non-trivial automorphisms of this trivial module R. Precisely these nontrivial automorphisms twist the R-cohomology of the space X.

3.1.2 Twisted cohomology

Recall that spectra describe generalized cohomology theories: if X is the fundamental ∞-groupoid of a topological space and R is a spectrum, the R-cohomology groups of X are defined to be

$$R^n(X) := \pi_0 \mathrm{Sp}\big(\Sigma^\infty_+ X, \Sigma^n R\big)$$

where $\Sigma^n R$ is the n-fold suspension of R (for $n < 0$ it is actually the $(-n)$-fold looping of R).

Using the left adjoint $(-) \wedge R : \mathrm{Sp} \to R\mathrm{Mod}$, we can equivalently write this as

$$R^n(X) := \pi_0 R\mathrm{Mod}\big(\Sigma^\infty_+ X \wedge R, \Sigma^n R\big)$$

More precisely, the R-cohomology spectrum of X is just the internal mapping space

$$\big[\Sigma^\infty_+ X \wedge R, R\big] \in R\mathrm{Mod}$$

One obtains the n-th cohomology group as the $(-n)$-th homotopy group of this spectrum.

Dually, we define the R-homology spectrum of X to be simply

$$\Sigma^\infty_+ X \wedge R.$$

The homotopy groups of this spectrum are the R-homology groups of X.

The following observation forms the basis of the abstract picture of twisted cohomology:

Proposition 3.1.12. *The R-homology spectrum of a space X is the homotopy colimit of the functor*

$$X \to \mathrm{RMod}$$

that is constant on R (seen as an R-module).

Proof. We use that X is an ∞-groupoid, so in particular we can consider functors from X into an ∞-category. Recall that any ∞-groupoid X is the colimit over the diagram

$$X \to \infty\mathrm{Gpd}$$

constant on the point. Since both $\Sigma^\infty_+(-)$ and $(-) \wedge R$ are left adjoints, they preserve colimits, so we obtain

$$\Sigma^\infty_+ X \wedge R \simeq \left(\Sigma^\infty_+ \operatorname*{colim}_X *\right) \wedge R \simeq \operatorname*{colim}_X \mathbb{S} \wedge R \simeq \operatorname*{colim}_X R$$

□

We can think of $\Sigma_+^\infty X \wedge R$ as the total space of a bundle of spectra over X: over each vertex in X, we have a copy of R and a 1-cell in X gives rise to an R-module map between these copies of R. Since the fibers are just the trivial R-modules, we obtain a line bundle over X, which is actually trivial. The R-cohomology spectrum of X is just the spectrum of R-linear maps from the total space $\Sigma_+^\infty \wedge R$ to R. Such maps can be interpreted as sections of the dual R-line bundle.

The whole idea of twisted cohomology is to replace the trivial R-bundle by any R-line bundle.

Definition 3.1.13 ([ABG+08])**.** Let X be an ∞-groupoid and $R \in \mathrm{CRing}_\infty$. We call a functor

$$\alpha\colon X \to \mathrm{Pic}(R)$$

into the Picard groupoid of R a *twist* for the R-cohomology of X. Given a twist $\alpha\colon X \to \mathrm{Pic}(R)$, we define the α-twisted R-homology groups of X as

$$R_{n+\alpha}(X) = \pi_n\big(\operatorname*{colim}_X \alpha\big)$$

and the α-twisted R-cohomology groups of X by

$$R^{n+\alpha}(X) = \pi_{-n}\big[\operatorname*{colim}_X \alpha, R\big]$$

where the homotopy colimit is taken in the ∞-category $R\mathrm{Mod}$.

In the case where α maps into $BGL_1(R)$, this produces the bundle picture mentioned above: each vertex in X gives rise to a copy of R, while higher cells in X give rise to cells between these copies of R which are twisted by $GL_1(R)$. By collapsing the copies of R to the point, we obtain a natural map of spectra

$$\operatorname*{colim}_X \alpha \to \operatorname*{colim}_X * \simeq X$$

whose homotopy fiber over each vertex is the spectrum R. We can interpret this as a line bundle over the space X, whose fiber is the ring R. The α-twisted homology groups of X are given by the stable homotopy groups of the total space of this bundle. The α-twisted cohomology is given by the homotopy groups of the spectrum of R-linear maps from $\operatorname{colim} \alpha$ into R. Such R-linear maps can be interpreted as sections of the dual R-line bundle.

Remark 3.1.14. More general, we interpret any twist $\alpha\colon X \to \mathrm{Pic}(R)$ as classifying an R-line bundle over X. The main difference with the previous discussion is that the fiber of this bundle need not be isomorphic to the ring R, but rather some invertible R-module (also called an R-line). For example, we can encorporate degree shifts in our twists by mapping X into the automorphism group of the suspension $\Sigma^n R \in \mathrm{Pic}(R)$, which is of course equivalent to the automorphism group of R itself.

The formation of twisted cohomology spectra gives rise to a monoidal functor.

Proposition 3.1.15. *Let $\infty\mathrm{Gpd}_{/\mathrm{Pic}(R)}$ be the ∞-category of functors $X \to \mathrm{Pic}(R)$, with X an ∞-groupoid. Then there is a natural functor*

$$\infty\mathrm{Gpd}_{/\mathrm{Pic}(R)} \to R\mathrm{Mod}^{op}$$

sending each twist $X \xrightarrow{\alpha} \mathrm{Pic}(R)$ to its twisted R-cohomology spectrum X^α. Moreover, if we endow $\infty\mathrm{Gpd}_{/\mathrm{Pic}(R)}$ with the symmetric monoidal structure from proposition 2.3.9, then this functor is monoidal.

Proof. We obtain this functor by taking the colimit functor

$$\operatorname{colim}\colon \infty\mathrm{Gpd}_{/\mathrm{Pic}(R)} \to R\mathrm{Mod}$$

and then applying the functor $[-, R] : R\mathrm{Mod} \to R\mathrm{Mod}^{op}$.

Given two twists $X \xrightarrow{\alpha} \mathrm{Pic}(R), Y \xrightarrow{\beta} \mathrm{Pic}(R)$, their product in $\infty\mathrm{Gpd}_{/\mathrm{Pic}(R)}$ is given by the functor $p_1^*\alpha \wedge_R p_2^*\beta \colon X \times Y \to \mathrm{Pic}(R)$. Then we have a natural equivalence

$$\begin{aligned}
\Big[\operatorname*{colim}_{X\times Y} p_1^*\alpha \wedge_R p_2^*\beta, R\Big] &\simeq \Big[\big(\operatorname*{colim}_X \alpha\big) \wedge_R \big(\operatorname*{colim}_Y \beta\big), R\Big] \\
&\simeq \lim_X \Big[\alpha \wedge_R \big(\operatorname*{colim}_Y \beta\big), R\Big] \\
&\simeq \big[\operatorname{colim} \alpha, R\big] \wedge_R [\operatorname*{colim}_Y \beta, R]
\end{aligned}$$

The last equivalence follows from the fact that α takes values in the Picard groupoid $\mathrm{Pic}(R)$: in that case, the monoidal dual of the value of α is simply $[\alpha, R]$ (see [ABG11]). In particular, the functor $(-) \wedge_R \alpha$ is (over each object of X) an equivalence of categories with inverse given by $(-) \wedge_R [\alpha, R]$. We then have

$$\begin{aligned}
R\mathrm{Mod}\Big(M, \lim_X [\alpha \wedge_R N, R]\Big) &\simeq \lim_X R\mathrm{Mod}\Big(M, \big[\alpha, [N,R]\big]\Big) \\
&\simeq \lim_X R\mathrm{Mod}\Big(M \wedge \alpha, [N,R]\Big) \\
&\simeq \lim_X R\mathrm{Mod}\Big(M, [\alpha \wedge R] \wedge_R [N,R]\Big) \\
&\simeq R\mathrm{Mod}\Big(M, [\operatorname*{colim}_X \alpha \wedge R] \wedge_R [N,R]\Big)
\end{aligned}$$

for all R-modules M and N. This shows that the functor is monoidal. $\square$

Now suppose that the ∞-group $B^{n-1}U(1)$ has a natural group homomorphism $B^{n-1}U(1) \to GL_1(R)$ into the group of units of some ∞-ring R. Then any circle n-bundle on a topological space X gives rise to a twist in R-theory via

$$X \xrightarrow{\alpha} B^nU(1) = BB^{n-1}U(1) \to BGL_1(R) \to \mathrm{Pic}(R)$$

This gives rise to an R-module spectrum $R^{*+\alpha}(X) = \big[\operatorname{colim}_X \alpha, R\big]$, the α-twisted R-cohomology spectrum of X.

We see that we can pass from geometrically discrete ∞-groupoids carrying a circle n-bundle to their R-cohomology spectra, twisted by that bundle. This allows us to linearize the correspondence diagrams we have considered in section 2.3 to certain cospan diagrams in the category of R-modules.

3.1.3 Linearizing correspondences in stable homotopy theory

We now use the above abstract machinery to pass from a geometric prequantum theory to its linearization. Let

$$\mathrm{Bord}_n \to \mathrm{Corr}_n(\infty\mathrm{Gpd}_{/B^nU(1)})$$

describe a prequantum field theory whose spaces of fields are geometrically discrete. Suppose we have picked a homomorphism of abelian ∞-groups

$$B^{n-1}U(1) \to GL_1(R)$$

into the units of an E_∞ ring spectrum R. We will discuss examples of such ring spectra in the next section 3.1.4.

Being a morphism of abelian ∞-groups, the above homomorphism deloops to a group homomorphism

$$BB^{n-1}U(1) \simeq B^nU(1) \to BGL_1(R) \to \mathrm{Pic}(R).$$

This gives rise to a functor

$$\infty\mathrm{Gpd}_{/B^nU(1)} \to \infty\mathrm{Gpd}_{/\mathrm{Pic}(R)}$$

by postcomposition along $B^nU(1) \to BGL_1(R) \to \mathrm{Pic}(R)$. Observe that this gives a monoidal functor since the monoidal structures involve the cartesian product and the group structures on $B^nU(1)$ and $\mathrm{Pic}(R)$, which are preserved. Since it also preserves pullbacks, we get an induced functor

$$\mathrm{Bord}_n \to \mathrm{Corr}_n(\infty\mathrm{Gpd}_{/BGL_1(R)})$$

linearizing our prequantum field theory, by embedding the values of the extended action functional into the group of units of the ring spectrum R.

This assigns to each k-dimensional cobordism Σ a k-fold correspondence diagram in $\infty\mathrm{Gpd}_{/BGL_1(R)}$. In turn, by applying the functor

$$\infty\mathrm{Gpd}_{/BGL_1(R)} \longrightarrow R\mathrm{Mod}^{op}$$

to such a diagram we obtain a k-fold cospan diagram in the category of R-modules, consisting of twisted R-cohomology spectra. In particular, a 1-dimensional cobordism Σ gives rise to a span in $\infty\mathrm{Gpd}_{/B^nU(1)}$ of the form

$$\begin{tikzcd}
& \mathrm{Fields}(\Sigma) \arrow[dl, "i"'] \arrow[dr, "f"] & \\
\mathrm{Fields}(\Sigma_{\mathrm{in}}) \arrow[dr, "\alpha"'] \arrow[rr, phantom, "\Rightarrow"] & & \mathrm{Fields}(\partial\Sigma) \arrow[dl, "\beta"] \\
& B^nU(1) &
\end{tikzcd}$$

which in turn gives rise to a cospan of the form

$$R^{*+\alpha}\Big(\mathrm{Fields}(\Sigma_{\mathrm{in}})\Big) \xrightarrow{i^*} R^{*+f^*\beta}\Big(\mathrm{Fields}(\Sigma)\Big) \xleftarrow{f^*} R^{*+\beta}\Big(\mathrm{Fields}(\Sigma_{\mathrm{out}})\Big) \tag{3.2}$$

The diagrams associated to higher dimensional cobordisms are obtained by taking products of such cospan diagrams.

Remark 3.1.16. Note that the above square is really divided into two triangles, as in diagram 3.3: the tip of the diagram forms an element in $\infty\mathrm{Gpd}_{/B^nU(1)}$ and therefore comes equipped with a map to $B^nU(1)$. The two 2-cells filling the two triangles enter explicitly in the construction of the above two maps i^*, f^* of R-modules.

Physically, the above cospan diagram consists of the spaces of ingoing and outgoing quantum states (or wave functions), together with maps that pull back these wave functions to $\mathrm{Fields}(\Sigma)$. The homotopy in the above square diagram provides an equivalence between the R-bundles over $\mathrm{Fields}(\Sigma)$ pulled back from the ingoing and outgoing boundary. This allows us to realize the ingoing and outgoing wave functions as sections of the same line bundle over the space of field trajectories.

In section 4 we decribe how such a single diagram gives rise to a single map of R-modules by pulling along i and forming the pushforward along f. This map of R-modules describes a propagator of the resulting quantum field theory: it is obtained by weighing the ingoing quantum state by the action functional and integrating this over the field trajectories.

With the goal in mind of quantizing the full pQFT, this is supposed to be a first step in the process. A well-developed theory of pull-push quantization of a single correspondence diagram is supposed to clear the way for a quantization procedure for k-fold correspondence diagrams. The result should describe how quantum states can propagate in k dimensions.

We can give a similar treatment of a boundary theory to a discrete n-dimensional field theory. Such a boundary theory was classified by a diagram of ∞-groupoids

$$\begin{tikzcd} & \mathrm{Fields}^{\partial} \arrow[dl] \arrow[dr, "f"] \arrow[dd] & \\ * \arrow[r, Rightarrow, "\xi"] \arrow[dr] & f^*\chi \arrow[r, Rightarrow] & \mathrm{Fields} \arrow[dl, "\chi"] \\ & B^nU(1) & \end{tikzcd} \tag{3.3}$$

The right 2-cell is the trivial cell that witnesses the composition of f and χ. On the other hand, the 2-cell ξ provides an explicit trivialization of the flat circle n-bundle $f^*\chi : \mathrm{Fields}^{\partial} \to B^nU(1)$. In other words, the 2-cell ξ describes a twisted circle $(n-1)$-bundle over $\mathrm{Fields}^{\partial}$, which may not be trivial itself.

This gives rise to a cospan diagram of the form

$$R \xrightarrow{\xi} R^{*+f^*\chi}(\mathrm{Fields}^{\partial}) \xleftarrow{f^*} R^{*+\chi}(\mathrm{Fields}). \tag{3.4}$$

The left map describes the class of the twisted circle $(n-1)$-bundle ξ in the twisted R-cohomology of the space $\mathrm{Fields}^{\partial}$, while the right map is simply the pullback map along f. In the next section we will push the twisted bundle ξ along the map f to obtain a twisted R-cohomology class over the space Fields. Such a class presents a state of the bulk field theory, which presents the holographic quantization of the boundary theory.

Finally, let us mention one more construction which gives us a single cospan diagram in a category of R-modules. This appears in some of the examples (see sections 5.1 and 5.2.3). Given an extended pQFT

$$\mathrm{Bord}_n \to \mathrm{Corr}_n(\infty\mathrm{Gpd}_{/B^nU(1)})$$

we can transgress to higher dimensional cobordisms. More precisely, recall that a closed k-dimensional manifold Σ gives rise to a k-fold span in $\infty\mathrm{Gpd}_{/B^nU(1)}$ in which all objects are trivial, except for the tip, which is $\mathrm{Fields}(\Sigma)$. The higher gauge transformations in the k-fold span diagram are then all encorporated in a single map

$$\mathrm{Fields}(\Sigma) \to B^{n-k}U(1)$$

Then a cobordism between two closed k-dimensional manifolds Σ_{in} and Σ_{out} gives rise to a correspondence

$$\begin{tikzcd} & \mathrm{Fields}(\Sigma) \arrow[dl, "i"'] \arrow[dr, "f"] & \\ \mathrm{Fields}(\partial\Sigma_{\mathrm{in}}) \arrow[dr, "\alpha"'] \arrow[rr, phantom, "\Rightarrow"] & & \mathrm{Fields}(\partial\Sigma_{\mathrm{out}}) \arrow[dl, "\beta"] \\ & B^{n-k}U(1) & \end{tikzcd}$$

which is really a k-fold correspondence diagram collapsed into a single correspondence diagram. If we view this as a single diagram over $B^{n-k}U(1)$, then we can also embed $B^{n-k-1}U(1)$ into the ring of units of a ring $\tilde{R}$. Note that this is a different ring then the one used above. Then the above 'transgressed' diagram gives rise to a map

$$\tilde{R}^{*+\alpha}\Big(\mathrm{Fields}(\Sigma_{\mathrm{in}})\Big) \xrightarrow{i^*} \tilde{R}^{*+f^*\beta}\Big(\mathrm{Fields}(\Sigma)\Big) \xleftarrow{f^*} \tilde{R}^{*+\beta}\Big(\mathrm{Fields}(\Sigma_{\mathrm{out}})\Big)$$

This is completely analogous to the diagram 3.2, except that the manifolds $\Sigma_{\mathrm{in}}, \Sigma_{\mathrm{out}}$ are not required to be 0-dimensional. The quantization of such a diagram can be interpreted as the quantization of the pQFT, *transgressed* to higher dimensions. More precisely, we can use such pull-push constructions to only quantize cobordisms of dimension $\geq k$, not the lower dimensional ones.

These give three different ways of obtaining cospan diagrams in categories of modules from pQFTs and their boundary theories. Of course, in each case we still have to choose the map $B^n U(1) \to BGL_1(R)$ to be able to pass to R-modules. We will discuss such choices for low values of n in the next section. Each of these choices appears in the literature on (topological) quantum field theories, and we will come back to their relevance to field theory in section 5.

3.1.4 Examples of twists

$\flat\mathbf{B}U(1)$-twist of ordinary complex cohomology

The easiest example of a twist in cohomology appears in ordinary cohomology. Let $\mathbb{C}$ be the *discrete* ring of complex numbers, which we identify with its Eilenberg-Maclane spectrum. Of course, the group of units of $\mathbb{C}$ is simply the discrete group $\mathbb{C}^\times$.

The discrete group $\flat U(1)$ embeds in the group of units of $\mathbb{C}$ and therefore we find a map

$$\flat\mathbf{B}U(1) \to \mathrm{Pic}(\mathbb{C})$$

describing a twist of complex ordinary cohomology by the classifying space $\flat\mathbf{B}U(1) = K(U(1), 1)$ of *flat* circle bundles. The ∞-category of $\mathbb{C}$-modules (where we view $\mathbb{C}$ as its Eilenberg-Maclane spaectrum) has a very simple presentation: it can be presented by the category of unbounded chain complexes of $\mathbb{C}$-modules (with $\mathbb{C}$ viewed as an ordinary ring), together with the projective model structure (see for example section 7.1.2 in [Lur12]).

Remark 3.1.17. In fact, this holds for any ordinary ring R. Taking $R = \mathbb{Z}$, we see that chain complexes of abelian groups give rise to spectra (which are modules over the Eilenberg-Maclane associated to $\mathbb{Z}$).

If X is an ∞-groupoid, modeled by a simplicial set, the homotopy colimit $\mathrm{colim}_X \mathbb{C}$ is given by the simplicial chain complex $C_*(X) := C_*(X, \mathbb{C})$ of X. Mapping this into $\mathbb{C}$, we obtain the simplicial cochain complex $C^*(X)$ (viewed as a chain complex in negative degree). The homotopy groups of these $\mathbb{C}$-modules are simply the ordinary complex (co)homology groups of the simplicial set X.

More generally, suppose X comes equipped with a map $X \to \flat\mathbf{B}U(1)$ classifying a flat $U(1)$-bundle over X. This factors over the fundamental groupoid $\Pi_{\leq 1}(X)$, since $\flat\mathbf{B}U(1)$ is 1-truncated. Then the homotopy colimit of the composite $X \to \mathbb{C}\mathrm{Mod}$ describes the cohomology with coefficients in the local system $\Pi_{\leq 1}(X) \to \mathbb{C}\mathrm{Mod}$.

A concrete example of this is the following: suppose X is connected and consider the universal 2-fold cover of X. This cover is classified by a map $X \to B\mathbb{Z}/2\mathbb{Z}$, which gives rise to a map $\alpha\colon X \to \flat\mathbf{B}U(1)$ by the inclusion of $\mathbb{Z}/2\mathbb{Z}$ in $U(1)$. Of course, the universal 2-fold cover of X is trivial iff X is an orientable space. In that case, as choice of orientation gives rise to a choice of trivialization of the 2-fold cover, which in turn induces a weak equivalence

$$C^{*+\alpha}(X) \simeq C^*(X)$$

between the α-twisted cohomology of X and the untwisted ordinary cohomology of X. We will come back to orientations in section 4.1, where they play a major role in the description of pushforward maps in cohomology.

Finally, the above homotopy theoretic discussion has a nice analogue in the case where $X = \Pi(M)$ is the homotopy type of a smooth manifold M. For smooth manifolds, we can use complex differential forms to describe the complex ordinary cohomology of M. Now suppose M comes equipped with a flat $U(1)$-bundle and let $L \to M$ be the associated line bundle. Being flat, this line bundle carries a flat connection ∇, which we can use to define the twisted de Rham complex

$$0 \longrightarrow \Gamma(L) \xrightarrow{\nabla} \Gamma(L \otimes T^*M) \xrightarrow{\nabla} \Gamma(L \otimes \textstyle\bigwedge^2 T^*M) \xrightarrow{\nabla} \cdots \longrightarrow \Gamma(L \otimes \textstyle\bigwedge^{top} T^*M) \longrightarrow 0$$

This is a cochain complex since the connection is flat. Then the twisted complex cohomology group of M are given by the homology groups of this cochain complex. In the case where $L = \mathbb{C}$ is the trivial bundle, then the connection is simply the de Rham differential d_{dR} and we obtain the ordinary complex cohomology of M.

$B^2U(1)$-twist of complex K-theory

Going up in dimension, we find a ring spectrum that allow for a twist by $B^2U(1)$: the ring spectrum KU classifying complex K-theory. We will give an extensive discussion of complex K-theory and its twist by $B^2U(1)$ in section 3.2, where we use the presentation of the K-theory spectrum in terms of Fredholm operators.

Instead, let us use this section to mention a purely homotopy theoretic characterization of the complex K-theory spectrum KU, due to Snaith [Sna81]. Consider the free ring spectrum $\mathbb{S}[BU(1)]$ (recall that this was just the suspension spectrum $\Sigma^\infty_+ BU(1)$). In degree 2, it contains a special element β, the Bott element, which corresponds to the map

$$S^2 \simeq \mathbb{C}P^1 \to BU(1)$$

classifying the tautological line bundle over $\mathbb{C}P^1$. Equivalently, we can present $BU(1) \simeq K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty$ as an infinite projective space, in which case the above map is simply the inclusion of the 2-skeleton. Multiplication by this element in degree two gives rise to maps of spectra

$$\mathbb{S}[BU(1)] \xrightarrow{\beta_*} \Sigma^{-2}\mathbb{S}[BU(1)] \xrightarrow{\beta_*} \Sigma^{-4}\mathbb{S}[BU(1)] \longrightarrow \cdots$$

We let $\big(\mathbb{S}[BU(1)]\big)[\beta^{-1}]$ be the homotopy colimit over the above diagram: it is the universal ring spectrum in which multiplication by β becomes an equivalence, i.e. it is the ring spectrum obtain by universally inverting the Bott element β.

Remarkably, this simple universal construction produces the complex K-theory spectrum:

Theorem 3.1.18 (Snaith, [Sna81])**.** *There is an equivalence of E_∞ ring spectra*

$$\big(\mathbb{S}[BU(1)]\big)[\beta^{-1}] \xrightarrow{\sim} KU$$

where KU is the complex K-theory spectrum as defined, for example, in section 3.2.

Remark 3.1.19. A proof of this result lies outside the scope of this text. One can construct a map of spectra

$$\mathbb{S}[BU(1)] \to KU$$

by sending each line bundle to its associated K-theory class. Since the Bott element is invertible in complex K-theory, this factors over the localization $\big(\mathbb{S}[BU(1)]\big)[\beta^{-1}]$ and the result is a morphism of ring spectra. An explicit computation shows that this morphism induces an isomorphism on homotopy groups (see for example the notes [Mat12]); this computation makes essential use of the fact that both spectra give rise to complex oriented multiplicative cohomology theories, which is well known for KU and true essentially by construction for $\big(\mathbb{S}[BU(1)]\big)[\beta^{-1}]$.

As a result of the adjunction 3.1.8, the map $\mathbb{S}[BU(1)] \to \big(\mathbb{S}[BU(1)]\big)[\beta^{-1}] \xrightarrow{\sim} KU$ corresponds to a morphism of abelian ∞-groups

$$BU(1) \to GL_1(KU).$$

This gives an abstract construction of the twist of complex K-theory by $B^2U(1)$, which can serve as a blueprint for a smooth refinement of the K-theory spectrum (in the sense of the discussion in section 3.1.5). In section 3.2, we will see how one can concretely realize this twist in terms of Hilbert bundles and Fredholm operators.

In the examples, we will see the twist of KU by $B^2U(1)$ appear in the context of two-dimensional field theory, describing the propagation of 1-dimensional objects (strings). This gives rise to a partition function of the boundary theories to such a 2d theory, which describe the propagation of particles. These are precisely the partition functions in (supersymmetric) quantum mechanics, which can be described in terms of K-theory via the index of the Dirac operator (see for example the contribution of Stolz-Teichner in [SS11]).

$B^3U(1)$**-twist of tmf**

Going up to dimension three brings us to a much more involved ring spectrum, known as tmf. Its homotopy groups form a graded ring of certain topological modular forms, from which it obtains its name. One of the major recent results in algebraic topology is the construction of this spectrum as the global sections of a certain sheaf of spectra over the stack of elliptic curves [Hop02].

It turns out that the classifying space $B^3U(1)$ of circle 3-bundles gives rise to a twist in tmf:

Proposition 3.1.20 ([ABG10], section 8)**.** *There is a map of abelian ∞-groups*

$$B^3U(1) \to BGL_1(\mathrm{tmf}).$$

Remark 3.1.21. This twist can be constructed from the string orientation for tmf [AHR06], which we also see later. The obstruction to a string structure is given by a circle 3-bundle (the geometric incarnation of the first fractional Pontryagin class), which results in a twist of tmf [ABG10].

A discussion of this result lies beyond the scope of this text. However, it produces an interesting example of our abstract discussion: just like the quantum mechanical particle on the boundary of a string has a partition function taking values in K-theory, we obtain the partition function of the string on the boundary of a brane taking values in modular forms (as discussed by Stolz-Teichner in [SS11]). This partition function is also known as the Witten genus: abstractly, it is just given by a pushforward in (twisted) tmf, as we will discuss in section 4.1.5.

3.1.5 Smooth refinement

We have seen in the previous paragraphs how to construct a linearized prequantum field theory

$$\mathrm{Bord}_n \to \mathrm{Corr}_n(\infty\mathrm{Gpd}_{/\Pi\left(\mathbf{B}^nU(1)\right)}) \to \mathrm{Corr}_n(\infty\mathrm{Gpd}_{/BGL_1(R)})$$

from an embedding of the geometrically discrete groups $B^{n-1}U(1) = \Pi\big(\mathbf{B}^{n-1}U(1)\big)$ into groups of units $GL_1(R)$ of (discrete) ring spectra. This allows us to associate cospan diagrams of R-modules to span diagrams of ∞-groupoids carrying a circle n-bundle.

This procedure applies only to the case where all objects involved are geometrically discrete: both the groups $B^{n-1}U(1)$ and the spaces of fields Fields have to be discrete ∞-groupoids, carrying the trivial smooth structure. Such field theories, where the stack **Fields** is geometrically discrete and carries a (flat) circle n-bundle, are also known as ∞-Dijkgraaf-Witten theories. A related discussion of their quantization is given in [FHLT10].

However, for most theories the classifying stack of fields **Fields** carries an actual smooth structure, while is comes equipped with a *smooth* higher circle bundle $\mathbf{Fields} \to \mathbf{B}^nU(1)$ which is not flat. For example, three dimensional Chern-Simons theory has as classifying stack the stack $\mathbf{B}G$ of G-bundles, where G is a Lie group, carrying a smooth circle 3-bundle $\mathbf{B}G \to \mathbf{B}^3U(1)$ induced by a level in

$$\pi_0\mathbf{H}(\mathbf{B}G, \mathbf{B}^3U(1)) \simeq H^4(BG, \mathbb{Z}) \simeq \mathbb{Z}.$$

The homotopy theoretic picture from the previous sections does not apply to this smooth situation.

Remark 3.1.22. In fact, this smooth circle 3-bundle refines to a smooth circle 3-bundle with connection on the moduli stack $\mathbf{B}G_{\mathrm{conn}}$ of G-bundles with connection. This differential refinement is what really gives Chern-Simons theory. In turn, one can realize this as a boundary to a certain 4d-theory whose action functional takes values in $\mathbf{B}^4U(1)$ [FSS].

There is a naive way to pass around this: first, given a map of smooth stacks $\mathbf{Fields} \to \mathbf{B}^nU(1)$, we can apply the functor $\Pi(-)\colon \mathbf{H} \to \infty\mathrm{Gpd}$ that sends each smooth stack to its homotopy type (traditionally also called its geometric realization). For example, we replace a smooth manifold M by (the homotopy type of) the underlying topological space of M, which looks like a reasonable approximation to the smooth manifold M. In this way we obtain a map of discrete ∞-groupoids

$$\mathrm{Fields} := \Pi(\mathbf{Fields}) \to B^nU(1).$$

One of the main properties of the functor $\Pi(-)$ is that it preserves finite products; this forms one half of the set of axioms for cohesion [Sch12]. Since the group structure of $B^nU(1)$ is the one induced by $\mathbf{B}^nU(1)$ under $\Pi(-)$, this then gives a monoidal functor

$$\Pi\colon \mathbf{H}_{/\mathbf{B}^nU(1)} \to \infty\mathrm{Gpd}_{/B^nU(1)}$$

Applying Π to the smooth action functional $\mathbf{Fields} \to \mathbf{B}^nU(1)$, we obtain a discrete action functional $\mathrm{Fields} \to B^nU(1)$ which classifies a discrete pQFT. In this case, we can apply the examples discussed in the previous section to linearize our theory. We use this perspective in the discussion of the string living on the boundary of a 2-brane in section 5.3.

When **Fields** is a smooth manifold, this provides a fairly good approximation to the smooth pQFT we start with. But if **Fields** is a true stack, then the (twisted) cohomology of its homotopy type is typically coarser than one would expect. For example, if M is a smooth manifold carrying a G-action, then we can form the quotient stack $M//G \in H$. We would expect the R-cohomology of this quotient stack to be the equivariant R-cohomology of the space M, but the R-cohomology of the homotopy type $\Pi(M//G)$ is not always given by the equivariant R-cohomology of $\Pi(M)$. For example, in the context of complex K-theory we have the following completion theorem, due to Atiyah and Segal:

Proposition 3.1.23 ([AS69])**.** *Let M be a compact space carrying the action of a compact Lie group G. Then the natural map induced by pullback along the G-equivariant projection $(M \times EG)/G \to M$*

$$K^*_G(M) \to K^*\big((M \times EG)/G\big) \simeq K^*_G(M)\hat{_I}.$$

realizes $K^*\big((M \times EG)/G\big)$ as the localization of $K^*_G(M)$ at the augmentation ideal I of the representation ring R (pulled back to $K^*_G(M)$). Here EG is the contractible space carrying a principal G-action.

In particular, if $M = *$ is the point, we find that

$$K^*(BG) \simeq K^*_G(EG) \simeq R(G)\hat{_I}$$

is the completion of the representation ring of G with respect to its augmentation ideal.

It is shown in [Sch12] that $\Pi(M//G) \simeq (M \times EG)/G$, so the above result shows that the K-theory of $\Pi(M//G)$ is a *completion* of the G-equivariant K-theory of M.

To avoid this kind of behaviour, we have to find an intrinsically smooth refinement of the abstract picture sketched in the previous sections. In fact, most of the things we have discussed in section 3.1.1 immediately extend to the smooth setting, by the general framework of higher algebra developed in [Lur12]:

Definition 3.1.24. Let $\mathrm{Sp}(\mathbf{H})$ be the ∞-category of spectra in the ∞-category of smooth stacks, i.e. sequences of pointed stacks $\{E_n\}$ with equivalences $E_n \xrightarrow{\sim} \Omega E_{n+1}$.

Remark 3.1.25. One can use the same construction as in section 3.1.1 to realize $\mathrm{Sp}(\mathbf{H})$ as a certain subcategory of $\mathrm{Fun}(\mathbb{Z} \times \mathbb{Z}, \mathbf{H})$. One obtains the same looping and suspension functors, which are given by the same formulas (as discussed in [Lur12]).

Equivalently, one can view spectra in $\mathbf{H}$ as sheaves of spectra over the sites SmMfd or CartSp. This follows immediately from the fact that homotopy pullbacks of stacks can be computed objectwise. The category of sheaves of spectra is a left exact localization of the ∞-category of presheaves of spectra.

The resulting category of sheaves of spectra carries a symmetric monoidal structure: it is simply given by first taking the objectwise smash product of spectra (which gives a presheaf of spectra over CartSp) and then mapping this to the associated sheaf of spectra (cf. section 1 in [Lur11]). Its monoidal unit is the constant sheaf whose value is the sphere spectrum $\mathbb{S}$. This procedure of forming the smash product of smooth spectra shows that it preserves homotopy colimits in both of its variables. Since $\mathrm{Sp}(\mathbf{H})$ is locally presentable, this means that it admits a right adjoint: the smooth mapping spectrum.

The general theory developed in [Lur12] described how the results in section 3.1.1 can be carried over to this setting. We have a notion of smooth E_∞-rings as E_∞ rings with respect to the smash product on $\mathrm{Sp}(\mathbf{H})$. Moreover, we can talk about smooth modules over a smooth ring spectrum $\mathbf{R}$, which carry a monoidal structure by taking the smash product over $\mathbf{R}$. This again preserves homotopy colimits, so it admits an $\mathbf{R}$-linear mapping spectrum. The functor forgetting the $\mathbf{R}$-module structure has a left adjoint which smashes a smooth spectrum with $\mathbf{R}$.

Finally, we have an adjunction

$$\mathrm{CRing}_\infty(\mathbf{H}) \underset{GL_1(-)}{\overset{\mathbb{S}[-]}{\leftrightarrows}} \mathrm{AbGrp}_\infty(\mathbf{H}) \tag{3.5}$$

between smooth abelian groups and smooth E_∞ ring spectra, just as in for discrete groups and rings.

In analogy to the discrete case, a sheaf of ring spectra $\mathbf{R}$ describes a smooth multiplicative cohomology theory. If M is an ordinary manifold, then one can form its suspsension spectrum and consider the (derived) mapping spaces

$$\mathrm{Sp}(\mathbf{H})\big(\Sigma^\infty M, \mathbf{R}\big).$$

The homotopy groups of this space are the smooth $\mathbf{R}$-cohomology groups of the manifold M. They are descibed by sheaf cohomology groups with coefficients in the sheaf of spectra $\mathbf{R}$. In analogy with remark 3.1.17, a chain complex of sheaves of abelian groups presents a sheaf of spectra $\mathbf{R}$. In that case, the above homotopy groups can be described by hypercohomology groups with values in that chain complex of sheaves of abelian group (see [Bro73]).

Remark 3.1.26. In parts of the literature 'smooth cohomology' is used for 'differential cohomology', which refines generalized cohomology by adding connections. Notice that what we discuss here is different: we genuinely refine from topological to (smooth) sheaf cohomology. Differential cohomology gives an even further refinement of this smooth sheaf cohomology.

For the abstract picture of twisted cohomology from section 3.1.2 to carry on in this smooth setting, we would need smooth analogues of the categories of modules over a ring spectrum. These would be sheaves of $(\infty, 1)$-categories, which become a bit hard to keep track of. Instead of using the abstract definition of twisted spectra in terms of homotopy colimits, we therefore use the original perspective in terms of *bundles*: recall that a twist $\alpha : X \to BGL_1(R)$ gave rise to a bundle of copies of R, whose total space we could model by the homotopy colimit $\mathrm{colim}_X\, \alpha$ in the category of R-modules.

In the smooth setting, we do not have a theory of homotopy colimits of maps of smooth ∞-groupoids, but we do have a well-developed theory of bundles. We can then give the following definition, which is the smooth analogue of the definition of the twisted R-homology spectrum given in [ABG+08].

Definition 3.1.27. Let $\mathbf{R}$ be a smooth E_∞ ring spectrum and let $GL_1(\mathbf{R}) \in \mathbf{H}$ be the associated smooth ∞-group of units of $\mathbf{R}$. Given a twist $\alpha : X \to \mathbf{B}GL_1(\mathbf{R})$ over a smooth ∞-groupoid X, let P be the associated principal ∞-bundle that fits into the homotopy cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d, "p"'] & * \arrow[d] \\ X \arrow[r, "\alpha"'] & \mathbf{B}GL_1(\mathbf{R}) \end{tikzcd}$$

Then we define the α-twisted smooth R-homology spectrum $R_{*+\alpha}(X)$ to be the smash product of smooth $\mathbb{S}(GL_1(\mathbf{R}))$-modules

$$\mathbf{R}_{*+\alpha}(X) := \Sigma^\infty_+ P \wedge_{\mathbb{S}[GL_1(\mathbf{R})]} \mathbf{R}.$$

Remark 3.1.28. We have not discussed the general abstract theory of principal ∞-bundles; essentially the result from proposition 2.2.32 remains true if we replace all manifolds by stacks. In particular, $\mathbf{B}GL_1(\mathbf{R})$ can be realized as the homotopy colimit over a simplicial diagram whose n-th space is the n-fold product of stacks $GL_1(\mathbf{R})$. A detailed treatment of the theory of principal ∞-bundles can be found for instance in [NSS12].

Remark 3.1.29. Both $\Sigma^\infty_+ P$ and $\mathbf{R}$ naturally form smooth modules over the smooth ring $\mathbb{S}[GL_1(\mathbf{R})]$. Indeed, the adjunction 3.5 gives rise to a counit map $\mathbb{S}[GL_1(\mathbf{R})] \to \mathbf{R}$ of smooth E_∞ ring spectra, which induces a $\mathbb{S}[GL_1(\mathbf{R})]$-module structure on $\mathbf{R}$.

On the other hand, the abelian ∞-group $GL_1(\mathbf{R})$ acts naturally on P, so the free smooth ring spectrum $\mathbb{S}[GL_1(\mathbf{R})]$ acts naturally on the suspension spectrum $\Sigma^\infty_+ P$.

Observe that this smooth spectrum has a natural $\mathbf{R}$-action, since $\mathbf{R}$ is a bimodule over $\mathbb{S}[GL_1(\mathbf{R})]$ and itself. The twisted cohomology is given by mapping into $\mathbf{R}$:

Definition 3.1.30. The α-twisted smooth $\mathbf{R}$-cohomology of X is given by the mapping spectrum into the smooth ring $\mathbf{R}$

$$\mathbf{R}^{*+\alpha}(X) := \left[\Sigma^\infty_+ P \wedge_{\mathbb{S}[GL_1(\mathbf{R})]} \mathbf{R}, \mathbf{R}\right].$$

Remark 3.1.31. Given a map of stack $f\colon X \to Y$, there is a natural map $f^*\colon \mathbf{R}^{*+\alpha}(Y) \to \mathbf{R}^{*+f^*\alpha}(X)$, obtained by a map of $GL_1(\mathbf{R})$-bundles covering the map f. We see that the twisted smooth twisted $\mathbf{R}$-cohomology defines functor from spaces in $\mathbf{H}$ carrying a smooth $GL_1(\mathbf{R})$-bundle to the category of smooth $\mathbf{R}$-modules:

$$\Gamma\colon \mathbf{H}_{/\mathbf{B}GL_1(\mathbf{R})} \longrightarrow \mathbf{R}\mathrm{Mod}^{op}$$

We cannot press the theory much further at this point; we are currently missing concrete examples of smooth spectra. However, Snaith's theorem 3.1.18 gives a suggestion for how to proceed, as was amplified by Thomas Nikolaus. Given the smooth stack $\mathbf{B}^nU(1)$, there is a universal smooth ring that has $\mathbf{B}^nU(1)$ in its group of units: we can simply take the free smooth ring spectrum $\mathbb{S}[\mathbf{B}^nU(1)]$. This certainly allows us to 'add up' the higher phases in $\mathbf{B}^nU(1)$, but they add up formally, so there is no interference. To force quantum interference, we have to pass to a certain quotient, or localization, of the free ring. For example, Snaith's theorem suggests that we might obtain a smooth refinement of complex K-theory by taking a localization

$$\mathbb{S}[\mathbf{B}U(1)][\beta^{-1}]$$

with respect to a certain smooth Bott element.

We expect that a good theory of quantization can be developed along these lines, although a further discussion of this issue lies beyond our scope. Following a different path, we can obtain a more tractable model of a generalized cohomology of stacks that does resolve the forementioned issues. This cohomology theory is given by the complex K-theory of differentiable stacks.

3.2 K-theory for local quotient stacks

In this section, we will shortly recall the description of twisted K-theory by Atiyah and Segal [AS04], and discuss an extension of this to the class of local quotient stacks, due to [FHT11]. This section serves mainly as a bridge between the homotopy theory in the previous section and the operator theoretic discussion in the next section.

3.2.1 Fredholm picture of twisted K-theory

There are several different descriptions of the complex K-theory spectrum KU. We give the presentation from [AS04] in terms of the following spaces of Fredholm operators:

Definition 3.2.1. Let $Cl(n)$ be the complexified Clifford algebra associated to the standard inner product on $\mathbb{R}^n$, with its $\mathbb{Z}/2\mathbb{Z}$-grading. We let S_n be an $\mathbb{Z}/2\mathbb{Z}$-graded irreducible $Cl(n)$-module.

Let H_0 be the separable $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space whose even and odd part are both infinite dimensional. Then $S_n \otimes H_0$ is a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space carrying an action of $Cl(n)$. We define the spaces

$$\mathrm{Fred}^{(n)} := \left\{F \in B(S_n \otimes H_0) : F \text{ odd, } F^* = F, F^2 - 1 \in K(S_n \times H_0) \text{ and } [F,\gamma] = 0 \text{ for all } \gamma \in Cl(n)\right\}$$

whose topology is the topology induced by the embedding

$$\mathrm{Fred}^{(n)} \to B(S_n \otimes H_0) \times K(S_n \otimes H_0); \quad F \mapsto (F, F^2 - 1)$$

where $B(S_n \otimes H_0)$ carries the compact-open topology and $K(S_n \otimes H_0)$ carries the norm topology.

It is well known that these spaces together form a spectrum satisfying a kind of Bott periodicity, as shown in [AS04]:

Lemma 3.2.2. *There are natural maps* $\mathrm{Fred}^{(n)} \to \Omega\mathrm{Fred}^{(n+1)}$ *which are weak equivalences.*

Lemma 3.2.3. *There is a natural weak equivalence* $\mathrm{Fred}^{(n)} \xrightarrow{\simeq} \mathrm{Fred}^{(n+2)}$.

Using this equivalence, we then define the complex K-theory spectrum by

Definition 3.2.4. Let KU be the spectrum given by

$$KU_n = \begin{cases} \mathrm{Fred}^{(0)} & n \text{ even} \\ \mathrm{Fred}^{(1)} & n \text{ odd} \end{cases}$$

where the maps $KU_n \to \Omega KU_{n+1}$ are induced by weak equivalences from the previous lemmas.

Since the spectrum KU satisfies Bott periodicity, one can easily compute its homotopy groups

Proposition 3.2.5. *The homotopy groups of the spectrum KU are given by*

$$\pi_n(KU) = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

The homotopy class of a Fredholm operator $F \in \mathrm{Fred}^{(0)}$ is uniquely described by its index $[\ker F] - [\mathrm{coker}\, F]$*. Such a virtual vector space is uniquely defined by its virtual rank, which gives the corresponding element in $\mathbb{Z}$.*

If M is a locally compact Hausdorff space, we define its complex K-theory spectrum to be the mapping spectrum

$$K^*(M) := \left[\Sigma^\infty_+ M, KU\right]$$

This slightly conflicts with the traditional notation for K-theory: the above definiton gives the *representable* K-theory of the space M, which is also denoted $RK^*(M)$. On the other hand, in the literature on often uses the *compactly* supported K-theory spectrum, whose spaces consist not of all maps $M \to \mathrm{Fred}^{(n)}$, but only those functions that map into the unitary operators outside of a compact in M. We will denote this spectrum by $K^*_c(M)$, its homotopy groups are given by

$$K^n_c(M) := \left\{F\colon M \to \mathrm{Fred}^{(n)} \middle| F \text{ unitary outside a compact}\right\} \Big/ \text{homotopy}$$

When M is compact, the two of course agree. Note that compactly supported K-theory is only functorial with respect to proper maps of spaces.

Given a compactly supported M-family of Fredholm operators $F\colon M \to \mathrm{Fred}^{(0)}$, taking their kernels and cokernels gives a formal difference of two vector spaces over each point. These do not quite give vector bundles, but by deforming F by a homotopy and adding trivial bundles to both the kernels and the cokernels, we do obtain a formal difference $[V] - [W]$ between two vector bundles over M. Since F is unitary outside of compact, V and W come equipped with an equivalence outside of a compact subset of M. This produces the classical description of $K^0_c(M)$:

Proposition 3.2.6 ([Ati89a])**.** *$K^0_c(M)$ is given by the abelian group of isomorphism classes of virtual vector bundles $[V] - [W]$ over M, together with an isomorphism $V \xrightarrow{\sim} W$ outside of a compact subset of M.*

This gives the relation between K-theory and vector bundles, which really forms the starting point of K-theory. Most K-classes that we encounter arise like this from (virtual) vector bundles.

Finally, we can use the language of Fredholm operators to give the twist in K-theory associated to a (topological) circle 2-bundle on a topological space $M \to B^2U(1)$. This uses an equivalent model for $B^2U(1)$ by the classifying space of the projective unitary group on H_0.

Lemma 3.2.7. *Let $U(H_0)$ be the group of unitary operators on the separable, infinite-dimensional Hilbert space H_0, with the compact-open topology. There is a short exact sequence of topological groups*

$$U(1) \to U(H_0) \to PU(H_0)$$

Since $U(H_0)$ is contractible, there is an weak homotopy equivalence of groups $PU(H_0) \simeq BU(1)$.

Remark 3.2.8. By the compact open topology on $U(H_0)$ we really means the topology induced by the inclusion

$$U(H_0) \to B(H_0) \times B(H_0); \quad u \mapsto (u, u^{-1})$$

where $B(H_0)$ carries the *compact open topology.* This means that a sequence u_n converges to u iff both that sequence and the sequence u_n^{-1} converge to u, resp. u^{-1} in the standard compact-open topology on $U(H_0)$.

The fact that $U(H_0)$ is contractible in the strong topology is well known (Kuiper's theorem). The same thing holds in the compact-open topology, as was shown in [AS04].

This allows us to associate to any circle 2-bundle a bundle with structure group $PU(H_0)$. We can let $PU(H_0)$ act on the spaces of Fredholm operators $\mathrm{Fred}^{(n)}$:

Lemma 3.2.9. *The action by conjugation of $U(H_0)$ on $\mathrm{Fred}^{(n)}(H_0)$ is continuous.*

Proof. $\mathrm{Fred}^{(n)}(H_0)$ obtains its topology from the inclusion into $B(H_0) \times K(H_0)$, where the first has the compact open topology and the second has the norm topology. Certainly the conjugation action on $B(H_0)_{c.o.}$ is continuous. It is shown in [AS04] that the conjugation action on $K(H_0)_{norm}$ is continuous as well. □

Since the group $PU(H_0)$ acts continuously by conjugation on each of the spaces $\mathrm{Fred}^{(n)}$, we can use a $PU(H_0)$-bundle to form a twisted bundle of Fredholm operators. The sections of this bundle describe twisted K-theory classes.

Definition 3.2.10 ([AS04])**.** Let $\chi : M \to \mathbf{B}^2U(1)$ classify a circle 2-bundle over the space M and let $P \to M$ be the corresponding $PU(H_0)$-bundle. Let

$$\mathrm{Fred}^{(n)+\chi} := P \times_{PU(H_0)} \mathrm{Fred}^{(n)}$$

be the associated bundle of Fredholm operators. Then the χ-twisted K-theory spectrum of M is the spectrum consisting of the spaces

$$\Gamma\big(M, \mathrm{Fred}^{(n)+\chi}\big)$$

of sections of this bundle. Similarly, the χ-twisted compactly supported K-theory spectrum consists of the spaces of sections that give unitaries outside of a compact subset of M.

This agrees with the abstract homotopy-theoretic construction of twisted K-theory and has all the expected properties [FHT11]: it satisfies homotopy invariance, excision, Bott periodicity and has a long exact sequence associated to a pair of spaces (for which we have the expected notion of relative twisted K-theory). Combining the last two properties, we obtain the Mayer-Vietoris sequence in twisted K-theory.

Example 3.2.11. Let $\alpha\colon S^3 \to \mathbf{B}^2U(1)$ classify the circle 2-bundle corresponding to the element $n \in H^3(S^3, \mathbb{Z})$. Then $K^{0+\alpha}(S^3) = 0$ and $K^{1+\alpha}(S^3) = \mathbb{Z}/n\mathbb{Z}$. This follows immediately from the Mayer-Vietoris sequence, where we cover S^3 by $U_+ = S^3 \setminus (1,0,0)$ and $U_- = S^3 \setminus (-1,0,0)$. Since the restriction of α to U_+, U_- and $U_+ \cap U_-$ is always trivializable, we find an exact sequence

$$0 \longrightarrow K^{0+\alpha}(S^3) \longrightarrow K^0(U_+) \oplus K^0(U_-) \longrightarrow K^0(U_+ \cap U_-) \longrightarrow K^{1+\alpha}(S^3) \longrightarrow 0$$

The middle two terms are both just $\mathbb{Z} \oplus \mathbb{Z}$. To identify the map between them, we have to remember how we trivialized the twist α on U_+ and U_-. The two twisted line bundles trivializing the circle 2-bundle differ by a line bundle over $U_+ \cap U_- \sim S^2$: this line bundle is precisely the n-fold tensor product of the tautological line bundle L over S^2. If we therefore trivialize α on $U_+ \cap U_-$ using the trivialization on U_+, then the above map is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} : K^0(U_+) \oplus K^0(U_-) = \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \cdot L \simeq K^0(S^2)$$

Indeed, the trivial line bundle on U_+ restricts to the trivial line bundle on $U_+ \cap U_-$, but the trivial bundle on U_- is twisted by $L^{\otimes n}$. The result then follows.

Since untwisted K-theory has a nice interpretation in terms of virtual vector bundles, we might expect the same thing to hold in this case: we might want to present K-theory classes by twisted vector bundles.

Definition 3.2.12. Let $M \to \mathbf{B}^2U(1)$ be a circle 2-bundle presented by a cocycle $g_{ijk} : U_{ijk} \to U(1)$ over the triple overlaps of a good open cover of the manifold M. A (complex) twisted vector bundle over M of rank n is given by a cocycle $h_{ij} : U_{ij} \to U(n)$ such that

$$h_{ij}h_{jk} \cdot g_{ijk} = h_{ik} \colon U_{ijk} \to U(n).$$

In other words, a twisted vector bundle is given by a $U(n)$-cocycle, up to the twist by the cocycle g_{ijk}. Any twisted vector bundle gives rise to a class in twisted K-theory and in fact, the twisted K-classes that we want to push in the examples are given by twisted line bundles.

However, usually there a no twisted vector bundles for a circle 2-bundle. Indeed, by taking the determinant in the above twisted cocycle condition, we see that

$$\det(h)_{ij} \det(h)_{jk} g_{ijk}^n = \det(h)_{ik}$$

which means that the n-fold power of the circle 2-bundle is trivialized by the determinant of the twisted vector bundle. We therefore only have twisted vector bundles if the circle 2-bundle is torsion. Because of this reason, it is necessary to work with infinite dimensional Hilbert spaces to produce twisted K-theory classes over manifolds: if the circle 2-bundle is non-torsion, there are no twisted vector bundles to realize twisted K-classes.

Summarizing, we see that the twisted K-theory of a manifold can be presented by sections of a bundle of Fredholm operators. By replacing this bundle of Fredholm operators by a suitable *equivariant* bundle of Fredholm operators, one can give a similar description of the K-theory of a certain class of stacks, known as local quotient stacks.

3.2.2 Extension to local quotient stacks

Following the discussion by Freed-Hopkins-Teleman [FHT11], we define a K-theory for a certain class of stacks using the Fredholm picture from the previous section. If M is an ordinary smooth manifold, this produces the (representable) K-theory of the homotopy type of M, but for a more general stack X, we find that it is more refined than the K-theory of its homotopy type.

First, we define the class of stacks for which this construction of a K-theory spectrum works. It is a subclass of the class of differentiable stacks:

Definition 3.2.13. A differentiable stack is called a local quotient stack if it can be presented by a Lie groupoid having a cover of open subgroupoids which are all equivalent to the action groupoids $M//G$ of a proper G-action on a manifold M.

The basic idea behind this construction of K-theory for such local quotient stacks is to replace the Hilbert space used in the previous section by a bundle of Hilbert spaces, in analogy with the construction of twisted K-theory. The groupoid structure should give an action on this bundle of Hilbert spaces. We then form the associated bundles of Fredholm operators and consider certain sections of these bundles.

Example 3.2.14. Let G be a compact Lie group and consider the stack $\mathbf{B}G$, presented by the delooping Lie groupoid $*//G$ of G. Then a bundle of Hilbert spaces over the space of objects of $*//G$ is just a Hilbert space, and G is required to act on it. The Hilbert space has to be 'infinite-dimensional enough', also as a G-module. We can do this by taking $H = H_0 \otimes L^2(G)$ with H_0 a separable Hilbert space, where G acts only on $L^2(G)$.

We now give the general definition of a Hilbert bundle over a Lie groupoid X:

Definition 3.2.15. A Hilbert bundle $H \to X$ over a Lie groupoid X is a locally trivial bundle $H \to X_0$ of $\mathbb{Z}/2$-graded separable Hilbert spaces H_0, together with an equivalence of bundles $s^*H \to t^*H$ over X_1, which for each morphism $g\colon x \to y$ gives a unitary map

$$H_x \xrightarrow{\phi_g} H_y$$

such that $\phi_h \phi_g = \phi_{hg}$ whenever the composition makes sense. In other words, it is given by an equivariant Hilbert bundle over the space X_0.

Remark 3.2.16. There is a stack **Hilb** of Hilbert bundles in the category **H**. Then a Hilbert bundle over a Lie groupoid X presents a map of stacks $X \to \mathbf{Hilb}$.

Remark 3.2.17. Each point $x \in X_0$ has an open neighbourhood U_x on which the bundle $H \to X_0$ looks like

$$\begin{array}{c} U_x \times H_0 \\ \Big\downarrow{\scriptstyle p_1} \\ U_x \end{array}$$

On intersections $U_x \cap U_y$, the trivializations induce transition functions

$$U_x \cap U_y \to U(H_0)$$

mapping into the unitary operators on H_0. Here $U(H_0)$ has the compact-open topology from remark 3.2.8. The equivalence of bundles $s^*H \to t^*H$ over X_1 also has to be continuous with respect to this topology on $U(H_0)$.

We can equivalently view a Hilbert bundle H as principal $U(H_0)$-bundle P over the stack X. Since $U(H_0)$ acts on Fredholm operators, we can form the associated bundle of Fredholm operators.

Definition 3.2.18. For H a Hilbert bundle over a Lie groupoid X, define the associated bundles of Fredholm operators by

$$\mathrm{Fred}^{(n)}(H) = P \times_{U(H_0)} \mathrm{Fred}^{(n)}(H_0).$$

This bundle of Fredholm operators is equivariant with respect to the groupoid structure on X.

In our definition of K-theory for topological spaces, we took an infinite dimensional separable Hilbert space H_0 and considered the Fredholm operators on this Hilbert space. By taking H_0 infinite dimensional, we could realize any formal difference $[V] - [W]$ of finite dimensional vector spaces as the index of some Fredholm operator. On the other hand, if we had taken H_0 to be finite dimensional, then all operators would have had index 0.

It is therefore important to take H_0 large enough, which is to say that it is locally universal.

Definition 3.2.19. A Hilbert bundle $H \to X$ on X is called universal if any other Hilbert bundle embeds in H. A Hilbert bundle H is called locally universal if its restriction to any open subgroupoid is universal.

Example 3.2.20. Any universal Hilbert bundle should have infinite dimensional fibers, so the structure group of the bundle $H \to X_0$ is the group $U(H_0)$ of unitary operators on an infinite dimensional H_0. Since $U(H_0)$ is contractible, the bundle $H \to X_0$ is equivalent to a trivial bundle.

In case that $X = M$ is a manifold, without any nontrivial morphisms, a locally universal Hilbert bundle is then simply $M \times H_0 \xrightarrow{p_1} M$. Since the associated Fredholm bundle is also trivial, its sections are precisely functions

$$M \to \mathrm{Fred}^{(n)}(H_0)$$

Taking homotopy classes of sections produces the (representable) K-theory of M.

When the stack X is not a manifold, we have to keep track of the morphisms of X acting on the Hilbert bundle H.

Theorem 3.2.21 ([FHT11])**.** *Any local quotient groupoid admits a locally universal Hilbert bundle. Moreover, such Hilbert bundle is unique up to (nontrivial) isomorphism.*

Proof. We first show uniqueness: if H is a universal Hilbert bundle, its fibers are infinite dimensional. This means that $H \otimes \ell^2 \simeq H$, by picking an equivalence $H_0 \otimes \ell^2 \simeq H_0$. Now let V be any other Hilbert bundle, and embed it in the universal bundle H so that $H = V \oplus V^\perp = V^\perp \oplus V$. We then have that

$$\begin{aligned} H &\simeq H \otimes \ell^2 \simeq H \oplus H \oplus ... \simeq (V \oplus V^\perp) \oplus (V \oplus V^\perp) \oplus ... \\ &\simeq V \oplus (V^\perp \oplus V) \oplus ... \simeq V \oplus (H \otimes \ell^2) \simeq V \oplus H \end{aligned}$$

If H' is now another universal Hilbert bundle, we have that

$$H \simeq H \oplus H' \simeq H'$$

which shows the uniqueness of universal Hilbert bundles.

We construct a locally universal Hilbert bundle on X by constructing them locally and adding up all these local bundles. Locally, X looks like an action groupoid $U//G$ of a compact Lie group G. Over such groupoid, consider the Hilbert bundle

$$H := U \times H_0 \otimes L^2(G) \xrightarrow{p_1} U$$

where H_0 is the $\mathbb{Z}/2\mathbb{Z}$-graded separable Hilbert space whose even and odd part are both infinite dimensional. The arrows in $U//G$ act on this via

$$(x, v \otimes f) \overset{\phi_g}{\longmapsto} (g \cdot x, v \otimes R_g^* f)$$

where $R_g : G \to G$ is the right multiplication by $g \in G$. It is shown in [FHT11] that this indeed defines a locally universal Hilbert bundle over $U//G$. Moreover, the authors give a construction of a locally universal Hilbert bundle from such Hilbert bundles on an open cover, essentially by adding them all up. □

Given such a locally universal Hilbert bundle H, we can form the associated bundles of Fredholm operators, $\mathrm{Fred}^{(n)}(H)$.

Definition 3.2.22. The K-theory spectrum of a local quotient groupoid is defined by

$$K^*(X) := \Gamma(X, \mathrm{Fred}^{(*)}(H))$$

where $\Gamma(X, \mathrm{Fred}^{(*)}(H))$ is the space of *equivariant* sections of the Fredholm bundle over X_0, endowed with the compact-open topology. If we denote by $\pi : X_0 \to X_0/X_1$ the projection onto the orbit space of the Lie groupoid, then the compactly supported K-theory spectrum of X is defined by

$$K_c^*(X) := \left\{ F \in \Gamma(X, \mathrm{Fred}^{(n)}(H) : F \text{ is unitary outside of } \pi^{-1}(K), \text{ with } K \subset X_0/X_1 \text{ compact} \right\}$$

Remark 3.2.23. As we already stated, if $X = M$ is an ordinary manifold, then this produces the K-theory of the space underlying the manifold M.

Example 3.2.24. If $X = M//G$ is the quotient of a manifold by the action of a compact Lie group G, then this definition agrees with the definition of the G-equivariant K-theory developed in [Seg68]. In particular, we have the analogue of proposition 3.2.6:

Proposition 3.2.25 ([FHT11])**.** *If M is a manifold carrying an action of a compact Lie group G, then $K_c^0(M//G)$ is naturally isomorphic to the group of virtual G-equivariant vector bundles $[V] - [W]$, such that $[V]$ and $[W]$ become isomorphic outside of the G-orbit of a compact subset.*

Example 3.2.26. As an immediate corollary, we obtain that for a compact Lie group G, the group $K^0_c(\mathbf{B}G)$ agrees with the representation ring $R(G)$ of G, i.e. with the ring of virtual (finite dimensional) G-representations.

We have the expected functoriality of complex K-theory of local quotient stacks.

Proposition 3.2.27 ([FHT11])**.** *A smooth functor of local action groupoids $f : X \to Y$ induces a map $f^*\colon K^*(Y) \to K^*(X)$. If f is proper, then it induces a map $f^*\colon K^*_c(Y) \to K^*_c(X)$.*

If the functor f describes an equivalence of stacks, then $f^*\colon K^*(Y) \to K^*(X)$ is an equivalence of spectra. Consequently, the above description of K-theory gives a functor

$$K^*(-)\colon \mathrm{LocQuoStack}^{op} \to KU\mathrm{Mod}.$$

Finally, a circle 2-bundle on a stack X can be described by an equivariant $PU(H_0)$-bundle on the groupoid presenting X. Then we define the twisted K-theory of X in analogy with definition 3.2.10.

Definition 3.2.28. Let X be a local quotient groupoid with a universal Hilbert bundle $H \to X_0$, and equipped with a circle 2-bundle $\chi\colon X \to \mathbf{B}^2U(1)$. Let $P \to X_0$ be equivariant $PU(H_0)$-bundle over X_0 associated to χ, and let $PU(H) \to X_0$ be the equivariant $PU(H_0)$-bundle associated to the universal Hilbert bundle H. Their product describes an equivariant $PU(H_0)$-bundle over X_0.

This gives rise to equivariant bundles of Fredholm opertors $\mathrm{Fred}^{(n)+\chi}(H) \to X_0$. The χ-twisted K-theory is given by the spaces of sections of these bundles

$$K^{*+\chi}(X) := \Gamma(X, \mathrm{Fred}^{(*)+\chi}(H))$$

and similarly the compactly supported K-theory is given by compactly supported sections.

We have given a rather geometric description of K-theory for local quotient stacks, closely resembling the picture in ordinary homotopy theory. In the next two sections, we will sketch another definition of K-theory of stacks, which is well-defined for *all* differentiable stacks. We will only describe K-theory with compact support, although there is also a (slightly more involved) definition of K-theory without support conditions.

The main advantage of this approach over the one just sketched is that it has a natural *bivariant* extension that allows us to neatly formalize the pull-push construction discussed in section 4. On the other hand, to construct pushforward maps we usually take a geometric perspective, which matches the decription of K-theory given in this section.

3.3 Operator algebraic K-theory

In this section we introduce another description of twisted K-theory of stacks, due to [TXLG04]. This uses the well-developed machinery of bivariant K-theory for operator algebras. The main advantage of this approach is that we pass from K-theory to its bivariant counterpart, which is a mixture of K-theory and K-homology. In particular, bivariant K-theory, or KK-theory for short, naturally encorporates index theory and allows for a more conceptual treatment of pushforward maps.

In the first section, we will show how to functorially assign to a differentiable stack X the algebra of functions on X, also called the convolution algebra of X. If the stack X comes equipped with a circle 2-bundle, one can twist the functions on X to obtain a the twisted convolution algebra of X. We show how one can functorially form such twisted algebras in section 3.3.2.

Having a transition from stacks to algebras, we can consider the K-theory of these algebras. This is done in section 3.3.3, where it forms a natural part of bivariant K-theory. In section 3.3.4 we then apply this to the algebras of functions on a differentiable stack to obtain the (twisted) K-theory of that stack, and discuss how this relates to the discussion of K-theory for stacks in the previous section. In particular, it turns out that there is a functor

$$\mathrm{KK} \longrightarrow \mathrm{ho}\big(KU\mathrm{Mod}\big)$$

that sends each C^*-algebra to its K-theory spectrum (in the homotopy category of KU-modules). This produces the twisted K-theory spectra considered in the previous section. We conclude with a summary of how corresponce diagrams of differentiable stacks give rise to cospan diagrams in KK-theory, and therefore to cospan diagrams in the homotopy category of KU-modules.

3.3.1 Convolution algebras

The main aim of this section is to show that there is a functorial way of passing from differentiable stacks to C^*-algebras of functions. One should really view a function algebra of a stack X as a representative for its category of modules (which morally correspond to vector bundles over the stack X). Therefore, the natural notion of a morphism between two such algebras is not that of a $*$-homomorphism, but of a bimodule.

The aim for this section is to prove the following result:

Theorem 3.3.1. *There is a functor of* $(2,1)$*-categories*

$$\mathrm{DiffStack}^{op}_{prop} \xrightarrow{C^*(-)} \mathrm{C^*Alg}_{bim} = \left\{ \begin{array}{c} \mathrm{C^*-algebras} \\ \text{Hilbert bimodules} \\ \text{Intertwiners} \end{array} \right\}$$

taking each stack to its algebra of continuous functions (vanishing at infinity). In fact, a proper morphism of stacks gives rise to a proper *Hilbert bimodule.*

Remark 3.3.2. To obtain C^*-algebras, we have to consider algebras of functions having compact support. Because such functions can only be pulled back along proper maps, we can only expect to obtain maps of C^*-algebras from proper maps of stacks. This remains the case if we consider the bimodule category of C^*-algebras.

The main steps in the proof have been known in the literature, most notably [Lan01] and [AG06]. We add the functoriality at the level of 2-morphisms, together with the extension to stacks carrying a circle 2-bundle in section 3.3.2. Before giving the construction, we recall the definition of the target category.

The bicategory of C^*-algebras.

Given a manifold M, its K-theory is concerned with Hermitean bundles over M. Passing to the dual picture of algebras, we have Swan's theorem that realizes every complex vector bundle E over M as a finitely generated module over the algebra $\mathcal{C}_0(M)$ of continuous functions on M, vanishing at ∞. The module is simply given by the space of sections $\Gamma_0(M,E)$, vanishing at infinity. If the vector bundle E comes equipped with an Hermitean metric, then $\Gamma_0(M,E)$ comes equipped with a pairing

$$\Gamma_0(M,E) \times \Gamma_0(M,E) \to \mathcal{C}_0(M)$$

obtained by taking the fiberwise inner product of sections. Algebraically, this map gives the space $\Gamma_0(M,E)$ the structure of a (right) Hilbert module over $\mathcal{C}_0(M)$.

Definition 3.3.3. Let A be a C^*-algebra. A (right) Hilbert module over A is a right A-module N, together with a map

$$\big\langle -,- \big\rangle : N \times N \to A$$

satisfying

(i) for all $v, w_1, w_2 \in N$ we have $\big\langle v, w_1 + w_2 \big\rangle = \big\langle v, w_1 \big\rangle + \big\langle v, w_2 \big\rangle$.

(ii) for all $a \in A$, $v, w \in N$ we have $\big\langle v, w \cdot a \big\rangle_A = \big\langle v, w \big\rangle_A \cdot a$.

(iii) for all $v, w \in N$, we have $\big\langle v, w \big\rangle = \big\langle w, v \big\rangle^*$, where $(-)^*$ is the involution of the C^*-algebra A.

(iv) for all $v \in N$, $\big\langle v, v \big\rangle_A \geq 0$ in A and $\big\langle v, v \big\rangle = 0$ iff $v = 0$.

(v) N is complete with respect to the norm induced by the norm on A:

$$||v||^2 := |\langle v, v \rangle|$$

where $|-|$ denotes the C^*-norm on A.

The first properties just say that the inner product behaves as the fiberwise inner product of a Hermitean vector bundle, while the last property gives the tools to do analysis on the module N.

Example 3.3.4. Any Hilbert bundle on a manifold M gives rise to a Hilbert module over $\mathcal{C}_0(M)$ of sections vanishing at infinity. However, there are more Hilbert modules: these correspond to more singular 'fields of Hilbert spaces', which are not locally trivial.

A morphism between two such Hilbert modules is a map admitting an adjoint:

Definition 3.3.5. Let A be a C^*-algebra. If M, N are right Hilbert modules over A, an adjointable map from M to N is a map $\phi : M \to N$ that admits an adjoint $\phi^* : N \to M$ such that

$$\langle v, \phi(w) \rangle_A = \langle \phi^*(v), w \rangle_A$$

for all $w \in M, v \in N$. Let $B(M, N)$ be the set of all such maps.

Remark 3.3.6. The set of adjointable maps $M \to N$ forms a closed linear subspace of the space of all bounded linear operators from M to N. This given $B(M, N)$ a norm. When $M = N$ we then have that $B(M) := B(M, M)$ is a C^*-algebra.

Example 3.3.7. For any two vectors $u \in M, v \in N$, there is an operator

$$\phi_{v,u} : M \to N; \quad w \mapsto v \cdot \langle u, w \rangle_A$$

The adjoint is given by $\phi_{u,v}$. The linear subspace of $B(M, N)$ spanned by those operators is the space of finite rank operators. Its closure is the space $K(M, N)$ of compact operators. If $M = N$, then the space $K(M) := K(M, M)$ forms a closed two-sided ideal in $B(N)$, so in particular it is a C^*-algebra itself.

For us, a C^*-algebra serves mainly as a representative of its category of Hilbert modules, from which we can construct its K-theory. The natural notion of morphism between two algebras is therefore that of a bimodule.

Definition 3.3.8. Let A, B be C^*-algebras. An $A-B$-bimodule ${}_AN_B$ is a right Hilbert module N over B, together with a $*$-homomorphism

$$A \to B(N)$$

to the C^*-algebra of adjointable operators on N. We will say a $A-B$-bimodule is proper if A acts on N by compact operators. A map $\phi : M \to N$ is a map of bimodules if it is adjointable and respects the left action by A.

Definition 3.3.9. Let $\mathrm{C}^*\mathrm{Alg}_{bimod}$ be the $(2,1)$-category whose objects are C^*-algebras, whose morphisms $A \to B$ are Hilbert bimodules ${}_AN_B$ and whose 2-morphisms are unitary intertwiners. The unit morphism $A \to A$ is the bimodule A over itself, where the inner product is $\langle a_1, a_2 \rangle = a_1^* a_2$.

The composition of bimodules is given by the tensor product. For ${}_AM_B$ and ${}_BN_C$, we define $M \otimes_B N$ as follows: take the vector space tensor product of $M \otimes_{\mathbb{C}} N$ and endow it with the symmetric form

$$\langle m_1 \otimes n_1, m_2 \otimes n_2 \rangle_A = \langle n_1, \langle m_1, m_2 \rangle_B n_2 \rangle_A$$

Then take the quotient by the submodule spanned by all vector have zero length with respect to this pairing and complete it with respect to the norm induced by the above inner product.

Since the compact operators form a two-sided ideal, there is a subcategory $\mathrm{C}^*\mathrm{Alg}_{prop}$ whose objects are C^*-algebras, whose morphisms are the *proper* bimodules and whose 2-morphisms are the unitary intertwiners.

Remark 3.3.10. Throughout this text, we will tacitly assume all C^*-algebras and Hilbert modules to be separable. Since all the algebras and modules we consider are those of functions on a smooth manifold, this poses no problems.

Remark 3.3.11. The composition of bimodules is only natural with respect to *unitary* intertwiners. The unitarity condition guarantees that the spaces of zero length vectors that we divide out are preserved by the intertwiners.

Example 3.3.12. Any $*$-homomorphism $\phi\colon A \to B$ gives rise to a proper bimodule from A to B. We take $H = B$, with B-valued inner product given by $\langle b_1, b_2 \rangle = b_1^* b_2$, and where the left action of A is given by $a \cdot b = \phi(a)b$. This gives a compact operator since A has an approximate unit (here we use the assumption that A is separable).

In fact, it is easily seen that this construction gives rise to a functor

$$\mathrm{C^*Alg}_{*-\mathrm{hom}} \to \mathrm{C^*Alg}_{prop}$$

from the category of C^*-algebras with $*$-homomorphisms between them. If we pass to the homotopy category of $\mathrm{C^*Alg}_{prop}$ (by identifying unitary equivalent bimodules), it turns out the above functor has an abstract categorical description.

Proposition 3.3.13 ([Mey08])**.** *The functor* $\mathrm{C^*Alg}_{*-\mathrm{hom}} \to ho\big(\mathrm{C^*Alg}_{prop}\big)$ *described above is the universal localization of the category* $\mathrm{C^*Alg}_{*-\mathrm{hom}}$ *at the maps*

$$A \to A \otimes K(\ell^2); a \mapsto a \otimes p$$

where p is the projection onto a 1-dimensional subspace in ℓ^2.

In other words, $\mathrm{C^*Alg}_{prop}$ (or its homotopy category) is the universal category that identifies the algebra of compact operators with the trivial algebra $\mathbb{C}$.

Proposition 3.3.14. $\mathrm{C^*Alg}_{bimod}$ *is a symmetric monoidal* $(2,1)$*-category under the minimal (or spatial) tensor product* $- \otimes -$*. The same thing holds for* $\mathrm{C^*Alg}_{prop}$*. The functor* $\mathrm{C^*Alg}_{*-\mathrm{hom}} \to \mathrm{C^*Alg}_{prop}$ *constructed above is monoidal.*

Proof. It is well known that the minimal tensor product of C^*-algebras makes $C^*\mathrm{Alg}_{*-\mathrm{hom}}$ into a symmetric monoidal category. We simply have to extend this operation from $*$-homomorphisms to bimodules.

If M and N are Hilbert modules over B_1 and B_2, then the algebraic tensor product $M \otimes_{alg} N$ has a Hermitean form taking values in the algebraic tensor product $B_1 \otimes_{alg} B_2$, and hence in the minimal tensor product $B_1 \otimes B_2$. It is simply given by

$$\big\langle m_1 \otimes n_1, m_2 \otimes n_2 \big\rangle = \big\langle m_1, m_2 \big\rangle \otimes \big\langle n_1, n_2 \big\rangle$$

The Hermitean form need not be definite, so we copy the procedure of definition 3.3.9 by dividing out the vectors of length zero and taking the completion. This is the right Hilbert module $M \otimes N$ over $B_1 \otimes B_2$.

There are natural isometric inclusions of C^*-algebras $B(M) \otimes B(N) \to B(M \otimes N)$ and similarly for compact operators (see e.g. [Lan95]). If M, N were (proper) bimodules, then the maps $A_1 \to B(M), A_2 \to B(N)$ assemble into a map $A_1 \otimes A_2 \to B(M \otimes N)$, so that $M \otimes N$ is a (proper) bimodule from $A_1 \otimes A_2$ to $B_1 \otimes B_2$.

This minimal tensor product of bimodules is easily seen to be natural with respect to unitary equivalences: we just take the algebraic tensor product of two unitary maps and extend this map to the completion. The associativity conditions are easily checked, analogous to the argument for the category $C^*\mathrm{Alg}_{*-\mathrm{hom}}$. This shows that $C^*\mathrm{Alg}_{bimod}$ and $C^*\mathrm{Alg}_{prop}$ are symmetric monoidal categories.

If $M = B_1$ and $N = B_2$ present $*$-homomorphisms, then their tensor product is simply the minimal tensor product $B_1 \otimes B_2$, with the left action induced by the tensor product of $*$-homomorphisms $A_1 \otimes A_2 \to B_1 \otimes B_2$. This shows that the functor from $C^*\mathrm{Alg}_{*-\mathrm{hom}}$ is monoidal. □

Example 3.3.15. If M is a smooth manifold, then the minimal tensor product $\mathcal{C}_0(M) \otimes A$ is equivalent to the C^*-algebra of A-valued functions on M, vanishing at ∞. In case $A = \mathcal{C}_0(N)$ for another manifold N, then $\mathcal{C}_0(M) \otimes \mathcal{C}_0(N) \simeq \mathcal{C}_0(M \times N)$.

If $E \to M$ is a Hermitean vector bundle over M and H is any other Hilbert module over A, then $\Gamma_0(M, E) \otimes H \simeq \Gamma_0(M, E \otimes H)$ is equivalent to the space of sections (vanishing at infinity) of the bundle with fibers $E_x \otimes H$ having an A-valued inner product. If $A = \mathcal{C}_0(N)$ and $H = \Gamma_0(N, F)$ for a Hermitean vector bundle $F \to N$, then $\Gamma_0(M, E) \otimes \Gamma_0(N, F) \simeq \Gamma_0(M \times N, p_1^* E \otimes p_2^* F)$.

We construct the functor from theorem 3.3.1 by using the representation of differentiable stacks from section 2.2.3. We will assign algebras to Lie groupoids and bimodules to proper bibundles between them. We will start by constructing the algebra of functions on a Lie groupoid, which goes under the name of the (reduced) convolution algebra.

Convolution algebra

Essentially we obtain the function algebra $C^*(G)$ of a Lie groupoid G by taking functions on the space of arrows G_1 and taking the convolution product along the source and target fibers of the groupoid. To make sense of this, we need a measure along those fibers.

Definition 3.3.16. Let G be a Lie groupoid. A (left) Haar system on G is given by a familiy of measures $\{\mu^x\}_{x \in G_0}$ along the target fibers $t^{-1}(x)$. These measures have to satisfy the following three conditions:

- left invariance: given an arrow $x \xrightarrow{g} y$ in G_1 and a compactly supported smooth function f on G_1, we have that
$$\int_{t(h)=x} d\mu^x f(gh) = \int_{t(h)=y} d\mu^y f(h).$$
- each measure μ^x on the smooth manifold $t^{-1}(x)$ should agree locally with the Lebesgue measure on $\mathbb{R}^n$.
- The system of measure should be smooth, in the sense that for any smooth, compactly supported function $f : G_1 \to \mathbb{C}$, we have that the function
$$G_0 \to \mathbb{C}; \quad x \mapsto \int_{t(g)=x} d\mu^x f(g)$$
is smooth.

Lemma 3.3.17. *Any Lie groupoid G admits a Haar system, which need not be unique.*

Proof. Such a Haar system is obtained by considering the tangent bundle long the target fibers and taking its density bundle. When restricted to the unit vectors, one obtains a density bundle over G_0. Take any section of this bundle and extend it equivariantly over G. ☐

The fact that the Haar system is not unique is only an annoying detail. It will turn out that the construction of the convolution algebra does not depend on this choice (up to Morita equivalence). We will anticipate the redundancy of the choice of Haar system by dropping it from all notation. If we integrate over all $g \in t^{-1}(x)$ for some $x \in G_0$, we will write this like
$$\int_{t(g)=x} f(g) := \int_{t(g)=x} d\mu^x f(g)$$
The integration along target fibers also allows us to integrate along source fibers, via
$$\int_{s(g)=x} f(g) := \int_{t(g)=x} f(g^{-1}).$$
With all this, we can construct the convolution algebra of a Lie groupoid G (with Haar system). A priori, we only obtain a $*$-algebra of functions on G.

Definition 3.3.18. Let G be a Lie groupoid with Haar system and let $\mathcal{C}_c^\infty(G_1)$ be the space of compactly supported smooth functions on G_1. It comes equipped with a product operation, called the convolution product

$$f_1 * f_2(g) := \int_{t(g')=t(g)} f_1(g')f_2(g'^{-1}g) = \int_{t(g')=s(g)} f_1(gg')f_2(g'^{-1})$$

The function $f_1 * f_2$ is smooth and has compact support. Moreover, $\mathcal{C}_c^\infty(G_1)$ comes with an involution defined by

$$f^*(g) = \overline{f}(g^{-1}).$$

Together, these turn $\mathcal{C}_c^\infty(G_1)$ into a $*$-algebra.

Example 3.3.19. If $G = M$ is a smooth manifold, then we obtain the ordinary commutative algebra of smooth, compactly supported functions on M.

Example 3.3.20. If $G = \mathbf{B}A$ is the delooping groupoid of a discrete group A, this produces the group algebra of A. Indeed, the compactly supported functions on A are spanned by the delta-functions $\{\delta_g\}_{g\in A}$ which are 1 on g and zero otherwise. The Haar measure is simply the counting measure.

Then the convolution product is given by

$$\delta_g * \delta_h(a) = \sum_{b\in A} \delta_g(ab^{-1})\delta_h(b) = \delta_{gh}(a)$$

so that $g \mapsto \delta_g$ provides a ring isomorphism with the group ring $\mathbb{C}[A]$.

To make this into a C^*-algebra, we embed $\mathcal{C}_c^\infty(G_1)$ in the C^*-algebra of adjointable maps on a Hilbert module and take its closure.

Definition 3.3.21. Endow $\mathcal{C}_c^\infty(G_1)$ with the $\mathcal{C}_0(G_0)$-valued inner product given by

$$\langle a, b\rangle(x) = \int_{s(g)=x} \overline{a}(g)b(g)$$

This turns $\mathcal{C}_c^\infty(G_1)$ into a pre-Hilbert module over $\mathcal{C}_0(G_0)$, i.e. a vector space that satisfies all conditions of a Hilbert module except completeness. The norm is given by

$$||f|| = \sup_{x\in G^0} \langle f, f\rangle(x)$$

We can then complete $\mathcal{C}_c^\infty(G_1)$ with respect to this norm. The result is a Hilbert module H_G over $\mathcal{C}_0(G_0)$.

The completion H_G is just the L^2-completion of $\mathcal{C}_0(G_1)$ along the source fibers. The inner product is obtained by taking the L^2-inner product along each such source fiber.

We have already seen that $\mathcal{C}_c^\infty(G_1)$ acts on itself by the convolution product. With respect to the pre-Hilbert norm on $\mathcal{C}_c^\infty(G_1)$, this action is by bounded operators and thus extends to the completion H_G. The adjoint of the action by $f \in \mathcal{C}_c^\infty(G_1)$ is the action by $f^* \in \mathcal{C}_c^\infty(G_1)$. It follows that all operators f are adjointable, so we get a well defined $*$-homomorphism

$$\mathcal{C}_c^\infty(G_1) \to B(H_G)$$

Since it is injective, we define

Definition 3.3.22. Let $C^*(G)$ be the completion of $\mathcal{C}_c^\infty(G_1)$ in the C^*-algebra $B(H_G)$ of bounded adjointable operators on H_G.

Remark 3.3.23. This construction is also called the *reduced* convolution algebra of the Lie groupoid G.

Remark 3.3.24. Any sequence of smooth functions converging in the inductive topology on $\mathcal{C}_c^\infty(G_1)$ also converges in the topology on $C^*(G)$. Recall that sequence of smooth functions f_n with compact support converges to f in the inductive topology if there is a compact K containing the supports of f and all f_n, while all derivatives of the f_n converge uniformly to the derivatives of f.

Our next step is to associate a bimodule to a proper bibundle.

Bimodule of a proper bibundle

Recall from section 2.2.3 that a morphism of differentiable stacks can be modeled by a bibundle 2.2.33

$$G_0 \xleftarrow{\sigma} P \xrightarrow{\tau} H_0$$

For a proper morphism of stacks, the right G-action on P is proper and the map $P \xrightarrow{\tau} H_0$ has the property that the inverse image of a compact is contained in the G-orbit of a compact subset of P.

Following the discussion in [Lan01], we will associate to such a bibundle a $C^*(H)$-$C^*(G)$ Hilbert bimodule. We first construct a bimodule which is not complete and then show that it can be completed to a Hilbert bimodule.

Definition 3.3.25. Consider the vector space $\mathcal{C}_c^\infty(P)$ of compactly supported functions on P. It carries a left action of $\mathcal{C}_c^\infty(H_1)$ given by

$$(f \cdot a)(p) = \int_{s(h)=\tau(p)} f(h^{-1})a(hp)$$

for $f \in \mathcal{C}_c^\infty(H_1)$ and $a \in \mathcal{C}_c^\infty(P)$.

Similarly, there is a right action by $\mathcal{C}_c^\infty(G_1)$, given by

$$(a \cdot f)(p) = \int_{t(g)=\sigma(p)} a(pg)f(g^{-1}).$$

This turns $\mathcal{C}_c^\infty(P)$ into a bimodule for these two $*$-algebras.

Remark 3.3.26. Both formulas only involve integration over G_1 and H_1. In particular, the bibundle P does not have to interact with the Haar measures on G and H in any way. This implies that the resulting convolution algebra is independent of the chosen Haar system, up to an isomorphism witnessed by a bimodule.

Since the map $\sigma\colon P \to G_0$ is the quotient of a left principal H-action, we have for any $x \in G_0$ that the fiber $\sigma^{-1}(x) \subseteq P$ is (non-canonically) diffeomorphic to a source fiber of the Lie groupoid H. To obtain such a diffeomorphism, we pick a point $p_0 \in \sigma^{-1}(x)$. The map

$$\Big\{h \in H_1 : s(h) = \tau(p_0)\Big\} \to \sigma^{-1}(x); \quad h \mapsto hp_0$$

then provides a diffeomorphism between $\sigma^{-1}(x)$ and the source fiber $s^{-1}(\tau(p_0))$ in H.

This diffeomorphism allows us to integrate over the fiber $\sigma^{-1}(x)$ by

$$\int_{\sigma(p)=x} f(p) := \int_{s(h)=\tau(p_0)} f(hp_0).$$

By principality of the H-action and invariance of the Haar system on H, this is independent of the chosen point p_0. The integral of a smooth (compactly supported) function over the σ-fibers gives a smooth function on G_0.

Given two functions $a, b \in \mathcal{C}_c^\infty(P)$, define their $C^*(G)$-valued inner product by

$$\big\langle a, b\big\rangle(g) = \int_{\sigma(p)=t(g)} \overline{a}(p)b(pg) \tag{3.6}$$

This gives a compactly supported smooth function on G_1 precisely because the right G-action is proper: the latter implies that the map

$$P \times_{G^0} G^1 \to \mathbb{C}; (p, g) \mapsto \overline{a}(p) \cdot b(pg)$$

has compact support. Integrating over p then gives a smooth function with compact support.

A direct computation shows that the pairing $\big\langle -, -\big\rangle$ is Hermitean and $\mathcal{C}_c^\infty(G_1)$-linear in the second argument. To conclude that it gives $\mathcal{C}_c^\infty(P)$ the structure of a pre-Hilbert module over $C^*(G)$, we show:

Lemma 3.3.27. *The pairing $\langle -,- \rangle$ is positive definite.*

Proof. We have to check that for any $a \in \mathcal{C}_c^\infty(P)$, we have that $\langle a,a \rangle \geq 0$ as an element in $C^*(G)$. Since $C^*(G)$ is contained in the algebra of adjointable operators on a Hilbert module, we can use the positivity criterion for bounded operators on a Hilbert C^*-module (see [WO93], 15.2.3). This says that an adjointable operator T acting on a Hilbert module is positive when $\langle Tv,v \rangle \geq 0$ for all v in (a dense subset of) the Hilbert module. This is completely anologous to the case where T acts on a Hilbert space, except that now the inner product takes values in a C^*-algebra.

We thus have to check that

$$\Big\langle \langle a,a \rangle * f, f \Big\rangle_{H_G} \in \mathcal{C}_c^\infty(G_0)$$

is a positive function, for all f in the dense subset $\mathcal{C}_c^\infty(G_1) \subseteq H_G$. Here the outer metric is the one on the Hilbert module H_G associated to the groupoid G, while $\langle a,a \rangle$ is the inner product on $\mathcal{C}_C^\infty(P)$. For a point $x \in G^0$, we compute

$$\begin{aligned}
\Big\langle \langle a,a \rangle * f, f \Big\rangle_{H_G}(x) = \quad & \Big\langle \langle a,a \cdot f \rangle, f \Big\rangle_{H_G}(x) \quad = \int_{s(g)=x} \overline{\langle a,a \cdot f \rangle}(g) f(g) \\
&= \int_{s(g)=x} \langle a \cdot f, a \rangle (g^{-1}) f(g) = \langle a \cdot f, a \rangle * f(\mathrm{id}_x) \\
&= \langle a \cdot f, a \cdot f \rangle (\mathrm{id}_x)
\end{aligned}$$

Here we use that we have chosen f in the dense subset $\mathcal{C}_c^\infty(G_1)$, so that it acts on $a \in \mathcal{C}_c^\infty(P)$. Of course, the result is just the L^2-inner product of the function $a \cdot f$ along the fiber $\sigma^{-1}(x) \subset P$, which is positive.

We conclude that $\langle a,a \rangle$ is a positive element of $C^*(G)$. Moreover, we have that $\langle a,a \rangle(\mathrm{id}_x)$ computes the L^2-norm of a along $\sigma^{-1}(x)$. It follows that $\langle a,a \rangle = 0$ iff $a = 0$. □

The pairing $\langle -,- \rangle$ endows $\mathcal{C}_c^\infty(P)$ with the structure of a pre-Hilbert module over $C^*(G)$. We let H_P be the Banach space obtained by closing $\mathcal{C}_c^\infty(P)$ with respect to the induced norm. The inner product on $\mathcal{C}_c^\infty(P)$ extends by continuity to an inner product on H_P taking values in $C^*(G)$.

We have to check that the actions of $\mathcal{C}_c^\infty(G_1)$ and $\mathcal{C}_c^\infty(H_1)$ extend to actions of $C^*(G)$ and $C^*(H)$. For $f \in \mathcal{C}_c^\infty(G_1)$ and $a \in \mathcal{C}_c^\infty(P)$, we have that

$$||a \cdot f||_{H_P}^2 := ||\langle a \cdot f, a \cdot f \rangle||_{C^*(G)} = ||f^* * \langle a,a \rangle * f||_{C^*(G)} \leq ||a||_P^2 \cdot ||f||_G^2$$

It follows that the action of $\mathcal{C}_c^\infty(G_1)$ on $\mathcal{C}_c^\infty(P)$ is continuous, so it extends to an action of $C^*(G)$ on H_P. Continuity also implies that the inner product on H_P satisfies $\langle a, b \cdot f \rangle = \langle a,b \rangle * f$ for all $a,b \in H_P$ and $f \in \mathcal{C}^*(G)$.

It remains to check that $\mathcal{C}_c^\infty(H_1)$ acts continuously on $\mathcal{C}_c^\infty(P)$ by bounded adjointable operators. For this, we use the result from [MRW87] that

$$||\langle f \cdot a, f \cdot b \rangle|| \leq ||f||_{C^*(H)}^2 \cdot ||\langle a,b \rangle||$$

We find a continuous action of $\mathcal{C}_c^\infty(H_1)$ on $\mathcal{C}_c^\infty(P)$ by bounded operators, which extends to a continuous action of $C^*(H)$ on H_P. As expected, f^* acts as the adjoint of the operator induced by f.

We therefore have that H_P becomes a bimodule over $C^*(G)$ and $C^*(H)$. The following lemma gives an easy criterion for convergence of functions in the topology on H_P, which is defined rather indirectly.

Lemma 3.3.28. *If $f_n \in \mathcal{C}_c^\infty(P)$ converge to $f \in \mathcal{C}_c^\infty(P)$ in the inductive topology, then $f_n \to f$ in the norm topology on H_P.*

Proof. Suppose we have a sequence f_n of compactly supported smooth functions on P converging to $f \in \mathcal{C}_c^\infty(P)$ in the inductive topology. For such a converging sequence, the integrals

$$\langle f - f_n, f - f_n \rangle (g) = \int_{s(h)=\tau(p)} \overline{(f-f_n)}(hp)(f-f_n)(hpg)$$

(which are continuous functions on G_1) converge to 0 in the inductive topology on $\mathcal{C}_c^\infty(G)$. By remark 3.3.24 the norm then also converges to 0. □

If the bibundle is proper, then the resulting Hilbert bimodule is also proper in the sense of definition 3.3.8, as was also noted in [AG06].

Proposition 3.3.29. *$C^*(H)$ acts by compact operators on H_P.*

Proof. We check that the dense subalgebra $\mathcal{C}_c^\infty(H_1)$ acts by compact operators. Let $f \in \mathcal{C}_c^\infty(H_1)$ have support inside a compact $K \subseteq H_1$. The image $s(K) \subset H_0$ is compact, so there is a compact $L \subseteq P$ whose G-orbit contains $\tau^{-1}(s(K))$, since we have assumed P to be a proper bibundle.

There is a smooth, compactly supported function ψ on P with the property that

$$\int_{t(g)=\sigma(p)} \psi(pg) = 1$$

for all points $p \in L$ (see the next lemma). The same thing then holds for all points p in the orbits of L. For any $a \in \mathcal{C}_c^\infty(P)$, we now have that

$$f \cdot a(p) := \int_{s(h)=\tau(p)} f(h^{-1})a(hp) = \int_{s(h)=\tau(p)} \int_{t(g)=\sigma(p)} f(h^{-1})a(hp)\psi(pg)$$

The second equality holds since $f(h^{-1}) \cdot a(hp)$, as a function of the point p, has its support inside the orbit of L.

Since the left H-action on P is principal, we obtain a smooth, compactly supported function

$$F : P \times_{G_0} P \simeq H_1 \times_{H_0} P \to \mathbb{C}; \quad (p, hp) \mapsto f(h^{-1}) \cdot \psi(p)$$

We can always extend this function to a smooth, compactly supported function on the whole product $P \times P$ and approximate this by products of functions on P only:

$$F(p, hp) := f(h^{-1}) \cdot \psi(p) = \sum_i \alpha_i(p)\beta_i(hp)$$

for $\alpha_i, \beta_i \in \mathcal{C}_c^\infty(P)$. This converges in the inductive topology on $\mathcal{C}_c^\infty(P \times P)$, so we have that

$$\begin{aligned} f \cdot a(p) &= \int_{s(h)=\tau(p)} \int_{t(g)=\sigma(p)} F(p, hpg)a(hp) \\ &= \sum_i \int_{s(h)=\tau(p)} \int_{t(g)=\sigma(p)} \alpha_i(pg)\beta_i(hpg)a(hp) \\ &= \sum_i \int_{t(g)=\sigma(p)} \alpha_i(pg)\langle \overline{\beta}_i, a\rangle(g^{-1}) \\ &= \sum_i \big(\alpha_i \cdot \langle \overline{\beta_i}, a\rangle\big)(p) \end{aligned}$$

The third line uses the definition of the $\mathcal{C}_c^\infty(G^1)$-valued inner product by picking the point $p \in \sigma^{-1}(g)$. This shows that $f \cdot a$ is approximated by $\sum_i \big(\alpha_i \cdot \langle \overline{\beta_i}, a\rangle\big)$. In fact, since $\sum_i \alpha_i(hp)\beta_i(p)$ converges to F in the inductive topology, we have that $f \cdot (-)$ is uniformly approximated by the sum of finite rank operators. This shows that $C^*(H)$ acts by compact operators. $\square$

Lemma 3.3.30. *Let P be a smooth manifold with a proper right G-action over $\sigma : P \to G_0$. Given a compact subset $L \subset P$, there exists a smooth, compactly supported function ψ on P such that*

$$\int_{t(g)=\sigma(p)} \psi(pg) = 1$$

for all points $p \in L$.

Proof. We can pick any smooth function $\tilde{\psi}$ with compact support which is positive on an open neighbourhood of L. Then the function

$$\psi(p) = \frac{\tilde{\psi}(g)}{\int_{t(g)=\sigma(p)} \tilde{\psi}(pg)}$$

gives a well-defined smooth, bounded function on the orbit of the compact L. It therefore extends to a smooth function to the closure of the orbit of L, which we can then extend by any smooth function on P. Clearly the resulting function ψ has the desired property. $\square$

Finally, we associate a unitary map of bimodules to an automorphism of bibundles.

On 2-morphisms

Let P_1 and P_2 be $G-H$-bimodules and $\phi : P_1 \to P_2$ a diffeomorphism intertwining the G- and H-action. Pullback along the diffeomorphism ϕ gives a map

$$\mathcal{C}_c^\infty(P_2) \xrightarrow{\phi^*} \mathcal{C}_c^\infty(P_1)$$

that respects the actions of $\mathcal{C}_c^\infty(G_1)$ and $\mathcal{C}_c^\infty(H_1)$. The fact that the inner product 3.6 is obtained by integrating over the target fibers of H immediately implies that ϕ^* preserves the inner product:

$$\langle \phi^* a_1, \phi^* a_2 \rangle_{H_{P_1}} = \langle a_1, a_2 \rangle_{H_{P_2}}$$

for all $a \in \mathcal{C}_c^\infty(P_2)$. It follows that ϕ^* is an isometry, so it extends to an isometry $\phi^* : H_{P_2} \to H_{P_1}$. The image of ϕ^* contains the dense subset $\mathcal{C}_c^\infty(P_1)$, so ϕ^* is unitary. In particular, it is adjointable with adjoint $(\phi^{-1})^*$.

Having defined the functor from theorem 3.3.1 on objects, morphisms and 2-morphisms, it remains to check that it is a 2-functor.

Functoriality

To show that the above defines a 2-functor, we have to provide equivalences witnessing that the above assignment respects units and compositions. We will start with the 2-morphism that shows that composition is respected.

Suppose we have a composition diagram

$$\begin{tikzcd}
 & & P \times_H Q \arrow[dl] \arrow[dr] \arrow[ddll, bend right, "\sigma"'] \arrow[ddrr, bend left, "\tau"] & & \\
 & P \arrow[dl, "\sigma_1"] \arrow[dr, "\tau_1"'] & & Q \arrow[dl, "\sigma_2"] \arrow[dr, "\tau_2"'] & \\
G_0 & & H_0 & & K_0
\end{tikzcd}$$

of bibundles from G to H to K.

Lemma 3.3.31. *There is a unitary map of bimodules*

$$(-) \times (-) : H_P \otimes_{C^*(H)} H_Q \to H_{P \otimes Q}$$

which is natural in P and Q.

Proof. First, we work just over the dense domain consisting of compactly supported functions. To start, we define a map

$$(-) \times (-) \colon \mathcal{C}_c^\infty(P) \otimes_{\mathbb{C}} \mathcal{C}_c^\infty(Q) \to \mathcal{C}_c^\infty(P \times_H Q) = \mathcal{C}_c^\infty(P \times_{H_0} Q/H).$$

Given points $p \in P, q \in Q$, let $[p, q]$ denote their image in $P \times_H Q$. We define

$$a \times b[p, q] = \int_{s(h)=\tau_1(p)} a(hp) b(qh^{-1}) \tag{3.7}$$

This function is obtained from the product function $a \times b \in \mathcal{C}_c^\infty(P \times Q)$ by integrating over the H-orbits. As such, it is smooth with compact support. A straightforward, but tedious computation now shows that

$$\langle a_1 \times b_1, a_2 \times b_2 \rangle_{H_{P\otimes Q}} = \langle a_1, \langle b_1, b_2 \rangle_{H_Q} \cdot a_2 \rangle_{H_P}$$

As a result, the map is continuous with respect to the norms on H_P and H_Q and sends all $a \otimes b \in H_P \otimes_{\mathbb{C}} H_Q$ for which $\langle a \cdot \langle b, b \rangle, a \rangle = 0$ to the zero element in $H_{Q\otimes P}$. The map then descends to a map

$$\times \colon H_P \otimes_{C^*(H)} H_Q \to H_{P\otimes Q}$$

which is an isometry. To show that it is unitary, it remains to check that its image forms a dense subset of $H_{P\otimes Q}$. By remark 3.3.28, the product functions $a \times b \in \mathcal{C}_c^\infty(P \times_{H_0} Q)$ lie dense with respect to the inductive topology. Moreover, the map

$$\mathcal{C}_c^\infty(P \times_{H_0} Q) \to \mathcal{C}_c(P \times_H Q); f(p, q) \mapsto \int_{s(h)=\tau_1(p)} f(hp, qh^{-1})$$

is surjective and continuous with respect to the inductive topology. Surjectivity follows from the existence of bump functions as in lemma 3.3.30. It follows that the image of $(-) \times (-)$ lies dense in $\mathcal{C}_c^\infty(P \times_H Q)$ with respect to the inductive topology, so by remark 3.3.28 its image also lies dense in $H_{P\otimes Q}$.

We have that $(-) \times (-)$ defines a unitary map of Hilbert bimodules. From the formula 3.7 it follows immediately that this map is natural with respect to P and Q. □

The equivalence witnessing the preservation of units can simply be taken to be the identity. Indeed, if $P = G_1$ is the identity bibundle on G, then the left and right action of $\mathcal{C}^\infty(G_1)$ are given by the convolution product. Moreover, the inner product 3.6 on $\mathcal{C}_c^\infty(G_1)$ reduces to

$$\langle a, b \rangle(g) = \int_{s(g')=t(g)} \overline{a}(g')b(g'g) = \int_{t(g')=t(g)} a^*(g')b(g'^{-1}g) = (a^* * b)(g)$$

The completion of $\mathcal{C}_c^\infty(G_1)$ with respect to this product is simply the C^*-algebra $C^*(G)$ itself, which is the unit in $C^*\mathrm{Alg}_{bim}$.

The coherence conditions for the above two equivalences are easily seen to be satisfied. This concludes the construction of the functor from theorem 3.3.1. We finish by showing that the functor $C^*(-)$ is a monoidal functor:

Proposition 3.3.32. *The functor $C^*(-)$ is a monoidal functor, where* DiffStack *carries the cartesian monoidal structure and $C^*\mathrm{Alg}_{prop}$ carries the monoidal structure described in proposition 3.3.14.*

Proof. The product of stacks is simply presented by the product of Lie groupoids, and similarly for bibundles and equivalences of bibundles. Let X and Y be two Lie groupoids. Then $\mathcal{C}_0(X_0 \times Y_0) \simeq \mathcal{C}_0(X_0) \otimes \mathcal{C}_0(Y_0)$ and the Hilbert module 3.3.21 used to define $C^*(X \times Y)$ satisfies

$$H_{X\times Y} \simeq H_X \otimes H_Y.$$

The closure of the algebraic tensor product $\mathcal{C}_c^\infty(X_1) \otimes_{alg} \mathcal{C}_c^\infty(Y_1)$ in $B(H_X) \otimes B(H_Y)$ is precisely the minimal tensor product $C^*(X) \otimes C^*(Y)$. Since $B(H_X) \otimes B(H_Y)$ is a closed subset of $B(H_X \otimes H_Y)$, this is also the closure of $\mathcal{C}_c^\infty(X_1) \otimes_{alg} \mathcal{C}_c^\infty(Y_1)$ in $B(H_{X\times Y})$. By remark 3.3.24, we know that this closure contains the closure of $\mathcal{C}_c^\infty(X_1 \times Y_1)$ in $B(H_{X\times Y})$. But the latter is precisely $C^*(X \times Y)$, so we conclude that there is a natural isomorphism

$$C^*(X) \otimes C^*(Y) \to C^*(X \times Y).$$

For $P : X_1 \to X_2$ a bibundle and $Q : Y_1 \to Y_2$ another bibundle, consider the algebraic tensor product $\mathcal{C}_c^\infty(P) \otimes_{alg} \mathcal{C}_c^\infty(Q)$ sitting inside $\mathcal{C}_c^\infty(P \times Q)$. On this subspace, the inner product 3.6 for $P \times Q$ gives

$$\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle = \langle a_1, a_2 \rangle \otimes \langle b_1, b_2 \rangle.$$

Its completion with respect to the induced norm therefore gives the minimal tensor product of bimodules $H_P \otimes H_Q$.

On the other hand, we know from remark 3.3.28 that the algebraic tensor product $\mathcal{C}_c^\infty(P)\otimes_{alg}\mathcal{C}_c^\infty(Q)$ is dense in $\mathcal{C}_c^\infty(P\times Q)$. It therefore follows that there is a natural isomorphism of completions $H_P\otimes H_Q \to H_{P\times Q}$.

Finally, given two equivalences ϕ, ψ of bibundles, we have that $(\phi\times\psi)^*$ acts on the algebraic tensor product $\mathcal{C}_c^\infty(P)\otimes_{alg}\mathcal{C}_c^\infty(Q)$ as the tensor product $\phi^*\otimes_{alg}\psi^*$. The extesions to the completion therefore agree. □

We next extend this construction of the function algebra of a stack to the case where the stack carries a circle 2-bundle.

3.3.2 Twisted convolution algebras

In the presence of a circle 2-bundle, we can twist the functions on the stack X by that bundle. The functor 3.3.1 from the previous section naturally extends to the case where we allow for such twists:

Theorem 3.3.33. *There is a functor of* $(2,1)$*-categories*

$$\left(\mathrm{DiffStack}_{prop}\right)^{op}_{/\mathbf{B}^2U(1)} \xrightarrow{C^*(-)} \mathrm{C^*Alg}_{prop} = \left\{\begin{array}{c}\mathrm{C^*-algebras}\\ \text{Proper Hilbert bimodules}\\ \text{Intertwiners}\end{array}\right\}$$

sending each stack carrying a circle 2-bundle to its twisted convolution algebra, whose restriction to the category of stacks with trivial circle 2-bundle is the functor from theorem 3.3.1.

Remark 3.3.34. If X is a differentiable stack carrying a circle 2-bundle $\alpha : X \to \mathbf{B}^2U(1)$, we denote its twisted convolution algebra by $C^*_\alpha(X)$.

Proof. We can present stacks with a circle 2-bundle by centrally extended Lie groupoids, according to the discussion in section 2.2.3. Suppose that

$$\mathbf{B}U(1) \to G^\chi \to G$$

is a central extension of Lie groupoids presenting the circle 2-bundle $\chi : G \to \mathbf{B}^2U(1)$. The functions $f\in\mathcal{C}_c^\infty(G_1^\chi)$ that are $U(1)$-equivariant form a sub-$*$-algebra of $\mathcal{C}_c^\infty(G_1^\chi)$. Let $C^*_\chi(G)$ be the closure of the set of $U(1)$-equivariant functions in the convolution algebra $C^*(G^\chi)$.

Let (P,ρ) is a proper bibundle from G^χ to H^ξ, together with a $U(1)$-action, so that it presents a morphism in the slice $\mathrm{DiffStack}^{prop}_{/\mathbf{B}^2U(1)}$. Then the space of $f\in\mathcal{C}_c^\infty(P)$ that are $U(1)$-equivariant is a sub-bimodule of $\mathcal{C}_c^\infty(P)$, together with the left and right action of $C^*_\xi(H)$ and $C^*_\chi(G)$. Its closure gives rise to a proper Hilbert bimodule from $C^*_\xi(H)$ to $C^*_\chi(G)$.

Finally, a $U(1)$-equivariant equivalence of bibundles preserves the $U(1)$-equivariant functions, hence gives rise to a unitary map of bimodules. The same arguments as in the previous section now show that this assignment gives rise to a functor. □

Example 3.3.35. Suppose M is an orindary smooth manifold, equipped with a circle 2-bundle $\alpha\colon M\to \mathbf{B}^2U(1)$. We can present M by the Cech groupoid of a good open cover $\{U_i\}$, for which the circle 2-bundle α is given by a the cocycle $\alpha_{ijk} : U_{ijk}\to U(1)$. Then the twisted convolution algebra $C^*_\alpha(M)$ is obtained as a completion of the algebra $\mathcal{C}_c^\infty(\coprod U_{ij})$, equipped with the convolution product

$$f_1 * f_2(x_{ik}) = \sum_j f_1(x_{jk})f_2(x_{ij})\alpha_{ijk}(x)$$

for any point $x_{ik}\in U_{ik}\subseteq M$.

In analogy with proposition 3.3.32, we have

Proposition 3.3.36. *The functor* $C^*(-)\colon \left(\mathrm{DiffStack}_{prop}\right)^{op}_{/\mathbf{B}^2U(1)} \to \mathrm{C^*Alg}_{prop}$ *is monoidal.*

Proof. The tensor product in $(\mathrm{DiffStack}_{prop})_{/\mathbf{B}^2U(1)}$ is given by the cartesian product of the underlying stacks, equipped with the product of circle 2-bundles

$$G \times H \xrightarrow{\chi\times\xi} \mathbf{B}^2U(1) \times \mathbf{B}^2U(1) \xrightarrow{\mu} \mathbf{B}^2U(1)$$

If we present the circle 2-bundles by multiplicative $U(1)$-bundles $G_1^\chi \to G_1$ and $H_1^\xi \to H_1$, then the tensor product is given by the product of line bundles over $G_1 \times H_1$, i.e. it is given by

$$G_1^\chi \times_{U(1)} H_1^\xi = G_1^\chi \times H_1^\xi / U(1) \to G_1 \times H_1$$

where we divide out by the diagonal action of $U(1)$. But now a $U(1)$-equivariant function on $G_1^\chi \times_{U(1)} H_1^\xi$ is equivalently just a function on $G_1^\chi \times H_1^\xi$ which is equivariant under both $U(1)$-actions on the left and the right side. The same argument as in 3.3.32 then shows that $C^*(-)$ preserves the tensor product of objects. One argues similarly for bimodules and maps of bimodules. $\square$

3.3.3 Bivariant K-theory

In the previous two sections we have seen how differentiable stacks carrying a circle 2-bundle give rise to algebras of 'twisted functions'. We now discuss how one can use this algebraic picture to describe the twisted K-theory of these stacks. This requires the K-theory of C^*-algebras, which nicely extends to a bivariant K-theory of C^*-algebras, also called KK-theory.

Definition 3.3.37. Let A, B be C^*-algebras. A Kasparov (A, B)-bimodule is a $\mathbb{Z}/2$-graded Hilbert bimodule H, together an odd operator $D \in B(H)$ so that

$$[D, a],\ (D^2 - 1)a,\ (D - D^*)a \in K(H)$$

for all $a \in A$.

In other words, D is a Fredholm operator which is self-adjoint and unitary, up to compact operators. The condition that $[D, a] = 0$ for all $a \in A$ expresses that D is a local operator over A.

Example 3.3.38. If H is a bimodule where A acts by compact operators, then $(H, D = 0)$ is a Kasparov bimodule.

Example 3.3.39. Let $E, F \to M$ be two Hermitean vector bundles over a manifold M. Let H be the $\mathbb{Z}/2$-graded Hilbert module whose even and odd parts are both $\Gamma_0(M, E) \oplus \Gamma_0(M, F)$. A self-adjoint elliptic operator

$$D\colon \Gamma_0(M, E) \to \Gamma_0(M, F)$$

gives rise to the odd operator

$$\begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} : \Gamma_0(M, E) \oplus \Gamma_0(M, F) \to \Gamma_0(M, E) \oplus \Gamma_0(M, F)$$

This gives H the structure of a Kasparov $(\mathbb{C}, \mathcal{C}_0(M))$-bimodule.

Example 3.3.40. More interesting, let $E, F \to M$ be two Hermitean vector bundles over a compact oriented manifold and let

$$D\colon L^2(M, E) \to L^2(M, F)$$

be an elliptic order zero pseudodifferential operator. Then $H = L^2(M, E) \oplus L^2(M, F)$ together with D gives a Kasparov $(\mathcal{C}_0(M), \mathbb{C})$-module.

Two Kasparov (A, B)-bimodules $(H_1, D_1), (H_2, D_2)$ are unitary equivalent if there is an even unitary equivalence $H_1 \xrightarrow{\phi} H_2$ such that $D_2 \circ \phi = \phi^{-1} \circ D_1$. However, we also want to consider two such bimodules to be equivalent if they are homotopic, just as homotopic families of Fredholm operators gave rise to equivalent K-theory classes in section 3.2.

To see how this comes about, suppose (H, D) is a Kasparov bimodule from A to $\mathcal{C}[0,1] \otimes B$. Let $B_t := \mathcal{C}(\{t\}) \otimes B$ be the Hilbert bimodule from $\mathcal{C}[0,1] \otimes B$ to B, with the expected actions and inner product. Then $H \otimes B_t$ is an (A, B)-bimodule, and it comes equiped with the operator

$$D_t = D \otimes 1\colon H \otimes B_t \to H \otimes B_t$$

The algebra A acts on $H \otimes B_t$ only via the copy of H, while the action of $\mathcal{C}[0,1] \otimes B$ on B_t is by compact operators. From this it follows that $(H \otimes B_t, D_t)$ is a Kasparov (A, B)-bimodule.

Definition 3.3.41. Two Kasparov (A, B)-bimodules (H_0, D_0) and (H_1, D_1) are said to be homotopic if there is a Kasparov bimodule (H, D) from A to $\mathcal{C}[0,1] \otimes B$, together with unitary equivalences

$$(H_0, D_0) \simeq (H \otimes_{\mathcal{C}[0,1]\otimes B} B_0, D_0) \qquad (H_1, D_1) \simeq (H \otimes_{\mathcal{C}[0,1]\otimes B} B_1, D_1).$$

Let $\mathrm{KK}(A, B)$ be the set of homotopy classes of Kasparov bimodules from A to B.

As an example of this homotopy relation, we have the following:

Lemma 3.3.42. *Suppose (H, D) is a degenerate Kasparov module, meaning that*

$$[D, a] = (D^2 - 1)a = (D - D^*)a = 0$$

for all $a \in A$. Then (H, D) is homotopic to the 0-module.

Proof. Consider the module $\mathcal{C}_0[0,1) \otimes_{\mathbb{C}} H$, consisting of continuous H-valued functions on $[0,1]$ vanishing at 1. It carries the pointwise left action of A and a right action from $\mathcal{C}[0,1] \otimes B$. If we equip it with the operator $1 \otimes D$ that just acts pointwise, then $\mathcal{C}_0[0,1) \otimes H$ becomes a Kasparov bimodule (which is again degenerate). At $t = 0$ the module is equivalent to H, while at $t = 1$ is is equivalent to the zero module. □

Lemma 3.3.43. *Let H be a $\mathbb{Z}/2$-graded right B-module. Suppose for each $t \in [0,1]$ we have a left action $A \ni a \mapsto a_t \in B(H)$ and an odd operator D_t such that (H, D_t) is a Kasparov bimodule. If the maps $t \mapsto D_t$, $t \mapsto D_t^*$ and $t \mapsto a_t$ are strongly continuous, then (H, D_0) is homotopic to (H, D_1).*

Proof. Take the module $\mathcal{C}[0,1] \otimes_{\mathbb{C}} H \simeq \mathcal{C}([0,1], H)$ with

$$(a \cdot f)(t) = a(t) \cdot f(t) \qquad (Df)(t) = D_t(f(t))$$

This turns the module in a Kasparov bimodule whose restrictions to $t = 0, 1$ give (H, D_0) and (H, D_1). □

Lemma 3.3.44. *The direct sum of bimodules turns $\mathrm{KK}(A, B)$ into an abelian group.*

Proof. Given two Kasparov (A, B)-bimodules (H_0, D_0) and (H_1, D_1), we can simply take their direct sum $(H_0 \oplus H_1, D_0 \oplus D_1)$. Clearly this respects the homotopy relation on such bimodules. The zero module acts as a unit for this addition.

The inverse of a bimodule (H, D) is given by $(H[1], -D)$, where $H[1]$ has opposite grading. Then $(H \oplus H[1], D \oplus -D)$ homotopic to $H \oplus H[1]$ with the operator that interchanges H and $H[1]$, via the 1-parameter family of operators

$$\begin{pmatrix} D\cos t & \sin t \\ \sin t & -D\cos t \end{pmatrix}.$$

The latter is a degenerate bimodule, so homotopic to the zero module. □

Definition 3.3.45. For A a C^*-algebra, the zeroth K-theory group of A is defined as $K_0(A) = \mathrm{KK}(\mathbb{C}, A)$. More generally, the n-th K-theory group of A is given by $K_n(A) = \mathrm{KK}(\mathbb{C}, \mathcal{C}_0(\mathbb{R}^n, A))$, when $n \geq 0$. When $n < 0$, we define $K_n(A) := \mathrm{KK}(\mathcal{C}_0(\mathbb{R}^{-n}), A)$.

Example 3.3.46. $K_{2i}(\mathbb{C}) \simeq \mathbb{Z}$ and $K_{2i+1}(\mathbb{C}) \simeq 0$. A KK-cocycle from $\mathbb{C}$ to $\mathbb{C}$ consists of Hilbert spaces H_0, H_1 and a Fredholm operator $H_0 \to H_1$. Two such Fredholm operators are homotopic if there indices agree, which gives the isomorphism $K_0(\mathbb{C}) \simeq \mathbb{Z}$.

For the other isomorphisms, we use the well-known fact that this operator K-theory agrees with topological K-theory when applied to $\mathcal{C}_0(M)$:

Proposition 3.3.47 ([WO93]). *If M is a locally compact topological space, then there are natural isomorphisms*

$$K_i(\mathcal{C}_0(M)) \simeq K_c^i(M).$$

The fundamental result in KK-theory is the following theorem by Kasparov:

Theorem 3.3.48 (Kasparov, [Kas88]). *There is a category* KK *enriched over* Ab*, whose objects are the separable C^*-algebras, such that $\mathrm{KK}(A, B)$ is the group of morphisms from A to B.*

The proof of the theorem is nontrivial: given a bimodule (H_1, D_1) from A to B and a bimodule (H_2, D_2) from B to C, one has to construct a Fredholm operator on the tensor product $H_1 \otimes H_2$. One has to modify the operator $D_1 \otimes 1 + 1 \otimes D_2$ so that it satisfies the conditions of definition 3.3.37, and show that these choices are unique up to homotopy. In the case that D_1 and D_2 are both zero, one can simply pick the zero operator on $H_1 \otimes H_2$.

An immediate corollary of this is that any element $f \in \mathrm{KK}(A, B)$ gives rise to a map

$$f_* : K_0(A) \to K_0(B)$$

by postcomposition. In the case of example 3.3.40, this is precisely the map obtained by taking the index of the elliptic operator.

There is a natural functor

$$C^*\mathrm{Alg}_{prop} \to \mathrm{KK}$$

from the bicategory of C^*-algebras with *proper* bimodules to the category KK. It sends a proper bimodule ${}_AH_B$ to the Kasparov bimodule having H as its even degree part, while it has no odd degree part. We equip it with the zero operator, which satisfies all conditions from 3.3.37 since A acts by compact operators on H. Unitary equivalent bimodules give rise to the same maps in KK-theory, so this factors over the homotopy category of $C^*\mathrm{Alg}_{prop}$.

Remarkably, it turns out that the composition of this functor with the one from proposition 3.3.13 gives a functor which can also be characterized universally:

Proposition 3.3.49 (Higson). *The above functor $C^*\mathrm{Alg}_{*-\mathrm{hom}} \to \mathrm{KK}$ is the universal functor such that*

- *its target is an additive category.*
- *it identifies homotopic maps of $C*$-algebras. Two $*$-homomorphisms $f_0, f_1 : A \to B$ are said to be homotopic if there is a $*$-homomorphism $f : A \to \mathcal{C}([0,1], B)$ such that $\mathrm{ev}_0 \circ f = f_0$ and $\mathrm{ev}_1 \circ f = f_1$.*
- *it inverts the maps $A \to A \otimes K(\ell^2); a \mapsto a \otimes p$ from proposition 3.3.13.*
- *it preserves* split *exact sequences $0 \to J \to A \to A/J \to 0$ of C^*-algebras.*

This universal characterization gives rise to many properties of the category KK. We mention two examples:

Lemma 3.3.50. *The category* KK *carries a symmetric monoidal structure induced by the minimal tensor product of C^*-algebras. The functor $C^*\mathrm{Alg}_{*-\mathrm{hom}} \to \mathrm{KK}$ is a monoidal functor, as is the functor $C^*\mathrm{Alg}_{prop} \to \mathrm{KK}$.*

Proof. This follows from the fact that the minimal tensor product

$$C^*\mathrm{Alg}_{*-\mathrm{hom}} \times C^*\mathrm{Alg}_{*-\mathrm{hom}} \to C^*\mathrm{Alg}_{*-\mathrm{hom}} \to \mathrm{KK}$$

sastisfies the above four properties in each of its two variables (see [Mey08] for an extensive discussion). Note that the resulting tensor product

$$- \otimes -: \mathrm{KK} \times \mathrm{KK} \to \mathrm{KK}$$

is an enriched functor in each of its entries. ☐

Proposition 3.3.51 (Bott periodicity). *For any C^*-algebra A, there is a natural isomorphism in* KK $A \simeq \mathcal{C}_0(\mathbb{R}^2, A)$.

Proof. One can show that any functor satisfying the above four conditions makes A and $\mathcal{C}_0(\mathbb{R}^2, A)$ isomorphic. This is discussed for instance in [Cun84]. □

This produces the familiar Bott periodicity of K-theory groups:

Corollary 3.3.52. *There is a natural equivalence of K-theory groups $K_n(A) \simeq K_{n+2}(A)$.*

We close our discussion of the bivariant K-theory of operator algebras with a homotopy theoretic remark. The description of KK in terms of homotopy classes of bimodules suggests that KK arises as the homotopy category of some ∞-category. Moreover, the relation with K-theory suggests that we might look for a stable ∞-category whose homotopy category is KK.

Indeed, there are many homotopical structures on the category of C^*-algebras whose homotopy categories are KK. See for example the treatment in [Uuy10]. In [Mey08] it is shown that KK carries the structure of a triangulated category, so that it may serve as the homotopy category of a stable ∞-category. The following result by [JS09] and [DEKM11] is of great importance for us since it brings us back to the realm of stable homotopy theory:

Proposition 3.3.53 ([JS09], [DEKM11]). *There is a functor*

$$\mathrm{KK} \to ho\big(KU\mathrm{Mod}\big)$$

from the category KK *to the homotopy category of KU-modules, which has the properties that*

- *its value on the algebra of functions $\mathcal{C}_0(M)$ on a locally compact space agrees with the compactly supported K-theory spectrum of the space M, as defined in 3.2.*
- *the homotopy groups of the spectrum assigned to an algebra A are precisely the K-theory groups of A.*

Furthermore, this functor is lax monoidal.

This functor brings us back to the setting of section 3.1, namely into the (homotopy) category of module spectra: it associates to each C^*-algebra A its K-theory spectrum $K_*(A)$.

Remark 3.3.54. This result gives a sharpening of the corollary we mentioned before: a map in $\mathrm{KK}(A, B)$ does not just induce a map on K-theory groups, but really a map of K-theory spectra (in the homotopy category of KU-modules). We will discuss in section 4.2.3 how this can be used to obtain pushforward maps of K-theory spectra from index theory.

All these results suggest that we might not have to pass to the homotopy category KK to construct the K-theory groups of our differentiable stacks; instead, we might be able to produce function algebras which live in a true ∞-category of C^*-algebras. But this lies beyond the scope of this text and we confine ourselves with working in the homotopy category KK, where we can already produce maps of KU-modules by the above result.

3.3.4 Twisted K-theory of differential stacks

The previous results immediately imply that there is a functor from differentiable stacks, with proper maps between them, to the category KK describing bivariant K-theory.

Corollary 3.3.55. *There is a functor*

$$\mathrm{DiffStack}^{prop}_{/\mathbf{B}^2U(1)} \xrightarrow{C^*(-)} \mathrm{KK}^{op}$$

that sends each stack to its twisted convolution algebra. Moreover, this functor is monoidal.

Together with the definition of the K-theory spectrum of an operator algebra provided by proposition 3.3.53, we obtain the following definition of the twisted K-theory spectrum of a differentiable stack X:

Definition 3.3.56. Let $X \xrightarrow{\chi} \mathbf{B}^2U(1)$ be a differentiable stack carrying a circle 2-bundle. We define the compactly supported, χ-twisted K-theory spectrum of the stack X to be

$$K_c^{*+\chi}(X) := K_*(C_\chi^*(X)) \in \mathrm{ho}\big(KU\mathrm{Mod}\big).$$

Remark 3.3.57. As expected, the construction of the (twisted) *compactly supported* K-theory spectrum of a stack is only functorial with respect to *proper* maps between differentiable stacks. In [EM09], a definition is given for the K-theory groups of a stack without any support conditions. Such groups are obtained as certain sets of arrows in equivariant KK-theory.

As an immediate result, we obtain that the K-theory of a differentiable stack has the natural structure of a ring. Note that this is not true for arbitrary C^*-algebras.

Corollary 3.3.58. *The untwisted K-theory of a differentiable stack X has a natural ring structure. More generally, there are maps of twisted K-theory groups*

$$K^{m+\alpha}(X) \otimes K^{n+\beta}(X) \to K^{m+n+\alpha+\beta}(X).$$

Proof. Any stack X has the natural structure of a coalgebra with respect to the cartesian monoidal structure on DiffStack^{prop}: the comultiplication is simply the diagonal map. Taking convolution algebras and using that $C^*(-)$ is monoidal, we obtain natural maps in the KK-category

$$C^*(X) \otimes C^*(X) \to C^*(X) \qquad\qquad C^*(*) = \mathbb{C} \to C^*(X).$$

Mapping $\mathbb{C}$ into these diagrams, and using that the tensor product respects the enrichment by abelian groups, we obtain maps

$$K^0(X) \otimes K^0(X) \to \mathrm{KK}\big(\mathbb{C}, C^*(X) \otimes C^*(X)\big) \to K^0(X) \qquad \mathbb{Z} = K^0(\mathbb{C}) \to K^0(X)$$

These give a ring structure on $K^0(X)$. By replacing $\mathbb{C}$ by $\mathcal{C}_0(\mathbb{R}^n)$, or $C^*(X)$ by $\mathcal{C}_0(\mathbb{R}^n) \otimes C^*(X)$, we obtain the graded ring structure.

The same argument can be used to construct the pairing of twisted cohomology groups: if X carries circle 2-bundles α, β, then the tensor product is $X \times X$ with the circle 2-bundle $p_1^*\alpha + p_2^*\beta$. The diagonal is then equipped with the sum of the two bundles, $\alpha + \beta$. $\square$

This indeed generalizes the twisted K-theory of local quotient stacks that we saw in section 3.2:

Proposition 3.3.59 ([TXLG04])**.** *Let X be a local quotient stack, together with a smooth circle 2-bundle $\chi : X \to \mathbf{B}^2U(1)$. Then the twisted K-theory spectrum defined above is homotopy equivalent to the K-theory spectrum defined in section 3.2.*

To show this, we have to give a description of the twisted convolution algebras in terms of projective unitary bundles, which gave the twists of K-theory in section 3.2. This is done in [TXLG04], whose discussion we summarize. Recall that a circle 2-bundle on a differentiable stack X can be presented by an equivariant $PU(H)$-bundle over the associated Lie groupoid $X_1 \rightrightarrows X_0$, i.e. a $PU(H)$-bundle $P \to X_0$ together with an equivalence $s^*P \to t^*P$ satisfying a cocycle and unitality condition.

Let X be a differentiable stack with a circle 2-bundle $\alpha : X \to \mathbf{B}^2U(1)$, presented by a central extension of Lie groupoids $X^\alpha \to X$. From this we can construct a field of infinite dimensional separable Hilbert spaces over X_0 by

$$\coprod_{x \in X_0} L^2(t^{-1}(x), \mathbb{C})^{U(1)} \otimes H_0 \to X_0.$$

Here $L^2(t^{-1}(x), \mathbb{C})^{U(1)}$ is the space of $U(1)$-equivariant L^2-functions on $t^{-1}(x) \subseteq X_1^\alpha$ with respect to a Haar system on the centrally extended Lie groupoid X^α. H_0 is a fixed infinite dimensional separable Hilbert space. This field of Hilbert spaces actually forms a locally trivial Hilbert bundle over X_0 (see [TXLG04]).

This Hilbert bundle does not form an equivariant Hilbert bundle over X_0, but it is an equivariant *projective* bundle over X_0. Indeed, let $x \xrightarrow{g} y$ be an arrow in X_1 and let $x \xrightarrow{\tilde{g}} y$ be a lift of this map to X_1^α. Then $\tilde{g}$ gives rise to a unitary map

$$L^2(t^{-1}(y), \mathbb{C})^{U(1)} \otimes H_0 \to L^2(t^{-1}(x), \mathbb{C})^{U(1)} \otimes H_0; \qquad f(-) \otimes v \mapsto f(g \cdot -) \otimes v$$

Different choices of lifts $\tilde{g}$ are related by the $U(1)$-action: if we pick another lift $\tilde{g} \cdot \lambda$ of the arrow $g \in G_1$, then we obtain the map

$$f(-) \otimes v \mapsto f(\tilde{g} \cdot \lambda \cdot -) \otimes v = \lambda \cdot f(\tilde{g} \cdot -) \otimes v$$

since f is an equivariant function on $t^{-1}(x)$. It follows that the arrow $g \in X_1$ gives rise to a projective unitary map of Hilbert spaces. The associated $PU(H_0)$-bundle $P^\alpha \to X_0$ then gives rise to an equivariant $PU(H_0)$-bundle on the Lie groupoid $X_1 \rightrightarrows X_0$.

Lemma 3.3.60 ([TXLG04], 2.38)**.** *This equivariant $PU(H_0)$-bundle $P^\alpha \to X_0$ presents the circle 2-bundle $\alpha\colon X \to \mathbf{B}^2U(1)$.*

Remark 3.3.61. If X is a local quotient stack, then the above projective Hilbert bundle is precisely the universal Hilbert bundle constructed in [FHT11]. Indeed, this follows immediately from comparing the above construction locally with the construction of the universal Hilbert bundle on an action groupoid.

Given this $PU(H_0)$-bundle, we can use the action of $PU(H_0)$ on the algebra $K(H_0)$ of compact operators (with the norm topology) to construct an equivariant bundle of algebras $P^\alpha \times_{PU(H_0)} K(H_0) \to X_0$. We can use this bundle of algebras to give a different presentation of the twisted convolution algebra $C^*_\alpha(X)$, at least for proper stacks, which can be presented by proper Lie groupoids:

Proposition 3.3.62 ([TXLG04], 4.3)**.** *Let X be a proper stack. Then the twisted convolution algebra $C^*_\alpha(X)$ is isomorphic to the C^*-algebra of equivariant sections of the bundle $P^\alpha \times_{PU(H_0)} K(H_0) \to X_0$, vanishing at infinity.*

Remark 3.3.63. If s is an equivariant section of this bundle, then $||s|| : X_0 \to \mathbb{R}_{\geq 0}$ is constant along the orbits of the Lie groupoid. Hence this function descends to a continuous function on the quotient space X_0/X_1. The section s is said to vanish at infinity if this function on X_0/X_1 vanishes at infinity in the traditional sense.

The K-theory classes of the algebra $K(H_0)$ of compact operators can be presented by homotopy classes of Fredholm operators on H_0. Applying this fiberwise, we obtain

Proposition 3.3.64 ([TXLG04], 4.6)**.** *Let X be a proper stack. Then there is a natural isomorphism between $K_i(C^*_\alpha(X))$ and the group of homotopy classes of sections of the associated Fredholm bundle*

$$P^\alpha \times_{PU(H_0)} \mathrm{Fred}^{(i)} \to X_0.$$

In particular, if X is a proper stack, then the operator-algebraic definition of the twisted compactly supported K-theory groups agrees with the definition in terms of Fredholm operators given in section 3.2.

This shows how the operator algebraic description of the twisted K-theory of a stack agrees with the description in terms of Fredholm operators. This allows us transfer geometric ideas from the Fredholm perspective to the operator algebraic picture. For example, we immediately obtain

Corollary 3.3.65. *Let M be a smooth manifold carrying an action of a compact Lie group G. Then the operator algebraic K-theory of the stack $M//G$ agrees with the G-equivariant compactly supported K-theory of M, as defined in section 3.2. In particular, if M is a compact manifold, then $K^0(M//G) = K^0(C^*(M//G))$ agrees with the Grothendieck group of equivariant G-bundles on M.*

Summarizing, we obtain a functor

$$\big(\mathrm{DiffStack}_{prop}\big)^{op}_{/\mathbf{B}^2U(1)} \longrightarrow \mathrm{KK}^{op} \longrightarrow \mathrm{ho}\big(KU\mathrm{Mod}\big)$$

which agrees with the description of K-theory for local quotient stacks in [FHT11], the twisted K-theory for differentiable stacks in [TXLG04] and on morphisms agrees with the constructions in [Lan01] and [AG06]. Moreover, this is a lax monoidal functor.

Using this picture of K-theory for differentiable stacks, we can linearize the correspondence diagrams that appear for a two-dimensional pQFT, just like did in the geometrically discrete setting in section 3.1.

3.3.5 Linearizing 2d corresponces using KK-theory

In section 3.1 we have discussed how one can use twisted generalized cohomology to linearize a prequantum theory, in case all spaces of fields involved are discrete ∞-groupoids. The KK-theoretic discussion allows us to also linearize the correspondence diagrams that appear in two-dimensional prequantum field theories where the spaces of fields are differentiable stacks. An example of such a theory is (the topological part of) the non-perturbative Poisson sigma model, which will be one of our main examples in section 5.

Such a prequantum theory is given by a monoidal functor

$$\mathrm{Bord}_2 \longrightarrow \mathrm{Corr}_2\big((\mathrm{DiffStack}_{prop})_{/\mathbf{B}^2U(1)}\big)$$

Remark 3.3.66. Since we only allow for *proper* maps between differentiable stacks, not every differentiable stack arises as the classifying stack of fields of a pQFT. The stack has to satisfy certain properness conditions that guarantee that maps like the diagonal or the map to the point are all proper maps of stacks. We will mostly ignore this technical requirement.

On the other hand, if we drop the properness condition on the morphisms, then every differentiable stack can arise as the classifying stack of a two-dimensional pQFT.

In complete analogy to the linearization in the discrete case 3.1.3, a k-fold correspondence diagram in $\mathrm{DiffStack}_{/\mathbf{B}^2U(1)}$ now gives rise to a k-fold cospan diagram in the KK-category of C^*-algebras. Passing to the associated K-theory spectra, we obtain a k-fold cospan diagram in the homotopy category of KU-modules. These k-fold cospan diagrams are built up out of single cospan diagrams, which we want to turn into actual maps of spectra to quantize the theory. This quantization of a single such cospan diagram is the topic of the next section.

Alternatively, in the same way as we discussed in section 3.1.3 we can obtain a single correspondence in $\mathrm{DiffStack}_{/\mathbf{B}^2U(1)}$ as the *transgression* of a higher dimensional pQFT to higher dimensional cobordisms. For example, if we have a three-dimensional pQFT, then a two-dimensional surface with as boundary a disjoint union of circles gives rise to a diagram of the form

$$\begin{tikzcd}
 & \big[\Pi(\Sigma), \mathbf{Fields}\big] \arrow[dl, "\rho\mathrm{in}"'] \arrow[dr, "\rho_{\mathrm{out}}"] & \\
\big[\Pi(S^1), \mathbf{Fields}\big]^{\times p} \arrow[dr, "\alpha"'] \arrow[rr, phantom, "\Rightarrow"] & & \big[\Pi(S^1), \mathbf{Fields}\big]^{\times q} \arrow[dl, "\beta"] \\
 & \mathbf{B}^2U(1) &
\end{tikzcd}$$

which is obtained by collapsing the 2-fold correspondence diagram to a single correspondence diagram. The we can pass to KU-modules to obtain a diagram

$$K^{*+\alpha}\big[\Pi(S^1)^{\amalg p}, \mathbf{Fields}\big] \to K^{*+\rho_{\mathrm{out}}^*\beta}\big[\Pi(\Sigma), \mathbf{Fields}\big] \leftarrow K^{*+\beta}\big[\Pi(S^1)^{\amalg q}, \mathbf{Fields}\big]$$

As we mentioned in section 3.1.3, we can apply our quantization to a single such diagram, but we inevitably forget about the data that is assigned to lower dimensional manifolds (in this case, what is assigned to the point). An important example of this situation is given by the discussion of Chern-Simons theory in [FHT10], which we discuss in section 5.2.3.

We can proceed in the exact same way for boundary theories. Suppose we have a diagram of differentiable stacks carrying a circle 2-bundle

$$\begin{tikzcd}
 & \mathbf{Fields}^\partial \arrow[dl] \arrow[dr, "f"] \arrow[dd, "f^*\chi"] & \\
* \arrow[dr] \arrow[r, phantom, "\overset{\xi}{\Longrightarrow}"] & {} \arrow[r, phantom, "\Longrightarrow"] & \mathbf{Fields} \arrow[dl, "\chi"] \\
 & \mathbf{B}^2U(1) &
\end{tikzcd}$$

The left 2-cell describes a trivialization of the circle 2-bundle $f^*\chi$, which we can interpret as a $f^*\chi$-twisted line bundle over $\mathbf{Fields}^\partial$. Then under the functor $C^*(-)$, this gets sent to a diagram in the KK-category

$$\mathbb{C} \xrightarrow{\xi} C^*_{f^*\chi}(\mathbf{Fields}^\partial) \xleftarrow{f^*} C^*_\chi(\mathbf{Fields})$$

Passing to the associated K-theory spectra, we obtain the analogue of diagram 3.4 in the setting of differentiable stacks:

$$KU \xrightarrow{\xi} K^{*+f^*\exp(iS)}(\mathbf{Fields}^\partial) \xleftarrow{f^*} K^{*+\exp(iS)}(\mathbf{Fields})$$

The left map gives the twisted K-theory class of the twisted line bundle ξ.

To actually quantize these boundary theories, we have to produce a map going from the left to the right. To do this, we turn the right map around: instead of pulling back along the map f, we push along it. Observe that this pushforward map is constructed only from the right hand side of the diagram. The left hand side of the diagram describes a twisted line bundle over $\mathbf{Fields}^\partial$, while the right hand side of the diagram will result in the 'integration' of this twisted line bundle along the fibers of the map $f\colon \mathbf{Fields}^\partial \to \mathbf{Fields}$. The resulting map of spectra describes a cocycle in the twisted cohomology of the stack **Fields**, which we interpret as the quantization of the boundary theory. The construction of a pushforward map along f will be the topic of the next section.

4 Quantization by pull-push

We have seen in the previous section how the embedding of the groups $\mathbf{B}^n U(1)$ into the group of units of a (smooth) ring spectrum allows us linearize the diagrams that describe trajectories and boundary theories. More precisely, in section 3.1.5 we obtained a functor

$$\mathbf{H}_{/\mathbf{B}^n U(1)} \longrightarrow \mathbf{H}_{/\mathbf{B}GL_1(\mathbf{R})} \longrightarrow \mathbf{R}\mathrm{Mod}^{op}$$

to the category of (smooth) $\mathbf{R}$-modules, assigning to each space with a twist its twisted R-cohomology spectrum. We have a relatively concrete description of this functor in two cases:

(i) all stacks are geometrically discrete (i.e. just ∞-groupoids), and R is a ring spectrum without any smooth structure. In that case we have given an extensive treatment in 3.1.

(ii) the stacks are differentiable stacks, equipped with a smooth circle two bundle. In that case, the above functor can be approximated by passing to KK-theory and taking the K-theory spectra of the (twisted) convolution algebras of these differentiable stacks

$$\big(\mathrm{DiffStack}_{prop}\big)_{/\mathbf{B}^2 U(1)} \xrightarrow{C^*(-)} \mathrm{KK}^{op} \longrightarrow \mathrm{ho}\big(KU\mathrm{Mod}\big)^{op}$$

This was discussed in section 3.3.

Whichever concrete description we choose, we always obtain a functor to a category of spectra, which reverses the direction of all arrows. We can now consider the image under this functor of diagrams consisting of a single correspondence

$$\begin{tikzcd}
 & \mathrm{Fields}(\Sigma) \arrow[dl, "i"'] \arrow[dr, "f"] \arrow[d, phantom, "\Rightarrow"] & \\
\mathrm{Fields}(\partial\Sigma_{\mathrm{in}}) \arrow[dr, "\alpha"'] & & \mathrm{Fields}(\partial\Sigma_{\mathrm{out}}) \arrow[dl, "\beta"] \\
 & \mathbf{B}^n U(1) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbf{Fields}^\partial \arrow[dl] \arrow[dr, "f"] \arrow[d, phantom, "\Rightarrow \xi"] & \\
* \arrow[dr] & & \mathbf{Fields} \arrow[dl, "\beta"] \\
 & \mathbf{B}^n U(1) &
\end{tikzcd}$$

describing either the trajectory of a field along a cobordism Σ between two closed manifolds, or classifying a boundary to a pQFT. In both cases, we are left with a cospan of R-modules: the left diagram gives rise to a diagram of like

$$R^{*+\alpha}\big(\mathrm{Fields}(\partial_{\mathrm{in}}\Sigma)\big) \xrightarrow{i^*} R^{*+f^*\beta}\big(\mathrm{Fields}(\Sigma)\big) \xleftarrow{f^*} R^{*+\beta}\big(\mathrm{Fields}(\partial_{\mathrm{out}}\Sigma)\big) \tag{4.1}$$

describing the linearized trajectories from $\partial_{\mathrm{in}}\Sigma$ to $\partial_{\mathrm{out}}\Sigma$. Recall from remark 3.1.16 that a correspondence in the slice over $\mathbf{B}^n U(1)$ also comes equipped with a vertical map that divides the square into two triangles. The maps i^* and f^* also depend on the 2-cells filling these two triangles.

The diagram classifying a boundary to a pQFT gives rise to a diagram of the form

$$R \xrightarrow{\xi} R^{*+f^*\beta}(\mathbf{Fields}^\partial) \xleftarrow{f^*} R^{*+\beta}(\mathbf{Fields}).$$

For simplicity, we will only consider the last diagram, where the right map is given by pullback along the map $f\colon \mathbf{Fields}^\partial \to \mathbf{Fields}$, while the left map is induced by the twisted circle $(n-1)$-bundle ξ over $\mathbf{Fields}^\partial$.

To get a map of spectra from left to right, we would want to turn the right map around and form the composition. Since we are working in a category of linear objects, there is an obvious way to turn a map around: we simply form dual objects and take the dual map. Indeed, there is a good theory of duality, both in the context of generalized cohomology (i) and in the context of KK-theory (ii). In the realm of generalized cohomology, this is called Spanier-Whitehead duality (see section 4.1.1), while in KK-theory, this is the theory of Poincaré dual algebras (section 4.2.4). In both cases, the existence of a dual requires some kind of smallness on the space (or C^*-algebra). One should compare this with the existence of a categorical dual to a vector space: such a dual exists only if the vector space is *finite dimensional*.

If both $R^{*+f^*\beta}(\mathbf{Fields}^\partial)$ and $R^{*+\beta}(\mathbf{Fields})$ are small (i.e. finite R-modules), then we can form their dual spectra and consider the dual map

$$R^{*+f^*\beta}(\mathbf{Fields}^\partial)^\vee \xrightarrow{(f^*)^\vee} R^{*+\beta}(\mathbf{Fields})^\vee \tag{4.2}$$

In fact, since we are only interested in turning around the map f^*, we do not have to assume both its domain and codomain to be dualizable; instead, we only require that the map f itself is small, i.e. that its (homotopy) fibers are small. In that case, we can dualize $R^{*+f^*\beta}(\mathbf{Fields}^\partial)$ along the fibers and get a map

$$\text{Fiberwise dual of } R^{*+f^*\beta}(\mathbf{Fields}^\partial) \xrightarrow{(f^*)^\vee} R^{*-\beta}(\mathbf{Fields}). \tag{4.3}$$

where we use that the dual of β is $-\beta$ if we work *over* the space **Fields**. However, in both cases we are facing the same problem: we have changed the domain of our map from $R^{*+f^*\beta}(\mathbf{Fields}^\partial)$ into its (fiberwise) dual. To fix this, we would like to pick an identification

$$R^{*+f^*\beta}(\mathbf{Fields}^\partial) \xrightarrow{\sim} \text{Fiberwise dual of } R^{*+f^*\beta}(\mathbf{Fields}^\partial)$$

This poses unnaturally strong conditions on the twist β. Instead, we observe that we can take the dual map $(f^*)^\vee$ for *any* choice of twist β. We might therefore hope to find an equivalence

$$R^{*+f^*\beta}(\mathbf{Fields}^\partial) \xrightarrow{\sim} \text{Fiberwise dual of } R^{*+f^*\gamma}(\mathbf{Fields}^\partial) \tag{4.4}$$

for *some* choice of twist γ. We then obtain a composition of maps

$$\begin{tikzcd} R \arrow[r, "\xi"] & R^{*+f^*\beta}(\mathbf{Fields}^\partial) \arrow[d, "\sim"] \arrow[dr, bend left, "f^!"] & \\ & \text{Fiberwise dual of } R^{*+f^*\gamma}(\mathbf{Fields}^\partial) \arrow[r, "(f^*)^\vee"] & R^{*-\gamma}(\mathbf{Fields}). \end{tikzcd}$$

The vertical equivalence and the right map together constitute a *pushforward map*, or Umkehr map, or Gysin map, which we will denote by $f^!$ (to distinguish it from the left Kan extension $f_!$ that we will encounter in section 4.1). This pushforward map plays the role of the path integral. The twisted circle $(n-1)$-bundle ξ gives rise to a twisted R-cocycle on $\mathbf{Fields}^\partial$, which we now integrate along the map f. The resulting class in the R-cohomology of **Fields** can be interpreted as the *partition function* of the boundary theory. Observe that the result does not necessarily lie in the β-twisted R-cohomology of **Fields**; the twist might be different.

In an analogous way can we form a single map of R-modules from the diagram 4.1 describing the linearized trajectories over a cobordism Σ between two closed manifolds. The resulting map can be interpreted as the quantum propagator along Σ.

This quantization procedure explicitly depends on the choice of equivalence in 4.4. To obtain such a choice, it is important to get a better understanding of the dual spectra that we are using. For the homotopy type of a manifold, there is a very nice description of its dual spectrum: it is given by the Thom spectrum associated to its stable normal bundle (see section 4.1.2). This allows us to analyze the choice of equivalence 4.4 in more detail. For manifolds, such a choice of equivalence is given by a choice of R-*orientation* on the stable normal bundle (or equivalently, on the tangent bundle).

This choice of orientation might not exist, and if it exists, it might not be unique. In particular, there are obstructions to quantization and the resulting theory depends on additional choices. Such obstructions often appear in quantization, and are called *anomalies*. On the other hand, the non-uniqueness of a choice of orientation is also not unexpected: the boundary theories to a TFT are often not topological, but require extra geometric information to construct such an orientation. One of the best studied examples of this is the boundary theory to Chern-Simons theory, which is a conformal field theory known as the WZW-model. In section 6 we will shortly come back to the dependence on the choice of orientation, but a treatment of this issue lies beyond our reach.

Using this, we can form pushforward maps of spectra in the discrete setting, as we discuss in section 4.1.4. We will then restrict our attention to duality and pushforward maps in complex K-theory, so that we can apply the same ideas in the smooth setting. Over smooth manifolds, the pushforward maps in complex K-theory have a very concrete description: we can use the smooth structure to construct elliptic differential operators, which give rise to maps in K-theory via index theory. In particular, a spin^c-structure on a smooth manifold gives rise to a Dirac operator, whose index describes a pushforward map in K-theory. This description of maps in K-theory via elliptic operators generalizes from manifolds to local quotient stacks, as long as the maps between them are very nice (namely presentable). We will give a brief account in section 4.2.1.

The use of index theory to produce maps in K-theory is the trademark of KK-theory; essentially KK-theory is set up precisely so that maps are given by differential operators. The slightly ad-hoc use of index theory in section 4.2.1 comes to a full-grown theory once we work in KK-theory; the monoidal structure on KK described in section 3.3.3 allows for a notion of duality, just as the Spanier-Whitehead duality in stable homotopy theory. In this case the duality is witnessed by differential operators, for example by the Dirac operator associated to a spin^c-structure. For manifolds the KK-theoretic dual and the Spanier-Whitehead dual are essentially the same, which provides some geometric intuition behind the algebraic definitions that arise in KK-theory. We will discuss duality and pushforward maps from this algebraic perspective in section 4.2.3.

With all necessary tools discussed, we conclude with an overview of how pull-push quantization now comes about. Essentially this just places the constructions from this section and the previous section in the context of quantization. In the next section 5, we will then see some examples of this.

4.1 Pushforward and duality in generalized cohomology

In this section we continue the abstract story from section 3.1. We follow the discussion of pushforwards in generalized cohomology as developed in [ABG11]. Recall that an R-twist over X was simply a functor $\alpha\colon X \to \mathrm{Pic}(R)$ into the Picard ∞-groupoid of the ring spectrum R. Such a twist gave rise to twisted homology and cohomology spectra

$$R_{*+\alpha}(X) = \operatorname*{colim}_X \alpha \qquad\qquad R^{*+\alpha}(X) = [\operatorname*{colim}_X \alpha, R]$$

where the colimit is taken in the ∞-category of R-module spectra. The homotopy groups of these spectra provide the α-twisted homology and cohomology groups of X.

Given a map of spaces $f\colon X \to Y$ and a twist $\beta : Y \to \mathrm{Pic}(R)$, there is the natural map of homology spectra

$$R_{*+f^*\beta}(X) = \operatorname*{colim}_X \beta\circ f \xrightarrow{f_*} \operatorname*{colim}_Y \beta = R_{*+\beta}(Y) \tag{4.5}$$

which gives a map of cohomology spectra by mapping into R

$$R^{*+\beta}(Y) \xrightarrow{f^*} R^{*+f^*\beta}(X).$$

A pushforward map in R-cohomology is supposed to be a map the other way around, from the cohomology on X to the cohomology on Y. We do not require the pushforward to map into the β-twisted cohomology of Y: in principle, any form of twisted cohomology on Y will do.

As we sketched on the previous pages, we obtain such a map by

(i) passing to the 'fiberwise dual' of the map $f^* : R^{*+\gamma}(Y) \to R^{*+f^*\gamma}(X)$ for some choice of twist γ, possibly different from the twist β we start with. The domain of this 'fiberwise dual' map is the fiberwise dual of the spectrum $R^{*+f^*\gamma}(X)$.

(ii) identifying this fiberwise dual of $R^{*+f^*\gamma}(X)$ with the spectrum $R^{*+f^*\beta}(X)$.

Let us first consider the first step more explicitly. We can view $f^*\gamma$ and γ as functors from X and Y into RMod, which are usually called parametrized R-modules. Associated to the map f are functors

$$\mathrm{Fun}(X, R\mathrm{Mod}) \underset{f_*}{\overset{f_!}{\underset{\xleftarrow{f^*}}{\rightrightarrows}}} \mathrm{Fun}(Y, R\mathrm{Mod}) \underset{p_*}{\overset{p_!}{\underset{\xleftarrow{p^*}}{\rightrightarrows}}} R\mathrm{Mod}$$

induced by the left and right Kan extension of the pullback functor f^*. The right functors are induced by the map $p : Y \to *$. The left adjoint $p_!$ is the functor that takes the (homotopy) colimit over Y, while $p_! f_!$ takes the homotopy colimit over X. Then the counit of the adjunction $f^* \vdash f_!$ gives a natural map

$$R_{*+f^*\gamma}(X) = p_! f_! f^* \gamma \to p_! \gamma = R_{*+\gamma}(Y)$$

in homology. By turning this map around and mapping into R, we obtain the map f^* in (twisted) R-cohomology induced by the map $f \colon X \to Y$.

The idea is to reverse the direction of the above map in homology before we have applied $p_!$, by dualizing in the functor category $\mathrm{Fun}(Y, R\mathrm{Mod})$. Instead of posing smallness conditions on both X and Y, this only poses smallness conditions on the (homotopy) fibers of the map $f \colon X \to Y$. In good cases, we obtain a map of R-modules parametrized over Y

$$\gamma^\vee \to (f_! f^* \gamma)^\vee.$$

This is the fiberwise dual map we need in the first step. Since γ is given by a functor $Y \to \mathrm{Pic}(R)$, its dual is simply its inverse. Applying the colimit functor and mapping into R, we then obtain a map of R-modules

$$\Big[p_! (f_! f^* \gamma)^\vee, R\Big] \to R^{*-\gamma}(Y).$$

The domain is obtained from the twist $f^*\gamma : X \to \mathrm{Pic}(R)$ by forming the dual spectrum along the (homotopy) fibers of the map f. In particular, it is still an R-module that we assign to a twist over X.

We now come to the second step: we have to give an equivalence of spectra

$$R^{*+f^*\beta}(X) = \Big[p_! f_! f^* \beta, R\Big] \xrightarrow{\sim} \Big[p_! (f_! f^* \gamma)^\vee, R\Big].$$

We can do this by providing an equivalence $f_! f^* \beta \simeq (f_! f^* \gamma)^\vee$ for some choice of γ.

In general it is not quite clear what dual spectra look like precisely, so it is not straightforward to describe such a map. However, Atiyah-Whitehead duality realizes the dual spectrum of a smooth manifold as the (twisted) Thom space construction of its stable normal bundle, as we will see in section 4.1.2. If X and Y are manifolds and f is a nice enough map between them, then the homotopy fibers will be smooth manifolds and we can apply the Thom space construction. In this case, we can construct an identification $f_! f^* \beta \simeq (f_! f^* \gamma)^\vee$ by a choice of orientation on the fibers of the map f, as we will discuss in section 4.1.3.

4.1.1 Spanier-Whitehead duality of spectra

Let R be a ring spectrum and consider the category RMod of R-modules. Recall that RMod carries a symmetric monoidal structure given by the smash product over R. Spanier-Whitehead duality is then simply duality with respect to this monoidal structure:

Definition 4.1.1. If S is an R-module, then a Spanier-Whitehead dual to S is an R-module $S^\vee$, together with maps

$$\mathrm{coev} \colon R \to S \wedge_R S^\vee \qquad\qquad \mathrm{ev} \colon S^\vee \wedge_R S \to R$$

satisfying the zigzag identities

$$\big(1_S \wedge_R \mathrm{ev}\big)\big(\mathrm{coev} \wedge_R 1_S\big) \simeq 1_S \qquad\qquad \big(\mathrm{ev} \wedge_R 1_{S^\vee}\big)\big(1_{S^\vee} \wedge_R \mathrm{coev}\big) \simeq 1_{S^\vee}.$$

If $R = \mathbb{S}$ is the sphere spectrum, this is dualizibility in the category of spectra. In that case, we have the following result, due to [SW55]:

Proposition 4.1.2. *A spectrum S is dualizable iff it is finite, i.e. if it is equivalent to a spectrum having a finite cell decomposition. The dual is given by the mapping spectrum*

$$S^\vee = [S, \mathbb{S}]$$

into the sphere spectrum.

In particular, note that passing to the dual spectrum is a *contravariant* functor.

Example 4.1.3. Since any closed manifold has the homotopy type of a finite CW complex, their suspension spectra admit a Spanier-Whitehead dual. In the next section we will give a concrete description of the Spanier-Whitehead dual of a closed manifold, using the Thom construction.

Of course, the functor

$$(-) \wedge R : \mathrm{Sp} \to R\mathrm{Mod}$$

is monoidal, so it preserves dualizable objects. However, more objects might be dualizable in RMod. In particular, we have:

Proposition 4.1.4 ([ABG11]). *If X is a finite ∞-groupoid and $\alpha\colon X \to \mathrm{Pic}(R)$ gives a twist, then the twisted homology spectrum $R_{*+\alpha}(X)$ admits a dual.*

We are mainly interested in the relative case where we consider spectra parametrized over a base Y. In that case, there is an analogous definition of duality:

Definition 4.1.5. Let Y be an ∞-groupoid and endow the ∞-category $\mathrm{Fun}(Y, \mathrm{Sp})$ with the symmetric monoidal structure of the objectwise smash product of spectra. Then a parametrized spectrum S over Y is dualizable in $\mathrm{Fun}(Y, \mathrm{Sp})$ iff it is objectwise finite. The same holds for parametrized R-modules.

Example 4.1.6. Of course the functor constant on R is dualizable in $\mathrm{Fun}(Y, R\mathrm{Mod})$. Also, all twists $Y \to \mathrm{Pic}(R)$ are dualizable, since the Picard group consists of invertible objects with respect to the tensor product. For such a twist, the dual is given by mapping objectwise into R, where we use that Y is an ∞-groupoid to obtain a functor $Y \simeq Y^{op} \to R\mathrm{Mod}$.

Example 4.1.7. Let $f\colon N \to M$ be a proper map of smooth manifolds. If we pass to their homotopy types, then the Grothendieck construction associates to this map a functor $\Pi(M) \to \infty\mathrm{Gpd}$, which assigns to each vertex in the homotopy type $\Pi(M)$ of M the homotopy fiber of f over that point. Since f is proper all homotopy fibers are finite. Then we can form the suspension spectrum of this parametrized space, which gives rise to a finite parametrized spectrum over M. As such, we see that the space N is dualizable over M.

Although we know that the suspension spectrum of a compact manifold has a dual, the mapping spectrum $[\Sigma^\infty_+ M, \mathbb{S}]$ hardly gives a concrete description of this dual spectrum. Remarkably, the dual spectrum of $\Sigma^\infty_+ M$ turns out to be a twisted version of the suspension spectrum $\Sigma^\infty_+ M$. The twist is given by the Thom construction.

4.1.2 Thom spectra

In the case of smooth manifolds, the above categorical definition of duality has a nice geometric description in terms of the so-called Thom space construction. In general, the Thom construction assigns a spectrum to each stable vector bundle over a space X. In the particular case when X is the homotopy type of a compact smooth manifold and we consider its stable normal bundle, then the Thom construction gives a model for the Spanier-Whitehead dual to X.

Definition 4.1.8. Let
$$\mathbb{Z} \times BO = \operatorname{colim}_n \mathbb{Z} \times BO(n)$$
be the classifying space of stable vector bundles. If X is a compact connected ∞-groupoid, then $X \to \mathbb{Z} \times BO$ maps into one of the connected components of BO labeled by n. We call n the virtual rank of the stable vector bundle.

We focus on the group $O = \operatorname{colim} O(n)$ sitting over the point 0 in $\mathbb{Z} \times BO$. We can define a homomorphism of ∞-groups $O \to gl_1(\mathbb{S})$ as follows: for each $g \in O(n) \subset O$, there is a natural action of g on $\mathbb{R}^n$ that can be extended to an action of g on S^n fixing the points 0 and ∞. In other words, g defines a map of pointed spaces from S^n to itself (with basepoint ∞). This gives a continuous map
$$O(n) \to \Omega^n S^n$$
Moreover, under the inclusion $O(n) \subset O(n+1)$, the element $g \in O(n)$ gives rise to a map $S^{n+1} \to S^{n+1}$ that is the identity along one of the coordinates on S^{n+1}. This means precisely that there is a commutative diagram
$$\begin{tikzcd} O(n) \arrow[r] \arrow[d] & \Omega^n S^n \arrow[d] \\ O(n+1) \arrow[r] & \Omega^{n+1} S^{n+1} \end{tikzcd}$$
where the right map sends a map $f : S^n \to S^n$ to its suspension $\Sigma f : S^{n+1} \to S^{n+1}$. We thus find a map
$$O \to \Omega^\infty \mathbb{S} = GL_1(\mathbb{S})$$
which is a group homomorphism. Delooping, we find a map
$$BO \to BGL_1(\mathbb{S}) \to \operatorname{Pic}(\mathbb{S})$$
which encodes the action of O on the sphere spectrum $\mathbb{S}$. We extend this to $\mathbb{Z} \times BO$ by sending the copy of $\{n\} \times BO$ to the automorphism ∞-group of the n-fold suspension $\Sigma^n \mathbb{S}$.

Note that the ∞-groupoid $\mathbb{Z} \times BO$ has a natural monoidal structure (the tensor product of vector bundles) and that the functor $\mathbb{Z} \times BO \to \operatorname{Pic}(\mathbb{S})$ is monoidal, where $\operatorname{Pic}(\mathbb{S})$ has the smash product of spectra as monoidal structure.

Definition 4.1.9. The monoidal functor $\mathbb{Z} \times BO \xrightarrow{J} \operatorname{Pic}(\mathbb{S})$ is called the J-homomorphism.

We can use the J-homomorphism to assign a twist to every (stable) vector bundle over an ∞-groupoid.

Definition 4.1.10. Let M be a smooth manifold and let $\xi \colon M \to \mathbb{Z} \times BO$ classify a stable vector bundle on M. Then the homotopy colimit
$$M^\xi := \operatorname{colim}_M J \circ \xi$$
in $\mathbb{S}\mathrm{Mod} = \mathrm{Sp}$ is called the Thom spectrum of the stable bundle ξ.

Remark 4.1.11. There is no reason to restrict to M being a manifold. We can do the same thing for any spectrum with a map to BO. In particular, doing this for BO itself gives the Thom spectrum MO. It has the property that its homotopy groups $\pi_n(MO)$ can be naturally identified with cobordisms classes of n-dimensional manifolds.

Similarly, we can consider the Thom spectrum of the classifying spaces $B\mathrm{Spin}^c$ or $B\mathrm{String}$ for spin^c or string-bundles. These have the property that $\pi_n(M\mathrm{Spin}^c)$ is given by cobordism classes of spin^c-manifolds (or string-manifolds).

Remark 4.1.12. Since any ring spectrum comes equipped with a ring map $\mathbb{S} \to R$, we see that the J-homomorphism allows us to form twists in R-cohomology for any ring R, simply by forming the composite
$$X \xrightarrow{\xi} BO \to BGL_1(\mathbb{S}) \to BGL_1(R) \to \operatorname{Pic}(R).$$

This abstract definition of the Thom spectrum has a very explicit construction. For simplicity, we only consider the case of a stable bundle of virtual rank 0, so that it is described just by a map $M \to BO$. When M is compact, M must map into some $BO(d)$, which classifies a vector bundle $V \to M$. The corresponding virtual bundle is given by $[V] - [\mathbb{R}^d]$. In general, a virtual rank 0 bundle $[V] - [W]$ is equivalent to one of the form

$$[V \oplus W^\perp] - [W \oplus W^\perp] \simeq [V \oplus W^\perp] - [\mathbb{R}^d]$$

where $W^\perp$ is a bundle such that $W \oplus W^\perp \simeq \mathbb{R}^d$ (such bundles exist over smooth manifolds). So we can indeed present any stable rank 0 bundle by a map $M \to BO$.

Let V be the rank d vector bundle corresponding to the map $M \to BO(d) \subset BO$. It comes equipped with a Riemannian metric, and we obtain a bundle $B(V) \to M$ of closed balls with radius 1, as well as a sphere bundle $S(V) \to M$ of radius 1. We define the Thom space

$$\mathrm{Th}(V) = B(V)/S(V)$$

to be the quotient of the total space $B(V)$ by its boundary, with the image of $S(V)$ as its basepoint. Essentially we take the vector bundle V and add a single point at the 'infinities' of all fibers.

Lemma 4.1.13. *The Thom space construction satisfies*

$$\mathrm{Th}(V \oplus \mathbb{R}) \simeq \Sigma \mathrm{Th}(V)$$

Proof. In each fiber over $m \in M$, we can construct a homeomorphism from the unit ball in $V_m \oplus \mathbb{R}$ to $B(V_m) \times [0,1]$, so that the unit sphere corresponds to

$$S(V_m) \times [0,1] \cup B(V_m) \times \{0,1\}$$

The quotient is precisely the description of the suspension $\Sigma\mathrm{Th}(V)$. □

The lemma shows that, up to a suspension, the Thom space is independent of the chosen representative of the stable bundle: if we had chosen a representative $[V \oplus \mathbb{R}] - [\mathbb{R}^{d+1}]$, then we would find the suspension $\Sigma\mathrm{Th}(V)$. In the following alternative definition of the Thom spectrum, we introduce an extra suspension to cancel this behaviour:

Proposition 4.1.14. *If $\xi\colon M \to \{n\} \times BO(d)$ describes the stable bundle $[V] - [\mathbb{R}^{d-n}]$, then the spectrum*

$$\Sigma^{n-d}\Sigma^\infty \mathrm{Th}(V)$$

agrees with the Thom spectrum in terms of the J-homomorphism.

The shift in degree is required to match the choice made in the construction of the J-homomorphism.

Proof. We only consider the case where the virtual dimension is 0. The others follow by suspending. Associated to the vector bundle V is its frame bundle with structure group $O(d)$. The orthogonal group acts on S^d (preserving both the point 0 and ∞), and the J-homomorphism gives the associated (stable) spherical fibration.

The homotopy colimit from definition 4.1.10 is essentially the total space of this spherical fibration. However, $O(d)$ acts on S^d as a pointed space (with basepoint ∞), and the homotopy colimit in definition 4.1.10 really glues all the fibers of the spherical fibration as pointed spaces. We therefore identity all the points at ∞ with a single point, which gives the Thom space $\mathrm{Th}(V)$.

Using this, we see that for $n = 0$, the Thom spectrum is given by

$$M^V = \operatorname*{colim}_M \mathbb{S} = \operatorname*{colim}_M \Sigma^{\infty-d} S^d = \Sigma^{\infty-d} \operatorname*{colim}_M S^d = \Sigma^{-d}\Sigma^\infty \mathrm{Th}(V)$$

For virtual rank n, we simply suspend the Thom spectrum n times. □

We are mainly interested in the stable normal bundle $-\tau$ to a compact smooth manifold M. If we embed the smooth manifold M in some $\mathbb{R}^N \subset S^N$, then the stable normal bundle $-\tau$ can be represented by the normal bundle ν of M inside S^N. Indeed, we have that $\nu \oplus \tau_M \simeq \mathbb{R}^N$ is the trivial stable vector bundle.

The normal bundle ν of M inside $\mathbb{R}^N$ is homeomorphic to a tubular neighbourhood of M inside $\mathbb{R}^N$. The disc bundle $D(\nu)$ corresponds to a massive tube around M in $\mathbb{R}^N$ and the Thom space collapses the boundary of that tube to a single point. Equivalently, we can just take any open tubular neighbourhood U of M inside $\mathbb{R}^N \subset S^N$ and collapse the complement $S^N - U$ to a single point. This is the classical description of the Thom space of a compact manifold M.

Proposition 4.1.15 (Atiyah-Whitehead, [Ati61])**.** *Let M be a smooth compact manifold with tangent bundle τ, and let $-\tau$ be the stable normal bundle. Then the Thom spectrum $M^{-\tau} = \Sigma^\infty_+ M^\vee$ is the Spanier-Whitehead dual to the suspension spectrum of M.*

The remarkable thing about Atiyah-Whitehead duality is that it relates the abstract categorical notion of duality to a very concrete geometric construction: the dual spectrum to $\Sigma^\infty M$ is not just any spectrum, but it is actually a spectrum that describes a form of twisted (co)homology of the manifold M itself.

Corollary 4.1.16. *Let R be an E_∞-ring spectrum and M a compact manifold. Then the spectrum $R \wedge \Sigma^\infty_+ M$ has dual spectrum*

$$[R \wedge \Sigma^\infty M, R] \simeq R \wedge M^{-\tau} =: R_{*-\tau}(M)$$

More generally, let $\alpha : M \to \mathrm{Pic}(R)$ be any twist. Then the α-twisted R-cohomology spectrum

$$[\operatorname*{colim}_M \alpha, R] \simeq R_{*-\alpha-\tau}(X)$$

is equivalent to the $(-\alpha-\tau)$-twisted R-homology spectrum.

Proof. From the fact that $R \wedge (-) : \mathbb{S}\mathrm{Mod} \to R\mathrm{Mod}$ is a monoidal functor, it immediately follows that $R \wedge M^{-\tau}$ is dual to $R \wedge M := R \wedge \Sigma^\infty_+ M$. To see that the dual can also be presented by mapping $R \wedge M$ into R, we use that $R \wedge M$ is a *free* R-module:

$$\begin{aligned}
(R \wedge M)^\vee &\simeq [\Sigma^\infty_+ M, \mathbb{S}] \wedge R \simeq [\operatorname*{colim}_M \mathbb{S}, \mathbb{S}] \wedge R \\
&\simeq \lim_M \mathbb{S} \wedge R \\
&\simeq \lim_M R \\
&\simeq [\operatorname*{colim}_M R, R] \simeq [R \wedge \Sigma^\infty M, R]
\end{aligned}$$

The second claim follows the exact same procedure, where we use that $\alpha : M \to \mathrm{Pic}(R)$ maps into the dualizable objects, whose dual is obtained by mapping into R. $\square$

Remark 4.1.17. All the above results generalize to the relative case, assuming all spaces involved are smooth manifolds. More precisely, let $f : M \to N$ be a proper surjective submersion of smooth manifolds. This gives a fibration of the underlying spaces, so that the homotopy fibers agree with the fibers of f, which are compact manifolds.

If α is a twist on M, then we can consider $f_!\alpha \in \mathrm{Fun}(N, R\mathrm{Mod})$: over each point $n \in N$, we have that $f_!\alpha$ is given by $R_{*+\alpha}(f^{-1}(n))$. The dual of $f_!\alpha$ is obtained by simply picking the dual over each point $n \in N$. The Thom construction allows us to identify the dual $f_!\alpha^\vee$ with the parametrized spectrum which over each $n \in N$ gives the twisted cohomology of the fiber $f^{-1}(n)$

$$R_{*-\alpha-\tau_f}(f^{-1}(n)).$$

where τ_f is the tangent bundle along the fiber.

Since all fiberwise tangent bundles can be joined in a single bundle over M, we can therefore naturally realize the fiberwise dual as

$$(f_!\alpha)^\vee \simeq f_!\big(-\alpha - \tau_f\big)$$

where $\tau_f = \ker(Tf)$ is the tangent bundle along the fibers of f.

The corollary shows that the α-twisted *cohomology* spectrum of a compact manifold M is equivalent to the $(-\tau-\alpha)$-twisted *homology* spectrum of M. If $R=\mathbb{C}$ is just the complex numbers (or any other ordinary ring), this produces a version of Poincaré duality in ordinary cohomology:

$$H^{-*}(M,\mathbb{C})\simeq H_{*-\tau}(M,\mathbb{C}).$$

Here the homology of the manifold M is twisted by the stable normal bundle $-\tau$: this includes both a shift in degree (by $\dim(M)$), and a further twist due to the fact that M might not be *orientable.* In case M is orientable, a choice of orientation provides a further isomorphism

$$H_{*-\tau}(M,\mathbb{C})\simeq H_{*-n}(M,\mathbb{C})$$

This generalizes to any generalized cohomology theory, as we discuss in the next section.

4.1.3 Orientations in generalized cohomology

In the simple case of ordinary cohomology, we have just seen that an orientation allows us to trivialize a certain twist. Abstractly, this is really just all an orientation is:

Definition 4.1.18. Let $X\in\infty\mathrm{Gpd}$, $R\in\mathrm{CRing}_\infty$ and let $\alpha : X\to BGL_1(R)$ be a twist over X, where we realize $BGL_1(R)$ as the automorphism group of the n-fold suspension $\Sigma^n R$. Then an orientation of the twist α is an homotopy between α and the functor that takes the constant value on the object $\Sigma^n R\in\mathrm{Pic}(R)$:

$$\begin{tikzcd} X \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\Sigma^n R"'] \arrow[r, phantom, "\Downarrow"] & BGL_1(R). \end{tikzcd}$$

It follows immediately from the definition that a choice of orientation gives rise to an equivalence of spectra

$$R_{*+n}(X)\xrightarrow{\sim}R_{*+\alpha}(X)$$

This isomorphism is called the Thom isomorphism. Traditionally, the Thom isomorphism refers to an equivalence in R-cohomology between M and the Thom space M^ξ for a vector bundle ξ carrying an orientation.

To relate this to the classical notion of an orientation on a vector bundle, we use the J-homomorphism: if $X\xrightarrow{\xi}BO$ classifies a stable vector bundle (of virtual rank 0) and $R\in\mathrm{CRing}_\infty$, we have the composite

$$X\xrightarrow{\xi}BO\xrightarrow{J}BGL_1(\mathbb{S})\to BGL_1(R).$$

An R-orientation is a choice of orientation for the R-twist described by the above diagram. Since we map into the automorphisms of the object R, an orientation will always be given by a homotopy to R (this means that virtual rank 0 bundles give no degree shifts in cohomology). More generally, an orientation on a stable vector bundle will be given by a homotopy to a suspension spectrum $\Sigma^n R$.

In lots of cases, there is a 'universal orientation' that tells us that an orientation on a vector bundle ξ is given by a reduction from its structure group to some group G.

Definition 4.1.19. Let $G\in\mathrm{Grp}_\infty$ come equipped with a group homomorphism $G\to O$. Then an R-orientation on G-bundles is a choice of homotopy in the diagram

$$\begin{tikzcd} BG \arrow[d] \arrow[rrd, bend left, "R"] & & \\ BO \arrow[r, "J"'] & BGL_1(\mathbb{S}) \arrow[r, "R\wedge(-)"'] \arrow[u, phantom, "\Leftarrow"] & BGL_1(R). \end{tikzcd}$$

Remark 4.1.20. For such an orientation to really be 'universal', we would also want the space of choices of homotopies in the above diagram to be contractible.

Having chosen such a homotopy, we obtain orientations on vector bundles by a reduction of the structure group: indeed, given a vector bundle $X \xrightarrow{\xi} BO$, a reduction of the structure group of ξ from O to G is a choice of lift and homotopy

$$\begin{tikzcd} & BG \arrow[d] \\ X \arrow[ur, dotted] \arrow[r, "\xi"'] & BO \end{tikzcd}$$

Then one obtains an orientation on ξ by gluing the diagrams

$$\begin{tikzcd} & BG \arrow[d] \arrow[rrr, bend left, "R"] & & & \\ X \arrow[ur, dotted] \arrow[r, "\xi"'] & BO \arrow[r, "J"'] & \mathrm{Pic}(\mathbb{S}) \arrow[r, "R\wedge(-)"'] & \mathrm{Pic}(R). \end{tikzcd}$$

Remark 4.1.21. The spectra discussed in section 3.1.4 have such 'universal' orientations. We will mention them in section 4.1.5, after we have discussed how to abstractly form pushforward maps.

We conclude with a discussion of orientations in the relative case. If we have a map $f\colon X \to Y$ and a twist $\alpha : X \to BGL_1(R)$, then we can give the following relative version of an orientation:

Definition 4.1.22. An orientation of the twist α over Y, or relative to Y, is a choice of $\gamma \in \mathrm{Fun}(Y, BGL_1(R))$, together with an equivalence of twists

$$\begin{tikzcd} X \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "f^*\gamma"'] \arrow[r, phantom, "\Downarrow"] & BGL_1(R). \end{tikzcd}$$

Remark 4.1.23. When $Y = *$ this produces definition 4.1.18 of an orientation. Moreover, if α carries an orientation, then it certainly carries an orientation over f, where the twist γ is simply the constant functor with value R, as in definition 4.1.18.

This expresses the idea that all fibers of $f\colon X \to Y$ carry an orientation, but there might still be a nontrivial twist along Y. As in the case where Y is a point, we obtain an equivalence

$$R^{*+\alpha}(X) \xrightarrow{\sim} R^{*+f^*\gamma}(X)$$

between the α-twisted R-cohomology of X and the cohomology of X twisted by the map $f^*\gamma$. With these notions of orientation and duality, we can construct pushforward maps in twisted generalized cohomology.

4.1.4 Pushforward maps in generalized cohomology

Following [ABG11], we use the discussion from the previous sections to describe pushforward maps in generalized cohomology. Suppose that we have a diagram of the form

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f^*\beta"'] & \arrow[d, phantom, "\Leftarrow \xi"] & Y \arrow[dl, "\beta"] \\ & BGL_1(R) & \end{tikzcd}$$

and recall that f gives rise to functors

$$\mathrm{Fun}(X, R\mathrm{Mod}) \underset{f_*}{\overset{f_!}{\rightleftarrows}} \mathrm{Fun}(Y, R\mathrm{Mod}) \underset{p_*}{\overset{p_!}{\rightleftarrows}} R\mathrm{Mod}$$

with $p : Y \to *$ the projection.

Definition 4.1.24. Let $\gamma\colon Y \to BGL_1(R) \to \mathrm{Pic}(R)$ be a twist over Y with the property that $f_! f^*\gamma \in \mathrm{Fun}(Y, R\mathrm{Mod})$ is dualizable, together with an equivalence in $\mathrm{Fun}(Y, R\mathrm{Mod})$

$$f_! f^*\beta \xrightarrow{\sim} (f_! f^*\gamma)^\vee.$$

Then the pushforward map, or Umkehr map, associated to this equivalence is given by the composite

$$\begin{tikzcd} R^{*+f^*\beta}(X) \arrow[r, equal] & \left[p_! f_! f^*\beta, R\right] \arrow[d, "\sim"] & \\ & \left[p_!(f_! f^*\gamma)^\vee, R\right] \arrow[r] & \left[p_!(\gamma^\vee), R\right] = R^{*-\gamma}(Y) \end{tikzcd}$$

We will denote it by $f^!\colon R^{*+f^*\beta}(X) \to R^{*-\gamma}(X)$.

Remark 4.1.25. Observe that the twist γ in the above definition is unique up to homotopy, if it exists. The map $f_! f^*\beta \xrightarrow{\sim} (f_! f^*\gamma)^\vee$ is not unique and the pushforward map depends on it.

Remark 4.1.26. In the smooth setting, we expect that one can give a similar definition of the pushforward map. Instead of working with the category $\mathrm{Fun}(Y, R\mathrm{Mod})$, one might use the slice category $\mathbf{R}\mathrm{Mod}_{/\Sigma^\infty_+ Y\wedge \mathbf{R}}$ to describe bundles of $\mathbf{R}$-modules. Since the ∞-category of such modules admits pullbacks, a map $f\colon X \to Y$ gives rise to the adjunction

$$\begin{tikzcd} \mathbf{R}\mathrm{Mod}_{/\Sigma^\infty_+ X\wedge \mathbf{R}} \arrow[r, shift left, "f_!"] & \mathbf{R}\mathrm{Mod}_{/\Sigma^\infty_+ Y\wedge \mathbf{R}} \arrow[l, shift left, "f^*"] \end{tikzcd}$$

required in the above definition. Furthermore, one can endow $\mathbf{R}\mathrm{Mod}_{/\Sigma^\infty_+ Y\wedge \mathbf{R}}$ with a monoidal structure as follows: given two $\mathbf{R}$-modules E, F over $\Sigma^\infty_+ Y \wedge \mathbf{R}$, we can take their smash product which comes equipped with a natural map

$$E \wedge_{\mathbf{R}} F \to \Sigma^\infty_+ (Y \times Y) \wedge \mathbf{R}$$

Pulling this back along the diagonal inclusion $\Delta : Y \to Y \times Y$ provides the fiberwise smash product of E and F over Y. This should provide all the ingredients for the above definition to also make sense in the smooth setting.

We now describe two ways of obtaining such an Umkehr map, in the case where X and Y are smooth manifolds. In the case where $f\colon M \to N$ is a proper surjective submersion of smooth manifolds, we have the following description of the pushforward map:

Proposition 4.1.27. *Let $f\colon M \to N$ be a proper surjective submersion of smooth manifolds and let $\tau_f : M \to \mathbb{Z} \times BO$ classify the tangent bundle along the fibers of f.*

Then a choice of orientation of $\tau_f \simeq f^*\gamma$ over the space N gives rise to a pushforward map in twisted cohomology

$$R^{*+f^*\beta}(M) \longrightarrow R^{*+\beta+\gamma}(N)$$

for any choice of twist β.

We thus see that an orientation of the fiberwise tangent bundle τ_f, relative to the space N, gives rise to a pushforward map in twisted R-cohomology.

Proof. An orientation on the bundle $\tau_f\colon M \to \{d\} \times BO$, relative to the space N, was given by a twist $\gamma : N \to \{\Sigma^d R\} \times BGL_1(R) \to \mathrm{Pic}(R)$, together with an equivalence $f^*\gamma \xrightarrow{\sim} \tau_f$, where we use τ_f to denote the associated twist

$$\tau_f\colon M \to \{d\} \times BO \xrightarrow{J} \{\Sigma^d R\} \times BGL_1(R).$$

Since f is a proper surjective submersion, all its homotopy fibers are compact manifolds. We can then apply the fiberwise Thom construction, together with the equivalence $f^*\gamma \xrightarrow{\sim} \tau_f$ to obtain a canonical equivalence

$$f_! f^*\beta \xrightarrow{\sim} f_!\big(-f^*\beta - \tau_f\big)^\vee \xrightarrow{\sim} f_!\big(-f^*\beta - f^*\gamma\big)^\vee.$$

The associated pushforward map then gives a map in twisted cohomology

$$f^! \colon R^{*+f^*\beta}(M) \longrightarrow R^{*+\beta+\gamma}(N).$$

$\square$

Remark 4.1.28. We have seen that an orientation of the fiberwise tangent bundle τ_f in R-cohomology always gives rise to an orientation of τ_f, relative to N. In particular, if we have a diagram

$$\begin{tikzcd} M \arrow[r, bend left, "\tau_f"] \arrow[r, bend right, "\Sigma^d R"'] \arrow[r, phantom, "\Downarrow"] & \{\Sigma^d R\} \times BGL_1(R). \end{tikzcd}$$

then we obtain a pushforward map

$$R^{*+f^*\beta}(M) \longrightarrow R^{*+\beta+d}(N)$$

for all choices of twists β. In this case, the map f is said to be R-oriented. In most cases of interest, we want to pick an R-orientation on τ_f so that the resulting pushforward map takes values in $R^{*+\beta}(N)$, up to a shift in degree.

Remark 4.1.29. On the other hand, we can obtain pushforward maps also when τ_f is only R-oriented relative to N. For example, observe that the tangent bundle τ_M of M can be written as the sum of the fiberwise tangent bundle τ_f and the pullback $f^*\tau_N$ of the tangent bundle to N. As a result, we have a natural equivalence of stable bundles

$$\tau_f \simeq \tau_M - f^*\tau_N.$$

If the tangent bundle τ_M carries an R-orientation, then the above decomposition induces an orientation on τ_f relative to the space N. There is an induced pushforward map

$$R^{*+f^*\beta}(M) \to R^{*+\beta+\dim(M)-\tau_N}(N).$$

If M and N are both compact manifolds and we have any smooth map $M \to N$, then we can easily construct pushforward maps:

Proposition 4.1.30. *Let $f \colon M \to N$ be a smooth map of compact manifolds. Then there is a natural pushforward map*

$$R^{*+f^*\beta+f^*\tau_N-\tau_M}(M) \to R^{*+\beta}(N).$$

In particular, if f is an R-oriented map, then we obtain a pushforward map

$$R^{*+f^*\beta}(M) \to R^{*+\beta}(N).$$

Proof. Since M and N are compact, the Thom construction gives us an equivalence

$$R^{*+f^*\beta+f^*\tau_N-\tau_M}(N) \simeq R_{*-f^*\beta-f^*\tau_N}(M)$$

Applying the natural map $R_{*-f^*\beta-f^*\tau_N}(M) \to R_{*-\beta-\tau_N}(N)$ in homology and dualizing once more, we obtain the desired map. $\square$

With this notion of pushforward map, we can describe our proposal for pull-push quantization in generalized cohomology. Recall from section 3.1.3 how we could linearize our prequantum field theory by passing to correspondence diagrams in the category of R-modules. For example, a codimension 1 boundary theory gave rise to a diagram

$$R \xrightarrow{\xi} R^{*+f^*\beta}(\mathrm{Fields}^\partial) \xleftarrow{f^*} R^{*+\beta}(\mathrm{Fields}).$$

of twisted R-cohomology spectra. Now if the space of boundary fields Fields^∂ is orientable relative to Fields, then we can form the pushforward along the map f. The result is a map

$$R \xrightarrow{f_!\xi} R^{*+\gamma}(\mathrm{Fields})$$

which gives a cocycle in the twisted R-cohomology of the ∞-groupoid Fields, with respect to some twist γ (which is unique up to homotopy). This twisted cohomology class describes the partition function of the boundary theory.

4.1.5 Examples of pushforwards

Here we provide some examples of orientations for the generalized cohomology theories we considered before. The simplest example is of course the case where R is an ordinary ring. This gives the classical definition of an orientation.

Example 4.1.31. Let R be an ordinary ring and let $SO = \mathrm{colim}_n\, SO(n)$. Both R and its group of units $GL_1(R)$ are discrete, i.e. just sets. The group homomorphism

$$SO \to O \to GL_1(\mathbb{S}) \to GL_1(R)$$

maps SO into the connected component of the identity in $GL_1(R)$. Since R is discrete, there is then a unique homotopy (up to a contractible space of choices) between the delooped map $BSO \to BGL_1(R)$ and the trivial map.

We see that a choice of orientation (in the classical sense) on a vector bundle gives rise to an abstract R-orientation. Also observe that for $R = \mathbb{Z}/2\mathbb{Z}$, we have that

$$GL_1(\mathbb{Z}/2\mathbb{Z}) \simeq 1$$

is just the one-element ring. In that case, there is also a universal orientation on BO itself. This produces the familiar fact that any vector bundle is $\mathbb{Z}/2\mathbb{Z}$-oriented.

If M is an oriented manifold, then a choice of orientation on the tangent bundle τ_M determines an equivalence

$$\mathrm{H}R^*(M) \simeq \mathrm{H}R_{*+n}(M)$$

of modules over the Eilenberg-Maclane spectrum HR. Passing to homology and cohomology, this is of course precisely the classical Poincaré duality theorem

$$H^k(M, R) \simeq H_{n-k}(M, R).$$

Finally, if we have a proper fiber bundle $f\colon X \to Y$ with d-dimensional fibers, then an oriention (in the classical sense) on the fibers determines a pushforward map. In the case where $R = \mathbb{R}$, this can be described in terms of de Rham cohomology by fiber integration:

$$H^k(X, \mathbb{R}) \mapsto H^{k-d}(Y, \mathbb{R}); \quad \omega \mapsto \int_{f^{-1}(y)} \omega.$$

Example 4.1.32. Let $G = \mathrm{Spin}^c$ be the spin^c-group, obtained as the $U(1)$ central extension of BO classified by the third integral Stieffel-Whitney class $W_3 : BO \to B^2U(1) \simeq B^3\mathbb{Z}$. Then there is a universal KU-orientation on $B\mathrm{Spin}^c$. This can be shown by abstract homotopy theoretic means (see [Hen08]). At the level of cohomology groups, one can use index theory to provide the KU-orientation of Spin^c. We will come back to this in the next section.

Example 4.1.33. Let String be the 2-group obtained as the $BU(1)$-central extension of Spin classified by the fractional Pontryagin class $\frac{1}{2}p_1 \colon B\mathrm{Spin} \to B^3U(1)$ (recall that any spin-bundle has a first Pontryagin class which is divisible by 2). The string group is the 4-connected cover in the Whitehead tower of the othogonal group O.

Then BString carries a famous orientation for tmf, constructed in [AHR06]. An overview is given in [Hen08]. If M is a compact string manifold, then the pushforward induced by the string structure on M

$$\mathrm{tmf}^*(M) \to \mathrm{tmf}$$

is called (the refinement to tmf of) the Witten genus. More generally, given any smooth map between two compact string manifolds $M \to N$, then by proposition 4.1.30 there is a natural pushforward map in tmf

$$\mathrm{tmf}^*(M) \to \mathrm{tmf}^*(N).$$

The tmf-orientation of BString only has a construction in homotopy theory and there is no known geometric construction of the associated pushforwards maps. It arises in this abstract form in the quantization of a string sitting at the boundary of a brane, as we discuss in section 5.

Remark 4.1.34. Using this string-orientation for tmf, we can easily construct the twist of tmf by $B^3U(1)$ mentioned in 3.1.4, following the discussion in [ABG10]. Indeed, the string orientation on tmf determines a homotopy commutative diagram of spectra

$$\begin{tikzcd} B\mathrm{String} \arrow[r] \arrow[d] & * \arrow[d] \\ B\mathrm{Spin} \arrow[r] & \mathrm{tmf} \end{tikzcd}$$

Now we can take the homotopy cofiber of the left and right vertical morphism. The cofiber of the left map is precisely $B^3U(1)$ (since BString is obtained as the homotopy fiber of BSpin over $B^3U(1)$), while the right homotopy cofiber is just tmf. We therefore find a homotopy commutative diagram

$$\begin{tikzcd} B\mathrm{Spin} \arrow[r] \arrow[d, "\frac{1}{2}p_1"'] & \mathrm{tmf} \arrow[d, "\mathrm{id}"] \\ B^3U(1) \arrow[r] & \mathrm{tmf} \end{tikzcd}$$

The bottom map precisely determines the twist of tmf by $B^3U(1)$ we mentioned in proposition 3.1.4.

The pushforward map in (complex) ordinary cohomology is very concrete: it can be realized by integrating differential forms. On the other hand, the pushforward maps in tmf are far from concrete, and arise by purely homotopical constructions. In the next section, we will give an overview of the pushforward constructions in (twisted) K-theory. These have traditionally been described in terms of index theory, which neatly fits in the picture of KK-theory.

4.2 Pushforward maps in twisted K-theory

In the previous section we have seen how one can use duality and a choice of orientation to produce a pushforward map in generalized cohomology. In this section, we shortly summarize how these abstract constructions come about in K-theory: in this case, one can often use families of elliptic operators to present pushforward maps in K-theory. We will give a short overview of the pushforward constructions in terms of differential operators in section 4.2.1. These constructions are naturally captured in KK-theory, where they fit in the picture of Poincaré dual algebras, which gives the analogue of the Spanier-Whitehead duality from the previous section.

4.2.1 Pushforward maps between manifolds

In this section we recall the main results on pushforward maps in twisted K-theory of smooth manifolds. An extensive account can be found in [CW08].

Thom construction

In the previous section, we have seen that a key tool in the construction of pushforward maps is given by the Thom construction. Here we give an account of the Thom construction in the Fredholm picture of K-theory. Moreover, we encorporate the twists in K-theory that arise from circle 2-bundles. The main result is then

Proposition 4.2.1 ([CW08]). *Let M be a smooth manifold carrying a circle 2-bundle $\alpha : M \to \mathbf{B}^2U(1)$ and let ξ be an oriented real vector bundle $p\colon V \to M$, of rank d. Then there are natural isomorphisms of K-theory groups*

$$K_c^{*+\alpha+\xi}(M) \simeq K_c^{*+p^*\alpha}(V).$$

Here the twist in K-theory induced by ξ is really via its third integral Stieffel-Whitney class $W_3(\xi)$, together with its rank d.

Remark 4.2.2. Note that we work with the compactly supported K-theory groups. If M is a compact manifold, then the left hand side agrees with the twisted K-theory of M. On the other hand, the right hand side then agrees with the reduced K-theory of the one-point compactification of V, which is precisely the Thom space $\mathrm{Th}(V)$.

The construction of this isomorphism is roughly as follows: the bundle $V \to M$ gives rise to its Clifford bundle $Cl(V)$. For $n \geq 0$, a class in $K_c^{n+\alpha+\xi}(V)$ can be presented by a section of the Fredholm bundle

$$P^\alpha \times_{PU(H_0)} \mathrm{Fred}'\big(Cl(V) \otimes Cl(n) \otimes H_0\big) \to M$$

twisted by the $PU(H_0)$-bundle P^α associated to the circle 2-bundle α. Here H_0 is an infinite dimensional separable Hilbert space. The bundle Fred' consists of Fredholm operators that commute with the pointwise Clifford action on $CL(V) \otimes CL(n)$. Note that the vector bundle ξ twists the Fredholm operators via the Clifford bundle $Cl(V)$.

Now let D be a section of this bundle, presenting a class in $K_c^{n+\alpha+\xi}$. Then D pulls back to a section p^*D of the bundle

$$p^*P^\alpha \times_{PU(H_0)} \mathrm{Fred}'\big(Cl(p^*V) \otimes Cl(n) \otimes H_0\big)$$

Now for each element $v \in V$ let $\gamma(v) \in Cl(V)$ be its image in the Clifford bundle. Then we can consider the Fredholm operator

$$\tilde{D}(v) = D(p(v)) + \gamma(v)$$

The action of this operator only commutes with the Clifford action on the copy $Cl(n)$ and becomes unitary outside of a compact in V. Consequently, it gives rise to a class in $K_c^{n+p^*\alpha}(V)$. This construction gives rise to a map

$$K_c^{*+\alpha+\xi}(M) \to K_c^{*+p^*\alpha}(V)$$

which turns out to be an isomorphism (see [CW08] for more details).

With the above incarnation of the Thom construction, we can construct the pushforward maps in complex K-theory associated to a map of smooth manifolds. We will show that a smooth map $f\colon M \to N$ of manifolds gives rise to a pushforward map in twisted K-theory

$$K_c^{*+f^*\beta+f^*\tau_N-\tau_X}(M) \to K_c^{*+\beta}(N).$$

where $f^*\tau_N - \tau_M$ is the twist by the stable normal bundle of M in N (via its Stieffel-Whitney class and its virtual rank). The idea is to factor this map into an embedding, followed by a fiber bundle over N, all whose fibers carry a spin^c-structure. We give an explicit construction of the pushforward maps for each of these classes of maps, and then compose the two to obtain a pushforward map in twisted K-theory.

Pushforward along embeddings

Let $i\colon M \to N$ be a codimension d embedding of M into N and let $\beta : N \to \mathbf{B}^2U(1)$ classify a circle 2-bundle over N. Let ν be the normal bundle to M in N. We then construct a pushforward map in compactly supported K-theory

$$i^! \colon K_c^{*+i^*\beta+\nu}(M) \to K_c^{*+\beta}(N).$$

Let V be the total space of the normal bundle. It can be identified with a tubular neighbourhood of M in N, so that there is an open inclusion $j : V \to N$. Observe that j is homotopic to the composition

$$V \to M \xrightarrow{i} N.$$

The Thom isomorphism 4.2.1 then gives us isomorphisms

$$K_c^{*+i^*\beta+\nu}(M) \to K_c^{*+j^*\beta}(V)$$

Classes in the compactly supported twisted K-theory of V can be identified with sections of the bundle

$$\mathrm{Fred}^{(n+d)} \times_{PU(H)} P^\beta \to N$$

over the open subset V, so that they become the identity outside of a compact subset of V. But those sections can just be extended to the whole of N, by taking them to be the identity map outside of V. This provides a map

$$K_c^{*+j^*\beta+d}(V) \to K_c^{*+\beta}(N)$$

The composite is the desired pushforward map $i^!$ on K-theory groups.

Pushforward along fiber bundles

Let $f\colon M \to N$ be a fiber bundle whose fiber F carries a spin^c-structure. We then construct a pushforward map of twisted K-theory groups

$$K_c^{*+f^*\beta}(M) \to K_c^{*+d+\beta}(N)$$

where d is the dimension of the fiber F. Let τ_f denote the tangent bundle to the fibers of $f$which we assume to carry a spin^c-structure depending smoothly on the fiber. Let S be the spinor bundle associated to this spin^c-structure on τ_f. We can endow S with a connection $\nabla : \Gamma(M,S) \to \Gamma(S \otimes \tau_f^*)$ taking values in the cotangent spaces to the fibers of the fibration f. For each point $y \in N$, we then obtain a Dirac operator

$$D_y : \Gamma(f^{-1}(y), S) \to \Gamma(f^{-1}(y), S)$$

by composing the connection with the Clifford multiplication. The resulting Dirac operator depends smoothly on y, in the sense that the Dirac operators D_y together form a single operator $D : \Gamma(M,S) \to \Gamma(M,S)$.

Since the Dirac operators D_y depend smoothly on y, the regularized Dirac operators

$$\tilde{D}_y := \frac{D_y}{\sqrt{1 + D_y^* D_y}} : L^2(f^{-1}(y), S) \to L^2(f^{-1}(y), S)$$

will depend continuously on y (see [CW08]). Since each Hilbert space $L^2(f^{-1}(y), S) \simeq L^2(F,S)$ is equivalent to the Hilbert space $\ell^2 \otimes S_d$, we obtain a continuous map

$$\tilde{D}\colon N \to \mathrm{Fred}^{(d)}$$

which assigns to each point $y \in N$ the regularized Dirac operator $\tilde{D}_y$.

Finally, we introduce the twist β on the base space N: let $\{U_i\}$ be a good open cover of the manifold N. Then the twist β is given by a $PU(H)$-bundle $P^\beta \to N$, with transition functions $g_{ij} : U_{ij} \to PU(H)$.

Over each U_i, the fiber bundle $M \to N$ trivializes. The pullback bundle $f^*P^\beta \to M$ is then trivial on each of the opens $U_i \times F \subseteq M$, with transition functions given by $\tilde{g}_{ij}(y,v) = g_{ij}(y)$ for $(y,v) \in U_{ij} \times F$.

Now we show how to use the operator $\tilde{D}$ to push twisted K-theory classes on M to classes on N. Let

$$Q : M \to f^*P^\beta \times_{PU(H)} \mathrm{Fred}^{(n)}$$

present a class in $K^{n+f^*\beta}(M)$. This means that G is given by local functions

$$Q_i : U_i \times F \to \mathrm{Fred}^{(n)}$$

such that $Q_j = \tilde{g}_{ij} Q_i \tilde{g}_{ij}^{-1}$ on intersections $U_{ij} \times F$. For a fixed point $y \in U_i$, we have that the family of Fredholm operators $G_i(y,-) : F \to \mathrm{Fred}^{(n)}$ determines a single Fredholm operator on $L^2(F) \otimes H$.

Fixing an isomorphism $L^2(F) \otimes H \simeq H$, these Fredholm operators then give rise to a map

$$\tilde{Q}_i \colon U_i \to \mathrm{Fred}^{(n)}.$$

Finally, we can couple this family of Fredholm operators to the Dirac operator defined before to obtain a family of operators

$$T_i = \tilde{D} \otimes 1 + 1 \otimes \tilde{Q}_i \colon U_i \to \mathrm{Fred}^{(d+n)}$$

One can show that these families of Fredholm operators satisfy the condition $T_j = g_{ij} T_i g_{ij}^{-1}$ on intersection U_{ij}. This means that the T_i define a class in the twisted K-theory group $K_c^{n+d+\beta}(N)$. This is the image of $Q \in K^{n+f^*\beta}(M)$ under the pushforward map

$$f^! \colon K_c^{*+f^*\beta}(M) \to K_c^{*+d+\beta}(N).$$

as discussed in [CW08].

Remark 4.2.3. This discussion of the pushforward map in K-theory is a bit technical, mainly due to the fact that we encorporate twists. We will give an easier description in the case where there are no twists and the manifold M is compact. In that case, suppose we have a class in $K_c^0(M)$ given by a vector bundle $V \to M$. The pushforward of this class in $K_c^{*+d}(N)$ is obtained by picking a connection $\nabla : \Gamma(V \otimes S) \to \Gamma(V \otimes S \otimes \tau_f^*)$ on the bundle V coupled to the spinor bundle. Then one can consider the family of Dirac operators coupled to V

$$D(V)_y \colon \Gamma(f^{-1}(y), V \otimes S) \xrightarrow{\nabla} \Gamma(f^{-1}(y), V \otimes S \otimes \tau_f^*) \xrightarrow{\gamma} \Gamma(f^{-1}(y), V \otimes S)$$

by composing with the Clifford multiplication. This provides the N-family of Fredholm operators presenting $f^!(V) \in K^d(N)$.

In the case where d is even, we can even give a presentation of this K-theory class in terms of virtual vector bundles. We have that the spinor bundle decomposes as a sum of an even and an odd part $S = S^+ \oplus S^-$, while the Dirac operator $D(V)_y$ gives a map

$$D(V)_y \colon \Gamma(f^{-1}(y), V \otimes S^+) \longrightarrow \Gamma(f^{-1}(y), V \otimes S^-)$$

Taking the index of this operator gives a virtual vector space over each point $y \in N$. In good cases, these form a virtual vector bundle over N, which describes the class $f^!(V) \in K^d(N)$. In general, this bundle is rather singular, but by adding trivial bundles to both the kernel and cokernels of the $D(V)_y$ we can make this into a true virtual vector bundle.

General case

Finally, suppose we have a general map $f : M \to N$ between smooth manifolds. Then we can factor f as an embedding followed by a fiber bundle: we simply embed the manifold M in $\mathbb{R}^N$ for N large enough and taking the composite

$$M \xrightarrow{(i,f)} \mathbb{R}^N \times N \xrightarrow{p_2} N$$

The normal bundle of the first embedding is stably isomorphic to the sum

$$f^*\tau_N + \mathbb{R}^N - \tau_M$$

The first bundle is the tangent bundle to N (pulled back to M), while $\mathbb{R}^N - \tau_M$ describes the normal bundle of M in $\mathbb{R}^N$ (note that it is indeed of virtual rank $N - \dim(M)$, so this makes sense). Consequently, we find maps in K-theory

$$K_c^{*+f^*\beta+f^*\tau_N-\tau_M}(M) \to K_c^{*+p_2^*\beta-N}(\mathbb{R}^N \times N) \to K_c^{*+\beta}(N).$$

The following result due to [CW08] tells us that the resulting pushforward map is independent of the chosen factorization into an embedding and a fibration:

Proposition 4.2.4. *The resulting pushforward map*

$$f^! : K_c^{*+f^*\beta-\tau_f}(M) \to K_c^{*+\beta}(N).$$

is independent of the chosen factorization into an embedding followed by a fibration. In case M and N are both compact, this presents the pushforward in K-theory as defined abstractly in section 4.1.4.

Moreover, the pushforward map depends functorially on f:

Proposition 4.2.5 ([CW08], section 4.3)**.** *If $f\colon M_1 \to M_2$ and $g\colon M_2 \to M_3$ are smooth maps of manifolds, then*

$$(gf)^! = g^! \circ f^!.$$

Remark 4.2.6. The functoriality of the pushforward map does not use any orientation. In particular, note that the domain of $g^! \circ f^!$ is twisted by $-f^*\tau_g - \tau_f = -\tau_{gf}$. Now if f and g carry an orientation, then we obtain a composite

$$g^! f^! \colon K^{*+f^*g^*\beta} \to K^{*+g^*\beta} \to K^{*+\beta}$$

This is precisely the pushforward along gf if we choose the orientation on gf to be the one induced from the orientations on g and f.

This gives us explicit tools for computing pushforward maps, especially in the situation where the map f is a fiber bundle whose fibers carry a spin^c-structure: in that case the pushforward map is constructed from the family of Dirac operators associated to the spinor bundle along the fiber. All these constructions have an immediate extension to the setting of local quotient stacks, as long as the maps between them are representable (which means that the fibers of the map are manifolds). We will give a very rough sketch of this in the next section, focussing mainly on the case where the local quotient stacks are actual quotient stacks. In that case, we obtain a pushforward map in *equivariant* (twisted) K-theory.

4.2.2 Pushforward maps between local quotient stacks

Following the discussion in [FHT11], we can extend the above pushforward maps in K-theory to the setting of local action stacks, at least as long as the map between the two stacks is presentable.

Definition 4.2.7. A morphism $X \xrightarrow{f} Y$ of stacks is presentable if $X \times_Y M$ is a smooth manifold for any M a smooth manifold with a map $M \to Y$.

Example 4.2.8. If M is a manifold and Y a differentiable stack, then the map $M \to Y$ is always presentable (essentially by the definition of a differentiable stack). On the other hand, a map whose domain is a stack is almost never representable. For example, the terminal map $\mathbf{B}G \to *$ is clearly not representable.

Example 4.2.9. Let M, N be manifolds carrying an action by a compact group G and let $M//G$ and $N//G$ be the respective quotient stacks. Then a G-equivariant map $f\colon M \to N$ determines a representable map of stacks $M//G \to N//G$. Indeed, to check this it suffices to check that the homotopy pullback of this map along the atlas $N \to N//G$ is presentable by a manifold. Indeed, we have a homotopy cartesian square

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \arrow[d] \\ M//G \arrow[r] & N//G. \end{tikzcd}$$

This really serves as the basic example in this section.

The main idea in the construction of the pushforward along a representable map of stacks $f\colon X \to Y$ is that we can effectively work with the map $X \times_Y M \to M$ instead of the map f. Indeed, suppose X and Y are local quotient stacks and let $Y_0 \to Y$ be an atlas for Y. Then the pullback

$$X_0 := f^* Y_0 \to X$$

forms an atlas for the stack X (this follows from the fact that effective epimorphisms are preserved by pullbacks). The map $f_0\colon X_0 \to Y_0$ now forms the map on objects of a smooth functor of Lie groupoids from the groupoid associated to $X_0 \to X$ to the groupoid associated to $Y_0 \to Y$.

The fact that f_0 is the map on objects of a smooth functor immediately implies that the stable normal bundle $-\tau_f := f^*\tau_{Y_0} - \tau_{X_0}$ determines an bundle on X_0 which is equivariant under the action of the space of arrows X_1; in other words, it gives rise to a map $-\tau_f\colon X \to \mathbf{B}O$. Since this bundle is equivariant, is defines a twist for the K-theory of the local quotient stack X.

We can now apply the same constructions as before: a class $Q \in K_c^{n+\alpha-\tau_f}(X)$ is presented by an equivariant section of an equivariant bundle of Fredholm operators. The pushforward over the map $f_0\colon X_0 \to Y_0$ produces a section of the equivariant section of Fredholm operators on Y_0. Since Q is equivariant, its pushforward will be equivariant as well (see [FHT11] for more details).

We conclude this section with an application of this to the above example where we have a G-equivariant map of manifolds. When the map is a fibration, the pushforward can be obtained by taking a G-equivariant Dirac operator.

Example 4.2.10. Let $f\colon M \to N$ be a G-equivariant fibration of compact smooth manifolds, whose fibers have dimension d. Suppose the fibers of f come equipped with a G-equivariant spin^c-structure. Associated to this spin^c-structure is a G-equivariant spinor bundle S over M. Then there is a straightforward description of the map $K^0(M//G) \to K^d(N//G)$: a class in $K^0(M//G)$ can be presented by a G-equivariant vector bundle $V \to M$ (see 3.2.25). Now we endow this G-equivariant bundle with a G-equivariant connection along the fibers of f, which induces a G-equivariant family of Dirac operators

$$D_y\colon \Gamma(f^{-1}(y), V\otimes S) \xrightarrow{\nabla} \Gamma(f^{-1}(y), V\otimes S\otimes \tau_f) \xrightarrow{\gamma} \Gamma(f^{-1}(y), V\otimes S)$$

In other words, pulling back a section $s\colon f^{-1}(y) \to V\otimes S$ along the action by an element g gives a section $L_g^* s\colon f^{-1}(g^{-1}y) \to V\otimes S$ and the Dirac operator satisfies

$$L_g^*\big(D_y s\big) = D_{g^{-1}y}(L_g^* s).$$

This equivariant Dirac operator presents a class in $K^d(N//G)$. Whenever d is even, we can obtain a G-equivariant virtual vector bundle over N by taking the kernel and cokernels of the Dirac operator, just like we did in remark 4.2.3.

In particular, if M is a compact even-dimensional manifold with a G-action, carrying a G-equivariant spin^c-structure, then there is a pushforward map

$$K^0(M//G) \to R(G)$$

This simply sends a vector bundle to the index of the above Dirac operator. We will use this pushforward map in the quantization of a Hamiltonian G-space, in section 5.2.1.

These constructions of pushforward maps are nicely captured in the algebraic framework of KK-theory. This will be the content of the next section.

4.2.3 Pushforward maps in operator algebraic K-theory

In the previous section we have seen how we can present pushforward maps in complex K-theory in terms of Fredholm operators. In particular, we saw how for a fiber bundle with spin^c-fibers we can use the spin^c Dirac operator to construct a pushforward map in twisted K-theory. This fits in a more coherent story if we pass to the operator algebraic description of K-theory, as we discussed in section 3.3.3. The category KK which described bivariant K-theory is constructed precisely in such a way that maps are given by such families of Fredholm operators (like Dirac operators).

In this section we will give an overview of how the above constructions of pushforward maps come about in terms of KK-theory. We will mainly focus on the construction of the pushforward along a fibration in terms of KK-theory. In the next section, we discuss the notion of duality in the category KK. This is the analogue of Spanier-Whitehead duality of spectra, and therefore puts the construction of pushforward maps in a more general context.

Thom isomorphism

The above version of the Thom isomorphism can be formulated in KK-theory as

Proposition 4.2.11. *Let ξ be an even dimensional vector bundle $p : V \to M$ over a smooth manifold M and let $\alpha : M \to \mathbf{B}^2U(1)$ be a smooth circle 2-bundle. Then there is an isomorphism in* KK

$$C^*_{\alpha+\xi}(M) \simeq C^*_{p^*\alpha}(V).$$

of twisted convolution algebras (see 3.3.34).

Remark 4.2.12. Throughout, we only consider twists by even dimensional vector bundles. This has to do with the fact that maps in KK give rise to maps between the even K-theory groups and between the odd K-theory groups; there is never a shift by odd degree. One can extend KK by adding odd parts to all sets of morphisms. In that case the odd morphisms will describe maps that change the degree in K-groups by 1.

One can construct the Kasparov bimodule witnessing this equivalence in a similar way as one constructs the Thom isomorphism on K-theory groups. A discussion can be found in [BMRS08].

This version of the Thom isomorphism is slightly more refined than its counterpart 4.2.1: indeed, under the functor

$$\mathrm{KK} \to \mathrm{ho}\big(KU\mathrm{Mod}\big)$$

we see that the above KK-isomorphism induces an equivalence of K-theory spectra, not just of K-theory groups.

Of course, using the above form of the Thom isomorphism we can easily describe the pushforward map along an embedding. Let $i : M \to N$ be an embedding of even codimension, whose normal bundle ν is given by $p : V \to M$. We can identify V with an open subset of N via the tubular neighbourhood theorem, with inclusion $j : V \to N$. Note that we have a homotopy $j \simeq i \circ p$, since V contracts onto the submanifold M. We find a map in KK-theory

$$i^! \colon C^*_{i^*\alpha+\nu}(M) \simeq C^*_{p^*i^*\alpha}(V) \simeq C^*_{j^*\alpha}(V) \to C^*_{\alpha}(N)$$

which describes the pushforward map along the inclusion i. Here the first isomorphism in KK is the Thom isomorphism and the second follows from the fact that ip and j are homotopic. The last map is given by 'extension by zero': all functions in $C^*_{j^*\alpha}(V)$ have to vanish at infinity, to they can be extended by zero to obtain functions in $C^*_{\alpha}(N)$.

Passing to K-theory spectra, we then obtain a pushforward map

$$i^! \colon K_c^{*+i^*\alpha+\nu}(M) \to K^{*+\alpha}(N)$$

which refines the pushforward map of K-theory groups that we constructed in the previous section.

Pushforward along fiber bundles

Let $f \colon M \to N$ be a smooth fiber bundle, with even dimensional compact fiber F carrying a spin^c-structure. We have already seen that M the carries a spinor bundle $S = S^+ \oplus S^-$, together with a fiberwise regularized Dirac operator

$$D_y \colon L^2(f^{-1}(y), S^+) \to L^2(f^{-1}(y), S^-)$$

depending continuously on the point $y \in N$. Moreover, we obtained the regularized Dira

We now interpret this in terms of Kasparov bimodules. Let $\mathcal{C}_0(M)$ and $\mathcal{C}_0(N)$ be the C^*-algebras corresponding to the spaces M and N. The spaces $\Gamma(M, S^+)$ and $\Gamma(M, S^-)$ are bimodules over $\mathcal{C}_0(M)$ and $\mathcal{C}_0(N)$. Moreover, we can give the spinor bundle S a metric, so that both spaces $\Gamma(M, S^+)$ and $\Gamma(M, S^-)$ come equipped with an inner product with value in $\mathcal{C}_0(N)$, simply by integration along the fibers of the map f.

The completion with respect to the induced norm is given by the $\mathbb{Z}/2$-graded space

$$\mathcal{C}_0(N, L^2(f^{-1}(y), S^\pm))$$

of sections of $S^\pm$ over M that are L_2 along the fibers of f. This naturally forms a $\mathbb{Z}/2$-graded Hilbert bimodule from $\mathcal{C}_0(M)$ to $\mathcal{C}_0(N)$, whose even part consists of the sections of S^+ and whose odd part consists of the sections of S^-.

Finally, the regularized Dirac operator gives an odd operator

$$\tilde{D} \colon \mathcal{C}_0(N, L^2(f^{-1}(y), S^+)) \to \mathcal{C}_0(N, L^2(f^{-1}(y), S^-))$$

Its adjoint with respect to the chosen metrics on S^+ and S^- gives the map the other way around. One can now check that the odd operator $\tilde{D}$ satisfies all conditions required in definition 3.3.37. Most importantly, the fact that $\tilde{D}$ is a local operator implies that

$$[\tilde{D}, a] \in K\Big(\mathcal{C}_0(N, L^2(f^{-1}(y), S^\pm))\Big)$$

An extensive treatment can be found in [Hig90]. The resulting Kasparov bimodule specifies a map in $\mathrm{KK}\big(\mathcal{C}_0(M), \mathcal{C}_0(N)\big)$. Then the composition with this map gives rise to a map of K-theory groups

$$K_c^0(M) = K^0(\mathcal{C}_0(M)) = \mathrm{KK}\big(\mathbb{C}, \mathcal{C}_0(M)\big) \longrightarrow \mathrm{KK}\big(\mathbb{C}, \mathcal{C}_0(N)\big) = K_c^0(N)$$

which precisely gives the pushforward map in K-theory. Notice how the setting of KK-theory makes the description of the pushforward map much simpler. Once we have the Dirac operator $\tilde{D}$, we can immediately write down the Kasparov bimodule that witnesses the pushforward map. We do not have to couple the Dirac operator to any other Fredholm operator by hand, like we did in the previous section: the fact that KK forms a category already guarantees that the Dirac operator $\tilde{D}$ gives rise to a pushforward map. Moreover, we can now pass to K-theory spectra to obtain a pushforward map of KU-modules

$$K_c^*(M) \to K_c^*(N)$$

The advantage of passing to the KK-category becomes even more apparent in the case where N carries a circle 2-bundle $\alpha : N \to \mathbf{B}^2U(1)$. In that case, we have that $C^*_\alpha(N)$ can be presented as a completion of the algebra $\mathcal{C}_c^\infty(\coprod U_{ij})$, together with the convolution product as in example 3.3.35.

Similarly, we can construct $C^*_{f^*\alpha}(M)$ as a completion of the algebra $\mathcal{C}_c^\infty(\coprod U_{ij} \times F)$, with the convolution product given by

$$f_1 * f_2(x_{ik}, v) = \sum_j f_1(x_{jk}, v) f_2(x_{ij}, v) \alpha_{ijk}(x)$$

for $x_{ik} \in U_{ik}, v \in F$. Then we can consider the bimodule

$$\mathcal{C}_0(\coprod U_{ij}, L^2(F, S^{\pm}))$$

with right action from $C^*_\alpha(N)$ given by

$$(s \cdot f)(x_{ik}, v) = \sum_j s(x_{jk}, v) f(x_{ij})$$

for $s \in \mathcal{C}_0(\coprod U_{ij}, L^2(F, S^{\pm}))$ and $f \in C^*_\alpha(N)$. The left action of $C^*_{f^*\alpha}(M)$ is constructed similarly. The same Dirac operator as in the untwisted case turns this into a Kasparov bimodule. This bimodule then induces the pushforward in twisted K-theory

$$K^{0+f^*\alpha}(M) = \mathrm{KK}(\mathbb{C}, C^*_{f^*\alpha}(M)) \longrightarrow \mathrm{KK}(\mathbb{C}, C^*_\alpha(N)) = K^{0+\alpha}(N)$$

and more generally, a pushforward of K-theory spectra

$$K^{*+f^*\alpha}(M) \to K^{*+\alpha}(N).$$

This produces a much cleaner description of the pushforward map in twisted K-theory, since we can present $C^*_\alpha(N)$ by a convolution algebra where the twist α only appears in the multiplication; we do not have to use any Hilbert bundle machinery to couple the Dirac operator to a section of a bundle of Fredholm operators.

One can apply essentially the same construction in G-equivariant K-theory, by using the G-equivariant Dirac operator. More precisely, if M and N are manifolds carrying an action of a compact Lie group, then their C^*-algebras of functions also carry an action of this Lie group. A G-equivariant spinor bundle with a G-equivariant Dirac operator then gives rise to a morphism in G-equivariant KK-theory

$$KK^G\big(\mathcal{C}_0(M), \mathcal{C}_0(N)\big)$$

Essentially this is just an extension of KK-theory where everything carries a G-action (see [Kas88] for more details). There is then a map

$$\mu \colon KK^G\big(\mathcal{C}_0(M), \mathcal{C}_0(N)\big) \to KK\big(C^*(M/\!/G), C^*(N/\!/G)\big) \tag{4.6}$$

which is known as the Baum-Connes assembly map (cf. [Bla98] for a textbook account). It is a main problem in operator K-theory, known as the Baum-Connes conjecture, to determine when this map is an isomorphism. All that is relevant for us is that such a map exists, which means that G-equivariant elliptic operators give rise to morphisms between the function algebras of the quotient stacks by the group action.

In particular, a G-equivariant spin^c-structure along the fibers of $f \colon M \to N$ determines a G-equivariant Dirac operator and therefore a map $C^*(M/\!/G) \to C^*(M/\!/G)$ in KK-theory. In fact, if N carries a G-equivariant circle 2-bundle, given by a map $\beta \colon N/\!/G \to \mathbf{B}^2U(1)$, then the same construction as in the non-equivariant case gives a map in KK-theory

$$C^*_{f^*\beta}(M) \to C^*_\beta(N)$$

which is G-equivariant, where both C^*-algebras carry a G-action. Under the assembly map, this gives a map $C^*_{f^*\beta}(M/\!/G) \to C^*_\beta(N/\!/G)$, which in term gives rise to a map of KU-modules

$$f^! \colon K^{*+f^*\beta}(M/\!/G) \to K^{*+\beta}(N/\!/G)$$

This is the pushforward map from the twisted K-theory spectrum of the quotient stack $M/\!/G$ to the twisted K-theory spectrum of $N/\!/G$.

It was shown in [EM10] that the pushforward along K-oriented G-equivariant maps is also functorial:

Proposition 4.2.13. *The formation of the pushforward along a K-oriented G-equivariant map determines a functor into G-equivariant KK-theory from the category of manifolds carrying an action of a compact Lie group G, with K-oriented G-equivariant smooth maps between them.*

In any case, most of the constructions in K-theory give rise to maps in KK-theory, essentially by the construction of KK-theory in terms of bimodules carrying a Fredholm operator. Using the functor that sends a C^*-algebra to its K-theory spectrum, we can use this to obtain pushforward maps of K-theory spectra from differential operators, like the Dirac operator.

4.2.4 Poincaré duality in bivariant K-theory

We have seen that the category KK has a natural symmetric monoidal structure given by the minimal tensor product of C^*-algebras. This monoidal structure gives rise to a notion of duality, which is called Poincaré duality in the literature (but note that this is really about duality in K-theory, not in ordinary cohomology).

Definition 4.2.14. Let A be a C^*-algebra. Then a Poincaré dual of A is an algebra $A^\vee$ such that

$$KK(A \otimes B, C) \simeq KK(A, A^\vee \otimes C)$$

for all C^*-algebras B and C. Equivalently, there are evalutation and coevalutation maps

$$\mathrm{ev}\colon A \otimes A^\vee \to \mathbb{C} \qquad\qquad \mathrm{coev}\colon \mathbb{C} \to A \otimes A^\vee$$

satisfying the zigzag identities.

Remark 4.2.15. Since the functor $\mathrm{KK} \to \mathrm{ho}\big(KU\mathrm{Mod}\big)$ from 3.3.53 is lax monoidal, it preserves dual objects. It follows that the K-theory spectrum of the Poincaré dual of a C^*-algebra A is the spectrum $K_*(A)^\vee$ dual to the K-theory spectrum of A itself.

In general it is not that easy to construct Poincaré dual algebras. The best-known example arises from the case where $A = \mathcal{C}(M)$ is given by the continuous functions on a compact manifold M. We have seen in the previous two sections how the K-theoretic dual of M is presented by the K-theory spectrum of M, twisted by the stable normal bundle to M. This also happens at the level of C^*-algebras, as noted for instance in [BMRS08].

Proposition 4.2.16. *Let M be a compact even dimensional oriented manifold. Then the dual algebra to $C^*(M)$ is given by the twisted convolution algebra*

$$C^*(M)^\vee \simeq C^*_{-\tau}(M)$$

where $-\tau$ is the twist induced by the circle 2-bundle corresponding to $-W_3(\tau) \in H^3(M, \mathbb{Z})$.

Remark 4.2.17. In the case where M is not an oriented manifold, then the stable normal bundle does not only give a twist by $-W_3(\tau)$, but also by the first Stieffel-Whitney class $w_1(\tau)$.

Remark 4.2.18. The same result remains true if we endow M with a circle 2-bundle $\chi : M \to \mathbf{B}^2U(1)$. In that case we have that $C^*_\chi(M)^\vee \simeq C^*_{-\tau-\chi}(M)$.

The proposition follows from the following two observations:

Lemma 4.2.19. *There is an equivalence of C^*-algebras $C^*_{-\tau}(M) \simeq \mathcal{C}_0(M, Cl(T^*M))$.*

This follows from an analogous argument as the Thom isomorphism 4.2.11. See [Kas80] for an extensive discussion. The proposition then follows from the following lemma:

Lemma 4.2.20 ([Kas80])**.** *Let M be a compact oriented manifold. Then $C^*(M) = \mathcal{C}_0(M)$ has a dual algebra $\mathcal{C}_0(M, Cl(T^*M))$ where $Cl(T^*M)$ is the (complex) Clifford bundle associated to the cotangent bundle of M.*

Proof. Fix a metric on the manifold M. We first describe the Kasparov bimodule that give the evaluation map
$$\mathcal{C}_0(M) \otimes \mathcal{C}_0(M, Cl(T^*M)) \to \mathbb{C}.$$

Let H be the Hilbert space $L^2(M, \bigwedge^* T^*M \otimes \mathbb{C})$ of complex differential forms, where the even (resp. odd) part is given by the even (odd) degree differential forms. It comes equipped with the obvious $\mathcal{C}_0(M)$-action given by pointwise multiplication.

Given a point $\xi \in T_p^*M$, we define an action of ξ on $\bigwedge^* T_p^*M \otimes \mathbb{C}$ by

$$\gamma'(\xi)\eta = \xi \wedge \eta + \iota_\xi \eta$$

for all $\eta \in \bigwedge^* T_p^*M \otimes \mathbb{C}$, where we use the metric to identify ξ with an element in the tangent space T_pM. It is easily seen that this action of T_p^*M on $\bigwedge^* T_p^*M \otimes \mathbb{C}$ satisfies the Clifford identities, so we get a pointwise action of the Clifford algebra on $\bigwedge^* T_p^*M \otimes \mathbb{C}$.

Then the sections $\mathcal{C}_0(M, Cl(T^*M))$ acts on H by the above pointwise action. This action commutes with the $\mathcal{C}_0(M)$-action, and one can check that H is a bimodule from $\mathcal{C}_0(M) \otimes \mathcal{C}_0(M, Cl(T^*M))$ to $\mathbb{C}$. Finally, we equip H with the operator $D = d + d^*$, which is elliptic (hence Fredholm). The resulting Kasparov bimodule describes the evaluation map.

For the coevaluation, take the space $\mathcal{C}_0(T^*M, p^*Cl(T^*M))$ of sections of the pullback of the Clifford bundle to T^*M. To give it the structure of a right $\mathcal{C}_0(M) \otimes \mathcal{C}_0(M, Cl(T^*M))$-module, we realize T^*M as a tubular neighbourhood of the diagonal $\Delta(M) \subseteq M \times M$ (for example by using the exponential map associated to the chosen metric, together with a rescaling). Then $p^*Cl(T^*M)$ is just the restriction of the Clifford bundle $p_1^*Cl(T^*M) \to M \times M$ to the tubular neighbourhood T^*M.

We get the action of $\mathcal{C}_0(M, Cl(T^*M))$ by Clifford multiplication from the first coördinate, and the action from $\mathcal{C}_0(M)$ by pointwise multiplication from the second coördinate. Finally, the Fredholm operator $\tilde{D}$ on $\mathcal{C}_0(T^*M, p^*Cl(T^*M))$ sends a section s to

$$\tilde{D}s(\xi) = \gamma(\xi)s(\xi)$$

where we use the embedding $\gamma : T^*M \to Cl(T^*M)$. An explicit computation shows that these evaluation and coevaluation maps satisfy the zigzag identities (see e.g. [Kas88]). □

A choice of orientation is a choice of identification between A and its dual $A^\vee$. The above result immediately implies that

Corollary 4.2.21. *Any spin^c-structure on a compact manifold M gives rise to an oriention of M in K-theory.*

Proof. Since the dual to $C^*(M)$ is given by the twisted convolution algebra $C^*_{-\tau}(M)$, any trivialization of the circle 2-bundle $W_3(\tau) : M \to \mathbf{B}^2U(1)$ gives rise to an equivalence between $C^*(M)$ and $C^*(M)^\vee$. □

Appart from the result for compact manifolds, not much examples of Poincaré dual algebras are known. This is analogous to the situation in stable homotopy theory: only for the suspension spectra of smooth manifolds do we have an explicit description of their dual, via the Thom construction.

In case where the manifold M carries a G-action, the above Poincaré duality has a natural G-equivariant analogue, due to [Tu09]:

Proposition 4.2.22. *Let M be a compact smooth manifold carrying an action of a compact Lie group G, together with a G-equivariant circle 2-bundle $\chi\colon M \to \mathbf{B}^2U(1)$. Then the dual to the G-equivariant algebra $C^*_\chi(M)$ in the G-equivariant KK-category is the G-equivariant C^*-algebra $C^*_{-\tau-\chi}(M)$.*

Although this is not quite duality of the algebra $C^*_\chi(M//G)$, this kind of duality does allow us to construct pushforward maps: we first dualize in G-equivariant KK-theory and then apply the assembly map 4.6 to obtain a pushforward map in KK-theory. This kind of duality gives rise to the G-equivariant pushforward maps we mentioned in the previous section.

We conclude this section with a summary of how these pushforward maps allow us to quantize correspondence diagrams. All the ingredients have already appeared by now, so we just combine them to give an overview of the quantization process by pull-push.

4.3 Overview of the quantization procedure

In section 2.3 we defined a prequantum field theory to be a symmetric monoidal (∞, n)-functor

$$\mathrm{Bord}_n \to \mathrm{Corr}_n(\mathbf{H}_{/\mathbf{B}^n U(1)})$$

into the monoidal category of n-fold correspondences in the slice category $\mathbf{H}_{/\mathbf{B}^n U(1)}$. The ultimate goal would be to functorially quantize each such n-fold correspondence, but as a first step, we quantize a *single* correspondence diagram. Since an n-fold correspondence diagram is built up from single spans, one might hope that a fully functorial quantization of a single correspondence will eventually result in the quantization of an n-fold span.

We would therefore like to quantize a single span in $\mathbf{H}_{/\mathbf{B}^n U(1)}$. In fact, such a single span does not only arise as a building block of the n-fold spans that describe the above pQFT:

- if Σ is a $k+1$-dimensional cobordism between two *closed* k-dimensional manifolds $\Sigma_{\text{in}}, \Sigma_{\text{out}}$, then the above pQFT assigns to Σ a k-fold span in which almost all objects are trivial. Then the k-fold span collapses to a single span diagram

$$\begin{tikzcd} & \mathrm{Fields}(\Sigma) \arrow[dl, "i"'] \arrow[dr, "f"] & \\ \mathrm{Fields}(\Sigma_{\text{in}}) \arrow[dr, "\alpha"'] \arrow[rr, phantom, "\Rightarrow"] & & \mathrm{Fields}(\Sigma_{\text{out}}) \arrow[dl, "\beta"] \\ & \mathbf{B}^{n-k}U(1) & \end{tikzcd}$$

  but now over $\mathbf{B}^{n-k}U(1)$.

  For $k > 0$, we can use such spans to quantize a part of the pQFT, transgressed to higher dimensional cobordisms. We cannot quantize all n-fold correspondence diagrams that arise in the description of the pQFT, but only the k-fold correspondences over $\mathbf{B}^n U(1)$ which collapse to a single correspondence over $\mathbf{B}^{n-k}U(1)$.

- on the other hand, we have seen how the cobordism hypothesis allows us to classify a boundary theory to an n-dimensional pQFT by a single correspondence diagram of the form

$$\begin{tikzcd} & \mathbf{Fields}^{\partial} \arrow[dl] \arrow[dr] & \\ * \arrow[dr] \arrow[rr, phantom, "\Rightarrow"] & & \mathbf{Fields} \arrow[dl, "\chi"] \\ & \mathbf{B}^n U(1) & \end{tikzcd}$$

  Similarly, we can use 2-fold span diagrams to describe codimension 2 defects of a pQFT.

In the examples, we will mostly consider the last source of correspondence diagrams, classifying boundary theories. But we also give an example of pull-push quantization in the first setting, in the context of the string topology operations and the '2-1-part' of Chern-Simons theory.

Depending on what type of span diagram one considers, the pull-push quantization of such a diagram can have two different interpretations:

- in the first case, the result describes a (quantum) propagator from Σ_{in} to Σ_{out} along the cobordism Σ.

- in the second case, the result of the quantization can be seen as the partition function of the boundary theory.

However, in both cases the process of quantization is the same. First we embed the smooth ring $\mathbf{B}^n U(1)$ into the group of units $GL_1(\mathbf{R})$ of a smooth ring spectrum $\mathbf{R}$. There is then a monoidal functor

$$\mathbf{H}_{/\mathbf{B}^n U(1)} \to \mathbf{R}\mathrm{Mod}^{op}$$

that sends a space carrying a circle n-bundle to its smooth twisted $\mathbf{R}$-cohomology spectrum, which forms a smooth $\mathbf{R}$-module (see section 3.1.5).

The above correspondence diagrams then give rise to cospan diagrams of smooth $\mathbf{R}$-modules. In the case of a cobordism from Σ_{in} to Σ_{out} we obtain a diagram

$$\mathbf{R}^{*+\alpha}\big(\mathrm{Fields}(\Sigma_{\mathrm{in}})\big) \xrightarrow{i^*} \mathbf{R}^{*+f^*\beta}\big(\mathrm{Fields}(\Sigma)\big) \xleftarrow{f^*} \mathbf{R}^{*+\beta}\big(\mathrm{Fields}(\Sigma_{\mathrm{out}})\big)$$

The diagram classifying a boundary to a pQFT gives rise to a diagram of the form

$$\mathbf{R} \xrightarrow{\xi} \mathbf{R}^{*+f^*\beta}(\mathrm{Fields}^{\partial}) \xleftarrow{f^*} \mathbf{R}^{*+\beta}(\mathrm{Fields}).$$

where the left map gives the R-cocycle induces by the twisted circle $(n-1)$-bundle on $\mathrm{Fields}^{\partial}$. We would now want to apply the procedure mentioned at the beginning of this section: from an orientation on the map f we can form the pushforward map along f in cohomology, and this we compose with the pullback map on the left to obtain a linear map of $\mathbf{R}$-modules.

Now we have some tools to do this in the two main cases we have considered so far:

- if all stacks involved are geometrically discrete, then the above diagrams can be interpreted in (parametrized) stable homotopy theory. This is particularly useful in the case where the spaces involved are manifolds.
- if all stacks are differentiable stacks, and the span sits over $\mathbf{B}^2 U(1)$, then we can apply the twisted K-theory for differentiable stacks.

In both cases, the pushforward map along f is given abstractly by some form of (fiberwise) duality: if all objects are discrete ∞-groupoids, then we obtain the map $f^!$ as described in the introduction to this section: we pick a twist $\gamma\colon$: Fields $\to B^n U(1)$ such that

$$f_! f^* \beta \simeq \big(f_! f^* \gamma\big)^{\vee}$$

in the category $\mathrm{Fun}(\mathrm{Fields}, R\mathrm{Mod})$ of parametrized spectra. We then constructed a pushforward map $f^!$ in section 4.1.4, which gives rise to the partition function for the boundary theory

$$f^!(\xi)\colon R \to R^{*-\gamma}(\mathrm{Fields})$$

which is obtained by integrating the twisted line bundle ξ over $\mathrm{Fields}^{\partial}$ in R-cohomology. In the case of trajectories, we obtain a map

$$f^! \circ i^*\colon R^{*+\alpha}\big(\mathrm{Fields}(\Sigma_{\mathrm{in}})\big) \to R^{*-\gamma}\big(\mathrm{Fields}(\Sigma_{\mathrm{out}})\big)$$

which describes the quantum propagator between the R-modules of quantum states on Σ_{in} and Σ_{out}. The same kind of structure appears for the story in terms of twisted K-theory of differentiable stacks.

If all spaces involved are smooth manifolds, and the tangent bundle to the fiber of the map f carries an R-orientation, then we obtain a partition function

$$f^!(\xi)\colon R \to R^{*+\chi}(\mathrm{Fields})$$

and a propagator

$$f^! \circ i^*\colon R^{*+\alpha}\big(\mathrm{Fields}(\Sigma_{\mathrm{in}})\big) \to R^{*+\beta}\big(\mathrm{Fields}(\Sigma_{\mathrm{out}})\big).$$

These are the two results of our quantization procedure, both of which appear in the examples: we find partition functions of boundary theories and propagators of quantum states.

To really quantize the pQFT as a whole, we have to make sure that we construct these propagators in a consistent way, so that we preserve functoriality of the field theory. In the next section, we will see two examples where this functoriality has been established. In the outlook section 6 we will comment the issue of extending the quantization of a single correspondence to the quantization of a whole pQFT.

5 Examples

We are now in the position to apply the procedures sketched in sections 3 and 4 to some concrete examples. Most of these examples have been well-studied in the literature. We show how they all become manifestations of a single procedure: the process of pull-push quantization in generalized cohomology.

5.1 Cohomological quantization in ordinary cohomology

We already encountered the pull-push construction in ordinary cohomology in the introduction: this really served as the motivating example of the whole process. The pull-push quantization in ordinary cohomology has a very nice application in the realm of string topology.

5.1.1 String topology operations

One of the best developed theories of pull-push quantization is provided by the string topology operations on the ordinary (co)homology of the free loop space LM of a compact oriented manifold M (see for example [CG04] for a treatment relevant for our perspective). Since this is usually treated in terms of ordinary homology, we will start off with the homological perspective, where the string product is most easily decribed. However, the construction of the string operations by a pull-push construction can be applied equally well to *co*homology, as we will see.

The first and motivating example of a string operation is given by the string product, due to [CS99], which gives a map on the (complex) homology of the free loop space of M

$$H_*(LM) \otimes H_*(LM) \to H_{*-\dim(M)}(LM)$$

This map can be constructed in the following way: pick a basepoint $* \in S^1$ and let $\mathrm{ev} : LM \to M$ be the evaluation at that basepoint. Now suppose we are given an n-chain α and an m-chain β of elements in LM. Under the evaluation map ev, these give two chains in the manifold M, which we can always arrange to intersect transversally. In that case, their intersection is given by an $m + n - \dim(M)$-chain on the manifold M.

Now for each point x in the intersection, we have two loops α_x, β_x whose basepoint is x. Concatenating these two loops gives a family of loops over the points x lying in a $m + n - \dim(M)$-dimensional chain inside M. We thus obtain an $m+n-\dim(M)$-chain $\alpha\cdot\beta$ in the loop space LM. As was shown in [CS99], this gives a well defined associative and graded-commutative product on the homology $H_*(LM)$, called the string product.

The main insight in [CG04] is that this product operation is a shadow of a (nearly) full structure of a (non-extended) 2-dimensional TQFT assigning $H_*(LM)$ to a circle, which also encodes higher string topology operations. One might view this TQFT as a linear approximation to the quantization of the (closed) string moving over the spacetime M. More precisely, one starts by considering the following prequantum theory: the space of configurations over a circle is given by the loop space LM, which describes the configurations of a closed string in M. To a cobordism Σ one assigns a certain space of maps into M

$$\mathrm{Map}(\Gamma_\Sigma, M)$$

The domain of these maps is a certain model of the surface Σ, its fat graph. If the surface Σ has p ingoing and q outgoing circles, then these circles are realized as cycles in the fat graph Γ_Σ. The restrictions to these cycles then give maps

$$LM^{\times p} \xleftarrow{\rho_{\mathrm{in}}} \mathrm{Map}(\Gamma_\Sigma, M) \xrightarrow{\rho_{\mathrm{out}}} LM^{\times q}$$

Note that this does not give a fully extended pQFT, since we do not assign any data to the point. Instead, this gives precisely the structure we obtain by transgressing a two-dimensional pQFT to two-dimensional cobordisms between 1-dimensional *closed* manifolds.

In particular, we endow all loop spaces with the trivial flat $U(1)$-bundle, and each space $\mathrm{Map}(\Gamma_\Sigma, M)$ with the trivial gauge transformation between these two bundles, so that we obtain correspondences like

$$\begin{tikzcd}
& \mathrm{Map}(\Gamma_\Sigma, M) \arrow[dl, "\rho_{\mathrm{in}}"'] \arrow[dr, "\rho_{\mathrm{out}}"] & \\
LM^{\times p} \arrow[dr, "1"'] \arrow[rr, phantom, "\overset{1}{\Longrightarrow}"] & & LM^{\times q} \arrow[dl, "1"] \\
& \flat\mathbf{B}U(1) &
\end{tikzcd} \tag{5.1}$$

We should really view such a correspondence as being obtained by collapsing a twofold correspondence diagram in $\infty\mathrm{Gpd}_{/\flat\mathbf{B}^2U(1)}$ to a single correspondence diagram.

In [CG04], the authors show how one can quantize such correspondences by passing to homology and doing a pull-push construction. We slightly reinterpret their results and instead pass to cohomology. The main point in the construction in [CG04] is that the restriction maps $\rho_{\mathrm{in}}, \rho_{\mathrm{out}}$ are embeddings of finite codimension.

Lemma 5.1.1. *When $p, q \geq 1$, the map $\rho_{\mathrm{out}} : \mathrm{Map}(\Gamma_\Sigma, M) \to LM^{\times q}$ is an embedding and has an open neighbourhood $\nu(\Sigma)$ which is diffeomorphic to the total space of a vector bundle of rank $-\chi(\Sigma) \cdot \dim(M)$. Moreover, the orientation on M induces a natural orientation on this vector bundle $\nu(\Sigma) \to \mathrm{Map}(\Gamma_\Sigma, M)$.*

This places us in the context where we can apply the pull-push formalism we have developed so far: we pass to cohomology to obtain a diagram

$$H^*(LM^{\times p}) \xrightarrow{\rho_{\mathrm{in}}^*} H^*\big(\mathrm{Map}(\Gamma_\Sigma, M)\big) \xleftarrow{\rho_{\mathrm{out}}^*} H^*(LM^{\times q})$$

If $q \geq 1$, then we obtain a Thom isomorphism in cohomology

$$H^*\big(\mathrm{Map}(\Gamma_\Sigma, M)^{\nu(\Sigma)}\big) \simeq H^{*-\chi(\Sigma)\cdot\dim(M)}\big(\mathrm{Map}(\Gamma_\Sigma, M)\big)$$

Finally, since the Thom space is obtained from the vector bundle $\nu(\Sigma)$ by collapsing the point at infinity, we obtain a collapse map

$$LM^{\times q} \to \mathrm{Map}(\Gamma_\Sigma, M)^{\nu(\Sigma)}.$$

Combined with the above Thom isomorphism, this provides the pushforward map in ordinary cohomology

$$\rho_{\mathrm{out}}^! : H^*\big(\mathrm{Map}(\Gamma_\Sigma, M)\big) \to H^{*+\chi(\Sigma)\cdot\dim(M)}(LM^{\times q})$$

The the pull-push quantization of the correspondence diagram 5.1 gives rise to a map

$$H^*(LM^{\times p}) \to H^{*+\chi(\Sigma)\cdot\dim(M)}(LM^{\times q})$$

Example 5.1.2. If Σ is a pair of pants from two circles to one circle, then its Euler characteristic is -1. Using that we are working with field coefficients, we then find a map

$$H^*(LM) \otimes H^*(LM) \to H^{*-\dim(M)}(LM)$$

which is precisely the map dual to the string product.

Furthermore, we can perform the pull-push construction along a cap D^2 from the circle to the empty set. In that case, the fat graph of D^2 consists of a single point, so that we obtain a diagram

$$LM \xleftarrow{\mathrm{const}} M \longrightarrow *$$

where the left map is the inclusion of the constant paths. Since M was a compact oriented manifold, we can push along the right map to obtain a map in complex cohomology

$$H^*(LM) \to H^*(M) \to H^{*-\dim(M)}(*)$$

corresponding to the cap D^2.

Remark 5.1.3. We cannot form the pull-push quantization along a cup from the emptyset to the circle. Indeed, this gives rise to a diagram

$$* \longleftarrow M \longrightarrow LM$$

where the right map is an inclusion of infinite codimension. This does not allow for a pushforward map.

We can thus obtain the string topology operations (or their duals in cohomology) by quantizing the two-dimensional trajectories of a closed string in a manifold M. Moreover, note that the orientation that we used in the Thom isomorphism was derived from the orientation on M. This turns out to give a 'consistent orientation', i.e. we have a canonical way to orient all the maps along which we want to integrate. This results in the following theorem:

Theorem 5.1.4 ([CG04], theorem 6)**.** *This pull-push construction is functorial with respect to gluing of cobordisms. Consequently, we obtain a functor to the category of chain complexes of complex vector spaces from the category of circles and diffeomorphism classes of cobordisms, whose connected components all have a nonempty ingoing boundary .*

Remark 5.1.5. Here a map of chain complexes is allowed to be of nonzero degree, since we have seen that the pull-push construction gives a change in degree by $-\chi(\Sigma)\dim(M)$.

Remark 5.1.6. There is an extension of this pull-push construction to the case of the open string moving on M. See for instance [Kup11] for a treatment of this case.

It turns out that this description of the string topology operations can also be extended to an extended TFT, up to the fact that we require the ingoing boundary to be nonempty. This is discussed in section 4.2 of [Lur09b] at the level of cochain complexes instead of cohomology groups. There it is shown that there is an $(\infty, 2)$-functor from the $(\infty, 2)$-category of cobordisms having nonempty ingoing boundary to the category of algebras and bimodules inside $\mathrm{Ch}(\mathbb{C})$, whose value on the point is the singular cochain complex $C^*(M, \mathbb{C})$. This requires the manifold M to be simply connected, so that the Hochschild homology of this cochain complex is

$$\int_{S^1} C^*(M, \mathbb{C}) \simeq C^*(LM, \mathbb{C}).$$

Then the two-dimensional cobordisms give rise to the string topology operations.

Summarizing, the string topology operations form one of the best-studied examples of pull-push quantization where the pull-push construction is actually shown to be functorial. In this case, the pull-push quantization indeed produces a functorial QFT (at least on cobordisms with nonempty ingoing boundary).

5.2 Cohomological quantization in K-theory

In this section we produce the main examples of pull-push quantization in the smooth context. This uses the description of twisted K-theory for differentiable stacks in terms of KK-theory. The first example of pull-push quantization arises from the quantization of symplectic manifolds. We will give a very short description of the pull-push quantization of a symplectic manifold from our perspective in section 5.2.1. A more detailed treatment can be found in [] **ref**, which also amplifies the relation with quantization by picking a polarization, which we mentioned in the introduction. We will discuss how this naturally extends to the case where the symplectic manifold carries a G-action; this gives a decription of the orbit method.

Our main example is the quantization of Poisson manifolds as the boundary theory of the two-dimensional Poisson sigma-model. In fact, the quantization of a symplectic manifold can be best viewed from this perspective. As we will discuss in section 5.2.2, this quantization of a Poisson manifold provides a stacky refinement of the quantization of Poisson manifolds developed in [Haw08].

One of the most interesting applications of the pull-push quantization procedure lies in the context of Chern-Simons theory, or rather its reduction to a two-dimensional theory, as discussed by [FHT10]. We give a very brief overview of this in section 5.2.3.

We conclude with an example arising from string theory: the D-brane charges that appear in (type II super-) string theory are precisely the result of quantizing the endpoint of a topological string moving along a D-brane Q. This re-interprets the KK-theoretic discussion of these D-brane charges as described in [BMRS08]. Moreover, this sets the scene for the quantization of the string as the boundary of a brane which we will see in the next section.

5.2.1 Quantization of a Hamiltonian G-space

Let us first recall the traditional picture of the spin^c-quantization of a symplectic manifold. After that, we will discuss how a Hamiltonian G-action gives rise to an action of G on the space of quantum states, by quantum operators.

Spin^c-quantization of symplectic manifolds

If (M, ω) is a symplectic manifold, then a prequantization of the symplectic form is given by a $U(1)$-bundle L with connection ∇, such the symplectic form is given by the curvature of that connection:

$$\omega = F_\nabla \in \Omega^2_{cl}(M).$$

In other words, a prequantization of a symplectic manifold is a choice of lift

$$\begin{tikzcd} & \mathbf{B}U(1)_{\text{conn}} \arrow[d, "F"] \\ M \arrow[ur, dotted, "\nabla"] \arrow[r, "\omega"'] & \Omega^2_{cl} \end{tikzcd}$$

where Ω^2_{cl} is the stack of closed differential 2-forms, $\mathbf{B}U(1)_{\text{conn}}$ is the classifying stack of $U(1)$-bundles with conenction and the map F takes the curvature of a connection. Recall that such a prequantization exists only if the class $[\omega] \in H^2(M, \mathbb{R})$ is an integral cohomology class.

Remark 5.2.1. In the end, for the spin^c-quantization the particular choice of connection will not be important, all we need is the $U(1)$-bundle L. This is essentially due to the fact that our quantization process is through cohomology, instead of a differential refinement thereof containing also connections. In the context of K-theory, such a differential refinement has been developed in [FL10]. However, (exponentiated) Hamiltonian actions can be naturally described in terms of the map $\nabla : M \to \mathbf{B}U(1)_{\text{conn}}$, so it is good to keep in mind that we also have a connection.

We will view the manifold M with its prequantum line bundle L as a boundary theory for the trivial two-dimensional TFT:

$$\begin{tikzcd} & M \arrow[dl] \arrow[dr] \arrow[dd] & \\ * \arrow[dr] \arrow[r, phantom, "\overset{L}{\Longrightarrow}"] & \arrow[r, phantom, "\Longrightarrow"] & * \arrow[dl, "1"] \\ & \mathbf{B}^2U(1) & \end{tikzcd}$$

where the right 2-cell is the 2-cell witnessing the composition of the right two maps. We will see in section 5.2.2 that this trivial theory is actually the topological part of the Poisson sigma model associated to the symplectic manifold M.

The left 2-cell defines a map of K-theory spectra

$$L\colon KU = K^*(*) \longrightarrow K^*(M)$$

which gives the K-theory class of the line bundle L. Under the assumption that the manifold M is compact and carries a K-orientation, the right 2-cell gives rise to a pushforward map

$$K^*(M) \xrightarrow{\text{ind}_D} K^*(*).$$

We can push the line bundle L along this map and obtain a cocycle in $KU = K^*(*)$: such a class is given by a (finite-dimensional) virtual vector space $\mathcal{H}$, which is the vector space of quantum states of the symplectic manifold.

To compute this space of quantum states, recall from section 4.2.1 that a K-orientation on M is a choice of spin^c-structure on M. Having chosen a spin^c-structure, we can form the associated spinor bundle $S \to M$. Then we obtain the space of quantum states $\mathcal{H}$ as the index of the operator

$$D_L \colon \Gamma(M, L \otimes S) \xrightarrow{\nabla \otimes 1} \Gamma(M, L \otimes T^*M \otimes S) \xrightarrow{1 \otimes \gamma} \Gamma(M, L \otimes S)$$

where γ describes the Clifford action of T^*M on the spinor bundle S. This is precisely the traditional picture of the spin^c-quantization of a compact symplectic manifold, as described for example in [Bon13]. Note that the index of D_L is independent of the chosen connection ∇ on the line bundle L.

Remark 5.2.2. In the case where M is a Kähler manifold, we have seen in the introduction that we can also quantize by picking a holomorphic polarization. If there are enough holomorphic section, then the result agrees with the spin^c-quantization. In general, instead of picking holomorphic sections of the line bundle L, one has to pick the alternating sum of the sheaf cohomology groups of the sheaf of holomorphic sections. For more details, we refer to [Bon13].

This procedure gives us a space of quantum states, but we usually also want to quantize observables. It turns out that it is not possible to quantize all classical observables in geometric quantization. If we quantize by choosing a polarization, we can only quantize those classical observables whose Hamiltonian flow preserves the polarization. We will see a similar thing in spin^c-quantization: we can quantize a Hamiltonian G-action only if it preserves the spin^c-structure (which is the analogue of a choice of polarization).

Quantization of a Hamiltonian G-action

Recall that a Hamiltonian G-action on a symplectic manifold (M, ω) is a G-action, together with a G-equivariant moment map $\mu : M \to \mathfrak{g}^*$ with the property that $d\langle \mu, X \rangle = \iota_X \omega$, where $X \in \mathfrak{g}$ gives rise to a vector field on M. When M is prequantized by a $U(1)$-bundle L with connection $\nabla \colon M \to \mathbf{B}U(1)_{\text{conn}}$, it is shown in [FRS13] that such a Hamiltonian G-action gives the infinitesimal version of an exponentiated Hamiltonian G-action, which acts by automorphisms of the map $\nabla : M \to \mathbf{B}U(1)_{\text{conn}}$: each element $g \in G$ gives rise to a diagram of the form

$$\begin{tikzcd} M \arrow[rr, "\sim"] \arrow[dr] & \arrow[d, phantom, "\Leftarrow"] & M \arrow[dl] \\ & \mathbf{B}U(1)_{\text{conn}} & \end{tikzcd}$$

where the horizontal arrow describes the associated diffeomorphism of M, while the 2-cell encodes the exponentiated moment map. In other words, we have a smooth G-action on the space X, together with equivalences $\nabla \xrightarrow{\sim} \rho(g)^*\nabla$ of $U(1)$-bundles with connection, depending smoothly on the element $g \in G$ acting by $\rho(g)$ on M. From this the following result, originally due to Atiyah-Bott [AB84], immediately follows:

Proposition 5.2.3. *An exponentiated Hamiltonian G-action on (M, ∇) is equivalently given by an extension of ∇ to a G-equivariant $U(1)$-bundle with connection on M*

$$\begin{tikzcd} M \arrow[r] \arrow[dr, "\nabla"'] & M//G \arrow[d, dotted, "\nabla//G"] \arrow[dl, phantom, "\Leftarrow"] \\ & \mathbf{B}U(1)_{\text{conn}} \end{tikzcd}$$

We now show how such an exponentiated Hamiltonian action quantizes to an action on the space of quantum states of the symplectic manifold M. We can interpret M, together with its Hamiltonian G-action, as the boundary theory to a two-dimensional theory whose fields are principal G-bundles

$$\begin{tikzcd} & M//G \arrow[dl] \arrow[dr] \arrow[d, phantom, "\Downarrow \nabla//G"] & \\ * \arrow[dr] & & \mathbf{B}G \arrow[dl, "1"] \\ & \mathbf{B}^2U(1) & \end{tikzcd}$$

As before, the pull-push quantization does not really depend on the differential structure of the connection on the $U(1)$-bundle. Passing to K-theory, we find a map of KU-modules

$$K^*(*) = KU \longrightarrow K^*(M//G)$$

which describes the class of the G-equivariant $U(1)$-bundle L over M (recall from corollary 3.3.65 that the K-theory of the stack $M//G$ agrees with the G-equivariant K-theory of M). To push this line bundle over $M//G$, we have to pick a G-equivariant spin^c-structure on M: the condition that the spin^c-structure be G-equivariant is the analogue of the condition that the Hamiltonian flow of a quantizable observable preserves the a chosen polarization.

Whenever M is compact and we have such an equivariant spin^c-structure, we can form the pushforward in G-equivariant K-theory (see section 4.2.2), which gives rise to a map

$$K^*(M//G) \xrightarrow{\mathrm{ind}_D} K^*(\mathbf{B}G).$$

Applying this to the line bundle L, we obtain a map $KU \to K^*(\mathbf{B}G)$ which describes an element of the representation ring of the Lie group G. Explicitly, it is given by by the index of the G-equivariant differential operator

$$D_L\colon \Gamma(M, L\otimes S) \xrightarrow{\nabla\otimes 1} \Gamma(M, L\otimes T^*M\otimes S) \xrightarrow{1\otimes\gamma} \Gamma(M, L\otimes S)$$

where ∇ is any G-equivariant connection on the line bundle L, for example the one we had before. Forgetting the G-equivariance, the index of the above operator gives us the space of quantum states of the symplectic manifold M. If the spin^c-structure is G-equivariant, then this space of quantum states lives in the representation ring of G, so it carries an action of the Lie group G by 'quantum operators'.

Example 5.2.4. We can apply this to the 2-sphere, carrying the action of $SU(2)$ by rotations in $\mathbb{R}^3$. Note that $H^2(S^2,\mathbb{Z}) \simeq \mathbb{Z}$ so that every line bundle on S^2 appears as a tensor product of the tautological line bundle $\gamma : S^2 \simeq \mathbb{C}P^1 \to \mathbf{B}U(1)$. The tautological line bundle γ is clearly $SU(2)$-equivariant, where we use the action of $SU(2)$ on $\mathbb{C}^2$. Moreover, the canonical complex structure on $S^2 = \mathbb{C}P^1$ gives rise to an $SU(2)$-equivariant spin^c-structure.

One can show that for $k \geq 0$, the quantization of the line bundle $\gamma^{\otimes k}$ gives a $k+1$-dimensional vector space (without a virtual part). This $k+1$-dimensional vector space carries an $SU(2)$-action, which in fact gives the irreducible representation of $SU(2)$ of dimension $k+1$. See [] **ref** for a discussion. The quantization therefore gives an irreducible spin representation.

In fact, one can realize S^2 as a (special) coadjoint orbit of the Lie group $SU(2)$, with its prequantum bundle $\gamma^{\otimes k}$ induced from the Lie-Poisson structure on $\mathfrak{g}^*$. This process of obtaining irreducible representations by quantizing coadjoint orbits is called the *orbit method*, of which a textbook account (by its main founder) is [Kir04]. We will come back to the orbit method in the next section, where we discuss the quantization of a Poisson manifold.

5.2.2 Quantization of Poisson manifolds

Our main example of quantization by pull-push is the quantization of a Poisson manifold as space of boundary fields for the Poisson sigma model (or rather, its topological part). We give a short description of the stack that describes the Poisson sigma model, together with its circle 2-bundle, and then describe how to use this to produce a quantization of a Poisson manifold. As a main example, we offer a description of the orbit method by means of the quantization of a Lie-Poisson structure.

Poisson sigma model

Let $(\mathfrak{P}, \pi)$ be a Poisson manifold and recall that this gives rise to a Lie algebroid

$$\begin{tikzcd} T^*\mathfrak{P} \arrow[rr, "\pi"] \arrow[dr] & & T\mathfrak{P} \arrow[dl] \\ & \mathfrak{P} & \end{tikzcd}$$

with Lie bracket given by

$$\Gamma(T^*\mathfrak{P}) \wedge \Gamma(T^*\mathfrak{P}) \to \Gamma(T^*\mathfrak{P}); \quad (\alpha, \beta) \mapsto L_{\pi(\alpha)}\beta - L_{\pi(\beta)}\alpha - d\big(\pi(\alpha \wedge \beta)\big).$$

Under suitable conditions (see [CF04]), this Lie algebroid integrates to a source-connected Lie groupoid $X_1 \rightrightarrows \mathfrak{P}$, called the symplectic groupoid integrating $\mathfrak{P}$. In addition to the conditions mentioned in [CF04], we also demand this Lie groupoid to be Hausdorff, so that it gives rise to a differentiable stack $X := \mathrm{SymplGpd}(\mathfrak{P})$, together with an atlas $\mathfrak{P} \to X$. For the rest of this story it is imortant that we do not just obtain the stack X, but also an *atlas* for this stack by the manifold $\mathfrak{P}$.

The Lie groupoid $X_1 \rightrightarrows \mathfrak{P}$ is called the *symplectic* groupoid integrating $\mathfrak{P}$ due to the fact that the space of arrows X_1 carries a multiplicative symplectic form ω with the property that the source map

$$s\colon (X_1, \omega^{-1}) \to (\mathfrak{P}, \pi)$$

is a Poisson map. In analogy with the symplectic case, a prequantization of this symplectic groupoid is now given by a multiplicative $U(1)$-bundle with connection on X_1, whose curvature is equal to the symplectic form ω.

If we forget about the connection, we obtain a multiplicative $U(1)$-bundle over the space X_1. As we have seen in section 2.2.3, such a multiplicative line bundle classifies a circle 2-bundle $X \to \mathbf{B}^2U(1)$ over the stack $X = \mathrm{SymplGpd}(\mathfrak{P})$. This circle 2-bundle serves as the action functional of the Poisson sigma model:

Definition 5.2.5. Let $(\mathfrak{P}, \pi)$ be a Poisson manifold integrating to a prequantized (Hausdorff) Lie groupoid. Then the two-dimensional Poisson sigma model is the pQFT whose value on the point is given by the circle 2-bundle

$$\chi\colon \mathrm{SymplGpd}(\mathfrak{P}) \to \mathbf{B}^2U(1).$$

Remark 5.2.6. In fact, this gives only the topological part of the Poisson sigma model. One obtains the full Poisson sigma model from the differential refinement of the above map

$$\mathrm{SymplGpd}(\mathfrak{P})_{\mathrm{conn}} \to \mathbf{B}^2U(1)_{\mathrm{conn}}.$$

Here a map from a manifold M into $\mathrm{SymplGpd}(\mathfrak{P})_{\mathrm{conn}}$ is given by a groupoid bundle over M, together with a connection on it. In case the bundle is trivial, such a connection is given by a map $\phi : M \to \mathfrak{P}$ together with a 1-form $\eta \in \Omega^1(M, \phi^*T\mathfrak{P})$. More general, the bundle can be nontrivial, which leads to nonperturbative effects.

The integration of the Poisson manifold $\mathfrak{P}$ also gave us an atlas $f\colon \mathfrak{P} \to X$ for the stack X by the manifold $\mathfrak{P}$. The pullback of the circle 2-bundle along this atlas is trivializable, since the circle 2-bundle was given by a multiplicative bundle on the space of arrows. In other words, we obtain a diagram of the form

$$\begin{tikzcd} & \mathfrak{P} \arrow[dl] \arrow[dr, "f"] & \\ * \arrow[dr] \arrow[rr, phantom, "\Rightarrow \xi"] & & X \arrow[dl, "\chi"] \\ & \mathbf{B}^2U(1) & \end{tikzcd} \tag{5.2}$$

These simple observations lead to the following conclusion:

Corollary 5.2.7. *Every Poisson manifold $(\mathfrak{P}, \pi)$ that integrates to a prequantized (Hausdorff) Lie groupoid gives rise to a boundary theory for its two-dimensional Poisson sigma model.*

Remark 5.2.8. The same perspective arises in the deformation quantization of Poisson manifolds. In [CF00], Cattaneo and Felder compute Kontsevich's formula for the star product [Kon03] as a certain correlator in the perturbative quantization of the Poisson sigma model. This is the correlator for three points on the boundary of a closed disc, where the fields on the boundary take value in $\mathfrak{P}$, rather than its symplectic groupoid.

We can now employ the theory developed in the previous sections to give the counterpart of the above result in geometric quantization: we quantize a Poisson manifold as the boundary of its Poisson sigma model, under compactness and orientability conditions similar to those that arise in the quantization of symplectic manifolds.

Quantization

Suppose $\mathfrak{P}$ is a compact Poisson manifold satisfying all of the above integrability assumptions, whose symplectic Lie groupoid is proper. Then we can linearize the diagram 5.2 to obtain a diagram in KK-theory

$$\mathbb{C} \xrightarrow{\xi} C^*_{f^*\chi}(\mathfrak{P}) \xleftarrow{f^*} C^*_\chi(X)$$

which in turn determines a map of K-theory spectra

$$KU \xrightarrow{\xi} K^{*+f^*\chi}(\mathfrak{P}) \xleftarrow{f^*} K^{*+\chi}(X)$$

The the quantization of the Poisson manifold is obtained by picking a K-orientation on the map f and forming the pushforward $f^!\xi$.

For example, suppose that $C^*_\chi(X)$ is dualizable in KK. In that case the right map admits a canonical dual

$$C^*_{-\tau-f^*\chi}(\mathfrak{P}) \simeq C^*_{f^*\chi}(\mathfrak{P})^\vee \to C^*_\chi(X)^\vee$$

Since the diagram 5.2 trivializes the circle 2-bundle $f^*\chi$ on $\mathfrak{P}$, the left hand side is canonically equivalent to $C^*_{-\tau}(\mathfrak{P})$. A choice of spin^c-structure on the manifold $\mathfrak{P}$ determines an equivalence $\mathcal{C}(\mathfrak{P}) \simeq \mathcal{C}_{-\tau}(\mathfrak{P})$, so that we obtain a map

$$\mathbb{C} \to \mathcal{C}(\mathfrak{P}) \xrightarrow{f^!} C^*_\chi(X)^\vee$$

This describes a K-theory class of $C^*_\chi(X)^\vee$, which we can interpret as the quantization of the Poisson manifold $\mathfrak{P}$.

Remark 5.2.9. The intepretation of the K-theoretic quantization in terms of operator algebraic K-theory forms a stacky refinement of the quantization of symplectic groupoids discussed by Hawkins in [Haw08]. There it was proposed that the convolution algebra of polarized functions on a symplectic groupoid can serve as the quantization of the corresponding Poisson manifold. However, this discussion is not invariant under equivalences of the stacks presented by these symplectic groupoids. By passing to KK-theory, we make the discussion in that paper be invariant under Morita equivalence of groupoids (i.e. equivalence of the associated stacks).

However, the stack associated to a Poisson manifold can be very degenerate, even though the Poisson manifold itself is not. Indeed, the symplectic groupoid of a Poisson manifold behaves roughly like the leaf space of the symplectic foliation of $\mathfrak{P}$. In particular, when $\mathfrak{P}$ is close to being symplectic, the symplectic groupoid is close to being a single point. To still obtain a reasonable quantization of a Poisson manifold, we therefore should not just consider its symplectic groupoid, but we should also remember the atlas $\mathfrak{P} \to \mathrm{SymplGpd}(\mathfrak{P})$. This is precisely captured by the picture of a Poisson manifold forming a boundary theory for its two-dimensional Poisson sigma model.

For example, we can consider the case where the Poisson manifold $\mathfrak{P}$ is symplectic, in which case the symplectic groupoid integrating $\mathfrak{P}$ is just the point:

Example 5.2.10. Let $(\mathfrak{P}, \pi) = (M, \omega^{-1})$ be a compact symplectic manifold, which we assume to be simply connected. This Poisson manifold integrates to the pair groupoid $\mathrm{Pair}(M)$, i.e. the groupoid $M \times M \rightrightarrows M$ where there is a unique arrow between any two objects. The symplectic form is given by $p_1^*\omega - p_2^*\omega$. If M is prequantizable, with prequantum bundle $\xi \colon M \to \mathbf{B}U(1)$, then the pair groupoid can be prequantized by the multiplicative line bundle $p_1^*\xi - p_2^*\xi$, which gives rise to the circle 2-bundle $\chi : \mathrm{Pair}(M) \to \mathbf{B}^2U(1)$.

Observe that the pair groupoid $\mathrm{Pair}(M)$ gives a presentation of the trivial stack $*$, since there is a unique morphism connecting any two objects. Consequently, the circle 2-bundle determined by $p_1^*\xi - p_2^*\xi$ is trivial, since the point carries only the trivial circle 2-bundle. Indeed, we obtain a diagram

$$\begin{tikzcd}
\mathrm{Pair}(M) \arrow[rr] \arrow[dr, "\chi"'] & \arrow[d, phantom, "\overset{\xi}{\Leftarrow}"] & * \arrow[dl] \\
& \mathbf{B}^2U(1) &
\end{tikzcd}$$

where the circle 2-bundle χ is trivialized by the circle bundle $\xi : M \to \mathbf{B}U(1)$. Pasting this into the above diagram, we see that a symplectic manifold gives rise to the diagram

$$\begin{tikzcd}
& M \arrow[dl] \arrow[dr] \arrow[dd] & \\
* \arrow[r, phantom, "\overset{\xi}{\Longrightarrow}"] \arrow[dr] & {} \arrow[r, phantom, "\Longrightarrow"] & * \arrow[dl] \\
& \mathbf{B}^2U(1) &
\end{tikzcd}$$

where the left 2-cell is given by the prequantum bundle $\xi : M \to \mathbf{B}U(1)$, while the right 2-cell is the trivial 2-cell witnessing the composition. We have seen that the quantization of such a diagram gives the traditional description of spin^c-quantization of a symplectic manifold.

Alternatively, observe that the atlas $f \colon \mathfrak{P} \to X$ is a representable map of stacks. If the symplectic groupoid is now a local quotient groupoid, then we can apply the pushforward construction for local quotient stacks which we mentioned in section 4.2.2. In that case, we form the homotopy cartesian square

$$\begin{tikzcd}
X_1 \arrow[r, "f_0"] \arrow[d] & X_0 = \mathfrak{P} \arrow[d] \\
X_0 = \mathfrak{P} \arrow[r, "f"'] & X
\end{tikzcd}$$

The left and right vertical arrows form the atlases of the stacks $\mathfrak{P}$ and X. The Lie groupoid corresponding to the atlas $s \colon X_1 \to \mathfrak{P}$ has as objects the space X_1 of arrows in the symplectic groupoid, while a morphism is given by a commutative diagram

$$\begin{tikzcd} & x \arrow[dl, "g"'] \arrow[dr, "h"] & \\ y \arrow[rr] & & z \end{tikzcd}$$

This gives an arrow from g to h. Clearly two elements of X_1 are identified via a unique morphism if their sources agree, so this indeed presents $\mathfrak{P}$. The map f_0 is now simply the target map of the Lie groupoid $X_1 \rightrightarrows X_0 = \mathfrak{P}$. When this is a local quotient groupoid, we can construct a pushforward in (twisted) K-theory by picking an equivariant spin^c-structure on the tangent bundle τ_t to the target fibers. Then we construct the associated equivariant spinor bundle and equip it with an equivariant Dirac operator D. The map

$$KU \xrightarrow{\xi} K^{*+f^*\chi}(\mathfrak{P})$$

is induced by the twisted line bundle that trivializes the circle 2-bundle $f^*\chi$ in diagram 5.2. The quantization of the Poisson manifold is the obtained by coupling the Dirac operator D to the twisted line bundle ξ and taking its index. This produces a map

$$KU \longrightarrow K^{*+\chi}(X)$$

i.e. a cocycle in the χ-twisted K-theory of X. This we interpret as the quantization, or partition function, of the Poisson manifold.

As an example of this construction, we conclude with the quantization of a Lie-Poisson structure. This closely relates to the orbit method already mentioned in the previous section.

Quantization of Lie-Poisson manifolds: the universal orbit method

Let $\mathfrak{g}$ the Lie algebra of a compact, simple, simply connected Lie group G. Then $\mathfrak{g}^*$ carries the Lie-Poisson structure induced by the Lie bracket on $\mathfrak{g}$. The associated symplectic groupoid presents the quotient stack $\mathfrak{g}^*//G$ of the (right) coadjoint action on $\mathfrak{g}^*$ [BC05]:

$$\begin{tikzcd} \mathfrak{g}^* \times G \arrow[d, shift right, "\mathrm{Ad}^*"'] \arrow[d, shift left, "p_1"] \\ \mathfrak{g}^* \end{tikzcd}$$

Under the identification $\mathfrak{g}^* \times G \simeq T^*G$, the multiplicative symplectic form on $\mathfrak{g}^*//G$ is just the canonical form ω on the cotangent bundle T^*G. Clearly this symplectic form is prequantizable, since it can be realized as the differential of the canonical 1-form on T^*G (which is also multiplicative).

We thus obtain the diagram

$$\begin{tikzcd} & \mathfrak{g}^* \arrow[dl] \arrow[dr] & \\ * \arrow[dr] \arrow[rr, phantom, "\Rightarrow"] & & \mathfrak{g}^*//G \arrow[dl] \\ & \mathbf{B}^2U(1) & \end{tikzcd}$$

which realizes $\mathfrak{g}^*$ as the boundary to its Poisson sigma model. In this diagram, all gerbes and trivializations of them are trivial. Note that we have to use K-theory without support conditions to pull back along the left map. On the other hand, the right map is proper (since G is a compact Lie group), so we can also form the pushforward in K-theory along if we have no support conditions.

The quantization of the above boundary theory proceeds by pushing the trivial K-theory class (the trivial line bundle over $\mathfrak{g}^*$) along the atlas $\mathfrak{g}^* \to \mathfrak{g}^*//G$. The computations from in the previous section show that $\mathfrak{g}^*$ is presented by the action groupoid $\mathfrak{g}^* \times G//G$. Here G acts on from the right on $\mathfrak{g}^*$ by the coadjoint action and on G by *right* translation, so that the projection map onto $\mathfrak{g}^*$ is G-equivariant.

The fibers of projection $\mathfrak{g}^* \times G//G \to \mathfrak{g}^*//G$ are given by the copies of $\{\mu\} \times G$ for $\mu \in \mathfrak{g}^*$. The tangent bundle to the fibers is then given by $\mathfrak{g}^* \times TG$. We trivialize this tangent bundle using *left* translation along G. In that case, the tangent bundle along the fibers is given by

$$\mathfrak{g}^* \times G \times \mathfrak{g} \to \mathfrak{g}^* \times G$$

where the Lie group G acts on the copy of $\mathfrak{g}$ by the adjoint action. This bundle comes equipped with a metric, given by the Killing form on $\mathfrak{g}$. We obtain a G-equivariant spin^c-structure of this bundle by choosing a lift for the adjoint action

$$\begin{tikzcd} & \mathrm{Spin}^c(\mathfrak{g}) \arrow[d] \\ G \arrow[r, "Ad"'] \arrow[ur, dotted] & O(\mathfrak{g}) \end{tikzcd}$$

to an action by spin^c-operators. This is possible since G is simply connected.

Let $Cl(\mathfrak{g})$ be the Clifford algebra associated to the Killing form on $\mathfrak{g}$ and let S be an irreducible spinor representation. Then the spinor bundle associated to the above spin^c-bundle is of course trivial

$$\mathfrak{g}^* \times G \times S \to \mathfrak{g}^* \times G.$$

where we have used left translation along G to trivialize it. It is G-equivariant, where G acts via $\mathrm{Spin}^c(\mathfrak{g}) \subseteq Cl(\mathfrak{g})$ on the copy of S. Over each point $\mu \in \mathfrak{g}^*$, we have a space of L^2-sections of the spinor bundle, which is simply

$$L^2(\{\mu\} \times G) \otimes S$$

This carries a left action of G by the spinor action on S and by pullback along right translation on $L^2(G)$.

To form the pushforward in K-theory, we now have to pick the G-equivariant family of Dirac operators along the $\{\mu\} \times G$ and take its index. We do not pick the same Dirac operator for each fiber, since this is not equivariant. Rather, we add an explicit dependence of the point $\mu \in \mathfrak{g}^*$ to the Dirac operators to make the family equivariant, following the discussion in [FHT13].

Let e_α be a basis for $\mathfrak{g}$ and e^α the dual basis for $\mathfrak{g}^*$. Then we introduce the following notation:

- Let $\gamma^\alpha \in Cl(\mathfrak{g})$ denote the spinor action of e^α on S (we use the Killing form $\langle -, - \rangle$ on $\mathfrak{g}$ to identify $\mathfrak{g} \simeq \mathfrak{g}^*$).
- Let σ_α denote the action of $e_\alpha \in \mathfrak{g}$ on S, via the derivative of the G-action on S induced by the coadjoint action.
- Finally, we let let R_α denote the left-invariant vector field on G induced by $e_\alpha \in \mathfrak{g}$.

Along the fiber of $\mu \in \mathfrak{g}^*$, we now define the Dirac operator on the trivial bundle $\{\mu\} \times G \times S$ given by

$$D_\mu = D_0 + \mu_\alpha \gamma^\alpha \qquad\qquad D_0 = i\gamma^\alpha R_\alpha + \frac{i}{3}\gamma^\alpha \sigma_\alpha$$

The smooth sections of the spinor bundle over $\{\mu\} \times G$ are just $\mathcal{C}^\infty(G) \otimes S$. The first term in the Dirac operator acts on the smooth function part, while the second term acts on the spinorial part, both via the CLifford action and the derivative of the G-action.

Lemma 5.2.11 ([FHT13])**.** *The collection of D_μ gives a G-equivariant family of Dirac operators on the spinor bundle $\mathfrak{g}^* \times G \times S \to \mathfrak{g}^* \times G$.*

The K-theory pushforward is obtained by taking the index of this Dirac operator: over each $\mu \in \mathfrak{g}^*$, we find an operator

$$D_\mu : L^2(G) \otimes S \to L^2(G) \otimes S$$

By the Peter-Weyl theorem, we have

$$L^2(G) = \bigoplus_{V \in \text{ irrep}(G)} V \otimes V^*$$

as a G-module. The Dirac operator respects this decomposition, and therefore gives rise to operators

$$D_\mu(V)\colon V \otimes V^* \otimes S \to V \otimes V^* \otimes S$$

To study the index of this Dirac operator, we extend the Poisson boundary to $\mathfrak{g}^*//G$ to a full defect diagram, as we discussed in section 2.3.3. Recall that such a defect diagram consists of a second two-dimensional pQFT with boundary, which meets the Poisson sigma model and its Poisson boundary at yet another boundary.

In this case, we can construct such a defect diagram as follows: let $\lambda \in \mathfrak{g}^*$ be an integral regular weight and let $O_\lambda \subseteq \mathfrak{g}^*$ be its coadjoint orbit. Then we can form the diagram

$$\begin{tikzcd}
* & * \arrow[l] \arrow[r] & * \\
O_\lambda//G \arrow[u] \arrow[d] & O_\lambda \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \mathfrak{g}^* \arrow[u] \arrow[d] \\
*//G & O_\lambda//G \arrow[l] \arrow[r] & \mathfrak{g}^*//G
\end{tikzcd}$$

In fact, this diagram lives over $\mathbf{B}^2U(1)$, but we will omit this from the notation. Observe that the bottom right square is homotopy cartesian. The right vertical correspondence describes $\mathfrak{g}^*$ as a boundary of its symplectic groupoid $\mathfrak{g}//G$. The left vertical correspondence describes the symplectic manifold O_λ together with the Hamiltonian action of G given by the coadjoint action. The coadjoint orbit of the regular weight λ forms a defect between these two boundary theories.

The bottom correspondence is studied in detail in [FHT13]. Their results can be summarized as follows:

Proposition 5.2.12. *For $\mathfrak{g}^*$ the dual of the Lie algebra of a compact, simply connected Lie group G, the pushforward in compactly supported K-theory from $\mathfrak{g}^*//G$ to $*//G$ along the G-equivariant map $\mathfrak{g}^* \to *$ produces the Thom isomorphism*

$$K_c^{\dim(G)}(\mathfrak{g}^*//G) \xrightarrow{\sim} K^0(*//G) \simeq R(G).$$

Moreover, if $i\colon O_\lambda \to \mathfrak{g}^*$ is the inclusion of a regular coadjoint orbit, then we have pushforward maps

$$R(G) \simeq K^0(*//G) \longleftarrow K^{\dim(G)}(O_\lambda//G) \xrightarrow{i^!} K_c^{\dim(G)}(\mathfrak{g}^*//G)$$

with the property that $i^!$ is surjective, and that the left pushforward map agrees with $i^!$ up to the above Thom isomorphism.

This allows us to investigate the Dirac operator on the right, at the level of $\mathfrak{g}^*$ at the boundary of its Poisson sigma model, in terms of its pullback to the coadjoint orbits O_λ: we can pull back the spinor bundle to the orbit O_λ and push this to $*//G$. This corresponds to a part of the pushforward of the Dirac operator along the right vertical map $\mathfrak{g}^* \to \mathfrak{g}^*//G$.

It is computed in [FHT13] that the index of the restriction of the Dirac operator to the regular coadjoint orbit O_λ precisely produces the G-equivariant quantization of that symplectic manifold. This shows that the quantization of the Lie-Poisson manifold $\mathfrak{g}^*$ serves as a kind of 'universal orbit method': we can obtain the quantization of each coadjoint orbit O_λ, together with its action of G, by quantizing O_λ as a defect of the Poisson sigma model. The quantization of defects of the Lie-Poisson sigma model produces the representations that we get from the orbit method.

5.2.3 In Chern-Simons theory transgressed to dimension 2

Recall that three dimensional Chern-Simons theory (at least without differential refinement) is the 3-dimensional pQFT classified by the map of stacks

$$\lambda\colon \mathbf{B}G \to \mathbf{B}^3U(1)$$

where G is a compact connected, simply connected simple Lie group. The cohomology class of λ in $\pi_0\mathbf{H}(\mathbf{B}G, \mathbf{B}^3U(1)) \simeq \mathbb{Z}$ is called the *level* of the theory. We only consider the value of this pQFT on circles and two-dimensional cobordisms between them. In that case, the 2-fold correspondence diagram over $\mathbf{B}^3U(1)$ associated to a 2-dimensional cobordism Σ reduces to a single correspondence diagram of the form

$$\begin{tikzcd}[column sep=small] & \left[\Pi(\Sigma), \mathbf{B}G\right] \arrow[dl, "\rho_{\mathrm{in}}"'] \arrow[dr, "\rho_{\mathrm{out}}"] & \\ \left[\Pi(S^1), \mathbf{B}G\right]^{\times p} \arrow[dr, "\chi^{\times p}"'] \arrow[rr, phantom, "\Longrightarrow"] & & \left[\Pi(S^1), \mathbf{B}G\right]^{\times q} \arrow[dl, "\chi^{\times q}"] \\ & \mathbf{B}^2U(1) & \end{tikzcd} \tag{5.3}$$

The stacks that arise in the above diagram are the stacks of flat bundles, either over the circle or over Σ. However, these stacks carry a nontrivial smooth structure: a map $U \to [\Pi(S^1), \mathbf{B}G]$ classifies a G-bundle over $U \times S^1$ which is flat only along S^1. All these stacks are actually differentiable stacks: for example, an easy computation shows that $[\Pi(S^1), \mathbf{B}G] \simeq G//G$ is the quotient of G by its adjoint action. This follows from the fact that each flat G-bundle over S^1 is characterized uniquely by its holonomy, and that a gauge transformation acts on this holonomy by the adjoint action. We will denote the stack of flat G-bundles on Σ by $\flat\mathrm{GBund}(\Sigma)$.

The map $\chi : G//G \to \mathbf{B}^2U(1)$ is known as the WZW-gerbe on the Lie group G. It is obtained by transgressing λ over the circle. Since G is a compact Lie group, all maps in the above diagram are proper maps of differentiable stacks, so we can apply the theory from section 3 to obtain a diagram of K-theory spectra:

$$K^{*+\chi}(G//G)^{\otimes p} \simeq K^{*+\chi^{\times p}}\big(G//G^{\times p}\big) \xrightarrow{\rho_{\mathrm{in}}^*} K^{*+\rho_{\mathrm{in}}^*\chi^{\times p}}\big(\flat\mathrm{GBund}(\Sigma)\big) \xleftarrow{\rho_{\mathrm{out}}^*} K^{*+\chi}(G//G)^{\otimes q}.$$

As was shown in [FHT10], one can form the pushforward map along ρ_{out}. The target of this pushforward map carries an extra twist, besides the twist by $\chi^{\times q}$. To compensate for this, one adds an extra twist to all twists χ, given by the dual Coxeter number of the Lie group G times a generator for $\pi_0(G//G, \mathbf{B}^2U(1)) \simeq \mathbb{Z}$. The resulting twist is denote by τ in the literature. Furthermore, one shifts the degree of the K-theory spectra by the dimension of G.

After introducing this extra twist, the fundamental result of [FHT10] is that we can consistently orient the maps ρ_{out}:

Theorem 5.2.13. *There is a natural orientation on the maps ρ_{out}, so that the maps of K-theory spectra*

$$\rho_{\mathrm{out}}^! \circ \rho_{\mathrm{in}}^* \colon K^{*+\tau+\dim(G)}(G//G)^{\otimes p} \to K^{*+\tau+\dim(G)}(G//G)^{\otimes q}$$

induced by pulling and pushing over the differentiable stack $\flat\mathrm{GBund}(\Sigma)$ constitute a functor from the category of 1-dimensional closed manifolds and two-dimensional oriented cobordisms between them to the category of KU-modules.

This functor describes the quantization of the reduction of 3d Chern-Simons theory to 2d cobordisms between circles. In particular, passing to the zeroth homotopy groups, we get a two-dimensional (non-extended, oriented) TQFT, which is classified by the structure of a Frobenius algebra on $K^{\tau+\dim(G)}(G//G)$.

Example 5.2.14. For example, the flat G-bundles on the pair of pants are given by $(G\times G)//G$ with the diagonal action: indeed, by drawing the pair of pants as a closed disc with two smaller discs removed, one sees that the flat bundles are classified by their holonomy around each of these two discs. The correspondence diagram induced by the pair of pants is then given by

$$G//G \times G//G \longleftarrow (G\times G)//G \xrightarrow{\mu} G//G$$

where the right map is given by multiplication. We have omitted the circle 2-bundles. By pulling and pushing along this diagram, we obtain the product of the Frobenius structure on $K^{\tau+\dim(G)}(G//G)$.

The famous Freed-Hopkins-Teleman theorem from [FHT13] relates this Frobenius algebra structure on $K^{\tau+\dim(G)}(G//G)$ to the Verlinde ring of the tensor category of loop group representations of level $\lambda \in H^4(BG,\mathbb{Z}) \simeq \mathbb{Z}$:

Theorem 5.2.15. *There is a natural isomorphism between $K^{\tau+\dim(G)}(G//G)$ and the Verlinde ring of the tensor category of loop group representations of level λ. Moreover, this identifies the natural Frobenius algebra structure on the Verlinde ring with the Frobenius algebra structure on $K^{\tau+\dim(G)}(G//G)$ induced by the above TQFT.*

This relates the pull-push quantization of the dimensional reduction of Chern-Simons theory to the purely categorical definition of (quantum) Chern-Simons theory via the Reshetikhin-Turaev construction: we see that the pull-push quantization produces the Grothendieck ring of the modular tensor category of loop group representations.

5.2.4 D-brane charges

We now consider a boundary theory in the literal sense: let X be a compact smooth manifold (which we think of as spacetime) and consider a string propagating in X. The string couples to a background field on X, called the B-field in the string theory literature, which is gives by a circle 2-bundle $\chi : X \to \mathbf{B}^2U(1)$. If the string world sheet is given by map $\phi\colon \Sigma \to X$ from a closed two-dimensional (oriented) surface, the the exponentiated action functional is given by

$$\exp(iS)(\phi) = \int_\Sigma \phi^*\chi.$$

This really describes a *topological* string moving on X: we do not include the kinetic terms that depend on a conformal structure on Σ. From the point of view of superstring theory, the spacetime X is supposed to be 10-dimensional, but since we only look at the topological part of the action, we do not need such an assumption.

Remark 5.2.16. More precisely, the B-field on X is given by a circle 2-bundle carrying a connection $(\chi, B)\colon X \to \mathbf{B}^2U(1)_{\mathrm{conn}}$. In the literature, it is usually the connection that is called the B-field, although one should also include the bundle. The action functional over a closed surface is then obtained by computing the higher holonomy of this connection along the surface.

In our description of a pQFT using spans, a map from Σ into X is really given by a map $\Pi(\Sigma) \to X$ from the discrete ∞-groupoid $\Pi(\Sigma)$ into X. Pulling back χ along this map gives a $U(1)$-bundle on $\Pi(\Sigma)$, which corresponds to a *flat* $U(1)$-bundle on Σ. Then the above integral is the holonomy of this flat bundle over Σ.

The extended action functional can therefore be presented by the map $\chi : X \to \mathbf{B}^2U(1)$. Now a string moving in X can have a (marked) boundary, which we confine to move in a closed submanifold $i\colon Q \to X$. In other words, some of the endpoints of the string are forced to move inside the submanifold Q, which is called a *D*-brane. These endpoint behave as (topological) particles moving along Q.

As we have seen in section 5.1, such topological particles moving on Q can be given an action functional from a line bundle ξ on Q. We therefore get two contributions to the action functional: one from the string moving in X and one from its endpoints moving in Q. In order for the resulting action functional to make sense, meaning that it assigns a number to each map $\phi\colon \Sigma \to X$ such that $\phi(\partial\Sigma) \subseteq Q$, we should not have a line bundle ξ on Q, but rather a line bundle ξ, *twisted* by the circle 2-bundle $i^*\chi$. This is known as the *Freed-Witten anomaly cancellation condition* in string theory.

From our perspective, it just says that we obtain a true boundary prequantum theory, specified by the diagram

$$\begin{tikzcd}
 & Q \arrow[dl] \arrow[dr, "i"] \arrow[dd, "i^*\chi"] & \\
{*} \arrow[dr] \arrow[r, phantom, "\overset{\xi}{\Longrightarrow}"] & {} \arrow[r, phantom, "\Longrightarrow"] & X \arrow[dl, "\chi"] \\
 & \mathbf{B}^2U(1) &
\end{tikzcd}$$

where the left 2-cell describes the $i^*\chi$-twisted line bundle over the D-brane Q. Again, this gives rise to a diagram in KK-theory

$$\mathbb{C} \xrightarrow{\xi} C^*_{i^*\chi}(Q) \xleftarrow{i^*} C^*_{\chi}(X)$$

The pullback map i^* also gives rise to a pullback map $i^*\colon C^*_{-\chi-\tau_X}(X) \to C^*_{-i^*\chi-i^*\tau_X}(Q)$. As we have seen in section 4.2.4, the KK-dual algebras of these twisted function algebras are given by taking the inverse twist and further twisting this by the stable normal bundles (via their third integral Stieffel-Whitney classes). Then we obtain the dual map

$$i_!\colon C^*_{i^*\chi+i^*\tau_X-\tau_Q}(Q) \to C^*_{\chi}(X)$$

The stable bundle $i^*\tau_X - \tau_Q$ is equivalent to the normal bundle ν_Q of Q inside X. Since the integral Stieffel-Whitney behave additively with respect to the direct sum of vector bundles, we have that the twist by $i^*\tau_X - \tau_Q$ is equivalently just the twist by the normal bundle ν_Q. A choice of spin^c-structure on the normal bundle ν_Q then gives rise to an equivalence in KK-theory

$$C^*_{i^*\chi}(Q) \xrightarrow{\sim} C^*_{i^*\chi+\nu_Q}(Q)$$

so that we can form the composite with ξ. This gives us a twisted K-theory class on spacetime X

$$[i_!\xi] \in K^{*+\chi}(X)$$

which is called the D-brane charge in the literature.

Remark 5.2.17. If we would not start from a prequantum boundary theory, then the map $i_!$ allows us to push any element $[\xi] \in K^{*+i^*\chi+\nu_Q}(Q)$. Such a twisted K-theory class ξ is called a Chan-Paton gauge field in the literature. If we have such a ξ, then the D-brane is said to satisfy the Freed-Kapustin-Witten anomaly cancellation condition.

5.3 Cohomological quantization in tmf

In the three-dimensional case, we can use the string orientation of tmf to form pushforward maps. We have no tools to explicitly construct such maps, but the general perspective already provides an interesting example in the context of string theory and M-theory.

For example, it was argued by Witten [Wit88] that for a closed string manifold X, the image of the unit element under the pushforward map in tmf

$$\mathrm{tmf}^*(X) \longrightarrow \mathrm{tmf}$$

can be interpreted as the index as the S^1-equivariant index of the Dirac operator on loop space LX. As such, it serves as the partition function of the 'heterotic string' moving over the string manifold M. Because of this string-theoretic perspective due to Witten, this resulting tmf-class, which we also discussed in example 4.1.33 is called the Witten genus of the string manifold X.

More generally, we can consider a string moving on a manifold which carries a certain *twisted* string structure. In this case, the string is interpreted as the boundary of a 2-brane moving in a larger spacetime.

5.3.1 String moving at the boundary of a 2-brane

In analogy with the string moving in a spacetime X, we can consider a 2-brane, called the M2-brane, moving in a spacetime Y. Such a situation has been studied in relation to string theory by Hořava-Witten [HW96]: in the case where Y is an 11-dimensional spin manifold carrying an action of $\mathbb{Z}_2 = \mathbb{Z}/2$ and X is its 10-dimensional $\mathbb{Z}_2$-fixed locus (sometimes called the M9-brane), then the boundary of the 2-brane ending on X is supposed to describe the heterotic string moving on the 10-dimensional spacetime X.

Moreover, the 11d spacetime Y comes equipped with a circle 3-bundle

$$\chi \colon Y \to \mathbf{B}^3 U(1)$$

in analogy with the B-field that arises in string theory. This is also called the C-field, which gives the topological part of the M2-brane moving over Y. More precisely, this bundle is supposed to satisfy some twisted form of equivariance with respect to the $\mathbb{Z}_2$-action, which can be encoded by saying that the C-field is a lift

$$\begin{tikzcd} & \mathbf{B}\mathrm{Aut}(\mathbf{B}^2 U(1)) \arrow[d] \\ Y/\!/\mathbb{Z}_2 \arrow[r] \arrow[ur, dotted] & \mathbf{B}\mathbb{Z}_2 \end{tikzcd}$$

where the bottom map classifies the $\mathbb{Z}_2$-action on Y and where $\mathrm{Aut}(\mathbf{B}^2 U(1))$ is a certain (higher) central extension of the group $\mathbb{Z}_2$ by $\mathbf{B}^2 U(1)$ called the automorphism 3-group of $\mathbf{B}^2 U(1)$. This kind of datum is also refered to as a 'higher orientifold' in the literature. When pulled back from the quotient $Y/\!/\mathbb{Z}_2$ to Y itself, the $\mathbb{Z}_2$-bundle vanishes and we obtain the above circle 3-bundle χ.

The C-field interacts with the gravity field on Y, which is given by a spin bundle $Y \to \mathbf{B}\mathrm{Spin}$. It turns out that the C-field χ should come equipped with an equivalence

$$\chi \simeq \frac{1}{2}\mathbf{p}_1(Y) + 2\mathbf{a}$$

where $\frac{1}{2}\mathbf{p}_1(Y) \colon Y \to \mathbf{B}^3 U(1)$ is the (smooth) fractional Pontryagin class associated to the spin structure on Y, while $\mathbf{a}$ is the characteristic class in $\mathbf{B}^3 U(1)$ of an auxiliary E_8-field on Y. See [FSS12] for a extensive treatment of the above setup from our stacky perspective.

Remark 5.3.1. As for the B-field, the C-field is really supposed to be a differential refinement of the definition we take here, i.e. the bundles also carry a connection. In that case the above maps are refined to their Chern-Simons invariants, which map into the stack $\mathbf{B}^3 U(1)_{\mathrm{conn}}$ describing circle 3-bundles with connection (see [FSS12] for more discussion).

The $\mathbb{Z}_2$-equivariance of the theory forces the C-field to trivialize over the fixed locus X of the $\mathbb{Z}_2$-action. Since we only know how to form the pushforward in tmf over smooth manifolds, we will forget about the $\mathbb{Z}_2$-action and just consider the circle 3-bundle χ, so that we have a boundary diagram of the form

$$\begin{tikzcd} & X \arrow[dl] \arrow[dd] \arrow[dr, "i"] & \\ {*} \arrow[r, phantom, "\xRightarrow{B}"] \arrow[dr] & {} \arrow[r, phantom, "i^*\chi \Longrightarrow"] & Y \arrow[dl, "\chi"] \\ & \mathbf{B}^3 U(1) & \end{tikzcd}$$

The trivialization of $i^*\chi$ over X gives rise to a twisted circle 2-bundle over the spacetime X, a twisted B-field. The condition that such a twisted B-field exists on X turns out to be the *Green-Schwarz anomaly cancellation condition*, which is precisely the anomaly cancellation condition that appears in in heterotic string theory, as was shown in [SSS12].

Now suppose that $\frac{1}{2}\mathbf{p}_1(X) \simeq i^*\chi$, which means that the spacetime X carries a twisted string structure, with the twist given by $i^*\chi = i^*\frac{1}{2}\mathbf{p}_1(Y) + i^*2\mathbf{a}$. As was pointed out in [ABG10], we can then form the pushforward in twisted tmf: indeed, the twist $i^* \frac{1}{2}\mathbf{p}_1(Y) - \frac{1}{2}\mathbf{p}_1(X)$ induced by the normal bundle ν of X in Y is equivalent to $-i^* 2\mathbf{a}$, so that we can apply the Thom construction to obtain

$$tmf^*(X) \simeq tmf^{*+i^* 2\mathbf{a}}\big(X^\nu\big) \to tmf^{*+2\mathbf{a}}(Y).$$

By analogy with the discussion of the D-brane charge in the previous section, the image of this map might be called the M9-brane charge. Twisted tmf charges of this kind induced from twisted string structures on branes in string/M-theory were originally suggested in [Sat10], by which the discussion of twisted Umkehr maps by ABG was motivated.

If we replace Y by the point, then we can form the pushforward map under the assumption that $\frac{1}{2}\mathbf{p}_1(X) \simeq 0$, which means that X is a string manifold. Then the resulting class in tmf is the Witten genus we saw above and in example 4.1.33, presenting the partition function of the heterotic string moving over a string manifold.

6 Outlook: functoriality of pull-push quantization

We have seen how a single correspondence diagram classifies a boundary theory for a local pQFT, and how one can quantize that boundary theory by using the pull-push formalism in twisted generalized cohomology. The result can then be interpreted as the partition function of the boundary theory. Many phenomena from the literature appear as these kind of pull-push quantizations of boundary theories: we have seen the K-theoretic quantization of a symplectic manifold, the D-brane charge as the quantization of a particle moving at the end of a string and the M9-brane charge as the quantization of a string moving at the boundary of a 2-brane.

The quantization of a symplectic manifold naturally extends to the quantization of a Poisson manifold as the boundary of its non-perturbative Poisson sigma model. This gives a cohomological formulation of the quantization of Poisson manifolds proposed in [Haw08], which fixes the issue that the quantization proposed there is not invariant under Morita equivalence. In particular, applying this K-theoretic quantization to Lie-Poisson structures gives an interpretation of the result from [FHT13] as providing a 'universal orbit method'.

On the other hand, the full structure of a local pQFT is also described in terms of correspondences: we have seen that a local prequantum field theory is decribed by a functor

$$\mathrm{Bord}_n \to \mathrm{Corr}_n(\mathbf{H}_{/\mathbf{B}^n U(1)})$$

from the (∞, n)-category of cobordisms to the (∞, n)-category of n-fold correspondences in the slice topos over the moduli stack of smooth circle n-bundles (where in principle one can replace $\mathbf{B}^n U(1)$ by a more general abelian ∞-group, like the group of flat line bundles in ordinary cohomology or super line 2-bundles in K-theory).

There is currently no formulation of quantization that allows for the quantization of such an (∞, n)-functor. The results from this thesis point to a possible general description of quantization in the fully extended setting: the quantization of a local pQFT requires the extension of the quantization of a single span to a *functorial* quantization of correspondence diagrams. In this outlook section, we will make some remarks on how the pull-push quantization decribed in the previous sections might fit into the broad scheme of quantizing the full pQFT.

We have seen two examples where the functoriality of the pull-push quantization has been established: in the context of the string topology operations 5.1.1, due to [CG04], and in the case of the transgression of Chern-Simons theory 5.2.3 to 2d-cobordisms between 1d closed manifolds [FHT10]. In both cases, the pull-push quantization of a composition of two correspondences is just the quantization of the composite span. The pull-push quantization then gave rise to a (not fully extended) two-dimensional TQFT.

In general, a functorial description of pull-push quantization should eventually give rise to a well-defined functor

$$\mathrm{Corr}_1(\mathbf{H}_{/\mathbf{B}^n U(1)})^{\mathrm{nice,or}} \longrightarrow \mathrm{Corr}_1(\mathbf{H}_{/BGL_1(R)})^{\mathrm{nice,or}} \longrightarrow R\mathrm{Mod}$$

Here the domain of the functor is supposed to be a category of suitable correspondences: the map along which we want to push should have small fibers and should carry a *chosen* orientation in R-cohomology.

Now there are roughly two ingredients that are needed to construct such a functor: first of all, we should require the formation of pushforward maps to be compatible with the composition. We have seen that the pushforward in K-theory satisfies such a condition (propositions 4.2.5 and 4.2.13). In general, this seems like a very reasonable result. Indeed, suppose we have two maps $f\colon X_1 \to X_2$ and $g\colon X_2 \to X_3$, both having small fibers and carrying an R-orientation. Then the R-orientations on f and g determine a canonical orientation on the composite gf, such that the pushforward along gf with respect to this orientation agrees with $g^! f^!$.

The second, more fundamental ingredient in the development of such a functorial pull-push construction is the so-called *Beck-Chevalley condition.* To see where this arises, suppose we have a composition of two trajectories

$$
\begin{tikzcd}[column sep=tiny]
 & & \mathrm{Fields}(\Sigma_1 \coprod_{\partial_{\mathrm{out}}\Sigma^1} \Sigma^2) \arrow[dl, "i_1"'] \arrow[dr, "f_2"] & & \\
 & \mathrm{Fields}(\Sigma^1) \arrow[dl] \arrow[dr, "f_1"] & & \mathrm{Fields}(\Sigma^2) \arrow[dl, "i_2"'] \arrow[dr] & \\
\mathrm{Fields}(\partial_{\mathrm{in}}\Sigma^1) & & \mathrm{Fields}(\partial_{\mathrm{out}}\Sigma^1) = \mathrm{Fields}(\partial_{\mathrm{in}}\Sigma^2) & & \mathrm{Fields}(\partial_{\mathrm{out}}\Sigma^2)
\end{tikzcd}
$$

where the top square is homotopy cartesian. We want to make sure that pulling and pushing along the top correspondence agrees with pulling and pushing along the bottom two correspondences, and taking their composite. This is now guaranteed by the Beck-Chevalley condition, which says that

$$i_2^* f_1^! = f_2^! i_1^*.$$

This condition also appears in the context of Grothendieck's six operations, which provides an abstract categorical formalism encorporating pullback and pushforward maps (see for example [CD09] for a discussion).

In the context of K-theory, the Beck-Chevalley condition has been verified for manifolds and foliation groupoids in [CS84] and in the G-equivariant setting in [EM09]. In those cases, one can actually realize the above functor from correspondences (of G-manifolds) over $\mathbf{B}^2U(1)$ to KU-modules. The results of [CG04] and [FHT10] can then be interpreted as stating that the transgressed prequantum field theories described in 5.1.1 and 5.2.3 give rise to functors

$$\mathrm{Bord}_{1,2} \longrightarrow \mathrm{Corr}_1(\mathbf{H}_{/BGL_1(KU)})^{\mathrm{nice,or}}$$

from the category of 2d cobordisms between 1d closed manifolds (with nonempty ingoing boundary for the string operations) to the category of spaces equipped with a twist in K-theory, with K-*oriented* spans between them. More precisely, there is a consistent way of picking K-theory orientations on the maps

$$\mathrm{Fields}(\Sigma) \longrightarrow \mathrm{Fields}(\Sigma_{\mathrm{out}})$$

Together with the Beck-Chevalley condition, this is what allows us to construct the TQFTs from sections 5.1.1 and 5.2.3 by the pull-push quantization. To quantize a pQFT we therefore have to be able to pick a consistent orientation, which should somehow be induced from an orientation on the classifying stack of fields, just like an orientation on the manifold M defines an orientation on all maps

$$\mathrm{Map}\big(\Gamma_\Sigma, M\big) \to LM^{\times q}$$

in section 5.1.1. Further development of the pull-quantization procedure along these line therefore requires a functorial way of forming pushforward maps in (twisted) R-cohomology, satisfying the Beck-Chevalley condition, together with the construction of a 'consistent orientation' for our pQFT, which is given by a lift

$$
\begin{tikzcd}
 & \mathrm{Corr}_n(\mathbf{H}_{/\mathbf{B}^nU(1)})^{\mathrm{nice,or}} \arrow[d] \\
\mathrm{Bord}_n \arrow[r] \arrow[ur, dotted] & \mathrm{Corr}_n(\mathbf{H}_{/\mathbf{B}^nU(1)})^{\mathrm{nice}}
\end{tikzcd}
$$

to the category of n-fold spans carrying an R-orientation. In fact, one would additionally want to use a *smooth* ring spectrum $\mathbf{R}$ to present the smooth $\mathbf{R}$-cohomology of a stack, along the lines of section 3.1.5.

Since k-fold correspondence diagrams are essentially just cubical diagrams of spans, one might expect a functorial pull-push construction to give rise to a functor

$$\mathrm{Corr}_n(\mathbf{H}_{/\mathbf{B}^nU(1)}) \longrightarrow R\mathrm{Mod}^{\square^{\leq n}}$$

into the category of cubical diagrams with values in RMod. This naturally forms an (∞, n)-category where a k-morphism is given by a k-cube between $(k-1)$-cubes, at least for $k \leq n$. Then the quantization of a pQFT, endowed with a consistent orientation, would be given by the composition

$$\mathrm{Bord}_n \longrightarrow \mathrm{Corr}_n(\mathbf{H}_{/\mathbf{B}^n U(1)})^{\mathrm{nice,or}} \longrightarrow R\mathrm{Mod}^{\square^{\leq n}}$$

This assigns a linear space of states to the point, while to a k-dimensional cobordism Σ it assigns a k-dimensional propagator, consisting of propagators in k different directions describing how the quantum field can move in the k directions of Σ.

Of course, this is just a vague sketch and there are various problems that arise on the way. But the above program provides an interesting possible extension of the pull-push construction described in this text, which eventually might serve to fully quantize the prequantum field theories of interest in modern physics.

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