Derived n-plectic geometry:
towards non-perturbative BV-BFV quantisation and M-theory

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Joint work with Charles Young to appear soon

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Main approaches to make classical (and quantum) BV-theory precise in the literature:

1. *NQP*-**manifolds approach.** [Jurčo, Raspollini, Sämann, Wolf, ...]
   Algebra of classical observables is given by a Poisson dg-Lie algebra of functions on an *NQP*-manifold, i.e. a differential-graded manifold (dg-manifold) equipped with a (-1)-shifted symplectic form. (Equivalently, a symplectic L_∞-algebroid.)

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3. **Perturbative Algebraic Quantum Field Theory (pAQFT).** [Rejzner, ...]
   Algebra of observables is given by a net of locally convex topological Poisson $*$-algebras on spacetime.

Approaches (1) & (2) very close, (2) & (3) related by [Schenkel, Benini, ...]

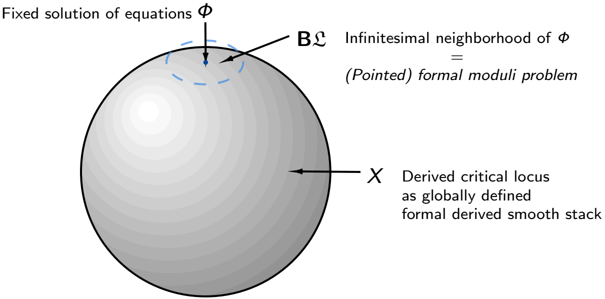
Motivation: towards global smooth BV-theory

Formal Moduli Problem: (algebraic) derived stack on Artinian dg-algebras, i.e.

$$F : \mathrm{dgArt}^{\leq 0} \longrightarrow \mathrm{sSet}$$

Artinian dg-algebras $\simeq$ algebras of function on "derived thickened points".

A (-1)-symplectic Formal Moduli Problem can be seen as the formal completion of a fully-fledged (-1)-symplectic derived stack at some given point.



Motivation: towards global smooth BV-theory

We have the following picture:

$$\text{Formal Moduli Problem} \longleftrightarrow \text{Perturbative physics}$$
$$\text{Formal derived smooth stack} \longleftrightarrow \text{Non-perturbative physics}$$

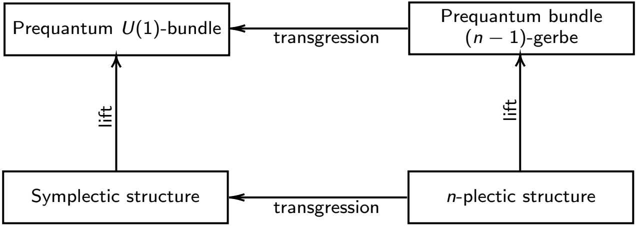
Example (Stack of G-bundles with connection)

$$\underbrace{[\Omega^1(M,\mathfrak{g})/\mathcal{C}^\infty(M,\mathfrak{g})]}_{L_\infty\text{-algebroid}} \neq \underbrace{\mathbf{Bun}_G^\nabla(M)}_{\text{stack of } G\text{-bundles}} := [M, \mathbf{B}G_{\mathrm{conn}}]$$

- M-theory includes (higher) gauge theories
  - Quantisation requires BV-theory, i.e. **derived geometry**
  - Finite (higher) gauge transformations and global properties require stacks, i.e. **higher geometry** (e.g. Aharonov-Bohm phase and magnetic charge for electromagnetic field)
- Moreover, we have global string (and M-)dualities and non-perturbative effects
- It's not totally clear how the 0-symplectic (BFV) structure at the boundary would fit in this derived geometric picture.

Motivation: higher geometric (pre)quantisation

***n*-plectic geometry** (or higher symplectic geometry) [Rogers, Baez, Saemann, Szabo, Bunk, Fiorenza, Schreiber, Sati, ...] naturally fits in the following picture:



Example (Closed string)

[Waldorf 2009]: transgression of a bundle gerbe on a smooth manifold M to a principal $U(1)$-bundle on the loop space $\mathcal{L}M = [S^1, M]$.

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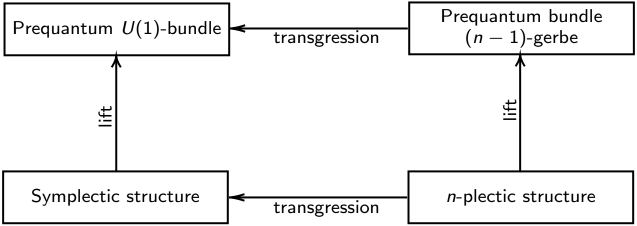

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- [Ševera 2000]: Courant 2-algebroid and Vinogradov n-algebroid are higher generalisations of the Poisson 1-algebroid (as symplectic L_∞-algebroids).
- [Rogers 2011], [Sämann, Ritter 2015]: relation between the L_∞-algebras of observables on n-plectic manifolds and Vinogradov n-algebroids.

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Geometry as theory of sheaves and stacks

- An ordinary geometric space can be encoded by its **functor of points**, i.e. a functor

$$\texttt{space} : \texttt{probing spaces}^{\mathrm{op}} \longrightarrow \texttt{sets}$$

which satisfies the sheaf condition.

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- In the same spirit, a higher geometric space can be defined as a **stack**, i.e. a functor

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which is fibrant-cofibrant respect to a certain simplicial model category structure.

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- A higher derived geometric space can be defined as a **derived stack**, i.e. a functor

$$\begin{matrix}\texttt{higher} \\ \texttt{derived space}\end{matrix} : \begin{matrix}\texttt{derived} \\ \texttt{probing spaces}\end{matrix}^{\mathrm{op}} \longrightarrow \infty\texttt{-groupoids}$$

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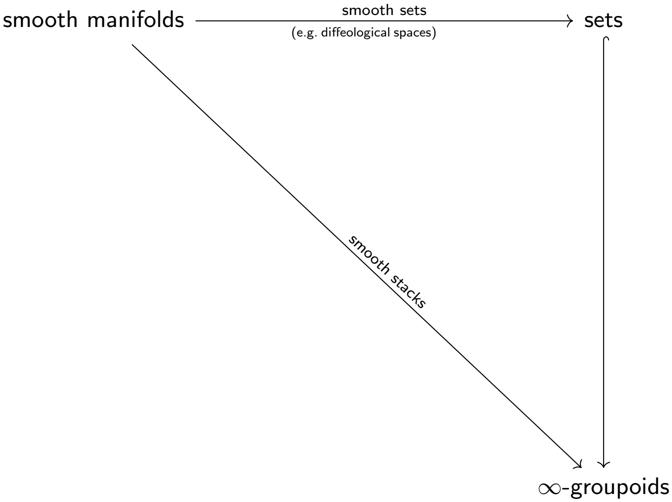
Family tree of smooth stacks



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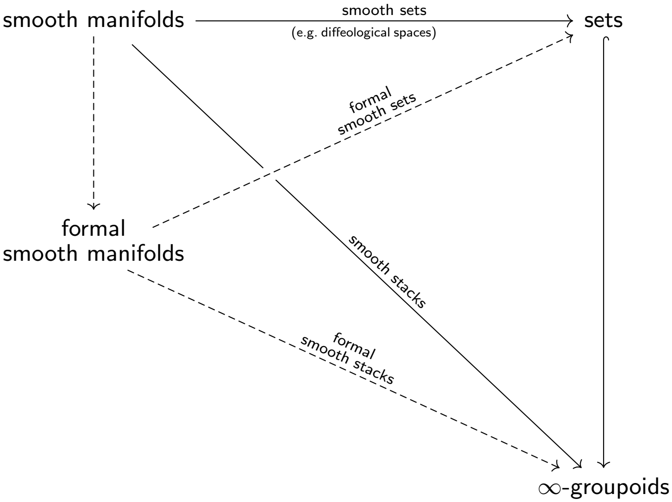

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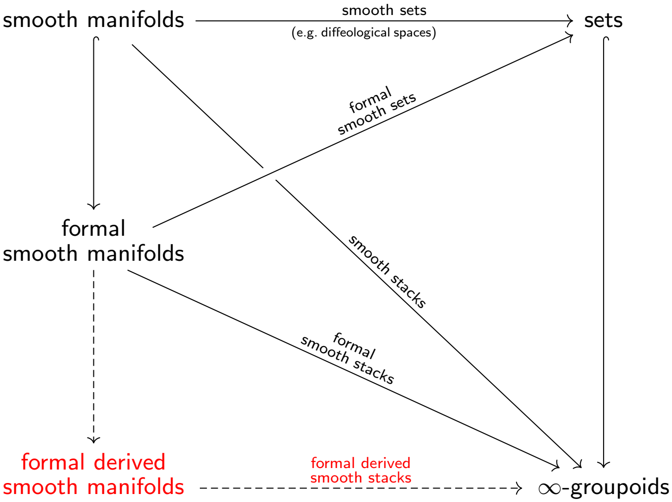

Formal derived smooth manifolds

Homotopy $\mathcal{C}^\infty$-algebras: simplicial $\mathcal{C}^\infty$-algebras with projective model structure, i.e.

$$\mathsf{hC^\infty Alg} \;\coloneqq\; [\Delta^{\mathrm{op}}, \mathsf{C^\infty Alg}]^{\circ}_{\mathrm{proj}},$$

where Δ is the simplex category.

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The following will be our effective definition of formal derived manifolds.

Theorem [Carchedi, Steffens 2019]

There is a canonical equivalence of $(\infty, 1)$-categories

$$\mathsf{dFMfd} \;\simeq\; \mathsf{hC^\infty Alg}^{\mathrm{op}}_{\mathrm{fp}}$$

between the $(\infty, 1)$-category of formal derived manifolds, and the opposite of the $(\infty, 1)$-category of homotopically finitely presented homotopy $\mathcal{C}^\infty$-algebras.

At an intuitive level, $U \in \mathsf{dFMfd}$ is a geometric object whose algebra of smooth function is a homotopically finitely presented homotopy $\mathcal{C}^\infty$-algebra modelled as

$$\mathcal{O}(U) \;=\; \left(\;\cdots\; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathcal{O}(U)_3 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathcal{O}(U)_2 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathcal{O}(U)_1 \substack{\longrightarrow\\ \longrightarrow} \mathcal{O}(U)_0 \right)$$

where each $\mathcal{O}(U)_i$ is an ordinary $\mathcal{C}^\infty$-algebra.

Formal derived smooth stacks

- We can define étale maps of formal derived smooth manifolds so that they truncate to local diffeomorphisms of ordinary manifolds.
- By using étale maps, we can make dFMfd into a $(\infty, 1)$-site.
- By [Toen, Vezzosi 2006], we can define formal derived smooth stacks by

$$\mathbf{dFSmoothStack} \;\coloneqq\; [\mathsf{dFMfd}^{\mathrm{op}}, \mathsf{sSet}]^{\circ}_{\mathsf{proj,loc}}.$$

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Formal derived smooth sets can be defined as those stacks whose underived-truncation happens to be an ordinary formal smooth set, i.e. as an element of the pullback

$$\mathbf{dFSmoothSet} \;\coloneqq\; \mathbf{dFSmoothStack} \times^{h}_{\mathbf{FSmoothStack}} \mathrm{FSmoothSet}$$

Thus, one has (co-)reflective embeddings

$$\begin{array}{ccc}
\mathbf{dFSmoothStack} & \underset{t_0}{\overset{\longleftarrow}{\longrightarrow}} & \mathbf{FSmoothStack} \\
\uparrow\;\downarrow{\scriptstyle \tau_0} & & \uparrow\;\downarrow{\scriptstyle \tau_0} \\
\mathbf{dFSmoothSet} & \underset{t_0}{\overset{i}{\rightleftarrows}} & \mathrm{FSmoothSet}.
\end{array}$$

On an affine derived formal smooth set $\mathbb{R}\mathrm{Spec}(R)$, these maps amount to

$$t_0\mathbb{R}\mathrm{Spec}(R) \simeq \mathrm{Spec}(\pi_0 R), \qquad i\mathrm{Spec}(R) \simeq \mathbb{R}\mathrm{Spec}(R)$$

Derived differential cohesion

Let $\mathsf{C}^\infty\mathsf{Alg}^{\mathrm{red}}$ be the sub-category of reduced $\mathcal{C}^\infty$-algebras, i.e. with no non-zero nilpotent elements. The reduction functor is defined by

$$\begin{aligned} (-)^{\mathrm{red}} : \mathsf{hC}^\infty\mathsf{Alg} &\longrightarrow \mathsf{C}^\infty\mathsf{Alg}^{\mathrm{red}} \\ R &\longmapsto R^{\mathrm{red}} := \pi_0 R/\mathfrak{m}_{\pi_0 R} \end{aligned}$$

where $\mathfrak{m}_{\pi_0 R} \subset \pi_0 R$ is the ideal of nilpotent elements of $\pi_0 R$.
This is right-adjoint to the natural embedding, i.e.

$$\mathsf{C}^\infty\mathsf{Alg}^{\mathrm{red}}_{\mathrm{fp}} \underset{\iota^{\mathrm{red}}}{\overset{(-)^{\mathrm{red}}}{\rightleftarrows}} \mathsf{hC}^\infty\mathsf{Alg}_{\mathrm{fp}}.$$

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These give rise to a quadruplet of adjoint functors:

$$\mathbf{dFSmoothStack} \;\begin{array}{c} \xleftarrow{\;\iota^{\mathrm{red}}_{!}\;} \\ \xrightarrow{\;\iota^{\mathrm{red}*}\simeq(-)^{\mathrm{red}}_{!}\;} \\ \xleftarrow{\;\iota^{\mathrm{red}}_{*}\simeq(-)^{\mathrm{red}*}\;} \\ \xrightarrow{\;(-)^{\mathrm{red}}_{*}\;} \end{array}\; \mathbf{SmoothStack},$$

where $\mathbf{SmoothStack} := \mathbf{Stack}(\mathsf{Mfd})$ is the $(\infty,1)$-topos of (non-formal) smooth stacks, i.e. of stacks on ordinary smooth manifolds.

This quadruplet is a differential cohesion structure, as defined by [Schreiber 2013]:

$$\mathbf{dFSmoothStack} \;\underset{(-)^{\mathrm{red}}_*}{\overset{\iota^{\mathrm{red}}_!}{\substack{\longleftarrow \\ \xrightarrow{\iota^{\mathrm{red}*}\simeq(-)^{\mathrm{red}}_!} \\ \xleftarrow{\iota^{\mathrm{red}}_*\simeq(-)^{\mathrm{red}*}} \\ \longrightarrow}}}\; \mathbf{SmoothStack} \;\substack{\xrightarrow{\Pi} \\ \xleftarrow{\mathrm{Disc}} \\ \xrightarrow{\Gamma} \\ \xleftarrow{\mathrm{CoDisc}}}\; \infty\mathbf{Grpd},$$

On an affine derived formal smooth set $\mathbb{R}\mathrm{Spec}(R)$, the crucial maps amount to

$$\iota^{\mathrm{red}}_!\mathrm{Spec}(R) \simeq \mathbb{R}\mathrm{Spec}(R) \qquad \iota^{\mathrm{red}*}\mathbb{R}\mathrm{Spec}(R) \simeq \mathrm{Spec}(R^{\mathrm{red}}),$$

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Infinitesimal shape modality

$$\begin{aligned} \Im : \mathbf{dFSmoothSet} &\longrightarrow \mathbf{dFSmoothSet} \\ X &\longmapsto \iota_*^{\mathrm{red}} \circ \iota^{\mathrm{red}*}(X). \end{aligned}$$

Adjunction $\iota^{\mathrm{red}*} \dashv \iota_*^{\mathrm{red}}$ implies that there is an adjunction unit (infinitesimal shape unit):

$$\mathfrak{i}_X : X \longrightarrow \Im(X)$$

Derived infinitesimal disks and jet bundles

Thanks to differential cohesion, we can do differential geometry on formal derived smooth sets, i.e. we can extend results of [Khavkine, Schreiber].

Roughly, this allows us to define derived infinitesimal disks by

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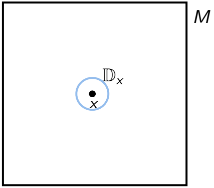

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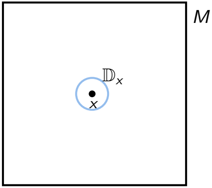


This allows us to study a number of geometric objects, including jet bundles:

$$\mathrm{Jet}_M \,:\, E \longmapsto \mathrm{Jet}_M E := (\mathrm{i}_M)^*(\mathrm{i}_M)_* E.$$

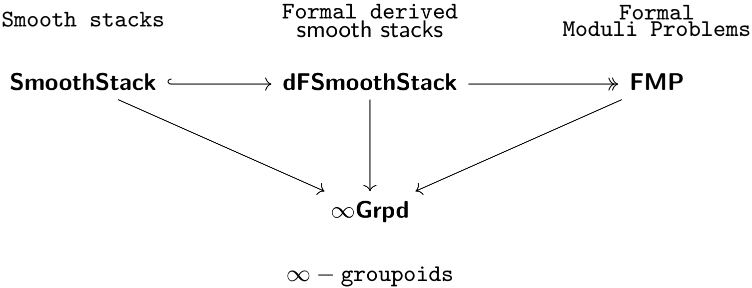
which are designed to satisfy the property

$$(\mathrm{Jet}_M E)_x \simeq \mathbf{\Gamma}(\mathbb{D}_x, E)$$

at any point $x \in M$.

Formal moduli problems as infinitesimal cohesion

Let **FMP** be the $(\infty, 1)$-category of Formal Moduli Problems, which can be seen as formal derived stacks on derived infinitesimal disks.



Differential forms

The complex of p-forms on a formal derived smooth set is

$$\mathrm{A}^p(X) \coloneqq \mathbb{R}\Gamma(X, \wedge^p_{\mathbb{O}_X}\mathbb{L}_X)$$

This gives rise to a bi-complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow Q & & \downarrow Q & & \downarrow Q & & \\
\mathrm{A}^0(X)_{-2} & \xrightarrow{\mathrm{d}_{\mathrm{dR}}} & \mathrm{A}^1(X)_{-2} & \xrightarrow{\mathrm{d}_{\mathrm{dR}}} & \mathrm{A}^2(X)_{-2} & \xrightarrow{\mathrm{d}_{\mathrm{dR}}} & \cdots \\
\downarrow Q & & \downarrow Q & & \downarrow Q & & \\
\mathrm{A}^0(X)_{-1} & \xrightarrow{\mathrm{d}_{\mathrm{dR}}} & \mathrm{A}^1(X)_{-1} & \xrightarrow{\mathrm{d}_{\mathrm{dR}}} & \mathrm{A}^2(X)_{-1} & \xrightarrow{\mathrm{d}_{\mathrm{dR}}} & \cdots \\
\downarrow Q & & \downarrow Q & & \downarrow Q & & \\
\mathrm{A}^0(X)_{0} & \xrightarrow{\mathrm{d}_{\mathrm{dR}}} & \mathrm{A}^1(X)_{0} & \xrightarrow{\mathrm{d}_{\mathrm{dR}}} & \mathrm{A}^2(X)_{0} & \xrightarrow{\mathrm{d}_{\mathrm{dR}}} & \cdots
\end{array}$$

$$\text{with} \quad \mathrm{d}_{\mathrm{dR}}^2 = Q^2 = \mathrm{d}_{\mathrm{dR}} \circ Q + Q \circ \mathrm{d}_{\mathrm{dR}} = 0$$

Closed differential forms

The complex of closed p-forms on a formal derived smooth set is

$$\mathrm{A}^p_{\mathrm{cl}}(X) \ \coloneqq \ \Big(\prod_{n\geq p} \mathrm{A}^n(X)[-n]\Big)[p]$$

with total differential $\mathrm{d}_{\mathrm{dR}} + Q$.

Definition (Closed form)

An *n-shifted closed p-form* on a derived formal smooth set X is defined as an n-cocycle $(\omega_i) \in \mathrm{Z}^n\mathrm{A}^p_{\mathrm{cl}}(X)$, i.e. as an element $\omega \in \mathrm{A}^p_{\mathrm{cl}}(X)$ such that $(\mathrm{d}_{\mathrm{dR}} + Q)\omega = 0$.

In other words, an n-cocycle in $\mathrm{A}^p_{\mathrm{cl}}(X)$ is given by a formal sum $\omega = (\omega_p + \omega_{p+1} + \dots)$, where each form $\omega_i \in \mathrm{A}^i(X)$ is an element of degree $n + p - i$, satisfying the equations

$$\begin{aligned} Q\omega_p \qquad\qquad\qquad &= 0, \\ \mathrm{d}_{\mathrm{dR}}\omega_p \quad + Q\omega_{p+1} &= 0, \\ \mathrm{d}_{\mathrm{dR}}\omega_{p+1} + Q\omega_{p+2} &= 0, \\ &\ \vdots \end{aligned}$$

or, more compactly, $(\mathrm{d}_{\mathrm{dR}} + Q)\omega = 0$.

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Ordinary n-plectic geometry

Definition (Ordinary n-plectic structure)

Given a formal smooth set $X \in \mathrm{SmoothSet}$, an *n-plectic structure* on X is a closed differential $(n+1)$-form $\Omega \in \Omega^{n+1}_{\mathrm{cl}}(X)$ such that the induced map

$$\Omega^{\sharp} \,:\, T_X \longrightarrow \wedge^n T^*_X$$

is a monomorphism.

Example (Symplectic structure)

A symplectic structure is a 1-plectic structure.

Poisson L_∞-algebra of observables

[Rogers 2011] Define Hamiltonian forms by

$$\Omega^{n-1}_{\mathrm{Ham}}(X) \;\coloneqq\; \left\{\alpha \in \Omega^{n-1}(X) \,\middle|\, \iota_{V_\alpha}\Omega = \mathrm{d}_{\mathrm{dR}}\alpha\right\}$$

We call V_α is the Hamiltonian vector of α.

The differential graded vector space

$$\mathrm{Ham}(X,\Omega) \;=\; \left(\mathcal{C}^\infty(X) \xrightarrow{\mathrm{d}_{\mathrm{dR}}} \Omega^1(X) \xrightarrow{\mathrm{d}_{\mathrm{dR}}} \dots \xrightarrow{\mathrm{d}_{\mathrm{dR}}} \Omega^{n-2}(X) \xrightarrow{\mathrm{d}_{\mathrm{dR}}} \Omega^{n-1}_{\mathrm{Ham}}(X)\right)$$

equipped with brackets for all $k > 1$:

$$\ell_1(\alpha) \;=\; \begin{cases} 0 & \text{if } |\alpha| = 0, \\ \mathrm{d}_{\mathrm{dR}}\alpha & \text{if } |\alpha| \neq 0, \end{cases}$$

$$\ell_k(\alpha_1,\dots,\alpha_k) \;=\; \begin{cases} (-1)^{\binom{k+1}{2}}\iota_{V_{\alpha_1}}\cdots\iota_{V_{\alpha_k}}\Omega & \text{if } |\alpha_1\otimes\cdots\otimes\alpha_k| = 0, \\ 0 & \text{if } |\alpha_1\otimes\cdots\otimes\alpha_k| \neq 0, \end{cases}$$

is an L_∞-algebra.

Variational bi-complex

On the jet bundle there is a canonical splitting horizontal/vertical

$$\mathrm{d}_{\mathrm{dR}} \;=\; \mathrm{d}_{\mathrm{h}} + \mathrm{d}_{\mathrm{v}}$$

which gives rise to the *variational bi-complex* [Anderson 1989], i.e.

$$\begin{array}{ccccccccc}
\mathcal{O}(\mathrm{Jet}E) & \xrightarrow{\mathrm{d_h}} & \Omega^{1,0}(\mathrm{Jet}E) & \xrightarrow{\mathrm{d_h}} & \Omega^{2,0}(\mathrm{Jet}E) & \xrightarrow{\mathrm{d_h}} & \cdots & \xrightarrow{\mathrm{d_h}} & \Omega^{m,0}(\mathrm{Jet}E) \\
\downarrow{\scriptstyle \mathrm{d_v}} & & \downarrow{\scriptstyle \mathrm{d_v}} & & \downarrow{\scriptstyle \mathrm{d_v}} & & & & \downarrow{\scriptstyle \mathrm{d_v}} \\
\Omega^{1,0}(\mathrm{Jet}E) & \xrightarrow{\mathrm{d_h}} & \Omega^{1,1}(\mathrm{Jet}E) & \xrightarrow{\mathrm{d_h}} & \Omega^{1,2}(\mathrm{Jet}E) & \xrightarrow{\mathrm{d_h}} & \cdots & \xrightarrow{\mathrm{d_h}} & \Omega^{m,1}(\mathrm{Jet}E) \\
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\downarrow{\scriptstyle \mathrm{d_v}} & & \downarrow{\scriptstyle \mathrm{d_v}} & & \downarrow{\scriptstyle \mathrm{d_v}} & & & & \downarrow{\scriptstyle \mathrm{d_v}} \\
\vdots & & \vdots & & \vdots & & & & \vdots
\end{array}$$

Pre-symplectic current of a field theory

Consider a Lagrangian density $\mathscr{L} \in \Omega^{m,0}(\mathrm{Jet}E)$.

[Anderson 1989] tells us that its differential can be decomposed by

$$\mathrm{d}_{\mathrm{dR}}\mathscr{L} \;=\; \delta_{\mathrm{EL}}\mathscr{L} - \mathrm{d}_{\mathrm{h}}\,\Theta_{\mathrm{pre}},$$

where

- $\delta_{\mathrm{EL}}\mathscr{L} \in \Omega^{m,1}(\mathrm{Jet}E)$ is a "source" $(m,1)$-form
- $\Theta_{\mathrm{pre}} \in \Omega^{m-1,1}(\mathrm{Jet}E)$ is a $(m-1,1)$-form.

Definition (Pre-symplectic current)

The *pre-symplectic current* $\Omega_{\mathrm{pre}} \in \Omega^{m-1,2}(\mathrm{Jet}E)$ of a classical field theory is defined by the vertical derivative

$$\Omega_{\mathrm{pre}} \;\coloneqq\; \mathrm{d}_{\mathrm{v}}\,\Theta_{\mathrm{pre}}$$

This form is not closed: in fact, one has

$$\mathrm{d}_{\mathrm{dR}}\Omega_{\mathrm{pre}} \;=\; -\mathrm{d}_{\mathrm{v}}(\delta_{\mathrm{EL}}\mathscr{L})$$

Euler-Lagrange critical locus

The following is an application of [Khavkine, Schreiber 2017].

Euler-Lagrange critical locus

The *Euler-Lagrange critical locus* $\mathrm{Crit}_{\mathrm{EL}}(\mathscr{L})$ can be defined as the the pullback of formal smooth sets

$$\begin{array}{ccc} \mathrm{Crit}_{\mathrm{EL}}(\mathscr{L}) & \hookrightarrow & \mathrm{Jet}_M E \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \Delta_E} \\ \mathrm{Jet}_M \ker(\delta_{\mathrm{EL}} L) & \xrightarrow{\mathrm{Jet}_M e} & \mathrm{Jet}_M(\mathrm{Jet}_M E), \end{array}$$

where $e : \ker(\delta_{\mathrm{EL}} L) \hookrightarrow \mathrm{Jet} E$ is the natural embedding.

This has the crucial property that its fiber at any point $x \in M$ is given by germs of solutions of the field equations, i.e

$$\mathrm{Crit}_{\mathrm{EL}}(\mathscr{L})_x \;\simeq\; \mathrm{Crit}(S)(\mathbb{D}_x)$$

Crucial example of n-plectic structure

Let $e_{\mathrm{EL}} : \mathrm{Crit}_{\mathrm{EL}}(\mathscr{L}) \hookrightarrow \mathrm{Jet}E$ the natural embedding and define the pullback

$$\Omega \; := \; e_{\mathrm{EL}}^{*}\Omega_{\mathrm{pre}}.$$

Example

The pair $(\mathrm{Crit}_{\mathrm{EL}}(\mathscr{L}), \, \Omega)$ is an n-plectic formal smooth set with $n = \dim(M)$.

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Example

The pair $(\mathrm{Crit}_{\mathrm{EL}}(\mathscr{L}),\, \Omega)$ is an n-plectic formal smooth set with $n = \dim(M)$.

Moreover, consider the transgression functor

$$\begin{aligned} \mathfrak{T}_{\Sigma} :\; \Omega^{\dim(\Sigma),p}(\mathrm{Crit}_{\mathrm{EL}}(\mathscr{L})) \;&\longrightarrow\; \Omega^{p}\big(\mathrm{Crit}(S)(\Sigma_{\mathrm{th}})\big) \\ \xi \;&\longmapsto\; \mathfrak{T}_{\Sigma}\xi \;\coloneqq\; \int_{\Sigma} j(-)^{*}\xi, \end{aligned}$$

which sends a $(n-1,p)$-form on $\mathrm{Crit}_{\mathrm{EL}}(\mathscr{L})$ to a p-form on the phase space $\mathrm{Crit}(S)(\Sigma_{\mathrm{th}})$ of the theory by integrating on a codimension 1 submanifold $\Sigma \subset M$.

This sends our n-plectic form to the honest symplectic form on the (infinite-dimensional) phase space of the theory, i.e.

$$\omega(\phi) \;=\; \int_{\Sigma} j(\phi)^{*}\Omega$$

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Derived n-plectic structure

Definition (Derived n-plectic geometry)

Let $X \in \mathbf{dFSmoothSet}$ be a formal derived smooth set. A p-shifted n-plectic form is a cocycle $\Omega \in \mathrm{Z}^p\mathrm{A}^{n+1}_{\mathrm{cl}}(X)$ such that the induced morphism of quasi-coherent sheaves

$$\Omega^\sharp \; : \; \mathbb{T}_X \;\longrightarrow\; \wedge^n \mathbb{L}_X[p]$$

gives rise to a monomorphism of the ∞-groupoids of their sections

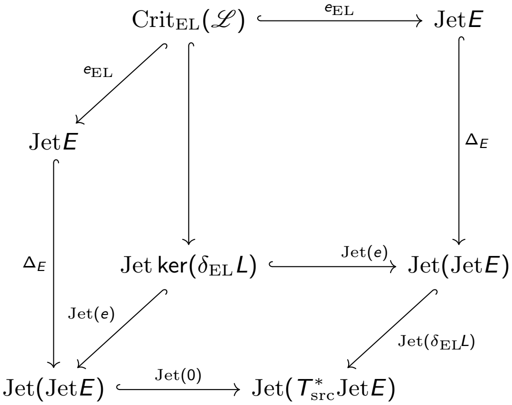
$$\Omega^\sharp \; : \; \mathfrak{X}(X,0) \;\hookrightarrow\; \mathcal{A}^n(X,p)$$

Example (Derived symplectic structure)

A derived symplectic structure is, in particular, a derived 1-plectic structure.

Euler-Lagrange critical locus as a zero locus

It is possible to show that there are pullback squares



Euler-Lagrange critical locus as a zero locus

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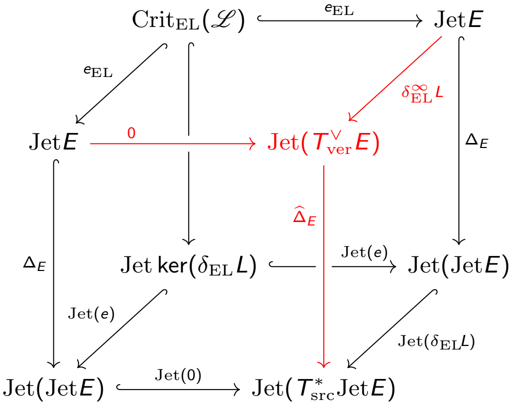


This recasts the Euler-Lagrange critical locus into the zero-locus of section $\delta^{\infty}_{\mathrm{EL}}L$, i.e.

$$\mathrm{Crit}_{\mathrm{EL}}(\mathscr{L}) \simeq \ker(\delta^{\infty}_{\mathrm{EL}}L)$$

Derived Euler-Lagrange critical locus

The *derived Euler-Lagrange critical locus* is the formal derived smooth set defined by the homotopy pullback

$$\begin{array}{ccc} \mathbb{R}\mathrm{Crit}_{\mathrm{EL}}(\mathscr{L}) & \longrightarrow & \mathrm{Jet}E \\ \Big\downarrow{\scriptstyle p_{\mathrm{EL}}} & & \Big\downarrow{\scriptstyle 0} \\ \mathrm{Jet}E & \xrightarrow{\ \delta^{\infty}_{\mathrm{EL}}L\ } & \mathrm{Jet}(T^{\vee}_{\mathrm{ver}}E) \end{array}$$

in the $(\infty,1)$-category of formal derived smooth sets.

Dually, we can compute the derived tensor product of $\mathcal{C}^{\infty}$-algebras

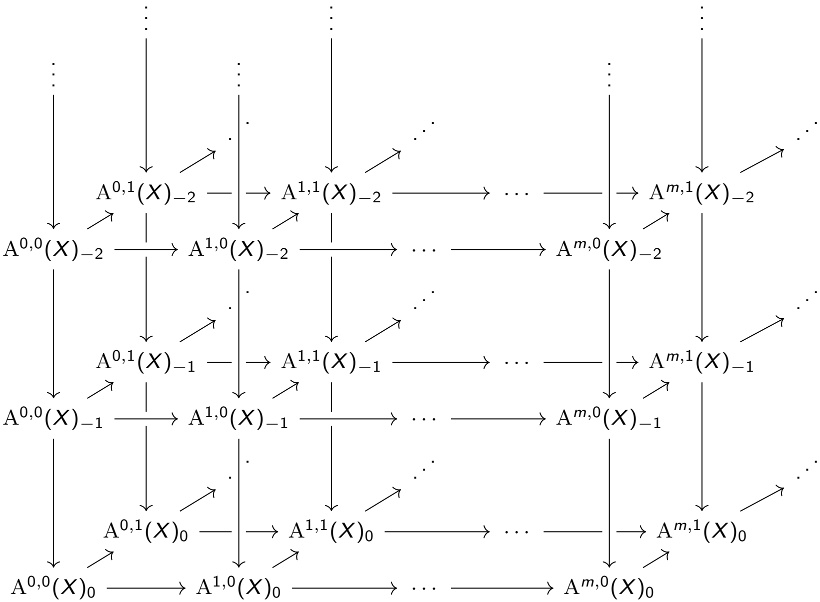
$$\mathcal{O}(\mathbb{R}\mathrm{Crit}_{\mathrm{EL}}(\mathscr{L})) \;\simeq\; \mathcal{O}\big(\mathrm{graph}(\delta^{\infty}_{\mathrm{EL}}L)\big)\,\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}(\mathrm{Jet}(T^{\vee}_{\mathrm{ver}}E))}\,\mathcal{O}(\mathrm{Jet}E)$$

The underlying dg-algebra is going to be of the form

$$\mathcal{O}(\mathbb{R}\mathrm{Crit}_{\mathrm{EL}}(\mathscr{L})) \;\simeq\; \Gamma(\mathrm{Jet}E,\, \wedge^{\bullet}\mathrm{Jet}^{\vee}(T^{\vee}_{\mathrm{ver}}E))$$

with differential given by contraction $Q \;=\; \langle \delta^{\infty}_{\mathrm{EL}}L, - \rangle$.

Derived variational *tri-complex* of $X = \mathbb{R}\mathrm{Crit}_{\mathrm{EL}}(\mathscr{L})$



Closed forms on the derived Euler-Lagrange critical locus

The complex of closed (p,q)-form on $X = \mathbb{R}\mathrm{Crit}_{\mathrm{EL}}(\mathscr{L})$ is

$$\mathrm{A}^{p,q}_{\mathrm{cl}}(X) \;\coloneqq\; \Big(\prod_{\substack{i\geq p\\ j\geq q}} \mathrm{A}^{i,j}(X)[-i-j]\Big)[p+q],$$

An n-cocycle in the complex $\mathrm{A}^{p,q}_{\mathrm{cl}}(X)$ is given by a formal sum of elements

$$\begin{array}{ccccccc}
 & & & \Omega^{p,q}_{n} & & & \\
 & & \Omega^{p+1,q}_{n-1} & & \Omega^{p,q+1}_{n} & & \\
 & \Omega^{p+2,q}_{n-2} & & \Omega^{p+1,q+1}_{n-2} & & \Omega^{p,q+2}_{n-2} & \\
\Omega^{p+3,q}_{n-3} & & \Omega^{p+2,q+1}_{n-3} & & \Omega^{p+1,q+2}_{n-3} & & \Omega^{p,q+3}_{n-3} \\
\iddots & & \vdots & & \vdots & & \vdots \quad \ddots
\end{array}$$

where $\Omega^{p',q'}_{n'} \in \mathrm{A}^{p',q'}(X)_{n'}$ for each p', q', n'.

To be a cocycle, these elements have to satisfy the following set of equations:

$$\begin{cases} Q\Omega_n^{p,q} &= 0, \end{cases}$$

$$\begin{cases} \mathrm{d_v}\Omega_n^{p,q} + Q\Omega_{n-1}^{p,q+1} &= 0, \\ \mathrm{d_h}\Omega_n^{p,q} + Q\Omega_{n-1}^{p+1,q} &= 0, \end{cases}$$

$$\begin{cases} \mathrm{d_v}\Omega_{n-1}^{p,q+1} + Q\Omega_{n-2}^{p,q+2} &= 0, \\ \mathrm{d_h}\Omega_{n-1}^{p,q+1} + \mathrm{d_v}\Omega_{n-1}^{p+1,q} + Q\Omega_{n-2}^{p+1,q+1} &= 0, \\ \mathrm{d_h}\Omega_{n-1}^{p+1,q} + Q\Omega_{n-2}^{p+2,q} &= 0, \end{cases}$$

$$\begin{cases} \mathrm{d_v}\Omega_{n-2}^{p,q+2} + Q\Omega_{n-3}^{p,q+3} &= 0, \\ \mathrm{d_h}\Omega_{n-2}^{p,q+2} + \mathrm{d_v}\Omega_{n-2}^{p+1,q+1} + Q\Omega_{n-3}^{p+1,q+2} &= 0, \\ \mathrm{d_h}\Omega_{n-2}^{p+1,q+1} + \mathrm{d_v}\Omega_{n-2}^{p+2,q} + Q\Omega_{n-3}^{p+2,q+1} &= 0, \\ \mathrm{d_h}\Omega_{n-2}^{p+2,q} + Q\Omega_{n-3}^{p+3,q} &= 0, \end{cases}$$

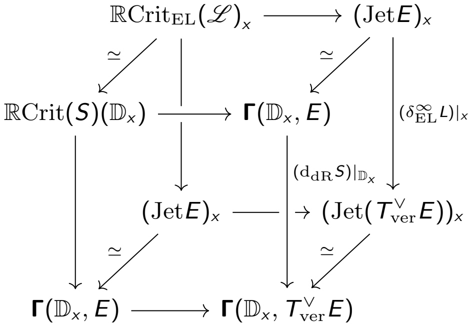
$$\vdots$$

Transgression of shifted closed forms

Crucially, the following property holds still in the derived setting:

$$\mathbb{R}\mathrm{Crit}_{\mathrm{EL}}(\mathscr{L})_x \;\simeq\; \mathbb{R}\mathrm{Crit}(S)(\mathbb{D}_x)$$

at any point $x \in M$.



- Roughly, this tells us that there exists a "derived transgression":

$$\mathfrak{T}_M \;:\; \mathcal{A}^{\dim(M),q}\big(\mathbb{R}\mathrm{Crit}_{\mathrm{EL}}(\mathscr{L}), n\big) \longrightarrow \mathcal{A}^{q}\big(\mathbb{R}\mathrm{Crit}(S)(M), n\big).$$

  from the derived Euler-Lagrange critical locus to the the critical locus of the action functional at M.

- Not too surprisingly, this derived transgression lifts to a map of closed forms

$$\mathfrak{T}_M \;:\; \mathcal{A}^{\dim(M),q}_{\mathrm{cl}}\big(\mathbb{R}\mathrm{Crit}_{\mathrm{EL}}(\mathscr{L}), n\big) \longrightarrow \mathcal{A}^{q}_{\mathrm{cl}}\big(\mathbb{R}\mathrm{Crit}(S)(M), n\big)$$

However, in the derived setting there is more!

- If $\partial M \simeq 0$ is trivial and $p \leq m$, we obtain a transgression map

$$\mathfrak{T}_M \;:\; \mathcal{A}_{\mathrm{cl}}^{\dim(M)-p,q}\big(\mathbb{R}\mathrm{Crit}_{\mathrm{EL}}(\mathscr{L}), n\big) \longrightarrow \mathcal{A}_{\mathrm{cl}}^{q}\big(\mathbb{R}\mathrm{Crit}(S)(M), n-p\big).$$

- If $\partial M \not\simeq 0$ is not trivial and $p \leq m$, we obtain a transgression map

$$\mathfrak{T}_M \;:\; \mathcal{A}_{\mathrm{cl}}^{\dim(M)-p,q}\big(\mathbb{R}\mathrm{Crit}_{\mathrm{EL}}(\mathscr{L}), n\big) \longrightarrow \mathcal{A}_{\mathrm{BFV}}^{q}\big(\mathbb{R}\mathrm{Crit}(S)(M), n-p\big),$$

where on the right-hand-side there is the ∞-groupoids whose elements are couples

$$\omega \in \mathrm{A}_{\mathrm{cl}}^{q}\big(\mathbb{R}\mathrm{Crit}(S)(M)\big)_{n-p} \qquad \varpi \in \mathrm{A}_{\mathrm{cl}}^{q}\big(\mathbb{R}\mathrm{Crit}(S)(\partial M_{\mathrm{th}})\big)_{n-p+1}$$

such that

$$\begin{aligned}(\mathrm{d}_{\mathrm{dR}} + Q)\omega + \pi_{\partial M}^{*}\varpi \;&=\; 0,\\ (\mathrm{d}_{\mathrm{dR}} + Q)\varpi \;&=\; 0.\end{aligned}$$

i.e. a shifted form ω whose failure to be closed amounts to the pullback of a closed form ϖ living on the boundary and 1 degree higher.

Canonical derived n-plectic structure of a classical field theory

Now, the derived Euler-Lagrange critical locus $\mathbb{R}\mathrm{Crit}_{\mathrm{EL}}(\mathscr{L})$

- comes with a canonical (-1)-shifted $(m, 2)$-form Ω_{BV},
- inherits a 0-shifted $(m-1, 2)$-form $\Omega_{\mathrm{BFV}} \coloneqq p_{\mathrm{EL}}^{*}\Omega_{\mathrm{pre}}$ from $\mathrm{Jet}E$.

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One can show that $\Omega_{\mathrm{BFV}} + \Omega_{\mathrm{BV}} \in \mathrm{Z}^0\mathrm{A}_{\mathrm{cl}}^{m-1,2}(\mathrm{Crit}_{\mathrm{EL}}(\mathscr{L}))$ is a closed form, i.e.

$$\begin{aligned} Q\Omega_{\mathrm{BFV}} &= 0, \\ \mathrm{d}_{\mathrm{dR}}\Omega_{\mathrm{BFV}} + Q\Omega_{\mathrm{BV}} &= 0, \\ \mathrm{d}_{\mathrm{dR}}\Omega_{\mathrm{BV}} &= 0. \end{aligned}$$

Example

$(\mathbb{R}\mathrm{Crit}_{\mathrm{EL}}(\mathscr{L}),\ \Omega_{\mathrm{BFV}} + \Omega_{\mathrm{BV}})$ is a 0-shifted n-plectic structure with $n = \dim(M)$.

By derived transgression map of closed forms

$$\mathfrak{T}_M \ :\ \mathcal{A}_{\mathrm{cl}}^{\dim(M)-1,2}\big(\mathbb{R}\mathrm{Crit}_{\mathrm{EL}}(\mathscr{L}),0\big) \longrightarrow \mathcal{A}^2_{\mathrm{BFV}}\big(\mathbb{R}\mathrm{Crit}(S)(M),-1\big)$$

one makes contact with BV-BFV theory:

$$\begin{aligned} (\mathrm{d}_{\mathrm{dR}} + Q)\omega_{\mathrm{BV}} + \pi_{\partial M}^{*}\varpi_{\mathrm{BFV}} &= 0, \\ (\mathrm{d}_{\mathrm{dR}} + Q)\varpi_{\mathrm{BFV}} &= 0. \end{aligned}$$

Extra: higher derived brackets?

- **Poisson structure**: bivector π_2 such that $[\pi_2, \pi_2] = 0$.

  Poisson algebroid $\mathfrak{Pois}(X, \pi_2) = T^*_X \xrightarrow{\pi_2^\flat} T_X$ so that

  $$\mathrm{CE}(\mathfrak{Pois}(X, \pi_2)) \;=\; \Big(\Gamma(X, \wedge^* T_X),\; \mathrm{d}_{\mathrm{CE}} = [\pi_2, -]\Big)$$

  "Derived" L_∞-bracket:

  $$\{f, g\} \;=\; [[\pi_2, f], g]$$

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- *k*-**shifted Poisson structure**: formal sum $\pi = \pi_2 + \pi_3 + \pi_4 + \dots$ such that each π_p is a $(k + p - 2)$-shifted p-vector and

  $$Q\pi + \frac{1}{2}[\pi . \pi] \;=\; 0.$$

  "Derived" Poisson algebroid $\mathfrak{Pois}(X, \pi)$ so that

  $$\mathrm{CE}(\mathfrak{Pois}(X, \pi)) \;=\; \Big(\mathbb{R}\Gamma(X, \wedge^* \mathbb{T}_X),\; \mathrm{d_{CE}} = Q + [\pi, -]\Big)$$

  "Higher derived" L_∞-bracket [Voronov 2004]:

  $$\ell_1(f) \;=\; Qf, \qquad \ell_p(f_1, f_2, \dots, f_p) \;=\; \mathrm{Proj}\, [\cdots[[\pi, f_1], f_2]\cdots, f_p]$$

$\Rightarrow$ Current work on Courant/Vinogradov version of this generalisation.

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Outlook

- Setting to go beyond BV-quantisation
  - [Bunk, Sämann, Szabo], [Fiorenza, Sati, Schreiber]: higher geometric prequantisation of n-plectic structures and prequantum bundle n-gerbes
  - [Safronov]: geometric quantisation of derived symplectic structures in derived algebraic geometry via bundle k-gerbes

$\Longrightarrow$ Beyond BV-quantisation by "higher derived" geometric (pre)quantisation?

- Setting to go beyond BV-BRST theory
  - Usually one would consider $\Omega^*(X, \mathfrak{g})$ with L_∞-structure and take shifted cotangent bundle $T^*[-1]\Omega^*(X, \mathfrak{g})$
  - We can consider $\mathbf{Bun}_G^\nabla(X) := [X, \mathbf{B}G_{\mathrm{conn}}]$ (or some concretification of this), and take derived critical locus $\mathbb{R}\mathrm{Crit}(S)(M)$ for a given $S : \mathbf{Bun}_G^\nabla(X) \to \mathbb{R}$

$\Longrightarrow$ Global geometric generalisation of BV-BRST theory?

- Investigate global and quantum aspects of dualities of string and M-theory

Thank you for your attention!