The Adams Spectral Sequence

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Introduction

The Adams spectral sequence is one of the most important tools in stable homotopy theory. It allows one to pass from homological information to "homotopical" information. In principle, it somehow streamlines and enhances the Serre spectral sequence computations of homotopy groups of spheres. Unlike the Serre spectral sequence, the Adams spectral sequence deals with stable homotopy groups only. To emphasize this, I will commit to working with spectra throughout the lecture. In addition to this, I will expand the definition the Adams spectral sequence to generalized cohomology theories. There are hypotheses that we are going to impose on our cohomology theories, which may seem a bit restrictive. However, they are not so bad, in light of the fact that the theories of interest to us mostly adhere to them (maybe after some modifications).

I will mainly be concerned by the construction of the spectral sequence. Therefore, let the scarcity (or maybe complete lack) of examples not discourage the audience. The approach is that of Haynes Miller's in [**HRM**]. It is referenced in [**COCTALOS**], which I am going to shamelessly copy here, along with [**RGB**]. The nice thing about the approach is that we do the necessary homological algebra over the spectra and obtain the spectral sequence by a simple application of π_* functor. Without further ado (I've already spent enough time with this introduction), let us begin.

1. Definitions

I will assume that people know about spectra, their relation to (co)homology theories, and smash products of spectra. Most of the information is in [**ABB**].

Definition 1.1. *(i) A sequence of spectra $A_1 \longrightarrow A_2 \longrightarrow \dots \longrightarrow A_n$ is exact if the sequence of homotopy functors it represents is exact.*
(ii) A map $A \longrightarrow B$ is a monomorphism if $* \longrightarrow A \longrightarrow B$ is exact.
(iii) A map $A \longrightarrow B$ is an epimorphism if $A \longrightarrow B \longrightarrow *$ is exact.
(iv) A sequence $A \longrightarrow B \longrightarrow C$ is short exact if $* \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow *$ is exact.

Remark 1.2. Here the homotopy functor that represents A is the functor $[A, -]$ (and not $[-, A]$). In particular, we can conclude that any cofiber sequence is exact. These two notions in some sense are very closely related.

The monomorphisms and epimorphisms end up being actually quite simple.

Lemma 1.3. *If $f : A \longrightarrow B$ is mono, then there is a map $g : C \longrightarrow B$, such that $f \vee g : A \vee C \longrightarrow B$ is a weak equivalence. If $g : A \longrightarrow B$ is epi, then there is a homotopy section $r : B \longrightarrow A$, i.e. $gr \simeq \mathbb{1}$, and a map $f : F \longrightarrow A$, such that $r \vee f : B \vee F \longrightarrow A$ is a weak equivalence.*

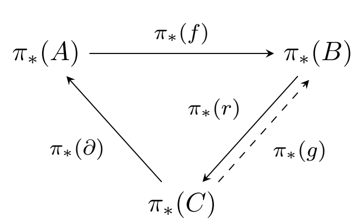
Proof. Note that $[\Sigma E, -] \cong [E, \Sigma^{-1}-]$, which implies that Σ preserves the exactness of sequences of spectra. Then we look at the following cofiber sequence of spectra

$$A \xrightarrow{f} B \xrightarrow{r} C \xrightarrow{\partial} \Sigma A \xrightarrow{-\Sigma f} \Sigma B$$

Note that $-\Sigma f$ is mono. Then we have an exact sequence

$$[C, \Sigma A] \xleftarrow{\partial_*} [\Sigma A, \Sigma A] \xleftarrow{(-\Sigma f)_*} [\Sigma B, \Sigma A]$$

$(-\Sigma f)_*$ is surjective, forcing $\partial_* = 0$. Thus, $\partial \simeq *$. Then recall that the sequence $B \xrightarrow{r} C \xrightarrow{*} \Sigma A$ is a fiber sequence. Then $r : [C, B] \longrightarrow [C, C]$ is surjective. Pick a lift for the identity and call it $g : C \longrightarrow B$. This is a homotopy section of r. Then we look at the exact triangle:



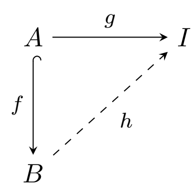
Note that $\pi_*(\partial) = 0$ and $\pi_*(r)$ admits a section (namely, $\pi_*(g)$), and we observe that we have a split short exact sequence. Thus, $\pi_*(f \vee r) = \pi_*(f) \oplus \pi_*(r)$ is an iso.

The statement about epis can be derived from the first part, by taking the fiber $f : F \longrightarrow A$ of $g : A \longrightarrow B$ and then observing that f is mono. ■

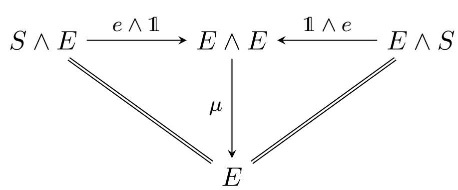
Observe that the splitting is not natural, since it involved a choice of a lift.

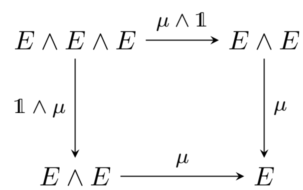
Definition 1.4. *A sequence of spectra is E-exact if the sequence becomes exact after smashing with E. The rest of the (E-)notions from 1.1 are defined similarly.*

Definition 1.5. *A spectrum I is E-injective if for each E-mono $f : A \longrightarrow B$, and each map $g : A \longrightarrow I$, there is an up to homotopy solution to the diagram*



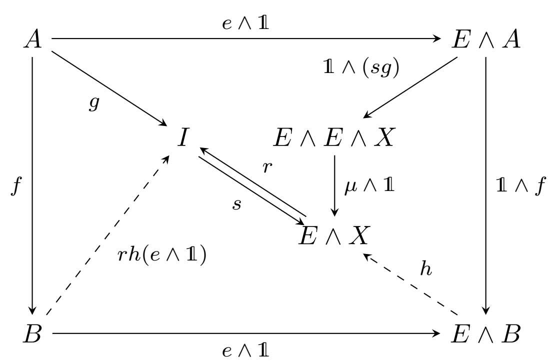
Remark 1.6. Miller's approach in defining these notions is different. Fortunately, one can reconcile some of the differences. Miller considers ring spectra right away, so let us do the same. From this point on we will assume that E is a homotopy associative ring spectrum, with structure maps $e : S \longrightarrow E$ (unit) and $\mu : E \wedge E \longrightarrow E$ (multiplication), such that the following diagrams commute up to homotopy





The diagram on left tells us that E is unital (up to homotopy), and the one on the right tells us that E is (homotopy) associative.

Miller calls a spectrum I E-injective if it is a retract of $E \wedge X$ for some spectrum X. Our notion of E-injectivity implies this. For any spectrum X, the map $e \wedge \mathbb{1} : X \longrightarrow E \wedge X$ is E-mono. Indeed, $\mathbb{1} \wedge e \wedge \mathbb{1} : E \wedge X \longrightarrow E \wedge E \wedge X$ admits a homotopy retraction, namely, the map $\mu \wedge \mathbb{1}$. Then, by E-injectivity of I, we obtain a homotopy retraction $r : E \wedge I \longrightarrow I$. The implication goes the other way too. To see this consider the diagram



The map h exists, since $E \wedge A \longrightarrow E \wedge B$ is a mono. Just to check: $rh(e \wedge \mathbb{1})f = rh(\mathbb{1} \wedge f)(e \wedge \mathbb{1}) = r(\mu \wedge \mathbb{1})(\mathbb{1} \wedge (sg))(e \wedge \mathbb{1}) = r(\mu \wedge \mathbb{1})(e \wedge (sg)) = rsg = g$.

Another thing that may be somewhat unclear is Miller's definition of E-exactness. By his definition, $A_1 \longrightarrow A_2 \longrightarrow \ldots \longrightarrow A_n$ is E-exact if we get an exact sequence when we apply the functor $[-, I]$ for E-injective spectra I. Our definition implies that of Miller. It suffices to look at the case $n = 3$. Suppose we are given a map, $f : A_2 \longrightarrow I$, such that $fj_1 \simeq *$, where j_k denotes the map $A_k \longrightarrow A_{k+1}$. Since, $E \wedge A_1 \longrightarrow E \wedge A_2 \longrightarrow E \wedge A_3$ is exact, we get a map $g : E \wedge A_3 \longrightarrow E \wedge I$, such that $g(\mathbb{1} \wedge j_2) = \mathbb{1} \wedge f$.

Then we consider, $rg(e \wedge \mathbb{1}) : A_3 \longrightarrow I$, where $r : E \wedge I \longrightarrow I$ is the retraction of $e \wedge \mathbb{1}$ obtained in the previous paragraph. This map extends f: $rg(e \wedge \mathbb{1})j_2 = rg(\mathbb{1} \wedge j_2)(e \wedge \mathbb{1}) = r(\mathbb{1} \wedge f)(e \wedge \mathbb{1}) = r(e \wedge \mathbb{1})f = f$. I am not sure whether the implication goes in the other direction or not. At this point we know that our assumptions provide us with a more general setting, and it is sufficient for our later discussion.

When people define injective objects after that they usually define resolutions.

Definition 1.7. *An E-Adams resolution (or an E-resolution) of a spectrum X is an E-exact sequence*

$$* \longrightarrow X \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1 \xrightarrow{i_2} \cdots$$

such that I_j's are E-injective.

Remark 1.8. In general, the fact that we have an E-exact sequence does not imply $i_{s+1}i_s \simeq *$. Indeed, suppose we have the obvious map $H\mathbb{Z} \longrightarrow H\mathbb{F}_2$. ($HG$ is the Eilenberg-MacLane spectrum for the abelian group G.) Smashing it with $H\mathbb{Q}$ produces $H\mathbb{Q} \wedge H\mathbb{Z} \longrightarrow H\mathbb{Q} \wedge H\mathbb{F}_2$. The target spectrum is trivial, since $\pi_*(H\mathbb{Q} \wedge H\mathbb{F}_2) = H_*(H\mathbb{F}_2; \mathbb{Q}) = 0$, so the map is forced to be trivial. However, in our setting, where the target objects are E-injective the statement does follow through. Namely, suppose I is E-injective, and we are given a map $\nu : A \longrightarrow I$, such that $\mathbb{1} \wedge \nu : E \wedge A \longrightarrow E \wedge I$ is null. If $\sigma : I \longrightarrow B$ is the cofiber of ν, then $\mathbb{1} \wedge \sigma : E \wedge I \longrightarrow E \wedge B$ is the cofiber of $\mathbb{1} \wedge \nu \simeq *$. This easily implies that $\mathbb{1} \wedge \sigma$ is mono, or that σ is E-mono. Then we get a retraction $r : B \longrightarrow I$. Thus, $\nu = r\sigma\nu \simeq *$. We conclude that in the above sequence, $i_{s+1}i_s \simeq *$ for all $s \geq 0$.

Whenever people define resolutions after that what they want to check is whether or not they lift maps.

Proposition 1.9. *Given a diagram of form*

$$\begin{array}{ccccccccc}
* & \longrightarrow & X & \xrightarrow{i_0} & I_0 & \xrightarrow{i_1} & I_1 & \xrightarrow{i_2} & \cdots \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \\
* & \longrightarrow & Y & \xrightarrow[j_0]{} & J_0 & \xrightarrow[j_1]{} & J_1 & \xrightarrow[j_2]{} & \cdots
\end{array}$$

where the horizontal sequences are E-exact, and f is any map, there is a lift of f to a map of resolutions. Furthermore, this lift is unique up to chain homotopy.

Proof. Suppose we have lifted the map up to n-th level (regard f at level -1). Then we have a diagram

$$\begin{array}{ccccc}
I_{n-1} & \xrightarrow{i_n} & I_n & \xrightarrow{i_{n+1}} & I_{n+1} \\
\downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} \\
J_{n-1} & \xrightarrow[j_n]{} & J_n & \xrightarrow[j_{n+1}]{} & J_{n+1}
\end{array}$$

Note that $j_{n+1}f_n i_n = j_{n+1}j_n f_{n-1} \simeq *$. Then according to 1.6, there exists f_{n+1}, such that $f_{n+1}i_{n+1} = j_{n+1}f_n$.

For the second part of the proposition, it suffices to prove that there is contracting chain homotopy for $f \simeq *$. Set $h_0 = *$. Then suppose we have constructed the chain homotopy h up to level n. We have the following diagram,

$$\begin{array}{ccccccc}
 & & I_{n-1} & \xrightarrow{i_n} & I_n & \xrightarrow{i_{n+1}} & I_{n+1} \\
 & {\scriptstyle h_{n-1}}\swarrow & \downarrow{\scriptstyle f_{n-1}} & {\scriptstyle h_n}\swarrow & \downarrow{\scriptstyle f_n} & \swarrow{\scriptstyle h_{n+1}} & \\
J_{n-2} & \xrightarrow[j_{n-1}]{} & J_{n-1} & \xrightarrow[j_n]{} & J_n & &
\end{array}$$

We know that $f_{n-1} = h_n i_n + j_{n-1}h_{n-1}$. Then we observe that $(f_n - j_n h_n)i_n = f_n i_n - j_n h_n i_n = j_n f_{n-1} - j_n f_{n-1} + j_n j_{n-1} h_{n-1} = *$. This implies that there exist h_{n+1}, such that $f_n = h_{n+1}i_{n+1} + j_n h_n$. ■

Another thing people would demand is the existence of resolutions. Let's show that there are E-resolutions. Here, we will definitely need that fact that E is an associative ring spectrum. The following resolution is called *canonical* or *standard*.

LEMMA 1.10. *Let $I_n = E^{\wedge(n+1)} \wedge X$ and let $\delta^i : I_n \longrightarrow I_{n+1}$ be the map $\mathbb{1}^{\wedge i} \wedge e \wedge \mathbb{1}^{\wedge(n+1-i)} \wedge \mathbb{1}_X$, for $i \in \{0, 1, \dots, n+1\}$. Then the sequence*

$$* \longrightarrow X \xrightarrow{\delta} I_0 \xrightarrow{\delta} I_1 \xrightarrow{\delta} \cdots$$

where $\delta : I_n \longrightarrow I_{n+1}$ is the map $\sum_{i=0}^{n+1}(-1)^i\delta^i$, is an E-resolution.

Proof. It is fairly clear that I_n's are injective. Thus, we need to show that the sequence is E-exact. In fact what we are dealing with here is a cosimplicial spectrum, i.e. a functor $\mathbf{\Delta} \longrightarrow \mathbf{Spec}$, where $\mathbf{\Delta}$ is the category of simplicial objects and **Spec** is the category of spectra. This cosimplicial spectrum sends $[n]$ to $E \wedge I_{n-1}$ (regard, $X = I_{-1}$). The coface maps are $d^i = \mathbb{1} \wedge \delta^i$, and codegeneracy maps are $s^j = \mathbb{1}^{\wedge(j-1)} \wedge \mu \wedge \mathbb{1}^{\wedge(n-j+1)} \wedge \mathbb{1}_X : I_{n+1} \longrightarrow I_n$. It is rather tedious to check the cosimplicial identities, so I'll skip doing that. Here are the identities,

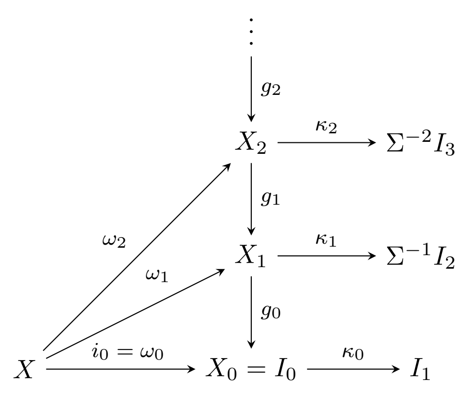
$$\begin{aligned}
d^j d^i &= d^i d^{j-1} \quad (i<j)\\
s^j d^i &= d^i s^{j-1} \quad (i<j)\\
&= \mathbb{1} \quad (i=j,j+1)\\
&= d^{i-1}s^j \quad (i>j+1)\\
s^j s^i &= s^{i-1}s^j \quad (i>j).
\end{aligned}$$

The fact that $\delta^2 = 0$ follows formally from these identities. Now if we define $\rho = \sum_{i=0}^{n}(-1)^i s^i$, then another tedious computation will reveal that $\rho\delta + \delta\rho = \mathbb{1}$, i.e. that ρ is a contracting homotopy. The moral of the "proof" is that the sequences constructed from cosimplicial objects give rise to exact sequences. ■

2. The Construction

Here is the plan. First we define the notion of an *E-Adams tower*, which should remind the audience of Postnikov towers. This tower will give rise to an exact couple, which will produce the Adams spectral sequence. Before constructing the exact couple, we will show that an E-Adams tower can be reconstructed from an E-resolution.

DEFINITION 2.1. *A diagram of the following form*



will be called a tower, if the sequences $X_{n+1} \longrightarrow X_n \longrightarrow \Sigma^{-n}I_{n+1}$ are fiber sequences. We can derive a sequence of the following form out of the tower:

$$* \longrightarrow X \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1 \xrightarrow{i_2} \cdots$$

where i_n is the composition $I_n \longrightarrow \Sigma^n X_n \longrightarrow I_{n+1}$. If the resulting sequence is an E-resolution, then we call the tower an E-Adams tower.

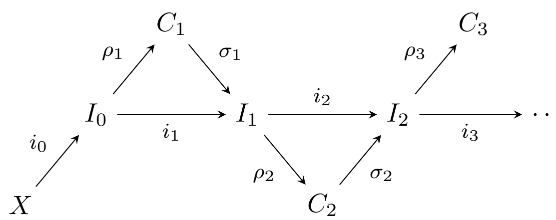
REMARK 2.2. One thing that we can infer from any tower, is the fact $i_{s+1}i_s \simeq *$. It follows from the following sequence,

$$I_n \longrightarrow \Sigma^n X_n \longrightarrow I_{n+1} \longrightarrow \Sigma^{n+1} X_{n+1} \longrightarrow I_{n+2}$$

The part of the sequence that is labeled green is a fiber sequence, hence the composition is null.

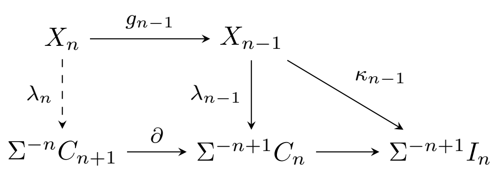
PROPOSITION 2.3. *Every E-resolution gives rise to an E-Adams tower.*

Proof. Suppose that we have an E-resolution of X as in 2.1. We can break this resolution into short exact sequences,

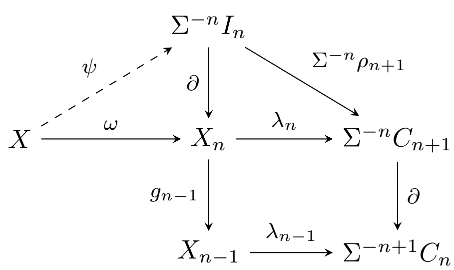


We define ρ_1 to be the cofiber of i_0, and let σ_1 be an induced map, such that $\sigma_1\rho_1 = i_1$. One can easily show that ρ_1 is E-epi and σ_1 is E-mono. Now if we show that $i_2\sigma_1 \simeq *$, we can iterate the construction and inductively construct the short exact sequence decomposition. Since, $i_2\sigma_1\rho_1 \simeq *$, we conclude that $i_2\sigma_1 \simeq *$, for ρ_1 is E-epi and I_2 is E-injective.

Now we construct the E-Adams tower inductively. Suppose that we have constructed the tower up to the level $n-1$, and it satisfies the following properties: (a) $\kappa_{n-1} = (\Sigma^{-n+1}\sigma_n)\lambda_{n-1}$, where $\lambda_{n-1} : X_n \longrightarrow \Sigma^{-n+1}C_n$; (b) $\lambda_{n-1}\omega_{n-1} \simeq *$. Now let X_n be the fiber of κ_{n-1} (we don't really have a choice here). Define λ_n via the following diagram:



Thus, we can define κ_n to be $(\Sigma^{-n}\sigma_{n+1})\lambda_n$. We are left to construct ω_n, such that $\lambda_n\omega_n \simeq *$. Note, $\kappa_{n-1}\omega_{n-1} = (\Sigma^{-n+1}\sigma_n)\lambda_{n-1}\omega_{n-1} \simeq *$. This implies that there is a map $\omega : X \longrightarrow X_n$, such that $\omega_{n-1} = g_{n-1}\omega$. Then let us look at the diagram,



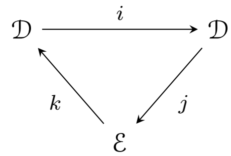
Notice that $\partial\lambda_n\omega = \lambda_{n-1}g_{n-1}\omega = \lambda_{n-1}\omega_{n-1} \simeq *$. Thus, there is ψ, such that $\lambda_n\omega = (\Sigma^{-n}\rho_{n+1})\psi$. Define ω_n to be $\omega - \partial\psi$. Let's check: $\lambda_n\omega_n = \lambda_n(\omega - \partial\psi) = \lambda_n\omega - \lambda_n\partial\psi = \lambda_n\omega - (\Sigma^{-n}\rho_{n+1})\psi = 0$. This finishes the inductive step of the construction of the tower. One can easily check that associated sequence to this tower is the one that we started off with. ■

Notice that we can extract another sequence from the E-Adams tower:

$$X = C_0 \xleftarrow{\gamma_0} \Sigma^{-1}C_1 \xleftarrow{\gamma_1} \Sigma^{-2}C_2 \xleftarrow{\gamma_2} \cdots$$

All the maps are the boundary maps of the appropriate cofiber sequences. We will refer to this sequence as *the associated inverse sequence*. Note that after smashing this sequence with E all the maps become trivial. Note also that the cofiber of γ_n is $\Sigma^{-n}I_n$. This sequence is what Ravenel calls an E-Adams resolution in [**RGB**].

Suppose we are given an E-Adams tower. To get the spectral sequence we construct an exact couple,



where $\mathcal{D} = \bigoplus_{s,t} \pi_{t-s}(X_s)$, $\mathcal{E} = \bigoplus_{s,t} \pi_t(I_s)$ are double graded groups. The maps are defined as follows,

$$i : \mathcal{D}^{s+1,t+1} = \pi_{t-s}(X_{s+1}) \xrightarrow{\pi_{t-s}(g_s)} \pi_{t-s}(X_s) = \mathcal{D}^{s,t}$$

$$j : \mathcal{D}^{s,t} = \pi_{t-s}(X_s) \xrightarrow{\pi_{t-s}(\kappa_s)} \pi_{t-s}(\Sigma^{-s} I_{s+1}) = \mathcal{E}^{s+1,t}$$

$$k : \mathcal{E}^{s+1,t} = \pi_{t-s}(\Sigma^{-s} I_{s+1}) \xrightarrow{\pi_{t-s}(\partial_s)} \pi_{t-s}(\Sigma X_{s+1}) = \mathcal{D}^{s+1,t}$$

It is not difficult at all to understand that we have an exact couple here. Thus, we obtain a spectral sequence $(\mathcal{E}_r, d_r)$, which we call the *Adams spectral sequence*.

Remark 2.4. Before going into the next section let us comment on the grading of the differentials. With care and the use of induction one can show that in r-th derived couple the maps have the following grading: $i_r : \mathcal{D}_r^{s+1,t+1} \longrightarrow \mathcal{D}_r^{s,t}$, $j_r : \mathcal{D}_r^{s,t} \longrightarrow \mathcal{E}_r^{s+r,t+r-1}$ and $k_r : \mathcal{E}_r^{s,t} \longrightarrow \mathcal{D}_r^{s,t}$. Thus, $d_r : \mathcal{E}_r^{s,t} \longrightarrow \mathcal{E}_r^{s+r,t+r-1}$. Note that $\mathcal{E}^{s,t} = \pi_t(I_s) = 0$ if $s < 0$, which implies that $\mathcal{E}_r^{s,t} = 0$ if $s < 0$. If $r > s$, the differentials entering into $\mathcal{E}_r^{s,t}$ are all clearly 0. Thus, we see that $\mathcal{E}_{s+1}^{s,t} \supset \mathcal{E}_{s+2}^{s,t} \supset \ldots$, and $\mathcal{E}_\infty^{s,t} = \bigcap_{r>s} \mathcal{E}_r^{s,t}$. Notice also that $\mathcal{E}_r^{s,t}$ may not ever stabilize to $\mathcal{E}_\infty^{s,t}$.

There is an alternate exact couple that gives rise to the same spectral sequence. This exact couple actually arises from the associated inverse sequence. We'll write $K_n = \Sigma^{-n} C_n$. Replace $\mathcal{D}$ with $\mathcal{F} = \bigoplus_{s,t} \pi_{t-s}(K_s)$. The maps are defined as follows:

$$i : \mathcal{F}^{s+1,t+1} = \pi_{t-s}(K_{s+1}) \xrightarrow{\pi_{t-s}(\gamma_s)} \pi_{t-s}(K_s) = \mathcal{F}^{s,t}$$

$$j : \mathcal{F}^{s,t} = \pi_{t-s}(K_s) \xrightarrow{\pi_{t-s}(\lambda_s)} \pi_{t-s}(\Sigma^{-s} I_s) = \mathcal{E}^{s,t}$$

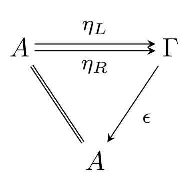
$$k : \mathcal{E}^{s,t} = \pi_{t-s}(\Sigma^{-s} I_s) \xrightarrow{\pi_{t-s}(\Sigma^{-s}\rho_{s+1})} \pi_{t-s}(\Sigma K_{s+1}) = \mathcal{F}^{s+1,t}$$

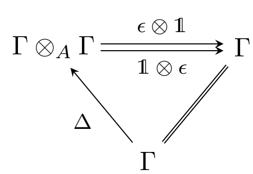
where ρ_s was defined in the proof of 2.3. There are maps from $\nu_s : K_s \longrightarrow X_s$, such that $\kappa_s \nu_s = \Sigma^{-s}(\sigma_{s+1})$, $g_s \nu_{s+1} = \gamma_s \nu_s$ and $(\Sigma^{-s+1}\rho_s)\nu_s = \partial_s$. This produces a map between the exact couples that induces the identity map on the $\mathcal{E}_1$-page. We will use this exact couple to prove statements about the convergence of the Adams spectral sequence.

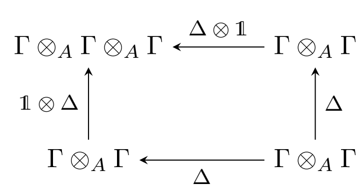
3. A Homological Algebra Detour

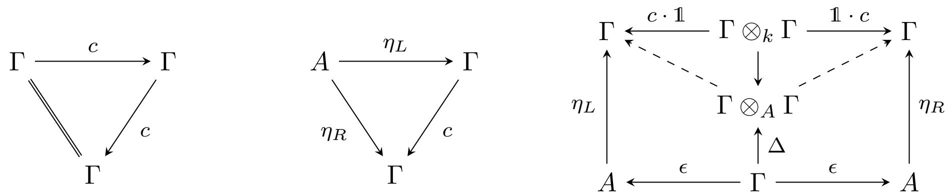
Before we embark onto our journey to investigate the properties of the Adams spectral sequence, I would like to pause to discuss some of the homological algebra machinery that we may need. Most of the discussion is taken from [**RGB**, A.1.1., A.1.2], and we make some use of [**MLH**].

We begin by defining the notion of a Hopf algebroid. Start with a commutative ring k, and two associative, commutative k-algebras A and Γ. The pair (A, Γ) along with maps $\eta_L, \eta_R : A \longrightarrow \Gamma$ (left and right units), $\epsilon : \Gamma \longrightarrow A$ (counit), $\Delta : \Gamma \longrightarrow \Gamma \otimes_A \Gamma$ (comultiplication) and $c : \Gamma \longrightarrow \Gamma$ (conjugation), will be called a Hopf algebroid if η, Δ are A-bimodule maps, with respect A-bimodule structure of Γ given by the multiplication, $(\eta_L \cdot \mathbb{1} \cdot \eta_R) : A \otimes_k \Gamma \otimes_k A \longrightarrow \Gamma$, and if the following diagrams commute,





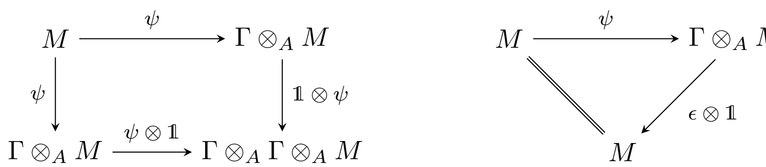




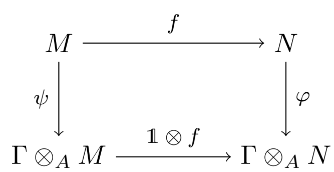
In the last diagram, we require the existence of dashed maps that make the diagram commute. A graded Hopf algebroid is simply a Hopf algebroid, such that A and Γ are graded, A is concentrated in dimension 0, and all the structure maps are grading preserving. A graded Hopf algebroid is *connected* if ϵ is an isomorphism on degree 0. We will often write Γ for the Hopf algebroid at hand, of course, keeping A in mind all the time.

Remark 3.1. One can observe that Hopf algebroids are cogroupoid objects in the category of associative, commutative k-algebras. One can backtrack and understand that this observation can be taken as the definition Hopf algebroids. For (slightly) more detail, see [**RGB**, A.1.1].

Definition 3.2. *A left Γ-comodule M, is a left A-module along with a comultiplication map $\psi : M \longrightarrow \Gamma \otimes_A M$, which is left A-linear, and such that the following diagrams commute,*



A map of left Γ-comodules is a left A-linear map $f : M \longrightarrow N$, such that the follwoing diagram commutes,

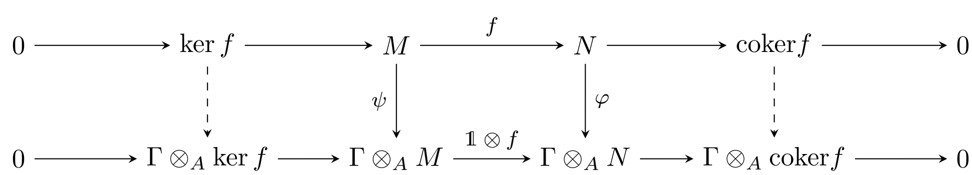


The notions of right Γ-comodules and right Γ-comodule maps are defined similarly.

Note that left Γ-comodules form an additive category. However, it is not always true that this category is abelian, due to lack of kernels. The following proposition shows that the category will be abelian under certain constraint.

Proposition 3.3. *If Γ is flat as a right A-module, then the category of left Γ-comodules is abelian.*

Proof. The functor $\Gamma \otimes_A -$ is exact. Then if $f : M \longrightarrow N$ is a Γ-comodule map, then we have the following commutative diagram,



The induced maps are uniquely determined and are left A-linear, since $\Gamma \otimes_A -$ can be viewed as a functor from the category of left A-modules to itself. A trivial check will show that $\ker f$ and $\mathrm{coker} f$ with respective comultiplications are the kernel and cokernel of f in the category of left Γ-comodules. ∎

Remark 3.4. Since A is commutative, then there is a canonical equivalence between left and right A-modules, and the functor $\Gamma \otimes_A -$ turns into $- \otimes_A \Gamma$ modulo a natural isomorphism that comes from the conjugation on Γ. Thus, Γ is left A-flat iff it is right A-flat. In a situation like this, we will call Γ *flat* over A.

We would like to show that there are enough injective Γ-comodules. We start off with a general lemma.

Lemma 3.5. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are abelian categories. Let $G : \mathcal{A} \rightleftarrows \mathcal{B} : F$ be an adjunction, where F and G are additive, and G is exact and faithful. Then if $\mathcal{B}$ has enough injectives, then so does $\mathcal{A}$.*

Proof. Let A be an object in $\mathcal{A}$. Then there is a monomorphism $i : G(A) \longrightarrow I$, where I is injective in $\mathcal{B}$. Then we have an adjoint map $i^* : A \longrightarrow F(I)$. We'll show two things: i) i^* is mono, ii) $F(I)$ is injective in $\mathcal{A}$. We have an exact sequence $\ker i^* \hookrightarrow A \longrightarrow F(I)$, then $G(\ker i^*) \hookrightarrow G(A) \longrightarrow I$ is exact. Thus, $G(\ker i^*) \hookrightarrow G(A)$ is null, and so is $\ker i^* \hookrightarrow A$, implying that $\ker i^* = 0$. $F(I)$ is injective iff the functor $[-, F(I)] : \mathcal{A} \longrightarrow \mathbf{Ab}$ sends monos to epis. The fact that this is the case follows from the following diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\ G\ } & \mathcal{B} \\ & {\scriptstyle [-,F(I)]} \searrow \quad \swarrow {\scriptstyle [-,I]} & \\ & \mathbf{Ab} & \end{array}$$

which commutes up to a natural isomorphism. ■

Proposition 3.6. *The pair $G : \Gamma - \mathbf{Comod} \rightleftarrows A - \mathbf{Mod} : F$, where G is the forgetful functor and $F(N) = \Gamma \otimes_A N$, with comultiplication $\Delta \otimes \mathbb{1}$, is an adjoint pair.*

Proof. We will give the natural transformations required by the adjunction, and leave it to the reader to verify that they are inverses of each other. The maps are $\theta : \mathrm{Hom}_A[M, N] \rightleftarrows \mathrm{Hom}_\Gamma[M, \Gamma \otimes_A N] : \phi$ − $\theta(f) = (\mathbb{1} \otimes f)\psi$ and $\phi(g) = (\epsilon \otimes \mathbb{1})g$. ■

Corollary 3.7. *The category of left Γ-comodules has enough injectives.* ■

Remark 3.8. We could have developed the discussion in a highly general setting. This is all part of what MacLane calls relative homological algebra. We will stick to left Γ-comodules in our discussion. To see a discussion on relative homological algebra, see [**MLH**, Ch. IX]. We will make a use of relative homological algebra later on in this section, when we develop the notion of relative Ext.

Thus, we are in a position of defining derived functors. We define $\mathrm{Ext}^s_\Gamma(M, -)$ to be the s-th derived functor of $\mathrm{Hom}_\Gamma(M, -)$. We can assemble these abelian groups into a single graded abelian group, $\mathrm{Ext}_\Gamma(M, N) = \bigoplus_s \mathrm{Ext}^s_\Gamma(M, N)$. Observe that $\mathrm{Ext}^0_\Gamma(M, N) = \mathrm{Hom}_\Gamma(M, N)$. We also write $\mathrm{Ext}^+_\Gamma(M, N) = \bigoplus_{s>0} \mathrm{Ext}^s_\Gamma(M, N)$. Note that if we are working in a graded setting $\mathrm{Ext}_\Gamma(M, N)$ inherits another grading from the grading in the category. The reason for the introduction of this graded Ext is the fact that it becomes a graded ring if N is a comodule algebra over Γ, and M is projective over A. N is a comodule algebra if it has a commutative, associative A-algebra structure, and the comultiplication φ is an A-algebra map. However, in order to talk about this structure we need to learn about resolutions by relative injectives.

Definition 3.9. *A relatively injective Γ-comodule is a direct summand of $\Gamma \otimes_A N$, for some left A-module N. A relatively injective resolution of M is a an exact sequence of form*

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \dots$$

such that I_i's are relatively injective and the sequence is split-exact over left A-modules.

These resolutions do behave like injective resolution. However, we first show that there are enough relatively injectives in $\Gamma - \mathbf{Comod}$. The notion of "enough" is used to accommodate the existence of relatively injective resolutions.

Lemma 3.10. *Given any left Γ-comodule M, there exists a relatively injective I and a monomorphism $M \longrightarrow I$ that is split over A.*

Proof. Actually this is really simple. We can take $I = \Gamma \otimes_A M$ and the monomorphism to be ψ coming from the Γ-comodule structure of M. Clearly, I is relatively injective. The map $\epsilon \otimes \mathbb{1} : \Gamma \otimes_A M \longrightarrow M$ provides us with a retraction of ψ over left A-modules. ■

Here is another lemma that somehow justifies the use of the nomenclature "injective" for relative injectives.

Proposition 3.11. *Suppose $i : M \longrightarrow N$ is a monomorphism of left Γ-comodules, and $f : M \longrightarrow I$ is any left Γ-comodule map, and I is relatively injective. Then there is a map $g : N \longrightarrow I$, such that $gi = f$. Conversely, any I with the lifting property given as above has to be relatively injective.*

Proof. The converse statement is easy. As in the previous lemma the Γ-comodule structure map $\iota : I \longrightarrow \Gamma \otimes_A I$ is split over A. Thus, we obtain a retraction $r : \Gamma \otimes_A I \longrightarrow I$ of ι, which proves the statement.

Now let's show the direct statement. Suppose that $j : I \longrightarrow \Gamma \otimes_A R$ is an inclusion and $\pi : \Gamma \otimes_A R \longrightarrow I$ is a corresponding projection, that comes from relative injectiveness of I. Here, of course, R is just a left A-module. So, we obtain a map $jf : M \longrightarrow \Gamma \otimes_A R$. If we take the adjoint of this map with respect to the adjunction in 3.6, then we get left A-module map $(jf)^* : M \longrightarrow R$. Let $s : N \longrightarrow M$ be the splitting of i over A. Thus, we have a map $(jf)^* s : N \longrightarrow R$. Take the adjoint of this map and compose it with π, then we get $\pi((jf)^* s)^* : N \longrightarrow I$. This is in fact the lift that we are looking for. The reader may verify the validity of the following derivation: $\pi((jf)^* s)^* i = \pi(\mathbb{1} \otimes (jf)^*) s^* i = \pi(\mathbb{1} \otimes (jf)^*)(\mathbb{1} \otimes s)\varphi i = \pi(\mathbb{1} \otimes (jf)^*)(\mathbb{1} \otimes s)(\mathbb{1} \otimes i)\psi = \pi(\mathbb{1} \otimes (jf)^*)\psi = \pi j f = f$. ■

The reader can easily verify the following proposition.

Proposition 3.12. *Given a diagram of form*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i_0} & I_0 & \xrightarrow{i_1} & I_1 & \xrightarrow{i_2} & \cdots \\ & & \downarrow f & & \downarrow f_0 & & \downarrow f_1 & & \\ 0 & \longrightarrow & N & \xrightarrow[j_0]{} & J_0 & \xrightarrow[j_1]{} & J_1 & \xrightarrow[j_2]{} & \cdots \end{array}$$

where the horizontal sequences are relatively injective resolutions, and f is any map of left Γ-comodules, there is a lift of f to a map of resolutions. Furthermore, this lift is unique up to chain homotopy.

Proof. This is an easy consequence of 3.11. ■

Remark 3.13. One ought to notice some the similarities between proofs presented in section 1 to the ones presented just above. In fact, we will see later that the relative injective resolutions have nice interaction with the Adams spectral sequence.

The previous discussion allows us to define relative derived functors. To be specific, given a left exact functor $F : \Gamma - \mathbf{Comod} \longrightarrow \mathbf{C}$, where $\mathbf{C}$ is some abelian category, we define $\widetilde{R}_n F(N) = H^n(\mathcal{I})$, where $\mathcal{I}$ denotes a relatively injective resolution of N, and call them relative right derived functors. A standard argument shows that everything is well-defined. We'll write $\widetilde{\mathrm{Ext}}^n_\Gamma(M, -)$ for $\widetilde{R}_n \mathrm{Hom}_\Gamma(M, -)$. Again, we can assemble these groups into a single one: $\widetilde{\mathrm{Ext}}_\Gamma(M, N) = \bigoplus_n \widetilde{\mathrm{Ext}}^n_\Gamma(M, N)$. The group $\widetilde{\mathrm{Ext}}^+_\Gamma(M, N)$ is defined similarly. The graded group $\widetilde{\mathrm{Ext}}_\Gamma(M, N)$ is not necessarily the same as $\mathrm{Ext}_\Gamma(M, N)$. Even though the injectives are relatively injective, an injective resolution may not be relatively injective resolution, since the maps may not split over A.

Proposition 3.14. *If M is projective as a left A-module, then $\widetilde{\mathrm{Ext}}_\Gamma(M, N) \cong \mathrm{Ext}_\Gamma(M, N)$.*

Proof. It will suffice to show that relative injectives are $\mathrm{Hom}_\Gamma(M, -)$ acyclic if M is projective over A. Suppose I is a relative injective. Suppose I is a direct summand of $\Gamma \otimes_A L$ for some A-module L. Then $\mathrm{Ext}^+_\Gamma(M, I)$ is a direct summand of $\mathrm{Ext}^+_\Gamma(M, \Gamma \otimes_A L)$. Thus, it will suffice to show that $\mathrm{Ext}^+_\Gamma(M, \Gamma \otimes_A L) = 0$ for all A-modules L. Let $\mathcal{J}$ denote an injective resolution of L, i.e. $H^+(\mathcal{J}) = 0$ and $H^0(\mathcal{J}) = L$. Then from 3.5, 3.6 we know that $\Gamma \otimes_A \mathcal{J}$ is an injective resolution of $\Gamma \otimes_A L$. Thus, $\mathrm{Ext}^+(M, \Gamma \otimes_A L) = H^+(\mathrm{Hom}_\Gamma(M, \Gamma \otimes_A \mathcal{J})) = \mathrm{Hom}_\Gamma(M, H^+(\Gamma \otimes_A \mathcal{J})) = \mathrm{Hom}_\Gamma(M, \Gamma \otimes_A H^+(\mathcal{J})) = 0$. ■

The advantage of looking at $\widetilde{\mathrm{Ext}}$ is that it provides manageable ways of computing Ext. It will make the discussion on the $\mathcal{E}_2$-page of the Adams spectral sequence much easier. Before stating and proving the proposition, we need a minor definition. Given two left Γ-comodules M and N, we can take their tensor product over A, $M \otimes_A N$, where the right A-module structure of N is given in the obvious way. We can give $M \otimes_A N$ left Γ-comodule structure via the comultiplication $M \otimes_A N \longrightarrow (\Gamma \otimes_A M) \otimes_A (\Gamma \otimes_A N) \longrightarrow \Gamma \otimes_A (M \otimes_A N)$, where the middle tensor product in the middle term is taken with respect to left A-module structure, and the last map sends $(\alpha \otimes m) \otimes (\beta \otimes n)$ to $(\alpha\beta) \otimes (m \otimes n)$.

Proposition 3.15. *Suppose we are given four left Γ-comodules M_1, M_2, N_1 and N_2. Then there is natural external product map*

$$\smile : \widetilde{\mathrm{Ext}}_\Gamma(M_1, N_1) \otimes_A \widetilde{\mathrm{Ext}}_\Gamma(M_2, N_2) \longrightarrow \widetilde{\mathrm{Ext}}_\Gamma(M_1 \otimes_A M_2, N_1 \otimes_A N_2).$$

Proof. Let $\mathcal{I}^i$ denote a relatively injective resolution for N_i, where $i \in \{1, 2\}$. We claim that $\mathcal{I}^1 \otimes_A \mathcal{I}^2$ is a split resolution for $N_1 \otimes_A N_2$, i.e. it is split acyclic as an A-module complex and $H^0(\mathcal{I}^1 \otimes_A \mathcal{I}^2) = N_1 \otimes N_2$. Let $\mathcal{I}$ be a relatively injective resolution for $N_1 \otimes_A N_2$. Then there is a map $\alpha : \mathcal{I}^1 \otimes_A \mathcal{I}^2 \longrightarrow \mathcal{I}$, unique up to chain homotopy. Then if $\nu_1 : M_1 \longrightarrow I^1_n$ and $\nu_2 : M_2 \longrightarrow I^2_m$, represent two elements of $\widetilde{\mathrm{Ext}}(M_1, N_1)$ and $\widetilde{\mathrm{Ext}}(M_2, N_2)$, respectively. Then we have an element $\alpha(\nu_1 \otimes \nu_2) : M_1 \otimes_A M_2 \longrightarrow I_{n+m}$. The element of $\widetilde{\mathrm{Ext}}(M_1 \otimes_A M_2, N_1 \otimes_A N_2)$ that it represents is precisely $[\nu_1] \smile [\nu_2]$. A tedious check can verify that everything is well-defined. ■

Now let's go back to lemma 3.10. The proof of this lemma not only tells us about the existence relatively injective resolutions, it also gives us a recipe of constructing relatively injective resolutions. In what follows we will use the bar notation instead of the tensor product.

Definition/Proposition 3.16. *Let M be a left Γ-comodule. Define the complex $D_\Gamma(M)$ as follows: $D^n_\Gamma(M) = \Gamma \otimes \overline{\Gamma}^{\otimes n} \otimes M$, where $\overline{\Gamma} \subset \Gamma$ is the kernel of ϵ, and with the differential $\partial : D^n_\Gamma(M) \longrightarrow D^{n+1}_\Gamma(M)$*

$$\partial(\gamma_0|\gamma_1|\dots|\gamma_n|m) = \sum_{k=0}^{n}(-1)^k(\gamma_0|\dots|\Delta(\gamma_k)|\dots|m) \;+\; (-1)^{n+1}\gamma_0|\dots|\psi(m),$$

where we use bars instead of tensor product signs. All the tensors are taken over A. The complex $D^*_\Gamma(M)$ is well defined, and is a relatively injective resolution of M, called the cobar resolution.

Proof. The first "problem" with the definition is that ∂ defines a map from $\Gamma \otimes \overline{\Gamma}^{\otimes n} \otimes M$ to $\Gamma^{\otimes(n+2)} \otimes M$. We need to show that it lands in $\Gamma \otimes \overline{\Gamma}^{\otimes n+1} \otimes M$. The observation to make here is that $\Gamma \otimes \overline{\Gamma}^{\otimes n+1} \otimes M$ is the kernel of the map $\sigma_n : \Gamma \otimes \Gamma^{\otimes n+1} \otimes M \longrightarrow \bigoplus_{k=1}^n \Gamma \otimes \Gamma^{\otimes n} \otimes M$, where

$$\sigma_n(\gamma_0|\dots|\gamma_n|m) = \bigoplus_k \gamma_0|\dots|\epsilon(\gamma_k)|\dots|\gamma_n|m.$$

We now have to verify that $\sigma_n\partial = 0$. The computation is straightforward, but tedious, so is left to the reader.

The second thing that we must verify is whether or not the differentials are Γ-comodule maps. This is easy to verify, the main ingredient being the coassociativity of Γ.

Finally, to show that the $D_\Gamma(M)$ is a relatively injective resolution we have to notice that it comes from a cosimplicial object in Γ-comodules. We can write it as $D_\bullet$. We set $D_0 = M$ and $D_n = \Gamma \otimes \Gamma^{\otimes(n-1)} \otimes M$ if $n > 1$. The face maps $d_k : D_n \longrightarrow D_{n+1}$ and the degeneracy maps are $s_k : D_n \longrightarrow D_{n-1}$:

$$d_k(\gamma_0|\dots|\gamma_n|m) = \begin{cases} \gamma_0|\dots|\Delta(\gamma_k)|\dots|\gamma_n|m & \text{if } k \neq n \\ \gamma_0|\dots|\gamma_n|\psi(m) & \text{if } k = n \end{cases}$$
$$s_k(\gamma_0|\dots|\gamma_n|m) = \gamma_0|\dots|\epsilon(\gamma_k)|\dots|\gamma_n|m.$$

One can check that this is a simplicial set. The maps d_k induce the differentials and s_k induce the contracting homotopy. In fact, over $D^n_\Gamma(M)$ the only non-vanishing s_k is s_0. ■

For any Γ-comodule M, we can construct a complex called the cobar complex, $C_\Gamma(M)$, such that its cohomology is equal to the relative Ext. We will define the cobar complex to be $\mathrm{Hom}_\Gamma(A, D_\Gamma(M))$. From the previous discussion it follows right away that the cohomology of this complex ought to give us $\widetilde{\mathrm{Ext}}_\Gamma(A, M) = \mathrm{Ext}_\Gamma(A, M)$. However, there is a way of expressing the cobar complex, which we will use in the next section. It is also useful for certain computations.

Proposition 3.17. *The A-module $C^n_\Gamma(M)$ is isomorphic to $\overline{\Gamma}^{\otimes n} \otimes M$, and the differentials are defined as in definition 3.15.*

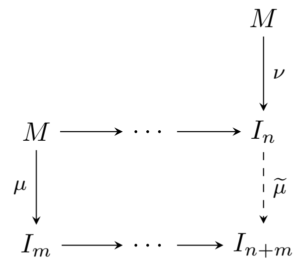
Proof. This is really the adjunction in proposition 3.7: $C^n_\Gamma(M) = \mathrm{Hom}_\Gamma(A, \Gamma \otimes \overline{\Gamma}^{\otimes n} \otimes A) \cong \mathrm{Hom}_A(A, \overline{\Gamma}^{\otimes n} \otimes A) \cong \overline{\Gamma}^{\otimes n} \otimes A$. ■

From this point of view one can also easily see that the graded A-algebra structure on $\mathrm{Ext}_\Gamma(A, M)$, if M is a Γ-comodule algebra, is the same as the A-algebra structure induced from the natural A-algebra structure on $C_\Gamma(M)$, obtained by concatenation.

If we have any Γ-comodule, M, we can impose a natural A-algebra structure on $\widetilde{\mathrm{Ext}}_\Gamma(M, M)$

$$\frown : \widetilde{\mathrm{Ext}}_\Gamma(M, M) \otimes_A \widetilde{\mathrm{Ext}}_\Gamma(M, M) \longrightarrow \widetilde{\mathrm{Ext}}_\Gamma(M, M).$$

Let $\nu : M \longrightarrow I_n$ and $\mu : M \longrightarrow I_m$ represent elements of $\widetilde{\mathrm{Ext}}_\Gamma(M, M)$. Then we define $[\mu] \frown [\nu]$ via the following diagram



Namely, $[\mu] \frown [\nu]$ is the class of $\widetilde{\mu} \circ \nu$. One can easily verify that the definition is valid. We will refer to this as *the Yoneda algebra structure*. Generally speaking it is hard to determine the Yoneda structure; however, in one case we can determine it, and in fact, we get something quite nice!

Proposition 3.18. *The multiplications $\frown$ and $\smile$ agree on $\widetilde{\mathrm{Ext}}_\Gamma(A, A)$. Furthermore, they are commutative.*

Proof. Applying an Eckmann-Hilton argument for these two products will yield the result. Notice that $\mathbb{1} : A \longrightarrow A$ yields the identity for $\frown$ and $\smile$. We are left to demonstrate that

$$(\kappa \smile \lambda) \frown (\nu \smile \mu) = (\mu \frown \lambda) \smile (\nu \frown \kappa).$$

Let us denote the degrees of the symbols above by their Latin equivalents. Let $\mathcal{I}$ be a relatively injective resolution of A. We will abuse the notation by identifying elements with the representing maps. Then $(\kappa \smile \lambda) \frown (\nu \smile \mu) = \widetilde{\kappa \smile \lambda} \circ (\nu \smile \mu) = \widetilde{\kappa \smile \lambda} \circ \alpha \circ (\nu \otimes \mu) = \alpha \circ (\widetilde{\kappa} \otimes \widetilde{\lambda}) \circ (\nu \otimes \mu) = \alpha \circ ((\widetilde{\kappa} \circ \nu) \otimes (\widetilde{\lambda} \circ \mu)) = \alpha \circ ((\kappa \frown \nu) \otimes (\lambda \frown \mu)) = (\mu \frown \lambda) \smile (\nu \frown \kappa)$. ■

4. The $\mathcal{E}_2$-Page & The Convergence

In order to have a nice spectral sequence with nice $\mathcal{E}_2$-page and decent convergence, we need some assumptions on our ring spectrum. We will state the assumptions later. Under the assumptions on the ring spectrum, we can show that E-completion of spectra exist and the functor Ext makes sense. All we need to know is that $H\mathbb{F}_p$ and BP satisfy those conditions. We'll talk about the conditions later.

Definition 4.1. *If we have a sequence*

$$X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} \cdots$$

then the homotopy limit of the sequence, $\underleftarrow{\mathrm{holim}}\, X_s$, is the fiber of the following map $\prod X_i \longrightarrow \prod X_j$, where $(j-1)$-th component of the map is $p_{j-1} - f_j p_j$.

Definition 4.2. *An E-Adams tower is simple if the associated inverse sequence has a trivial homotopy limit, i.e. $\underleftarrow{\mathrm{holim}}\, K_n \simeq *$.*

Definition 4.3. *An E-completion of a spectrum X is another spectrum $\widehat{X}$, with a map $X \longrightarrow \widehat{X}$ that induces an isomorphism on E_*-homology, and $\widehat{X}$ has a simple E-Adams tower.*

Before stating the main theorem, let me mention something about E_*E that we will need. If we assume that E_*E is flat over $\pi_*(E)$, then the pair $(\pi_*(E), E_*E)$ is a graded Hopf algebroid. This assumption on E_*E will be referred to as *flatness of E*. To specify the Hopf algebroid structure, we need to provide the structure maps – $\eta_L, \eta_R : \pi_*(E) \longrightarrow E_*E$, $\Delta : E_*E \longrightarrow E_*E \otimes_{\pi_*(E)} E_*E$, $\varepsilon : E_*E \longrightarrow \pi_*(E)$, and $c : E_*E \longrightarrow E_*E$. We can define some of these maps right away: $\eta_L = \pi_*(e \wedge \mathbb{1})$, $\eta_R = \pi_*(\mathbb{1} \wedge e)$, $\varepsilon = \pi_*(\mu)$, and $c = \pi_*(\tau)$, where $\tau : E \wedge E \longrightarrow E \wedge E$ denotes the twist map. To define Δ we'll need the following lemma, which will also be used later.

Lemma 4.4. *There is a natural map*

$$E_*E^{\otimes n} \otimes_{\pi_*(E)} E_*(X) \longrightarrow \pi_*(E^{\wedge(n+1)} \wedge X),$$

which an isomorphism.

Proof. We will define the map in due course. Actually, let's look at the case $n = 1$. We define

$$m : E_*E \otimes_{\pi_*(E)} E_*(X) \longrightarrow \pi_*(E^{\wedge 2} \wedge X),$$

so that if $\alpha \in \pi_*(E \wedge E)$, $\beta \in \pi_*(E \wedge X)$, then $m(\alpha \otimes \beta) = (\mathbb{1} \wedge \mu \wedge \mathbb{1})(\alpha \wedge \beta)$. This map is an isomorphism for the following reason. The functors $E_*E \otimes_{\pi_*(E)} E_*(-)$ and $\pi_*(E^{\wedge 2} \wedge -) = (E \wedge E)_*(-)$ are homology theories that agree on S^0 via m (m can be thought of as a map between homology theories). Thus, they ought to be naturally isomorphic via m. Note that we implicitly used the flatness of E to conclude that $E_*E \otimes_{\pi_*(E)} E_*(-)$ is a homology theory. The construction of the rest of the isomorphisms is done via induction. ■

We define Δ as $\pi_*(\mathbb{1} \wedge e \wedge \mathbb{1}) : \pi_*(E \wedge E) = E_*E \longrightarrow \pi_*(E \wedge E \wedge E) \cong E_*E \otimes_{\pi_*(E)} E_*E$. It is a routine check to verify the axioms of Hopf algebroid.

Theorem 4.5. *If E is flat, and X has an E-completion, the Adams spectral sequence $(\mathcal{E}_r, d_r)$ for X converges to $\pi_*(\widehat{X})$ and*

$$\mathcal{E}_2^{s,t} = \operatorname{Ext}^{s,t}_{E_*E}(\pi_*E, E_*(X)),$$

where $\operatorname{Ext}^{s,t}_{\Gamma}(M, -)$ denotes the t-th graded piece of the s-th derived functor of $\operatorname{Hom}_\Gamma(M, -)$ over the category of graded left Γ-comodules.

We will discuss the filtration of $\pi_*(\widehat{X})$ in due course.

Remark 4.6. We can show that $\mathcal{E}_2$-page has the above-mentioned form after a discussion on "uniqueness" of Adams spectral sequence. Recall that we have constructed the Adams spectral sequence from the E-Adams tower, and there could be lots of them, and in principle, they may give us different spectral sequences. Theorem 3.5 hints us that starting from $\mathcal{E}_2$-page, the spectral sequences must be isomorphic. This is what we mean by "uniqueness" of the spectral sequence. Let $\mathcal{T}^1_X$ and $\mathcal{T}^2_X$ be two E-Adams towers for X. These two towers have corresponding E-resolutions, $\mathcal{R}^1_X$ and $\mathcal{R}^2_X$ for X. Lifting the identity, provides us with a chain homotopy equivalence $\rho : \mathcal{R}^1_X \longrightarrow \mathcal{R}^2_X$. We can lift this map to a map of towers $\tau : \mathcal{T}^1_X \longrightarrow \mathcal{T}^2_X$. This induces a map between the corresponding spectral sequences, $\tau_* : \mathcal{E}^{(1)}_* \longrightarrow \mathcal{E}^{(2)}_*$. Now lets see what τ_* does on $\mathcal{E}_1$-page. The differentials are easy to compute:

$$\begin{array}{ccccc}
\mathcal{E}_1^{s,t} & \longrightarrow & \mathcal{D}_1^{s,t} & \longrightarrow & \mathcal{E}_1^{s+1,t} \\
\Vert & & \Vert & & \Vert \\
\pi_t(I_s) & \longrightarrow & \pi_{t-s}(X_{s+1}) & \longrightarrow & \pi_t(I_{s+1})
\end{array}$$

$$d_1 = \pi_t(j_{s+1}) : \pi_t(I_s) \longrightarrow \pi_t(I_{s+1})$$

The induced maps between the $\mathcal{E}_1$-pages are, $\pi_t(\rho_s) : \pi_t(I^{(1)}_s) \longrightarrow \pi_t(I^{(2)}_s)$. Since ρ is chain homotopy equivalence, then so is $\pi_t(\rho_*)$ for all t. Thus, $\pi_t(\rho_*)$ induces an isomorphism on cohomology of $\pi_t(I^{(k)}_*)$. However, $\mathcal{E}_2^{(k)\,s,t} = H^s(\pi_t(I^{(k)}_*))$. A similar discussion proves that any map $X \longrightarrow Y$ induces a natural map between the spectral sequences, "modulo $\mathcal{E}_1$-page". This statement applies in general and no assumptions were needed on E other than the ones made in section 2.

From the previous remark we can conclude that we can use any E-resolution to compute $\mathcal{E}_2$-page of the Adams spectral sequence. We will use the one that we already know of, i.e. the canonical resolution. The $\mathcal{E}_1$-page look as follows, $\mathcal{E}_1^{s,*} = \pi_*(I_s) = \pi_*(E^{\wedge(s+1)} \wedge X) = E_*E^{\otimes s} \otimes_{\pi_*(E)} E_*(X)$. After some examination one realizes that the sequences we get on $\mathcal{E}_1$-page

$$0 \longrightarrow E_*(X) \longrightarrow E_*E \otimes_{\pi_*(E)} E_*(X) \longrightarrow E_*E^{\otimes 2} \otimes_{\pi_*(E)} E_*(X) \longrightarrow \cdots$$

is the cobar complex. The cohomology of this complex is known to be $H^*(C_{E_*E}(E_*(X))) = \operatorname{Ext}^{*,*}_{E_*E}(\pi_*(E), E_*(X))$ for section 3.

Now we'll discuss the convergence of the Adams spectral sequence. Recall that $X \longrightarrow \widehat{X}$ induces an isomorphism on E_*-homology. This implies that $E \wedge X \longrightarrow E \wedge \widehat{X}$ is an equivalence. The map $X \longrightarrow \widehat{X}$

induces a (natural) map between the canonical resolutions. Thus, we obtain a map from one spectral sequence to the other one $\mathcal{E}_* \longrightarrow \widehat{\mathcal{E}}_*$. On page 1, the map looks as follows, $\mathcal{E}_1^{s,t} = \pi_t(E^{\wedge(s+1)} \wedge X) \longrightarrow \pi_t(\widehat{E}^{\wedge(s+1)} \wedge \widehat{X}) = \widehat{\mathcal{E}}_1^{s,t}$. This is clearly an isomorphism. Therefore, $\mathcal{E}_* \longrightarrow \widehat{\mathcal{E}}_*$ is an isomorphism. If we pick any two resolutions of X and $\widehat{X}$, respectively, we are guaranteed by 3.6, that their spectral sequences are isomorphic, in a natural way, starting from page 2. Thus, we will study the convergence of the spectral sequence of $\widehat{X}$.

It will be convenient to phrase the convergence in terms of proposition. There, we will also specify the filtration of $\pi_*(\widehat{X})$.

PROPOSITION 4.7. *Suppose $\widehat{X}$ has a simple E-Adams tower. Then*

$$\mathcal{E}_\infty^{s,t} \cong \operatorname{im} \pi_{t-s}(K_s)/\operatorname{im} \pi_{t-s}(K_{s+1})$$

where the images are taken in $\pi_{t-s}(\widehat{X})$, and $\bigcap \operatorname{im} \pi_*(K_n) = 0$.

Proof. We first show that the intersection of the filtration pieces is 0. Let $\widehat{X} \longleftarrow K_1 \longleftarrow K_2 \longleftarrow \dots$ be the inverse sequence associated to a simple E-Adams tower of $\widehat{X}$. Then, by definition, $\underleftarrow{\operatorname{holim}}\, K_n = *$. This implies, among other things, that $\underleftarrow{\lim}\, \pi_*(K_n) = 0$. We will write for the L_n for $\bigcap \operatorname{im} \pi_*(K_{n+r}) \subset \pi_*(K_n)$. Then we have a sequence, $L_0 \longleftarrow L_1 \longleftarrow L_2 \longleftarrow \dots$, where the maps are the restrictions of $\pi_*(\gamma_s)$'s. These restrictions are surjective. We are trying to show that $L_0 = 0$. Suppose that $x_0 \in L_0$. Then there is $x_1 \in L_1$, such that it maps to x_0. Similarly, there is an element $x_2 \in L_2$, that maps to x_1. If we continue this way, we obtain an element $(x_0, x_1, x_2 \dots)$ of $\underleftarrow{\lim}\, \pi_*(K_n)$. However, this element must be 0, since the inverse limit is trivial. This implies that $x_0 = 0$.

I'll define a map $\eta : \mathcal{E}_\infty^{s,t} \longrightarrow \mathcal{G}^{s,t}/\mathcal{G}^{s+1,t+1}$, where $\mathcal{G}^{s,t} = \operatorname{im} \pi_{t-s}(K_s) \subset \pi_{t-s}(\widehat{X})$. Let $[\alpha] \in \mathcal{E}_\infty^{s,t}$, where $\alpha \in \mathcal{E}^{s,t}$ is an element that represents $[\alpha]$. We would like to show that $k(\alpha) = 0$. Notice that if $r > s$, then $d_r([\alpha]) = 0$. Recall that $d_r = j_r k_r$; thus, $k_r([\alpha]) = k(\alpha) \in \ker j_r = \operatorname{im} i_r$. If we take into account the grading we can show that $k(\alpha) \in \operatorname{im} \pi_{t-s-1}(K_{s+r}) \subset \pi_{t-s-1}(K_{s+1})$. Thus, $k(\alpha) \in \bigcap_{r>s} \operatorname{im} \pi_{t-s-1}(K_{s+r}) = 0$, which shows the claim.

Thus, by exactness of the sequence $\mathcal{F}^{s,t} \longrightarrow \mathcal{E}^{s,t} \longrightarrow \mathcal{F}^{s+1,t}$, we see that there is $\beta \in \mathcal{F}^{s,t}$, such that $j(\beta) = \alpha$. There is a quotient map $\varphi : \mathcal{F}^{s,t} \longrightarrow \mathcal{G}^{s,t} \longrightarrow \mathcal{G}^{s,t}/\mathcal{G}^{s+1,t+1}$. Thus, we define $\eta([\alpha]) = \varphi(\beta)$. We need to show that the definition is independent of the choice of α and β. This is equivalent to stating that if $[\alpha] = 0$, then for any lift β of α, $\varphi(\beta) = 0$. I would like to show first that in this case the choice of β does not matter. Choose $\overline{\beta}$, such that $j(\overline{\beta}) = \alpha$. Then $j(\beta - \overline{\beta}) = 0$, which implies that $\beta - \overline{\beta} \in \operatorname{im} i$, implying that $\varphi(\beta) = \varphi(\overline{\beta})$. The fact that $[\alpha] = 0$ implies that $[\alpha] \in \operatorname{im} d_r$ for some $r \leq s$. One can show by induction that the set of α's in $\mathcal{E}^{s,t}$ that satisfy this property is $j(h^{-1}(k(\mathcal{E}^{s-r,t-r+1})))$, where the maps are shown in the diagram:

$$\begin{array}{ccccc} & & \pi_{t-s}(K_s) & \xrightarrow{\quad j \quad} & \mathcal{E}^{s,t} \\ & & \Big\downarrow h & & \\ \mathcal{E}^{s-r,t-r+1} & \xrightarrow{\ k\ } & \pi_{t-s}(K_{s-r+1}) & & \end{array}$$

Thus, we can find $\gamma \in \mathcal{E}^{s-r,t-r+1}$ and $\beta \in \pi_{t-s}(K_s)$, such that $h(\beta) = k(\gamma)$ and $j(\beta) = \alpha$. Then $ih(\beta) = ik(\gamma) = 0$, implying that $\varphi(\beta) = 0$, since $\pi_{t-s}(K_s) \longrightarrow \pi_{t-s}(X)$ factors through ih.

Now suppose that $[\alpha] \neq 0$. If $\eta([\alpha]) = 0$, then there is a maximal r, such that the image of β, λ, in $\pi_{t-s}(K_{s-r+1})$ that is nonzero. Then $i(\lambda) = 0$. That means there is $\gamma \in \mathcal{E}^{s-r,t-r+1}$, such that $k(\gamma) = \lambda$. This implies that $d_r([\gamma]) = [\alpha]$, thus, contradicting the non-triviality of $[\alpha]$. This shows that η is injective.

Now let's show that η is surjective. Suppose we are given $\alpha = j(\beta) \in \mathcal{E}^{s,t}$. This equivalent to saying that $k(\alpha) = 0$. If α survives (i.e. is a cycle) up till $(r-1)$-th page, then $k_r([\alpha]) = k(\alpha) = 0$. This implies that $d_r([\alpha]) = 0$. Thus, α is cycle on r-th page as well. This proves that α defines a class in $\mathcal{E}_\infty^{s,t}$. Clearly, $\eta([\alpha]) = \varphi(\beta)$, which proves the surjectiveness. ■

The convergence may look a bit weird. However, it ends up being nice in the cases we may be interested in. Actually, one thing we need is the existence of the E-completion. These assumptions are taken from [**RGB**], and they guarantee the existence of the E-completion.

ASSUMPTIONS 4.8. *(a) E is commutative and associative.*

(b) E is connective, i.e. $\pi_r(E) = 0$ for $r < 0$.

(c) The map $\mu_* : \pi_0(E) \otimes \pi_0(E) \longrightarrow \pi_0(E)$ is an isomorphism.

(d) Let $\theta : \mathbb{Z} \longrightarrow \pi_0(E)$ be the unique ring homomorphism, and let $R \subset \mathbb{Q}$ be the largest subring to which θ extends. Then $H_r(E; R)$ is finitely generated over R for all r.

THEOREM 4.9. *If X is connective and E satisfies the conditions in 3.8, then $\widehat{X} = XG$, if $\pi_0(E) = G$ for the cases $G = \mathbb{Q}$, $\mathbb{Z}_{(p)}$, and $\mathbb{Z}$. If $G = \mathbb{F}_p$ and $\pi_*(X)$ are finitely generated, then $\widehat{X} = X\mathbb{Z}_p$, where $\mathbb{Z}_p$ denotes the p-adic integers.*

Proof. [**ABB**, 14.6, 15.$*$]. ■

Let me comment on the Adams spectral sequence for $E = H\mathbb{F}_p$. The $\mathcal{E}_2$-page, is that of classical Adams spectral sequence, $\mathrm{Ext}^{*;*}_{\mathcal{A}_{p*}}(\mathbb{F}_p, H_*(X; \mathbb{F}_p))$. Where $\mathcal{A}_{p*}$ is the dual Steenrod algebra. If X is finite CW-spectrum, the spectral sequence converges to $\pi_*(X) \otimes \mathbb{Z}_p$, which is $\pi_*(X)$ modulo the non-p-torsion.

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