COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
56. DIFFERENTIAL GEOMETRY, EGER (HUNGARY), 1989

Characteristic Classes of Loop Group Bundles and Generalized String Classes

A. ASADA

§0. Introduction

In a talk at the last colloquium the author treated the differential geometric and non-abelian cohomological meanings of the logarithm of (complex) matrix valued functions ([5]). In the continuation of that research we get the following bijection:

$$B_0 : H^0(M, G_d)/\exp(H^0(M, \mathfrak{g}_d)) \cong H^1(M, \Omega G_d).$$

Here, M is a smooth (Hilbert) manifold, $G = GL(n, \mathbb{C})$ (or $U(n)$), $\mathfrak{g}$ its Lie algebra, ΩG the (based) loop group over G. G_d, etc., mean the sheaves of germs of smooth G, etc., valued functions over M. The bijection B_0 gives natural meaning and examples of loop group bundles. Another important example of a loop group bundle is the tangent bundle of the (based) loop space ΩM over M (cf. [11], [15], [22], [27]). On ΩM the Dirac–Ramond operator (loop space version of the Dirac operator) is defined if and only if the structure group of the tangent bundle of ΩM is lifted on $\widetilde{\Omega G}$, the basic central extension of ΩG ([2], [15], [22]). The obstruction for this lifting was named string class ([22], cf. [11], [15]). Its free part belongs to $H^3(\Omega M, \mathbb{C})$ and is mapped to the first (rational) Pontrjagin class of M by transgression. (The torsion part needs a more delicate discussion, cf. [22], [25], [26].)

In this article we give differential geometric descriptions of the string class and its generalization (higher dimensional string classes) as follows: Let $\xi(\{g_{ij}\}$ be an ΩG-bundle over M, $\{\theta_i\}$ and $\{\Theta_i\}$ a connection and its curvature of ξ. Then, for any $p \geq 1$, there is a 1-cochain of $2p$-forms $\{\Psi_i\}$ such that

$$\int_0^1 \operatorname{tr}(\Theta_i^p g_{ij}' g_{ij}{}^{-1})\, dt = \Psi_j - \Psi_i.$$

Here, t is the loop variable, $g' = dg/dt$ and $\Theta^p = \overbrace{\Theta \wedge \ldots \wedge \Theta}^{p}$. By using this $\{\Psi_i\}$, we obtain

$$\int_0^1 \operatorname{tr}(\Theta_i{}^p \wedge \theta_i') - d\Psi_i = \int_0^1 \operatorname{tr}(\Theta_j{}^p \wedge \theta_j')dt - d\Psi_j,$$

on $U_i \cap U_j$. This form is closed and its de Rham class $\tilde{c}^p(\xi)$ is determined by ξ. Especially, $\tilde{c}^1(\xi)$ is the original string class and it vanishes if and only if ξ has an $\widetilde{\Omega\mathfrak{g}}$-valued connection. Here $\widetilde{\Omega\mathfrak{g}}$ means the basic central extension of $\Omega\mathfrak{g}$, the based loop algebra over $\mathfrak{g}$.

An ΩG-bundle ξ over M induces a G-bundle ξ^b over $M \times S^1$ by the correspondence

$$\xi = \{g_{ij}\} \to \xi^b = \{g_{ij}{}^b\}, \qquad g_{ij}{}^b(x,t) = (g_{ij}(x))(t).$$

On the other hand, a G-bundle ξ over M induces an LG-bundle ξ^L over ΩM by the correspondence

$$\xi = \{g_{ij}\} \to \xi^L = \{g_{ij}{}^L\}, \qquad (g_{ij}{}^L(\gamma))(t) = g_{ij}(\gamma(t)).$$

Here LG means the free loop group over G. It is shown that ξ^L is equivalent to an ΩG-bundle. By definition we get

$$(\xi^L)^b = ev^*(\xi).$$

Here $ev : \Omega M \times S^1 \to M$, $ev(\gamma, t) = \gamma(t)$ is the evaluation map ([9]). We also define the Gysin map
$\gamma : H^0(M, G_d)/\exp(H^0(M, \mathfrak{g}_d)) \to H^1(M \times S^1, G_d)$ by

$$\gamma(g) = (B_0(g))^b.$$

Since the inverse of the transgression $\tau^{-1} : H^{q+1}(M, \mathbb{C}) \to H^q(\Omega M, \mathbb{C})$ is the composition of the evaluation map and integration along S^1 ([9], [10]), together with the properties of the Gysin map ([14], cf. [3], [4]), we obtain a *trinity of β-classes (Chern–Simons classes), string classes and transgressed Chern classes.* All of these results are formulated in terms of non-abelian de Rham theory ([5], [6]). The use of non-abelian de Rham theory is essential in these studies. For example, geometric studies of integrable forms are mostly devoted to their monodromies. We can define the Gysin map for integrable forms with non-trivial monodromies. Their images are not G-bundles, but belong to $H^1(M \times S^1, \mathcal{M}^1)$, the two-dimensional non-abelian de Rham set of $M \times S^1$. On the other hand, as we have pointed out in our talk at the last Colloquium, one-dimensional non-abelian de Rham theory treats global properties of the equation

$$d^e f = df + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!} (\operatorname{ad} f)^n (df) = \theta,$$
$$d\theta + \theta \wedge \theta = 0, \qquad (\operatorname{ad} f)(\zeta) = f\zeta - \zeta f.$$

The local properties of this equation and of the equation

$$dg = g\theta \qquad (g = e^f),$$

are the same, but global properties differ. Differences are measured by (integral) β-classes (g^*)-images of generators of $H^*(G, \mathbb{Z})$) and β-classes are the origins of Chern classes via transgression (cf. [7]). These results together with the Grassmannian model of the loop group ([23]) suggest that we may identify Q.F.T. on M with the Chern–Simons actions and Q.F.T. on $M \times S^1$ with the topological actions or Q.F.T. on ΩM with the (stable) Yang–Mills actions on the one hand, and Q.F.T. on M with the topological actions and Q.F.T. on $M \times S^1$ with the (stable) Chern–Simons actions, or Q.F.T. on ΩM with the (stable) Chern–Simons actions on the other hand (cf. [8], [13], [24], [24]', [28]). It also suggests that complex representations of the group of paths $[\Omega M]$ ([7], [18], [19], [21]) divides two-classes, one defines Q.F.T. on M with the topological actions and the other defines Q.F.T. on M with the Chern–Simons actions (cf. [19]).

This article is outlined as follows. In Sect. 1 we study $\Omega\mathfrak{g}$ and $\widetilde{\Omega\mathfrak{g}}$-valued integrable forms and cohomologies with coefficients in $\mathcal{M}^1{}_{\Omega\mathfrak{g}}$ and $\mathcal{M}^1{}_{\widetilde{\Omega\mathfrak{g}}}$, the sheaves of germs of $\Omega\mathfrak{g}$ and $\widetilde{\Omega\mathfrak{g}}$-valued integrable forms. In Sect. 2 we define B_0 and the Gysin map. Connections and $\widetilde{\Omega\mathfrak{g}}$-valued connections of ΩG-bundles are defined in Sect. 3. Properties of $\widetilde{\Omega\mathfrak{g}}$-valued connections and their curvatures give a prototype of the definitions of general string classes which are also defined in Sect. 3. It seems that similar discussions may be possible for the basic abelian extension valued connections of Map(s^{2n-1}, G)-bundles by using the results in [20]. (cf. [20]') In Sect. 4, we show that string classes of ξ^L are the inverse-images of transgression of the Chern characters of ξ. The equivalence of the β-classes of g and the string classes of $B^0(g)$ and the existence of the Bott map

$$\widetilde{B} : H^0(M, \mathcal{M}^1)_0/d^e(h^0(M, \mathfrak{g}_d)) \to H^2(M, \mathcal{M}^1),$$

where $\mathcal{M}^1$ is the sheaf of germs of $\mathfrak{g}$-valued integrable forms and $H^0(M, \mathcal{M}^1)_0$ is a suitable subset of $H^0(M, \mathcal{M}^1)$, are shown in Sect. 5.

Acknowledgement. In an interesting study of the geometry of curves in a space, Prof. K. Abe (Dep. Math., General Education, Sinsyu Univ.) obtained a non-linear equation with a parameter ([1], cf. [7]). In the study of this equation, we recognized that it is more natural to consider this equation to be a solvable loop algebra valued integrable form. It was one of the starting points of this research. Another starting point is the relation of geometry of loop spaces and non-abelian de Rham theory which was suggested by S. I. Andersson and A. Connes. I would like to thank them. Mensky's book [19] gave many suggestions in this research. I would like to thank Dr. Terazawa (Dep. Phys., Fac. Sci., Sinsyu Univ.) who taught the author Mensky's book.

§1. Non-abelian de Rham theory with respect to loop groups

1. Let $G = G_n$ be $GL(n, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{g}_n$ its Lie algebra. $LG = LG_n$, $\Omega G = \Omega G_n$, $L\mathfrak{g} = L\mathfrak{g}_n$ and $\Omega\mathfrak{g} = \Omega\mathfrak{g}_n$ are free and based loop groups and loop algebras over G and $\mathfrak{g}$. The basic central extensions of LG, ΩG, $L\mathfrak{g}$ and $\Omega\mathfrak{g}$ are denoted by $\widetilde{L}G$, $\widetilde{\Omega}G$, $\widetilde{L}\mathfrak{g}$ and $\widetilde{\Omega}\mathfrak{g}$, respectively.

If g (or ζ) is an LG-valued function (or an $L\mathfrak{g}$-valued form) on a smooth Hilbert manifold M, we define a G-valued function g^b (or a $\mathfrak{g}$-valued form ζ^b) on $M \times S^1$ by

$$g^b(x,t) = (g(x))(t), \quad \zeta^b(x,t) = (\zeta(x))(t), \quad x \in M, \quad t \in S^1. \tag{1}$$

Convention. *We call g (or ζ) to be smooth on M if and only if g^b (or ζ^b) is smooth on $M \times S^1$.*

dg/dt and $d\zeta/dt$ are often denoted by g' and ζ'. They are smooth if g and ζ are smooth. An $\widetilde{L}G$-valued function $\widetilde{g}$ induces an LG-valued function g. $\widetilde{g}$ is said to be smooth if $\widetilde{g}$ is a smooth map to $\widetilde{L}G$ and g is smooth in the above sense. An $\widetilde{L}\mathfrak{g}$-valued differential form $\widetilde{\zeta}$ is written

$$\widetilde{\zeta} = (\zeta, \beta), \quad \zeta \text{ is an } L\mathfrak{g}\text{-valued form, } \beta \text{ is a usual form.}$$

We call $\widetilde{\zeta}$ to be smooth if ζ is smooth in the above sense and β is smooth.

A smooth $L\mathfrak{g}$-valued 1-form θ is said to be integrable (or flat) if it satisfies

$$d\theta + \theta \wedge \theta = 0. \tag{2}$$

A smooth $\widetilde{L}\mathfrak{g}$-valued 1-form $\widetilde{\theta} = (\theta, \beta)$ is said to be integrable (or flat) if it satisfies

$$d\widetilde{\theta} + \frac{1}{2}[\widetilde{\theta}, \widetilde{\theta}] = 0, \quad \text{i.e. } d\theta + \theta \wedge \theta = 0, \quad d\beta + \frac{1}{2}\int_0^1 \mathrm{tr}(\theta \wedge \theta')dt = 0. \tag{2$'$}$$

Lemma 1. (i) *Let ζ be an $L\mathfrak{g}$-valued 1-form on M, then*

$$\int_0^1 \mathrm{tr}(\zeta^{2p} \wedge \zeta')dt = 0, \quad p \geq 0, \quad \zeta^q = \overbrace{\zeta \wedge \ldots \wedge \zeta}^{q}. \tag{3}$$

(ii) *If θ is an integrable $L\mathfrak{g}$-valued 1-form, then*

$$d\left(\int_0^1 \mathrm{tr}(\theta^{2p+1} \wedge \theta')\right)dt = 0, \quad p \geq 0. \tag{4}$$

Proof. Since $\int_0^1 \mathrm{tr}(\zeta^{2p} \wedge \zeta')dt = 1/(2p+1)\int_0^1 (\mathrm{tr}(\zeta^{2p+1}))'dt$, we have (3). If θ is integrable, then $d\theta = -\theta \wedge \theta$ and $d(\theta') = (d\theta)'$ by Convention, so we get

$$\begin{aligned} d\left(\int_0^1 \mathrm{tr}(\theta^{2p+1} \wedge \theta')\right)dt &= -\int_0^1 \mathrm{tr}\left(\theta^{2p+1} \wedge (d\theta)'\right)dt \\ &= \int_0^1 \mathrm{tr}\left((\theta^{2p+1})' \wedge d\theta\right)dt \\ &= \frac{2p+1}{2p+3}\int_0^1 (\mathrm{tr}(\theta^{2p+3}))'dt = 0. \end{aligned}$$

■

Definition 1. *Let θ be an integrable L-valued 1-form on M. Then we set*

$$\alpha^p(\theta) = \text{the de Rham class of } \int_0^1 \mathrm{tr}(\theta^{2p-1} \wedge \theta')dt \in H^{2p}(M, \mathbb{C}).$$

By definition and $(2)'$, we obtain

Proposition 1. (i) *If $\theta = g^{-1}dg$ on M, we have*

$$\alpha^p(\theta) = c_p g^*(\widehat{e}_p). \tag{5}$$

Here $\widehat{e}_p$ is the $2p$-dimensional generator of $H^*(\Omega G, \mathbb{C})$ (cf. [23]) and c_p is a non-zero constant determined by the choice of $\widehat{e}_p$.

(ii) *If $\theta = g^{-1}dg$ and $g = e^f$ on M, then $\alpha^p(\theta) = 0$ for all p.*

(iii) *$\alpha^1(\theta) = 0$ if and only if there exists an $\widetilde{L\mathfrak{g}}$-valued integrable 1-form $\widetilde{\theta}$ such that $\widetilde{\theta} = (\theta, \zeta)$.*

2. Let $\zeta = \sum_i \zeta_i dx_i$ be an $L\mathfrak{g}$-valued 1-form on a (starlike) neighborhood U of the origin of a (separable) Hilbert space. Then we set

$$(I\zeta(x))(t) = \int_0^1 \sum_i sx_i(\zeta_i(sx))(t)ds,$$

$$P_\zeta(f) = \sum_{n=0}^{\infty} I_\zeta{}^n(f), \quad I_\zeta{}^0(f) = f, \quad I_\zeta(g) = I(\zeta g).$$

If ζ is integrable, $df = 0$ and $(f(x))(0) = ((x))(1)$, we get

$$dP_\zeta(f) = \zeta P_\zeta(f), \qquad (p_\zeta(f)(x))(0) = (P_\zeta(f)(x))(1).$$

Hence an integrable $L\mathfrak{g}$-valued 1-form θ is locally integrable, that is locally written as $\theta = g^{-1}dg$.

For a (scalar valued) 2-form $\zeta = \sum_{ij} \zeta_{ij} dx_i \wedge dx_j$, $I\zeta$ is defined by

$$I\zeta = \sum_i I\zeta_i dx_i, \qquad I\zeta_i = \int_0^1 \sum_j s^2 x_i \zeta_{ij}(sx)ds.$$

So we can define canonical local integration β_0 of $-1/2\int_0^1 \mathrm{tr}(\theta \wedge \theta')dt$ on U if θ is an integrable $L\mathfrak{g}$-valued 1-form. Hence, if $\widetilde{\theta} = (\theta, \beta)$ is an integrable $\widetilde{L\mathfrak{g}}$-valued 1-form on U, we can set

$$(\theta, \beta) = (g^{-1}dg, c^{-1}dc + \beta_0), \qquad c \text{ is a smooth } \mathbb{C}^*\text{-valued function.}$$

Therefore we can associate a smooth $\widetilde{LG}$-valued function (g, c), $g^{-1}dg = \theta$, to $\widetilde{\theta}$. We note that since $g(U)$ is contractible, $p^{-1}(g(U)) = g(U) \times \mathbb{C}^*$, where $p : \widetilde{LG} \to LG$ is the projection. We set

(6) $\rho(g) = g^{-1}dg$, $\quad g$ is an LG-valued (or a G-valued) function,

$$\widetilde{\rho}_I((g, C)) = \left(g^{-1}dg, c^{-1}dc - \frac{1}{2}I\left(\int_0^1 \mathrm{tr}\left(g^{-1}dg \wedge (g^{-1}dg)'\right)dt\right)\right).$$

By definition, ρ is defined globally, but $\widetilde{\rho}_I$ is defined only locally.

In the sequel, we use the following notations.

$\mathbb{C}^{*t}$, LG_t *and* $\widetilde{LG}_t$*: the sheaves of germs of constant $\mathbb{C}^*$, LG and $\widetilde{LG}$ valued functions over M.*

$\mathbb{C}^*{}_d$, LG_d *and* $\widetilde{LG}_d$*: the sheaves of germs of smooth $\mathbb{C}^*$, LG and $\widetilde{LG}$ valued functions over M.*

Φ^p*: the sheaf of germs of closed p-forms over M.*

$\mathcal{M}^1{}_{L\mathfrak{g}}$ *and* $\mathcal{M}^1{}_{\widetilde{L\mathfrak{g}}}$*: the sheaves of germs of integrable $L\mathfrak{g}$ and $\widetilde{L\mathfrak{g}}$ valued 1-forms over M.*

Stalks of these sheaves at x are denoted by $\mathbb{C}^*{}_{t,x}$, etc. Then we have the following commutative diagram with exact lines and columns.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Phi^1 & \xrightarrow{i} & \mathcal{M}^1{}_{\widetilde{L\mathfrak{g}}} & \xrightarrow{j} & \mathcal{M}^1{}_{L\mathfrak{g}} & \longrightarrow & 0 \\
 & & \uparrow & (\widetilde{\rho}_i) & \rho\uparrow & & \rho\uparrow & & \\
0 & \longrightarrow & \mathbb{C}^*{}_d & \xrightarrow{i} & \widetilde{LG}_d & \xrightarrow{j} & LG_d & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbb{C}^*{}_t & \xrightarrow{i} & \widetilde{LG}_t & \xrightarrow{j} & LG_t & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Here $\widetilde{\rho}_I$ is not a sheaf map (not continuous). But we have

$$\widetilde{\rho}_I(i(G)) = i(\rho(g)) = (0, g^{-1}dg),$$

$$\rho(j(g, c)) = j(\widetilde{\rho}_I(g, c)) = g^{-1}dg.$$

ρ and $\widetilde{\rho}_I$ are right logarithmic derivations of g and (g, c). Corresponding left logarithmic derivations ρ_L and $\widetilde{\rho}_{I,L}$ are given by

$(6)_L$ $\rho_L(\mathfrak{g}) = (dg)g^{-1}$,

$$\widetilde{\rho}_{I,L}((g, c)) = \left(\rho_L(g), c^{-1}dc + \frac{1}{2}I\left(\int_0^1 \mathrm{tr}\left(\rho_L(g) \wedge (\rho_L(g))'\right)dt\right)\right).$$

3. **Definition 2.** *Let $\widetilde{\zeta} = (\zeta, \beta)$ be an $\widetilde{L}\mathfrak{g}$-valued differential form and g is an LG-valued function. Then we define the adjoint action $\widetilde{\zeta}^g$ of g on $\widetilde{\zeta}$ by*

$$\widetilde{\zeta}^g = \left(\zeta^g, \beta + \int_0^1 \mathrm{tr}(\zeta g' g^{-1})dt\right), \qquad \zeta^g = g^{-1}\zeta g. \tag{7}$$

We also define the left adjoint action ${}^g\widetilde{\zeta}$ by

$$ {}^g\widetilde{\zeta} = \widetilde{\zeta}^{g^{-1}}. \tag{7$_L$}$$

By definition, if $\widetilde{\theta}$ is right integrable and locally takes the form $\widetilde{\theta} = (g^{-1}dg, \beta)$, then ${}^g\widetilde{\theta}$ is left integrable, that is, we have

$$d({}^g\widetilde{\theta}) - \frac{1}{2}[{}^g\widetilde{\theta}, {}^g\widetilde{\theta}] = 0.$$

Let $\mathfrak{U} = \{U_i\}$ be a locally finite open covering of M, $C^p(\mathfrak{U}, \mathcal{M}^1{}_{L\mathfrak{g}})$ and $C^p(\mathfrak{U}, \mathcal{M}^1{}_{\widetilde{L}\mathfrak{g}})$ the sets of p-cochains with coefficients in $\mathcal{M}^1{}_{L\mathfrak{g}}$ and $\mathcal{M}^1{}_{\widetilde{L}\mathfrak{g}}$. If $\{\omega_{ij}\}$ belongs to $C^1(\mathfrak{U}, \mathcal{M}^1{}_{L\mathfrak{g}})$, then we define

$$\delta\omega_{ijk} = \omega_{jk} - \omega_{ik} + \omega_{ij}{}^{g_{jk}}, \qquad \omega_{ij} = g_{ij}{}^{-1}dg_{ij}.$$

Similarly, if $\{\widetilde{\omega}_{ij}\}$ belongs to $C^1(\mathfrak{U}, \mathcal{M}^1{}_{\widetilde{L}\mathfrak{g}})$, then we define

$$\delta\widetilde{\omega}_{ijk} = \widetilde{\omega}_{jk} - \widetilde{\omega}_{ik} + \widetilde{\omega}_{ij}{}^{g_{jk}}, \qquad \widetilde{\omega}_{ij} = (g_{ij}^{-1}dg_{ij}, \beta_{ij}).$$

Then we can define the cohomology sets $H^1(M, \mathcal{M}^1{}_{L\mathfrak{g}})$ and $H^1(M, \mathcal{M}^1{}_{\widetilde{L}\mathfrak{g}})$ (cf. [5], [6]).

Lemma 2. *If $\{\omega_{ij}\} \in C^1(\mathfrak{U}, \mathcal{M}^1{}_{L\mathfrak{g}})$ is a cocycle, that is $\delta\omega_{ijk} = 0$, then*

$$\begin{aligned}\frac{1}{2}\int_0^1 &\mathrm{tr}\left(\omega_{jk} \wedge \omega_{jk}' - \omega_{ik} \wedge \omega_{ik}' + \omega_{ij}{}^{g_{jk}} \wedge (\omega_{ij}{}^{g_{jk}})'\right) dt \\ &= d\left(\int_0^1 \mathrm{tr}(\omega_{ij} g_{jk}' g_{jk}{}^{-1})dt\right),\end{aligned} \tag{8}$$

$$\omega_{ij} = g_{ij}{}^{-1}dg_{ij}, \quad g_{ij}g_{jk}g_{ki} = c_{ijk}, \quad \text{a constant } (\in LG).$$

Proof. Since $\delta\omega_{ijk} = 0$, we have

$$\begin{aligned}&\mathrm{tr}\left(\omega_{jk} \wedge \omega_{jk}' - \omega_{ik} \wedge \omega_{ik}' + \omega_{ij}{}^{g_{jk}} \wedge (\omega_{ij}{}^{g_{jk}})'\right) \\ &\quad = 2\,\mathrm{tr}(\omega_{ij}{}^{g_{jk}}(g_{jk}{}^{-1}g_{jk}'\omega_{ik} - g_{jk}{}^{-1}d(g_{jk}'))).\end{aligned}$$

Then, since $g_{jk}{}^{-1}g_{ij}{}^{-1} = g_{ki}c_{ikj}$, we get

$$\begin{aligned}&\mathrm{tr}(\omega_{ij}{}^{g_{jk}}(g_{jk}{}^{-1}g_{jk}'\omega_{ik} - g_{jk}{}^{-1}d(g_{jk}'))) \\ &\quad = -\mathrm{tr}(c_{ikj}(dg_{ij}d(g_{jk}'g_{ki}))) = d(\mathrm{tr}(\omega_{ij}{}^{g_{jk}}g_{jk}{}^{-1}g_{jk}'). \qquad \blacksquare\end{aligned}$$

Corollary. *If $\delta\omega_{ijk} = 0$ and a 1-cocycle of scalar 1-forms $\{\alpha_{ij}\}$ satisfies*

$$d\alpha_{ij} + \frac{1}{2}\int_0^1 \mathrm{tr}(\alpha_{ij} \wedge \omega_{ij}')dt = 0,$$

then the 2-cochain $\{\beta_{ijk}\}$ given by

$$\beta_{ijk} = \alpha_{jk} - \alpha_{ik} + \alpha_{ij} + \int_0^1 \mathrm{tr}(\omega_{ij}g_{jk}'g_{jk}{}^{-1})dt, \tag{9}$$

belongs to $\mathbb{Z}^2(\mathfrak{U}, \Phi^1)$, that is we have

$$d\beta_{ijk} = 0, \qquad \beta_{ijk\ell} = \beta_{jk\ell} - \beta_{ik\ell} + \beta_{ij\ell} - \beta_{ijk} = 0, \tag{10}$$

provided $c_{ijk}' = 0$.

By this Corollary we can define the coboundary map $\delta : H^1(M, \mathcal{M}^1{}_{L\mathfrak{g}}) \to H^2(M, \Phi^1)$ by $\delta((\langle\{\omega_{ij}\}\rangle) = \langle\{\beta_{ijk}\}\rangle$. Here $\langle\{\zeta\}\rangle$ means the cohomology class of $\{\zeta\}$. Then we obtain

Proposition 2. *The following diagram is commutative and each line is exact:*

$$\begin{array}{ccccccccc}
0 \longrightarrow & H^0(M, \Phi^1) & \xrightarrow{i} & H^0(M, \mathcal{M}^1{}_{\widetilde{L}\mathfrak{g}}) & \xrightarrow{j} & H^0(M, \mathcal{M}^1{}_{L\mathfrak{g}}) & \xrightarrow{\delta} & H^1(M, \Phi^1) & \\
 & \uparrow & & & & \rho\uparrow & & \uparrow & \\
0 \longrightarrow & H^0(M, \mathbb{C}^*{}_d) & \xrightarrow{i} & H^0(M, \widetilde{L}G_d) & \xrightarrow{j} & H^0(M, LG_d) & \xrightarrow{\delta} & H^1(M, \mathbb{C}^*{}_d) & \\
\xrightarrow{i^*} & H^1(M, \mathcal{M}^1{}_{\widetilde{L}\mathfrak{g}}) & \xrightarrow{j^*} & H^1(M, \mathcal{M}^1{}_{L\mathfrak{g}}) & \xrightarrow{\delta} & H^2(M, \Phi^1) \cong H^3(M, \mathbb{C}) & & & \\
 & \uparrow & & \rho\uparrow & & \uparrow & & & \\
\xrightarrow{i^*} & H^1(M, \widetilde{L}G_d) & \xrightarrow{j^*} & H^1(M, LG_d) & \xrightarrow{\delta} & H^2(M, \mathbb{C}^*{}_d) \cong H^3(M, \mathbb{Z}) & & & \\
 & & & \uparrow & & \uparrow & & & \\
 & & \mathrm{Hom}(\pi_1(M), LG) \cong & H^1(M, LG_t) & \xrightarrow{\delta} & H^2(M, G_t). & & &
\end{array}$$

Note. We get the same commutative diagram with exact lines replacing LG, $L\mathfrak{g}$, etc., by ΩG, $\Omega\mathfrak{g}$, etc.

§2. Geometric meanings of loop group bundles and Gysin map in non-abelian de Rham theory

4. For a complex matrix A, we define linear maps $F_A : \mathfrak{g} \to \mathfrak{g}$ and $G_A : \mathfrak{g} \to \mathfrak{g}$ by

$$F_A(X) = X + \sum_{n=2}^{\infty} \frac{1}{n!}\left(\sum_{s=0}^{n-1} A^s X A^{n-s-1}\right),$$
$$G_A(X) = X + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!}(\mathrm{ad}\,A)^n(X),$$
$$(\mathrm{ad}\,A)(X) = [A, X] = AX - XA.$$

In [5] we showed

Lemma 3. (i) *$F_A(X)$ is equal to $e^A G_A(X)$. It is also shown that*

$$F_A(X) = G_{A,L}(X)e^A, \quad G_{A,L}(X) = X + \sum_{n=1}^{\infty} \frac{1}{(n+1)!}(\mathrm{ad}\,A)^n(X).$$

(ii) *The Jacobian of* $\exp : \mathfrak{g} \to G$ *at A is F_A.*

Corollary. (i) *The Jacobian of* $\exp : \mathfrak{g} \to G$ *is non-degenerate at A if and only if A satisfies the following condition:*

(∗) *If λ_i and λ_j are distinct proper values of A, then $\frac{1}{2\pi\sqrt{-1}}(\lambda_i - \lambda_j)$ is not integer.*

(ii) *A smooth G-valued function g on M is locally written as $g = e^f$, where f is smooth $\mathfrak{g}$-valued function on some open set of M.*

We denote by $\mathfrak{g}_d$ and G_d the sheaves of germs of smooth $\mathfrak{g}$ and G-valued functions over M. Then $\exp : \mathfrak{g} \to G$ induces a sheaf map $\exp : \mathfrak{g}_d \to G_d$. Its kernel sheaf is denotes by $N_{\mathfrak{g},d}$. In [5], we defined the first cohomology set of M with coefficients in $N_{\mathfrak{g},d}$ as follows: Let

$$\delta_P n_{ijk} = n_{jk} - n_{ik} + n_{ij}{}^{P_{jk}}, \qquad \{n_{ij}\} \in C^1(\mathfrak{U}, N_{\mathfrak{g},d}),$$
$$\{P_{ij}\} \in C^1(\mathfrak{U}, G_d).$$

By using this coboundary map, we can define $H^1(M, N_{\mathfrak{g},d})$. Then we have the following exact sequence:

$$0 \longrightarrow H^0(M, N_{\mathfrak{g},d}) \longrightarrow H^0(M, \mathfrak{g}_d) \xrightarrow{\exp} H^0(M, G_d) \xrightarrow{\delta} H^1(M, N_{\mathfrak{g},d}).$$

If $\delta_P n_{ijk} = 0$, the relation

$$U_i \times N_{\mathfrak{g}} \ni (x, n(x)) \sim \left(x, P_{ij}(x)n(x)P_{ij}(x)^{-1} + n_{ij}(x)\right) \in U_j \times N_{\mathfrak{g}},$$

is an equivalence relation. The quotient space of $\bigcup U_i \times N_{\mathfrak{g}}$ by this relation is a fibre bundle over M with the fibre $N_{\mathfrak{g}}$. Hence we have

Lemma 4. *If M is contractible, then* $\exp : H^0(M, \mathfrak{g}_d) \to H^0(M, G_d)$ *is onto.*

Corollary 1. *Let $i : M \to E$ be a smooth imbedding of M into a contractible space E. Then we have*

(10) $$\exp(H^0(M, \mathfrak{g}_d)) = i^*(H^0(E, G_d)).$$

Corollary 2. $\exp(H^0(M, \mathfrak{g}_d))$ *is a normal subgroup of* $H^0(M, G_d) = \mathrm{Map}(M, G)$.

Note. In general $\exp(H^0(M, \mathfrak{g}_d))$ is *not* a closed subgroup of $\mathrm{Map}(M, G)$ (cf. [23]).

5. Let g be a smooth G-valued function on M. Then by Corollary of Lemma 3, there is a locally finite open covering $\mathfrak{U} = \{U_i\}$ of M such that $g(x) = \exp\left(2\pi\sqrt{-1}f_i(x)\right)$, f_i is a smooth $\mathfrak{g}$-valued function on U_i.

On $(U_i \cap U_j) \times \mathbb{C}^*$, we set

$$g_{ij}(x, z) = e^{f_i(x)\log z} e^{-f_j(x)\log z}, \quad x \in U_i \cap U_j, \quad z \in \mathbb{C}^*.$$

By definition $g_{ij}(x, z)$ is single-valued and $g_{ij}(x, 1) = I$, the unit matrix. Hence we can define a smooth ΩG-valued function $g_{ij}{}^{\Omega}$ on $U_i \cap U_j$ by

$$(g_{ij}{}^{\Omega}(x))(t) = g_{ij}\left(x, e^{2\pi\sqrt{-1}t}\right).$$

If $\exp\left(2\pi\sqrt{-1}f_{i,1}(x)\right) = \exp\left(2\pi\sqrt{-1}f_{i,2}(x)\right)$, then we have

$$g_{ij,2}(x) = h_i(x)g_{ij,1}(x)h_j(x)^{-1},$$
$$(h_i(x))(t) = e^{2\pi\sqrt{-1}f_{i,2}(x)t} e^{-2\pi\sqrt{-1}f_{i,1}(x)t}.$$

Hence $\{g_{ij}\}$ defines an ΩG-bundle $B_0(g)$ over M and its equivalence class as an ΩG-bundle is determined by g. By definition, $B_0(g)$ is trivial if $g = e^f$, f a smooth $\mathfrak{g}$-valued function on M.

On the other hand, if g is an ΩG-valued function, we define a $\mathrm{Map}(\mathbb{R}, G)$-valued function $\widetilde{g}$ by

$$(\widetilde{g}(x))(t) = (g(x))\left(e^{2\pi\sqrt{-1}t}\right), \qquad t \in \mathbb{R}.$$

We note that $(\widetilde{g}(x))(0) = I$ for all x. If $\xi = \{g_{ij}\}$ is an ΩG-bundle, then we define a $\mathrm{Map}(\mathbb{R}, G)$-bundle $\widetilde{\xi}$ by $\{\widetilde{g}_{ij}(x)\}$. Then, since $\mathrm{Map}(\mathbb{R}, G)$ is a contractible group, we can set

$$\widetilde{g}_{ij}(x) = \widetilde{h}_i(x)\widetilde{h}_j(x)^{-1}, \qquad \widetilde{h}_i(x) \text{ is a smooth } \mathrm{Map}(\mathbb{R}, G)\text{-valued function on } U_i.$$

By definition we have $(\widetilde{g}_{ij}(x))(t) = (\widetilde{g}_{ij}(x))(t+1)$. Hence we have

$$(\widetilde{h}_i(x)(t))^{-1}(\widetilde{h}_i(x)(t+1)) = ((\widetilde{h}_j(x)(t))^{-1}(\widetilde{h}_j(x)(t+1)).$$

Therefore we can define a smooth G-valued function g on M by

$$g(x) = (\widetilde{h}_i(x)(0))^{-1}(\widetilde{h}_i(x)(1)) = \widetilde{h}_i(x)(1), \qquad x \in U_i.$$

By the definition of B_0 we get $B_0(g) = \xi$. Hence we obtain

Theorem 1. *There is a bijection*

$$B_0 \cdot H^0(M, G_d)/\exp(H^0(M, \mathfrak{g}_d)) \cong H^1(M, \Omega G_d).$$

Note. If $g(x) = \exp\left(2\pi\sqrt{-1}f(x)\right)$, we have $g(x)\exp(f(x)\log z) = \exp(f(x)\log z)g(x)$. Hence

$$g_{ij,L}(x, z) = e^{-f_i(x)\log z}e^{f_j(x)\log z}$$

is also a single valued smooth function on $(U_1 \cap U_j) \times C^*$. By using this $\{g_{ij,L}\}$, we can define an alternative bijection

$$B_{0L} : H^0(M, G_d)/\exp(H^0(M, \mathfrak{g}_d)) \cong H^1(M, G_d).$$

The relation between B_0 and B_{0L} is given by

(11) $$B_0(g^{-1}) = B_{0L}(g) \quad (= (B_0(g))^{-1}).$$

We denote the connected component of the identity of $H^0(M, G_{n,d}) = H^0(M, G_d)$ by $H^0(M, G_{n,d})_0$. It contains $\exp(H^0(M, \mathfrak{g}_{n,d}))$ and there is a map $k : H^0(M, G_{n,d})/H^0(M, G_{n,d})_0 \to K^1(M)$ $(= K^{-1}(M))$. k is an isomorphism if n is sufficiently large. Hence there is a homomorphism

$$k^1 : H^1(M, G_{n,d}) \to K^1(M),$$

such that $\ker k^1 = B^0(H^0(M, G_{n,d})_0/\exp(H^0(M, \mathfrak{g}_{n,d}))$. k^1 is onto if n is sufficiently large.

6. We denote by $\mathcal{M}^1$ and $M^1{}_L$ the sheaves of germs of right and left integrable 1-forms. We have

$$\mathcal{M}^1 = \rho(G_d) = d^e(\mathfrak{g}_d), \qquad \mathcal{M}^1{}_L = \rho_L(G_d) = d^e{}_L(\mathfrak{g}_d).$$

Here $\rho(g) = g^{-1}dg$, $\rho_L(g) = (dg)g^{-1}$. d^e and $d^e{}_L$ are given by

$$d^e f = e^{-f}d(e^f) = df + \sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)!}(\mathrm{ad}\, f)^n(df),$$

$$d^e{}_L f = (d(e^f))e^{-f} = df + \sum_{n=1}^{\infty}\frac{1}{(n+1)!}(\mathrm{ad}\, f)^n(df).$$

If θ belongs to $H^0(M, \mathcal{M}^1)$, $\pi^*(\theta)$ is integrated on $\widetilde{M}$, the universal covering space of M. Here $\pi : \widetilde{M} \to M$ is the projection.

Definition 3. *We set*

$$H^0(M, \mathcal{M}^1)_0' = \{\theta | \pi^*(\theta) = \rho(g),\ \rho_L(g) \in \pi^*(H^0(M, \mathcal{M}^1{}_L))\}.$$

By definition θ belongs to $H^0(M, \mathcal{M}^1)_0$ if and only if $\pi^*(\theta) = \rho(g)$ and g satisfies

(12) $$g^\sigma = \chi_\sigma g = g\chi_\sigma, \qquad \sigma \in \pi_1(M), \quad \chi \in \mathrm{Hom}(\pi_1(M), G).$$

Theorem 1'. *There is a map $\widetilde{B}_0 : H^0(M, \mathcal{M}^1)_0/d^e(H^0, \mathfrak{g}_d)) \to H^1(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}})$ such that the following diagram becomes commutative:*

$$\begin{array}{ccc} H^0(M, \mathcal{M}^1)_0/d^e(H^0(M, \mathfrak{g}_d)) & \xrightarrow{\widetilde{B}_0} & H^1(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}}) \\ \rho\uparrow & & \rho\uparrow \\ H^0(M, G_d)_0/\exp(H^0(M, \mathfrak{g}_d)) & \xrightarrow{B_0} & H^1(M, \Omega G_d). \end{array}$$

Proof. By (1) we can take local integrations $\{g_i\}$ of $\theta \in H^0(M, \mathcal{M}^1)_0$ to be

$$g_i = c_{ij} g_j = g_j c_{ij}, \qquad \text{on } U_i \cap U_j. \tag{12$'$}$$

We assume $g_i = \exp\left(2\pi\sqrt{-1} f_i\right)$ on U_i and set

$$h_{ij}(x, z) = e^{f_i(x)\log z} e^{-f_j(x)\log z}.$$

Then by $(12)'$ we have

$$h_{ij}(x, z) h_{jk}(x, z) h_{ki}(x, z) = c_{ij} c_{jk} c_{ki}.$$

Hence if we define $\rho_x(h_{ij})^\Omega$ by

$$\left(\rho_x(h_{ij})^\Omega(x)\right)(t) = \rho_x\left(h_{ij}\left(x, e^{2\pi\sqrt{-1}t}\right)\right),$$
$$\rho_x(f(x,t)) = ((x,t)^{-1} d_x f(x,t),$$

$\{\rho_x(h_{ij})\}$ becomes a cocycle. Hence we can define $\widetilde{B}_0$ by

$$\widetilde{B}_0([\theta]) = \left\{\rho_x(h_{ij})^\Omega\right\} \quad (= \{\rho(h_{ij}{}^\Omega)\}).$$

Then we have the Theorem by the definitions of B_0 and $\widetilde{B}_0$. ■

$\widetilde{B}_0$ is not onto. We set $H^1(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}})_0$ to be the subset of $H^1(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}})$ whose representing cocycle $\{\omega_{ij}\}$ satisfies

$$\omega_{ij} = g_{ij}{}^{-1} dg_{ij}, \qquad \frac{d}{dt}(g_{ij}(x) g_{jk}(x) g_{ki}(x)) = 0.$$

Then we have

$$\widetilde{B}^0(H^0(M, \mathcal{M}^1)_0 / d^e(H^0(M, \mathfrak{g}_d)) \subset H^1(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}})_0.$$

If g is a smooth LG-valued function on M, we define a smooth G-valued function g^b on $M \times S^1$ by

$$g^b(x, t) = (g(x))(t). \tag{13}$$

Since $d(g^b)$ contains derivation in t, $(g^b)^{-1} d(g^b)$ is not determined by $g^{-1}dg$. If $\{\omega_{ij}\}$ is a representing cocycle of an element of $H^1(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}})_0$ and $\{g_{ij,1}\}$ and $\{g_{ij,2}\}$ are integrations of $\{\omega_{ij}\}$ such that

$$\frac{d}{dt}(g_{ij,1}(x) g_{jk,1}(x) g_{ki,1}(x)) = \frac{d}{dt}(g_{ij,2}(x) g_{jk,2}(x) g_{ki,2}(x)) = 0,$$

the difference between $\left\{(g_{ij,1}{}^b)^{-1} dg_{ij,1}{}^b\right\}$ and $\left\{(g_{ij,2}{}^b)^{-1} dg_{ij,2}{}^b\right\}$ comes from a representing cocycle of an element of $H^1(S^1, \mathcal{M}^1)$. Since an element of $H^1(S^1, \mathcal{M}^1)$ is determined by a representation of (the universal covering group of) $[\Omega S^1{}_e]$, the group of zero homotopic paths of S^1 ([7], cf. [18], [19], [21]), $H^1(S^1, \mathcal{M}^1)$ vanishes. Hence we can define the map $\natural : H^1(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}})_0 \to H^1(M \times S^1, \mathcal{M}^1)$ by

$$\langle\{\omega_{ij}\}\rangle^\natural = \left\langle\left\{(g_{ij}{}^b)^{-1} d(g_{ij}{}^b)\right\}\right\rangle.$$

Here $\{g_{ij}\}$ is assumed to be $(g_{ij} g_{jk} g_{ki})' = 0$. On the other hand, we define $b : H^1(M, \Omega G_d) \to H^1(M \times S^1, G_d)$ by $\{g_{ij}\}^b = \{g_{ij}{}^b\}$. Then we have the commutative diagram

$$\begin{array}{ccc} H^1(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}})_0 & \xrightarrow{\natural} & H^1(M \times S^1, \mathcal{M}^1) \\ {\scriptstyle\rho}\uparrow & & {\scriptstyle\rho}\uparrow \\ H^1(M, \Omega G_d) & \xrightarrow{b} & H^1(M \times S^1, G_d). \end{array}$$

Definition 4. *We define Gysin maps* $\gamma : H^0(M, G_d)/\exp(H^0(M, \mathfrak{g}_d)) \to H^1(M \times S^1, G_d)$ *and* $\widetilde{\gamma} : H^0(M, \mathcal{M}^1)_0 / d^e(H^0(M, \mathfrak{g}_d)) \to H^1(M \times F^1, \mathcal{M}^1)$ *by*

$$\gamma([g]) = (B_0([g]))^b, \qquad \widetilde{\gamma}(\langle\omega\rangle) = (\widetilde{B}_0(\langle\omega\rangle))^\natural.$$

By the definitions the following diagram is commutative:

$$\begin{array}{ccc} H^0(M, \mathcal{M}^1)_0 / d^e(H^0(M, \mathfrak{g}_d)) & \xrightarrow{\widetilde{\gamma}} & H^1(M \times S^1, \mathcal{M}^1) \\ {\scriptstyle\rho}\uparrow & & {\scriptstyle\rho}\uparrow \\ H^0(M, G_d)/\exp(H^0(M, \mathfrak{g}_d)) & \xrightarrow{\gamma} & H^1(M \times S^1, G_d). \end{array}$$

§3. Connections of loop group bundles and string classes

7. We can define connections and curvatures of loop group bundles and elements of $H^1(M,\mathcal{M}^1{}_{\Omega\mathfrak{g}})$ similarly as connections and curvatures of G-bundles and elements of $H^1(M,\mathcal{M}^1)$ (cf. [5], [6]). If M has a smooth partition of unity subordinate to any locally finite open covering of M, connections always exist. Next we define the connection form $\{\widetilde{\theta}_i\}$ of an element of $H^1(M,\mathcal{M}^1{}_{\widetilde{\Omega\mathfrak{g}}})$ whose representing cocycle is $\{\widetilde{\omega}_{ij}\}$ by the relation

$$(14)\qquad \widetilde{\omega}_{ij}=\widetilde{\theta}_j-\widetilde{\theta}_i{}^{g_{ij}},\qquad \widetilde{\omega}_{ij}=({g_{ij}}^{-1}dg_{ij},\beta_{ij}).$$

Definition 5. *Let $\widetilde{\xi}$ be an $\widetilde{\Omega}G$-bundle and $\langle\{\widetilde{\omega}\}\rangle$ an element of $H^1(M,\mathcal{M}^1{}_{\widetilde{\Omega\mathfrak{g}}})$ such that*

$$j^*(\langle\{\widetilde{\omega}\}\rangle)=\rho^*(j^*(\widetilde{\xi})).$$

Then we say a connection of $\langle\widetilde{\omega}\rangle$ to be a connection of $\widetilde{\omega}$.

Note. Connections of elements of $H^1(M,\mathcal{M}^1{}_{\widetilde{L\mathfrak{g}}})$ and $\widetilde{L}G$-bundles are similarly defined.

The curvature $\{\widetilde{\Theta}_i\}$ of $\{\widetilde{\theta}_0\}$ is defined by

$$(15)\qquad \widetilde{\Theta}_i=d\widetilde{\theta}_i+\frac{1}{2}[\widetilde{\theta}_i,\widetilde{\theta}_i].$$

We set $\widetilde{\theta}_i=(\theta_i,\psi_i)$ and $\widetilde{\Theta}_i=(\Theta_i,\Psi_i)$. Then (14) and (15) mean

$$(14)'\qquad \begin{aligned}\omega_{ij}&=\theta_j-{g_{ij}}^{-1}\theta_i g_{ij},\\ \beta_{ij}&=\psi_j-\left(psi_i+\int_0^1\operatorname{tr}(\theta_i g_{ij}'{g_{ij}}^{-1})dt\right),\end{aligned}$$

$$(15)'\qquad \begin{aligned}\Theta_i&=d\theta_i+\theta_i\wedge\theta_i,\\ \Psi_i&=d\psi_i+\frac{1}{2}\int_0^1\operatorname{tr}(\theta_1\wedge\theta_i')dt.\end{aligned}$$

Proposition 3. (i) *If $\{\widetilde{\omega}_{ij}\}$ is a representing cocycle of an element of $H^1(M,\mathcal{M}^1{}_{\widetilde{\Omega\mathfrak{g}}})$, then it has a connection.*

(ii) *If $\{\widetilde{\theta}_i\}=\{(\theta_i,\psi_i)\}$ and $\{\widetilde{\theta}_{i,1}\}=\{(\theta_{i,1},\psi_{i,1})\}$ are connections of $\{\widetilde{\omega}_{ij}\}$, then*

$$(16)\qquad \begin{aligned}\theta_{i,1}&=\theta_i+\eta_i,\quad \eta_j={g_{ij}}^{-1}\eta_i g_{ij},\\ \psi_{i,1}&=\psi_i+\phi,\quad \phi \text{ is a global 1-form on } M.\end{aligned}$$

(iii) *If a collection of $\widetilde{L}\mathfrak{g}$-valued differential forms $\{\widetilde{\phi}_i\}=\{(\Phi_i,\zeta_i)\}$ satisfies $\widetilde{\phi}_i{}^{g_{ij}}=\widetilde{\phi}_j$, then we have*

$$(17)\qquad d\widetilde{\phi}_j+[\widetilde{\theta}_j,\widetilde{\theta}_j]=(d\widetilde{\phi}_i+[\widetilde{\theta}_i,\widetilde{\phi}_i])^{g_{ij}}.$$

Proof. Let $\{e\}$ be a smooth partition of unity subordinate to $\{U_i\}$. Then if we set $\widetilde{\omega}_{ki}=(\omega_{ki},\beta_{ki})$ and define

$$\theta_i=\sum_{U_i\cap U_k\neq\emptyset}e_k\omega_{ki},\qquad \psi_i=\sum_{U_i\cap U_k\neq\emptyset}e_k\beta_{ki},$$

we get $\omega_{ij}=\theta_j-{g_{ij}}^{-1}\theta_i g_{ij}$ and $\beta_{ij}=\psi_j-\psi_i+\sum e_k\int_0^1\operatorname{tr}(\omega_{ki}g_{ij}'{g_{ij}}^{-1})dt=\psi_j-(\psi_i+\int_0^1\operatorname{tr}(\theta_i g_{ij}'{g_{ij}}^{-1})dt$. Hence we have (i). (ii) follows from (14)$'$.

Since $\{\theta_i\}$ is a connection of $\{\omega_{ij}\}$, we have $d\phi_j+[\theta_j,\phi_j]={g_{ij}}^{-1}(d\phi_i+[\theta_i,\phi_i])g_{ij}$. Since $\theta_j={g_{ij}}^{-1}\theta_i g_{ij}+{g_{ij}}^{-1}dg_{ij}$, we get

$$\begin{aligned}&d\left(\int_0^1\operatorname{tr}(\phi_i g_{ij}'{g_{ij}}^{-1})dt\right)-\int_0^1\operatorname{tr})\theta_j'\phi_j)dt+\\ &+\int_0^1\operatorname{tr}(\theta_i'\phi_i)dt-\int_0^1\operatorname{tr}(d\phi_i g_{ij}'{g_{ij}}^{-1})dt\\ &=\int_0^1\operatorname{tr}([\theta_i,\phi_i]g_{ij}'{g_{ij}}^{-1})dt.\end{aligned}$$

Hence we have

$$\begin{aligned}&d\zeta_j+\int_0^1\operatorname{tr}(\theta_j\phi_j')dt\\ &=d\zeta_i+d\left(\int_0^1\operatorname{tr}(\phi_i g_{ij}'{g_{ij}}^{-1})dt\right)+\int_0^1\operatorname{tr}(\theta_j\phi_j')dt\\ &=d\zeta_i+\int_0^1\operatorname{tr}(\theta_i\phi_i')dt+\int_0^1\operatorname{tr}((d\phi_i+[\theta_i,\phi_i])g_{ij}'{g_{ij}}^{-1})dt.\end{aligned}$$

This shows (iii).

8. By straightforward calculations we obtain

Lemma 5. *Let $\{\theta_i\}$ be a connection form of $\{\omega_{ij}\} = g_{ij}{}^{-1}dg_{ij}$, a representing cocycle of an element of $H^1(M, \mathcal{M}^1{}_{L\mathfrak{g}})$, then*

$$(18)\quad \int_0^1 \mathrm{tr}\,(\theta_i{}^{g_{ij}} \wedge \omega_{ij}{}' + \omega_{ij} \wedge (\theta_i{}')^{g_{ij}} - \omega_{ij} \wedge [g_{ij}{}^{-1}g_{ij}{}', \theta_i{}^{g_{ij}}] - 2\theta_i d(g_{ij}{}')g_{ij}{}^{-1} + 2\theta_i g_{ij}{}' g_{ij}{}^{-1}(dg_{ij})g_{ij}{}^{-1})\, dt = 0.$$

Proposition 4. *The coordinate transformation law $\widetilde{\Theta}_i{}^{g_{ij}} = \widetilde{\Theta}_j$ and Bianchi identity $d\widetilde{\Theta}_i + [\widetilde{\theta}_i, \widetilde{\Theta}_i] =$ hold for the $\widetilde{L}\mathfrak{g}$-valued curvature form $\{\widetilde{\Theta}_i\}$. That is, we have*

$$(19)\quad \Theta_j = g_{ij}{}^{-1}\Theta_i g_{ij}, \quad \Psi_j = \Psi_i + \int_0^1 \mathrm{tr}(\Theta_i g_{ij}{}' g_{ij}{}^{-1})dt,$$

$$(20)\quad d\Theta_i + [\theta_i, \Theta_i] = 0, \quad d\Psi_i + \int_0^1 \mathrm{tr}(\theta_i \wedge \Theta_i{}')dt = 0.$$

Proof. We need only to show the second equalities of (19) and (20). Since we obtain

$$d(\psi_j - \psi_i) = -\frac{1}{2}\int_0^1 \mathrm{tr}\,\omega_{ij} \wedge \omega_{ij}{}')dt + \int_0^1 \mathrm{tr}(d\theta_i g_{ij}{}' g_{ij}{}^{-1} - \theta_i d(g_{ij}{}')g_{ij}{}^{-1} + \theta_i g_{ij}{}' g_{ij}{}^{-1}(gd_{ij})g_{ij}{}^{-1})dt,$$

we have

$$2(\Psi_j - \Psi_i) = \int_0^1 \mathrm{tr}(\theta_j \wedge \theta_j{}' - \theta_i \wedge \theta_i{}' - \omega_{ij}\omega_{ij}{}')dt + 2\int_0^1 \mathrm{tr}(d\theta_i g_{ij}{}' g_{ij}{}^{-1} - \theta_i d(g_{ij})g_{ij}{}^{-1} + \theta_i g_{ij}{}' g_{ij}{}^{-1}(fg_{ij})g_{ij}{}^{-1})dt.$$

Since $\theta_j = g_{ij}{}^{-1}\theta_i g_{ij} + \omega_{ij}$, we get the second equality of (19) by this equality and Lemma 5.

By the definition of Ψ_i we have $d\Psi_i = \int_0^1 \mathrm{tr}(d\theta_i \wedge \theta_i{}')dt$. Hence by Lemma 1, we obtain the second equality of (20). ■

The second equalities of (19) and (20) are rewritten as

$$(19)'\quad \int_0^1 \mathrm{tr}(\Theta_i g_{ij}{}' g_{ij}{}^{-1})dt = \Psi_j - \Psi_i,$$

$$(20)'\quad d\Psi_i = \int_0^1 \mathrm{tr}(\Theta_i \wedge \theta_i{}')dt.$$

We generalize (19)′ and (20)′ as follows: Let $\{\Theta_i\}$ be a curvature form of $\{\omega_{ij}\} = \{g_{ij}{}^{-1}dg_{ij}\}$. Then we set

$$\phi_{p,ij} = \int_0^1 \mathrm{tr}(\Theta_i{}^p g_{ij}{}' g_{ij}{}^{-1})dt, \qquad \Theta^p = \overbrace{\Theta \wedge \ldots \wedge \Theta}^{p}.$$

$\{\phi_{p,ij}\}$ is a 1-cochain of $2p$-forms, and we have by straightforward calculations

Lemma 6. *As a 1-cochain of $2p$-forms, we obtain*

$$\delta\phi_{p,ijk} = \int_0^1 \mathrm{tr}(\Theta_k{}^p c_{kij}{}' c_{kij}{}^{-1})dt, \quad c_{ijk} = g_{ij}g_{jk}g_{ki}.$$

Corollary. *If $\{\omega_{ij}\}$ is a representing cocycle of an element of $H^1(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}})_0$, then there exists a 0-chain of $2p$-forms $\{\Psi_{p,i}\}$ such that*

$$(21)\quad \int_0^1 \mathrm{tr}(\Theta_i{}^p g_{ij}{}' g_{ij}{}^{-1})dt = \Psi_{p,j} - \Psi_{p,i}.$$

Lemma 7. *Let $\{\theta_i\}$ be a connection form of $\{\omega_{ij}\} = \{g_{ij}{}^{-1}dg_{ij}\}$, $\{\Theta_i\}$ the curvature form of $\{\theta_i\}$. Then we have*

$$(22)\quad d\left(\int_0^1 \mathrm{tr}(\Theta_i{}^p g_{ij}{}' g_{ij}{}^{-1})dt\right) = \int_0^1 \mathrm{tr}(\Theta_j^p \wedge \theta_j{}')dt - \int_0^1 \mathrm{tr}(\Theta_i{}^p \wedge \theta_i{}')dt.$$

Proof. By the Bianchi identity we get $d(\Theta_i{}^p) + [\theta_i, \Theta_i{}^p] = 0$. Hence we have

$$d\left(\int_0^1 \mathrm{tr}(\Theta_i{}^p g_{ij}{}' g_{ij}{}^{-1})dt\right) = \int_0^1 \mathrm{tr}([\Theta_i^p, \theta_i]g_{ij}{}' g_{ij}{}^{-1} + \Theta_i{}^p d(g_{ij}{}' g_{ij}{}^{-1} - \Theta_i{}^p g_{ij}{}' \omega_{ij} g_{ij}{}^{-1})dt.$$

Then, since $\omega_{ij} = \theta_j - g_{ij}{}^{-1}\theta_i g_{ij}$, this right hand side is equal to

$$\int_0^1 \mathrm{tr}([\Theta_i{}^p, \theta_i]g_{ij}{}' g_{ij}{}^{-1} - ((\Theta_j{}^p)' \wedge \theta_j + (\Theta_j{}^p)' g_{ij}{}^{-1}\theta_i g_{ij})\, dt = \int_0^1 \mathrm{tr}\,((\Theta_i{}^p)' \wedge \theta_i - (\Theta_j{}^p)' \wedge \theta_j)\, dt.$$

Hence we have (22). ■

Corollary. *Let $\{\Psi_{p,i}\}$ be the 0-cochain determined by the Corollary of Lemma 6. Then the 0-cochain of $(2p+1)$-forms $\{\phi_{p,i}(\{\omega\})\}$ defined by*

$$\phi_{p,i}(\{\omega\}) = \int_0^1 \mathrm{tr}(\Theta_i{}^p \wedge \theta_i{}')dt - d\Psi_{p,i},$$

gives a global closed $(2p+1)$-form $\phi_p(\{\omega\})$ on M.

Proof. We need only to show $d\left(\int_0^1 \mathrm{tr}(\Theta_i{}^p \wedge \theta_i{}')dt\right) = 0$. Since $\int_0^1 \mathrm{tr}\left((\Theta_i{}^p)' \wedge \Theta_i\right) dt = 0$, we have $\int_0^1 \mathrm{tr}\left((\Theta_i{}^p)' \wedge d\theta_i\right) dt = -\int_0^1 \mathrm{tr}\left((\Theta_i{}^p)' \wedge \theta_i{}^2\right) dt$. Hence we get

$$d\left(\int_0^1 \mathrm{tr}(\Theta_i{}^p \wedge \theta_i{}')dt\right)$$
$$= -\int_0^1 \mathrm{tr}\left([\theta_i, \Theta_i{}^p] \wedge \theta_i{}' - (\Theta_i{}^p)' \wedge \theta_i^2\right) dt$$
$$= \int_0^1 \mathrm{tr}\left(\Theta_i{}^p \wedge (\theta_i{}' \wedge \theta_i + \theta_i \wedge \theta_i{}') - \Theta_i{}^p \wedge (\theta_i{}^2)'\right) dt = 0.$$

9. **Theorem 2.** *Let $\{\omega_{ij}\}$ be a representing cocycle of an element of $H^1(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}})_0$, and $\mathrm{Ch}^p\left(\langle\omega\rangle^\natural\right)$ the p-th Chern character of $\langle\omega\rangle^\natural \in H^p(M \times S^1, \mathcal{M}^1)$ ([5], [6]). Then we have*

$$(23) \qquad \langle\phi_{p,i}(\{\omega\})\rangle = -\left(2\pi\sqrt{-1}\right)^{p+1} p! \int_{S^1} \mathrm{Ch}^{p+1}\left(\langle\omega\rangle^\natural\right) dt.$$

Proof. For an $\Omega\mathfrak{g}$-valued differential form $\zeta = \sum \zeta_{i_1,\ldots,i_p} dx_{i_1} \wedge \ldots \wedge dx_{i_p}$, we set

$$\zeta^b = \sum \zeta_{i_1,\ldots,i_p}{}^b dx_{i_1} \wedge \ldots \wedge dx_{i_p}.$$

By using smooth partition of unity, for a connection form $\{\theta_i\}$ of $\{\omega_{ij}\}$, we can construct a connection form $\{\theta_i{}^\natural\}$ of $\{\omega_{ij}{}^\natural\}$ such that

$$\theta_i{}^\natural = \theta_i{}^b + f_i\, dt.$$

The curvature form $\{\Theta_i{}^\natural\}$ of $\{\theta_i{}^\natural\}$ takes the form

$$\Theta_i{}^\natural = \Theta_i{}^b + \left(df_i + [\theta_i{}^b, f_i] - \frac{\partial}{\partial t}\theta_i{}^b\right) dt.$$

By the Bianchi identity, we have $\mathrm{tr}\left((\Theta_i{}^b)^p \wedge [\theta_i{}^b, f_i]\right) = \mathrm{tr}\left(d(\Theta_i{}^p)^b f_i\right)$. Hence we have

$$\int_0^1 \mathrm{tr}\left((\Theta_i{}^\natural)^{p+1}\right) dt$$
$$= -(p+1)\int_0^1 \mathrm{tr}(\Theta_i{}^p \wedge \theta_i{}')dt + (p+1)\int_0^1 \mathrm{tr}((\Theta_i{}^b)^p \wedge (df_i + [\theta_i{}^b, f_i]))dt$$
$$= -(p+1)\int_0^1 \mathrm{tr}(\Theta_i{}^p \wedge \theta_i{}')dt + (p+1)d\left(\int_0^1 \mathrm{tr}(\Theta_i{}^p f_i{}^\#)dt\right).$$

Here $f^\#$ is the LG-valued function defined by

$$(24) \qquad (f^\#(x))(t) = f(x,t).$$

Since f_i satisfies $(g_{ij}{}^b)^{-1}\partial/\partial t(g_{ij}{}^b) = f_j - (g_{ij}{}^b)^{-1} f_i g_{ij}{}^b$, we get

$$\int_0^1 \mathrm{tr}(\Theta_j{}^p f_j{}^\#)dt - \int_0^1 \mathrm{tr}(\Theta_i{}^b f_i{}^\#)dt = \int_0^1 \mathrm{tr}(\Theta_i{}^p g_{ij}{}' g_{ij}{}^{-1})dt.$$

Hence we obtain

$$\int_0^1 \mathrm{tr}(\Theta_j{}^p f_j{}^\#)dt - \Psi_{p,j} = \int_0^1 \mathrm{tr}(\Theta_i{}^p f_i{}^\#)dt - \Psi_{p,i},$$

on $U_i \cap U_j$. Therefore we have (23). ■

Definition 6. *The de Rham class of $\phi_p(\{\omega\})$ is called the p-th string class of $\langle\omega\rangle$ and denoted $\widetilde{c}^p(\langle\omega\rangle)$. If ξ is an ΩG-bundle over M, then we denote $\widetilde{c}^p(\rho^*(\xi))$ by $\widetilde{c}^p(\xi$ and call it the p-th string class of ξ.*

String classes of the elements of $H^1(M, \mathcal{M}^1{}_{L\mathfrak{g}})$ and LG-bundles are similarly defined. By using the notation $\widetilde{c}^p(\langle\omega\rangle)$, (23) is rewritten as

$$(23)' \qquad \widetilde{c}^p(\langle\omega\rangle) = -\left(2\pi\sqrt{-1}\right)^{p+1} p! \int_{S^1} \mathrm{Ch}^{p+1}\left(\langle\omega\rangle^\natural\right) dt.$$

By the definition of $\widetilde{c}^1(\langle\omega\rangle)$ and Proposition 2 we have

Theorem 3. *The image of $\langle\omega\rangle$ by the coboundary map δ : $H^1(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}}) \to H^2(M, \Phi^1) = H^3(M, \mathbb{C})$ is $\widetilde{c}^1(\langle\omega\rangle)$.*

Corollary 1. *$\langle\omega\rangle$ is in the image of j^* : $H^1(M, \mathcal{M}^1{}_{\widetilde{\Omega\mathfrak{g}}}) \to H^1(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}})$ if and only if $\widetilde{c}^1(\langle\omega\rangle) = 0$.*

Corollary 2. *If the structure group of an ΩG-bundle ξ can be lifted up $\widetilde{\Omega G}$, then $\widetilde{c}^1(\xi) = 0$.*

This is the H-field

§4. Lifting of G-bundles on loop spaces

10. We denote by LM and ΩM the free and the based loop spaces over M, smooth Hilbert manifold modelled V. We assume that LM consists of Sobolev 1-loops (cf. [7]). Then if we set $\widetilde{H}^1(S^1) = \{\gamma \in H^1(S^1), \gamma(0) = \gamma(1)\}$ and $\widetilde{H}^1(S^1)_0 = \{\gamma \in H^1(S^1), \gamma(0) = \gamma(1) = 0\}$, LM and ΩM are Hilbert manifolds modeled by $\widetilde{H}^1(S^1)\otimes V$ and $\widetilde{H}^1(S^1)_0 \otimes V$. The connected component of the unit loop of ΩM is denoted by ΩM_e.

If g is a smooth G-valued function on M, we define a smooth LG-valued function g^L on LM by

$$(g^L(\gamma))(t) = g(\gamma(t)). \tag{25}$$

We also define ΩG-valued functions $g^\Omega = g^\Omega{}_R$ and $g^\Omega{}_L$ on M by

$$g^\Omega(\gamma) = g(\gamma(0))^{-1} g^L(\gamma), \qquad g^\Omega{}_L(\gamma) = g^L(\gamma) g(\gamma(0))^{-1}. \tag{25$'$}$$

Since $g(\gamma(0))$ is a constant, we have

Lemma 8. *If $\theta = g^{-1}dg = h^{-1}dh$, then we have*

$$(g^L)^{-1} d(g^L) = (h^L)^{-1} d(h^L),$$
$$(g^\Omega)^{-1} d(g^\Omega) = (g^L)^{-1} d(g^L).$$

Corollary. *If θ is an integrable form on M, then we can define an $\Omega\mathfrak{g}$-valued integrable form θ^L on ΩM_e by*

$$\theta|\Omega U = (g^L)^{-1} g(g^L), \qquad \theta = g^{-1}dg \text{ on } U. \tag{26}$$

Proof. We need only to show that there exists an open covering $\{U\}$ of M such that θ is integrated on U and $\{\Omega U\}$ covers ΩM_e. If γ belongs to ΩM_e, then γ is homotopic to 0. Hence it has a neighborhood $U(\gamma)$ such that θ is integrated on $U(\gamma)$. Since $\gamma \in \Omega U(\gamma)$, we have the Corollary. ■

By this Corollary we get a map $\widetilde{L} : H^0(M, \mathcal{M}^1) \to H^0(\Omega M_e, \mathcal{M}^1{}_{\Omega\mathfrak{g}})$. We can also define the maps $L : H^1(M, G_d) \to H^1(\Omega M_e, \Omega G_d)$ and $\widetilde{L} : H^1(M, \mathcal{M}^1) \to H^1(\Omega M_e, \mathcal{M}^1 \Omega\mathfrak{g})$. Then we have

Lemma 9. (i) *$\widetilde{L}(H^1(M, \mathcal{M}^1))$ is contained in $H^1(\Omega M_e, \mathcal{M}^1{}_{\Omega\mathfrak{g}})_0$.*
((ii) *We can define L to be the map $L : H^1(M, G_d) \to H^1(\Omega M, LG_d)$.*
(iii) *The following diagrams are commutative:*

$$\begin{array}{ccccc} H^0(M,\mathcal{M}^1) & \xrightarrow{\widetilde{L}} & H^0(\Omega M_e, \mathcal{M}^1{}_{\Omega\mathfrak{g}}) & \xrightarrow{i^*} & H^0(\Omega M_e, \mathcal{M}^1{}_{L\mathfrak{g}}) \\ \rho\uparrow & & & & \rho\uparrow \\ H^0(M, G_d) & & \xrightarrow{\quad L \quad} & & H^0(\Omega M_e, LG_d), \end{array}$$

$$\begin{array}{ccccc} H^1(M,\mathcal{M}^1) & \xrightarrow{\widetilde{L}} & H^1(\Omega M_e, \mathcal{M}^1{}_{\Omega\mathfrak{g}}) & \xrightarrow{i^*} & H^1(\Omega M_e, \mathcal{M}^1{}_{L\mathfrak{g}}) \\ \rho\uparrow & & & & \rho\uparrow \\ H^1(M, G_d) & & \xrightarrow{\quad L \quad} & & H^1(\Omega M_e, LG_d). \end{array}$$

Proof. (i) and (iii) follow from the definitions. Since a complex vector bundle over S^1 is always trivial, we have (ii). ■

Lemma 10. *Let $ev : \Omega M_e \times S^1 \to M$ be the evaluation map given by*

$$ev(\gamma, t) = \gamma(t)$$

([9]). Then we have

$$(g^L)^b = ev^*(g), \qquad (\theta^L)^\natural = ev^*(\theta). \tag{27}$$

Proof. Since $(g^L)^b(\gamma, t) = (g^L(\gamma))(t) = g(\gamma(t))$, we have the first equality. Then we obtain the second equality by the definitions of θ^L and $\natural$. ■

11. Since $\zeta(x) \in \Lambda^p V^* \otimes \mathfrak{g}$ if ζ is a $\mathfrak{g}$-valued p-form, if we define ζ^L by

$$(\zeta^L(\gamma))(t) = \zeta(\gamma(t)),$$

$\zeta^L(\gamma)$ belongs to $\mathrm{Map}(S^1, \Lambda^p V^* \otimes \mathfrak{g})$. Since $\mathrm{Map}(S^1, \Lambda^p V^* \otimes \mathfrak{g})$ is contained in $\Lambda^p(\mathrm{Map}(S^1, V^*)) \otimes \mathfrak{g}$, we may regard ζ^L as a p-form on LM. Since we get

$$f^L(\gamma + s\eta)(t) = f(\gamma(t)) + s\langle df(\gamma(t)), \eta(t)\rangle + o(s),$$

we have

$$d(f^L) = (df)^L. \tag{28}$$

If $\zeta = \sum \zeta_{i_1,\dots,i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p}$, we may write

$$\zeta^L = \sum \zeta_{i_1,\dots,i_p}{}^L dx_{i_1}{}^L \wedge \dots \wedge dx_{i_p}{}^L, \qquad dx^L = d(x^L).$$

Hence we obtain

Lemma 11. *Let $\{\theta_i\}$ be a connection of $\langle\{\omega_{ij}\}\rangle \in H^1(M, \mathcal{M}^1)$. Then $\{\theta_i{}^L\}$ becomes a connections form of $\{\omega_{ij}{}^L\}$ and its curvature is $\{\Theta_i{}^L\}$, where $\{\Theta_i\}$ is the curvature of $\{\theta_i\}$.*

Theorem 4. *Let $\langle\omega\rangle$ be an element of $H^1(M, \mathcal{M}^1)$. Then we have*

$$\widetilde{c}^p\left(\langle\omega\rangle^L\right) = -\left(2\pi\sqrt{-1}\right)^{p+1} p!\tau^{-1}(\mathrm{Ch}^{p+1}(\langle\omega\rangle)). \tag{29}$$

Here $\tau^{-1} : H^{q+1}(M, \mathbb{C}) \to H^q(\Omega M_e, \mathbb{C})$ is the inverse of the transgression map.

Proof. Since the diagram

$$\begin{array}{ccc} H^{q+1}(M,\mathbb{C}) & \xrightarrow{ev^*} & H^{q+1}(\Omega M_e \times S^1, \mathbb{C}) \\ & \searrow^{\tau^{-1}} & \downarrow \int_{S^1} \\ & & H^q(\Omega M_e, \mathbb{C}) \end{array} \qquad \left(\int_{S^1} \phi\right)(\gamma) = \int_{\gamma \times S^1} \phi,$$

is commutative ([9], [10]), we have by (23) and (27)

$$\begin{aligned} \widetilde{c}^p\left(\langle\omega\rangle^L\right) &= -\left(2\pi\sqrt{-1}\right)^{p+1} \int_{S^1} \mathrm{Ch}^{p+1}\left(\left(\langle\omega\rangle^L\right)^\natural\right) \\ &= -\left(2\pi\sqrt{-1}\right)^{p+1} \int_{S^1} \mathrm{Ch}^{p+1}(ev^*(\langle\omega\rangle)) \\ &= -\left(2\pi\sqrt{-1}\right)^{p+1} p!\tau^{-1}(\mathrm{Ch}^{p+1}(\langle\omega\rangle)). \end{aligned}$$

■

Corollary. *Let $c_p(\langle\omega\rangle)$ be the p-th Chern class of $\langle\omega\rangle$ (cf. [5], [6]). Then $\langle\omega\rangle^L$ is in the j^*-image if and only if*

$$c_1{}^2(\langle\omega\rangle) = 2c_2(\langle\omega\rangle).$$

Especially, ξ^L has an $\widetilde{\Omega\mathfrak{g}}$-valued connection if and only if $c_1{}^2(\xi) = 2c_s(\xi)$.

12. The map L is also defined for real vector bundles (In this case, L is only defined as the map from $H^1(M, GL(\mathbb{R})_d) \to H^1(\Omega M_e, LGL(\mathbb{R})_d)$). We denote by TM the tangent bundle of M. Then we have

$$(TM)^L = T(\Omega M_e). \tag{30}$$

Therefore, denoting by $T^{\mathbb{C}}M$ the complexification of TM, we get

$$(T^{\mathbb{C}}M)^L = T^{\mathbb{C}}(\Omega M_e). \tag{30'}$$

By (30)′ and the Corollary of Theorem 4 we obtain (cf. [11], [15], [22], [26], [27])

Theorem 5. *ΩM_e has an $\widetilde{\Omega\mathfrak{g}}$-valued connection if and only if $p_1(M) = 0$. Here $p_1(M)$ means the first (rational) Pontrjagin class of M.*

Note. The condition that ΩM_e has an $\widetilde{\Omega\mathfrak{g}}$-valued connection is weaker than the condition $T^{\mathbb{C}}(\Omega M_e)$ comes from an $\widetilde{\Omega G}$-bundle. Since we are working in de Rham cohomology, torsion parts of cohomology classes are ignored.

Theorem 4 shows that the $c_p(\langle\omega\rangle)$ are recovered from $\langle\omega\rangle^L$ if $p \geq 2$. We can also recover $c_1(\langle\omega\rangle)$ from $\langle\omega\rangle^L$ as follows: We denote by ΩG_0 the connected component of the identity of ΩG. The sheaf of germs of smooth ΩG_0-valued functions is denoted by $\Omega G_{0,d}$ and set $\rho(\Omega G_{0,d}) = \mathcal{M}^1{}_{\Omega\mathfrak{g}_0}$. Then if we set

$$\widetilde{r}(\theta) = \frac{1}{2\pi\sqrt{-1}} \int \mathrm{tr}(\theta)dt, \tag{31}$$

we have the following commutative diagram with exact lines:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M}^1_{\Omega\mathfrak{g}_0} & \xrightarrow{i} & \mathcal{M}^1{}_{\Omega\mathfrak{g}} & \xrightarrow{\widetilde{r}} & \mathbb{C}_t & \longrightarrow & 0 \\ & & \rho\uparrow & & \rho\uparrow & & \uparrow & & \\ 0 & \longrightarrow & \Omega G_{0,d} & \xrightarrow{i} & \Omega G_d & \xrightarrow{r} & \mathbb{Z}_t & \longrightarrow & 0. \end{array}$$

By this diagram, we obtain the following commutative diagram of cohomology sets:

$$\begin{array}{ccccc} H^1(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}_0}) & \xrightarrow{i^*} & H^1(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}}) & \xrightarrow{\widetilde{r}^*} & H^1(M, \mathbb{C}) \\ \uparrow & & \uparrow & & \uparrow \\ H^1(M, \Omega G_{0,d}) & \xrightarrow{i^*} & H^1(M, \Omega G_d) & \xrightarrow{\widetilde{r}^*} & H^1(M, \mathbb{Z}). \end{array}$$

Therefore we can define characteristic classes $\widetilde{r}^*(\langle\omega\rangle) \in H^1(M, \mathbb{C})$ and $R^*(\xi) \in H^1(M, \mathbb{Z})$ for $\langle\omega\rangle \in H^1(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}})$ and $\xi \in H^1(M, \Omega G_d)$.

Theorem 4'. (i) *$\widetilde{r}^*(\langle\omega\rangle)$ is equal to 0 if and only if $\langle\omega\rangle$ is in the i^*-image.*

(ii) *$\widetilde{r}^*(\rho(\xi))$ is an integral class.*

(iii) *Let $\tau^{-1} : H^2(M, \mathbb{C}) \to H^1(\Omega M_e, \mathbb{C})$ be the inverse of the transgression. Then we have*

$$\widetilde{r}^*\left(\langle\omega\rangle^L\right) = 2\pi\sqrt{-1}\tau^{-1}(c_1(\langle\omega\rangle)). \tag{29'}$$

Proof. (i) and (ii) follow from the definition. If $\{\omega_{ij}\}$ is a representing cocycle of $\langle\omega\rangle \in H^1(M,\mathcal{M}^1)$, $\{\mathrm{tr}\,\omega_{ij}\}$ represents $c_1(\langle\omega\rangle)$ in $H^1(M,\Phi^1)$. Hence we have $(29)'$ by (30) and the definition of τ^{-1}.

Theorems 4 and 4' show that $L : H^1(M,G_d) \to H^1(\Omega M_e, \Omega G_d)$ is injective if M is torsionfree. At this stage we do not know whether L is injective or not in general.

§5. The relation between β-classes and string classes and the Bott map in non-abelian de Rham theory

13. For an integrable form θ on M, we have defined its Chern–Simons type characteristic classes $\beta^p(\theta) \in H^{2p-1}(M,\mathbb{C})$ as the de Rham class of $(p-1)!/\left(2\pi\sqrt{-1}\right)^p(2p-1)!\,\mathrm{tr}(\theta^{2p-1})$ (cf. [7]). We know

(32) $$\beta^p(\theta) = g^*(e_p) \qquad \text{if } \theta = g^{-1}dg,$$

where e_p is the $(2p-1)$-th generator of $H^*(G,\mathbb{Z})$. By (32), $\beta^p(\theta)$ is equal to 0 if $g = e^f$ on M. Hence β-classes are defined as characteristic classes of the elements of $H^0(M,\mathcal{M}^1)/d^e(H^0(M,\mathfrak{g}_d))$ (cf. [7]).

If θ belongs to $H^0(M,\mathcal{M}^1)_0$, then its Gysin-image $\widetilde{\gamma}(\theta) \in H^1(M \times S^1, \mathcal{M}^1)$ is defined. By the definition of $\widetilde{\gamma}$, $\widetilde{\gamma}(\theta)|M \times (S^1 - \{0\})$ is trivial and it has a connection form $\{\zeta_1\}$ such that

$$\zeta_i(x,t) = d\left(h_i(x)^t\right)h_i(x)^{-t}, \qquad \text{on } U_i \times (S^1 - \{0\}) = U_i \times (0,1).$$

Here $\theta = h_i^{-1}dh_i$ holds on U_i. Since $\|\theta_i\|$ is bounded on $U_i \times (0,1)$, ζ_i defines a current on $U_i \times S^1$. Moreover, $\{\mathrm{tr}\,((d\zeta_i + \zeta_i\wedge\zeta_i)^p)\}$ defines a current on $M \times S^1$, It is computed as follows:

$$\begin{aligned}
&[\mathrm{tr}\,(d\zeta_i + \zeta_i \wedge \zeta_i)^p](\Psi)\\
&= \lim_{\varepsilon\to 0}\int_M\int_\varepsilon^{1-\varepsilon}(d(\mathrm{tr}(\zeta_i \wedge (d\zeta_i)^{p-1} + p{\zeta_i}^3 \wedge (d\zeta_i)^{p-2} + \dots\\
&\qquad + p\zeta^{2p-1}) \wedge \Psi) + \mathrm{tr}({\zeta_i}^{2p}) \wedge \Psi)\\
&= \lim_{\varepsilon\to 0}\int_M\int_\varepsilon^{1-\varepsilon} d(\mathrm{tr}({\zeta_i}^{2p-1}) \wedge \Psi\\
&= \lim_{\varepsilon\to 0}\int_M(\mathrm{tr}({\zeta_i}^{2p-1}(x,1-\varepsilon)) \wedge \Psi(x,1-\varepsilon) - \mathrm{tr}({\zeta_i}^{2p-1}(x,\varepsilon)) \wedge \Psi(x,\varepsilon))\\
&= \int_M(\mathrm{tr}\,{\theta_i}^{2p-1}) \wedge \Psi(x,0).
\end{aligned}$$

As a collection of currents $\{\zeta_i\}$ gives a connection of $\widetilde{\gamma}(\theta)$. Therefore the de Rham class of $\{\mathrm{tr}\,((d\zeta_i + \zeta_i \wedge \zeta_i)^p)\}$ (as a current) is $\left(2\pi\sqrt{-1}\right)^p p!\,\mathrm{Ch}^p(\widetilde{\gamma}(\theta))$. On the other hand, by the residue exact sequence ([4]), we have the following exact sequence

$$H^{2p-1}(M,\mathbb{C}) \xrightarrow{\delta_M} H^{2p}(M \times S^1,\mathbb{C}) \xrightarrow{i^*} H^{2p}(M \times (S^1 - \{0\}),\mathbb{C}).$$

By Künneth' formula, denoting e^p the generator of $H^p(S^1,\mathbb{C})$, we get

$$H^{2p})M \times S^1,\mathbb{C}) = H^{2p-1}(M,\mathbb{C}) \otimes e^1 \oplus H^{2p}(M,\mathbb{C}) \otimes e^0.$$

Therefore we obtain by the definitions of δ_M and i (cf. [4])

(33) $$\begin{aligned}&\delta_M : H^{2p-1}(M,\mathbb{C}) \cong H^{2p-1}(M,\mathbb{C}) \otimes e^1 \quad (\subset H^{2p}(M \times S^1,\mathbb{C})),\\ &i : H^{2p}(M,\mathbb{C}) \otimes e^0 \cong H^{2p}(M \times (S^1 - \{0\}),\mathbb{C}).\end{aligned}$$

Since $\delta_M(\langle\zeta\rangle)$ is represented by the current $T_{M,\zeta}$, $T_{M,\zeta}(\Psi) = \int_M \zeta(x) \wedge \Psi(x,0)$, we get

(34) $$\mathrm{Ch}^p(\widetilde{\gamma}(\theta)) = \frac{p!(p-1)!}{(2p-1)!}\delta_M(\beta^p(\theta)).$$

Theorem 6. *We have*

(35) $$\widetilde{c}^p(\widetilde{B}_0([\theta])) = -\left(2\pi\sqrt{-1}\right)^{p+1}\frac{(2p+1)!}{(p+1)!}\beta^{p+1}(\theta).$$

Proof. Since $\widetilde{\gamma}(\theta) = \widetilde{B}_0([\theta])^\natural$ and δ_M is an isomorphism by (33), we have Theorem 6 by (34) and (23)$'$. ■

Corollary. *If ξ is a loop group bundle and g is a G-valued function on M such that $\xi = B_0(g)$, then*

$$\widetilde{c}^p(\xi) = -\left(2\pi\sqrt{-1}\right)^{p+1} \frac{(2p+1)!}{(p+1)!} g^*(e_{p+1}). \tag{36}$$

Note. In [7], we defined maps

$$X : H^1(M, G_d) \to H^0(\Omega M_e, G_d)/\exp(H^0(\Omega M_e, \mathfrak{g}_d)),$$
$$\widetilde{X} : H^1(M, \mathcal{M}^1) \to H^0(\Omega M_e, \mathcal{M}^1)/d^e(H^0(\Omega M_e, \mathfrak{g}_d)).$$

$X(\xi)$ and $\widetilde{X}(\xi)$ are represented by representative functions of ΩM_e and $\widetilde{\Omega} M_e$, the universal covering space of ΩM_e, respectively ([7]). On the other hand, $(B_0)^{-1}(\xi^L) = g$ is given by

$$g = h_i(\gamma^2)(h_i(\gamma))^{-1}, \qquad \text{on } U_i,$$
$$\widetilde{g}_{ij}{}^L = h_i h_j{}^{-1}, \quad \widetilde{g}_{ij}{}^L(\gamma)(t) = g_{ij}(\gamma(t \bmod 1)).$$

Therefore we obtain

$$X(\xi) = (B_0)^{-1}(\xi^L), \qquad \widetilde{X}(\langle\omega\rangle) = \left(\widetilde{B}_0\right)^{-1}\left(\langle\omega\rangle^L\right). \tag{37}$$

14. We set $H_+ = \left\{\sum_{n>0} c_n e^{2n\pi\sqrt{-1}t}\right\}$ and $H_- = \left\{\sum_{n<0} c_n e^{2n\pi\sqrt{-1}t}\right\}$. Then we have $\widetilde{H}^1(S^1)_0 = H_+ \otimes H_-$. The algebra of all bounded linear operators on H_+ is denoted by $B(H_+)$. The ideal of all compact operators of $B(H_+)$ is denoted by $C(H_+)$. We also use the following notations (cf. [12]):

$$\mathrm{Cal} = B(H_+)/C(H_+),$$
$$F = F(H_+)/C(H_+), \qquad F(H_+) \text{ is the set of Fredholm operators.}$$

The connected component of the identity of F is denoted by F_0. By the imbedding of ΩG in GL_{res} the restricted general linear group on $\widetilde{H}^1(S^1)_0$ (cf. [23]), the $(1,1)$-component of $g \in \Omega G$ represents an element of F_0, that is, (1.1)-component of g has an inverse by a compact perturbation, if and only if $g \in \Omega G_0$. We also set

$$K = \{T \in GL(H_+) \mid T = I + C, \ C \text{ is a compact operator}\}.$$

K contains $GL(\infty) = \bigcup GL_n$ as a dense subgroup and the following sequence is exact:

$$0 \longrightarrow K \longrightarrow GL(H_+) \longrightarrow F_0 \longrightarrow 0.$$

Since $H^1(M, GL(H_+)) = \{0\}$ by a theorem of Kuiper ([17]), $\delta : H^1(M, F_{0,d}) \to H^2(M, K_d)$ is injective by this sequence (for the definitions of δ and $H^2(M, K_d)$ cf. [5], [6]). Here $F_{0,d}$ and K_d are the sheaves of germs of smooth K and F_0 valued functions.

Definition 7. *Let $q : H^1(M, G_{0,d}) \to H^1(M, F_{0,d})$ be the map induced by the projection to the $(1,1)$-component of th elements of ΩG_0 and $\delta_L : H^1(M, F_{0,d}) \to H^2(M, K_d)_L$ the coboundary map in the left handed non-abelian cohomology sets (cf. [6]). Then we set*

$$B^1{}_L = \delta_L q. \tag{38}$$

$B^1{}_R$ is similarly defined.

We lift $B^1{}_R$ to be a map between non-abelian de Rham sets as follows: Let $\mathcal{M}^1{}_K$, $\mathcal{M}^1{}_{\mathfrak{gl}(H_+)}$ and $M^1{}_{\mathrm{Cal}}$ be the image sheaves of K_d, $GL(H_+)_d$ and $F_{0,d}$ by $\rho : \rho(g) = g^{-1}dg$, and its induced map $\overline{\rho}$. We also set $\mathcal{M}^1{}_{K,L} = \rho_L(K_d)$, where $\rho_L(g) = (dg)g^{-1}$, and the induced map of q by $\widetilde{q}$. Then we have the following commutative diagram:

$$\begin{array}{ccccc}
H^1(M, \Omega G_{0,d}) & \xrightarrow{q} & H^1(M, F_{0,d}) & \xrightarrow{\delta} & H^2(M, K_d)_L \\
\rho\uparrow & & \overline{\rho}\uparrow & & \rho_L\uparrow \\
H^1(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}}) & \xrightarrow{\widetilde{q}} & H^1(M, M^1{}_{\mathrm{Cal}}) & \xrightarrow{\delta_L} & H^2(M, \mathcal{M}^1{}_{K,L}).
\end{array}$$

Hence if we define $\widetilde{B}^1{}_L : H^1(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}}) \to H^2(M, \mathcal{M}^1{}_{K,L})$ by

$$\widetilde{B}^1{}_L = \rho_L \widetilde{q}, \tag{38$'$}$$

we have the following commutative diagram:

$$\begin{array}{ccc}
H^1(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}}) & \xrightarrow{\widetilde{B}^1{}_L} & H^2(M, \mathcal{M}^1{}_{K,L}) \\
\rho\uparrow & & \rho_L\uparrow \\
H^1(M, \Omega G_{0,d}) & \xrightarrow{B^1{}_L} & H^2(M, K_d)_L.
\end{array}$$

Definition 8. *We define the maps*

$$B_L : H^0(M, G_d)/\exp(H^0(M, \mathfrak{g}_d)) \to H^2(M, K_d)_L$$
$$\widetilde{B}_L : H^0(M, \mathcal{M}^1)_0/d^e(H^0(M, \mathfrak{g}_d)) \to H^2(M, \mathcal{M}^1{}_{K,L})$$

by

(39) $$B_L = B^1{}_L B_0, \qquad \widetilde{B}_L = \widetilde{B}^1{}_L \widetilde{B}_0.$$

Here B_0 means B_{0R}. B_R and $\widetilde{B}_R$ are similarly defined.

It seems that we may replace K by GL_∞ $(= GL(\infty))$ and $\mathcal{M}^1{}_K$ by $\mathcal{M}^1{}_\infty$ $(= \mathcal{M}^1{}_{\mathfrak{gl}(\infty)})$. If this is true, $\widetilde{B}_L$ maps (a subset of) the first non-abelian de Rham set of M into the (stable) third non-abelian de Rham set of M. Hence we may say $\widetilde{B}_L$ to be *the (left) Bott map* in non-abelian de Rham theory. In [6], to get a good de Rham correspondence in the third non-abelian de Rham theory, pairing of the elements of the right handed and left handed third non-abelian de Rham sets was considered. We expect that the meaning of this pairing will be clarified via $\widetilde{B}$ and the definition of $H^0(M, \mathcal{M}^1)_0$.

15. By using the Grassmannian model of ΩG ([23]) we define the maps

$$gr : H^0(M, \Omega G_d) \to H^1(M, G_{\infty,d}),$$
$$\Omega : H^1(M, G_d) \to H^0(M, G_{\infty,d})/H^0(M, G_{\infty,0,d}).$$

These maps are lifted as the maps

$$\widetilde{gr} : H^0(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}}) \to H^1(M, M^1{}_\infty),$$
$$\widetilde{\Omega} : H^1(M, \mathcal{M}^1) \to H^0(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}_\infty})/H^0(M, \rho(\Omega G_{\infty,0,d})).$$

By using gr and Ω we define $\omega^b : H^1(M, \Omega G_d) \to H^1(\Omega M_e, G_{\infty,d})$ by

(40) $$\omega^b(\xi) = gr.\left((B_0{}^{-1}(\xi))^L\right).$$

By Theorem 1' we can define the lift $\widetilde{\omega}^b$: $H^1(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}})^\natural \to H^1(\Omega M_e, \mathcal{M}^1_\infty)$ of ω^b by

(40)' $$\widetilde{\omega}^b(\langle\omega\rangle) = \widetilde{gr}\left((\widetilde{B}_0{}^{-1}(\langle\omega\rangle))^L\right), \qquad H^1(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}})^\natural = \mathrm{Im}\,\widetilde{B}_0.$$

Then by (2) and (35) we have

(41) $$\widetilde{c}^p(\langle\omega\rangle) = -\left(2\pi\sqrt{-1}\right)^{p+1} \frac{(2p+1)!}{p!(p+1)!} \tau^{-1}(\mathrm{Ch}^{p+1}(\widetilde{\omega}^b(\langle\omega\rangle))).$$

In conclusion, our results are summarized as *the trinity of β-classes (Chern–Simons classes), string classes and transgressed Chern classes* on the one hand, and the following two types of trinities of non-abelian de Rham sets (with characteristic classes) on the other hand.

(I). (a). *The first non-abelian de Rham set over M.*

(b). *The second non-abelian de Rham set over $M \times S^1$.*

(c). *The stable second non-abelian de Rham set over ΩM_e.*

From (a), (b) is mapped by the Gysin map and (c) is mapped by the inverse of transgression. Their composition is the evaluation map.

(II). (a). *The second non-abelian de Rham set over M.*

(b). *The stable first non-abelian de Rham set over $M \times S^1$.*

(c). *The first non-abelian de Rham set over ΩM_e.*

Note. In both cases trinities are not in the strict sense. In fact, we do not know whether the Gysin map, etc., are bijective or not.

(I) and (II) are visualized as the commutativity of the following diagrams:

$$\begin{array}{ccc}
H^0(M, \mathcal{M}^1)_0/d^e(H^0(M, \mathfrak{g}_d)) & \xrightarrow{\widetilde{\gamma}} & H^1(M \times S^1, \mathcal{M}^1) \\
\downarrow & \searrow^{\widetilde{B}_0} & \uparrow \\
H^0(M, \mathcal{M}^1{}_\infty)/d^e(H^0(M, \mathfrak{g}_{\infty,d})) & & H^1(M, \mathcal{M}^1{}_{\Omega\mathfrak{g}})^\natural \\
\downarrow{\scriptstyle \widetilde{L}} & & \downarrow{\scriptstyle \widetilde{\omega}^b} \\
H^0(M_e, \mathcal{M}^1{}_{\Omega\mathfrak{g}})/d^e(\Omega M_e, \rho(\Omega G_{0,d})) & \xrightarrow{\widetilde{gr}} & H^1(\Omega M_e, \mathcal{M}^1{}_\infty),
\end{array}$$

$$\begin{array}{ccc}
H^1(M, \mathcal{M}^1) & \longrightarrow & H^0(M \times S^1, \mathcal{M}^1{}_\infty)/ d^e(H^0(M \times S^1, \mathfrak{g}_{\infty,d})) \\
\downarrow & \searrow & \uparrow \\
H^0(\Omega M_e, \mathcal{M}^1)/d^e(H^0(\Omega M_e, \mathfrak{g}_d)) & & H^0(M, \mathcal{M}^1{}_{\Omega\mathfrak{g},\infty})/ H^0(M, \rho(\Omega G_{\infty,0,d})) \\
\widetilde{B}_0 \downarrow\uparrow & \searrow & \downarrow \\
H^1(\Omega M_e, \mathcal{M}^1{}_{\Omega\mathfrak{g}})^\natural & \longrightarrow & H^0(\Omega M_e, \mathcal{M}^1{}_\infty)/ d^e(H^0(\Omega M_e, \mathfrak{g}_{\infty,d})).
\end{array}$$

Note. The Bott map relates the third non-abelian de Rham set and the first non-abelian de Rham set. Hence we may regard the third non-abelian de Rham theory to be a gauge theory on ΩM_e (Loop gauge theory) or on $M \times S^1$ (cf. [8]).

References

[1]. ABE, K., On tortal torsion and a generic property of closed regular curves in Riemannian manifolds, Preprint.

[2]. ALVAREZ, O., KILLINGBACK, T. P., MANGANO, M. and WINDEY, P., String theory and loop space index theorems, *Commun. Math. Phys.,* bf 111 (1987), 1–10.

[3]. ANDERSSON, S. I., Pseudodifferential operators and characteristic classes for non-abelian cohomology, *Lect. Notes in Math.,* **1045** (1984), 1–10.

[4]. ASADA, A., Currents and residue exact sequences, *Journ. Fac. Sci. Shinshu univ.,* **3** (1968), 85–151.

[5]. ASADA, A., Non abelian de Rham theories, Coll. Math. Soc. János Bolyai 46, Topics in Differential Geometry, 83–115, North-Holland, 1988.

[6]. ASADA, A., Non abelian de Rham theory, Proc. Prospects of Math. Sci., World Sci., 1988. 13–40.

[7]. ASADA, A., Integrable forms on iterrated loop spaces and higher dimensional non abelian de Rham theory, *Lect. Notes in Math.,* **1910** (1989), 27–51.

[8]. AWADA, M. A., The exact equivalence of Chern–Simons theory with fermionic string theory, *Phys. Lett.,* **B221** (1989), 21–26.

[9]. BONOLA, L., COTTA–RAMUSIANO, P., RINALDI, M. and STASHEFF, J., The evaluation map in field theory, sigma-models and strings, I, II, *Commun. Math. Phys.,* **112** (1987), 237–282, **114** (1988), 381–437.

[10]. BOTT, R., The space of loops on a Lie group, *Michigan Math. Journ.,* **5** (1958), 35–61.

[11]. COQUEREAUX, R. and PILCH, K., String structures on loop bundles, *Commun. Math. Phys.,* **120** (1989), 353–378.

[12]. DOUGLAS. R. G., Banach Algebra Techniques in Operator Theory, Academic Press, 1972.

[13]. FLORATOS, E. G., ILIOPOULOS, J. and TIKTOPOULOS, G., A note on $SU(\infty)$ classical Yang–Mills theories, *Phys. Lett.,* **B217** (1989), 285–288.

[14]. GYSIN, W., Zur Homologie Theorie des Abbildungen und Faserungen von Mannigfaltigkeiten, *Commet. Math. Helv.,* **14** (1941), 61–121.

[15]. KILLINGBACK, T. P., World-sheet anomalies and loop geometry, *Nucl. Phys.,* **B288** (1987), 578–588.

[16]. KILLINGBACK, T. P., Quantum Chern–Simons theory, *Phys. Lett.,* **B219** (1989), 448–456.

[17]. KUIPER, N. H., The homotopy type of the unitary group of Hilbert space, *Topology* **3** (1965), 19–30.

[18]. LASHOF, R., Classification of fibre bundles by the loop space of the base, *Ann. Math.,* **64** (1956), 436–446.

[19]. MENSKY, M. B., Group of paths, Observations, Fields and Particles, Moscow, 1983 (in Russian).

[20]. MICKELSSON, J. and RAJEEV, S. G., Current algebras in $D+1$-dimensions and determinant bundles over infinite-dimensional Grassmannians *Commun. Math. Phys.,* **116** (1988), 365–400.

[20]'. TANAKA, M. and FUJI, K., Universal Schwinger cocycles of current algebras in $(D+1)$-dimensions, Preprint.

[21]. MILNOR, J., Construction of universal bundles, I. *Ann. Math.,* **63** (1956), 272–284.

[22]. PILCH, K. and WARNER, N. P., String structure and the index of the Dirac–Ramond operator on orbifolds, *Commun. Math. Phys.,* **115** (1988), 191–212.

[23]. PRESSLEY, A. and SEGAL, G., Loop Groups, Oxford, 1986.

[24]. RAJEEV, S. G., An exactly integrable algebraic model for $(3+1)$-dimensional Yang–Mills theory, *Phys. Lett.,* **209** (1988), 53–58.

[24]'. TANAKA, M. and FUJII, K., Note on algebraic analogue of Yang–Mills–Higgs theory, Preprint.

[25]. TAUBES, C. H., S^1 actions and elliptic genera, *Commun. Math. Phys.* **122** (1989), 455–526.

[26]. VAFA, C., Modular invariance and discrete torsion on orbifolds, *Nucl. Phys.,* **B273** (1986), 592–606.

[27]. WITTEN, E., Global anomalies in string theory, *Anomalies, Geometry and Topology,* 61–99, World Sci., 1985.

[28]. WITTEN, E., Quantum field theory and the Jones polynomial, *Commun. Math. Phys.,* **121** (1989), 351–399.

Added in proof. Prof. Michor kindly remarked:

(i) By his result, we need not the *Convention.*

(ii) Lemma 3 occurs in Varadarajan's book Lie groups, Lie algebras and their Representations.