# Accessible categories and models of linear logic

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## Introduction

The connection between $*$-autonomous categories and linear logic has been pointed out in several places, including [Seely, 1989] and [Barr, to appear]. The only essential difference between them is the question of the existence of cofree coalgebras (counitary, coassociative, cocommutative, to be precise). In my previous paper, I gave a construction that did lead to models of linear logic, but only by beginning with a category that was already cartesian closed (with additional properties).

This paper follows quite a different tack. We begin with a category that is locally presentable in the sense of [Gabriel & Ulmer, 1971], which is equivalent to its being accessible in the sense of [Makkai & Paré, 1990] together with being either complete or cocomplete, as well as being autonomous and show that in that case the $*$-autonomous category constructed in [Chu, 1979] has cofree coalgebras and is thus a model of full linear logic.

## 1 Generalities on accessible categories

The most important conceptual tool we need is that of the accessible categories of [Makkai & Paré, 1990]. In fact, a locally presentable category can be defined as a complete (or cocomplete, see below) accessible category. We thus begin with some discussion of this important class of categories.

If $\kappa$ is a regular cardinal, a diagram $D: \mathscr{I} \longrightarrow \mathscr{C}$ is called **$\kappa$-filtered** if

1. Given any set of objects $I_\alpha$ of $\mathscr{I}$ of cardinality less than $\kappa$, there is an object $I$ of $\mathscr{I}$ and arrows $I_\alpha \longrightarrow I$ in $\mathscr{I}$.

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2. For any set of arrows $f_\alpha : I \longrightarrow I'$ of $\mathscr{I}$ of cardinality less than $\kappa$, there is an arrow $g: I' \longrightarrow I''$ of $\mathscr{I}$ such that all the composites $Dg \circ Df_\alpha$ are all the same.

An object $C$ of a category is said to be **$\kappa$-presentable** if $\mathrm{Hom}(C, -)$ commutes with the colimits of $\kappa$-filtered diagrams.

A category is **$\kappa$-accessible** if:

1. Every $\kappa$-filtered diagram has a colimit;

2. Every object is a colimit of a $\kappa$-filtered diagram of $\kappa$-presentable objects.

No assumption is made of the existence of any particular finite limits but the condition implies, for example, that a $\kappa$-filtered colimit of monomorphisms is a monomorphism. The fact that every object is a colimit of $\kappa$-presentable objects implies that those objects generate. As mentioned below, there is only a set of them.

A functor is **$\kappa$-accessible** if it preserves the colimit of $\kappa$-filtered diagrams.

A category or functor is **accessible** if it is $\kappa$-accessible for some regular cardinal $\kappa$. If a category is $\kappa$-accessible, it is not necessarily $\lambda$-accessible for all $\lambda > \kappa$, but there are arbitrary large values of $\lambda$ for which it is $\lambda$-accessible.

We record here some properties of accessible categories and functors which are found in [Makkai & Paré, 1990].

1. An accessible category is complete if and only if it is cocomplete.

2. An accessible category is well-powered and, provided it has coequalizers, is well-copowered.

3. If $U: \mathscr{B} \longrightarrow \mathscr{C}$ is an accessible functor, it satisfies the solution set condition.

4. A colimit or finite limit of accessible functors is accessible.

5. For any cardinal $\lambda$, there is only a set of isomorphism classes of $\lambda$-accessible objects.

6. The full subcategory of $\kappa$-accessible objects in a $\kappa$-accessible category is closed under finite limits (whenever they exist) and colimits taken over diagrams with fewer than $\kappa$ nodes.

Let $R, S: \mathscr{C} \longrightarrow \mathscr{D}$ be functors. We define a category $(R:S)$ to be the category whose objects are pairs $(C, c)$ where $C$ is an object of $\mathscr{C}$ and $c: RC \longrightarrow SC$ is an arrow of $\mathscr{D}$. A morphism $f: (C, c) \longrightarrow (C', c')$ is an arrow $f: C \longrightarrow C'$ such that

$$\begin{tikzcd} RC \arrow[r, "c"] \arrow[d, "Rf"'] & SC \arrow[d, "Sf"] \\ RC' \arrow[r, "c'"'] & SC' \end{tikzcd}$$

commutes.

**1.1 Theorem.** *Suppose that $R$ and $S$ are accessible functors between accessible categories. Then $(R\colon S)$ is accessible.*

This is an example of a weighted bilimit and thus follows from Theorem 5.1.6 of [Makkai & Paré, 1990]. □

**1.2 Corollary.** *Let $\mathscr{C}$ be locally presentable and suppose that in addition to $R$ and $S$ being accessible, either $R$ preserves colimits or $S$ preserves limits. Then $(R\colon S)$ is locally presentable.*

Proof. An accessible category is complete if and only if it is cocomplete and hence is locally presentable if it is either. If $\mathscr{C}$ is cocomplete and $R$ preserves colimits, then it is a triviality to show that the obvious underlying functor $(R\colon S) \longrightarrow \mathscr{C}$ creates colimits and so $(R\colon S)$ is cocomplete. If $S$ preserves limits, then the same underlying functor creates limits and $(R\colon S)$ is complete. □

The most important special case is when $\mathscr{C} = \mathscr{D}$ and one of $R$ or $S$ is the identity functor. If $R = \mathrm{id}$, the category $(R\colon S)$ is called the category of $S$-coalgebras and denoted $\mathscr{C}_S$. If $S$ is the identity, it is called the category of $R$-algebras and denoted $\mathscr{C}^R$.

**1.3 Corollary.** *If $R$ is an accessible endofunctor on a locally presentable category $\mathscr{C}$, then both $\mathscr{C}_R$ and $\mathscr{C}^R$ are locally presentable.* □

**1.4 Theorem.** *Let $R\colon \mathscr{C} \longrightarrow \mathscr{D}$ be an accessible functor between locally presentable categories. Then $R$ has a left adjoint if and only if it preserves limits; it has a right adjoint if and only if it preserves colimits.*

Proof. The first claim is implicit in the fact that any accessible functor between accessible categories satisfies the solution set condition. As for the second, it is an immediate consequence of the special adjoint functor theorem since accessible categories have generators and are well-powered. □

## 2 Coalgebras and cartesian closed categories

By an **autonomous category**, we mean a closed symmetric monoidal category. We denote the tensor product by $\otimes$ and the internal hom by $\multimap$, so that basic adjointness becomes

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, B \multimap C)$$

If $\mathscr{C}$ is an autonomous category, let us say that an object $C$ equipped with a map $C \longrightarrow C \otimes C$ and a map $C \longrightarrow \top$ is a **pre-coalgebra** and a coalgebra if those operations are counitary, coassociative and cocommutative, these notions defined dually as they are for algebras.

The reason that linear logic requires cofree coalgebras is to model ordinary (or even intuitionistic) logic inside itself. If you begin with an autonomous category, the category of counitary, coassociative, cocommutative coalgebras has a tensor product, which is given by the original tensor product and which turns out to be the cartesian product in this coalgebra category. As a result, this coalgebra category will be cartesian closed if it is closed, and for this it is sufficient that cofree coalgebras and equalizers exist. In fact, it is also the case that the Kleisli category (that is the full subcategory of cofree coalgebras) is cartesian closed and for that you don't even need the equalizers.

Actually, you don't need it to be the category of coalgebras either. What you need is a cotriple $\mathbf{G} = (G, \epsilon, \delta)$ for which $G(A \times B)$ is naturally isomorphic to $GA \otimes GB$. The reason is the following.

**2.1 Theorem.** *Let $\mathscr{C}$ be an autonomous category and $\mathbf{G} = (G, \epsilon, \delta)$ a cotriple on $\mathscr{C}$ such that the functor $G(A \times -)$ is naturally isomorphic to the functor $GA \otimes G-$ for all objects $A$ of $\mathscr{C}$. Then the Kleisli category of the cotriple is cartesian closed.*

Proof. Let $\mathscr{K}$ be the Kleisli category of the cotriple. Then one description of $\mathscr{K}$ is that it has the same objects as $\mathscr{C}$ and

$$\mathrm{Hom}_{\mathscr{K}}(A, B) = \mathrm{Hom}_{\mathscr{C}}(GA, B)$$

First we have that

$$\begin{aligned} \mathrm{Hom}_{\mathscr{K}}(A, B \times C) &\cong \mathrm{Hom}_{\mathscr{C}}(GA, B \times C) \cong \mathrm{Hom}_{\mathscr{C}}(GA, B) \times \mathrm{Hom}_{\mathscr{C}}(GA, C) \\ &\cong \mathrm{Hom}_{\mathscr{K}}(A, B) \times \mathrm{Hom}_{\mathscr{C}}(A, C) \end{aligned}$$

This shows that the cartesian product in $\mathscr{K}$ is the same as that of $\mathscr{C}$, which is just another way of saying that the right adjoint preserves products. Then we have that for any objects $A$, $B$ and $C$,

$$\begin{aligned} \mathrm{Hom}_{\mathscr{K}}(A \times B, C) = \mathrm{Hom}_{\mathscr{K}}(G(A \times B), C) &\cong \mathrm{Hom}_{\mathscr{C}}(GA \otimes GB, C) \\ &\cong \mathrm{Hom}_{\mathscr{C}}(GB, GA \multimap C) \cong \mathrm{Hom}_{\mathscr{K}}(B, GA \multimap C) \end{aligned}$$

and this isomorphism is natural in $B$. This means that for all objects $A$ and $C$, the functor $\mathrm{Hom}(A \times -, C)$ is representable, which is what is required for cartesian closedness. $\square$

As for the category of coalgebras, one easily sees that the tensor product of two coalgebras is their cartesian product (just dualize the argument that the tensor product is the sum in the category of commutative rings). This is just a way a way of saying that the underlying functor takes the product of two coalgebras to the tensor product of the underlying objects. Since the cofree functor is the right adjoint to the underlying functor, it preserves products so the composite functor takes products to tensor products. Thus in this case, not only is the Kleisli category cartesian closed, but so is the Eilenberg-Moore category.

There are certainly other instances than that of coalgebras of cotriples on an autonomous category that take product to the tensor product. Nonetheless, the cofree coalgebra cotriple, when it exists, is the only one that I know of that one "expects". Others may exist on an *ad hoc* basis.

## 3 The $*^V$ functor

For the rest of this paper, $\mathscr{V}$ will denote a locally presentable autonomous category, that is, one equipped with a closed symmetric monoidal structure. In view of the preceding theorem, it is necessary to suppose only that the monoidal functor, is cocontinuous. We suppose chosen once and for all a fixed object we will denote $\perp$ and called the dualizing object. We will use the results, terminology and notation of [Barr, to appear] throughout. In particular, for any object $V$, we will denote $V \multimap \perp$ by $V^\perp$.

The category $\mathscr{A} = \mathscr{V}_\perp$ is described in [Barr, to appear] as consisting of triplets $(V, V', v)$ where $V$ and $V'$ are objects of $\mathscr{V}$ and $v: V \otimes V' \longrightarrow \perp$ is a morphism. Such a map induces by adjointness maps $V \longrightarrow V'^\perp$ and $V' \longrightarrow V^\perp$. A morphism between such structures is a pair $(f, f'): (V, V', v) \longrightarrow (W, W', w)$ where $f: V \longrightarrow W$ and $f': W' \longrightarrow V'$ are maps in $\mathscr{V}$ such that any (and hence all) of the following equivalent diagrams commutes.

$$\begin{tikzcd}
V \otimes W' \arrow[r, "V \otimes f'"] \arrow[d, "f \otimes W'"'] & V \otimes V' \arrow[d, "v"] \\
W \otimes W' \arrow[r, "w"'] & \perp
\end{tikzcd}$$

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d] & W \arrow[d] \\
V'^\perp \arrow[r, "f'^\perp"'] & W'^\perp
\end{tikzcd}
\qquad
\begin{tikzcd}
W' \arrow[r, "f'"] \arrow[d] & V' \arrow[d] \\
W^\perp \arrow[r, "f^\perp"'] & V^\perp
\end{tikzcd}$$

To simplify notation, we will henceforth suppress the mention of the structure map $v$, except when necessary, so that the objects of $\mathscr{A}$ are pairs $(V, V')$.

In light of the preceding section, we are interested in the category of coalgebras in $\mathscr{A}$. Let $\mathrm{PC}(\mathscr{A})$ and $\mathrm{PC}(\mathscr{V})$ denote the categories of pre-coalgebras over $\mathscr{A}$ and $\mathscr{V}$ respectively. An object of $\mathrm{PC}(\mathscr{A})$ is an object $(V, V')$ of $\mathscr{A}$ together with maps $(V, V') \longrightarrow (V, V') \otimes (V, V')$ and $(V, V') \longrightarrow (\top, \perp)$. This is equivalent to maps $V \longrightarrow V \otimes V$, $V$

$\longrightarrow \top$, $\mathscr{V}((V,V'),(V',V)) \longrightarrow V'$ and $\perp \longrightarrow V'$ such that

$$\begin{tikzcd} \mathscr{V}((V,V'),(V',V)) \arrow[r] \arrow[d] & V' \arrow[d] \\ (V\otimes V)^{\perp} \arrow[r] & V^{\perp} \end{tikzcd} \qquad \begin{tikzcd} \perp \arrow[r] \arrow[d] & V' \arrow[d] \\ \perp \arrow[r] & V^{\perp} \end{tikzcd} \tag{1}$$

commute. It will be convenient to look at this condition in a slightly different way.

If $(V,V')$ and $(W,W')$ are objects of $\mathscr{A}$, then we have already seen in the definition of their tensor product that there is a map $\mathscr{V}((V,V'),(W',W)) \longrightarrow (V\otimes W)^{\perp}$. If $V = W$, this specializes to a map $\mathscr{V}((V,V'),(W',V)) \longrightarrow (V\otimes V)^{\perp}$. If there is, in addition, a comultiplication $V \longrightarrow V\otimes V$, then this can be followed by the dual map $(V\otimes V)^{\perp} \longrightarrow V^{\perp}$ to give a map $\mathscr{V}((V,V'),(W',V)) \longrightarrow V^{\perp}$. An object $(V,V')$ can be alternately described as giving an object $V' \longrightarrow V^{\perp}$ of $\mathscr{V}/V^{\perp}$. What we have said amounts to describing a biproduct $V',W' \mapsto \mathscr{V}((V,V'),(W',V))$ on $\mathscr{V}/V^{\perp}$. We will denote this biproduct by $V' *^{V} W'$. Since both diagrams of (1) end in $V^{\perp}$, the pre-coalgebra structure on $(V,V')$ can be summarized—given the pre-coalgebra structure on $V$—as saying that there are maps $V' *^{V} V' \longrightarrow V'$ and $\perp \longrightarrow V'$ in the category $\mathscr{V}/V^{\perp}$. Let us say that an object $V'$ equipped with $V' *^{V} V' \longrightarrow V'$ and $\perp \longrightarrow V'$ is a pre-algebra in $\mathscr{V}/V^{\perp}$. Let $\mathrm{PA}(\mathscr{V}/V^{\perp})$ denote the category of pre-algebras in $\mathscr{V}/V^{\perp}$ with the definition of maps that follows.

If $V' \longrightarrow V^{\perp} \longleftarrow W'$ are objects of $\mathscr{V}/V^{\perp}$ with pre-algebra structure maps $V' *^{V} V' \longrightarrow V'$, $\perp \longrightarrow V'$, $W' *^{V} W' \longrightarrow W'$ and $\perp \longrightarrow W'$, then a map $f\colon W' \longrightarrow V'$ is a map of pre-algebras provided the following squares commute:

$$\begin{tikzcd} W' *^{V} W' \arrow[r] \arrow[d, "f *^{V} f"'] & W' \arrow[d, "f"] \\ V' *^{V} V' \arrow[r] & V' \end{tikzcd} \qquad \begin{tikzcd} \perp \arrow[r] \arrow[d] & W' \arrow[d, "f"] \\ \perp \arrow[r] & W' \end{tikzcd}$$

On the other hand, the arrow $(\mathrm{id}, f)\colon (V,V') \longrightarrow (V,W')$ is a map of pre-coalgebras if and only if the squares

$$\begin{tikzcd} (V,V') \arrow[r] \arrow[d] & (V\otimes V, V' *^{V} V') \arrow[d] \\ (V,W') \arrow[r] & (V\otimes V, W' *^{V} W') \end{tikzcd} \qquad \begin{tikzcd} (V,V') \arrow[r] \arrow[d] & (\top,\perp) \arrow[d] \\ (V,W') \arrow[r] & (\top,\perp) \end{tikzcd}$$

commute. It is immediate that these conditions are exactly the same.

This development can be summarized as follows.

**3.1 Theorem.** *There is a functor* $K\colon \mathrm{PC}(\mathscr{A}) \longrightarrow \mathrm{PC}(\mathscr{V})$ *and the fiber over a pre-coalgebra* $V \longrightarrow V \otimes V$ *and* $V \longrightarrow \top$ *is isomorphic to the category* $\mathrm{PA}(\mathscr{V}/V^{\perp})$. □

Suppose now that $\mathscr{V}$ is locally presentable. Let $\Phi\colon \mathscr{V} \longrightarrow \mathscr{V}$ be the functor defined by $\Phi(V) = \top \times V \otimes V$. Then a $\Phi$-coalgebra is exactly what we have called a pre-coalgebra. Since $\Phi$ is a finite product of accessible functors, it is accessible. It follows that the underlying functor $U\colon \mathrm{PC}(\mathscr{V}) \longrightarrow \mathscr{V}$ has a right adjoint $R$.

For a pre-coalgebra $V$, let $\Psi_V$ be the endofunctor on the category $\mathscr{V}/V^{\perp}$ defined by $\Psi_V(V') = \perp + V' *^V V'$.

**3.2 Proposition.** *The category* $\mathscr{V}_V$ *is locally presentable and the functor* $\Psi_V$ *is accessible.*

Proof. The category $\mathscr{V}/V^{\perp}$ has the form $(R\colon S)$, where $R$ is the identity functor and $S$ is constant at $V$. As for $\Psi_V$, it is the sum of a constant functor and the functor that takes $V' \longrightarrow V$ to the pullback

$$\begin{tikzcd}
V' *^V V' \arrow[r] \arrow[d] & V \multimap V' \arrow[d] \\
V \multimap V' \arrow[r] & (V \otimes V)^{\perp}
\end{tikzcd}$$

and if the functor that takes $V'$ to $V \multimap V'$ is accessible, $\Psi_V$ certainly is. Let $\lambda$ be a cardinal sufficiently large that as $G$ ranges over a set of generators for $\mathscr{V}$ all the objects $G$ and $G \otimes V$ as well as $\mathscr{V}$ are $\lambda$-accessible. Suppose $V' = \operatorname{colim} V_i$, a $\lambda$-filtered diagram. Then since

$$\mathrm{Hom}(G \otimes V, V') \cong \operatorname{colim} \mathrm{Hom}(G \otimes V, V_i)$$

it follows that

$$\mathrm{Hom}(G, V \multimap V') \cong \operatorname{colim} \mathrm{Hom}(G, V \multimap V_i) \cong \mathrm{Hom}(G, \operatorname{colim}(V \multimap V_i))$$

for each such $G$ and so

$$V \multimap V' \cong \operatorname{colim}(V \multimap V_i)$$

□

It follows that the underlying functor $\mathrm{PA}(\mathscr{V}/V^{\perp}) \longrightarrow \mathscr{V}/V^{\perp}$ has a left adjoint, denoted $L_V$.

**3.3 Proposition.** *Let* $V$ *be a pre-coalgebra in* $\mathscr{V}$. *Then* $(V, V^{\perp})$ *(equipped with the evaluation map* $V \otimes V^{\perp} \longrightarrow \perp$*) can be given the structure of a pre-coalgebra in* $\mathscr{A}$. *Moreover, the resulting functor* $\mathrm{PC}(\mathscr{V}) \longrightarrow \mathrm{PC}(\mathscr{A})$ *is left adjoint to the functor that takes first components.*

Proof. It is shown in [Barr, to appear] that

$$(V, V^\perp) \otimes (W, W^\perp) \cong (V \otimes W, (V \otimes W)^\perp)$$

It follows that given $V \longrightarrow V \otimes V$ and $V \longrightarrow \top$, we automatically get $(V, V^\perp) \longrightarrow (V \otimes V, (V \otimes V)^\perp)$ and $(V, V^\perp) \longrightarrow (\top, \perp)$. If $(W, W')$ is a pre-coalgebra in $\mathscr{A}$ and $V \longrightarrow W$ is a morphism of pre-coalgebras, then we have $W' \longrightarrow W^\perp \longrightarrow V^\perp$, which gives an arrow $(V, V^\perp) \longrightarrow (W, W')$ which is easily seen to be a pre-coalgebra morphism. $\square$

**3.4 Theorem.** *Let $\mathscr{V}$ be locally presentable and $\mathscr{A} = \mathscr{V}_\perp$ the Chu category corresponding to an object $\perp$ of $\mathscr{V}$. The underlying functor $PC(\mathscr{A}) \longrightarrow \mathscr{A}$ has a right adjoint that takes the object $(V, V')$ to $(RV, L_{RV}(V'))$.*

Proof. Suppose $(W, W')$ is a pre-coalgebra over $\mathscr{A}$, $(V, V')$ is an object of $\mathscr{A}$ and $(f, f')\colon (W, W') \longrightarrow (V, V')$ is a map in $\mathscr{A}$. Then $f\colon W \longrightarrow V$ is a map in $\mathscr{V}$ and there is thus a unique coalgebra morphism $\widetilde{f}\colon W \longrightarrow RV$ such that the composite $W \longrightarrow RV \longrightarrow V$ is $f$. Define $W''$ so the diagram

$$\begin{tikzcd}
W'' \arrow[r] \arrow[d] & (RV)^\perp \arrow[d, "(\widetilde{f})^\perp"] \\
W' \arrow[r] & W^\perp
\end{tikzcd}$$

is a pullback. It is shown in [Barr, to appear] that colimits in $\mathscr{A}$ are calculated by taking colimits in the first component and limits in the second. It follows that the diagram

$$\begin{tikzcd}
(W, W^\perp) \arrow[r] \arrow[d] & (RV, (RV)^\perp) \arrow[d] \\
(W, W') \arrow[r] & (RV, W'')
\end{tikzcd}$$

is a pushout in $\mathscr{A}$. But the maps $(W, W^\perp) \longrightarrow (W, W')$ and $(W, W^\perp) \longrightarrow (RV, (RV)^\perp)$ are easily seen to be pre-coalgebra morphisms (the latter being an instance of the functor in the preceding proposition and the former being the back adjunction morphism). It is standard that the underlying functor $\mathrm{PC}(\mathscr{A}) \longrightarrow \mathscr{A}$ creates colimits and so there is a unique coalgebra structure on $(RV, W'')$ so that the above square is a pushout.

In [Barr, to appear], it is shown that there is a morphism $(g, g')\colon (RV, W'') \longrightarrow (V, V')$ so that the composite $(W, W') \longrightarrow (RV, W'') \longrightarrow (V, V')$ is $f$. The composite $V' \longrightarrow V^\perp \longrightarrow (RV)^\perp$ gives $V'$ the structure of an object of $\mathscr{V}/(RV)^\perp$ and $g'\colon V' \longrightarrow W''$ is a map in that category. Since $(RV, W'')$ is a coalgebra, $W''$ is a $\Psi_{RV}$-algebra and so there is a unique arrow $\widehat{g}\colon L_{RV}V' \longrightarrow W''$ such that the composite

$$V' \longrightarrow L_{RV}V' \longrightarrow W''$$

is $g'$. Thus we have that $f$ factors as

$$(W, W') \longrightarrow (RV, W'') \longrightarrow (RV, L_{RV}V') \longrightarrow (V, V')$$

and the first two arrows are pre-coalgebra morphisms.

The argument up to here suffices to establish that every arrow factors through $(RV, L_{RV}V')$ and hence this object by itself is a solution set for the adjoint. However, the uniqueness of the constructed arrow is not hard to prove and shows that that object is the adjoint. For suppose that $(g, g'), (h, h'): (W, W') \longrightarrow (RV, L_{RV}V')$ are two morphisms of pre-coalgebras whose composites with the constructed map $(RV, L_{RV}V') \longrightarrow (V, V')$ are each $(f, f')$. Then $g, h: W \longrightarrow RV$ both have the property that their composite with the back adjunction $RV \longrightarrow V$ is $f$. The uniqueness of that adjunction implies that $g = h$. It follows that if $U \longrightarrow L_{RV}V'$ is the equalizer of $g'$ and $h'$, then there is a pre-coalgebra structure on $(RV, U)$ so that

$$(W, W') \rightrightarrows (RV, L_{RV}V') \longrightarrow (RV, U)$$

is a coequalizer in the category of pre-coalgebras. Since the underlying functor preserves colimits, the same diagram is a coequalizer in $\mathscr{A}$. Since $(g, g')$ and $(h, h')$ have the same composite into $(V, V')$, it follows that $V' \longrightarrow L_{RV}V'$ factors through the pre-coalgebra subobject $U \subseteq L_{RV}V'$, which is impossible for a front adjunction. For plainly any subobject with that property would have the same universal mapping property and, in particular, the inclusion would split. $\square$

## 4 Equations

Let $\mathscr{C}$ be a category. By an **equation** on $\mathscr{C}$ we mean two functors $F, G: \mathscr{C} \longrightarrow \mathscr{D}$ and two natural transformations $\phi, \psi: F \longrightarrow G$. We say that $C$ satisfies the equation $\phi = \psi$ if $\phi C = \psi C$ The full subcategory of all objects $C$ of $\mathscr{C}$ that satisfy an equation is called an **equational subcategory** of $\mathscr{C}$. For example, suppose $\mathscr{V}$ is a monoidal category with tensor product $\otimes$, and $\mathscr{C}$ is the category of $\otimes$-algebras. Let $G$ be the underlying functor and $F$ be the functor that assigns to an algebra $(V, m: V \otimes V \longrightarrow V)$ the object $V \otimes V \otimes V$. Then for $\phi = m \circ m \otimes V$ and $\psi = m \circ V \otimes m$, the equation $\phi = \psi$ is just the associative law. Similarly, the commutative law can be made by an equation. If we modifying the algebra to put in a nullary operation, then the unitary law can also be described by an equation.

A set of equations $\phi_i, \psi_i: F_i \longrightarrow G_i: \mathscr{C} \longrightarrow \mathscr{D}_i$ can be combined into a single equation $\phi, \psi: F \longrightarrow G: \mathscr{C} \longrightarrow \prod \mathscr{D}_i$ by letting $F$ and $G$ have components $F_i$ and $G_i$ respectively and similarly for $\phi$ and $\psi$. No $\mathscr{D}_i$ can be empty unless $\mathscr{C}$ is so that an object satisfies that single equation if and only if it satisfies each one. Thus there is no loss of generality in supposing that there is just one equation.

In order to simplify the exposition, we will suppose a single fixed equation and say that an object of $\mathscr{C}$ that satisfies it is **admissible** and call the full subcategory of admissible objects, the **admissible subcategory**.

**4.1 Proposition.** *If $\phi, \psi \colon F \longrightarrow G$ is an equation and $G$ preserves limits, then the admissible subcategory is closed under limits; if $F$ preserves colimits, then the admissible subcategory is closed under colimits.*

The proof is left as an exercise. In fact the hypotheses are even too strong. It suffices for the first that the map from $\lim F \longrightarrow F \lim$ be monic and dually for the second that the induced map be epic. We note that a functor into a product of non-empty categories preserves products if and only if each component does, so that there is no loss of generality in supposing there is just one equation.

**4.2 Proposition.** *If $G$ preserves epics, then any quotient of an admissible object is admissible; dually if $F$ preserves monics, then any subobject of an admissible object is admissible.*

Proof. Both statements can be read off from the serially commutative diagram, for a map $f \colon A \longrightarrow B$

$$\begin{tikzcd}
FA \arrow[r, shift left, "\phi A"] \arrow[r, shift right, "\psi A"'] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\
FB \arrow[r, shift left, "\phi B"] \arrow[r, shift right, "\psi B"'] & FD
\end{tikzcd}$$

□

In any well-powered complete category, every morphism can be factored as an epimorphism followed by an extremal monomorphism. Dually, in a well-copowered cocomplete category, every morphism can be factored as an extremal epimorphism followed by a monomorphism.

**4.3 Theorem.** *Let $\mathscr{C}$ be complete and well-copowered and let $\phi, \psi \colon F \longrightarrow G$ be an equation. If $F$ preserves limits, then the admissible category is reflexive. Dually, if $\mathscr{C}$ is cocomplete and well-powered and $G$ preserves colimits, then the admissible category is coreflective.*

Proof. We will prove the second statement, it being conceptually easier to deal with subobjects than quotient objects. The condition that $G$ preserves colimits implies that the admissible objects are closed under colimits and, in particular, that the union of any set of admissible subobjects of an object is admissible. It follows that the union of all the admissible subobjects of an object is also admissible. If $f \colon A \longrightarrow B$ is a map from an admissible object, then it factors $A \twoheadrightarrow B_0 \rightarrowtail B$ where the first arrow is epic and the second extremal monic. But then $B_0$ is an admissible subobject of $B$ and hence contained in the union of all of them. Thus every map from an admissible object to $B$ factors through this largest admissible subobject, evidently uniquely. $\square$

A locally presentable category satisfies all these conditions and hence we have:

**4.4 Corollary.** *Let $\mathscr{C}$ be a locally presentable category and let $\phi, \psi: F \longrightarrow G$ an equation on $\mathscr{C}$. If $F$ preserves limits, then the admissible category is reflexive; if $G$ preserves colimits, then the admissible category is coreflexive.* $\square$

Unfortunately, a $*$-autonomous category cannot be locally presentable (unless it is a poset) because only a poset can be locally presentable and colocally presentable. Hence we have to work to apply the above.

**4.5 Proposition.** *If $(f, f'): (V, V') \longrightarrow (W, W')$ is a monomorphism in the category $\mathrm{PC}(\mathscr{A})$, then $f: V \longrightarrow V'$ is monic in the category $\mathrm{PC}(\mathscr{V})$ and $W' \longrightarrow V'$ has the property that the least $*^V$-sub-coalgebra of $V'$ that includes the image of $W'$ maps epimorphically to $V'$.*

Proof. To take the first point, suppose that $f$ is not monic in $\mathrm{PC}(\mathscr{A})$. Then there is a pair of $\Phi$-coalgebra homomorphisms $g, h: U \longrightarrow V$ such that $g \neq h$ and $f \circ g = f \circ h$. Let $v: V' \longrightarrow V$ denote the transpose of the structure map of the object $(V, V')$. Define $g'$ and $h'$ as the upper and lower composites in the diagram

$$V' \xrightarrow{\ v\ } V^{\perp} \underset{h^{\perp}}{\overset{g^{\perp}}{\rightrightarrows}} U^{\perp}$$

As we have seen, $(U, U^{\perp})$ has the structure of a coalgebra and from the adjunction $\mathrm{PC}(\mathscr{A}) \longrightarrow \mathrm{PC}(\mathscr{V})$ we have two pre-coalgebra morphisms $(g, g'), (h, h'): (U, U^{\perp}) \longrightarrow (V, V')$ that have the same composite with $(f, f')$. Hence $f$ is a monomorphism.

Now suppose there is a $*^V$-subalgebra $V'' \subseteq V'$ such that $W' \longrightarrow V'$ factors through $V''$. If the inclusion $V'' \longrightarrow V'$ is not an epimorphism in the category of $*^V$-algebras, there are two morphisms $g'', h'': V' \longrightarrow V'''$ of $*^V$-algebras such that $g'' \neq h''$, but $g''|V'' = h''|V''$. But then $(f, f') \circ (\mathrm{id}, g'') = (f, f') \circ (\mathrm{id}, h'')$, which contradicts the fact that $(f, f')$ is monic. $\square$

**4.6 Corollary.** *The category $\mathrm{PC}(\mathscr{A})$ is well-powered.*

Proof. Fix an object $(W, W')$. As an object of $\mathrm{PC}(\mathscr{V})$, $W$ has only a set of subobjects $V$. Since $\mathrm{PC}(\mathscr{V})$ is locally presentable, it is well-powered, so it is sufficient to show that for each such $V$ there is only a set of subobjects of the form $(V, V')$. But for any such subobject, the map $W' \longrightarrow V'$ has to have the property that the $*^V$-subalgebra it generates maps epimorphically to $V'$. But it follows from the accessibility that $W'$ can generate only a set of $*^V$-algebras (recall that $W'$ is fixed and, in this part of the argument, so is $V$). Since the category of $\mathscr{V}$-algebras is also locally presentable, it is well-copowered and so any of the $*^V$-algebras generated by $V'$ has only a set of quotients. Thus there is only a set of possibilities for $W'$. $\square$

**4.7 Proposition.** *The underlying functor from the category of counitary, coassociative, cocommutative coalgebras over $\mathscr{A}$ to $\mathscr{A}$ has a right adjoint.*

Proof. Since the equations to be satisfied have the form $\rho, \sigma\colon \Gamma \longrightarrow \Lambda$, where $\Gamma$ is the underlying functor that preserves colimits, it follows that the subcategory is closed under colimits, in particular unions. Thus the union of all the subcoalgebras that satisfy the equations does as well. Any arrow factors as an epi followed by a regular mono and a quotient of an object that satisfies the equation does as well. Hence any map from an admissible coalgebra to another factors through an admissible subcoalgebra and hence through that union. Thus that union defines a left adjoint to the inclusion of the admissible subcoalgebras. $\square$

Putting this all together, we have:

**4.8 Theorem.** *Suppose that $\mathscr{V}$ is a closed symmetric monoidal category that is locally presentable. Suppose $\perp$ is an internal cogenerator in $\mathscr{V}$ and $\mathscr{A}$ is the Chu category $\mathscr{V}_\perp$. Then $\mathscr{A}$ is a model of the full linear logic.* $\square$

## 5 Examples

**5.1 Complete lattices.** A category that is locally presentable and colocally presentable is a complete lattice. Therefore a $*$-autonomous category cannot be locally presentable unless it is a poset. For this reason, it seems interesting to consider that case, even though the resultant models of linear logic are probably not interesting. A complete lattice that is autonomous could use the inf operation as the tensor, in which case it is a Heyting algebra. However, it can be closed instead under another operation $\otimes$. Also, it is necessary to choose a dualizing object $\perp$. The bottom element can be chosen, but so could any other.

For example, the lattice of ideals of a commutative ring (or even a commutative monoid) is an autonomous category with the operations of ideal multiplication and division. If $I$ and $J$ are ideals, then $I/J = J \multimap I$ consists of all $x$ such that $xJ \subseteq I$. In the case of a ring, one possibility for the dualizing object is the zero ideal, although the unit ideal is also possible (in which case the Chu category consists of all pairs of ideals). In the case of a monoid, there is not necessarily a zero element, but any ideal can be chosen.

So let $P$ be a complete lattice with a symmetric biproduct $\otimes$, a unit element for the biproduct $\top$ and a right adjoint $\multimap$ for $\otimes$. Choose an element $\perp$ for the duality. Then the Chu category has for objects all pairs $(x, x')$ of elements of $P$ such that $x \otimes x' \le \perp$. The set of morphisms from $(x, x')$ to $(y, y')$ is a subset of $\mathrm{Hom}(x, y) \times \mathrm{Hom}(y', x')$ and evidently contains at most one element. Thus this category is again a poset, a lattice in fact, since it is also complete. Then $(x, x') \le (y, y')$ if and only if $x \le y$ and $y' \le x'$.

Also since pullbacks in a lattice are just infs,

$$P((x,x'),(y,y')) = (x \multimap y) \wedge (y' \multimap x')$$

Thus we have

$$(x,x') \otimes (y,y') = (x \otimes y, (x \multimap y') \wedge (y \multimap x'))$$

and

$$(x,x') \multimap (y,y') = ((x \multimap y) \wedge (y' \multimap x'), x \otimes y')$$

Note that the dualizing object determines the objects of the category, but has no effect on the structure.

It is interesting to see what the coalgebra category is in this case. Since we are in a poset, an object either is or isn't a coalgebra. Since $(x,x') \otimes (x,x') = (x \otimes x, x \multimap x')$ and $\top = (\top, \perp)$, we need that $x \le \top$, $\perp \le x'$, $x \le x \otimes x$ and $x \multimap x' \le x'$, in addition to $x \wedge x' \le \perp$ that every object must satisfy. Then $x \le \top$ implies that $x \otimes x \le x$ so we must have $x \otimes x = x$. Also $x \otimes x' \le \perp$ implies $x' \le \times^{\perp}$ while $x \otimes x^{\perp} \le \perp \le x'$ implies that $x^{\perp} \le x \multimap x' = x'$ so that $x' = x^{\perp}$. Hence the coalgebras are all of the form $(x, x^{\perp})$ such that $x \le \top$ and $x = x \otimes x$. The condition $\perp \le \times^{\perp}$ is automatic. The bottom of the lattice 0 is automatically a coalgebra, as is the tensor unit $\top$. The cofree coalgebra assigns to each object $(x,x')$ the object $(y, y^{\perp})$ which is the union of all the coalgebras included in $(x,x')$.

**5.2 Other examples.** For other examples of accessible autonomous categories, it is harder to find a complete autonomous category that isn't accessible than one that is. (Of course, as observed above, a complete $*$-autonomous category can't be accessible.) So this class includes all the common autonomous categories such as modules over a commutative ring, $M$-sets for a commutative monoid $M$ and the natural tensor product in that category. Any Grothendieck topos with its cartesian closed structure is also accessible. Of course, this means that $M$-sets has two quite different structures.

This example of $M$-sets is not entirely familiar, so I will give some more details. First off, if $M$ is any monoid, commutative or not, the category of left $M$-sets is a topos and thus has the cartesian closed structure of any topos. The tensor product is the cartesian product and the internal hom is quite complicated. It simplifies in the case that $M$ is a group to the set of all functions (not just the equivariant ones) between the two sets, with the group acting by conjugation (so that the points of the homset are the equivariant maps).

If $M$ is an arbitrary monoid $X$ a right $M$-set and $Y$ a left $M$ set then you can define a set $X \otimes_M Y$ quite analogous to the tensor product of modules as the coequalizer in

$$X \times M \times Y \rightrightarrows X \times Y \longrightarrow X \otimes_M Y$$

For any set $Z$, the adjunction relation

$$\mathrm{Hom}(X \otimes_M Y, Z) \cong \mathrm{Hom}_{M^{\mathrm{op}}}(X, \mathrm{Hom}(Y, Z))$$

is valid, where $\mathrm{Hom}(Y, Z)$ is made into a right $M$-set in the usual way. When $M$ is commutative, then we can just speak of $M$-sets and we get a closed monoidal category, just like modules over a commutative ring.

In the case that $M$ is a group, the internal hom consists of just the equivariant maps, made into an $M$-set by translation. So the structure is quite "linear", in contrast to the cartesian closed structure.

An interesting example that is $\aleph_1$ accessible, but not $\aleph_0$ accessible is the category of Banach spaces. The internal hom is the set of bounded linear maps with the sup on the unit ball norm. The tensor product is given the greatest cross-norm and completed.

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