Separability of tensor in Chu categories of vector spaces

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1 Introduction

If $\mathcal{V}$ is a symmetric tensored closed category and $\perp$ is an object of $\mathcal{V}$, there is a construction described in [Chu, 1979], of a $*$-autonomous category based on $\mathcal{V}$. Very briefly, an object of $\mathrm{Chu}(\mathcal{V}, \perp)$ is a pair (V, V') of objects of $\mathcal{V}$ together with a pairing $\langle -, - \rangle : V \otimes V' \longrightarrow \perp$. These objects have been called *Chu spaces* because in concrete examples one can think of V' as being a kind of topology on V ([Pratt, 1993]).

An object (V, V') is called *separated* if the induced map $V \longrightarrow V' \multimap \perp$ is monic and *extensional* if the induced $V' \longrightarrow V \multimap \perp$ is monic. It is not true in general that the tensor product of two separated Chu spaces is separated nor that the internal hom of two extensional spaces is extensional. The purpose of this note is to show that these claims (which are equivalent to each other) are true when the ground category is vector spaces.

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A complete introduction to Chu spaces and $*$-autonomous categories can be found in [Barr, 1991]. See also [Barr, to appear]. We give a quick sketch.

1.1 The Chu construction

A morphism $(f, f') : (V, V') \longrightarrow (W, W')$ is a pair of arrows $f : V \longrightarrow W$ and $f' : W' \longrightarrow V'$ such that the diagram

$$\begin{tikzcd}
V \otimes W' \arrow[r, "f \otimes W'"] \arrow[d, "V \otimes f'"'] & W \otimes W' \arrow[d, "\langle -, - \rangle"] \\
V \otimes V' \arrow[r, "\langle -, - \rangle"'] & \perp
\end{tikzcd}$$

If terms of elements, this says that for all $v \in V$ and $w' \in W'$, $\langle fv, w' \rangle = \langle v, f'w' \rangle$.

Another way of expressing this is that the diagram

$$\begin{tikzcd}
\mathrm{Hom}((V, V'), (W, W')) \arrow[r] \arrow[d] & \mathrm{Hom}(V, W) \arrow[d] \\
\mathrm{Hom}(W', V') \arrow[r] & \mathrm{Hom}(V \otimes W', \perp)
\end{tikzcd} \tag{1}$$

is a pullback.

The category $\mathrm{Chu}(\mathcal{V}, \perp)$ is a $*$-autonomous category, with the following structures. First, for Chu spaces (V, V') and (W, W'), define a Vect-valued hom $(V, V') \mathrel{-\!\!\bullet} (W, W')$ so that

$$\begin{tikzcd}
(V, V') \mathrel{-\!\!\bullet} (W, W') \arrow[r] \arrow[d] & V \multimap W \arrow[d] \\
W' \multimap V' \arrow[r] & (V \otimes W') \multimap K
\end{tikzcd}$$

is a pullback. Note that this diagram is simply the obvious strengthening of (1) to $\mathcal{V}$. Now we define

$$(V, V') \otimes (W, W') = (V \otimes W, (V, V') \mathrel{-\!\!\bullet} (W', W))$$

The pairing is most easily given in terms of elements. For $v \in V$, $w \in W$, $f : V \longrightarrow W'$ and $f' : W \longrightarrow V'$, we let $\langle v \otimes w, (f, f')\rangle = \langle w, fv\rangle = \langle v, f'w\rangle$, The internal hom is given similarly by

$$(V, V') \multimap (W, W') = ((V, V') \multimap\!\bullet\, (W, W'), V \otimes W')$$

The duality is $(V, V')^* = (V, V') \multimap (\perp, \top) = (V', V)$, where $\top$ is the tensor unit.

It is shown in [Barr, 1991] (where "separated" is called "left separated" and "extensional" is called "right separated") that in general the tensor product of extensional spaces is extensional, but, as shown in the example of the introduction, the tensor product of separated spaces may fail to be separated. The opposite happens for the internal hom, where the internal hom of two separated spaces is separated, but the internal hom of extensional spaces may not be extensional. We will see that this does not happen in $\mathrm{Chu}(\mathrm{Vect}, K)$. In light of the duality between the tensor and internal hom, it will follow, when we show that the tensor product of separated spaces is separated, that the internal hom of extensional spaces is extensional.

1.2 An example: abelian groups

Consider the category $\mathrm{Chu}(\mathrm{Ab}, T)$ where $T = \mathbf{R}/\mathbf{Z}$ is the circle group. There is a Chu space $(\mathbf{Q}, \mathbf{Z})$ using any pairing $\mathbf{Q} \otimes \mathbf{Z} \cong \mathbf{Q} \longrightarrow T$ that embeds $\mathbf{Q}$ into T. Multiplication by any irrational number will do. Such a pairing is both separated and extensional. On the other hand, $(\mathbf{Q}, \mathbf{Z}) \otimes (\mathbf{Q}, \mathbf{Z}) = (\mathbf{Q} \otimes \mathbf{Q}, (\mathbf{Q}, \mathbf{Z}) \multimap\!\bullet\, (\mathbf{Q}, \mathbf{Z}))$ with the latter being the pullback

$$\begin{tikzcd}
(\mathbf{Q}, \mathbf{Z}) \multimap\!\bullet\, (\mathbf{Q}, \mathbf{Z}) \arrow[r] \arrow[d] & \mathbf{Q} \multimap \mathbf{Z} \arrow[d] \\
\mathbf{Q} \multimap \mathbf{Z} \arrow[r] & \mathbf{Q} \otimes \mathbf{Q} \multimap T
\end{tikzcd}$$

which is evidently 0 since there are no non-zero homomorphisms $\mathbf{Q} \longrightarrow \mathbf{Z}$. Thus $(\mathbf{Q}, \mathbf{Z}) \otimes (\mathbf{Q}, \mathbf{Z}) = (\mathbf{Q}, 0)$, which is evidently not separated.

2 Vector spaces

Let K be a field and $\mathrm{Vect} = \mathrm{Vect}_K$ be the category of vector spaces over a field. We let $\mathcal{A}$ denote the category $\mathrm{Chu}(\mathrm{Vect}, K)$.

2.1 Theorem *In* $\mathrm{Chu}(\mathrm{Vect}_K, K)$ *the tensor product of separated spaces is separated and the internal hom of extensional spaces is extensional.*

Proof. This means that the map $V \otimes W \longrightarrow ((V, V') \multimapdot (W', W))^*$ must be shown to be injective. If not, there are finite dimensional subspaces V_0 and W_0 of V and W such that the composite

$$V_0 \otimes W_0 \longrightarrow V \otimes W \longrightarrow ((V, V') \multimapdot (W', W))^*$$

is not injective either. We will show that this is impossible by first showing that (V_0, V_0^*) is a subobject, in fact split subobject, of (V, V') and similarly for $(W_0, W_0^*) \longrightarrow (W, W')$ and that the upper and right arrows in

$$\begin{tikzcd}
V_0 \otimes W_0 \arrow[r] \arrow[d] & ((V_0, V_0^*) \multimapdot (W_0^*, W_0))^* \arrow[d] \\
V \otimes W \arrow[r] & ((V, V') \multimapdot (W', W))^*
\end{tikzcd} \tag{2}$$

are injective. Actually, the top arrow is an isomorphism.

From $V_0 \longrightarrow V$, we have $V' \longrightarrow V^* \longrightarrow V_0^*$ which gives us a morphism $(V_0, V_0^*) \longrightarrow (V, V')$. I claim that $V' \twoheadrightarrow V_0^*$. If not, the arrow factors through a proper subspace, say $U \subseteq V_0^*$ and then from vector space duality, we have $V_0 \twoheadrightarrow U^* \longrightarrow V'^*$. Then we have a commutative square

$$\begin{tikzcd}
V_0 \arrow[r, two heads] \arrow[d] & U^* \arrow[d] \\
V \arrow[r, tail] & V'^*
\end{tikzcd}$$

and the diagonal fill-in gives $U^* \longrightarrow V$ such that the upper triangle commutes, which implies, along with $V_0 \rightarrowtail V$, that $V_0 = U^*$, and then that $U = V_0^*$.

Next, I claim that the injection $(f, f') : (V_0, V_0^*) \rightarrowtail (V, V')$ is split. Let $v_1, \ldots, v_n$ be a basis of V_0 and let $v_1^*, \ldots, v_n^*$ be the dual basis of V_0^*. This means that $\langle v_i, v_j^* \rangle = \delta_{ij}$. Since f' is surjective, let $v_1', \ldots, v_n' \in V'$ be vectors such that $f'v_i' = v_i^*$, for $i = 1, \ldots, n$. Now define $g : V \longrightarrow V_0$ by

$$g(v) = \sum_{i=1}^{n} \langle v, v_i' \rangle v_i$$

and $g' : V_0^* \longrightarrow V'$ by $g'v_i^* = v_i'$. We want to show that $(g, g') : (V, V') \longrightarrow (V_0, V_0^*)$ is a map in the category and that it splits (f, f'). For the first, we have, for $v \in V$ and $i = 1, \ldots, n$,

$$\langle gv, v_i^* \rangle = \langle \sum_{j=1}^{n} \langle v, v_j' \rangle v_j, v_i^* \rangle = \langle v, v_i' \rangle = \langle v, g'v_i^* \rangle$$

so that (g, g') is a morphism. Then we have for $i = 1, \ldots, n$,

$$gf(v_i) = \sum_{j=1}^{n} \langle fv_i, v_j' \rangle v_i = \sum_{j=1}^{n} \langle v_i, f'v_j' \rangle v_i = \sum_{j=1}^{n} \langle v_i, v_j^* \rangle v_i = v_i$$

We could similarly show that $f'g' = \mathrm{id}$ but it is unnecessary since in the subcategory of separable extensional objects, the two halves of a map determine each other.

Now the diagram

$$\begin{tikzcd}
(V_0, V_0^*) \multimapdotbothB (W_0^*, W_0) \arrow[r] \arrow[d] & V_0 \multimap W_0^* \arrow[d] \\
W_0 \multimap V_0^* \arrow[r] & (V_0 \multimap W_0)^*
\end{tikzcd}$$

is a pullback while the bottom and right arrows are isomorphisms and hence

$$(V_0, V_0^*) \multimapdotbothB (W_0^*, W_0) = (V_0 \multimap W_0)^*$$

so that

$$((V_0, V_0^*) \multimapdotbothB (W_0^*, W_0))^* = V_0 \multimap W_0$$

which shows that the top map in (2) is an isomorphism, as claimed. As for the right hand map in (2), we observe that $(V_0, V_0^*) \longrightarrow (V, V')$ is split monic and, dually, $(W', W) \longrightarrow (W_0^*, W_0)$ is split epic so that

$$(V, V') \multimapdotbothB (W', W) \longrightarrow (V_0, V_0^*) \multimapdotbothB (W_0^*, W_0)$$

is split epic and then

$$((V_0, V_0^*) \multimap (W_0^*, W_0))^* \longrightarrow ((V, V') \multimap (W', W))^*$$

is (split) monic, as required.

References

M. Barr (1979), **∗-Autonomous categories**. Lecture Notes in Mathematics **752**, Springer-Verlag, Berlin, Heidelberg, New York.

M. Barr (1991), *∗-Autonomous categories and linear logic.* Math. Structures Comp. Sci., 1, 159–178.

M. Barr (to appear), *Non-symmetric ∗-autonomous categories.* Theoretical Computer Science.

P.-H. Chu (1979), *Constructing ∗-autonomous categories.* Appendix to [Barr, 1979].

V. R. Pratt (1993), *The Second Calculus of Binary Relations.* In Proceedings of MFCS '93, Gdańsk, Poland, 142-155, Springer-Verlag.