A COHOMOLOGY THEORY FOR COMMUTATIVE ALGEBRAS. II[1]

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1. **Introduction.** In the first paper of this series [1], henceforth referred to as I, we define a cohomology theory for a commutative algebra R with coefficients in an R-module M for which $H^2(R, M)$ is the group of singular extensions of R by M. In this paper we generalize to a cohomology $H(S, \phi, M)$ where (S, ϕ) is a singular extension of R and M is an R-module. The second group is again a group of extensions and when $(S, \phi) = (R, 1_R)$, $H(S, \phi, M) = H(R, M)$ as defined in I. This generalization leads to a surprising result: a connected sequence in the first variable.

2. **Definitions.** Let K be a commutative ring with identity. All rings considered will be unitary K-algebras and all modules will also be unitary K-modules. Let R be an algebra and M an R-module. In I we defined a singular extension $0 \to M \to S \to^{\phi} R \to 0$ of R by M and morphism of two such extensions. This extension will also be denoted by (S, ϕ) or just S. If Φ: $(S', \phi') \to (S, \phi)$ is a morphism we will also use Φ: $S' \to S$ to denote the implied algebra morphism. It will be called a surjection or an extension if the implied morphism is a surjection. In any case, ker Φ, ker ϕ, and ker ϕ' are all R-modules. We let $\mathcal{C}_R$ or just $\mathcal{C}$ denote the category of extensions of R and morphisms of extensions. Then an extension of (S, ϕ) is an exact sequence $0 \to M \to (S', \phi') \to^{\Phi} (S, \phi) \to 0$ where Φ is a surjection and M is its kernel. If M and (S, ϕ) are held fixed, the notions of equivalence and Baer composition make sense as in I. Then the equivalence classes can be shown to form a group which is denoted by $H^2(S, \phi, M)$. We denote by $H^1(S, \phi, M)$ the group of derivations of S to M (made into an S-module by ϕ).

We define an n-long exact sequence over $\mathcal{C}$ to be an exact sequence

$$0 \to M \to M_{n-2} \to M_{n-3} \to \cdots \to M_1 \to (S', \phi') \xrightarrow{\Phi} (S, \phi) \to 0$$

where $0 \to \ker \Phi \to (S', \phi') \to (S, \phi) \to 0$ is an extension and $0 \to M \to M_{n-2} \to \cdots \to M_1 \to \ker \Phi \to 0$ is an ordinary exact sequence of R-modules. An ω-long sequence is one like that which has no left end. Again we define equivalence, Baer composition and morphism of long exact

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sequences. An extension will also be called a short exact sequence. The classes of n-long sequences beginning with M, ending with (S, ϕ) can also be shown to form a group, denoted by $H^n(S, \phi, M)$.

Just as in I, a long (resp. short) exact sequence is called generic if it admits a morphism to all others with the same right end (and with the identity map on the right). This will also be called a generic resolution (resp. generic extension).

3. **Main results.** In this section we indicate how to compute the cohomology just defined as the cohomology of a complex, thus proving, among other things, that it is a group. Two computational theorems are also proved here.

LEMMA 1. *Suppose (S, ϕ) is in $\mathcal{C}$ with* $\ker \phi = M$ *and (F, σ) is a generic extension of R. Let X_0 be an R-projective mapping onto M and G be the inessential extension of F by X_0. Then there is a $\tau: G \to R$ and $\theta: (G, \tau) \to (S, \phi)$ which are in $\mathcal{C}$ and are a generic extension of (S, ϕ).*

PROOF. τ is just the composition $G \to F \to R$. θ is the sum of $X_0 \to M \to S$ and any map of G to S which commutes with ϕ. These are easily seen to be in $\mathcal{C}$. We now must show that $\theta: (G, \tau) \to (S, \phi)$ is generic. So suppose $\Phi: (S', \phi') \to (S, \phi)$ is in $\mathcal{C}$. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_0 & \underset{\overset{\rho}{\longleftarrow - - -}}{\longrightarrow} & G & \xrightarrow{\pi} & F & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \sigma} & & \\
0 & \longrightarrow & M & \longrightarrow & S & \longrightarrow & R & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \Phi} & & \uparrow{\scriptstyle \Phi} & & \updownarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & S' & \longrightarrow & R & \longrightarrow & 0
\end{array}$$

where $\rho: G \to X_0$ is the projection. We begin by choosing $\alpha: F \to S'$ with $\phi'\alpha = \sigma$. Then $\phi\Phi\alpha\pi = \phi'\alpha\pi = \sigma\pi = \phi\theta$ so that $\phi(\Phi\alpha\pi - \theta) = 0$. This means that $\Phi\alpha\pi - \theta$ maps G to M and its restriction to X_0 is easily seen to be an R-morphism. Since X_0 is projective we can find $\beta: X_0 \to M'$ so that $\Phi\beta = \Phi\alpha\pi - \theta$ on X_0. Then restricted to X_0, $\theta = \Phi(\alpha\pi - \beta\rho)$. Thus θ and $\Phi(\alpha\pi - \beta\rho)$ agree on X_0 and agree when followed by ϕ. Thus their difference induces a map $\gamma: F \to M$ which is easily seen to be a derivation. But since F is generic this means that we can find a derivation $\delta: F \to M'$ with $\Phi\delta = \gamma$. Then $\Phi\delta\pi = \Phi(\alpha\pi - \beta\rho) - \theta$ or $\theta = \Phi(\alpha\pi - \beta\rho - \delta\pi)$ and so $\theta' = \alpha\pi - \beta\rho - \delta\pi$ is the desired map. The reader should check that it is an algebra homomorphism. Recall in doing so that $M'^2=0$.

LEMMA 2. *Let* $0\to M\to G\to S\to 0$ *a generic extension of* S *and* $X\to M\to 0$ *an* R*-projective resolution of* M, *then* $X\to G\to S\to 0$ *is a generic resolution of* S.

PROOF. This is proved exactly as in I.

THEOREM 3. *Generic resolutions exist.*

PROOF. This follows from the existence of generic resolutions of R, shown in I, together with Lemmas 1 and 2.

THEOREM 4. *Let* $X\to G\to S\to 0$ *a generic resolution of* S, *then* $H(S,\phi,M)$ *is just the homology of*

$$0\to \mathrm{Der}(G,M)\to \mathrm{Hom}_R(X,M).$$

PROOF. The proof is left to the reader. It is a standard computation in homological algebra.

THEOREM 5. *Let* $0\to M\to S'\to S\to 0$ *be a short exact sequence, then we have connecting morphisms* $\mathrm{Ext}_R^{i-1}(M,N)\to H^{i+1}(S,\phi,N)$ *such that the following sequence is exact,*

$$\begin{aligned} 0&\to \mathrm{Der}(S,N)\to \mathrm{Der}(S',N)\to \mathrm{Hom}_R(M,N)\to H^2(S,\phi,N)\\ &\to H^2(S',\phi',N)\to\cdots\to H^i(S,\phi,N)\to H^i(S',\phi',N)\\ &\to \mathrm{Ext}_R^{i-1}(M,N)\to H^{i+1}(S,\phi,N)\to\cdots. \end{aligned}$$

PROOF. Let $X\to M\to 0$ be an R-projective resolution of M and $Y\to G\to S\to 0$ be a generic resolution of S. Then we have

$$\begin{array}{ccccccccc} & & \downarrow & & & & \downarrow & & \\ & & X_1 & & & & Y_1 & & \\ & & \downarrow & & & & \downarrow & & \\ & & X_0 & & & & G & & \\ & & \downarrow{\scriptstyle\epsilon} & & & & \downarrow{\scriptstyle\theta} & & \\ 0 & \longrightarrow & M & \longrightarrow & S' & \longrightarrow & S & \longrightarrow & 0. \end{array}$$

Now let G' be the inessential extension of G by X_0. Then G' is a generic resolution of S', exactly as was proved in Lemma 1. Let L be the kernel of this extension. We have $0 \to \ker \epsilon \to L \to \ker \theta \to 0$ is exact as is well known (or see Lemma 6 below). We can let $Z_i = X_i \oplus Y_i$ and $\cdots \to Z_i \to \cdots \to Z_2 \to Z_1 \to L \to 0$ will be a projective resolution of L. The remainder is a standard computation.

Before proceeding further, we need three lemmas.

LEMMA 6. *Suppose we have a commutative diagram*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_{11} & \to & A_{12} & \to & A_{13} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_{21} & \to & A_{22} & \to & A_{23} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_{31} & \to & A_{32} & \to & A_{33} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

in which all three columns, the middle row and one other row are exact, then the remaining row is exact.

PROOF. This is easily proved by diagram chasing.

LEMMA 7. *Suppose we have a commutative diagram with exact rows and columns,*

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & A_{11} & \to & A_{12} & \to & A_{13} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & A_{21} & \to & A_{22} & \to & A_{23} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_{31} & \to & A_{32} & \to & A_{33} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

then there is naturally induced an exact sequence,

$$0 \to A_{13} \oplus A_{31} \to A_{22}/A_{11} \to A_{33} \to 0.$$

PROOF. This can be proved by diagram chasing together with some computation. The key step is showing that

$$\mathrm{Im}(A_{21} \to A_{22}) \cap \mathrm{Im}(A_{12} \to A_{22}) = \mathrm{Im}(A_{11} \to A_{22}).$$

LEMMA 8. *Suppose* $0 \to A_1 \to A_2 \to A_3 \to 0$ *and* $0 \to B_1 \to B_2 \to B_3 \to 0$ *are exact sequences of R-modules and* $\mathrm{Tor}_1^R(A_3, B_3) = 0$, *then we have an exact sequence* $0 \to A_1 \otimes_R B_3 \oplus A_3 \otimes_R B_1 \to A_2 \otimes_R B_2 / A_1 \otimes_R B_1 \to A_3 \otimes_R B_3 \to 0$.

PROOF. This is an easy application of Lemma 7.

In what follows, $\otimes$, Tor, "flat," will mean $\otimes_K$, Tor^K, and "K-flat" respectively.

THEOREM 9. *Suppose that R' and R'' are flat algebras and that (S', ϕ') and (S'', ϕ'') are extensions of R' and R'' with kernels M' and M'' respectively. Suppose* $\mathrm{Tor}_1(S', S'') = 0$. *Then if we let*

$$S = S' \otimes S''/M' \otimes M'', \quad \textit{and} \quad R = R' \otimes R''$$

then $\phi' \otimes \phi''$ induces a map ϕ: $S \to R$ which is a singular extension and for any R-module M we have $H(S, \phi, M) \approx H(S', \phi', M) \oplus H(S'', \phi'', M)$.

PROOF. The first assertion is clear and we have from Lemma 8 that $\ker \phi = M' \otimes R'' \oplus R' \otimes M''$. If $0 \to L' \to F' \to R' \to 0$ and $0 \to L'' \to F'' \to R'' \to 0$ are generic resolutions then it is shown in I that $0 \to L' \otimes R'' \oplus R' \otimes L'' \to F' \otimes F''/L' \otimes L'' \to R \to 0$ is also generic. Let $X' \to M' \to 0$ and $X'' \to M'' \to 0$ be R'-projective and R''-projective resolutions of M' and M'' respectively. Then if G and G' are the inessential extensions of F' by X_0' and F'' by X_0'' respectively, they are generic extensions of S' and S''. Let $N' = \ker(G' \to S')$ and $N'' = \ker(G'' \to S'')$. Then

$$(*) \quad 0 \to N' \otimes S'' \oplus S' \otimes N'' \to G' \otimes G''/N' \otimes N'' \to S' \otimes S'' \to 0$$

is exact by Lemma 8. Also one can check that $\ker(G' \to R')$ is $X_0' \oplus L'$ and $\ker(G'' \to R'')$ is $X_0'' \oplus L''$. Then we have

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & N' & & & & \\
 & & & & \downarrow & & & & \\
0 & \to & X_0' \oplus L' & \to & G' & \to & R' & \to & 0 \\
 & & \downarrow & & \downarrow & & \updownarrow & & \\
0 & \to & M' & \longrightarrow & S' & \to & R' & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

from which, by Lemma 6, $N' = \ker(X_0' \oplus L' \rightarrow M')$ and similarly $N'' = \ker(X_0'' \oplus L'' \rightarrow M'')$. Now $\mathrm{Tor}_1(S', S'') = 0$, a fact we have already used. But together with the fact that R' and R'' are flat and the following two exact sequences

$$0 = \mathrm{Tor}_2(M', R'') \rightarrow \mathrm{Tor}_1(M', M'') \rightarrow \mathrm{Tor}_1(M', S'') \rightarrow \mathrm{Tor}_1(M', R'') = 0,$$

$$0 = \mathrm{Tor}_2(R', S'') \rightarrow \mathrm{Tor}_1(M', S'') \rightarrow \mathrm{Tor}_1(S', S'') \rightarrow \mathrm{Tor}_1(R', S'') = 0$$

we conclude also that

$$\mathrm{Tor}_1(M', M'') = 0.$$

But this permits another application of Lemma 8 to have an exact sequence

$$(**)\qquad \begin{aligned} &0 \rightarrow N' \otimes M'' \oplus M' \otimes N'' \\ &\quad \rightarrow (X_0' \oplus L') \otimes (X_0'' \oplus L'')/N' \otimes N'' \rightarrow M' \otimes M'' \rightarrow 0. \end{aligned}$$

Now we may combine (*) and (**) to get a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \rightarrow & N' \otimes M'' & \oplus M' \otimes N'' & \rightarrow & (X_0' \oplus L') \otimes (X_0'' \oplus L'')/N' \otimes N'' & \rightarrow & M' \otimes M'' \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \rightarrow & N' \otimes S'' & \oplus S' \otimes N'' & \longrightarrow & G' \otimes G''/N' \otimes N'' & \longrightarrow & S' \otimes S' \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \rightarrow & N' \otimes R'' & \oplus R' \otimes N'' & \rightarrow & G' \otimes G''/(X_0' \oplus L') \otimes (X_0'' \oplus L'') & \rightarrow & S' \otimes S''/M' \otimes M'' \\
 & & \downarrow & & \downarrow & & \downarrow \qquad \rightarrow 0 \\
 & & 0 & & 0 & & 0
\end{array}$$

The top two rows have been shown to be exact, the left column is exact because R' and R'' are flat and the other two columns are obviously exact. Hence, by Lemma 6, the bottom row is also exact. Now we know that $F' \otimes F''/L' \otimes L''$ is a generic extension of $R' \otimes R''$ and that the ker of ϕ is $M' \otimes R'' \oplus R' \otimes M''$. An $R' \otimes R''$-projective mapping onto this is $X_0' \otimes R'' \oplus R' \otimes X_0''$. (See proof of Theorem 7 of I for details.) Then a generic extension of S is the inessential extension of $F' \otimes F''/L' \otimes L''$ by $X_0' \otimes R'' \oplus R' \otimes X_0''$. We can map $G' \otimes G''/(X_0' \oplus L') \otimes (X_0'' \oplus L'')$ to this by sending $(x', f') \otimes (x'', f'') \rightarrow (x' \otimes \theta''(f''), \theta'(f') \otimes x'', f' \otimes f'')$ where $\theta' \colon F' \rightarrow R'$ and $\theta'' \colon F'' \rightarrow R''$ are the generic extensions. This map can be checked to be an algebra homomorphism and the following calculation shows it is an isomorphism,

$$\begin{aligned} G' \otimes G''/(X_0' \oplus L') \otimes (X_0'' \oplus L'') &\approx (X_0' \oplus F') \otimes (X_0'' \oplus F'')/(X_0' + L') \otimes (X_0'' + L'') \\ &\approx \frac{(X_0' \otimes X_0'' \oplus X_0' \otimes F'' \oplus F' \otimes X_0'' \oplus F' \otimes F'')}{(X_0' \otimes X_0'' \oplus X_0' \otimes L'' \oplus L' \otimes X_0'' \oplus L' \otimes L'')} \\ &\approx X_0' \otimes R'' \oplus R' \otimes X_0'' \oplus (F' \otimes F''/L' \otimes L''). \end{aligned}$$

Hence the bottom row of the above diagram is a generic extension of S. The remainder follows exactly as in Theorem 7 of I.

THEOREM 10. *Let (S, ϕ) be an extension of R with kernel N. Suppose that A is a multiplicatively closed subset of R not containing* 0 *and that it does not have any zero divisors of R. Let $B=\phi^{-1}(A)$. Then ϕ induces a map $\phi_B\colon S_B \to R_A$ which is a singular extension and for any R_A module M we have $H(S, \phi, M) \approx H(S_B, \phi_B, M)$.*

PROOF. From the proof of Theorem 8 of I we have an exact sequence $0 \to N \otimes_R R_A \to S_B \to^{\phi_B} R_A \to 0$ which is a singular extension and defines ϕ_B. Now let $0 \to L \to G \to^{\theta} S \to 0$ be a generic singular extension of S. In exactly the same way we have $0 \to L \otimes_S S_B \to G_C \to S_B \to 0$ where $C=\theta^{-1}(B)$ is exact. (An examination of the proof of Theorem 8 of I will show that it is not required that B contain no zero divisor of S.) Now

$$L \otimes_S N \otimes_R R_A \to L \otimes_S S_B \to L \otimes_S R_A \to 0$$

is exact. But the image in S_B of $N \otimes_R R_A$ is just $N \cdot S_B$ and $L \otimes_S N' S_B = L \cdot N \otimes_S S_B = 0$ since L is an R-module. Thus $L \otimes_S S_B \approx L \otimes_S R_A$. But since L and R_A are R-modules, $L \otimes_S R_A \approx L \otimes_R R_A$. Hence, $0 \to L \otimes_R R_A \to G_C \to S_B \to 0$ is a generic extension of S_B. The rest follows exactly as in I, Theorem 8.

4. **Remarks.** There still remains to consider the relative theory. The problem is this: Given a subring $K' \subset K$ describe the cohomology group $H^n_{K'}(S, \phi, M)$ of all n-long sequences which are K'-split and with two being equivalent if there is an equivalence which is K'-split. (An arbitrary $f\colon A \to B$ is called split if each of the sequences $0 \to \ker f \to A \to \operatorname{coim} f \to 0$ and $0 \to \operatorname{im} f \to B \to \operatorname{coker} f \to 0$ is a split exact sequence.) At one extreme we could take $K'=K$ and get the K-relative theory. At the other extreme we could take K' as the subring generated by 1 to get the additively split theory.

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