MODELS OF HORN THEORIES

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Abstract. This paper explores the connection between categories of models of Horn theories and models of finite limit theories. The first is a proper subclass of the second. The question of which categories modelable by FL theories are also models of a Horn theory is related to the existence of enough projectives.

1. Introduction

Since both logical theories and sketches are used to present types of categories, the question naturally arises as to what classes of logical theories are equivalent to what kinds of sketches. In general, they give such different ways of organizing things that direct comparisons are not possible. It has been assumed, for example, that universal Horn theories classify the same kinds of theories as finite limit (also known as left exact) sketches. It turns out that this is not quite the case. In fact, all categories classified by universal Horn theories can also be classified by FL sketches, but not conversely. The basic difference is that a left exact theory allows one to define an operation whose domain is a limit, while a Horn theory allows one only to say that one limit is a subset of another.

Results similar to these have been obtained, using logical methods, by Volger [1987] and Makowsky [1985].

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2. Equational Horn theories

By an **equational Horn theory** we mean an equational theory augmented by a set of conditions of the form

$$[\phi_1(x) = \psi_1(x)] \wedge \cdots \wedge [\phi_n(x) = \psi_n(x)] \Rightarrow [\phi(x) = \psi(x)]$$

where ϕ, ψ, ϕ_i and ψ_i are operations in the theory and x stands for an element of a product of sorts to which they all apply. Note that since we are using the whole theory (that is the full clone), irrelevant arguments may be added to operations so that the operations in

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the above clause have the same arguments. Of course, if a sort is empty, any equational sentence stated in terms of that sort is satisfied.

By a **generalized equational Horn theory** we mean an equational theory augmented by a set of conditions of the form

$$[\bigwedge(\phi_i(x) = \psi_i(x)] \Rightarrow [\phi(x) = \psi(x)],$$

allowing a possibly infinite conjunction.

We begin with,

2.1. THEOREM. *The category of models of a (generalized) equational Horn theory is sketchable by a finite limit (resp. limit) sketch.*

PROOF. We give the proof in the finite case. It is well known that the models of a multi-sorted equational theory can be sketched by an FP sketch, that is one that uses only the predicates of finite products. For any equational Horn sentence

$$(\bigwedge(\phi_i = \psi_i)) \Rightarrow (\phi = \psi)$$

the operations ϕ_i, ψ_i, ϕ and ψ are all operations on some object in the theory. Next we observe that it is only conventional notation that puts a meet in the antecedent. In fact, that antecedent is the same as the single equation

$$\langle \phi_1, \phi_2, \ldots, \phi_n \rangle = \langle \psi_1, \psi_2, \ldots, \psi_n \rangle \circ$$

In a sketch, there are no distinguished generating sorts anyway. Thus a Horn sentence can be given by $(\phi = \psi) \Rightarrow (\phi' = \psi')$. Now in any model, the set of solutions to $\phi = \psi$ is simply the equalizer of two arrows. If we add those two arrows (along with their source and target) to the sketch, we can also add their equalizer. We can do the same for $\phi' = \psi'$. We can then add an operation from the one equalizer to the other along with commutative diagrams that ensure that this operation is merely an inclusion between two subobjects of the same object. ∎

The following theorem appears to be closely related to the main result of [Banaschewski & Herrlich, 1976], except that the condition on filtered colimits is replaced by closure under ultraproducts, a trivial modification. See also the paper of Shafaat [1969].

2.2. THEOREM. *Let $\mathscr{C}$ be an equational category and $\mathscr{D} \subseteq \mathscr{C}$ be a full subcategory. A necessary and sufficient condition that $\mathscr{D}$ be the category of models of a generalized equational Horn theory based on the operations of $\mathscr{C}$ is that $\mathscr{D}$ is closed under subobjects and products. If the theory of $\mathscr{C}$ is finitary, then $\mathscr{D}$ is the category of models for a Horn theory if and only if it is also closed under filtered colimits.*

Of very similar import is,

2.3. THEOREM. *Let $\mathscr{C}$ be the category of models of an FL sketch (or of an FL theory). Then $\mathscr{C}$ is the category of models of a generalized equational Horn theory if and only if $\mathscr{C}$ has a regular generating class of regular projectives. The Horn theory can be taken to be finitary if and only if this generating set can be taken to be finite.*

PROOF. It is left to the reader to show that if $\mathscr{A}$ is the category of algebra for a multi-sorted equational theory and $\mathscr{C}$ is the full subcategory defined by an equational Horn theory, then $\mathscr{A}$ has enough projectives and their reflections in $\mathscr{C}$ are a sufficient set of projectives there.

Let I be a sufficient set of projectives. There is a functor $U: \mathscr{C} \longrightarrow \mathbf{Set}^I$ that sends an object C of $\mathscr{C}$ to the I-indexed set $\mathrm{Hom}(P, C)$, $P \in I$. Since $\mathscr{C}$ is the category of models of an FL theory, it is complete and cocomplete, which implies that this functor has a left adjoint F, which takes the I-indexed set $\{S_P\}$ to the algebra $\sum_P S_P \cdot P$ where we let $S \cdot P$ denote the sum of S copies of P. The adjoint pair induces a triple on the category $\mathbf{Set}^I$ whose category of algebras we denote by $\mathscr{A}$. We have the standard structure/semantics comparison $\Phi: \mathscr{C} \longrightarrow \mathscr{A}$ with left adjoint Ψ. Since I includes a generating family, the embedding is faithful. Also it is a full, by a theorem of Beck's which asserts that Φ is full and faithful if and only if every object of $\mathscr{C}$ is a regular quotient of a free algebra. (See [Barr & Wells, 1985], Section 3.3, Theorem 9 for single-sorted theories; the general case offers no additional problem.)

Now we have $\mathscr{C}$ embedded as a full reflective subcategory of a multi-sorted equational category; moreover, $\mathscr{C}$ contains the free algebras. What we will do is show that given an algebra A which is not (isomorphic to an) algebra of the subcategory $\mathscr{C}$, then there is a generalized Horn clause that is not satisfied by A and is satisfied by every object of $\mathscr{C}$.

Consider a presentation of A,

$$F_1 \underset{d^1}{\overset{d^0}{\rightrightarrows}} F_0 \xrightarrow{d} A$$

with F_0 and F_1 free, say F_0 free on the I-indexed set X and F_1 free on the I-indexed set Y. The pair of arrows d^0 and d^1 induces one arrow $\langle d^0, d^1 \rangle: F_1 \longrightarrow F_0 \times F_0$. Each generator y of F_1 is taken by this map to a pair $\langle \phi_y(x), \psi_y(x) \rangle$ of elements of F_0. These elements are words in the free algebra generated by the elements $x \in X$. Let K be the kernel pair of the induced arrow $F_0 \longrightarrow \Phi\Psi(A)$. Then since A is not in the image of Φ, there is an element $\langle \phi(x), \psi(x) \rangle \in K$, which is not in the image of $\langle d^0, d^1 \rangle$. Thus A does not satisfy the generalized Horn clause

$$\bigwedge_y (\phi_y(x) = \psi_y(x)) \Rightarrow (\phi(x) = \psi(x) \circ \qquad (*)$$

I claim that this clause is satisfied by every object of $\mathscr{C}$. In fact, let C be an object of $\mathscr{C}$. Choose a set of elements $c_x \in C$ in such a way that c_x has the same sort as x and so that every one of the equations $\phi_y(c_x) = \psi_y(c_x)$ is satisfied. There is a homomorphism $f: F_0 \longrightarrow C$ that takes x to c_x by the property of free algebras. The fact that $\phi_y(c_x) = \psi_y(c_x)$

is equivalent to $(f \circ d^0)(y) = (f \circ d^1(y)$. Since this is true for all the generators y of F_1, it follows that $f \circ d^0 = f \circ d^1$ and so f factors through A. But the adjunction property of Ψ implies that f factors through $\Phi\Psi(A)$ and hence that $f(\phi(x) = f(\psi(x)$ and hence that $\phi(c_x) = \psi(c_x)$. Thus any tuple of elements that satisfies all $\phi_y(x) = \psi_y(x)$ also satisfies $\phi(x) = \psi(x)$, so that (*) is satisfied by every object of $\mathscr{C}$.

If the projectives are small, $\mathscr{C}$ is closed in $\mathscr{A}$ under filtered colimits. To see this, we first note that the projectives are small in $\mathscr{C}$ and so the functors they represent commute with filtered colimits. But then the triple induce also commute with filtered colimits and then the underlying functors they represent on $\mathscr{A}$ also commute with filtered colimits and hence the representing objects are still small. Now if $C = \operatorname{colim} C_i$ is a filtered colimit in $\mathscr{C}$ and $A = \operatorname{colim} \Phi C_i$ is the colimit in $\mathscr{A}$ then for any $P \in X$, we have

$$\begin{aligned}&\operatorname{Hom}(\Phi P, A) \cong \operatorname{Hom}(\Phi P, \operatorname{colim} \Phi C_i) \cong \operatorname{colim} \operatorname{Hom}(\Phi P, \Phi C_i) \cong \operatorname{colim} \operatorname{Hom}(P, C_i) \cong \\ &\quad \operatorname{Hom}(P, \operatorname{colim} C_i) \cong \operatorname{Hom}(P, C) \cong \operatorname{Hom}(\Phi P, \Phi C)\end{aligned}$$

whence $\Phi C \cong A$.

This implies that Φ commutes with filtered colimits, while Ψ, being a left adjoint, commutes with all colimits.

Repeat the proof using only finitely presented algebras A with the free algebras all free on finite sets. We will find a set of finite (since the elements y range over a finite set) Horn clauses that are satisfied by a finitely presented algebra if and only if the algebra is isomorphic to an algebra coming from $\mathscr{C}$.

Now let A be an arbitrary object of $\mathcal{A}$ that satisfies all the Horn sentences. Like any object of $\mathcal{A}$, A is a filtered colimit of finitely presented algebras. Write $A = \operatorname{colim} A_i$ with each A_i finitely presented. Exactly as in the proof of the first part, that fact that A satisfies all those Horn clauses implies that each $A_i \longrightarrow A$ factors $A_i \longrightarrow \Phi\Psi(A) \longrightarrow A$. Then we have a diagram for each i

$$\begin{tikzcd} A_i \arrow[r, two heads, "f_i"] \arrow[d] & \Phi\Psi A_i \arrow[dl] \arrow[d] \\ A \arrow[r, two heads, "f"'] & \Psi\Phi A \end{tikzcd}$$

where the horizontal arrows are epic. The upper triangle commutes and the lower one does too because the upper arrow is epic. Taking colimits, we get

$$\begin{tikzcd} \operatorname{colim} A_i \arrow[r, two heads, "f_i"] \arrow[d] & \operatorname{colim} \Phi\Psi A_i \arrow[dl] \arrow[d] \\ A \arrow[r, two heads, "f"'] & \Psi\Phi A \end{tikzcd}$$

From the fact that both vertical arrows are isomorphisms, it follows by an easy diagram chase that the diagonal is too. Thus A is a filtered colimit of objects of $\mathscr{C}$ and hence is one too. ∎

As a consequence, we can conclude that neither the category of posets nor of small categories can be the category of models of a generalized equational Horn theory. In fact, in each case it is easy to see that the only projectives are discrete (posets or categories). For if P is a projective, let a be an object of P that is the target of a non-identity arrow. Construct a new category Q by adding a new object b and a single arrow $a \longrightarrow b$, subject to no equations so that for any object c of P, the induced $\mathrm{Hom}(c, a) \longrightarrow \mathrm{Hom}(c, b)$ is an isomorphism, while $\mathrm{Hom}(b, c) = \emptyset$. If dir is the category with two objects and one arrow between and no other non-identity arrow, then Q is the regular quotient of $P + dir$ gotten by identifying a with the head of the non-identity arrow of dir. The functor $P \longrightarrow Q$ that takes the object a to b and everything else to itself does not lift to an arrow $P \longrightarrow P + dir$. On the other hand, any regular quotient of a discrete is discrete and so the regular projectives are not a regular generating family.

2.4. For the record, here is an example of a finitary equational theory and a full subcategory closed under products and subobjects not closed under filtered colimits and hence not the category of models of an ordinary Horn theory, at least not one based on the given operations.

The equational theory is the category of commutative rings with countably many constants $a_1, a_2, \ldots$ adjoined. Consider the generalized Horn theory

$$(\bigwedge a_i x = 0) \Rightarrow (x = 0)$$

Let R be the free ring on one variable, which may be thought of as the ring of polynomials $R = \mathbf{Z}[x, a_1, a_2, \ldots]$. In the filtered sequence of rings

$$R/(a_1 x) \longrightarrow R/(a_1 x, a_2 x) \longrightarrow \cdots$$

each of these rings is a model of the theory. It satisfies neither the antecedent nor the consequent of the sentence. The colimit, on the other hand, satisfies the antecedent, but continues not to satisfy the consequent and is thus not a model.

3. Relational Horn theories

We saw in the last section that the discrete posets (resp. categories) are a generating family (in the sense of representing a faithful family of functors), but not a regular generating family, for the category of posets, but not for categories. Since posets are the models of a Horn theory, this suggests it may be possible to characterize the categories of models of a Horn theory by the existence of a family of regular projectives that generate. This may be so, but until now I have been able to verify only one half of this conjecture (but it is the half that establishes that **Cat** is not the category of models of a Horn theory and demonstrates that FL theories have more expressive power).

In general, a Horn theory allows relations and predicates involving those relations. I am unaware of any intrinsic characterization (one that is independent of particular choices of sorts) of such categories analogous to those for equational categories. That is the main reason the results are only partial. C. Lair claims that the answer to every interesting question involving sketches in a series of articles appearing in the "publication" **Diagrammes**, so all the answers can no doubt be found there.

3.1. THEOREM. *The category of models of a universal Horn theory has a generating family of regular projectives.*

PROOF. Let $\mathscr{C}$ be the category of models of a universal Horn theory. Let $\mathscr{A}$ and $\mathscr{B}$ be the categories of models of the corresponding equational and relational theories, resp. That is, the algebras in $\mathscr{A}$ have all the operations and satisfy all the equations that of the theory and those of $\mathscr{B}$ have additionally been equipped with the relations. We note that we have $\mathscr{C} \subseteq \mathscr{B} \subseteq \mathscr{A}$, with the first inclusion being full. Now let $C_1 \twoheadrightarrow C_2$ be a regular epi among objects of $\mathscr{C}$. I claim that this is also a regular epi in $\mathscr{A}$. In fact, for each sort s of the theory, let $C_3(s)$ be the image of the arrow $C_1(s) \longrightarrow C_2(s)$. This image is, as is well known, a model of the underlying equational theory. It can be made into a model of the relational theory by letting, for a relation $\rho \subseteq s_1 \times \cdots \times s_n$,

$$C_3(\rho) = C_1(\rho) \cap [C_3(s_1) \times \cdots \times C_3(s_n)] \circ$$

Call such a subalgebra of an algebra in $\mathscr{B}$ *full*. Then it is easy to see that every full subalgebra of an algebra in $\mathscr{C}$ is still in $\mathscr{C}$. Moreover the arrow $C_1 \longrightarrow C_2$ factors through C_3 and since it was assumed a regular epi, it follows that $C_3 = C_2$.

Next we observe that the underlying functors for the sorts of the operations are representable (and in fact, have adjoints). Since these functors preserve regular epis, the representing objects are regular projectives.

Finally we note that an arrow is monic if and only if it is injective on all sorts. In other words that the inclusion of $\mathscr{C}$ into $\mathscr{A}$ reflects monos. It certainly preserves them, in fact has a left adjoint. But since arrows in $\mathscr{C}$ are functions of the sorts, a non-mono is also not injective on a sort. But then the kernel pair (which is preserved by the inclusion) provides two distinct arrows that are identified by the given arrow. Thus the evaluations at sorts, represented by the regular projectives, are collectively faithful and so the representing objects are a generating family. ∎

3.2. As observed above, the only projectives in the category of small categories are the discrete ones, which represent the set of objects functor (and its powers) and these are not faithful since two functors can agree on objects and not on arrows. A homomorphism on a poset is determined by its value on elements.

The converse could be verified if I could show that given a category of models of a coherent theory and homomorphisms that preserve positive sentences and given a faithful set of representable functors, there is a coherent theory *on that given set of functors* for which the given category is equivalent to the category of models. The point is that in model theory, the sorts are generally prescribed beforehand, whereas we are looking into an intrinsic characterization.

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