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TOPOSES WITHOUT POINTS

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0. Introduction

A topos – even a Grothendieck topos – need not have any points. There is an example due to P. Deligne in [10, IV.7.4]. An even simpler example can be constructed as follows. Let B be a complete boolean algebra, considered as a partially ordered set, considered as a category, considered as a site with the canonical topology (a cover of a $b \in B$ is a set of elements whose sup is b), and let $\mathcal{F}B$ be the category of sheaves. The set of subobjects of 1 in $\mathcal{F}B$ is just B. A point of $\mathcal{F}B$ (a cocontinuous, left exact functor to the category S of sets) takes 1 to 1, subobjects to subobjects and preserves sups. Hence it restricts to a complete boolean homomorphism $B \to 2$ whose kernel is a *principal* maximal ideal generated by the complement of an atom. If B has no atoms, there are no points. (Conversely, each atom of B does induce a point of $\mathcal{F}B$.)

Since a complete atomic boolean algebra is just (the algebra of subsets of) a set, a complete atomless boolean algebra may be thought of as a "set without points" and more generally an arbitrary complete boolean algebra as a generalized set.

You can even talk about generalized topological spaces in this context. A topology on the set X is an exact retract (inclusion has an exact right adjoint) of 2^X. Done more generally for a complete boolean algebra we arrive at the notion of complete Heyting algebra. (Conversely, it can be shown that every complete Heyting algebra arises in this way.) When X is a topological space and $\mathbf{Top}(X)$ is the category of sheaves on X, the collection of points of $\mathbf{Top}(X)$ gives rise to an exact cotripleable functor $\mathbf{Top}(X) \to \mathcal{F}B$, where B is the algebra of subsets of X. This is because $\mathcal{F}B$ is just S^X, the X-indexed families of sets.

These considerations have led F.W. Lawvere to make the following conjecture: "Every Grothendieck topos has an exact cotripleable functor to a topos $\mathcal{F}B$ for some complete boolean algebra B." It is this conjecture which we establish here.

Lawvere would add the following to the above observations:

"A Grothendieck topos is of the form $\mathcal{F}B$ for some complete boolean algebra B if and only if it satisfies the condition that every epimorphism splits (by [8] and unpublished work of Diaconescu). Such toposes are thus a very natural class of extensions of the category of sets and one is thus led to conjecture that any Grothendieck topos has enough boolean-valued points. There is an analogy here with algebraic varieties which may have no points defined over the ground field but always have enough points defined over extension fields of the ground fields. Of course, unlike fields, complete boolean algebras are closed with respect to infinite products so that a "sufficient" set X of points valued in complete boolean algebras B_x can be combined into one "faithful point" valued in a complete Boolean algebra B. Thus the conjecture takes the form [given above]. The result is closely related to a Boolean-valued completeness theorem for suitable infinitary theories."

In connection with the last remark, G. Reyes has recently informed me that a new proof of Lawvere's conjecture can be derived from such a completeness theorem.

This conjecture could have been arrived at from an entirely different direction. In the spatial case, Van Osdol [11] has used the induced exact triple in **Top**(X) to carry over all the standard homological algebra (as in, for example, [3]) to the category of sheaves. The existence of an analogous triple will permit these results to be carried over to an arbitrary Grothendieck topos.

There are two points to observe in this connection. The first is that, since the triple will preserve finite products, it will extend to any category of finitary algebras over the topos. If E is the topos and $\mathbf{T} = (T, \eta, \mu)$ is an exact (or even finite-product preserving) triple, then for any ring object R in E, TR is a ring and ηR is a ring homomorphism. If M is an R-module, then TM is a TR-module which becomes an R-module using ηR to transport the structure. When this is done, ηM becomes a homomorphism of R-modules.

The second point is that neither B nor the cotripleable functor is canonical and hence neither is the triple. But by the dual of [3, (5.2)] the homological properties of the triple will depend only on the injective class of objects which, in turn, depends only on the injective class of maps. Adapting the discussion at the bottom of p. 395 of [4] to the non-additive case, we see that the injective maps are those which are sent to split monos in the underlying category. But regardless of B, a morphism in $\mathcal{F}B$ is a split mono if and only if it is monomorphism between objects of the same support (exercise: use the fact that $\mathcal{F}B$ is boolean and that supports split). Since an exact functor preserves support, this means that for any $\mathbf{T}$ of the type considered, the injective class of maps will be support preserving monomorphisms, and hence that any such $\mathbf{T}$'s are homologically equivalent.

D. Van Osdol has been using the results of this paper to show that such things as Verdier's assumption [10, Exp. V] of enough points is not necessary to do homological algebra in a topos.

I would like to thank A. Joyal for many valuable discussions and thank B. Banaschewski and G. Grätzer for telling me how to embed complete Heyting algebras into complete boolean algebras.

After this paper was written, I received a reprint of [7] in which the idea of a complete Heyting algebra (there called a complete Brouwerian lattice) as a generalized topological space-without-points is developed at length. Included in that paper are several further references to the same idea.

1. Topologies in an elementary topos

1.1. Let E be an elementary topos. There are many natural topologies on E, but in this paper we will be considering primarily two. The first is the canonical topology in which the covers are the universally regular epimorphic families (see [2, p. 106] for the definition). The second, which for want of a better term I will call the strong topology, has as covers all epimorphic families $\{E_i \xrightarrow{e_i} E\}$. Of course such a sieve is an epimorphic family if, given $f, f': E \to E'$, $fe_i = f'e_i$ for all i implies $f = f'$. It follows from elementary properties of toposes that such families are stable under pullback and are extremal (i.e. factor through no proper subobject of their common domain).

1.2. The canonical topology is usually described as the strongest topology for which the representable functors are sheaves. The strong topology may be analogously defined as the strongest topology for which the representable functors are separated presheaves.

It is shown in [2, (A.23)] that in a Grothendieck topos the two topologies coincide. We will investigate below a condition which guarantees that they do coincide.

1.3. If C is a site, we let $\mathcal{P}(C)$, $\mathcal{S}(C)$ and $\mathcal{F}(C)$ denote the categories of presheaves, separated presheaves and sheaves, respectively. For an object $C \in C$ we will denote the corresponding representable functor of $\mathcal{P}(C)$ by $(-, C)$ and the sheaf associated to $(-, C)$ by $\epsilon(C)$. Provided that representable functors are separated (which will always be the case), the functor ϵ is faithful, being the composite

$$C \to \mathcal{S}(C) \to \mathcal{F}(C)$$

in which the first is full and faithful and the second is (as always) faithful.

1.4. Theorem. *Suppose C is a site for which every cover is an extremal epimorphic family. Suppose also that given any cover $\{C_i \to C\}$ which can be factored $\{C_i \to C' \to C\}$, then the sieve $\{C_i \to C'\}$ covers some (necessarily unique) subobject $C'' \rightarrowtail C'$. Then the strong topology is the canonical one.*

Remark. C'' is necessarily unique because the covers are extremal epimorphic families.

Proof. This amounts to showing that every representable functor is a sheaf in the strong topology. Or equivalently that given a cover $\{C_i \xrightarrow{f_i} C: i \in I\}$ and a family of maps $g_i : C_i \to C'$ such that $C_i \times_{C'} C_j \supset C_i \times_C C_j$ for all $i, j \in I$, then there is a map $h : C \to C'$ such that $hf_i = g_i$, $\forall i \in I$ (h is necessarily unique because we are supposing covers to be epimorphic families). This is not the way the sheaf condition is normally phrased, but it is easily seen to be equivalent, and is the form most convenient for the proof.

Now consider the family of maps $\{C_i \xrightarrow{(f_i, g_i)} C \times C'\}$, which when followed by the projection $C \times C' \to C$ is a cover of C. Hence by our condition this family covers a unique subobject $R \rightarrowtail C \times C'$. We will show that R is the graph of a function $h: C \to C'$ which can easily be shown to have the desired property. This is equivalent to showing that the composite $R \to C \times C' \to C$ is an isomorphism. It is certainly an extremal epimorphism since if it factored through any subobject of C, so would every f_i. To see that it is a monomorphism, observe that $\forall i, j \in I$,

$$C_i \times_R C_j = C_i \times_{C \times C'} C_j = C_i \times_C C_j \cap C_i \times_{C'} C_j = C_i \times_C C_j.$$

Thus for any sheaf F we have a commutative diagram

$$\begin{tikzcd}
FR \arrow[r] \arrow[d] & \prod FC_i \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"] & \prod F(C_i \times_R C_j) \arrow[d, "\cong"] \\
FC \arrow[r] & \prod FC_i \arrow[r, shift left] \arrow[r, shift right] & \prod F(C_i \times_C C_j)
\end{tikzcd}$$

with both lines equalizers, which implies that $FR \xrightarrow{\cong} FC$.

But both the Yoneda embedding and sheaf reflection preserve finite limits, so that if

$$R \times_C R \overset{d^0}{\underset{d^1}{\rightrightarrows}} R \to C$$

is a kernel pair diagram, it remains so when ϵ is applied. Thus $\epsilon d^0 = \epsilon d^1$. But as observed in 1.2, ϵ is faithful, hence $d^0 = d^1$, and $R \to C$ is a monomorphism. □

1.5. Corollary (Joyal). *If a category with pullbacks has the property that every map factors uniquely as an epimorphism followed by a monomorphism and the epimorphisms are stable under pullbacks, then they are regular.*

Proof. Simply take the topology in which covers are the epimorphisms. The uniqueness of the factorization guarantees that the epimorphisms are extremal. □

Remark. This corollary is rather easy to prove if you assume that coequalizers exist but becomes remarkably more delicate without that assumption.

1.6. Corollary. *If E is an elementary topos in which the subobject lattices of each object are complete, then the strong topology is the same as the canonical one.*

Proof. For then one can satisfy the hypothesis of 1.4 by taking C'' to be the union of the images of the C_i. □

Of course these lattices are always internally complete but as observed, e.g., in [5, (2.7)], the internal union just will not necessarily work.

2. Staunch functors

2.1. Definition. Let E and E' be elementary toposes. A functor $F : E \to E'$ is called a *logical morphism* if it preserves finite limits, finite colimits, Ω and exponentiation. (Actually, following the work of Christian Juul Mikkelsen, it is unnecessary to suppose that it preserve the finite colimits.)* A functor $F : E \to E'$ is called *strong* if it preserves epimorphic families. Finally, we shall call a functor $F : E \to E'$ *staunch* if it is faithful, strong and logical.

2.2. Proposition. *For every object E of an elementary topos E, the functor $E \times - : E \to E/E$ is a strong logical morphism. It is staunch if and only if in addition E has support* 1.

Having support 1 means that the (unique) map $E \to 1$ is an epimorphism.

Proof. The first part follows directly from [5, (2.31)]. Since $E \times -$ reflects isomorphisms if and only if E has support 1 [2, III (2.11)], the second part follows from:

2.3. Lemma. *Let X and Y be balanced categories. A functor $F : X \to Y$ which preserves finite limits and colimits is faithful if and only if it reflects isomorphisms.*

Balanced is a practically archaic word used to describe categories in which maps that are simultaneously mono and epi are isomorphisms.

Proof. Let F reflect isomorphisms and $f, g: X \to X'$ with $Ff = Fg$. This implies that the equalizer of f and g becomes an isomorphism, hence is one, hence $f = g$. To go the other way, let Ff be an isomorphism. This means that the two maps in the kernel pair of f are equal, hence f is mono. Similarly it is epi. □

2.4. Theorem. *Let A be a directed set, and let $D : A \to$* **Comp Top** *be a functor into the category of small toposes and staunch morphisms. Then for each $\alpha \in A$ the transition map $D\alpha \to \varinjlim E$ is a staunch functor.*

* This has been beautifully simplified in a forthcoming paper of R. Paré.

Remark. Something of this sort is implicit in the proof [5, (3.211)]. The only trouble is that the part of the proof asserting that T is strong is based on his offhand assumption that a colimit of strong functors is strong. We show by counter-example later that this assertion is false. (The correction, including a complete revision of his Section 5.6, is available from Freyd.)

Proof. Let us denote $D\alpha$ by $\boldsymbol{E}_\alpha$ and for $\alpha < \beta$ the value of $D(\alpha < \beta)$ by $F_{\beta\alpha} : \boldsymbol{E}_\alpha \to \boldsymbol{E}_\beta$. Let $\boldsymbol{E}$ be the direct limit and $F_\alpha : \boldsymbol{E}_\alpha \to \boldsymbol{E}$ the canonical map. The argument of [5, (3.211)], dependent on the "essentially algebraic" nature of toposes, can be readily repeated to show that $\boldsymbol{E}$ is a topos and that the F_α are logical morphisms. That they are faithful is completely trivial. The only point at issue is whether or not they are strong. This is based on:

2.5. Lemma. *Suppose $F : \boldsymbol{E} \to \boldsymbol{E}'$ is staunch and $f : E_0 \to E$ a map in $\boldsymbol{E}$ which does not factor through a given subobject $E_1 \to E$. Then Ff does not factor through $FE_0 \rightarrowtail FE$.*

Proof. The map f factors through E_0 if and only if $\operatorname{Im} f \subset E_1$, if and only if $\operatorname{Im} f \cap E_1 = \operatorname{Im} f$. A logical morphism preserves images, so that we want to show that $\operatorname{Im} f \cap E_1 \subsetneqq \operatorname{Im} f$ implies that

$$F(\operatorname{Im} f) \cap FE_1 \subsetneqq F(\operatorname{Im} f),$$

or, since F preserves finite intersections, that $E_2 \subsetneqq E_1$ implies $FE_2 \subsetneqq FE_1$. But since monos are regular, there is an equalizer diagram

$$E_2 \to E_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} E_1 +_{E_2} E_1$$

with $d_0 \neq d_1$ and, since F is faithful, $Fd_0 \neq Fd_1$, and hence $FE_2 \subsetneqq FE_1$.

To continue the proof of 2.3, let $\{E_i \to E : i \in I\}$, be an epimorphic family in $\boldsymbol{E}_\alpha$. A map in $\boldsymbol{E}$ whose codomain is $F_\alpha E$ is represented by a $\beta \geqslant \alpha$ and a map $f : E' \to F_{\beta\alpha}E$ in $\boldsymbol{E}_\beta$. If this is not a monomorphism in $\boldsymbol{E}_\beta$, then since F_β is faithful we can repeat the above argument with the cokernel pair of f to conclude that the map $F_\beta f$ which f is representing is not mono either. Thus a proper subobject of $F_\alpha E$ is represented by a subobject, evidently proper, $E' \to F_{\beta\alpha}E$ for some $\beta \geqslant \alpha$. Since $F_{\beta\alpha}$ is assumed to be staunch, $\{F_{\beta\alpha}E_i \to F_{\beta\alpha}E\}$ is a cover in $\boldsymbol{E}_\beta$, and hence there is some $i \in I$ such that $F_{\beta\alpha}E_i \to F_{\beta\alpha}E$ does not factor through E'. By (2.4), $F_{\gamma\alpha}E_i \to F_{\gamma\alpha}E$ does not factor through $F_{\gamma\beta}E' \rightarrowtail F_{\gamma\alpha}E$ for any $\gamma \geqslant \beta$. Hence $F_\alpha E_i \to F_\alpha E$ does not factor through $F_\beta E' \rightarrowtail F_\alpha E$. □

The following seems to be the best substitute for [5, (3.21)] available.

2.7. Theorem. *For every small boolean topos $\boldsymbol{B}$ there is a boolean topos $\hat{\boldsymbol{B}}$ in which supports split and a staunch functor $T : \boldsymbol{B} \to \hat{\boldsymbol{B}}$.*

Proof. Follow the "proof" offered by Freyd [5] except stick to those A whose support is 1. The result will be a $\hat{B}$ and a staunch $T : B \to \hat{B}$ in which 1 is projective. Being boolean, all its subobjects are summands and are hence projective as well. □

By dropping the hypothesis of being boolean we obtain the following theorem, originally due to André Joyal (unpublished, but with substantially the same proof).

2.8. Theorem. *For every small topos* $\boldsymbol{B}$ *there is a topos* $\boldsymbol{B}$ *in which* 1 *is projective together with a staunch* $T : \boldsymbol{B} \to \hat{\boldsymbol{B}}$.

2.9. Remarks. Note that 1 being projective is equivalent to all its complemented subobjects being projective. In the boolean case, of course, this means that all subobjects of 1 are projective, in which case they can easily be seen to also generate. (Given $E' \rightarrowtail E$ a proper subobject, split the map from $\neg E'$ to its support.) If (and only if) it is also complete, then such a category is $\mathcal{F}B$, the category of sheaves for the canonical topology on the complete boolean algebra B. Needless to say, $B = (1, \Omega)$ is the lattice of subobjects of 1.

Note also that if a topos is complete (and then, evidently, not small), a logical functor is strong if and only if it is cocontinuous. For let $E = \amalg E_i$. Then $\{E_i \to E\}$ is an epimorphic family, which implies the same for $\{TE_i \to TE\}$. Furthermore,

$$TE_i \times_{TE} TE_j = T(E_i \times_E E_j) = T(\delta_{ij} E_i) = \delta_{ij} TE_i,$$

so that in fact $TE = \amalg TE_i$. If, additionally, E is a Grothendieck topos, then the special adjoint functor implies that T has a right adjoint R which is obviously a geometric morphism. Lawvere calls a geometric morphism whose left adjoint is logical a local homeomorphism, presumably because, as is easily shown, the geometric morphism induced by a local homeomorphism between topological spaces does not have that property.

3. Induced functors on sheaf categories

A sieve $\{E_i \to E\}$ in a category is called an *extremal epimorphic family* if no proper subobject of E factors all the E_i. It is *universal* if it remains extremal when pulled back with any $E' \to E$. In an elementary topos every epimorphic family is extremal and universal, the first because all monos are regular and the second because pullbacks have right adjoints. In a partially ordered set an *extremal family* is a family $\{b_i \leqslant b\}$ whose sup is b. In a Heyting algebra every such family is universal, again since $\wedge$ has a right adjoint.

All the categories we consider in this section will have the property that extremal epimorphic families are universal, and we consider them as sites with those families as covers. We let, for such a category $\boldsymbol{E}$, $\mathcal{P}\boldsymbol{E}$, $\mathcal{S}\boldsymbol{E}$ and $\mathcal{F}\boldsymbol{E}$ denote the categories of set-valued presheaves on $\boldsymbol{E}$ (functors $\boldsymbol{E}^{\mathrm{op}} \to \boldsymbol{S}$), separated presheaves and sheaves, respectively.

A functor is called *conservative* if it reflects isomorphisms. A functor between toposes is conservative if and only if it is faithful. A functor between lattices is conservative if and only if it is mono.

3.1. Theorem. *Let $F : \boldsymbol{E} \to \boldsymbol{D}$ be continuous and exact. Then F extends to an exact functor $F^s : \mathcal{S}\boldsymbol{E} \to \mathcal{S}\boldsymbol{D}$. F^s has a right adjoint $F_s : \mathcal{S}\boldsymbol{D} \to \mathcal{S}\boldsymbol{E}$. If F is conservative and $\boldsymbol{C}$ has complete subobject lattices, then F^s is also conservative.*

Proof. Everything except the last sentence follows from [10, Exp. I, 5.4(4)], together with [10, Exp. III, 1.2, 1.3, 1.6]. To show the last line, it will be necessary to analyse the proofs of these propositions more carefully. The functor F induces a functor $\mathcal{P}F : \mathcal{P}\boldsymbol{D} \to \mathcal{P}\boldsymbol{E}$ which has a left adjoint $F_! : \mathcal{P}\boldsymbol{E} \to \mathcal{P}\boldsymbol{D}$ (as well as a right adjoint $F^!$). When $\boldsymbol{E}$ has and F preserves finite limits, so does $F_!$. The continuity of F (in the topology) implies (rather easily) that $\mathcal{P}F$ takes sheaves to sheaves, so that we have a commutative diagram

$$\begin{tikzcd}
\mathcal{P}\boldsymbol{D} \arrow[r, "\mathcal{P}F"] & \mathcal{P}\boldsymbol{E} \\
\mathcal{S}\boldsymbol{D} \arrow[u, "I'"] \arrow[r, "F_s"'] & \mathcal{S}\boldsymbol{E} \arrow[u, "I"']
\end{tikzcd}$$

from which it follows that $F^s = \hat{I}'F_! I$ is left adjoint to F_s [1, Theorem 3, dual]. Here, of course, $\hat{I}'$ is the associated sheaf functor, well known to be exact. Since all three $\hat{I}'$, $F_!$ and I are exact, so is F^s. The rub is that $\hat{I}'$ is not faithful, so we cannot use the same argument to show that F^s is.

3.2. Proposition. *Under the hypotheses of Theorem 3.1, $F_!|\partial\boldsymbol{E}$ is faithful.*

Proof. It is clearly sufficient to show that if $R : \boldsymbol{E}^{\mathrm{op}} \to \boldsymbol{S}$ is a separated presheaf and $\widetilde{R} : \boldsymbol{D}^{\mathrm{op}} \to \boldsymbol{S}$ is its left Kan extension, then the natural map $R \to \widetilde{R}F$ is a monomorphism. If $D \in \boldsymbol{D}$, then an element of $\widetilde{R}D$ is a pair (f, x), where $f : D \to FE$ and $x \in FE$. If $f_1 : D \to FE_1$ and $f_2 : D \to FE_2$ are maps in $\boldsymbol{E}$, then $(f_1, x_1) \sim (f_2, x_2)$, provided there is an object $E \in \boldsymbol{E}$ and maps $g_1 : E \to E_1$, $g_2 : E \to E_2$ and $h : D \to FE$ such that

$$\begin{tikzcd}
 & D \arrow[dl, "f_1"'] \arrow[d, "h"] \arrow[dr, "f_2"] & \\
FE_1 & FE \arrow[l, "Fg_1"] \arrow[r, "Fg_2"'] & FE_2
\end{tikzcd}$$

commutes and such that $x_1|E = x_2|E$. Note that we will often write things like $x_1|E$ instead of the cumbersome $Rg_1(x_1)$ when the functor and morphism are clear.

The fact that E has and F preserves finite limits implies that this defines an equivalence relation. The contravariant functor defined by $\widetilde{R}$ takes a pair (x, f) and a map $g : D' \to D$ to (x, fg). The map $R \to \widetilde{R}F$ takes an element $x \in RE$ to (FE, x). In order that $(FE, x_1) \sim (FE, x_2)$, we need a commutative diagram

$$\begin{tikzcd}
 & FE \arrow[dl, "FE"'] \arrow[d, "h"] \arrow[dr, "FE"] & \\
FE & FE' \arrow[l, "Fg_1"] \arrow[r, "Fg_2"'] & FE
\end{tikzcd}$$

such that $Rg_1(x_1) = Rg_2(x_2)$. Since $Fg_1 \cdot h = Fg_2 \cdot h$, h factors through the equalizer of Fg_1 and Fg_2. Since F preserves equalizers. h factors through Fg', where

$$\begin{tikzcd}
E'' \arrow[r, "g'"] & E' \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & E
\end{tikzcd}$$

is an equalizer. This gives, with $g = g_1g' = g_2g'$, the diagram

$$\begin{tikzcd}
 & FE \arrow[dl, "FE"'] \arrow[d, "R"] \arrow[dr, "FE"] & \\
FE & FE'' \arrow[l, "Fg"] \arrow[r, "Fg"'] & FE
\end{tikzcd}$$

with $x_1|E'' = x_2|E''$. But Fg is a split, hence extremal epi, and since F is conservative, this implies that g is an extremal epi as well and, since R is a separated presheaf that Rg is mono. But $Rg(x_1) = Rg(x_2)$, and hence $x_1 = x_2$. □

3.3. Proposition. *Under the hypotheses of the theorem, $F_!(\mathcal{S}E) \subset \mathcal{S}D$.*

Proof. In other words, we are claiming that if $R \in \mathcal{S}E$, its left Kan extension $\widetilde{R} \in \mathcal{S}D$. If $(f_1, x_1), (f_2, x_2)$ represent two elements of $\widetilde{R}D$, we can represent them both with the same map, using

$$\begin{tikzcd}
 & D \arrow[dl, "f_1"'] \arrow[d, "{f_1, f_2}"] \arrow[dr, "f_2"] & \\
FE_1 & FE_1 \times FE_2 \arrow[l] \arrow[r] & FE_2
\end{tikzcd}$$

Thus we can suppose we are given $(f, x_1), (f, x_2) \in \widetilde{R}X$ and an extremal epimorphic family $\{D_i \xrightarrow{g} D : i \in I\}$ such that for all $i \in I$,

$$(f, x_1)|D_i = (f, x_2)|D_i.$$

The definition of equality means that we have a commutative diagram

$$\begin{tikzcd} & D_i \arrow[dl, "fg_i"'] \arrow[d, "f_i"] \arrow[dr, "fg_i"] & \\ FE & FE_i \arrow[l, "Fh_{i1}"] \arrow[r, "Fh_{i2}"'] & FE \end{tikzcd}$$

such that $Rh_{i1}(x_1) = Rh_{i2}(x_2)$. By replacing E_i by the equalizer of h_{i1} and h_{i2}, we may, in fact, suppose that $h_{i1} = h_{i2} = h_i$ and that we have a commutative diagram

$$\begin{tikzcd} D_i \arrow[r, tail, "g_i"] \arrow[d, "f_i"'] & D \arrow[d, "f"] \\ FE_i \arrow[r, tail, "Fh_i"] & FE \end{tikzcd}$$

such that $x_1|E_i = x_2|E_i$. Now let $E' \rightarrowtail E$ be the union of the images of the E_i. Clearly we have an extremal epimorphic family $\{E_i \xrightarrow{k_i} E'\}$ which gets sent to an extremal epimorphic family $\{FE_i \xrightarrow{Fk_i} FE'\}$ since F is continuous. Next I claim that $D \to FE$ factors through FE'. To see this, consider the pullback diagram

$$\begin{tikzcd} FE' \times_{FE} D \arrow[r, tail] \arrow[d] & D \arrow[d] \\ FE' \arrow[r, tail] & FE \end{tikzcd}$$

One easily sees that every $D_i \to D$ factors through $FE' \times_E D \to D$, and then extremality implies that the top arrow is an isomorphism. This gives the desired map $D \to FE'$. Since $x_1|E_i = x_2|E_i$ for all i, $\{E_i \to E'\}$ is a cover, and R is separated, $x_1|E' = x_2|E'$, which then implies that (f, x_1) and (f, x_2) represent the same element of $\tilde{R}D$. □

Now to return to the proof of Theorem 3.1. What we have shown is that there is a commutative diagram

$$\begin{tikzcd} \mathcal{P}E & \mathcal{P}D \arrow[l, "F_!"'] \\ \mathcal{S}E \arrow[u] & \mathcal{S}D \arrow[l, "F_\#"'] \arrow[u] \end{tikzcd}$$

and that $F_\#$ is faithful. This means that we could have as well defined F^s as the composite

$$\begin{tikzcd} \mathcal{S}E \arrow[d] & \mathcal{S}D \arrow[l, "F_\#"'] \\ \mathcal{F}E & \mathcal{F}D \arrow[u] \end{tikzcd}$$

each of which is faithful, and hence so is F^S. Notice that the associated sheaf functor is faithful when restricted to separated presheaves, since such a presheaf is contained in its associated sheaf. F^S is then a faithful functor between toposes and, as we noted above, this implies that it is conservative. □

Note that F_S is a geometric morphism. A geometric morphism whose left adjoint is faithful is described by Lawvere as "surjective". It is an easy consequence of Beck's tripleableness criteria (actually the co-CTT, see [9, p. 151, excercises 6, 7]) that F^S is cotripleable if and only if it is conservative if and only if it is faithful. Thus we have:

3.4. Corollary. *Let $F : \boldsymbol{E} \to \boldsymbol{D}$ be a strong exact faithful functor, and suppose that the subobject lattices of objects of $\boldsymbol{E}$ are complete. Then $F^S : \mathcal{F}\boldsymbol{E} \to \mathcal{F}\boldsymbol{D}$ is cotripleable.*

4. The main theorem

4.1. Theorem. *Let $\boldsymbol{E}$ be a Grothendieck topos. Then there is a complete boolean algebra B and an exact cotripleable functor $\boldsymbol{E} \to \mathcal{F}B$.*

The proof will be divided into two parts. First we shall find such a functor $\boldsymbol{E} \to \mathcal{F}H$ for a complete Heyting algebra H and then use a known theorem of lattice theory which provides an embedding $H \to B$ with suitable properties.

The first question to settle is when a Grothendieck topos is of the form $\mathcal{F}H$. The answer seems to be widely known, but I have not been able to find a direct reference in print.

4.2. Proposition. *A Grothendieck topos $\boldsymbol{E}$ is equivalent to an $\mathcal{F}H$ if and only if the subobjects of 1 are a set of generators of $\boldsymbol{E}$ in which case H is the subobject lattice of 1. If $\boldsymbol{C}$ is a site (with finite limits) in which every object has a cover by subobjects of 1, then the subobjects of 1 are a set of generators in $\mathcal{F}\boldsymbol{C}$.*

Proof. It is shown in [2, (A.22)] that if $\boldsymbol{E}$ is a Grothendieck topos and $\boldsymbol{C}$ is any subcategory of $\boldsymbol{E}$ containing a set of generators for $\boldsymbol{E}$ and closed under finite products and subobjects, then $\boldsymbol{E} \cong \mathcal{F}\boldsymbol{C}$ for the canonical topology on $\boldsymbol{C}$. Note that by being closed under subobjects, $\boldsymbol{C}$ will have complete subobject lattices and the strong topology is the same as the canonical one. In particular, if the Heyting algebra H of subobjects of 1 contains a set of generators, it certainly satisfies the other conditions, so that $\boldsymbol{E} \cong \mathcal{F}H$. The converse is a special case of the second sentence of the proposition, to whose proof we now turn. Let $F \rightarrowtail G$ be sheaves such that $FS \cong GS$ for all $S \rightarrowtail 1$. Let $C \in \boldsymbol{C}$, and choose a cover $\{S_i \to C\}$, where the $S_i \rightarrowtail 1$. Then consider the commutative diagram

$$\begin{tikzcd} FC \arrow[r] \arrow[d, tail] & \Pi FS_i \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"] & \Pi F(S_i \times_C S_j) \arrow[d, tail, "\cong"] \\ GC \arrow[r] & \Pi GS_i \arrow[r, shift left] \arrow[r, shift right] & \Pi G(S_i \times_C S_j) \end{tikzcd}$$

whose rows are equalizers. An easy diagram chase shows that $FC \cong GC$. □

We leave to the reader the statement of the more general proposition contained in the above proof.

Now let E be an arbitrary Grothendieck topos. Let G be an object of E (the sum of a set of generators will do). Let C be a small subcategory of E which is closed under finite products and sums, subobjects and quotient objects, and exponentiation, and which contains both G and Ω. Such a subcategory certainly exists, for you can begin with G and Ω, repeatedly close up under all those operations, and take a countable union. Clearly C is a small topos whose objects have complete subobject lattices. This implies, in particular, that the strong topology on C is the canonical one. Observe also that the object G must have support 1, otherwise its subobjects could not generate 1. Hence by 2.2 the functor $G \times - : C \to C/G$ is staunch, and similarly for

$$G^{n+1} \times_{G^n} - = G \times - : C/G^n \to C/G^{n+1}.$$

Here $G^{n+1} \to G^n$ is projection on the last n coordinates. Hence if we let D be the direct limit of the sequence

$$C \to C/G \to \ldots \to C/G^n \to \ldots,$$

the induced map $C \to D$ is competent by 2.4. From 3.4, we see that the induced functor $\mathcal{F}C \to \mathcal{F}D$ is cotripleable. From [2, (A.22)], we infer that $E \cong \mathcal{F}C$.

4.3. Proposition. *The subobjects of* 1 *generate* D.

Proof. An object of D is a map $C \to G^n$ for some n, subject to the identification of $C \to G^n$ with $G \times C \to G^{n+1}$. A map $C \to G^n$ to $C' \to G^m$ consists of a 3-tuple (i, j, f), where i and j are integers such that $i + n = j + m$ and $f : G^i \times C \to G^j \times C'$ over G^{i+n}, subject to the identification of (i, j, f) with $(i + 1, j + 1, G \times f)$. The map (i, j, f) represents a monomorphism in D if and only if f is a monomorphism in C (this depends on the fact that the transition functors are faithful; see the proof of 2.4 for details). In particular, a subobject of $f : C \to G^n$ is a subobject of $G^i \times C$ thought of as the object $G^i \times C \to G^{i+n}$ of E/G^{i+n}. Let $C' \rightarrowtail G^i \times C$. Since the subobjects of G generate C, there is a $u : S \rightarrowtail G$ and a map $g : S \to G^i \times C$ which does not factor through C'. In C/G^{1+i+n} we have the diagram

$$\begin{tikzcd} & S \arrow[d, "{(u,g)}"] \\ G \times C' \arrow[r, tail] & G^{i+1} \times C \arrow[d, "G\times f"] \\ & G^{1+i+n} \end{tikzcd}$$

and it follows from 2.5 that (u, g) does not factor through $G \times C'$ in $\boldsymbol{D}$. The structure map of S as an object of C/G^{1+i+n} is

$$(G \times f) \cdot (u, g) = (u, fg),$$

which is a monomorphism since its first coordinate is, and hence, as noted above, is a monomorphism in $\boldsymbol{D}$. Thus $(u, fg) : S \rightarrowtail G^{1+i+m}$ is a subobject of the terminal object of $\boldsymbol{D}$, which has a map to $G^{i+1} \times C$ that does not factor through the subobject $G \times C'$. □

4.4. Corollary. *Every object of* $\boldsymbol{D}$ *has a cover by subobjects of* 1. *Hence* $\mathcal{F}\boldsymbol{D} \cong \mathcal{F}H$ *for a complete Heyting algebra* H.

Proof. What we have shown is that for all $D \in \boldsymbol{D}$, and proper subobject $D' \rightarrowtail D$, there is an $S \rightarrowtail 1$ and a map $S \to D$ which does not factor through D'. Hence the family $\{S_i \to D\}$ of all maps to D whose domain is a subobject of 1 is an extremal epimorphic family. The last sentence follows from 4.2. □

4.5. Corollary. *Every Grothendieck topos* $\boldsymbol{E}$ *has an exact cotripleable functor* $\boldsymbol{E} \to \mathcal{F}H$ *for some complete Heyting algebra* H.

I am indebted to B. Banaschewski and G. Grätzer for pointing out the following.

4.6. Proposition. *Let* H *be a complete Heyting algebra. Then there is a boolean algebra* B *and an embedding* $H \to B$ *which preserves finite infs and arbitrary sups.*

Proof. For details refer to Grätzer [6, Section 10]. In that section he shows how every distributive lattice L has a standard embedding into a boolean algebra $B(L)$. He then states (his Lemma 13) that if the complete distributive lattice L has both infinite distributivities – of finite infs over all sups and of finite sups over all infs – then this embedding preserves all sups as well as all infs. But in fact the two parts are separate. He actually proves that if finite infs distribute over all sups, then this standard embedding preserves all sups, and this is exactly the result we need. □

4.7. Corollary. *For every complete Heyting algebra H, there is a complete boolean algebra B and an exact cotripleable functor $\mathcal{F}H \to \mathcal{F}B$.*

Proof. Use the embedding above followed by the embedding of $B(L)$ into its completion. This gives a faithful functor $H \to B$ between complete categories which preserves finite limits and epimorphic families, and hence, by 3.4, an exact cotripleable functor $\mathcal{F}H \to \mathcal{F}B$. Note than an exact tripleable functor is co-CTT (see [9]).

5. Some counter-examples

In this section we give a few assorted counter-examples.

5.1. *Example to show that 2.4 may fail if "faithful" is omitted (from both hypothesis and conclusion); that is, if "staunch" is replaced by strong and logical.* Referring to the notation used in the proof of 2.4, let $A = I$ be the set of positive integers and $\boldsymbol{E}_i$ be the category of sheaves on the open interval $(0, 1/i)$ (in the usual topology on $\mathbb{R}$). For $i < j$, the functor $\boldsymbol{E}_i \to \boldsymbol{E}_j$ is just restriction to that subinterval which of course is an instance of [5, (2.31)] and therefore strong and logical. Let $E_i \in \boldsymbol{E}_i$ be the functor represented by the open interval $(1/(i+2), 1/i)$. Then it is clear that $\{E_i \to 1\}$ is an epimorphic family (1 is the functor represented by the interval $(0, 1)$) while the inclusion $0 \to E_i$ becomes an isomorphism in $\boldsymbol{E}_{i+2}$, so that $F_1 E_i = 0$. On the other hand, $0 \to 1$ never becomes an isomorphism at any stage and hence does not in the limit. One way of looking at this is to observe that the two maps $1 \rightrightarrows 1 + 1$ remain distinct at every step and hence in the limit. Thus $\{0 \to 1\}$ is not a cover and the epimorphic family is not preserved.

Actually the existence of an example of this kind is implicit in the next example, but we included it anyway because it is so simple and direct.

5.2. *Example to show that* [5, *Theorem* 3.21] *is wrong.* (As mentioned above, a correction to this effect is available from Freyd, but it is hard to figure out from that an explicit counter-example.) Let B be a complete atomless boolean algebra. Let $C \subset \mathcal{F}B$ be a subcategory closed under finite products and sums, equalizers and coequalizers, Ω and exponentiation, just as in the proof of 4.1. Then it is evident that C is a topos and that $\mathcal{F}C \cong \mathcal{F}B$. If there were a topos $\hat{C}$, generated by 1 and a strong logical $T : C \to \hat{C}$, we would get functors $T^S : \mathcal{F}C \to \mathcal{F}\hat{C}$ and its right adjoint $T_S : \mathcal{F}\hat{C} \to \mathcal{F}C$ just as in the first part of the proof of 3.1, where faithfulness is not used. From 4.2 it follows readily that $\mathcal{F}\hat{C}$ is a Grothendieck topos in which 1 is a generator. The object 1 can have no proper subobjects because it could not generate them. More precisely, if $S \overset{\neq}{\rightarrowtail} 1$, then no map $1 \to S$ exists to distinguish $0 \rightarrowtail S$. Hence by 4.2, $\mathcal{F}\hat{C} \cong \mathcal{F}H$, where H is the Heyting algebra $\{0, 1\}$; i.e. $\mathcal{F}H \cong S$. But then T^S is a point of $\mathcal{F}B$, which is a contradiction.

5.3. *An example of a topos in which the strong topology is stronger than the canonical topology.* Let E_0 be the category of finite sets, and let $E_i = E_0/2^i$. Then we have a sequence of toposes and staunch functors

$$E_0 \to E_1 \to E_3 \to \dots \to E_n \to \dots,$$

in each case the functor describable as $2 \times \cdot$. Let E be the direct limit and $F_i : E_i \to E$ be the canonical functor. The terminal object of E_i is 2^i, which can be thought of as i-tuples of 0's and 1's. Let $S_i \rightarrowtail 2^i$ be the set of all i-tuples except $(0, \dots, 0)$. Then I claim first that $\{F_i S_i \to 1\}$ is an epimorphic family and second that it is not regular.

To see the first claim, let $S \rightarrowtail 1$ be a proper subobject of 1 in E. Then S is represented in some E_n, say by the subobject $A \rightarrowtail 2^n$, which is the same as $2 \times A \rightarrowtail 2^{1+n}$. Or, more precisely,

$$F_n(A) = F_{n+1}(2 \times A) = S.$$

Now $\#(A) \leqslant 2^n - 1$, since A is proper, so that $\#(2 \times A) \leqslant 2^{n+1} - 2$, which means that $S_{n+1} \not\subset 2 \times A$, and hence that

$$F_{n+1} S_{n+1} \not\subset F_{n+1}(2 \times A) = S.$$

This shows that the family is extremal epimorphic. To see that it is not regular, define a sequence of maps $f_i : S_i \to 2^i \times 2$, which represent maps $F_i S_i \to F_0 2$ in E. $S_0 = \emptyset$, so we need not worry about it. $f_1 : S_1 \to 2^2$ is defined by $f_1(1) = (1, 1)$, which is a map over 2. (The way things are set up the map $2^2 \to 2$ is the first coordinate projection.) Having defined $f_{i-1}: S_{i-1} \to 2^i$, we define $f_i : S_i \to 2^{i+1}$ by

$$f_i(a_1, \dots, a_i) = (a_1, f_{i-1}(a_2, \dots, a_i))$$

unless $(a_2, \dots, a_i) = (0, \dots, 0) \notin S_i$, and

$$f_i(1, 0, \dots, 0) = (1, 0, \dots, 0, b_i),$$

where b_i is 0 or 1 according as i is even or odd. From their construction the f_i are a coherent family of maps. Suppose there were some map $g : 1 \to 2$ whose restriction to $F_i S_i$ was $F_i f_i$. Say that g is represented by some map g: $2^n \to 2^{n+1}$ over 2^n, and suppose, for the sake of argument, that n is even. Then let

$$g(0, \dots, 0) = (0, \dots, 0, b),$$

from which

$$f_{n+1}(1, 0, \dots, 0) = (1, 0, \dots, 0, b) = (1, 0, \dots, 0, 1).$$

Hence $b = 1$. But also

$$f_{n+2}(1, 0, \dots, 0) = (1, 0, \dots, 0, b) = (1, 0, \dots, 0),$$

so that $b = 0$. Thus b cannot be chosen, and no such map g exists.

References

[1] M. Barr, The point of the empty set, Cahiers Topologie Géom. Différentielle 13 (1973) 357–368.
[2] M. Barr, Exact categories, Lecture Notes in Mathematics 236 (Springer, Berlin, 1971), 1–120.
[3] M. Barr and J. Beck, Homology and standard constructions, Lecture Notes in Mathematics 80 (1969) 245–335.
[4] S. Eilenberg and J.C. Moore, Adjoint functors and triples, Ill. J. Math. 9 (1965) 381–398.
[5] P. Freyd, Aspects of topoi, Bull. Austral. Math. Soc. 7 (1972) 1–76.
[6] G. Grätzer, Lattice Theory (Freeman, San Franciso, Calif., 1971).
[7] J.R. Isbell, Atomless parts of spaces, Math. Scand. 31 (1972) 5–32.
[8] F.W. Lawvere, Quantifiers and sheaves, in: Actes Congr. Intern. des Mathematiciens, 1970, tome 1, pp. 329–334.
[9] S, MacLane, Categories for the Working Mathematician (Springer, Berlin, 1971).
[10] SGA4, Théorie des Topos et Cohomologie Etale des Schémas, Lecture Notes in Mathematics 269 (Exposés I-IV), 270 (Exposés V-VIII) (Springer, Berlin 1972).
[11] D.H. Van Osdol, Remarks concerning Ext*(M, -), Bull. Am. Math. Soc. 76 (1970) 612–617.