Hamburg TFT

Note Title — 1/31/2009

Goal: explain some relations between gauge theory & representation theory.

Sketch of n-dimensional TFT:

cpt oriented
M n-manifold $\longmapsto Z(M) \in \mathbb{C}$: "$\int_{\mathcal{F}(M)} e^{-S(\varphi)} d\varphi$"

multiplicative: $\amalg \longmapsto \cdot$

$\phi \longmapsto 1$

invariant under diffeomorphisms

N n-1 manifold $\longmapsto Z(N) \in \mathrm{Vect}$:

Hilbert space of the theory on $N \times \mathbb{R}_{\text{time}}$

Rough idea: $Z(N)$ = functionals of some kind on fields on N : prescribe boundary values define path integral

M N φ_0

$$Z(M)(\varphi_0) = \int_{\varphi|_N = \varphi_0} e^{-S(\varphi)} d\varphi$$

N_1 M N_2

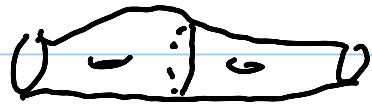
$$\mathcal{F}(N_1) \longleftarrow \mathcal{F}(M) \longrightarrow \mathcal{F}(N_2)$$

path integral defines linear operator $Z(N_1) \to Z(N_2)$

Properties: multiplicative

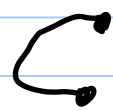
$\sqcup \longmapsto \otimes \qquad op \longmapsto * \qquad \emptyset \mapsto \mathbb{C}$

Locality of field theory $\Rightarrow$ sewing law



$Z(M_2) \circ Z(M_1) = Z(M_1 \sqcup_{N_2} M_2)$

Extend further: express locality on N_y

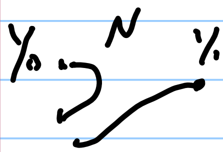


$Z(N)(\varphi_0) =$ functionals on fields $\varphi|_Y = \varphi_0$

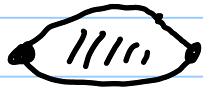
assign vector space to field on Y

$\rightsquigarrow Z(Y) = \left\{ \begin{array}{c} \text{vector bundles or sheaves} \\ \text{on } \mathcal{F}(Y) \end{array} \right\}$

– a category [topological D-branes]



functors $\mathcal{F}(Y_0) \rightarrow \mathcal{F}(Y_1)$



cobordisms between cobordisms

idea: level of complexity goes up but geometry gets much easier as we cut further & further.

Hopkins-Lurie: complete structure theory for fully extended TFTs – "freely determined by $Z(pt)$"

Example 2d gauge theory, G finite

Space of fields $\mathcal{F}(M) = \mathcal{M}_G(M)$:
G-bundles = G-coverings =
Reps $\pi_1(M) \longrightarrow G$.

$\mathcal{M}_G(\cdot) = \cdot / G$ trivial G-bundle, with automorphism group G

$\mathcal{M}_G(S^1) = \frac{G}{G}$ monodromy $\in G$/conjugation

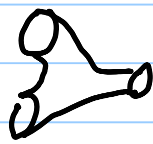
$\mathcal{M}_G(\Sigma_g) = \left\{ \begin{matrix} A_1 \dots A_g \\ B_1, \dots B_g \end{matrix} \in G \quad \prod [A_i, B_i] = 1 \right\} / G$

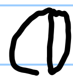
$Z_G(\Sigma_g) = \# \mathcal{M}_g(\Sigma_g)$ number (w/ multiplicity)
$= \#$ solutions above $/ |G|$

$Z_G(S^1) = \mathrm{Fun}\left(\frac{G}{G}\right) =$ class functions on G

$Z_G(\cdot) = \mathrm{Vect}_G\left(\frac{\cdot}{G}\right) = \mathrm{Rep}_{\mathbb{C}} G$
$= (\mathrm{Fun}(G), *)$-mod modules for group algebra

Some structures:

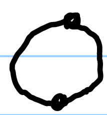
 :  $\mathbb{C}[\frac{G}{G}]$ has a (commutative, associative) multiplication (group convolution)

 - unit (δ_1)

 → trace (nondegenerate) $\Big\}$ commutative Frobenius algebra

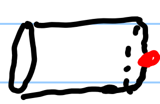
$f \longmapsto f(1)$

– this structure is equivalent to all 1 & 2 dim TFT operations – eg get Frobenius-Schur mass formula for $\# \mathcal{M}_g(\Sigma)$.

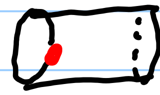
 decomposition of circle as
Vect ⟶ Rep G ⊗ Rep G ⟶ Vect

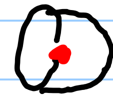
⇒ identify $Z(S^1)$ as <u>center</u> of $Z(\bullet)$ (a.k.a. Hochschild cohomology)

... endomorphisms of identity – in our case $Z(\cdot) = \mathbb{C}G$-mod, $\mathbb{C}\frac{G}{G} = \text{Center}(\mathbb{C}G)$

⇒  M maps $Z(S^1) \longrightarrow \text{End}\, M$ for any representation M.

Dually $Z(S^1)$ is also the <u>trace</u> of $Z(\cdot)$ (aka Hochschild homology), ie have a universal trace $\mathbb{C}G \longrightarrow \mathbb{C}\frac{G}{G}$

- trace map  $\mathrm{End}\, M \longrightarrow Z(S^1)$

factoring the trace  $\mathrm{End}\, M \longrightarrow \mathbb{C}$

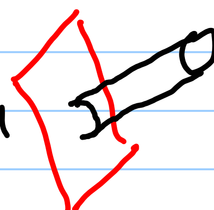
... this is the <u>character</u> of a representation

$\mathrm{Id}_M \longmapsto \chi_M \in \mathbb{C}\frac{G}{G}$.

[Digression: these structures make $Z(\cdot)$ into a Calabi-Yau category ...

B-model: $Z(\cdot)$ = B-branes = $D_{coh}(X)$

X Calabi Yau.

M  open-closed string transitions

character $\rightsquigarrow$ Chern character of a bundle.]

Three dimensional gauge theory & representations of Lie groups

Now G complex semisimple/reductive Lie group.

Describe some pieces of a would-be extended TFT in 3 dim, partially defined

- comes from maximally supersymmetric gauge theory ($\mathcal{N}=8$ SYM)

Rough idea: 3d TFT $\mathcal{X}_G$

<u>3-manifolds</u> Assign Euler characteristic or some Poincaré polynomial of moduli of monopole equations ..... least understood part

<u>2-manifolds</u> $\mathcal{M}_G(\Sigma)$ = moduli of flat G-bundles on Σ = $\{\text{Reps } \pi_1(\Sigma) \longrightarrow G\}/\sim$

- assign the de Rham cohomology $H^*(\mathcal{M}_G(\Sigma))$, perhaps with extra structure (Hodge). Subject of beautiful conjectures of Hausel-Rodriguez-Villegas, relations to Macdonald polynomials etc

& <u>Langlands duality</u>: relate for G & $G^\vee$.

Strongest relations to representation theory – and simplest structures! – come from going down to 1 & 0 dimensions

General idea – replace <u>functions</u> on G finite by <u>$\mathcal{D}$-modules</u>: algebraic systems of diff. eqs $\Longleftrightarrow$ vector bundles / sheaves with flat connections.

e.g $e^{\lambda x}$ not algebraic function but satisfies algebraic diff eq $(\partial - \lambda)f = 0$

More generally f function, $\mathcal{D}$ = ring of polynomial diffops, $\mathcal{D}\cdot f \subset C^\infty$ or $C^{-\infty}$ or ....
is a <u>module</u> for $\mathcal{D}$ expressing all algebraic diff eqs f satisfies.

functions $\longmapsto$ $\mathcal{D}$-modules
<u>spaces</u> of functions $\longmapsto$ <u>categories</u> of $\mathcal{D}$-modules
harmonic analysis $\longmapsto$ categorified / geometric harmonic analysis –
Langlands program $\longmapsto$ geometric Langlands

<u>So</u> replace group algebra $\mathbb{C}G$ by $\mathcal{D}G$ – (category of) $\mathcal{D}$-modules on G. Has associative multiplication via integration along $\mu: G \times G \longrightarrow G$

To a point instead of G-reps $= \mathbb{C}G$-modules assign smooth G-categories $:=$ $\mathcal{D}G$-modules (this is now a 2-category!)

To a circle assign $\mathcal{D}$-class functions on G — ie G-invariant systems of diff eqs on G. $\mathcal{D}\frac{G}{G}$.

Motivation: Two relations to rep theory:

1. Harish-Chandra: V ∞-dim (admissible) representation of G real or complex Lie group $\Rightarrow$ can define a character for V: initially G-invariant distribution on G but in fact satisfies strong (regular holonomic) system of diffeqs. like $e^{\lambda x}$ $\Rightarrow$ strong regularity properties (analytic fn. w/ prescribed singularities)

— nice $\mathcal{D}$-module on $\frac{G}{G}$, example of Lusztig's character sheaves: categorified analog of characters! — objects in theory $Z_G(S^1)$

2. Beilinson-Bernstein: categories of representations of Lie algebra $\mathfrak{g}$ are examples of smooth G-categories:

$\mathrm{Rep}\,\mathfrak{g} \simeq \mathcal{D}(G/B)$ flag manifold
$G \circlearrowright$ (ignoring infinitesimal character)

Reps of real forms $G_{\mathbb{R}}$ of G (H-C really) are $G_{\mathbb{R}}$-invariants in here!

Roughly speaking $\mathrm{Rep}\, G_{\mathbb{R}} \in \mathcal{X}_G(\cdot)$
- theory knows all rep. theory of real forms of G!

<u>B-Z-Nadler</u> • Develop some basics of formalism of G-cats using Lurie's homotopical algebra

• Prove characters of "highest weight" smooth G-cats (like $G_{\mathbb{R}}$-rep) are precisely Lusztig's character sheaves!

• Prove Langlands duality statements relating $\mathcal{X}_G$ & $\mathcal{X}_{G^\vee}$ - in particular character sheaves for $G, G^\vee$ are identified

Program: recover Rep $G_{\mathbb{R}}$ from Rep G $\Longrightarrow$ get Langlands duality for reps of real groups (Soergel's conjecture, Vogan character duality)

Source of all duality results: electro-magnetic duality for 4d SUSY gauge theory $\longleftrightarrow$ geometric Langlands conjectures.