1. Lecture: Gauss-Manin Connections

1.1. **Introduction.** This first lecture will review various facts about connections. An excellent reference is [**?**].

Let S be a smooth algebraic variety over $\mathbb{C}$, and let E be a vector bundle on S. We write $\mathcal{O}_S$ for the (Zariski)-sheaf of functions (in the sense of algebraic geometry) on S. (Note, however, it will sometimes be convenient to shift and work with holomorphic functions and analytic manifolds. To be precise, we should write $\mathcal{O}_{X^{an}}$ in this case, but precision is an ideal not always achieved. In the same spirit we write Ω^1_S for the sheaf of algebraic (or sometimes analytic) differential 1-forms on S.

Facts 1.1.1. *(i) Ω^1_S is the coherent sheaf generated by symbols df for f a section of $\mathcal{O}_S$, subject to the relations $df + dg = d(f+g)$ and $d(fg) = fdg + gdf$.*
(ii) $\Omega^1_S = T_S^\vee$, the dual of the tangent bundle.
(iii) analytically locally, if $s_1, \ldots, s_n$ are local coordinates, Ω^1_S is free on generators $ds_1, \ldots, ds_n$.

A *connection* ∇ on E, is a $\mathbb{C}$-linear map

$$\nabla : E \to E \otimes_{\mathcal{O}_S} \Omega^1_S, \tag{1.1.1}$$

satisfying the following condition

$$\nabla(fe) = f\nabla(e) + e \otimes df. \tag{1.1.2}$$

If D is a derivation (i.e. D is a section of the tangent bundle T_S) then by (ii), we may define

$$\nabla_D := (D \otimes id_E) \circ \nabla : E \to E. \tag{1.1.3}$$

$$\rho_\nabla : T_S \to \mathrm{End}_{\mathbb{C}}(E). \tag{1.1.4}$$

$$\mathrm{Curvature}(\nabla) := \nabla^2 = (\nabla \otimes id_E) \circ \nabla : E \to E \otimes \Omega^2_S. \tag{1.1.5}$$

Here $\Omega^2 := \bigwedge^2 \Omega^1$ is the sheaf of 2-forms. The connection is said to be *flat* or *integrable* if it has curvature 0.

Remark 1.1.2. *In the analytic category there is a one to one correspondence between integrable connections and local systems of $\mathbb{C}$-vector spaces. Indeed, given a local system $\mathcal{E}$ of $\mathbb{C}$-vector spaces, the bundle $E := \mathcal{E} \otimes_{\mathbb{C}} \mathcal{O}_{S^{an}}$ is coherent and admits a natural integrable connection given by exterior derivation on $\mathcal{O}_{S^{an}}$. For the converse, one has to show that an integrable connection on E has locally a basis of horizontal sections, viz. $E^{an} = E^{an,\nabla=0} \otimes \mathcal{O}_S^{an}$. This is a standard Taylor series type result. We omit the proof.*

We will see that even in the very basic case of an affine curve, the interplay between representations of the fundamental group and connections can be quite subtle.

An integrable connection (E, ∇) can be coupled to the de Rham complex Ω_S^* (for more on the de Rham complex, see below) to yield a complex of sheaves

$$E \xrightarrow{\nabla} E \otimes_{\mathcal{O}_S} \Omega_S^1 \xrightarrow{\nabla} E \otimes \Omega_S^2 \xrightarrow{\nabla} \cdots \tag{1.1.6}$$

The *de Rham cohomology* $H_{dR}^*(S, E)$ of E is the hypercohomology of this complex.

Exercise 1.1.3. *(i) The curvature is an $\mathcal{O}_S$-linear map $E \to E \otimes \Omega_S^2$.*
(ii) ∇ is flat $\Leftrightarrow \rho_\nabla : T_S \to End_{\mathbb{C}}(E)$ is a map of Lie algebras, i.e.

$$\nabla_{[D_1,D_2]} = \nabla_{D_1} \circ \nabla_{D_2} - \nabla_{D_2} \circ \nabla_{D_1}.$$

Example 1.1.4. *(i) $E = \mathcal{O}_S$ with $\nabla := d : \mathcal{O}_S \to \Omega_S^1$ the exterior derivative. Flatness amounts to the fact that $d^2 = 0$.*

(ii) Take S to be an abelian variety of dimension n. Then $\Omega_S^1 \cong \mathcal{O}_S^n$ is free of rank n with generators $\mu_1, \ldots, \mu_n \in \Gamma(S, \Omega^1)$ which are closed, i.e. $d\mu_i = 0 \in \Omega_S^2$. Let $M \in M_{n,\mathbb{C}}$, be an $n \times n$ matrix with entries in $\Gamma(S, \Omega_S^1) = \bigoplus \mathbb{C}\mu_i$, and define a connection on $\bigoplus_n \mathcal{O}_S$ by

$$\nabla(f_1, \ldots, f_n) = (df_1, \ldots, df_n) + \vec{f}M \tag{1.1.7}$$

The curvature is the matrix of 2-forms $\bigwedge^2 M = (\sum_k m_{ik} \wedge m_{kj})$. It is not in general zero if $n \geq 2$, so projective varieties can carry non-integrable connections.

(iii) Let E_i, ∇_i be connections, $i = 1, 2$. Then $E_1 \otimes_{\mathcal{O}_S} E_2$ admits a connection $\nabla(e_1 \otimes e_2) = \nabla_1(e_1) \otimes e_2 + e_1 \otimes \nabla_2(e_2)$. Similarly, the internal Hom sheaf $\underline{Hom}(E_1, E_2)$ has a connection given by $\nabla(\phi)(e_1) = -\phi(\nabla_1(e_1)) + \nabla_2(\phi(e_1))$.

1.2. **Connections and linear differential equations.** Any two connections $\nabla_1, \nabla_2 : E \to E \otimes \Omega_S^1$ differ by a function-linear map,

$$(\nabla_1 - \nabla_2)(f \cdot e) = f \cdot (\nabla_1 - \nabla_2)(e).$$

Suppose, for example, that $E = \mathcal{O}_S^n$. In this case exterior differentiation $d(f_1, \ldots, f_n) = (df_1, \ldots, df_n)$ is a connection, so any other connection can be written

$$\nabla = d + \Gamma; \quad \Gamma \in M_n(\Omega_S^1) \tag{1.2.1}$$

Suppose $S = \mathrm{Spec}\,(K)$ where $K = \mathbb{C}(x)$ is a function field in one variable (or one can take also $K = \mathbb{C}((x))$, the field of laurent series in one variable). Let $\nabla : K^n \to K^n \otimes \Omega^1_K = K^n dx$ be a connection. We have from (1.2.1)

$$\nabla_{\frac{d}{dx}} = \frac{d}{dx} + N; \quad N \in M_n(K). \tag{1.2.2}$$

The matrix N depends, of course, on the choice of basis for $E = K^n$. Suppose we have $e \in E$ such that $e, \nabla_{\frac{d}{dx}}(e), \ldots, \nabla^{n-1}_{\frac{d}{dx}}(e)$ form a K-basis for E. With this basis, we get

$$N = \begin{pmatrix} 0 & 1 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & 1 & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -b_0 & -b_1 & -b_2 & -b_3 & \ldots & -b_{n-1} \end{pmatrix} \tag{1.2.3}$$

In terms of vectors $(1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1)$ we have

$$\nabla_{\frac{d}{dx}} (\underbrace{0, \ldots, 1}_{p}, 0, \ldots, 0) = (\underbrace{0, \ldots, 1}_{p+1}, 0, \ldots, 0) \tag{1.2.4}$$

$$\nabla_{\frac{d}{dx}} (0, \ldots, 0, 1) = (-b_0, \ldots, -b_{n-1}) \tag{1.2.5}$$

To relate this to classical differential equations, Let $\mathcal{A}$ be some "large" K-algebra, and suppose the derivation $\frac{d}{dx}$ is extended as a derivation to $\mathcal{A}$. We may ask for a K-linear map $\phi : E \to \mathcal{A}$ which is compatible with the connection in the sense that $\frac{d}{dx} \circ \phi = \phi \circ \nabla_{\frac{d}{dx}}$.

Proposition 1.2.1. *The assignment $\phi \mapsto f := \phi(1, 0, \ldots, 0)$ is a bijective correspondence between K-linear maps $\phi : E \to \mathcal{A}$ compatible with the connection, and elements $f \in \mathcal{A}$ satisfying the linear differential equation*

$$\frac{d^n f}{dx^n} + b_{n-1} \frac{d^n f}{dx^{n-1}} + \cdots + b_1 \frac{df}{dx} + b_0 f = 0. \tag{1.2.6}$$

Proof. Straightforward. ∎

1.3. **Gauß-Manin connections.** The most important examples of connections for us are *Gauß* $-$ *Manin* connections. Let $f : X \to S$ be a smooth, proper map of varieties with S assumed smooth over $\mathbb{C}$. Let $\Omega^1_{X/S}$ be the sheaf of relative 1-forms,

$$\Omega^1_{X/S} = \Omega^1_X / f^* \Omega^1_S. \tag{1.3.1}$$

It is a locally free sheaf on X with local bases $dx_1, \ldots, dx_n$ where fibres of X/S have dimension n and $x_1, \ldots, x_n$ are local coordinates along the fibre. Exterior differentiation defines a structure of differential graded algebra on the exterior algebra $\bigwedge^* \Omega^1_{X/S}$. By definition, the de Rham complex of X over S is the complex $\Omega^*_{X/S}$:

$$(1.3.2) \qquad 0 \to \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/S} \xrightarrow{d} \Omega^2_{X/S} \cdots \xrightarrow{d} \Omega^n_{X/S}.$$

When $S = \mathrm{Spec}\,(\mathbb{C})$, the *Poincaré lemma* says that in the analytic topology, $\Omega^*_{X^{an}} := \Omega^*_{X^{an}/\mathbb{C}}$ is a resolution of the constant sheaf $\mathbb{C}_X$. In particular, the *Betti cohomology* $H^*(X, \mathbb{C})$ is identified with the hypercohomology of the complex $\Omega^*_{X^{an}}$. When X admits a projective embedding, this coincides with the algebraic hypercohomology of the corresponding complex of Zariski sheaves of algebraic differential forms.

More generally, when S is not a point, $\Omega^*_{X^{an}/S^{an}}$ is a resolution of the inverse image $f^{-1}\mathcal{O}_S$. (Nb. the sheaf-theoretic inverse image $f^{-1}\mathcal{O}_S$ is not the same as $\mathcal{O}_X = f^*\mathcal{O}_S$.) It follows that the hypercohomology along the fibres is

$$(1.3.3) \qquad R^*f_*(\Omega^*_{X^{an}/S^{an}}) = R^*f_*(\mathbb{C}_X) \otimes_{\mathbb{C}_S} \mathcal{O}_{S^{an}}.$$

The expression on the right exhibits an integrable $\mathcal{O}_S$-connection on relative de Rham cohomology, namely

$$(1.3.4) \quad \nabla(\alpha \otimes f) := \\ \alpha \otimes df \in R^*f_*(\mathbb{C}_X) \otimes_{\mathbb{C}_S} \Omega^1_S = R^*f_*(\Omega^*_{X^{an}/S^{an}}) \otimes_{\mathcal{O}^{an}_S} \Omega^1_{S^{an}}.$$

Definition 1.3.1. *The above integrable connection is called the Gauß-Manin connection on X/S.*

In the projective case, known theorems relating algebraic and analytic cohomology enable one to construct the Gauß-Manin connection on the relative algebraic de Rham cohomology, but this approach is unsatisfactory for several reasons. Firstly it is difficult in practice to make concrete the locally constant section decomposition as in (1.3.3). And secondly, there is no real reason to assume f has compact fibres. The Gauß-Manin connection is defined for any smooth $f : X \to S$.

Construction 1.3.2. *Assume simply that S is smooth and $f : X \to S$ is smooth as well. Define a decreasing filtration on Ω^*_X by defining*

$$(1.3.5) \qquad fil^p\Omega^q_X := Image(f^*\Omega^p_S \otimes_{\mathcal{O}_X} \Omega^{q-p}_X \to \Omega^q_X).$$

Thus, sections of fil^p have at least p ds's coming from S. We have an exact sequence of complexes

$$(1.3.6) \qquad 0 \to (\Omega^{*-1}_{X/S}) \otimes \Omega^1_S \to \Omega^*_X/fil^2 \to \Omega^*_{X/S} \to 0$$

obtained from the exact sequence

$$0 \to fil^1/fil^2 \to fil^0/fil^2 \to fil^0/fil^1 \to 0.$$

The boundary map on higher derived images from (1.3.6) *yields*

$$\nabla : R^p f_* \Omega^*_{X/S} \to R^p f_* \Omega^*_{X/S} \otimes_{\mathcal{O}_S} \Omega^1_S. \tag{1.3.7}$$

This is the algebraic construction [].

Example 1.3.3. *Suppose $f : X \to S$ is étale. Then $\Omega^*_{X/S} = \mathcal{O}_X$ and the connection ∇ coincides with the exterior derivative on X*

$$\nabla : \mathcal{O}_X \xrightarrow{d} \Omega^1_X \cong \mathcal{O}_X \otimes_{\mathcal{O}_S} \Omega^1_S. \tag{1.3.8}$$

1.4. **Regular singular points.** Typically, a proper map $f : X \to S$ of smooth varieties is itself smooth only over some open $S^0 \subset S$. Writing $X^0 = f^{-1}(S^0)$, the Gauß-Manin connection yields an integrable connection on $R^* f_*(\Omega^*_{X^0/S^0})$. An important theorem of Griffiths, [], [], says that these Gauß-Manin connections extend across infinity with *regular singular points.*

Classically, a linear differential equation

$$\frac{d^n f}{dx^n} + b_{n-1}(x)\frac{d^{n-1} f}{dx^{n-1}} + \cdots + b_1(x)\frac{df}{dx} + b_0(x) f = 0 \tag{1.4.1}$$

is said to have a *regular singular point* at $x = 0$ if $x^{n-i} b_i(x)$ is regular at x. Equivalently, the equation can be rewritten in terms of the derivation $x\frac{d}{dx}$:

$$\begin{aligned}(x\frac{d}{dx})^n(f) + c_{n-1}(x)(x\frac{d}{dx})^{n-1}(f)\ldots \\ + c_1(x)(x\frac{d}{dx})(f) + c_0(x) f = 0.\end{aligned} \tag{1.4.2}$$

where the $c_i(x)$ are regular at $x = 0$.

Example 1.4.1. *(i) A connection on the trivial bundle $\mathcal{O}_S$ is written in the form $\nabla = d + \gamma$ where $\gamma = \nabla(1) \in \Gamma(S, \Omega^1_S)$. When $S = \operatorname{Spec} K$, $K = \mathbb{C}(x)$ as above, the associated differential equation is $\frac{df}{dx} - \gamma_x = 0$, where $\gamma = \gamma_x dx$. There are three cases. If γ_x has no pole at $x = 0$, the equation and the connection extend across $x = 0$. If γ_x has a pole of order 1, the equation and the connection are said to have a regular singular point. If γ_x has a pole of order ≥ 2 at $x = 0$ the equation and the connection have an irregular singularity. Looking at solutions, the calculation becomes*

(1.4.3)

$$\nabla(1) = g(x)dx,\ \nabla(f(x)) = df + f \cdot g(x)dx,\ f = \exp(-\int g(x)dx).$$

The solution f has an exponential singularity at $x = 0$ if g has a pole of order ≥ 2. If $g(x) = c/x + h(x)$ where $h(x)$ has no pole at $x = 0$, *the solution looks like* $x^{-c}F(x)$ *with* $F(x)$ *smooth at* $x = 0$. *More concretely, for* $S = \mathbb{G}_m = \mathbb{P}^1 - \{0, \infty\}$ *one has the regular singular point connection on* $\mathcal{O}_S$ *given by* $\nabla(1) = cdt/t$ *with c constant. It has solution* t^c. *For* $S = \mathbb{A}^1$ *and* $P(t) \in \mathbb{C}[t]$ *a non-constant polynomial, the connection* $\nabla(1) = dP$ *has an irregular singular point at infinity with solution* $e^P(x)$.

(ii)(Bessel equation) For any integer n, *the Bessel equation of order* n *([], §17.11), is given by*

$$\frac{d^2 f}{dx^2} + \frac{1}{x}\frac{df}{dx} + (1 - \frac{n^2}{x^2})f = 0 \tag{1.4.4}$$

This equation has a regular singular point at $x = 0$. *At* ∞, *writing* $y = 1/x$, *we find* $\frac{d^2}{dx^2} = 2y^3\frac{d}{dy} + y^4\frac{d^2}{dy^2}$ *and the equation becomes*

$$y^4\frac{d^2 f}{dy^2} + y^3\frac{df}{dy} + (1 - n^2y^2)f = 0 \tag{1.4.5}$$

which has an irregular singular point at $y = 0$.

Let $D = \bigcup D_i \subset S$ be a normal crossings divisor, which means that for any point $s \in S$ there will be local coordinates $\{x_j\}_{j \in J}$ such that in some neighborhood of s, the divisor D will be defined by $\prod_{j \in J_D} x_j = 0$, the product taken over some subset $J_D \subset J$. The sheaf of 1-forms with *log poles* on D, $\Omega^1_X(\log D) \supset \Omega^1_X$ is generated locally by dx_j, $j \notin J_D$ and dx_j/x_j, $j \in J_D$. The sheaf of logarithmic 1-forms is locally free of rank $m = \dim S$, and one defines log forms of degree p, $\Omega^p_S(\log D) := \bigwedge^p \Omega^1_S(\log D)$. The exterior derivative extends to log forms, so one gets a log de Rham complex

$$\mathcal{O}_S \xrightarrow{d} \Omega^1_S(\log D) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m_S(\log D) \tag{1.4.6}$$

A connection with log poles along D for a coherent sheaf E on S is a map $\nabla : E \to E \otimes_{\mathcal{O}_S} \Omega^1_S(\log D)$ satisfying the derivation property (1.1.2).

Let $K = \mathbb{C}(x)$ (or $\mathbb{C}((x))$) as above, and let $\Lambda \subset K$ be the local ring of elements with no pole at $x = 0$. Let $\nabla : E \to E \otimes \Omega^1_S(\log 0)$ be a connection on $S = \mathrm{Spec}\,(\Lambda)$ with a log pole at 0. We assume $E \cong \Lambda^n$ is a free module.

Exercise 1.4.2. *(i) Assume* E *has a "cyclic vector", i.e.* $\exists e \in E$ *such that* $e, x\nabla_x(e), \ldots, (x\nabla_x)^{n-1}(e)$ *span* E *as a* Λ*-module. Associate to* (E, ∇, e) *a linear differential equation with regular singular points.*
(ii) Conversely, given a differential equation with regular singular point at $x = 0$, *construct a connection on* $\mathrm{Spec}\,\Lambda$ *with a log pole at the origin.*

Consider now $f : X \to S$ with X, S smooth and f proper. Let $S^0 \subset S$ be open, dense such that $f^0 : X^0 = f^{-1}(S^0) \to S^0$ is smooth. Griffiths regularity theorem says that $R^* f^0_*(\Omega^*_{X^0/S^0})$, which is a locally free sheaf with integrable connection on S^0 has regular singular points on $S - S^0$. One has to be a bit careful about what that means, however. For a precise discussion in the general case, see []. When $\dim S = 1$, the situation is easier

Theorem 1.4.3. *In the above situation, assume* $\dim S = 1$*. Then* $R^* f^0_*(\Omega^*_{X^0/S^0})$ *extends to a connection over* S *with log poles on* $S - S^0$*. (The extension is not unique, however.)*

2. Lecture: Periods

2.1. **Introduction.** Let X be a smooth variety. There are a number of ways one can extract information from the basic topological duality

$$H_p(X, \mathbb{C}) \times H^p(X, \Omega^*_X) \to \mathbb{C}. \tag{2.1.1}$$

(i) As a consequence of the duality, de Rham cohomology is identified with $\mathbb{C}$-Betti cohomology, and hence it carries a $\mathbb{Q}$-structure given by $\mathbb{Q}$-Betti cohomology. It also carries a Hodge filtration and a (rationally defined) weight filtration. These data constitute a *Hodge structure.*
(ii) When X is defined as an algebraic variety over a subfield $k \subset \mathbb{C}$, the *periods* $\int_\sigma \omega$ can be thought of as entries in a double coset in

$$GL_n(\mathbb{Q})\backslash GL_n(\mathbb{C})/GL_n(k).$$

These periods are of fundamental importance in diophantine geometry and number theory.
(iii) Of greater interest to us will be the variational structure. Given $f : X \to S$ smooth, a locally constant Betti homology class $\{\sigma_s\}_{s\in S}$ defines a map of sheaves ($p = \dim_{\mathbb{R}} \sigma_s$)

$$\int_{\sigma_s} : R^p f_*(\Omega^*_{X/S}) \to \widetilde{\mathcal{O}}_{S^{an}}. \tag{2.1.2}$$

Here $\widetilde{\mathcal{O}}$ is the sheaf of multivalued analytic functions (i.e. the push-forward of the sheaf of analytic functions on the universal cover of S. When S is contractible, $\widetilde{\mathcal{O}} = \mathcal{O}$.) The rule in calculus for differentiating under the integral sign translates in this context as

$$d_{\widetilde{S}} \circ \int_{\sigma_s} = \int_{\sigma_s} \circ \nabla_{GM} \tag{2.1.3}$$

(where ∇_{GM} is the Gauß-Manin connection for X/S.) This means that $\int_{\sigma_s}$ is a solution for the Gauß-Manin connection.

Example 2.1.1. *(i) Let $f : T \to S$ be a finite, étale cover, and take $E := f_*\mathcal{O}_T$. Endow E with the structure of connection as in example 1.3.3. Writing $\widetilde{S}$ for the universal cover and fixing basepoints, one has $\widetilde{S} \to T \to S$ and hence a map $E \to \widetilde{\mathcal{O}}_S$ which can be interpreted locally on S as zero dimensional integration over the family of points on T given by a local analytic section $\sigma : S \to T$ through the basepoint on T.*

(ii) (Picard-Fuchs equations). Let $f : X \to S$ be smooth and proper and assume $\dim S = 1$ with local coordinate z. The $\mathcal{O}_S$-module $R^p f_*(\Omega^*_{X/S})$ is locally free of finite rank r. Let ω be a section. Then there will be a relation with coefficients in $\mathcal{O}_S$ between $\omega, \nabla_{\frac{d}{dz}}(\omega), \ldots, (\nabla_{\frac{d}{dz}})^r(\omega)$. Localizing on S one can arrange that the relation take the form

$$P_0\omega + P_1\nabla_{\frac{d}{dz}}(\omega) + \cdots + P_{r-1}(\nabla_{\frac{d}{dz}})^{r-1}(\omega) + (\nabla_{\frac{d}{dz}})^r(\omega) = 0.$$

Given a family of p-cycles σ_s as above, one deduces that the function $F(s) = \int_{\sigma_s} \omega$ satisfies the differential equation (Picard-Fuchs equation)

$$(\frac{d}{dz})^r F(z) + P_{r-1}(z)(\frac{d}{dz})^{r-1} F(z) \cdots + P_0(z)F(z) = 0. \tag{2.1.4}$$

2.2. **Periods for connections on curves.** Let S be an open smooth curve, and write $S \subset \overline{S}$ for the completion. To simplify the exposition, I assume $S = \mathrm{Spec}\,(A)$ is affine, so $\overline{S} - S \neq \emptyset$.

Let (E, ∇) be a algebraic connection on S. Write $\mathbb{E} = \Gamma(S, E)$, so $H^*_{dR}(E)$ is calculated from the two-term complex

$$\mathbb{E} \xrightarrow{\nabla} \mathbb{E} \otimes \Omega^1_A. \tag{2.2.1}$$

In [], a dual homology theory is introduced, $H_*(S, E^\vee, \nabla_{E^\vee})$ (written as $H_*(\overline{S}, \overline{S} - S, E^\vee, \nabla^\vee)$ in that reference. The notation used here is to be preferred). The main point is a perfect pairing

$$H_i(S, E^\vee, \nabla_{E^\vee}) \times H^i_{dR}(S, E) \to \mathbb{C} \tag{2.2.2}$$

Let $\mathcal{E} = (E^{an})^{\nabla=0}$ (resp. $\mathcal{E}^\vee$) be the sheaf for the analytic topology of horizontal sections of E (resp. $E^\vee$). One defines a chain complex by coupling sections of $\mathcal{E}^\vee$ to topological chains. (Notice that local sections of $\mathcal{E}^\vee$ are solutions of the connection.)

For example, to define 1-chains one takes topological 1-chains σ on $\overline{S}$ and sections of $\mathcal{E}^\vee|_\sigma$ which have rapid decay approaching $\overline{S} - S$. (The result from op. cit. is limited to dimension 1 and is fairly elementary. Analogous results in higher dimension are much deeper.)

Example 2.2.1. *If E has at worst regular singular points on $\overline{S} - S$, then horizontal sections never have rapid decay so chains are compact* *chains supported on S coupled to horizontal sections of $\mathcal{E}$, and we have the familiar homology associated to the local system $\mathcal{E}^\vee$.*

Suppose that $S = \mathbb{G}_m$ with coordinate x and $\nabla : \mathcal{O}_S \to \Omega^1_S$ is defined by $\nabla(1) = cdx/x$ for $c \in \mathbb{C}$. A solution is given by x^{-c}. Let $\sigma \subset \mathbb{G}_m$ be a circle around $x = 0$, and consider the solution $x^{-c}|_\sigma$. If $c \in \mathbb{Z} \subset \mathbb{C}$ then x^{-c} is single-valued. The solution closes up and $x^{-c}|_\sigma$ represents a class in H_1. In this case, the local system in trivial and we have just constructed the non-trivial class in $H_1(\mathbb{G}_m, \mathbb{C})$.

If $c \notin \mathbb{Z}$ then $x^c|_\sigma$ is not single-valued, we do not get a co-cycle, and indeed $H_1(\mathbb{G}_m, \mathcal{E}^\vee) = (0)$.

(ii) Take $S = \mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$ and $E = \mathcal{O}_S$. Let $f \in \mathbb{C}[x]$ be a non-constant polynomial, and define $\nabla(1) = df$. We have

$$(2.2.3)\quad H^1_{dR}(E) \cong \mathit{coker}(\mathbb{C}[x] \to \mathbb{C}[x]dx); \quad P(x) \mapsto (\frac{dP}{dx} + P\frac{df}{dx})dx.$$

One checks that $\dim H^1_{dR} = \deg f - 1$. Suppose for example $f(x) = x^2$. The connection is then $P \mapsto (\frac{dP}{dx} + 2xP)dx$. The solution is $\exp(-x^2)$. To define a class in $H_1(\mathcal{E}^\vee)$, we take a circle σ through $x = \infty$ in such a way that the solution has rapid decay at infinity. Rapid decay occurs in the two sectors $\frac{-\pi}{4} < \arg x < \frac{\pi}{4}$ and $\frac{3\pi}{4} < \arg x < \frac{5\pi}{4}$. Take $\sigma = [-\infty, \infty]$, the real axis. the chain becomes $\exp(-x^2)|_{[-\infty,\infty]}$. The Poincaré pairing $H^1_{dR}(E) \times H_1(\mathcal{E}^\vee) \to \mathbb{C}$ is associates to a 1-form $g(x)dx$ representing a class in $H^1_{dR}(E)$ the integral

$$\int_{-\infty}^{\infty} \exp(-x^2)g(x)dx.$$

In fact, we can in this case take $g(x) = 1$ so the integral becomes simply $\int_{-\infty}^{\infty} \exp(-x^2)dx = \sqrt{\pi}$.

(iii) One can couple (tensor) connections. for example, on $\mathbb{G}_m$ define

$$(2.2.4)\qquad \nabla(1) = dx + cdx/x; \quad c \notin \mathbb{Z}.$$

One checks that $H^1_{dR} \cong \mathbb{C}$. The solution is $x^{-c}\exp(-x)$. Take the path σ to run from $\varepsilon > 0$ to $1/\varepsilon$ along the real axis, then to circle counterclockwise around ∞ and return to ε and circle counterclockwise around 0. Then $x^{-c}\exp(-x)|_\sigma$ represents a class in $H_1(\mathcal{E}^\vee)$. If we take dx/x as generator of $H^1_{dR}(E)$ we get as period

$$(2.2.5)\qquad \int_\sigma \exp(-x)x^{-c}dx/x = \Gamma(c)(\exp(-2\pi i c) - 1).$$

Our path is not allwed to pass through the origin because the connection has a regular singular point there. Note also that the period does not *have a pole for c a negative integer.*

(iv) (Bessel function again) The connection $\nabla(1) = \frac{ndx}{x} + \frac{z}{2}d(x - \frac{1}{x})$ *on* $\mathcal{O}_S$ *for* $S = \mathbb{G}_m$ *has* H^1_{dR} *of dimension* 2. *(Here, for the moment, we take* $n \in \mathbb{Z}$*, and* z *is a parameter. the factor of* $1/2$ *is traditional.) We can view this as a family parametrized by* z *of connections on* $\mathbb{G}_{m,x}$*, but we can also lift this relative connection for* $\mathbb{G}_{m,x} \times \mathbb{G}_{m,z}/\mathbb{G}_{m,z}$ *to an absolute integrable connection on* $\mathbb{G}_{m,x} \times \mathbb{G}_{m,z}$ *given by*

$$\nabla(1) = \frac{ndx}{x} + d(\frac{z}{2}(x - \frac{1}{x})). \tag{2.2.6}$$

We then can couple this absolute connection to the de Rham complex for $\mathbb{G}_m \times \mathbb{G}_m$ *and argue as in equation* (1.3.6) *to construct a twisted Gauß-Manin connection*

$$H^1_{dR}(\mathit{Bessel}(n, z)) \xrightarrow{\nabla_{GM}} H^1_{dR}(\mathit{Bessel}(n, z)) \otimes \Omega^1_{\mathbb{G}_{m,z}}. \tag{2.2.7}$$

Our periods now become functions of z*. To define the class in* H_1 *we couple the solution* $x^{-c}\exp(\frac{-z}{2}(x - \frac{1}{x}))$ *to the path* σ *which we take to be a circle around* 0 *on* $\mathbb{G}_{m,x}$*. The Poincaré pairing then associates to an element* $\omega \in H^1_{dR}(\mathit{Bessel}(n, z))$ *the integral* $\int_\sigma x^{-c}\exp(\frac{-z}{2}(x - \frac{1}{x}))\omega$*. Classically, the Bessel function* $J_c(z)$ *is defined by (note the sign change for* z*)*

$$J_c(z) := \int_\sigma x^{-c}\exp(\frac{z}{2}(x - \frac{1}{x}))\frac{dx}{x}. \tag{2.2.8}$$

When Re $z > 0$ *one can extend this definition to* $c \in \mathbb{C}$ *by cutting* σ *where it meets* $[-\infty, 0]$ *(say at* $x = -\varepsilon$*) and adding to* σ *the segments* $[-\infty, -\varepsilon]$ *and* $[-\varepsilon, -\infty]$*. Since our solution has rapid decay at* $-\infty$ *along this path, the resulting "keyhole" path is legitimate. (Note when* $c \notin \mathbb{Z}$*, our solution is not single-valued on* σ*.) Finally, differentiating* (2.2.8) *with respect to* z *one verifies that the Bessel differential equation* (1.4.4) *coincides with the equation obtained by applying the twisted Gauß-Manin connection* (2.2.7) *to the solution integral* (2.2.8)*.*

3. $\mathcal{D}$-modules

Lex S be a smooth algebraic variety over $\mathbb{C}$. We write $\mathcal{D}$ for the Zariski sheaf of differential operators on S. It is a sheaf of non-commutative rings contained in $\underline{End}_{\mathbb{C}}(\mathcal{O}_S)$ generated by multiplication by functions and by derivations of $\mathcal{O}_S$. Integrable connections on S yield $\mathcal{D}$-modules, but the notion of $\mathcal{D}$-module is more general and more flexible. In particular, the derived category of $\mathcal{D}$-modules carries Grothendieck's six functorialities. Building up the full theory would require more time than we have, so I will frequently short-circuit things by restricting to the case of $\mathcal{D}$-modules on curves. The sheaf $\mathcal{D}$ is noetherian, and all our $\mathcal{D}$-modules will coherent as $\mathcal{D}$-modules (and hence quasi-coherent as $\mathcal{O}_S$-modules). The basic notion of *holonomic* $\mathcal{D}$-module is a bit technical to explain in general, but for S a curve, a $\mathcal{D}$-coherent $\mathcal{D}$-module M is holonomic iff for any (local) section $m \in M$, the annihilator $Ann(m) \subset \mathcal{D}$ is non-trivial. For the most part, our modules will be holonomic. (Notice, however, that $\mathcal{D}$ itself is not holonomic.)

3.1. **Functoriality.** Following [], I list some of the more elementary and useful functoriality properties for $\mathcal{D}$-modules. These are defined on the level of the derived category, so one is dealing with complexes of $\mathcal{D}$-modules.

For $f : X \to Y$ smooth of relative dimension d, one defines

$$f_*M := Rf_*(M \otimes_{\mathcal{O}_X} \Omega^*_{X/Y})[d] \tag{3.1.1}$$

Here a $\mathcal{D}$-module is coupled to the de Rham complex in much the same way as an integrable connection. The $\mathcal{D}$-module structure on f_*M comes from a Gauß-Manin connection.

For $f : X \to Y$ arbitrary and M a $\mathcal{D}$-module on Y, one defines $f^!M$ a $\mathcal{D}$-module on X as $f^!M = \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}M[\dim X - \dim Y]$, where $\mathcal{D}_{X\to Y}$ is the sheaf of differential operators $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. More concretely, locally on Y if y_i are a system of coordinates, and $\frac{\partial}{\partial y_i}$ are the corresponding vector fields, we can identify

$$f^!M = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}M[\dim X - \dim Y]. \tag{3.1.2}$$

The action of $\mathcal{D}_X$ is given by

$$\rho(g \otimes m) = \rho g \otimes m; \quad \rho \in \mathcal{O}_X \tag{3.1.3}$$

(3.1.4)

$$\xi(g \otimes m) = \xi(g) \otimes m + \sum_i g\xi(y_i) \otimes \frac{\partial}{\partial y_i} m; \quad \xi \in T_X \text{ a derivation.}$$

One checks that this is independent of the choice of the y_i, and that sections of $f^{-1}\mathcal{O}_Y$ pass through the tensor product as necessary.

In the basic case M a connection, $f^!M$ coincides with the $\mathcal{O}$-module pullback f^*M endowed with the pullback connection.

Definition of pushforward f_* for a closed immersion $f : X \hookrightarrow Y$ is slightly delicate because one has to worry about left versus right $\mathcal{D}$-modules. Let $I \subset \mathcal{O}_Y$ be the ideal defining X. Let $\mathcal{D}_Y^I \subset \mathcal{D}_Y$ be the subring stabilizing I^k for all k. One checks that $\mathcal{D}_Y^I$ is generated by $\mathcal{O}_Y$ and by the subsheaf T_Y^I of derivations stabilizing I. Then $\mathcal{D}_Y^I$ maps to

$\mathcal{D}_X$ and it acts on I/I^2. It therefore also acts on $M \otimes \det(I/I^2)^\vee$ for M a $\mathcal{D}_X$-module. By definition

$$f_*M = \mathcal{D}_Y \otimes_{\mathcal{D}_Y^I} (M \otimes \det(I/I^2)^\vee). \tag{3.1.5}$$

Suppose, e.g. $X = \{0\} \hookrightarrow Y = \mathbb{A}^1$. Then $y\frac{d}{dy}$ acts as -1 on $\det(I/I^2)^\vee$. It follows that

$$f_*\mathbb{C}(0) = \mathcal{D}_Y/\mathcal{D}_Y(y, y\frac{d}{dy} + 1) \cong \mathbb{C}[y, y^{-1}]/\mathbb{C}[y]. \tag{3.1.6}$$

Note the quotient $M := \mathbb{C}[y, y^{-1}]/\mathbb{C}[y]$ is a $\mathcal{D}$-module which is generated as a $\mathcal{D}$-module by $1/y$. We have $(y\frac{d}{dy} + 1)(1/y) = 0$, so we get the above isomorphism.

Example 3.1.1. *(Convolution) Let G be an algebraic group over $\mathbb{C}$, and let M, N be $\mathcal{D}$-modules on G. The convolution $M * N$ is defined by*

$$M * N = \mu_*(M \boxtimes N). \tag{3.1.7}$$

Here $M \boxtimes N$ is the exterior tensor product on $G \times G$ and $\mu : G \times G \to G$ is the group law.

3.2. **Hypergeometric $\mathcal{D}$-modules.** We will be particularly interested in $\mathcal{D}$-modules on $\mathbb{A}^n$ and on $\mathbb{G}_m^n$. In such affine situations, there is no need to work with sheaves of differential operators, it suffices to consider modules for the rings ($\partial_i = \frac{d}{dx_i}$. The ring of differential operators on affine space is called the *Weyl algebra.*)

$$\mathbb{C}[x_1, \ldots, x_d, \partial_1, \ldots, \partial_d]; \quad \mathbb{C}[x_1, x_1^{-1}, \ldots, x_d, x_d^{-1}, x_1\partial_1, \ldots, x_d\partial_d] \tag{3.2.1}$$

Indeed we shall spend most of our time on the case $d = 1$. Our basic references are [], [], and [].

Example 3.2.1. *(Fourier transform) The map*

$$FT : \mathcal{D} \to \mathcal{D}; \quad x_i \to \partial_i,\ \partial_i \to -x_i \tag{3.2.2}$$

is an automorphism of the Weyl algebra. The fourier transform of a $\mathcal{D}$-module M on $\mathbb{A}^n$ is the $\mathcal{D}$-module

$$FT(M) := \mathcal{D}_{FT} \otimes_{\mathcal{D}} M \tag{3.2.3}$$

where the notation means the $\mathcal{D}$ on the left of the tensor is viewed as a right $\mathcal{D}$-module via $FT : \mathcal{D} \to \mathcal{D}$. The most important example is the case $d = 1$, $M = \mathcal{D}/\mathcal{D}P$ where $P = \sum_k f_k(x)\partial^k$. Then $FT(M) = \mathcal{D}/\mathcal{D}P^*$ where $P^* = \sum_k f_k(\partial)(-x)^k$. Thus, e.g.

$$FT(\mathcal{O}) = FT(\mathcal{D}/\mathcal{D}\partial) = \mathcal{D}/\mathcal{D}x = \mathbb{C}[x, x^{-1}]/\mathbb{C}[x] =: \delta_0.$$

Definition 3.2.2. *Let P, Q be non-constant polynomials of degrees n, m in one variable x. Write*

$$Hyp(P,Q) := P(D) - xQ(D); \quad D := x\partial. \tag{3.2.4}$$

The corresponding hypergeometric $\mathcal{D}$-module on $\mathbb{G}_m$ is

$$\mathcal{H}(P,Q) := \mathbb{C}[x, x^{-1}, D]/\mathbb{C}[x, x^{-1}, D]Hyp(P,Q).$$

Scaling $Hyp(P,Q)$ we may assume Q is monic. We write

$$\begin{gathered} Q = \prod(t-\beta_j),\ P = \lambda\prod(t-\alpha_i); \quad \alpha_i, \beta_j, \lambda \in \mathbb{C} \\ Hyp(P,Q) = Hyp_\lambda(\alpha,\beta),\ \mathcal{H}(P,Q) = \mathcal{H}_\lambda(\alpha,\beta). \end{gathered} \tag{3.2.5}$$

Proposition 3.2.3. *If $n \neq m$, $\mathcal{H}(P,Q)$ is a connection on $\mathbb{G}_m$. If $n = m$, the equation is regular singular at $x = 0, \lambda, \infty$ and has no other singularities.*

Proof. If $n \neq m$, we may multiply on the left by x^{-1} and replace x by x^{-1} to arrange that $n > m$. We have then $P(D) - xQ(D) = a_0 D^n + a_1(x)D^{n-1} + \cdots + a_n(x)$ where $0 \neq a_0 \in \mathbb{C}$ and $a_i(x)$ has degree ≤ 1 in x for $i \geq 1$. It follows (cf. the discussion in section 1.4) that at 0 this equation has at worst a singular point at ∞.

Suppose now $n = m$. Replacing x by λx and factoring out the constant λ, we can arrange

$$Hyp(P,Q) = (x-1)(D^n + \frac{\ell_1(x)}{x-1}D^{n-1} + \cdots + \frac{\ell_n(x)}{x-1}),$$

where $\ell_i(x)$ has degree ≤ 1 in x. Now we have at worst regular singularities at $0, \infty$, and in addition we may have a regular singularity at $x = 1$. Translating back, this becomes a regular singularity at λ for $\mathcal{H}_\lambda(P,Q)$. $\square$

To fix ideas, take $\lambda = 1$ and consider $Hyp_1(\alpha,\beta)$ as a differential equation with at worst regular singular points at $x = 0, 1, \infty$. Let $V(\alpha,\beta)$ be the local solutions of $Hyp_1(\alpha,\beta)$, viewed as a subspace of the space of analytic functions on some disk in $\mathbb{G}_m - \{1\}$. If we write $P(D) = (D-\alpha_1)R(D)$ with $R(D)$ of degree $n-1$, we get

$$\begin{aligned} &(P(D) - xQ(D))(D - \alpha_1 + 1) \\ &\qquad = (D-\alpha_1)((D-\alpha_1+1)R(D) - xQ(D)). \end{aligned} \tag{3.2.6}$$

As a consequence multiplication by $D - \alpha_1 + 1$ intertwines the two local systems

$$D - \alpha_1 + 1 : V(\alpha_1 - 1, \alpha_2, \ldots, \beta_1, \ldots) \to V(\alpha_1, \ldots, \beta_1, \ldots) \tag{3.2.7}$$

If $(D-\alpha_1+1)v=0$, then $Dv=(\alpha_1-1)v$. The only way we can have $v \in V(\alpha_1-1,\alpha_2,\dots,\beta_1,\dots)$ is if some $\beta_j=\alpha_1-1$. The most precise result in this direction is ([], proposition 3.2, and corollary 3.2.1)

Proposition 3.2.4. *(i) $\mathcal{H}_\lambda(\alpha,\beta)$ is an irreducible $\mathcal{D}$-module on $\mathbb{G}_m$ if and only if $\forall i,j;\ \alpha_i \not\equiv \beta_j \mod \mathbb{Z}$.*
(ii) The isomorphism class of $\mathcal{H}_\lambda(\alpha,\beta)$ for λ fixed depends only on the α's and β's $\mod \mathbb{Z}$.

Proof. Omitted. □

3.3. **Hypergeometric Group.** We focus on the hypergeometric

$$\mathcal{H}_1(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n)$$

with $\alpha_i-\beta_j\notin\mathbb{Z}$ for all i,j. Consider a connection with a log pole

$$\nabla : E \to E\otimes\Omega^1_S(\log D), \tag{3.3.1}$$

where $D\subset S$ is a smooth divisor. Residue defines a map $res : \Omega^1_S(\log D)\to\mathcal{O}_D$ and we can define an endomorphism $res(\nabla)$ of $E\otimes\mathcal{O}_D$ via the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\nabla} & E\otimes\Omega^1_S(\log D) \\ \downarrow & & \downarrow{\scriptstyle res} \\ E\otimes_{\mathcal{O}_S}\mathcal{O}_D & \xrightarrow{res(\nabla)} & E\otimes_{\mathcal{O}_S}\mathcal{O}_D. \end{array} \tag{3.3.2}$$

Suppose now that S is a curve, and $D\subset S$ is a point. A classical result ([**?**], corollaire 1.17.2) says that $\exp(-2\pi i Res(\nabla))$ has the same characteristic polynomial as the monodromy matrix for the local system of horizontal sections of ∇ over a punctured disk around D. The eigenvalues of $Res(\nabla)$ are called *local exponents*.

Proposition 3.3.1. *The local exponents for $\mathcal{H}_1(\alpha,\beta)$ are*

$$\begin{cases} \alpha_1,\dots,\alpha_n & x=0 \\ \beta_1,\dots,\beta_n & x=\infty \\ 0,1,2,\dots,n-2,\gamma := \sum_{i=1}^n\beta_i-\sum_{j=1}^n\alpha_j+n-1 & x=1. \end{cases} \tag{3.3.3}$$

Proof. The computations at $0,\infty$ are left for the reader. At $x=1$, we can use the indicial equation as calculated in [**?**, **?**], lemma 1.3.2. The equation

$$\begin{aligned} &\prod_1^n(x\frac{d}{dx}-\alpha_i)-x\prod_1^n(x\frac{d}{dx}-\beta_j) \\ &\quad=(1-x)(x\frac{d}{dx})^n+(-\sum\alpha_i+x\sum\beta_j)(x\frac{d}{dx})^{n-1}+R(x,x\frac{d}{dx}) \end{aligned} \tag{3.3.4}$$

where R has degree $\leq n-2$ in $x\frac{d}{dx}$. Dividing through by $x^n(1-x)$ yields

$$(\frac{d}{dx})^n + \frac{(-\sum \alpha_i + x\sum \beta_j)}{(1-x)x}(\frac{d}{dx})^{n-1} + \mathcal{R} \tag{3.3.5}$$

Here $\mathcal{R}$ involves terms T such that either T has degree $\leq n-2$ in $\frac{d}{dx}$ and $(1-x)T$ is regular at $x=1$ or $(1-x)T$ vanishes at $x=1$. It now follows from Beukers calculation that the indicial polynomial in this case is

$$t(t-1)\cdots(t-n+1) - (-\sum \alpha_i + \sum \beta_j)t(t-1)\cdots(t-n+2) \tag{3.3.6}$$

The zeroes of the indicial polynomial, in this case

$$t = 0, 1, \ldots, n-2, (-\sum \alpha_i + \sum \beta_j) + n - 1,$$

are the local exponents. □

We want to understand local solutions to (3.3.4) near $x = 1$.

Proposition 3.3.2. *Consider a differential equation of the form*

$$p_n(x)y^{(n)} + \cdots + p_1(x)y'(x) + p_0(x)y(x) = 0. \tag{3.3.7}$$

Suppose all the $p_i(x)$ are analytic at $x = a$ and that $p_n(x)$ has a simple zero there. Then the differential equation has $n-1$ independent holomorphic solutions in a neighborhood of $x = a$.

Proof. One writes $s(x) = \sum_{k\geq 0} f_k(x-a)^k$. Let $p_i(x) = \sum_{j\geq 0} p_{ij}(x-a)^j$ be Taylor series expansions of the coefficients of the equation. Solving recursively for the f_k making $s(x)$ a solution, one finds an $n-1$-dimensional solution space. Then one checks using classical estimates that these solutions are analytic at $x = a$. Details are left for the reader. □

Consider now the action of $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, p_0)$ on the solution space $V(\alpha, \beta)$ as above. Let h_0, h_1, h_∞ be the automorphisms of V corresponding to loops around $0, 1, \infty$, oriented in such a way that $h_0h_1 = h_\infty$. The eigenvalues of h_0 and h_∞ are $\exp(2\pi i\alpha_j)$ and $\exp(2\pi i\beta_k)$ respectively.

Definition 3.3.3. *A pseudo-reflection in $Aut(V)$ is an element h such that $h - Id$ has rank 1.*

As a consequence of proposition 3.3.2, $h_1 - Id$ has rank ≤ 1. Assuming that the last eigenvalue $\exp(2\pi i\gamma)$ calculated in proposition 3.3.1 is not 1, we will have that h_1 is a pseudo-reflection.

Lemma 3.3.4. *Let $H \subset GL_n(\mathbb{C})$ be a group generated by matrices A, B. Assume AB^{-1} is a pseudo-reflection. Then H acts irreducibly on $\mathbb{C}^n$ if and only if A and B have distinct sets of eigenvalues.*

Proof. Our assumption implies that $A - B$ has rank 1. If we have $0 \subsetneq V_1 \subsetneq \mathbb{C}^n$ stabilized by A and B, we must have that either $A - B|V_1$ has rank 1 or $A = B$ on V_1. In the latter case, A and B have a common eigenvalue. In the former case, A and B agree on $\mathbb{C}^n/V_1$ and again they have a common eigenvalue.

Conversely, if A, B have a common eigenvalue λ, write $W = \ker(A - B) \cong \mathbb{C}^{n-1}$. Let $v \in \mathbb{C}^n$ be an eigenvector of A with eigenvalue λ. If $v \in W$ then v is also an eigenvector of B with eigenvalue λ, so $\mathbb{C} \cdot v \subset \mathbb{C}^n$ is stable under H. If no such eigenvector lies in W, then $\ker(A - \lambda) \cap W = (0)$. Since $\ker(A - \lambda) \neq (0)$ by assumption and $W \subset \mathbb{C}^n$ has codimension 1, we must have $(0) \neq \mathrm{image}(A - \lambda) = (A - \lambda)(W)$. Similarly we can assume $\ker(B - \lambda) \cap W = (0)$, so $(0) \neq \mathrm{image}(B - \lambda) = (B - \lambda)(W)$. But then $(A - \lambda)W = (B - \lambda)W = (A - \lambda)\mathbb{C}^n = (B - \lambda)\mathbb{C}^n$ is stable under H, so H does not act irreducibly. □

Proposition 3.3.5. *(Levelt) Let $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{C}^\times$ be such that $a_i \neq b_j$, $\forall i, j$. Then there exists a pair $A, B \in GL_n(\mathbb{C})$ with eigenvalues the a's (resp. b's) such that AB^{-1} is a pseudo-reflection. The pair A, B is unique upto conjugation.*

Proof. Let A, B be given by the matrices

$$\begin{pmatrix} 0 & 0 & \dots & 0 & -A_n \\ 1 & 0 & \dots & 0 & -A_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -A_1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \dots & 0 & -B_n \\ 1 & 0 & \dots & 0 & -B_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -B_1 \end{pmatrix}. \tag{3.3.8}$$

It is straightforward to check that these matrices have the desired properties.

To check uniqueness, given A, B satisfying the conditions of the proposition, we exhibit a basis in which A, B have the above shape. Take $W = \ker(A - B) \cong \mathbb{C}^{n-1}$. define

$$V := W \cap A^{-1}W \cap \dots \cap A^{-(n-2)}W.$$

If $\dim V \geq 2$, then $\exists\, 0 \neq v \in V \cap A^{-(n-1)}W$. The vectors $v, Av, \dots, A^{n-1}v$ would span an A-stable subspace $S \subset W$. Since $A = B$ on W, S would be stable under A, B and $A = B$ on S. Thus, A, B would have identical eigenvalues on S, contradicting our assumption. It follows that $\dim V = 1$. Taking $0 \neq v \in V$, we have that $v, Av, \dots, A^{n-1}v$ form a basis of $\mathbb{C}^n$ with respect to which A, B have the form (3.3.8). □

The idea now is to apply these results to the hypergeometric group $H(a, b)$ generated by $A = h_\infty$ and $B = h_0^{-1}$. In particular, we would like to know when $H(a, b)$ is a finite group. Of course, for this to happen, a necessary condition is that the eigenvalues a_j, b_k be roots of unity.

The following two linear algebra lemmas appear in [**?**] (Lemmas 4.1, 4.2).

Lemma 3.3.6. *Let $P, Q \in GL_n(\mathbb{C})$ be invertible matrices which have the same characteristic polynomial. Assume that P is regular, so $\exists v \in \mathbb{C}^n$, $v, Pv, P^2v, \ldots, P^{n-1}v$ span $\mathbb{C}^n$. Then the space of $X \in M_n(\mathbb{C})$ such that $QX = XP$ has dimension $\geq n$.*

Proof. Exercise. $\square$

Lemma 3.3.7. *Let*

$$g = \begin{pmatrix} 0 & 0 & \ldots & g_n \\ 1 & 0 & \ldots & g_{n-1} \\ 0 & 1 & \ldots & g_{n-2} \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & g_1 \end{pmatrix}$$

Let $X = (X_{ij}) \in M_n(\mathbb{C})$ satisfy $g^t X \overline{g} = X$. Then the entries X_{ij} depend only on $i - j$.

Proof. Write $g = T + \gamma$ where $T = \sum_1^{n-1} e_{i+1,i}$ and γ has the g_i in the last column and zeroes elsewhere. (Here e_{ij} is the $n \times n$ matrix with 1 in the (i, j)-th place and zeroes elsewhere.) Thus for any $n \times n$ matrix Y, $\gamma^t Y$ has non-zero entries only in the last row. By the same token, $Y\overline{\gamma}$ has non-zero entries only in the last column. It follows that

$$g^t X \overline{g} = T^t X T + Z$$

where Z has non-zero entries only in the n-th row and column. The assertion follows by computing $W := T^t X T - X$ and setting all entries $w_{ij} = 0$ for $i, j \leq n - 1$. $\square$

Theorem 3.3.8 ([], theorem 4.3)**.** *Let $H(a, b) \subset GL_n(\mathbb{C})$ be the hypergeometric group with parameters $\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_n\}$. Assume both sets are invariant under the substitution $z \mapsto \overline{z}^{-1}$. Then there exists a non-degenerate hermitian form $F(x, y)$ on $\mathbb{C}^n$ which is invariant under $H(a, b)$, viz. $F(hx, hy) = F(x, y)$ for all $h \in H(a, b)$ and $\overline{F} = F^t$.*

Proof. It suffices to find an $F = (F_{ij})$ invariant for $h = h_0^{-1}, h_\infty$. Writing h_0^{-1}, h_∞ in the form given in proposition 3.3.5, we find that the entries of (F_{ij}) depend only on $i-j$. That means that there is a subspace of $M_n(\mathbb{C})$ of $\mathbb{C}$-dimension $2n-1$ containing any F invariant under h_∞ and any F invariant under h_0^{-1}. The condition for F can be written

$$F\bar{h} = (h^t)^{-1}F; \quad h = h_\infty, h_0^{-1}. \tag{3.3.9}$$

Since the eigenvalues are assumed invariant under $z \mapsto \bar{z}^{-1}$, it follows that $\bar{h}$ and $(h^t)^{-1}$ have the same characteristic polynomials. By lemma 3.3.6, it follows that the solutions for $h = h_0^{-1}, h_\infty$ each have dimension $\geq n$. Since they lie in a space of dimension $2n-1$, they must have a non-trivial intersection. Thus, there exists an F which is invariant under both h_0^{-1} and h_∞. Finally, if F satisfies (3.3.9), so does $\overline{F}^t$. Thus, both $F + \overline{F}^t$ and $i(F - \overline{F}^t)$ are invariant and hermitian. One of these, at least, is nonzero. □

We next investigate the signature of the hermitian form. From proposition 3.3.1, the non-trivial eigenvalue of monodromy at 1 is $c := b_1 b_2 \cdots b_n a_1^{-1} \cdots a_n^{-1}$. Let ζ be a solution of $\zeta^2 c = -1$, and define $D := \zeta(h_0 - 1)$.

Lemma 3.3.9. *There exists $u \in \mathbb{C}^n$ such that $D(x) = \pm F(x,u)u$.*

Proof. Exercise. (See [?], Prop. 4.4.) □

Theorem 3.3.10. *Assume $a_i \neq b_j$ and $|a_i| = |b_j| = 1$ for all $1 \leq i, j \leq n$. Write $a_j = \exp(2\pi i \alpha_j)$ and $b_j = \exp(2\pi i \beta_j)$ with $\alpha_j, \beta_j \in [0,1)$. Reordering, we may assume $0 \leq \alpha_1 \leq \cdots \leq \alpha_n < 1$ and $0 \leq \beta_1 \leq \cdots \leq \beta_n < 1$. Define*

$$m_j := \#\{k \mid \beta_k < \alpha_j\}.$$

Then the signature (p,q) of the invariant hermitian form for the hypergeometric group $H(a,b)$ is given by $|p-q| = |\sum_1^n (-1)^{j+m_j}|$.

Proof. Assume first the a_j are all distinct. Note the invariant form F is necessarily non-degenerate. (The null space would be stable under $H(a,b)$, contradicting proposition 3.3.4.) It follows that the eigenspace decomposition of $\mathbb{C}^n$ according to the eigenvalues a_j of h_0 is orthogonal for F. Let u be as in lemma 3.3.9, and write $u = u_1 + \cdots + u_n$ where $h_0(u_i) = a_i u_i$. Now we write with D as in lemma 3.3.9 (note

$h_1 = BA^{-1}$.)

$$\text{(3.3.10)} \quad \prod_{k=1}^{n}(b_k - t)(a_k - t)^{-1} = \det((B-t)(A-t)^{-1}) =$$
$$\det(1+\zeta^{-1}D(1-tA^{-1})^{-1}) = 1 \pm F((1-tA^{-1})^{-1}u, u) =$$
$$1 \pm \zeta^{-1}F(\sum_1^n a_j(a_j-t)^{-1}u_j, \sum_1^n u_j) = 1 \pm \zeta^{-1}\sum_1^n \frac{a_j}{a_j - t}F(u_j, u_j).$$

(Here we have used the fact that for M a rank 1 matrix given by $M(x) = w(x)u$ for some vector u and some linear form w, we have $\det(1+M) = 1 + w(u)$.)

Taking residues at $t = a_j$ yields

$$F(u_j, u_j) = \pm\zeta(b_j - a_j)a_j^{-1}\prod_{k\neq j}(b_k - a_j)(a_k - a_j)^{-1}.$$

Finally, substituting $\mp\zeta = ia_1^{1/2}\cdots a_n^{1/2}b_1^{-1/2}\cdots b_n^{-1/2}$ one finds

$$\text{(3.3.11)} \qquad F(u_j, u_j) = 2\sin\pi(\beta_j - \alpha_j)\prod_{k\neq j}\frac{\sin\pi(\beta_k - \alpha_j)}{\sin\pi(\alpha_k - \alpha_j)}.$$

The assertion of the theorem follows (in the case a_j all distinct) by computing the signs of these products. By continuity, the assertion remains true when the a_j are not all distinct. □

Corollary 3.3.11. *With assumptions as above, the hermitian form F is definite if and only if the a_j and b_k interlace on the unit circle, i.e. each a has two b's as nearest neighbors and vice versa.*

Proof. By the theorem, F is definite if and only if the numbers $j + m_j$ all have the same parity. □

We now consider the question of finiteness for the hypergeometric group $H(a, b)$. Of course, a necessary condition for finiteness is that all the eigenvalues a_j, b_k must be roots of 1. Assume this is the case, and write $\mathbb{Q}(a_j, b_k) = \mathbb{Q}(\exp(2\pi i/N))$. An immediate consequence of proposition 3.3.5 is that the matrices defining the action of $H(a, b)$ on $\mathbb{C}^n$ have coefficients in $\mathbb{Q}(\exp(2\pi i/N))$, so the galois group $(\mathbb{Z}/N\mathbb{Z})^\times$ acts in the sense that given k with $(k, N) = 1$, the $n \times n$-matrix representing an element in $H(a^k, b^k)$ is the transform by the element σ_k in the galois group of the corresponding element in $H(a, b)$. In particular, $H(a, b)$ is finite if and only if $H(a^k, b^k)$ is.

Theorem 3.3.12. *With notation as above, $H(a, b)$ is finite if and only if for all k with $(k, N) = 1$ the a_j^k, b_ℓ^k interlace.*

Proof. If $H(a,b)$ is finite, there will exist a definite F which is invariant, so the a's and b's interlace. Also $H(a,b)$ finite if and only if $H(a^k,b^k)$ finite, so the hypotheses of the theorem are necessary.

Conversely, assume the a^k, b^k interlace for all $(k,N)=1$. Consider the map ($\phi(N) = \#(\mathbb{Z}/N\mathbb{Z})^\times$)

$$\prod_k \sigma_k : H(a,b) \to \prod_k H(a^k,b^k) \subset GL_{n\phi(N)}(\mathbb{C}) \tag{3.3.12}$$

for a suitable basis of $\mathbb{C}^{n\phi(N)}$, the image is contained in $GL_{n\phi(N)}(\mathbb{Z})$. This image leaves invariant a definite hermitian form, so it lies in a compact unitary group. It is thus discrete and contained in a compact, hence finite. $\square$