Products of CW complexes
the full story

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For $n \in \omega$, let

- D^n denote the closed ball of radius 1 about the origin in $\mathbb{R}^n$ (the *n-disc*),
- $\mathring{D}^n$ its interior (the open ball of radius 1 about the origin), and
- S^{n-1} its boundary (the $n-1$-*sphere*).

CW complexes

Definition

A Hausdorff space X is a *CW complex* if there exists a set of functions $\varphi^n_\alpha : D^n \to X$ (*characteristic maps*), for α in an arbitrary index set and $n \in \omega$ a function of α, such that:

1. $\varphi^n_\alpha \restriction \mathring{D}^n$ is a homeomorphism to its image, and X is the disjoint union as α varies of these homeomorphic images $\varphi^n_\alpha[\mathring{D}^n]$.

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We denote $\varphi^n_\alpha[\mathring{D}^n]$ by e^n_α and refer to it as an *n-dimensional cell*.

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The Cartesian product of two CW complexes X and Y, with the product topology, need not be a CW complex.

Since $D^m \times D^n \cong D^{m+n}$, there is a natural cell structure on $X \times Y$, but the product topology is generally not as fine as the weak topology.

Convention

In this talk, $X \times Y$ is always taken to have the product topology, so "$X \times Y$ is a CW complex" means "the product topology on $X \times Y$ is the same as the weak topology".

Example (Dowker, 1952)

Let X be the "star" with a central vertex e^0_X and countably many edges $e^1_{X,n}$ ($n \in \omega$) emanating from it (and the countably many "other end" vertices of those edges).

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Consider the subset of $X \times Y$

$$H = \left\{ \left(\frac{1}{f(n)+1}, \frac{1}{f(n)+1} \right) \in e^1_{X,n} \times e^1_{Y,f} : n \in \omega, f \in \omega^\omega \right\}$$

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Since every cell of $X \times Y$ contains at most one point of H, H is closed in the weak topology.

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Let $k \in \omega$ be sufficiently large that $\frac{1}{g(k)+1} \in e^1_{Y,g} \cap V$.

Then $\left(\frac{1}{g(k)+1}, \frac{1}{g(k)+1}\right) \in U \times V \cap H$. So overall, we have that in the product topology, $(e^0_X, e^0_Y) \in \bar{H}$.

Improving Dowker's example

The *unbounding number* $\mathfrak{b}$

For $f, g \in \omega^\omega$, write $f \leq^* g$ if for all but finitely many $n \in \omega$, $f(n) \leq g(n)$.

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For $f, g \in \omega^\omega$, write $f \leq^* g$ if for all but finitely many $n \in \omega$, $f(n) \leq g(n)$. Then $\mathfrak{b}$ is the least size of a set of functions such that no one g is $\geq^*$ them all, ie,

$$\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge \forall g \in \omega^\omega \exists f \in \mathcal{F}(f \not\leq^* g)\}.$$

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$\aleph_1 \leq \mathfrak{b} \leq 2^{\aleph_0}$, and each of

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$$\aleph_1 = \mathfrak{b} = 2^{\aleph_0} \text{ (CH)}$$

is consistent.

Improving Dowker's example

For Dowker's example, it suffices for the bigger star to have only $\mathfrak{b}$ many edges, indexed by an unbounded set of functions $\mathcal{F}$.

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Example (*Folklore based on* Dowker, 1952)

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Let $g \colon \omega \to \omega \setminus \{0\}$ be an increasing function such that $[0, 1/g(n)) \subset e^1_{X,n} \cap U$ for every $n \in \omega$. *Take $f \in \mathcal{F}$ such that $f \not\leq^* g$.*

Let $k \in \omega$ be *such* that $\frac{1}{f(k)+1} \in e^1_{Y,f} \cap V$ *and* $f(k) > g(k)$.

Then $\left(\frac{1}{f(k)+1}, \frac{1}{f(k)+1} \right) \in U \times V \cap H$. So overall, we have that in the product topology, $(e^0_X, e^0_Y) \in \bar{H}$.

More preliminaries: subcomplexes

A *subcomplex* A of a CW complex X is a subspace which is a union of cells of X, such that if $e^n_\alpha \subseteq A$ then its closure $\bar{e^n_\alpha} = \varphi^n_\alpha[D^n]$ is in A.

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Definition

Let κ be a cardinal. We say that a CW complex X is *locally less than κ* if for all x in X there is a subcomplex A of X with fewer than κ many cells such that x is in the interior of A. We write *locally finite* for locally less than $\aleph_0$, and *locally countable* for locally less than $\aleph_1$.

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If X and Y are both locally countable, the $X \times Y$ is a CW complex.

Theorem (Y. Tanaka, 1982)

If neither X nor Y is locally countable, then $X \times Y$ is not a CW complex.

What was known, beyond ZFC

Theorem (Liu Y.-M., 1978)

Assuming CH, $X \times Y$ is a CW complex if and only if one of them is locally finite, or both are locally countable.

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Theorem (Y. Tanaka, 1982)

Assuming $\mathfrak{b} = \aleph_1$, $X \times Y$ is a CW complex if and only if one of them is locally finite, or both are locally countable.

A complete characterisation

Theorem (B.-T.)

Let X and Y be CW complexes. Then $X \times Y$ is a CW complex if and only if one of the following holds:

1. *X or Y is locally finite.*
2. *One of X and Y is locally countable, and the other is locally less than $\mathfrak{b}$.*

Proof

Most cases are dealt with by following result of Tanaka.

Theorem (Tanaka)

The following are equivalent.

1. $\kappa \geq \mathfrak{b}$
2. *If* $X \times Y$ *is a CW complex, then either*
   - A. X *or* Y *is locally finite, or*
   - B. X *or* Y *is locally countable and the other is locally less than* κ.

So taking $\kappa = \mathfrak{b}$, it suffices to show that (A)$\vee$(B) implies $X \times Y$ is a CW complex. We know that (A) implies $X \times Y$ is a CW complex, so it suffices to show that (B) implies $X \times Y$ is a CW complex.

So suppose X is locally countable and Y is locally less than $\mathfrak{b}$. We shall show that $X \times Y$ is a CW complex, ie, that the product topology on it is the same as the weak topology.

Topologies

Any compact subset of a CW complex X is contained in finitely many cells, and each closed cell $\bar{e}^n_\alpha$ is compact. So requiring X to have the weak topology is equivalent to requiring that the topology be *compactly generated*: a set is closed if and only if its intersection with every compact set is closed.

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Any sequential space is compactly generated. Since D^n is sequential for every n, we have that CW complexes are sequential.

On with the proof

We shall show that our $X \times Y$ is sequential. So suppose $H \subset X \times Y$ is sequentially closed, and $(x_0, y_0) \in X \times Y \setminus H$; we shall find open neighbourhoods U of x_0 in X and V of y_0 in Y such that $U \times V \cap H = \emptyset$.

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By moving if necessary to subcomplexes with x_0 and y_0 in their respective interiors, we may assume that X has countably many cells and Y has fewer than $\mathfrak{b}$ many. Enumerate the cells of X as $e^{m(i)}_{X,i}$ for $i \in \omega$, and the cells of Y as $e^{n(\alpha)}_{Y,\alpha}$ for ordinals $\alpha \in \mu$ for some $\mu < \mathfrak{b}$.

A neighbourhood base for x_0

Neighbourhoods in a single cell

Suppose $n \in \omega$ and w is in a cell e^d with characteristic map φ of a CW complex W. Let $\vec{z} = \varphi^{-1}(w) \in D^d \subset \mathbb{R}^d$. We define $B_n^\varphi(w)$ to be the image under φ of the open ball $B_r(\vec{z})$ in $\mathbb{R}^d$, where r is the minimum of $1/(n+1)$ and half the distance from $\vec{z}$ to the boundary of D^d.

Let $e^{m(i_0)}_{X,i_0}$ be the open cell of X containing x_0.

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- As $U_f \cap e^{m(i_0)}_{X,i_0}$, we take $B^{\varphi^{m(i_0)}_{X,i_0}}_{f(i_0)}(x_0)$.

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- As $U_f \cap e_{X,i_0}^{m(i_0)}$, we take $B_{f(i_0)}^{\varphi_{X,i_0}^{m(i_0)}}(x_0)$.
- For cells of dimension $\geq m(i_0)$ we proceed by induction on dimension:

A neighbourhood base for x_0

Suppose $U_f \cap X^m$ has been defined and e_i^{m+1} is an $m+1$-cell of X. Let $V_i = (\varphi_i^{m+1})^{-1}[U_f \cap X^m] \subseteq S^m \subset D^{m+1} \subset \mathbb{R}^{m+1}$. Then let

$$W_i = \{t \cdot \vec{z} \in D^{m+1} : t \in (1 - \frac{1}{f(i)+1}, 1] \wedge \vec{z} \in V_i\}$$

(where the multiplication $\cdot$ is scalar multiplication in the real vector space $\mathbb{R}^{m+1}$), and take $U_f \cap \bar{e}_i^{m+1} = \varphi_i^{m+1}[W_i]$.

Since this defines an open set in every cell of X, it defines an open set U_f in X.

A neighbourhood base for x_0

These neighbourhoods do define a neighbourhood base: given $x \in U \subseteq X$, we may inductively (on dimension m) choose values of $f(i)$ such that $U_f \cap X^m$ has *closure* contained in $U \cap X^m$, and then local compactness ensures the process can continue.

Back to the proof

Recall H sequentially closed, $(x_0, y_0) \notin H$, $x_0 \in e_{X,i_0}^{m(i_0)}$. Say $y_0 \in e_{Y,\alpha_0}^{n(\alpha_0)}$.

We shall actually construct a $g\colon \omega \to \omega$ and an open V in Y such that $(x_0, y_0) \in U_g \times V \subset X \times Y$ and $\bar{U}_g \times \bar{V} \cap H = \emptyset$.

The construction is by induction on dimension.

The base case

$e_{X,i_0}^{m(i_0)}$ and $e_{Y,\alpha_0}^{n(\alpha_0)}$ lie in finite subcomplexes X_0 and Y_0 of X and Y respectively.

Since $X_0 \times Y_0$ is a CW complex, it is sequential, so $H \cap X_0 \times Y_0$ is closed in $X_0 \times Y_0$. So there is an $f(i_0) \in \omega$ and a $V_{\alpha_0} \subset e_{Y,\alpha_0}^{n(\alpha_0)}$ open in $e_{Y,\alpha_0}^{n(\alpha_0)}$ such that

$$(x_0, y_0) \in B_{f(i_0)}^{\varphi_{X,i_0}^{m(i_0)}}(x_0) \times V \subset e_{X,i_0}^{m(i_0)} \times e_{Y,\alpha_0}^{n(\alpha_0)}$$

and

$$H \cap \bar{B}_{f(i_0)}^{\varphi_{X,i_0}^{m(i_0)}}(x_0) \times \bar{V} = \emptyset.$$

The inductive step

Suppose $U_f \cap X^{m(i_0)+k}$ and $V \cap Y^{n(\alpha_0)+k}$ have been defined such that

$$(U_f \cap \bar{X}^{m(i_0)+k}) \times (V \cap \bar{Y}^{n(\alpha_0)+k}) \cap H = \emptyset.$$

Consider those $(m(i_0) + k + 1)$-cells of X whose boundaries intersect $U_f \cap X^{m(i_0)+k}$ — there are countably many of them.

Young Set Theory 2017

Registration is now open (until March 31) for Young Set Theory 2017!

New directions in the higher infinite, ICMS Edingburgh, 10–14 July 2017

http://www.icms.org.uk/workshop.php?id=415
(Google "higher infinite ICMS")