MULTIPLICATION OF DISTRIBUTIONS

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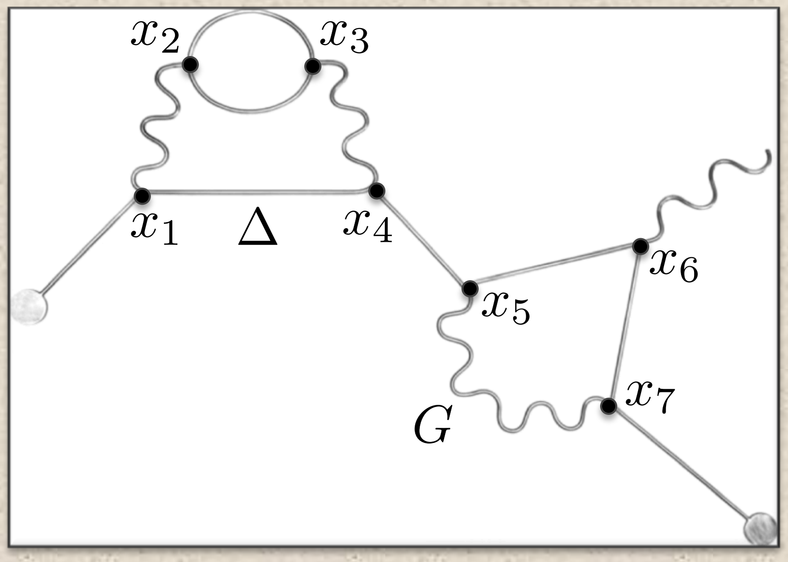
Quantum Field Theory

- Feynman diagram

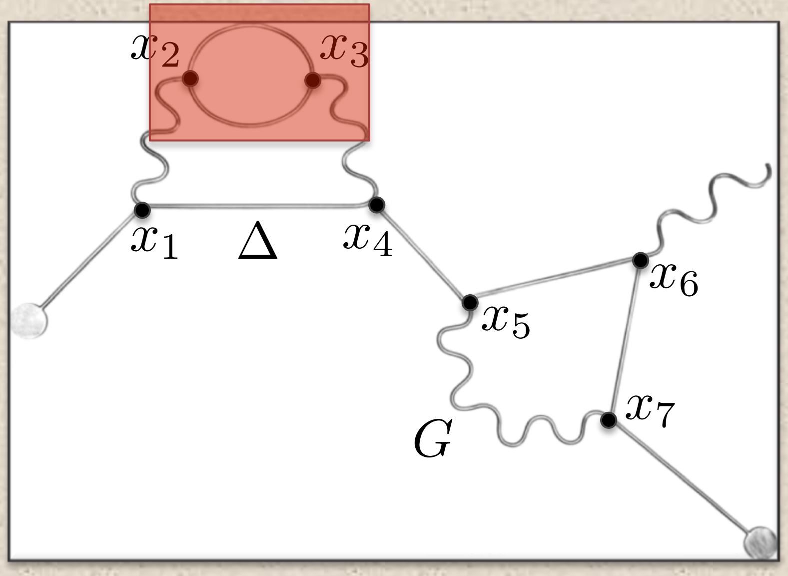


- Feynman amplitude

$$G(x_1, x_2)\Delta(x_2, x_3)^2 G(x_3, x_4)\Delta(x_1, x_4)\Delta(x_4, x_5)\Delta(x_5, x_6)\Delta(x_6, x_7)G(x_5, x_7)$$

Quantum Field Theory

- Feynman diagram

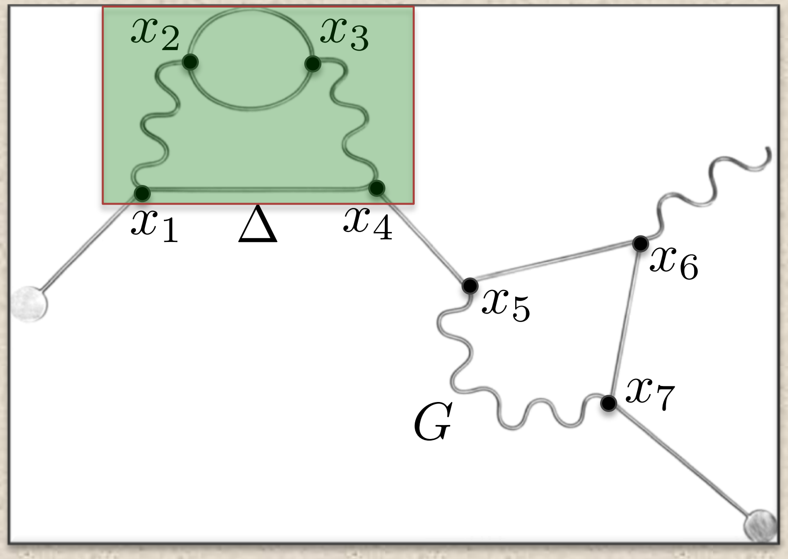


- Feynman amplitude

$$G(x_1, x_2)\underline{\Delta(x_2, x_3)}^2 G(x_3, x_4)\Delta(x_1, x_4)\Delta(x_4, x_5)\Delta(x_5, x_6)\Delta(x_6, x_7)G(x_5, x_7)$$

Quantum Field Theory

- Feynman diagram

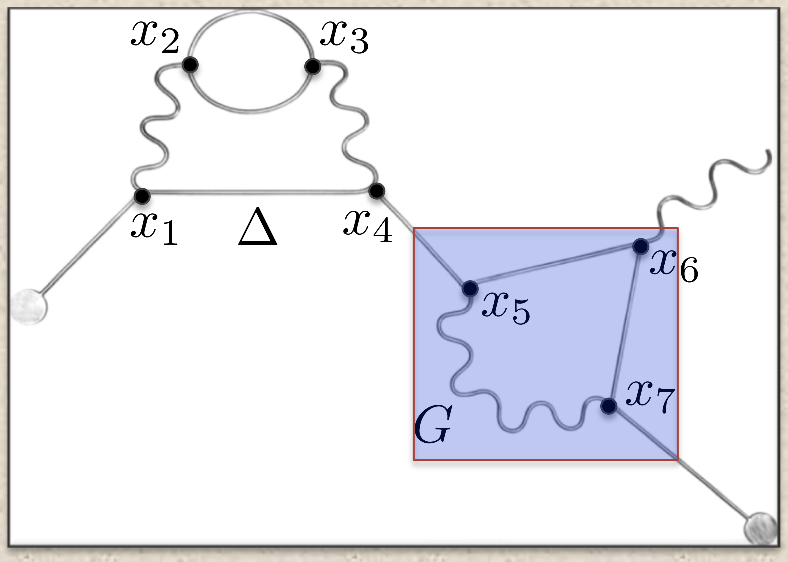


- Feynman amplitude

$$\underline{G(x_1,x_2)\Delta(x_2,x_3)^2 G(x_3,x_4)\Delta(x_1,x_4)}\Delta(x_4,x_5)\Delta(x_5,x_6)\Delta(x_6,x_7)G(x_5,x_7)$$

Quantum Field Theory

- Feynman diagram

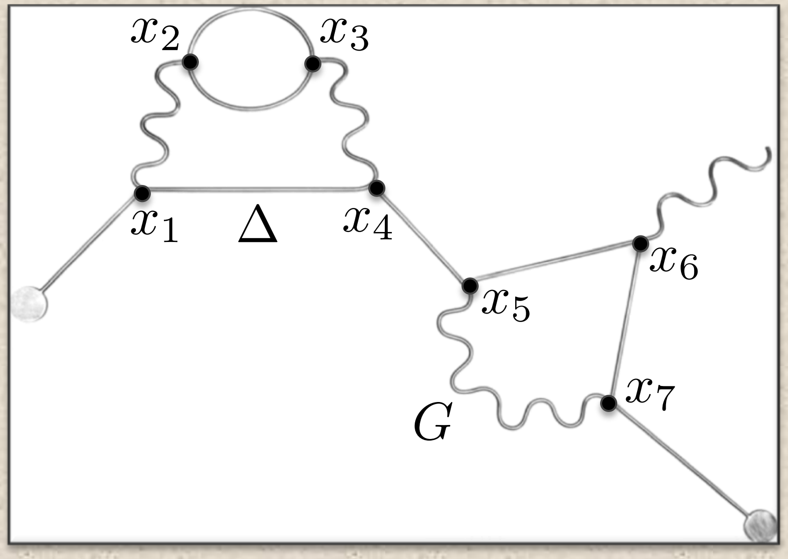


- Feynman amplitude

$$G(x_1,x_2)\Delta(x_2,x_3)^2 G(x_3,x_4)\Delta(x_1,x_4)\Delta(x_4,x_5)\underline{\Delta(x_5,x_6)\Delta(x_6,x_7)G(x_5,x_7)}$$

Quantum Field Theory

- Feynman diagram



- Feynman amplitude

$$G(x_1, x_2)\Delta(x_2, x_3)^2 G(x_3, x_4)\Delta(x_1, x_4)\Delta(x_4, x_5)\Delta(x_5, x_6)\Delta(x_6, x_7)G(x_5, x_7)$$

- Multiply distributions on the largest domain where this is well defined $\mathcal{D}(\mathbb{R}^{7d} \setminus \{x_i = x_j\})$
- Renormalization: extend the result to $\mathcal{D}(\mathbb{R}^{7d})$

Algebraic Quantum Field Theory

- Multiplication of distributions
  - Motivation
  - The wave front set of a distribution
  - Application and topology
- Extension of distributions (Viet)
  - Renormalization as the solution of a functional equation
  - The scaling of a distribution
  - Extension theorem
- Renormalization on curved spacetimes (Kasia)
  - Epstein-Glaser renormalization
  - Algebraic structures (Batalin-Vilkovisky, Hopf algebra)
  - Functional analytic aspects

- **Joint work with Yoann Dabrowski, Nguyen Viet Dang and Frédéric Hélein**



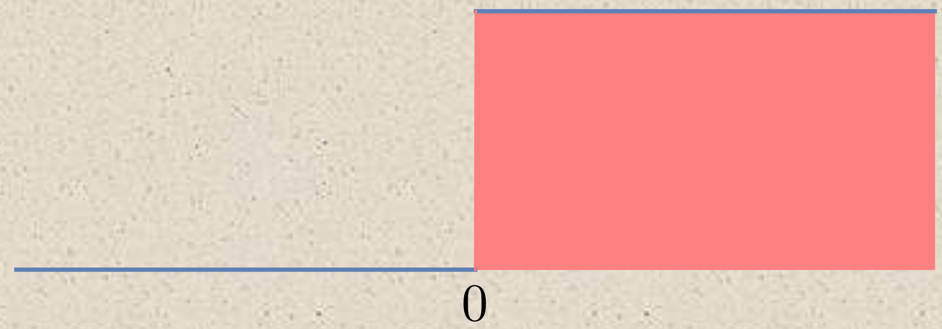
Outline

- Trying to multiply distributions
  - Singular support
  - Fourier transfom
- The wave front set
  - Examples
  - Characteristic functions
  - Hörmander’s theorem for distribution products
- Examples in quantum field theory
- Topology

MULTIPLY DISTRIBUTIONS

- Heaviside step function

$$\theta(x) = 0 \text{ for } x < 0,$$
$$\theta(x) = 1 \text{ for } x \geq 0.$$



- As a function $\theta^n = \theta$
- Heaviside distribution

$$\langle \theta, f \rangle = \int_{-\infty}^{\infty} \theta(x) f(x) dx = \int_{0}^{\infty} f(x) dx$$

- If $\theta^n = \theta$ then $n\theta^{n-1}\delta = \delta$ and $n\theta\delta = \delta$ for $n \geqslant 2$

REGULARIZATION

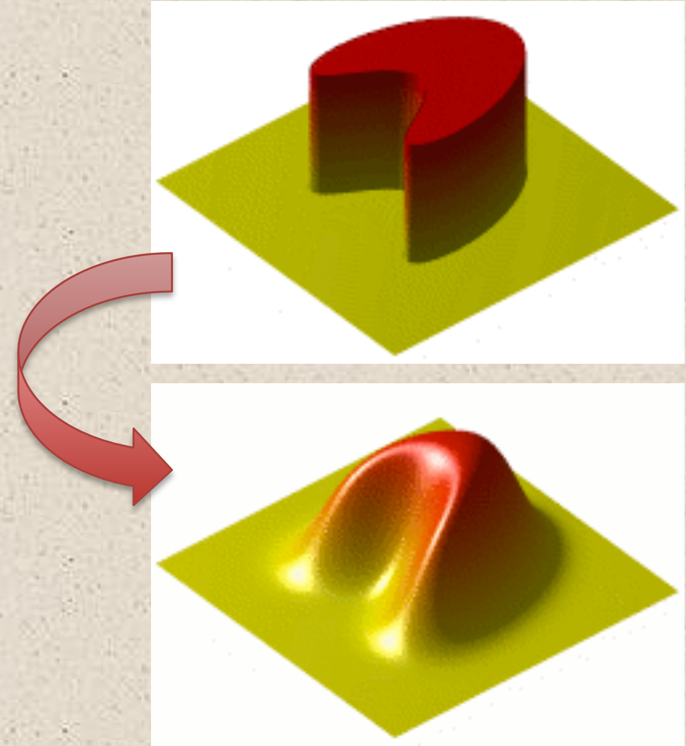
- *Mollifier* φ such that $\int \varphi(x)dx = 1$
- Distributions are *mollified* by convolution with $\delta_\epsilon(x) = \frac{1}{\epsilon^d}\varphi\left(\frac{x}{\epsilon}\right)$
- Mollified Heaviside distribution

$$\theta_\epsilon(x) = \int_{-\infty}^{x} \delta_\epsilon(y)dy$$

- Then,

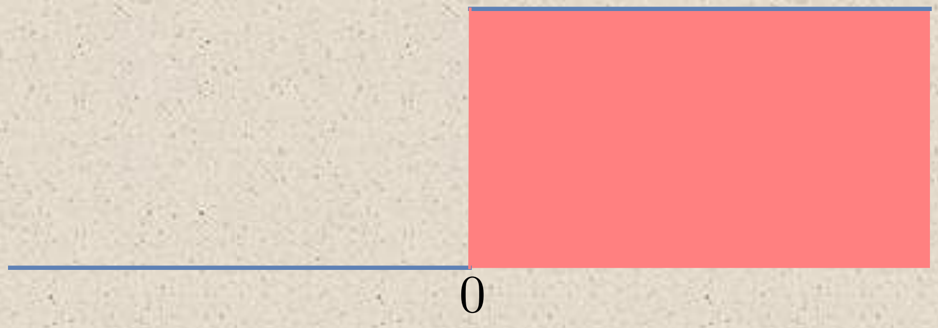
$$\theta\delta = \lim_{\epsilon \to 0} \theta_\epsilon \delta_\epsilon = \frac{1}{2}\delta$$

- But $\delta^2 = \lim_{\epsilon \to 0} \delta_\epsilon^2$ diverges
- Very heavy calculations (Colombeau generalized functions)

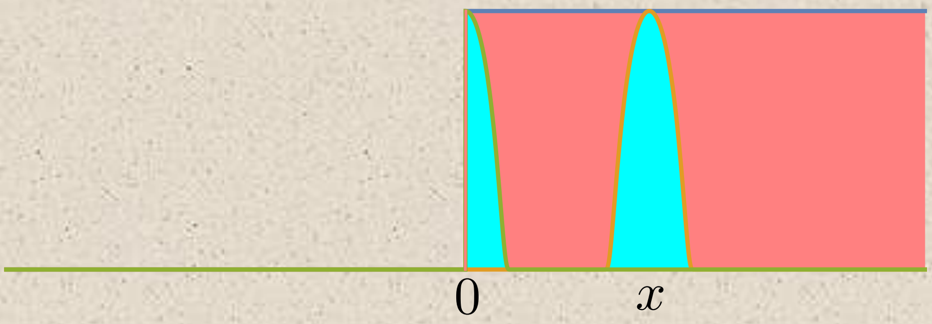


Singular support

- How detect a singular point in a distribution u ?



- Multiply by a smooth function $g \in \mathcal{D}(M)$ around $x \in M$



- Look whether gu is smooth or not

Singular support

- Let u be a distribution on $M = \mathbb{R}^d$ and $g \in \mathcal{D}(M)$ such that gu is a smooth function. For $e_\xi(x) = e^{i\xi\cdot x}$

$$g(x)u(x) = \langle gu, \delta_x \rangle = \int \frac{d\xi}{(2\pi)^d} \langle gu, e_\xi \rangle e^{-i\xi\cdot x}$$

- All the derivatives of gu exist:

$$\forall N, \exists C_N, s.t. \forall \xi, \quad |\langle gu, e_\xi \rangle| \leqslant \boxed{C_N(1+|\xi|)^{-N}}$$

- The **singular support** of u is the complement of the set of points $x \in M$ such that there is a $g \in \mathcal{D}(M)$ with gu a smooth function and $g(x) \neq 0$

EASY PRODUCTS

- You can multiply a distribution u and a smooth function f

$$\langle fu, g\rangle = \langle u, fg\rangle$$

- You can multiply two distributions u and v with disjoint singular supports

$$\langle uv, g\rangle = \langle u, vfg\rangle + \langle v, u(1-f)g\rangle$$

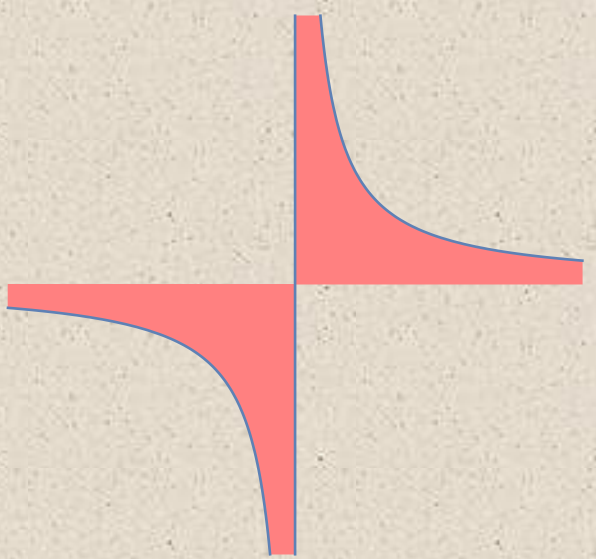
where

  - $f = 0$ on a neighborhood of the singular support of v
  - $f = 1$ on a neighborhood of the singular support of u

HARD PRODUCTS

- Product of distributions with common singular support
- Consider

$$u_+(x) = \frac{1}{x - i0^+} = i\int_0^\infty e^{-ik\xi}d\xi$$



- More precisely

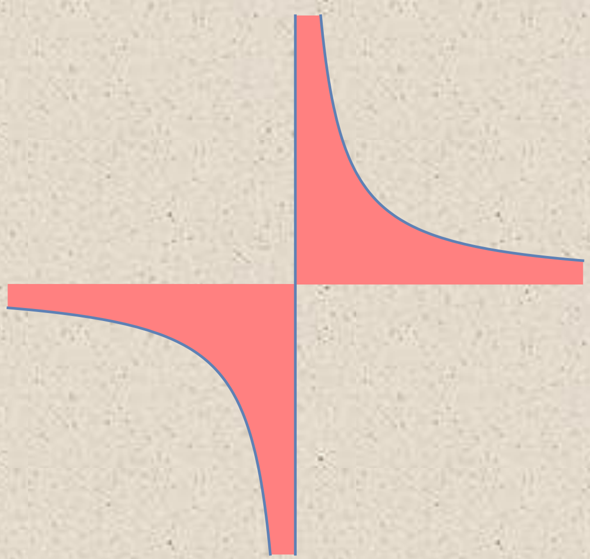
$$\langle u_+, g\rangle = i\int_0^\infty \hat{g}(-\xi)d\xi$$

- Its singular support is $\Sigma(u_+) = \{0\}$

Hard Products

- Product of distributions with common singular support
- Consider also

$$u_-(x) = \frac{1}{x + i0^+} = -i\int_0^\infty e^{ik\xi}d\xi$$



- More precisely

$$\langle u_-, g\rangle = -i\int_0^\infty \hat{g}(\xi)d\xi$$

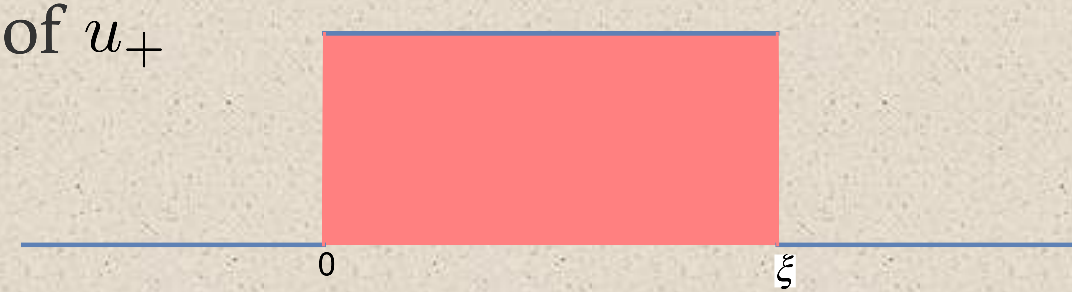
- Its singular support is $\Sigma(u_-) = \{0\}$

Fourier Transform

- Convolution theorem $\widehat{uv} = \widehat{u} \star \widehat{v}$
- Define the product by $uv = \mathcal{F}^{-1}(\widehat{u} \star \widehat{v})$
- Example

$$u_+(x) = \frac{1}{x - i0^+} \qquad \widehat{u_+}(\xi) = 2i\pi\theta(\xi)$$

- Square of u_+



$$\widehat{u_+^2}(\xi) = -2\pi \int_{\mathbb{R}} \theta(\eta)\theta(\xi - \eta) d\eta = -2\pi\xi\theta(\xi)$$

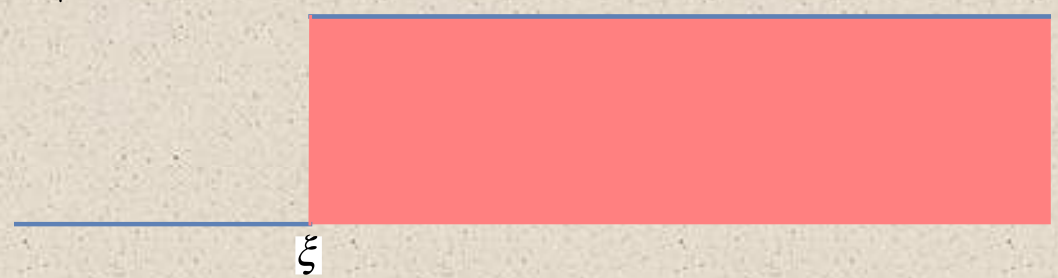
Fourier Transform

- Example

$$u_+(x) = \frac{1}{x - i0^+} \qquad \widehat{u_+}(\xi) = 2i\pi\theta(\xi)$$

$$u_-(x) = \frac{1}{x + i0^+} \qquad \widehat{u_-}(\xi) = -2i\pi\theta(-\xi)$$

- Product $u_+ u_-$

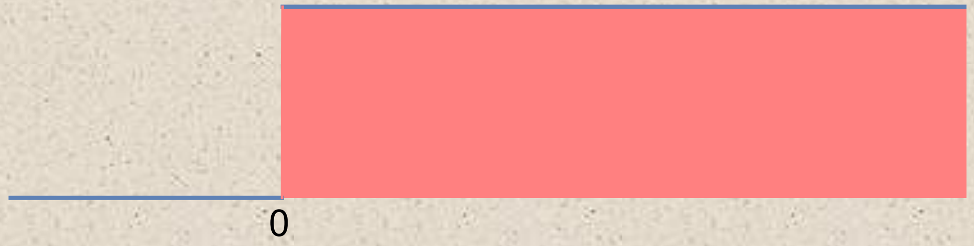


$$\widehat{u_+ u_-}(\xi) = 2\pi \int_{\mathbb{R}} \theta(\eta)\theta(\eta - \xi)d\eta \quad \text{diverges}$$

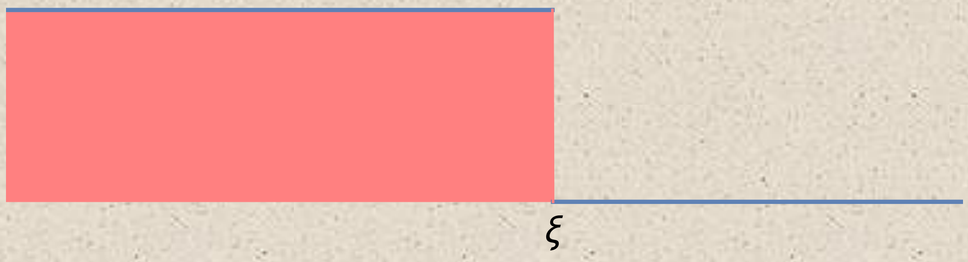
FOURIER TRANSFORM

- Interpretation

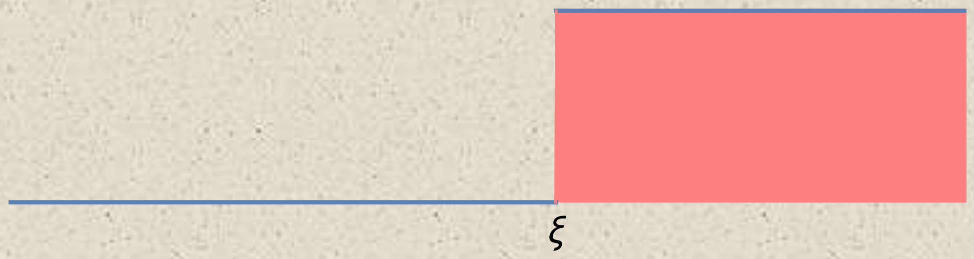
$$\widehat{u_+}(\eta)$$



$$\widehat{u_+}(\xi - \eta)$$



$$\widehat{u_-}(\xi - \eta)$$

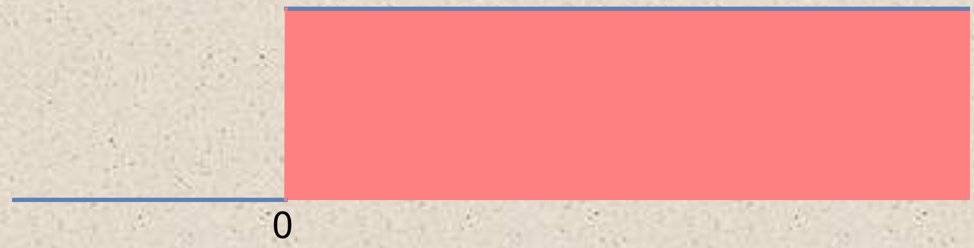


- $\widehat{u}(\eta)$ can be integrable in **some** direction
- The non-integrable directions of $\widehat{u}(\eta)$ can be compensated for by the integrable directions of $\widehat{v}(\xi - \eta)$

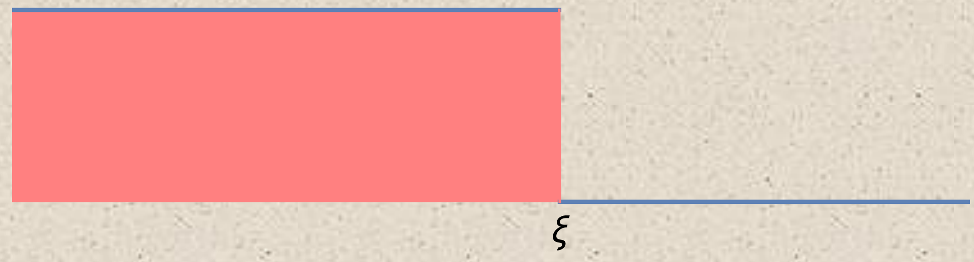
FOURIER TRANSFORM

- Interpretation

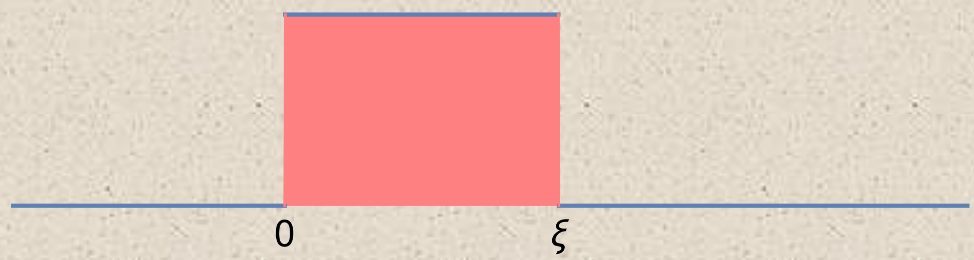
$$\widehat{u_+}(\eta)$$



$$\widehat{u_+}(\xi - \eta)$$



$$\widehat{u_+}(\eta)\widehat{u_+}(\xi - \eta)$$

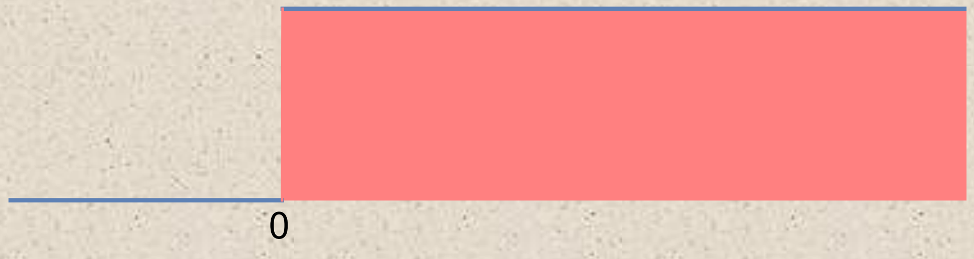


- Integrable : u_+^2 is well-defined

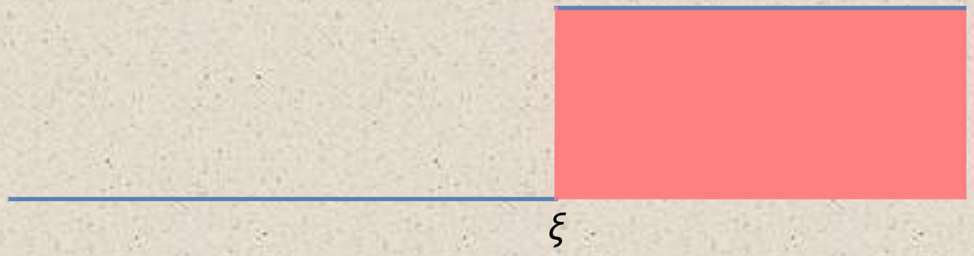
FOURIER TRANSFORM

- Interpretation

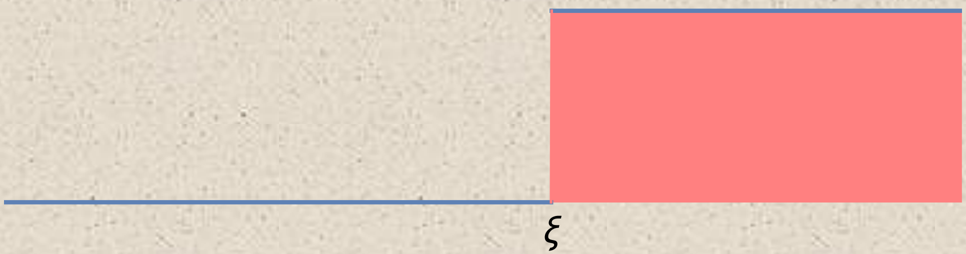
$$\widehat{u_+}(\eta)$$



$$\widehat{u_-}(\xi - \eta)$$



$$\widehat{u_+}(\eta)\widehat{u_-}(\xi - \eta)$$



- Not integrable : $u_+ u_-$ is not well-defined

Fourier Transform

- Define the product by $uv = \mathcal{F}^{-1}(\widehat{u} \star \widehat{v})$
- What if the distributions have no Fourier transform?
- The product of distributions is local: $w = uv$ near x if

  $\widehat{f^2 w} = \widehat{fu} \star \widehat{fv}$ for $f = 1$ in a neighborhood of x
- How should the integral converge?

$$\widehat{f^2 uv}(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{fu}(\eta)\widehat{fv}(\xi - \eta) d\eta$$

- Absolute convergence is not enough if we want the Leibniz rule to hold

FOURIER TRANSFORM

- How can the integral converge?

$$\widehat{f^2uv}(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{fu}(\eta)\widehat{fv}(\xi - \eta) d\eta$$

- The order of fu is finite: $|\widehat{fu}(\eta)| \leq C(1 + |\eta|)^m$
- If $\widehat{fu}(\eta)$ does not decrease along direction η , then $\widehat{fv}(\xi - \eta)$ must decrease **faster than any inverse polynomial**
- Conversely, $\widehat{fu}(\eta)$ must compensate for the directions along which $\widehat{fv}(\xi - \eta)$ does not decrease fast

OUTLINE

- Trying to multiply distributions
  - Singular support
  - Fourier transfom
- The wave front set
  - Examples
  - Characteristic functions
  - Hörmander’s theorem for distribution products
- Examples in quantum field theory
- Topology

The Wave Front Set

Mikio Sato
1928-

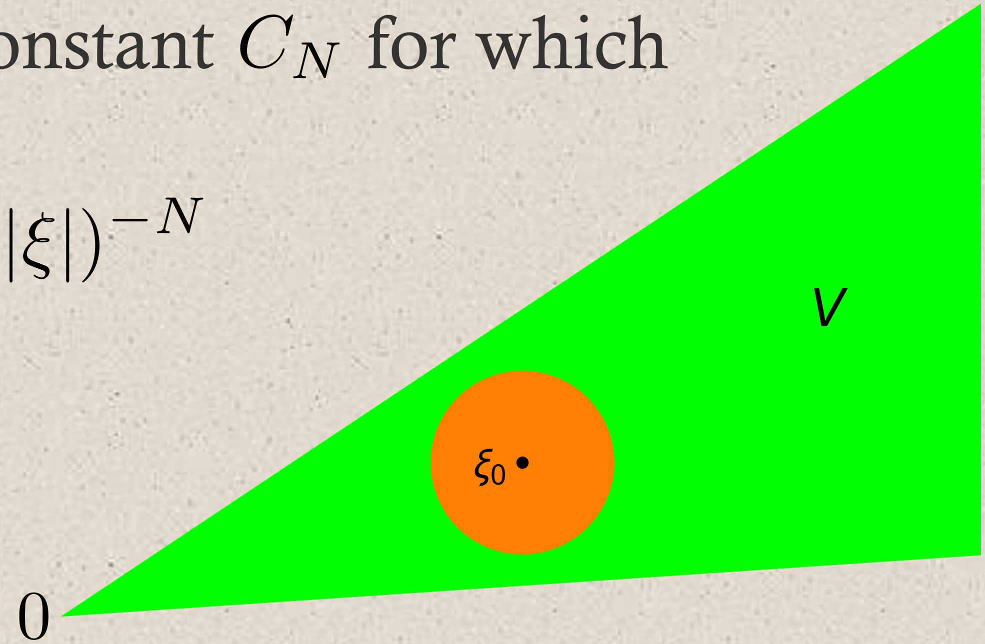
Lars Valter Hörmander
1931-2012

Wave Front Set

- A point $(x_0, \xi_0) \in T^*\mathbb{R}^d$ **does not belong** to the **wave front set** of a distribution u if there is a test function f with $f(x_0) \neq 0$ and a conical neighborhood $V \subset \mathbb{R}^d$ of ξ_0 such that, for every integer N there is a constant C_N for which

$$|\widehat{fu}(\xi)| \leq C_N (1 + |\xi|)^{-N}$$

for every $\xi \in V$



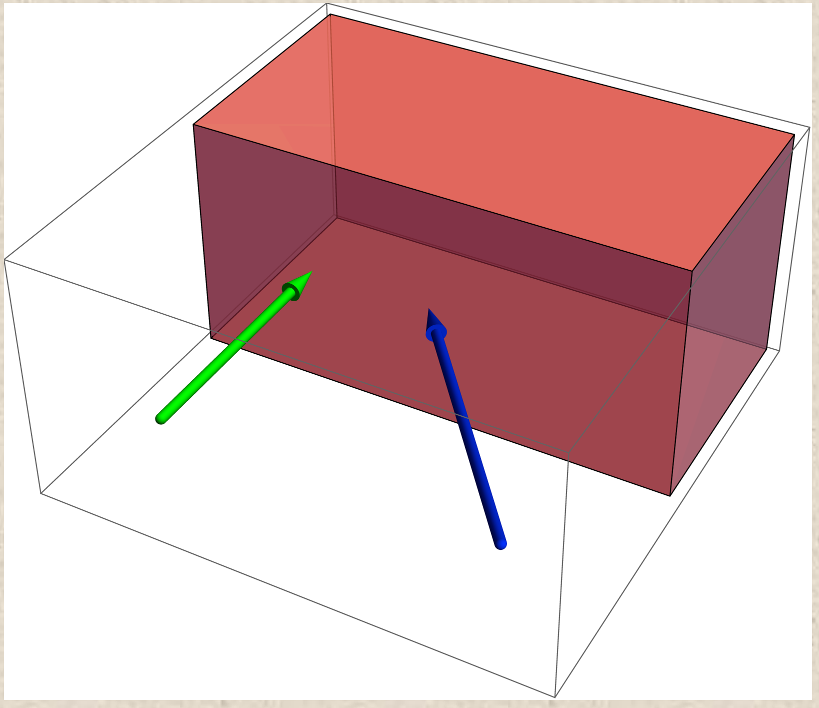
Wave Front Set

- The wave front set is a cone: if $(x, \xi) \in \mathrm{WF}(u)$, then $(x, \lambda\xi) \in \mathrm{WF}(u)$ for every $\lambda > 0$
- The wave front set is closed
- $\mathrm{WF}(u + v) \subset \mathrm{WF}(u) \cup \mathrm{WF}(v)$
- The singular support of u is the projection of $\mathrm{WF}(u)$ on the first variable

EXAMPLES

- The wavefront set describes in which direction the distribution is singular above each point of the singular support
- The Dirac δ function is singular at $x = 0$ and its Fourier transform is $\widehat{\delta}(\xi) = 1$
- Its wave front set is $\mathrm{WF}(\delta) = \{(0, \xi); \xi \neq 0\}$
- The distribution $u_+(x) = (x - i0^+)^{-1}$ is also singular at $x = 0$ but its Fourier transform is $\widehat{u_+}(\xi) = 2i\pi\theta(\xi)$
- Its wave front set is $\mathrm{WF}(u_+) = \{(0, \xi); \xi > 0\}$

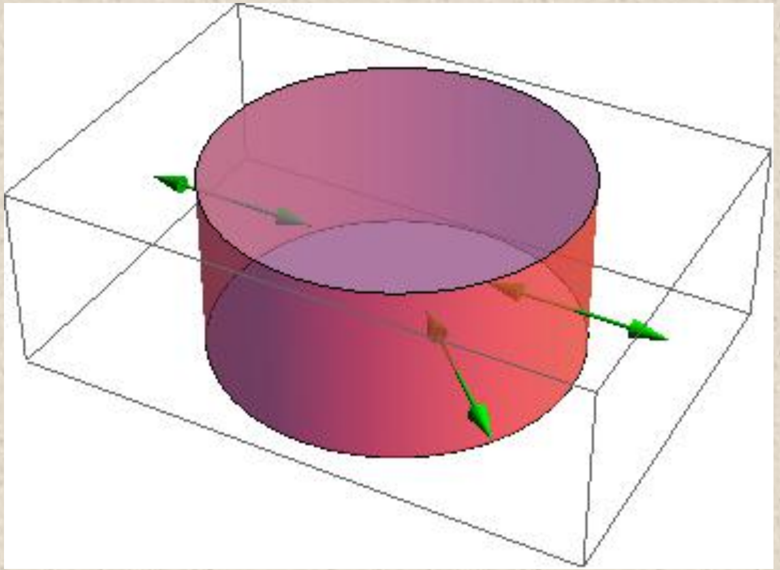
Characteristic Function



- Relation to the Radon transform

Characteristic function

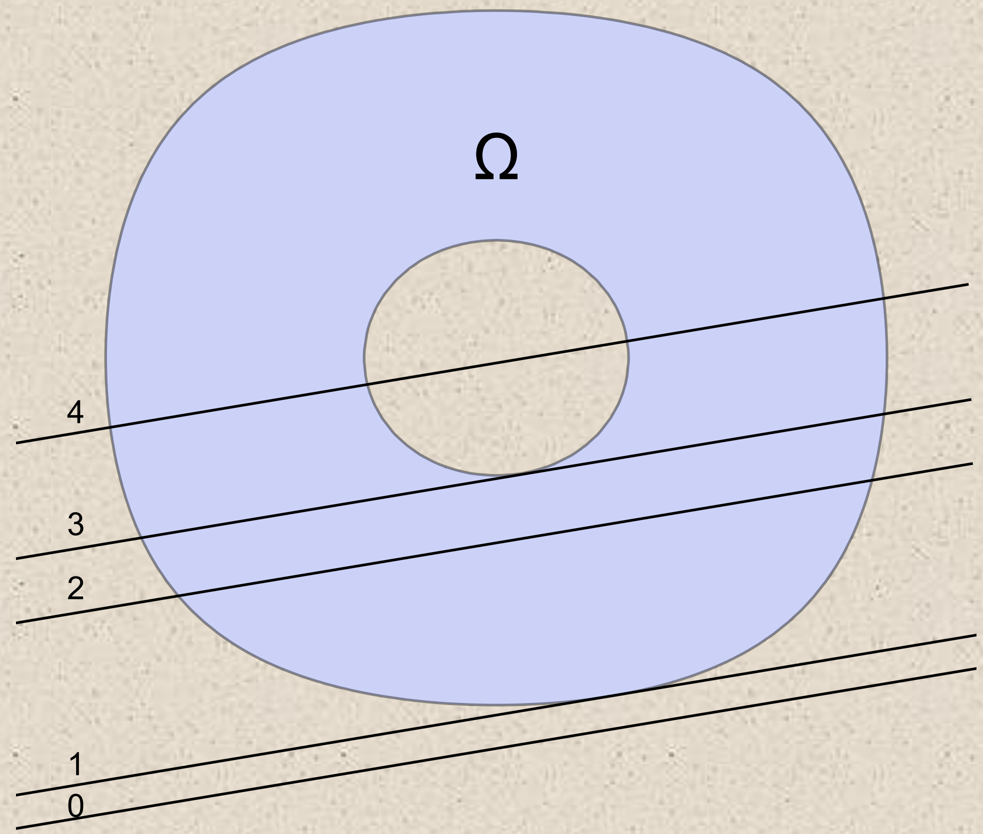
- Characteristic function of a disk: the wave front set is perpendicular to the edge



- The wave front set is used in edge detection for machine vision and image processing

Characteristic function

- Shape and wave front set detection by counting intersections



Distribution product

- Product of distributions

$$\widehat{f^2uv}(\xi) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \widehat{fu}(\eta)\widehat{fv}(\xi-\eta)d\eta$$

- Hörmander thm: The product of two distributions u and v is well defined if there is not point $(x,\xi)\in \mathrm{WF}(u)$ such that $(x,-\xi)\in \mathrm{WF}(v)$
- The wave front set of the product is

$$\mathrm{WF}(uv)\subset \mathrm{WF}(u)\oplus \mathrm{WF}(v)\cup \mathrm{WF}(u)\cup \mathrm{WF}(v)$$

$$\mathrm{WF}(u)\oplus \mathrm{WF}(v) = \{(x,\xi+\eta); (x,\xi)\in \mathrm{WF}(u) \text{ and } (x,\eta)\in \mathrm{WF}(v)\}$$

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QFT: The Causal Approach

Stueckelberg

Bogoliubov

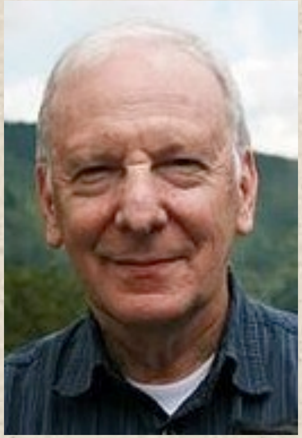
Radzikowski

Klaus Fredenhagen

Romeo Brunetti

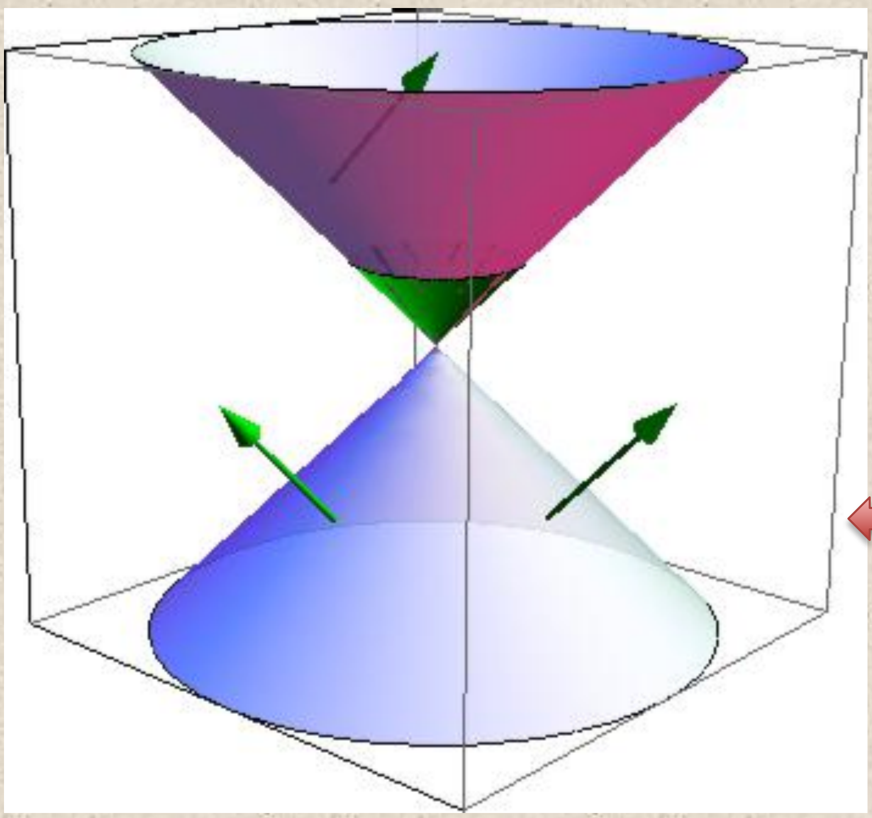
Stefan Hollands

Robert Wald



Katarzyna Rejzner

PROPAGATOR



Wightman propagator

- Product of fields

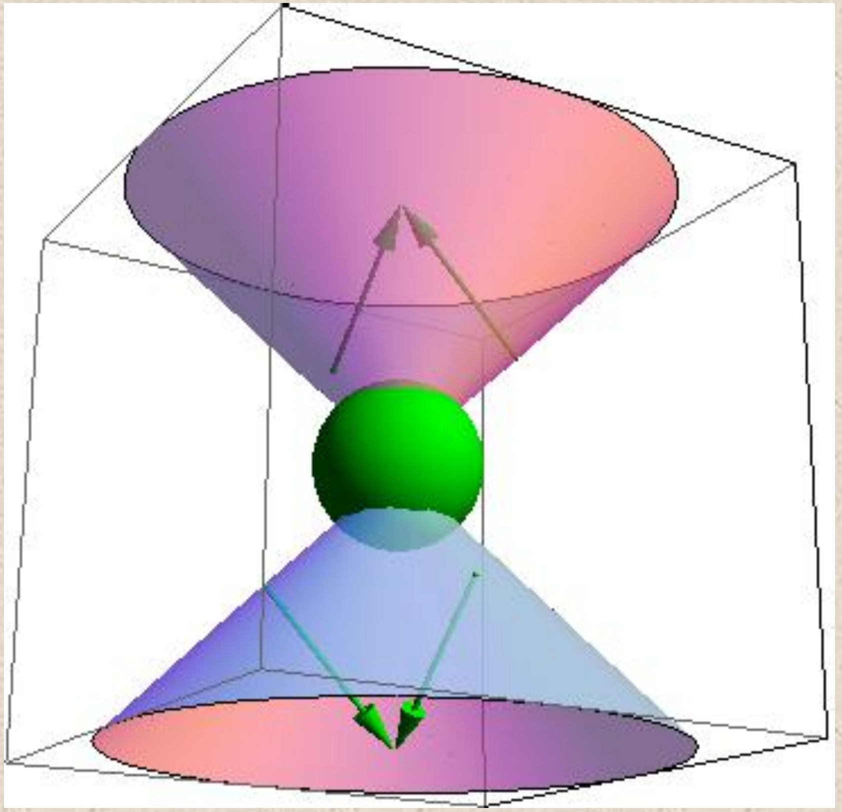
  $$\Delta_+(x) = \langle 0|\varphi(x)\varphi(0)|0\rangle$$

- Singular support

  $$\{(x,y,t); t^2 - x^2 - y^2 = 0\}$$

- Wavefront set
- Powers Δ_+^n are allowed
- Quantization does not need renormalization

PROPAGATOR



Feynman propagator

- Time-ordered product of fields
  $\Delta_F(x) = \langle 0 | T\big(\varphi(x)\varphi(0)\big) | 0 \rangle$
- Singular support
  $\{(x, y, t); t^2 - x^2 - y^2 = 0\}$
- Wavefront set
- Powers Δ_F^n are allowed away from $x = 0$
- Powers Δ_F^n are forbidden at $x = 0$
- Renormalize only at $x = 0$

Wave Front Set

- Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets and $f : U \to V$ a smooth map.
- The pull-back of a distribution $v \in \mathcal{D}'(V)$ by f is determined by the wave front set
- The dual space of a distribution is determined by its wave front set
- The restriction of a distribution to a submanifold is determined by the wave front set

EXAMPLES

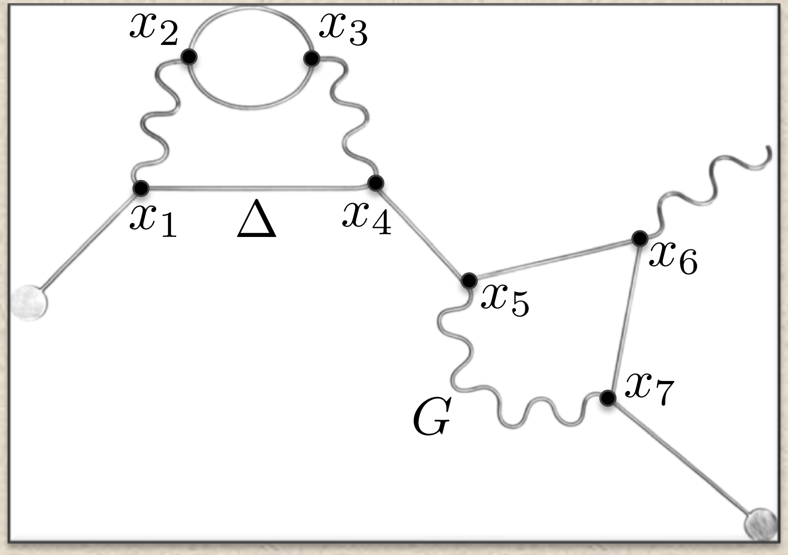
- The true propagator is $G(x, y) = \Delta_F(x - y)$
- By pull-back by $f(x, y) = x - y$, its wave front set is

$$\mathrm{WF}(G) = \{\big((x, y), (\xi, -\xi)\big); (x - y, \xi) \in \mathrm{WF}(\Delta_F)\}$$

- In curved space time, the wave front set of the propagator is obtained by pull-back:
  - either $\big((x, x), (\xi, -\xi)\big)$ for arbitrary $\xi \neq 0$
  - or $\big((x, y), (\xi, -\eta)\big)$ such that there is a null geodesic between x and y, and η is obtained by parallel transporting ξ along the geodesic

Quantum Field Theory

- Feynman diagram



- Feynman amplitude

$$G(x_1,x_2)\Delta(x_2,x_3)^2G(x_3,x_4)\Delta(x_1,x_4)\Delta(x_4,x_5)\Delta(x_5,x_6)\Delta(x_6,x_7)G(x_5,x_7)$$

- The wave front set of the amplitude is obtained by a recursive construction
- The amplitude is well defined, except on the diagonals
- It remains to renormalize to define the product on the diagonals

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TOPOLOGY

- For a closed cone $\Gamma \subset T^*M$ we define

$$\mathcal{D}'_\Gamma(U) = \{u \in \mathcal{D}'(U); \mathrm{WF}(u) \subset \Gamma\}$$

- We furnish $\mathcal{D}'_\Gamma(U)$ with a locally convex topology
- Let E be a vector space over $\mathbb{C}$. A *semi-norm* on E is a map $p : E \to \mathbb{R}$ such that
  - $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{C}$ and $x \in E$
  - $p(x+y) \leq p(x) + p(y)$ for all $x, y \in E$
- A locally convex space is a vector space E equipped with a family $(p_i)_{i \in I}$ of semi-norms on E
- The sets $V_{i,\epsilon} = \{x \in E; p_i(x) < \epsilon\}$ form a sub-base of the topology generated by the semi-norms

TOPOLOGY

- The seminorms of $\mathcal{D}'_\Gamma(U)$ are:
  - $p_B(u) = \sup_{f \in B} |\langle u, f \rangle|$ where B is bounded in $\mathcal{D}(U)$ are the seminorms of the strong topology of $\mathcal{D}'(U)$
  - $||u||_{N,V,\chi} = \sup_{k \in V} (1 + |k|)^N |\widehat{u\chi}(k)|$ for all integers N, closed cones V and functions $\chi \in \mathcal{D}(U)$ s.t. $\mathrm{supp}\chi \times V \cap \Gamma = \emptyset$
- The second set of seminorms is used to ensure that the Fourier transform of $u \in \mathcal{D}'_\Gamma(U)$ around $x \in \mathrm{supp}(\chi)$ decreases faster than any inverse polynomial: the wave front set of $u \in \mathcal{D}'_\Gamma(U)$ is in Γ

TOPOLOGY

Thm.

- $\mathcal{D}'_\Gamma(U)$ is complete
- $\mathcal{D}'_\Gamma(U)$ is semi-Montel (its closed and bounded subsets are compact)
- $\mathcal{D}'_\Gamma(U)$ is semi-reflexive
- $\mathcal{D}'_\Gamma(U)$ is nuclear
- $\mathcal{D}'_\Gamma(U)$ is a normal space of distributions

TOPOLOGY

Thm. With the topology of $\mathcal{D}'_{\Gamma}(U)$

- The pull-back is continuous
- The push-forward is continuous
- The multiplication of distributions is hypocontinuous
- The tensor product of distributions is hypocontinuous
- The duality pairing is hypocontinuous

FOR YOUR ATTENTION