HOLOMORPHIC GERBES AND THE BEILINSON REGULATOR

by Jean-Luc BRYLINSKI

Introduction

For X a smooth complex projective variety, Beilinson has defined regulator maps $c_{m,i} : K_i(X) \to H^{2m-i}(X, \mathbb{Z}(m)_D)$ from algebraic K-theory to Deligne cohomology [**Be1**]. For a variety over $\mathbb{Q}$, the conjectures of Beilinson express the leading term of the expansion of the Hasse-Weil L-functions at an integer in terms of this regulator.

Many computations of the regulator have been performed by Beilinson himself [**Be1**] [**Be2**] and by other authors [**D-W**] [**Ra2**]. There are however few cases where the regulator map has been described geometrically, the main reason being that the Deligne cohomology groups themselves do not have an easy global geometric interpretation. There is an important case where a geometric interpretation has been obtained by Bloch, Deligne and Ramakrishnan, namely that of the regulator $c_{2,2} : K_2(X) \to H^2(X, \mathbb{Z}(2)_D)$. For X projective, this goes as follows: Deligne showed that the group $H^2(X, \mathbb{Z}(2)_D)$ identifies with the group of isomorphism classes of holomorphic line bundles over X equipped with a holomorphic connection. Then Bloch [**Bl**] and Deligne [**De2**], constructed a holomorphic line bundle associated to a pair of invertible holomorphic functions, and Bloch showed that this gives a regulator map from $K_2(X)$ to the group of isomorphism classes of holomorphic line bundles with connection. Ramakrishnan gave an interpretation of this construction in terms of the three-dimensional Heisenberg group [**Ra1**].

Line bundles can however be used only in describing this special case of the Beilinson regulator. Other Deligne cohomology groups are in fact related to higher analogs of line bundles, which are called gerbes [**G**] (with band the sheaf of invertible holomorphic functions), 2-gerbes [**Bre**], etc.... In this paper, we give a geometric description for the regulator map $c_{2,1} : K_1(X) \to H^3(X, \mathbb{Z}(2)_D)$. This is based on the interpretation of the Deligne cohomology group $H^3(X, \mathbb{Z}(2)_D)$ as the group of equivalence classes of holomorphic gerbes equipped with a holomorphic connective structure. These notions were developed in [**Bry1**] and [**Bry2**] where they were applied to the geometry of loop spaces and of the space of knots in a three-manifold. For X projective, the Deligne cohomology group in question is the quotient of the dual of $H_2(X, \mathbb{R})$ by the linear forms of the type $\gamma \mapsto \Im(\int_\gamma \omega)$, for ω a holomorphic 2-form, and $\Im$ denotes the imaginary part. In terms of holomorphic gerbes, the linear form on $H_2(X, \mathbb{R})$ thus obtained is the *holonomy* of the holomorphic gerbe. In fact, the geometric significance of gerbes is that they give rise to such holonomy functionals for mappings of surfaces into the ambient manifold.

Underlying this is a theory of curvings compatible with the holomorphic structure of a gerbe. These curvings which are flat (i.e., have zero 3-curvature) are unique precisely up to a holomorphic 2-form; this is our geometric explanation for the ambiguity of a holomorphic 2-form in the regulator map.

In principle such ideas will lead to a geometric description of all regulator maps, once the categorical aspects have been cleared up. Hopefully this would lead to a better understanding of algebraic K-theory itself.

To make this paper self-contained, we have included a discussion of holomorphic gerbes and their differential geometry (connective structure, curving and 3-curvature). We only discuss Deligne cohomology, as opposed to the more delicate Deligne-Beilinson cohomology, except for some comments related to growth structures on gerbes, at the end of §2.

Finally in §5 we give the geometric description of the regulator $K_1(X) \to H^3(X, \mu_m^{\otimes 2})$ for m odd. This uses an analog of the line bundle

(f,g) of Bloch and Deligne, associated to a pair f,g of invertible regular functions. This analog is a gerbe, which is the obstruction to lifting an abelian covering with group $\mu_m \times \mu_m$ to a covering whose group is a finite Heisenberg group. The obstruction vanishes when $g = 1 - f$, due to the existence of an embedding of $\mu_m^{\otimes -1}$ into the jacobian of the Fermat curve $x^m + y^m = 1$. This is closely related to work of Deligne [**De3**] and Ihara [**I**].

I am grateful to Pierre Deligne and to Christophe Soulé for many interesting discussions. I am grateful to the referee for interesting comments. It is a pleasure to thank Christian Kassel, Jean-Louis Loday and Norbert Schappacher for organizing a very interesting and pleasant conference in Strasbourg.

This research was supported in part by a grant from the NSF.

1. Construction of holomorphic gerbes

We recall the notion of *gerbe* $\mathcal{C}$ on a space X, with band a commutative sheaf of groups A. This is a sheaf of categories (or stack) over X in the following sense. For every continuous map $f : Y \to X$, which is a local homeomorphism, there is category $\mathcal{C}(Y \xrightarrow{f} X)$ (or simply $\mathcal{C}(Y)$). Given a diagram $Z \xrightarrow{g} Y \xrightarrow{f} X$ of local homeomorphisms, there is a pull-back functor $g^* : \mathcal{C}(Y \xrightarrow{f} X) \to \mathcal{C}(Z \xrightarrow{gf} X)$. We do not require that $(hg)^* = g^*h^*$, since this does not hold in geometric situations. We do assume that there is a given invertible natural transformation $\theta_{g,h} : g^*h^* \xrightarrow{\sim} (hg)^*$, such that some commutative diagram commutes, for any diagram $V \xrightarrow{k} W \xrightarrow{h} Z \xrightarrow{g} Y \xrightarrow{f} X$ (we refer the reader to [**Bry2**] for details). Given a diagram of local homeomorphisms $Z \xrightarrow{g} Y \xrightarrow{f} X$, with g surjective, one has a descent category, whose objects are pairs (P, ϕ), where P is an object of $\mathcal{C}(Z)$, and $\phi : p_1^*P \xrightarrow{\sim} p_2^*P$ is an isomorphism between objects of $\mathcal{C}(Z \times_Y Z)$. We say that $\mathcal{C}$ is a *sheaf of categories* if the natural functor from $\mathcal{C}(Y)$ to the above descent category is an equivalence of categories, for any such diagram $Z \xrightarrow{g} Y \xrightarrow{f} X$.

Let now A be a sheaf of abelian groups over X. A *gerbe* over X with band A is a sheaf of categories $\mathcal{C}$ over X, together with an isomorphism $\alpha_P : A\tilde{\rightarrow}\underline{Aut}(P)$ for any object P of $\mathcal{C}(Y)$, such that the following properties are satisfied:

(1) The categories $\mathcal{C}(Y)$ are groupoids (i.e., every morphism is invertible).

(2) The isomorphisms α_P commute with all morphisms in $\mathcal{C}$.

(3) Two objects of $\mathcal{C}(Y)$ are locally isomorphic.

(4) There exists a surjective local homeomorphism $f : Y \rightarrow X$ such that $\mathcal{C}(Y)$ is non-empty.

A gerbe on X with band A leads to a cohomology class in $H^2(X, A)$. In fact, Giraud proved [**G**] that $H^2(X, A)$ identifies with the group of equivalence classes of gerbes over X with band A.

We will study gerbes over a complex-analytic manifold X, with band equal to the sheaf of groups $\mathcal{O}_X^*$. Such gerbes will be called *holomorphic gerbes.*

Recall briefly how a divisor D on X leads to a line bundle $\mathcal{O}(D)$. Here $D = \sum_i n_i D_i$ is a formal combination, with integer coefficients, of irreducible subvarieties of codimension one. There are several descriptions of $\mathcal{O}(D)$. First we can describe the space of sections $\Gamma(U, \mathcal{O}(D))$ for any open set U, as comprised of all meromorphic functions f on U such that $div(f) + D \geq 0$ in U, where $div(f)$ is the divisor of f. If we wish merely to describe the class of $\mathcal{O}(D)$ in $Pic(X) = H^1(X, \mathcal{O}_X^*)$, we may use the exact sequence of sheaves on X

$$0 \rightarrow \mathcal{O}_X^* \rightarrow \mathcal{O}_X(*Y)^* \xrightarrow{v_Y} \nu_{Y\,*}\mathbb{Z}_{\tilde{Y}} \rightarrow 0 \qquad (1-1)$$

where $Y = \cup_{i:n_i \neq 0} D_i$ is the support of D, $\nu_Y : \tilde{Y} \rightarrow Y$ is the normalization of Y. Note that $\tilde{Y} = \coprod_i \tilde{D}_i$, where $\tilde{D}_i$ is the normalization of D_i. The sheaf of algebras $\mathcal{O}_X(*Y)$ is the direct limit of the $\mathcal{O}_X(n \cdot Y)$ for $n \geq 1$; in other words, a section of $\mathcal{O}_X(*Y)$ over an open subset U of X is a holomorphic function on $U \setminus Y$, which is meromomorphic over U. The homomorphism v_Y associates to a meromorphic function f its polar divisor. More precisely, the function $v_Y(f)$ takes on $\tilde{D}_i \subseteq \tilde{Y}$ a value equal to the order m_i of the pole of f along D_i. Then we have a boundary map $\delta : H^0(\tilde{Y}, \mathbb{Z}) \to H^1(X, \mathcal{O}_X^*)$. The divisor D induces a function $\underline{n}$ on $\tilde{Y}$, whose value on $\tilde{D}_i$ is equal to n_i. Then we have

Proposition 1.1. *The class of $\mathcal{O}(D)$ in $H^1(X, \mathcal{O}_X^*)$ is equal to $\delta(\underline{n})$.*

We want to use the exact sequence (1-1) to construct holomorphic gerbes over X. We will consider the boundary map $\delta_1 : H^1(\tilde{Y}, \mathbb{Z}) \to H^2(X, \mathcal{O}_X^*)$. According to [G], it has the following geometric interpretation. To explain it, we will use the notion of a torsor F under a sheaf A of groups over X (also called an A-torsor). This means a sheaf F, together with a left action of A on F, which is locally simply transitive. The group of isomorphism classe of A-torsors over X identifies with $H^1(X, A)$. If $f : A \to B$ is a homomorphism of sheaves of groups, and if F is an A-torsor, then the contracted product $B \times^A F$, quotient of $B \times F$ by the diagonal action of A, is a B-torsor. Let F be a torsor under the sheaf of groups $\nu_{\tilde{Y}\,*}\mathbb{Z}_{\tilde{Y}}$ over X. Then we have a holomorphic gerbe $\mathcal{C}$ over X. We will describe the category $\mathcal{C}(U)$, for an open set U in X. An object of $\mathcal{C}(U)$ is a pair (H, ϕ), where H is a torsor under the sheaf $\mathcal{O}_X(*Y)^*_{/U}$ over U, and ϕ is an isomorphism of $\nu_{Y\,*}\mathbb{Z}_{\tilde{Y}}$-torsors $\phi : F \tilde{\to} \nu_{Y\,*}\mathbb{Z}_{\tilde{Y}} \times^{\mathcal{O}_X(*Y)^*} H$. A morphism from (H_1, ϕ_1) to (H_2, ϕ_2) is an isomorphism $\psi : H_1 \tilde{\to} H_2$ of $\mathcal{O}_X(*Y)^*$-torsors, which makes the following diagram commute:

$$\begin{array}{ccc} F & \xrightarrow{\phi_1} & \nu_{Y\,*}\mathbb{Z}_{\tilde{Y}} \times^{\mathcal{O}_X(*Y)^*} H_1 \\ \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \psi} \\ F & \xrightarrow{\phi_2} & \nu_{Y\,*}\mathbb{Z}_{\tilde{Y}} \times^{\mathcal{O}_X(*Y)^*} H_2 \end{array}$$

It is easy to see that $\mathcal{C}$ is a holomorphic gerbe. To obtain a more concrete description of $\mathcal{C}$, we will assume that the torsor F under $\nu_{Y\,*}\mathbb{Z}_{\tilde{Y}}$ is given by a family (f_i) of invertible holomorphic functions on D_i. We note that a torsor on X under $\nu_{Y\,*}\mathbb{Z}_{\tilde{Y}}$ is exactly the same thing as a torsor on $\tilde{Y}$ under $\mathbb{Z}_{\tilde{Y}}$. Hence we will describe a $\mathbb{Z}_{\tilde{Y}}$-torsor over $\tilde{Y}$. For an open subset V of $\tilde{Y}$, a section of F over V will be a family of functions h_i over

$V \cap \tilde{D}_i$ such that $exp(h_i) = f_i$. A section $\underline{n} = (n_i)$ of $\mathbb{Z}_{\tilde{Y}}$ over V acts on F by $\underline{n} \cdot (h_i) = (h_i + 2\pi\sqrt{-1} \cdot n_i)$.

We now need the line bundle (f, g) of Bloch [**Bl**] and Deligne [**De2**] associated to a pair of invertible holomorphic functions f and g over a complex manifold M. This line bundle is characterized by the fact that any local branch $Log(f)$ of a logarithm for f defines a section of L, denoted by $\{Log(f), g\}$. One imposes a relation between local sections defined by different branches of the logarithm:

$$\{Log(f) + 2\pi\sqrt{-1} \cdot n, g\} = g^n \cdot \{Log(f), g\}. \qquad (1-2)$$

Equivalently, one considers the trivial line bundle over the infinite cyclic covering $\tilde{M}$ of M, which is the fiber product $\tilde{M} = M \times_{\mathbb{C}^*} \tilde{\mathbb{C}}^*$, where $\tilde{\mathbb{C}}^* \to \mathbb{C}^*$ is the universal covering, and M maps to $\mathbb{C}^*$ via f. One descends the trivial line bundle to a line bundle on M via the "monodromy automorphism" which is multiplication by g.

We are now ready to describe the category $\mathcal{C}(U)$ over an open subset U of X. An object of $\mathcal{C}(U)$ will be a pair $(L, \mathcal{K})$, where

(1) L is a holomorphic line bundle over $U \setminus Y$.

(2) $\mathcal{K}$ is a sheaf on $Y \cap U$, consisting of meromorphic trivializations of the line bundle

$$\otimes_i \, (F_i, g_i)^{\otimes -1} \otimes L$$

defined in a neighborhood of $D_i \cap U$, where g_i is a local equation of D_i and F_i is an invertible holomorphic function which extends $f_i \in \mathcal{O}^*(D_i \cap U)$. We require that the sheaf $\mathcal{K}_i$ is a torsor under $\mathcal{O}_U^*$, which acts on meromorphic trivializations by multiplication.

We have to explain (2) in a little more detail. First if we replace g_i by another equation g_i', we have: $g_i' = g_i \cdot u_i$, where u_i is an invertible holomorphic function. Then the line bundle (F_i, g_i') is the tensor product of (F_i, g_i) with (F_i, u_i). The line bundle (F_i, u_i) is in fact defined over U; hence if we have a sheaf $\mathcal{K}$ as in (2), there results a sheaf $\mathcal{K}'$ of local trivializations of

$$\otimes_i \, (F_i, g_i')^{\otimes -1} \otimes L = \otimes_i \, (F_i, u_i)^{\otimes -1} \otimes \otimes_i \, (F_i, g_i)^{\otimes -1} \otimes L$$

consisting of the products $\alpha \otimes \beta$, where α is an invertible section of $\otimes_i (F_i, u_i)^{\otimes -1}$, and $\beta \in \mathcal{K}$.

Similarly, if we replace F_i by another function F_i', we have $F_i' = F_i + g_i \cdot q_i$. For our purpose, we may assume that q_i is invertible and $1 + g_i F_i^{-1} q_i$ is invertible, since any function is locally a linear combimation of functions with these properties. Therefore (F_i', g_i) is the tensor product of (F_i, g_i) with the line bundle $(1 + g_i F_i^{-1} q_i, g_i)$. Now we have

Proposition 1.2. *(Bloch [**Bl**] Deligne [**De2**]) Let h be an invertible holomorphic function on M such that $1 - h$ is invertible. Then the line bundle $(1 - h, h)$ is canonically trivialized.*

Thus $(1 + g_i F_i^{-1} q_i, g_i)$ is isomorphic to $(1 + g_i F_i^{-1} q_i, -F_i q_i^{-1})$, which extends to a holomorphic line bundle in a neighborhood of D_i. Hence the change from F_i to F_i' is accompanied by an isomorphism of the corresponding $\mathcal{O}_X^*$-torsors.

A morphism from $(L_1, \mathcal{K}_1)$ to $(L_2, \mathcal{K}_2)$ in $\mathcal{C}(U)$ is an isomorphism $\phi : L_1 \to L_2$, which transforms a local trivialization in $\mathcal{K}_1$ to one in $\mathcal{K}_2$. Then we have:

Proposition 1.3. *The presheaf of categories $\mathcal{C}$ is a holomorphic gerbe over X. The corresponding element of $H^2(X, \mathcal{O}_X^*)$ is equal to $\delta_1([(f_i)])$, where $[(f_i)] \in H^1(\tilde{Y}, \mathbb{Z}) = \oplus_i H^1(\tilde{D}_i, \mathbb{Z})$ is the class of the (f_i).*

We now discuss the *connective structure* on this holomorphic gerbe. We recall from [**Bry1**] [**Bry2**] that a connective structure on $\mathcal{C}$ consists in the following:

(C1) For any local homeomorphism $Y \to X$ and any object P of $\mathcal{C}(Y)$, we have a torsor $Co(P)$ under Ω_Y^1.

(C2) For a diagram $Z \xrightarrow{h} Y \to X$ of local homeomorphisms and any object P of $\mathcal{C}(Y)$, we have an isomorphism of Ω_Z^1-torsors $h^* Co(P) \xrightarrow{\sim} Co(h^* P)$.

(C3) Any isomorphism $\phi : P \xrightarrow{\sim} Q$ in $\mathcal{C}(Y)$ unduces a functorial isomorphism $\phi_* : Co(P) \xrightarrow{\sim} Co(Q)$, such that $g_*(D) = D - g^{-1} dg$ for the automorphism induced by a section g of $\mathcal{O}_X^*$.

There are some required compatibilities between (C2) and (C3), for which we refer to [**Bry2, Chapter 5**].

To give the connective structure on the gerbe $\mathcal{C}$, we will need the holomorphic connection ∇ on the Deligne line bundle (f, g). It is characterized by the equation:

$$\nabla(\{Log(f), g\}) = \frac{1}{2\pi\sqrt{-1}} \, Log(f) \cdot \frac{dg}{g} \otimes \{Log(f), g\}, \qquad (1-3)$$

Then for an open set U in X, and for an object $(L, \mathcal{K})$ of $\mathcal{C}(U)$, we define the Ω^1_U-torsor $Co(L, \mathcal{K})$ as a subsheaf of the sheaf of holomorphic connections D on L over $U \backslash Y$ as follows. Note that D induces a connection $-\sum_i \nabla_i + D$ on $\otimes_i (F_i, g_i)^{\otimes -1} \otimes L$, where ∇_i is the connection on (F_i, g_i). This connection should send a section in $\mathcal{K}$ to a holomorphic 1-form. Then we have:

Proposition 1.4. *The assignment $(L, \mathcal{K}) \mapsto Co(L, \mathcal{K})$ defines a holomorphic connective structure on $\mathcal{C}$.*

2. Holomorphic gerbes and Deligne cohomology

For a complex manifold X, and a subring A of $\mathbb{R}$, the Deligne complex $A(m)_D$ is the complex of sheaves

$$Cone \;\; (A(m) \oplus F^m\Omega^\bullet_X \to \Omega^\bullet_X) \, [-1] \qquad (2-1)$$

where $\Omega^\bullet_X$ is the holomorphic de Rham complex of sheaves, and $F^m(\Omega^\bullet_X)$ is the truncation (*filtration bête*)

$$\Omega^m_X \to \Omega^{m+1}_X \cdots \to \cdots.$$

The Deligne complex is the complex of sheaves

$$A(m) \to \mathcal{O}_X \to \Omega^1_X \to \cdots \Omega^{m-1}_X \qquad (2-2)$$

The Deligne cohomology groups are the hypercohomology groups $H^p(X, A(m)_D)$.

For $A = \mathbb{Z}$, there is a quasi-isomorphism of complex of sheaves

$$\begin{array}{ccccccccc} \mathbb{Z}(m) & \to & \mathcal{O}_X & \to & \Omega^1_X & \to & \cdots & \to & \Omega^{m-1}_X \\ & & \downarrow \alpha & & \downarrow (2\pi\sqrt{-1})^{1-m} & & & & \downarrow (2\pi\sqrt{-1})^{1-m} \\ 0 & \to & \mathcal{O}^*_X & \xrightarrow{d\ \log} & \Omega^1_X & \to & \cdots & \to & \Omega^{m-1}_X \end{array} \quad (2-3)$$

where $\alpha(f) = exp((2\pi\sqrt{-1})^{1-m} \cdot f)$. This of course requires the choice of a square root of -1.

For X a smooth complex algebraic variety, Beilinson [**Be1**] has introduced a version of Deligne cohomology which involves a compactification of X. This modified cohomology is called Deligne-Beilinson cohomology; for X projective, it coincides with Deligne cohomology. Beilinson constructed a regulator map $c_{i,p} : K_p(X) \to H^{2i-p}(X, A(i)_D)$ from the algebraic K-theory of X to its Deligne-Beilinson cohomology. His famous conjectures on values of L-functions of algebraic varieties over $\mathbb{Q}$ involve the regulator map for $A = \mathbb{R}$.

Beilinson did some beautiful computations of his regulator in the case of a curve X, mostly for an elliptic curve with complex multiplication and for a modular curve [**Be1**] [**D-W**]. The regulator which is relevant for the value at $s = 2$ of the L-function $L(H^1(X), s)$ is $c_{2,2} : K_2(X) \to H^2(X, \mathbb{Z}(2)_D)$. The group $H^2(X, \mathbb{Z}(2)_D)$ has a nice geometric interpretation. The complex of sheaves $\mathbb{Z}(2)_D$ is quasi-isomorphic to $\mathcal{O}^*_X \xrightarrow{d\ \log} \Omega^1_X$, so that $H^2(X, \mathbb{Z}(2)_D)$ is isomorphic to $H^1(X, \mathcal{O}^*_X \xrightarrow{d\ \log} \Omega^1_X)$. This latter group was shown by Deligne [**De2**] to be identical to the group of isomorphism classes of holomorphic line bundles equipped with a holomorphic connection. On the other hand, for X projective, the Deligne cohomology group $H^2(X, \mathbb{R}(2)_D)$ is isomorphic to $H^1(X, \mathbb{C})/H^1(X, \mathbb{R}(2)) = \sqrt{-1} \cdot H^1(X, \mathbb{R})$. The map $c_{2,2}$ has the following geometric interpretation, for X a curve. In that case the line bundle is flat, and the regulator map (for $A = \mathbb{R}$) associates to a holomorphic line bundle with holomorphic connection the logarithm of the absolute value of the monodromy of the line bundle. We refer to [**Be1**] [**E-V**] for an explicit formula for this monodromy, as an iterated integral.

Our purpose is to give a similar interpretation for another regulator map, namely $c_{2,1} : K_1(X) \to H^3(X, \mathbb{Z}(2)_D)$. The Deligne cohomology group $H^3(X, \mathbb{Z}(2)_D)$ is isomorphic to $H^2(X, \mathcal{O}_X^* \xrightarrow{d\ \log} \Omega_X^1)$, using the quasi-isomorphism (2-3). We have the following interpretation of the latter group.

Theorem 2.1. *([**Bry2, Chapter 5**]) The group $H^2(X, \mathcal{O}_X^* \xrightarrow{d\ \log} \Omega_X^1)$ is canonically isomorphic to the group of equivalence classes of holomorphic gerbes over X equipped with a holomorphic connective structure.*

We will write down a Čech cocycle correspond to a holomorphic gerbe $\mathcal{C}$ equipped with a connective structure Co. Let (U_i) be an open covering of X, with all non-empty intersections Stein and contractible. Let $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$, and so on. Pick an object P_i of $\mathcal{C}(U_i)$ and isomorphisms $u_{ij} : (P_j)_{/U_{ij}} \tilde{\to} (P_i)_{/U_{ij}}$. Then $h_{ijk} = u_{ik}^{-1} u_{ij} u_{jk}$ is an automorphism of P_k over U_{ijk}, hence gives a section of $\mathcal{O}_X^*$ over U_{ijk}. Then (h_{ijk}) is a Čech 2-cocycle with coefficients in $\mathcal{O}_X^*$. Next pick a section ∇_i of $Co(P_i)$, and define a section α_{ij} of Ω_X^1 over U_{ij} by

$$\alpha_{ij} = \nabla_i - u_{ij\ *}\nabla_j$$

(which makes sense since ∇_i and $u_{ij\ *}\nabla_j$ are both sections of the Ω_X^1-torsor $Co(P_i)$). Then (h_{ijk}, α_{ij}) is a Čech 2-cocycle of the covering (U_i) with coefficients in the complex of sheaves $\mathcal{O}_X^* \xrightarrow{d\ \log} \Omega_X^1$. This represents the cohomology class of the pair $(\mathcal{C}, Co)$.

Thus the holomorphic gerbe with connective structure of §1, associated to a family (f_i) of invertible holomorphic functions on the divisors D_i, yields a class in the Deligne cohomology group $H^3(X, \mathbb{Z}(2)_D)$. We want to identify this cohomology class geometrically. We note that there exists a push-forward map $H^1(D_i, \mathbb{Z}(1)_D) \to H^3(X, \mathbb{Z}(2)_D))$ in Deligne cohomology. This follows formally from the existence of a Deligne homology theory such that all the axioms of Bloch and Ogus [**B-O**] are satisfied: this is explained in [**Be1**] and in more details in [**J**].

To describe concretely this map, we first reduce attention to the case where the divisors D_i are smooth and do not intersect.

Lemma 2.2. *Let X be a complex manifold, and let $Z \subset X$ be a closed complex-analytic subvariety of codimension ≥ 2. Then the restriction map $H^3(X, A(2)_D) \to H^3(X \setminus Z, A(2)_D)$ is injective.*

Proof. The exact sequence of complexes of sheaves $0 \to \Omega_X^\bullet / F^2\Omega_X^\bullet \to A(2)_D \to A(2) \to 0$ gives rise to an exact sequence

$$H^2(X, A(2)) \to H^1(X, \Omega_X^2 \to \Omega_X^3 \to \cdots) \to H^3(X, A(2)_D) \to H^3(X, A(2))$$

and a similar exact sequence for $X \setminus Z$. Note that given a morphism of exact sequences of groups

$$\begin{array}{ccccccc} A_1 & \to & B_1 & \to & C_1 & \to & D_1 \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta \\ A_2 & \to & B_2 & \to & C_2 & \to & D_2 \end{array},$$

if α is surjective, and β and δ are injective, then γ is injective.

In our case, the restriction map $H^2(X, A) \to H^2(X \setminus Z, A)$ is bijective, and the restriction map $H^3(X, A) \to H^3((X \setminus Z, A)$ is injective, because the cohomology group with support $H_Z^p(X, A)$ is 0 for $p \leq 4$. Lastly we have to prove that the restriction map $H^1(X, \Omega_X^2 \to \Omega_X^3 \to \cdots) \to H^1(X \setminus Z, \Omega_X^2 \to \Omega_X^3 \to \cdots)$ is injective. Since $H^1(X, \Omega_X^2 \to \Omega_X^3 \to \cdots)$ injects into $H^1(X, \Omega_X^2 \to \Omega_X^3)$, it suffices to show that $H^1(X, \Omega_X^2 \to \Omega_X^3)$ maps injectively to $H^1(X \setminus Z, \Omega_X^2 \to \Omega_X^3)$. We have an exact sequence

$$H^0(X, \ O_X^2) \to H^0(X, \Omega_X^3) \to H^1(X, \Omega_X^2 \to \Omega_X^3) \to H^1(X, \Omega_X^2).$$

The restriction maps $H^0(X, \Omega_X^p) \to H^0(X \setminus Z, \Omega_X^p)$ are bijective for all p, by Hartogs' theorem. The map $H^1(X, \Omega_X^2) \to H^1(X \setminus Z, \Omega_X^2)$ is injective because $H_Z^1(X, \Omega_X^2)$ vanishes [**S-T**]. This finishes the proof. ∎

This lemma implies that for the purpose of checking that some formula for a class in $H^2(X, \mathcal{O}_X^* \to \Omega_X^1)$ is correct, we may delete from X any closed analytic subvariety of codimension ≥ 2. Returning to the invertible holomorphic functions f_i on D_i, we have the following

Proposition 2.3. *The class in $H^2(\mathcal{O}_X^* \to \Omega_X^1)$ of the holomorphic gerbe $\mathcal{C}$ with holomorphic connective structure $\mathcal{C}o$ is the opposite of the sum over i of the push-forward of $[f_i] \in H^1(D_i, \mathbb{Z}(1)_D)$.*

Proof. Using Lemma 2.2, we may assume that each D_i is smooth, and that $D_i \cap D_j = \emptyset$ for $i \neq j$. We may also assume that there exists a neighborhood V_i of D_i in X over which there exists an invertible holomorphic function F_i such that $(F_i)_{/D_i} = f_i$, and there exists an equation g_i of D_i over V_i. Without loss of generality, we may assume $V_i \cap V_j = \emptyset$ for $i \neq j$. Then we cover each V_i by open sets $U_{i,a}$ such that there exists a branch $Log_a(F_i)$ of a logarithm of F_i over $U_{i,a}$. Let $Log_a(f_i)$ be the induced logarithm function of f_i over $D_i \cap U_{i,a}$. Then, for each i, we have a Čech 1-cocycle of the covering $(D_i \cap U_{i,a})_a$ of D_i, with coefficients in $\mathbb{Z}(1)_D$, which represents $[f_i]$, namely $(Log_a(f_i) - Log_b(f_i), Log_a(f_i))$. The corresponding class in $H^3(X, \mathbb{Z}(2)_D) \xrightarrow{\sim} H^2(X, \mathcal{O}_X^* \to \Omega_X^1)$ then has a concrete Čech representative for the open covering consisting of the $U_{i,a}$ and of $U_0 = X \setminus Y$. This Čech cocycle is a pair $(\underline{g}, \underline{\alpha})$, where $\underline{g}$ is a Čech 2-cocycle with coefficients in $\mathcal{O}_X^*$, and $\underline{\alpha}$ is a Čech 1-cochain with coefficients in Ω_X^1. We have: $g_{0,(i,a),(i,b)} = g_i^{\frac{Log_a(F_i) - Log_b(F_i)}{2\pi\sqrt{-1}}}$ and $g_{(i,a),(i,b),(i,c)} = 1$; this defines $\underline{g}$ since all other possible intersections of three open sets are empty. And we have: $\alpha_{0,(i,a)} = (2\pi\sqrt{-1})^{-1} \cdot Log_a(F_i) \cdot \frac{dg_i}{g_i}$, and $\alpha_{(i,a),(i,b)} = 0$. Now we get the opposite cocycle corresponding to the gerbe $\mathcal{C}$ with connective structure, if we make the following choices the following choices: over U_0, we take the object $\mathcal{O}_{U_0}$ of $\mathcal{C}(U_0)$. Over $U_{(i,a)}$, we take for $P_{(i,a)}$ the pair $((F_i, g_i), can)$, where *can* consists of the nowhere vanishing holomorphic sections of the trivial line bundle $(F_i, g_i)^{\otimes -1} \otimes (F_i, g_i)$. We have to give isomorphisms u_{ij} of objects over the intersection of two open sets. When these open sets are $U_{(i,a)}$ and $U_{(i,b)}$, we take the isomorphism to be the identity. When the open sets are U_0 and $U_{(i,a)}$, we take $u_{(i,a),0}$ to be given by the section $\{Log_a(F_i), g_i\}$ of (F_i, g_i) over $U_0 \cap U_{(i,a)}$ Next we need a section of $Co((F_i, g_i), can)$, which will be given by the standard connection on (F_i, g_i). The Čech cocycle for these data is exactly equal to the opposite of $(\underline{g}, \underline{\alpha})$. ∎

We will now discuss the relation of holomorphic gerbes on a smooth quasi-projective algebraic variety X, to Deligne-Beilinson cohomology. This requires introducing a smooth compactification $j : X \hookrightarrow \overline{X}$ such that $Z = \overline{X} \setminus X$ is a divisor with normal crossings in $\overline{X}$. Then the *Deligne-Beilinson cohomology* group $H^3(X, \mathbb{Z}(2)_{DB})$ is defined as the hypercohomology group

$$H^3(\overline{X}, Cone\ (\mathbb{R}j_*\mathbb{Z}(2)_X \oplus F^2\Omega^\bullet_{\overline{X}}(Log\ Z) \to \mathbb{R}j_*\mathbb{C}_X)) \qquad (2-5)$$

where $\Omega^\bullet_{\overline{X}}(Log\ Z)$ is the complex of sheaves over $\overline{X}$ consisting of meromorphic differential forms with *logarithmic poles* along Z, and $F^2\Omega^\bullet_{\overline{X}}(Log\ Z)$ is the Hodge filtration (truncation in degrees ≥ 2).

Let $K^\bullet$ be the cone of the morphism of complexes of sheaves occurring in (2-2). It is hard to write down an explicit complex representing $K^\bullet$, because there is no easy realization of the object $\mathbb{R}j_*\mathbb{Z}(2)_X$ of a derived category (this situation has however recently been remedied by Karoubi [**K**], who uses non-commutative differential forms). However, there is an explicit logarithmic complex which maps to this cone. Indeed we have the exact sequence of sheaves on $\overline{X}$:

$$0 \to \mathbb{Z}(1)_{\overline{X}} = j_*\mathbb{Z}(1)_X \to \mathcal{O}_{\overline{X}} \xrightarrow{exp} j_{*\ mer}\mathcal{O}^*_X \to R^1j_*\mathbb{Z}_X(1) \to 0 \quad (2-6)$$

where $j_{*\ mer}\mathcal{O}^*_X$ denotes the meromorphic direct image. It follows that the cone of the morphism $\sigma_{\leq 1}\mathbb{R}j_*\mathbb{Z}(1)_X \to \mathcal{O}_{\overline{X}}$ is quasi-isomorphic to $j_{*\ mer}\mathcal{O}^*_X$. Here $\sigma_{\leq q}$ denotes Deligne's truncation of a complex in degrees $\leq q$. Let then $L^\bullet$ be the cone of the morphism $\sigma_{\leq 1}\mathbb{R}j_*\mathbb{Z}(1)_X \oplus$

$F^2\Omega^\bullet_{\overline{X}}(Log Z) \to \Omega^\bullet_{\overline{X}}(Log Z)$, so that there is a canonical morphism of complexes of sheaves from $L^\bullet$ to $K^\bullet$. We may view $L^\bullet$ as the cone of a morphism of complexes of sheaves $\sigma_{\leq 1}\mathbb{R}j_*\mathbb{Z}(1)_X \to [\mathcal{O}_{\overline{X}} \to \Omega^1_{\overline{X}}(Log\ Z)]$, which can be identified with the complex of sheaves $j_*\mathcal{O}^*_X \to \Omega^1_{\overline{X}}(Log\ Z)$. So we obtain

Proposition 2.4. *There is a natural morphism of complex of sheaves from* $j_{*\ mer}\mathcal{O}^*_X \to \Omega^1_{\overline{X}}(Log\ Z)$ *to the cone of the morphism* $\mathbb{R}j_*\mathbb{Z}(2)_X \oplus F^2\Omega^\bullet_{\overline{X}}(Log\ Z) \to \mathbb{R}j_*\mathbb{C}_X$.

The hypercohomology group $H^2(\overline{X}, j_{*\ mer}\mathcal{O}^*_X \to \Omega^1_{\overline{X}}(Log\ Z))$ has a description in terms of gerbes. First we define a meromorphic gerbe $\mathcal{C}$ over X: this is a gerbe over $\overline{X}$, with band equal to the sheaf of groups $j_{*\ mer}\mathcal{O}^*_X$. This gives in particular a holomorphic gerbe over X, together with the notion of meromorphic object of $\mathcal{C}$ over $U \cap X$, for U any open set in $\overline{X}$. However, there are topological obstructions to extend such a holomorphic gerbe $\mathcal{C}$ over X to a meromorphic gerbe over $\overline{X}$. The first obstruction is an element of $H^0(Z, i^*R^3j_*\mathbb{Z}(1))$, where $i : Z \hookrightarrow \overline{X}$ is the inclusion; this class is the obstruction to finding a local object of the sheaf of groupoids $j_*\mathcal{C}$. The second obstruction lives in $H^0(Z, i^*R^2j_*\mathbb{Z}(1))$; it is the obstruction to showing that two objects of $j_*\mathcal{C}$ are locally isomorphic.

Then a connective structure with logarithmic poles on the meromorphic gerbe $\mathcal{C}$ associates to any object P of $\mathcal{C}(U)$ a torsor $Co(P)$ under $\Omega^1_{\overline{X}}(Log\ Z)$. This must satisfy axioms similar to those of ordinary connective structures. Then we have easily

Proposition 2.5. *The group of equivalence classes of meromorphic gerbes over* X *equipped with a connective structure with logarithmic poles is isomorphic to the hypercohomology group* $H^2(\overline{X}, j_{*\ mer}\mathcal{O}^*_X \to \Omega^1_{\overline{X}}(Log\ Z))$, *which maps to the Deligne-Beilinson cohomology group* $H^3(X, \mathbb{Z}(2)_{DB})$.

Now if we start from invertible regular functions f_i on divisors D_i, the question arises to construct a compactification $\overline{X}$ such that the corresponding holomorphic over X has a meromorphic extension to $\overline{X}$. We these open sets are $U_{(i,a)}$ and $U_{(i,b)}$, we take the isomorphism to be the identity. When the open sets are U_0 and $U_{(i,a)}$, we take $u_{(i,a),0}$ to be given by the section $\{Log_a(F_i), g_i\}$ of (F_i, g_i) over $U_0 \cap U_{(i,a)}$ Next we need a section of $Co((F_i, g_i), can)$, which will be given by the standard connection on (F_i, g_i). The Čech cocycle for these data is exactly equal to the opposite of $(\underline{g}, \underline{\alpha})$. ∎

We will now discuss the relation of holomorphic gerbes on a smooth quasi-projective algebraic variety X, to Deligne-Beilinson cohomology. This requires introducing a smooth compactification $j : X \hookrightarrow \overline{X}$ such that $Z = \overline{X} \setminus X$ is a divisor with normal crossings in $\overline{X}$. Then the *Deligne-Beilinson cohomology* group $H^3(X, \mathbb{Z}(2)_{DB})$ is defined as the hypercohomology group

$$H^3(\overline{X}, Cone\ (\mathbb{R}j_*\mathbb{Z}(2)_X \oplus F^2\Omega^\bullet_{\overline{X}}(Log\ Z) \to \mathbb{R}j_*\mathbb{C}_X)) \qquad (2-5)$$

where $\Omega^\bullet_{\overline{X}}(Log\ Z)$ is the complex of sheaves over $\overline{X}$ consisting of meromorphic differential forms with *logarithmic poles* along Z, and $F^2\Omega^\bullet_{\overline{X}}(Log\ Z)$ is the Hodge filtration (truncation in degrees ≥ 2).

Let $K^\bullet$ be the cone of the morphism of complexes of sheaves occurring in (2-2). It is hard to write down an explicit complex representing $K^\bullet$, because there is no easy realization of the object $\mathbb{R}j_*\mathbb{Z}(2)_X$ of a derived category (this situation has however recently been remedied by Karoubi [**K**], who uses non-commutative differential forms). However, there is an explicit logarithmic complex which maps to this cone. Indeed we have the exact sequence of sheaves on $\overline{X}$:

$$0 \to \mathbb{Z}(1)_{\overline{X}} = j_*\mathbb{Z}(1)_X \to \mathcal{O}_{\overline{X}} \xrightarrow{exp} j_{*\ mer}\mathcal{O}^*_X \to R^1j_*\mathbb{Z}_X(1) \to 0 \quad (2-6)$$

where $j_{*\ mer}\mathcal{O}^*_X$ denotes the meromorphic direct image. It follows that the cone of the morphism $\sigma_{\leq 1}\mathbb{R}j_*\mathbb{Z}(1)_X \to \mathcal{O}_{\overline{X}}$ is quasi-isomorphic to $j_{*\ mer}\mathcal{O}^*_X$. Here $\sigma_{\leq q}$ denotes Deligne's truncation of a complex in degrees $\leq q$. Let then $L^\bullet$ be the cone of the morphism $\sigma_{\leq 1}\mathbb{R}j_*\mathbb{Z}(1)_X \oplus$

$F^2\Omega^\bullet_{\overline{X}}(Log Z) \to \Omega^\bullet_{\overline{X}}(Log Z)$, so that there is a canonical morphism of complexes of sheaves from $L^\bullet$ to $K^\bullet$. We may view $L^\bullet$ as the cone of a morphism of complexes of sheaves $\sigma_{\leq 1}\mathbb{R}j_*\mathbb{Z}(1)_X \to [\mathcal{O}_{\overline{X}} \to \Omega^1_{\overline{X}}(Log\ Z)]$, which can be identified with the complex of sheaves $j_*\mathcal{O}^*_X \to \Omega^1_{\overline{X}}(Log\ Z)$. So we obtain

Proposition 2.4. *There is a natural morphism of complex of sheaves from* $j_{*\ mer}\mathcal{O}^*_X \to \Omega^1_{\overline{X}}(Log\ Z)$ *to the cone of the morphism* $\mathbb{R}j_*\mathbb{Z}(2)_X \oplus F^2\Omega^\bullet_{\overline{X}}(Log\ Z) \to \mathbb{R}j_*\mathbb{C}_X$.

The hypercohomology group $H^2(\overline{X}, j_{*\ mer}\mathcal{O}^*_X \to \Omega^1_{\overline{X}}(Log\ Z))$ has a description in terms of gerbes. First we define a meromorphic gerbe $\mathcal{C}$ over X: this is a gerbe over $\overline{X}$, with band equal to the sheaf of groups $j_{*\ mer}\mathcal{O}^*_X$. This gives in particular a holomorphic gerbe over X, together with the notion of meromorphic object of $\mathcal{C}$ over $U \cap X$, for U any open set in $\overline{X}$. However, there are topological obstructions to extend such a holomorphic gerbe $\mathcal{C}$ over X to a meromorphic gerbe over $\overline{X}$. The first obstruction is an element of $H^0(Z, i^*R^3j_*\mathbb{Z}(1))$, where $i : Z \hookrightarrow \overline{X}$ is the inclusion; this class is the obstruction to finding a local object of the sheaf of groupoids $j_*\mathcal{C}$. The second obstruction lives in $H^0(Z, i^*R^2j_*\mathbb{Z}(1))$; it is the obstruction to showing that two objects of $j_*\mathcal{C}$ are locally isomorphic.

Then a connective structure with logarithmic poles on the meromorphic gerbe $\mathcal{C}$ associates to any object P of $\mathcal{C}(U)$ a torsor $Co(P)$ under $\Omega^1_{\overline{X}}(Log\ Z)$. This must satisfy axioms similar to those of ordinary connective structures. Then we have easily

Proposition 2.5. *The group of equivalence classes of meromorphic gerbes over* X *equipped with a connective structure with logarithmic poles is isomorphic to the hypercohomology group* $H^2(\overline{X}, j_{*\ mer}\mathcal{O}^*_X \to \Omega^1_{\overline{X}}(Log\ Z))$, *which maps to the Deligne-Beilinson cohomology group* $H^3(X, \mathbb{Z}(2)_{DB})$.

Now if we start from invertible regular functions f_i on divisors D_i, the question arises to construct a compactification $\overline{X}$ such that the corresponding holomorphic over X has a meromorphic extension to $\overline{X}$. We discuss this when the D_i are smooth. Then we can pick $X \hookrightarrow \overline{X}$ such that the closure $\overline{D}_i$ of D_i is smooth, and transverse to all the components of Z and to their pairwise intersections. Under these conditions, the topological obstructions vanish and we have a meromorphic gerbe with connective structure, hence a class in Beilinson-Deligne cohomology.

3. The regulator map from $H^1(X,\underline{K}_2)$ to $H^3(X,\mathbb{Z}(2)_D)$.

We wish to give a geometric description for the Beilinson regulator map $c_{1,2} : K_1(X) \to H^3(X, A(2)_D)$, for X a smooth projective algebraic variety over $\mathbb{C}$. All sheaves and groups in this section are taken in the algebraic sense, except when we put the superscript *an* as in $\mathcal{O}_X^{an}$. We recall that the group $K_1(X)\otimes\mathbb{Q}$ decomposes into simultaneous eigenspaces of the Adams operations, each one of which is identified with a term of the Gersten-Quillen spectral sequence [**Q**] [**So**]:

$$K_1(X)\otimes\mathbb{Q} = \sum_{i\geq 0} H^i(X,\underline{K}_{i+1})\otimes\mathbb{Q}.$$

The term of this decomposition which is relevant for the regulator map $c_{1,2}$ is $H^1(X,\underline{K}_2)$. Therefore we will describe geometrically the regulator map $c_{1,2} : H^1(X,\underline{K}_2) \to H^3(X,\mathbb{Z}(2)_D)$. The group $H^1(X,\underline{K}_2)$ is the first cohomology group of the *Gersten complex [***Q***]*

$$K_2(\mathbb{C}(x)) \xrightarrow{\delta_1} \oplus_{x\in X^{(1)}} \mathbb{C}(x)^* \xrightarrow{\delta_0} \oplus_{x\in X^{(2)}} \mathbb{Z}. \qquad (3-1)$$

Here $X^{(p)}$ denotes the set of points of X of codimension p, and for a point x, $\mathbb{C}(x)$ is the field of rational functions on the corresponding subvariety of X.

Our task is therefore as follows: we start with a finite family (D_i) of irreducible codimension one subvarieties in X, and a family f_i of non-zero meromorphic functions on the D_i, such that $\sum_i f_i$ is in the kernel of δ_0. For these data, we must construct a holomorphic gerbe over X, with a holomorphic connective structure. If all f_i are actually invertible holomorphic functions on D_i, then we have produced such a gerbe in §1. We now wish to generalize this construction. This depends on the following "moving lemma for $H^1(X, \underline{K}_2)$".

Lemma 3.1. *Let $a \in H^1(X, \underline{K}_2)$, and let $x \in X$. Then there exists some Zariski open set U containing x, such that a has zero restriction to $H^1(U, \underline{K}_2)$.*

Proof. This in fact a general result about higher cohomology of any sheaf. ∎

We will now associate a holomorphic gerbe $\mathcal{C}$ on X to a cocycle $\sum_i f_i$ in $\oplus_{x \in X^{(1)}} \mathbb{C}(x)^*$, where each f_i is a non-zero meromorphic function on the irreducible codimension one subvariety D_i of X. Let $Y = \cup_i D_i$. For an open subset U of X, an object of the category $\mathcal{C}(U)$ will be a holomorphic line bundle L over $U \setminus Y$, together with the following data at any point x of Y. For U a Zariski open set containing x, and $\alpha \in K_2(\mathbb{C}(X))$ such that $\delta_1(\alpha) = \sum_i f_i$ over U, let $\mathcal{L}(\alpha)$ be the corresponding Deligne line bundle over $U \setminus Y$ (it is indeed holomorphic outside of the support of $\delta_1(\alpha)$). Then, just as in §1, we should have a subsheaf $\mathcal{K}(\alpha)$ of the sheaf of meromorphic local trivializations of $\mathcal{L}(\alpha)^{\otimes -1} \otimes L$, which is a torsor under $\mathcal{O}_U^{an\ *}$.

These sheaves $\mathcal{K}(\alpha)$ should satisfy two sorts of conditions:

(i) compatibility with restriction to smaller open sets;

(ii) independence of α.

We explain the meaning of (ii). Let $\beta \in K_2(\mathbb{C}(X))$ with $\delta_1(\beta) = \delta_1(\alpha)$ in U. Then the difference $\beta - \alpha$ belongs to $H^0(U, \underline{K}_2)$, using the Gersten resolution. The line bundle $\mathcal{L}(\beta - \alpha) = \mathcal{L}(\beta) \otimes \mathcal{L}(\alpha)^{\otimes -1}$ therefore is holomorphic over U. We ask that a section s of $\mathcal{L}(\alpha)^{\otimes -1} L$ belongs to $\mathcal{K}(\alpha)$ if and only if, for a non-vanishing local section σ of $\mathcal{L}(\alpha - \beta)$, the section $s \otimes \sigma$ of $\mathcal{L}(\beta)^{\otimes -1} \otimes L$ belongs to $\mathcal{K}(\beta)$.

We then obtain

Theorem 3.2. *The sheaf of groupoids $\mathcal{C}$ is a holomorphic gerbe over X. Furthermore, if to any object $(L, \mathcal{K}(\alpha))$ of $\mathcal{C}(U)$ we assign the ${\Omega^1_X}^{an}$-torsor of meromorphic connections ∇ on L such that the induced connection on $\mathcal{L}(\alpha)^{\otimes -1} \otimes L$ is holomorphic, we obtain a holomorphic connective structure on $\mathcal{C}$.*

The proof is very similar to that of Proposition 1.3.

Now clearly the holomorphic gerbe with connective structure depends additively on the element of $Ker(\delta_0)$. We will show that an element of $Im(\delta_1)$ gives a trivial gerbe with connective structure. Indeed given $\delta_1(\alpha)$ for $\alpha \in K_2(\mathbb{C}(X))$, there is a global object of $\mathcal{C}$, given by the line bundle $\mathcal{L}(\alpha)$ over $X \backslash Y$, together with the obvious choice of $\mathcal{K}(\alpha)$. The ${\Omega^1_X}^{an}$-torsor has a global section, given by the Deligne connection on $\mathcal{L}(\alpha)$.

It follows that the equivalence class of the holomorphic gerbe with holomorphic connective structure depends only on a cohomology class in $H^1(X, \underline{K}_2)$.

We now come to the main result of this paper

Theorem 3.3. *The map which to an element of $H^1(X, \underline{K}_2)$ assigns the equivalence class of the above holomorphic gerbe with connective structure, is equal to the Beilinson regulator $c_{1,2}$.*

Proof. Given an element $\sum_i f_i$ of $Ker(\delta_0)$, we have to show that two elements of $H^3(X, \mathbb{Z}(2)_D)$ coincide. According to Lemma 2.2, we may verify this after removing from X some algebraic subvarieties of codimension at least 2. Thus we may assume that each f_i is an invertible regular function on D_i, in which case Proposition 2.3 describes the class of the holomorphic gerbe with connective structure as $\sum_i j_{i\,*}[f_i]$, where $j_i : D_i \hookrightarrow X$ is the inclusion, and $j_{i\,*}$ is the push-forward map $H^1(D_i, \mathbb{Z}(1)_D) \rightarrow H^3(X, \mathbb{Z}(2)_D)$ in Deligne cohomology. Now we use the fact that the Beilinson regulator maps are part of a morphism of pairs of homology and cohomology theories satisfying the axioms of [**B-O**]. Thus the regulator map we have a commutative diagram

$$\begin{array}{ccc} K_1(D_i) & \xrightarrow{j_{i\,*}} & K_1(X) \\ \big\downarrow{\scriptstyle c_{1,1}} & & \big\downarrow{\scriptstyle c_{2,1}} \\ H^1(D_i,\mathbb{Z}(1)_D) & \xrightarrow{j_{i\,*}} & H^3(X,\mathbb{Z}(2)_D) \end{array}$$

Since the regulator map $c_{1,1}$ simply maps f_i to $[f_i]$, this concludes the proof. ∎

4. Holonomy of gerbes and the Beilinson regulator

In the last section we showed that, for X a projective complex manifold, the Beilinson regulator map $c_{1,2} : H^1(X,\underline{K}_2) \to H^3(X,\mathbb{Z}(2)_D)$ may be described by associating to an element of $H^1(X,\underline{K}_2)$ a holomorphic gerbe over X equipped with a holomorphic connective structure. We will give a description of the real regulator $c_{1,2} : H^1(X,\underline{K}_2) \to H^3(X,\mathbb{R}(2)_D)$ using the *holonomy* of this gerbe. First of all, we recall the computation of $H^3(X,\mathbb{R}(2)_D)$ from classical Hodge theory [**Be1**]. We have the exact sequence

$$\cdots \to F^2H^2(X,\mathbb{C}) \oplus H^2(X,\mathbb{R}(2)) \to H^2(X,\mathbb{C}) \to H^2(X,\mathbb{R}(2)_D) \to \\ F^2H^3(X,\mathbb{C}) \oplus H^3(X,\mathbb{R}(2)) \to H^3(X,\mathbb{C})$$

The map $F^2H^3(X,\mathbb{C}) \oplus H^3(X,\mathbb{R}(2)) \to H^3(X,\mathbb{C})$ is injective, and the cokernel of the map $F^2H^2(X,\mathbb{C}) \oplus H^2(X,\mathbb{R}(2)) \to H^2(X,\mathbb{C})$ identifies with $\sqrt{-1}\cdot H^{1,1}(X)_{\mathbb{R}} := \sqrt{-1}\cdot H^{1,1}(X) \cap H^2(X,\mathbb{R})$. Therefore we see that $H^3(X,\mathbb{R}(2)_D) \simeq \sqrt{-1}\cdot H^{1,1}(X)_{\mathbb{R}}$.

We will use another equivalent description of this vector space, as the quotient of $Hom(H_2(X,\mathbb{Z}),\mathbb{R}(1))$ by the homomorphisms of the type $\sigma \mapsto \Im(\int_\sigma \omega)$, for ω a holomorphic 2-form.

We will consider gerbes $\mathcal{G}$ on a smooth manifold X with band equal to the sheaf $\underline{\mathbb{C}}^*_X$ of smooth $\mathbb{C}^*$-valued functions. Note that such gerbes are classified, up to equivalence, by the cohomology group $H^2(X,\underline{\mathbb{C}}^*_X) \simeq$

$H^3(X, \mathbb{Z}(1))$. For a gerbe $\mathcal{G}$ with band $\underline{\mathbb{C}}_X^*$, we have the notion of smooth connective structure Co on $\mathcal{G}$, which is defined similarly as in the holomorphic case, except that for a local object P of $\mathcal{C}$, $Co(P)$ is now an $\underline{A}^1_{X,\mathbb{C}}$-torsor, for $\underline{A}^p_{X,\mathbb{C}}$ the sheaf of smooth p-forms with complex coefficients. Then we have the notion of curving of the connective structure, as defined in [**Bry1**] and in [**Bry2**]. A curving of Co associates to a section ∇ of $Co(P)$ a complex-valued 2-form $K(\nabla)$ such that

(1) For any isomorphism $\phi : P \to P'$ and any $\nabla \in Co(P)$, we have:

$$K(\phi_* \nabla) = K(\nabla)$$

(2) We have

$$K(\nabla + \alpha) = K(\nabla) + d\alpha$$

for any complex-valued 1-form α.

The differential forms $K(\nabla)$ are not closed in general. In fact, there is a closed 3-form Ω, the so-called 3-*curvature* of the curving, such that $\Omega = d\, K(\nabla)$ for any local section ∇ of $Co(P)$, where P is a local object of $\mathcal{G}$. One of the main results of [**Bry2, §5.3**] is that the cohomology class of Ω is an integral class multiplied by $2\pi\sqrt{-1}$, and is the image in $H^3(X, \mathbb{C})$ of the $\mathcal{G} \in H^3(X, \mathbb{Z}(1))$.

Given a curving of the connective structure, we can define the *holonomy* of the gerbe $\mathcal{G}$ around a closed surface mapping into X. This is discussed in [**Bry2, Chapter 6**]. Let $\Phi : \Sigma \to X$ be a smooth mapping from the closed oriented surface Σ to X. We will associate a number $S(\Phi) \in \mathbb{C}^*$ (*action functional* or *holonomy*) as follows. We have the notion of the pull-back gerbe $\Phi^*\mathcal{G}$ over Σ, with band $\underline{\mathbb{C}}_\Sigma^*$, equipped with a pull-back connective structure Φ^*Co and a pull-back curving. Pick a global object P of $\Phi^*\mathcal{G}$ and a section ∇ of $(\Phi^*Co)(P)$. Then we set

$$S(\Phi) = exp(\int_\Sigma K(\Phi^*\nabla)).$$

If we change ∇ to $\nabla + \alpha$, for α some 1-form, then the integral $\int_\Sigma K(\Phi^*\nabla)$ does not change. If we change P to another object, obtained by twisting

P by some $\mathbb{C}^*$-torsor Q over Σ, then $\int_\Sigma K(\Phi^*\nabla)$ gets transformed into $\int_\Sigma K(\Phi^*\nabla) + 2\pi\sqrt{-1}\cdot c_1(Q)$, so the exponential of the integral does not change. Therefore the holonomy (4-3) depends only on Φ (and of course on the gerbe, connective structure and curving).

Now given a holomorphic gerbe $\mathcal{C}$ on the complex manifold X, since we have a morphism of sheaves $\mathcal{O}_X^* \to \underline{\mathbb{C}}_X^*$, there is a corresponding gerbe $\mathcal{G}$ with band $\underline{\mathbb{C}}_X^*$, and a holomorphic connective structure Co on $\mathcal{C}$ induces a connective structure Co^∞ on $\mathcal{G}$. Given a local object P of $\mathcal{C}$, a section ∇ of $Co^\infty(P)$ is called holomorphic if it belongs to the subsheaf $Co(P)$ of $Co^\infty(P)$. Then a curving of $\mathcal{G}$ will be said to be *compatible with the holomorphic structure* if it satisfies the following condition

(3) For any local object P of $\mathcal{C}$ and any section ∇ of $co(P)$, we have:

$$K(\nabla) \in \underline{A}^{2,0}.$$

Then we have the following

Lemma 4.1. *(a) There exists a curving of $\mathcal{G}$ compatible with the holomorphic structure.*

(b) If $\nabla \mapsto K(\nabla)$ is a curving which is compatible with the holomorphic structure, and if ω is a 2-form of type $(2,0)$, then $\nabla \mapsto K(\nabla)+\omega$ is another curving compatible with the holomorphic structure. Any such curving arises in this way from some 2-form of type $(2,0)$.

This is very similar to a result of [**Bry2**], so the proof is omitted.

We then come to the main result of this section

Proposition 4.2. *(a) Let $\mathcal{C}$ be a holomorphic gerbe on the complex projective manifold X, and let Co be a holomorphic connective structure on $\mathcal{C}$. Let $\mathcal{G}$ be the corresponding gerbe with band $\underline{\mathbb{C}}_X^*$. Then there exists a curving $\nabla \mapsto K(\nabla)$ of Co such that*

(1) the curving is compatible with the holomorphic structure;

(2) the curving is flat, that is to say, its 3-curvature is 0.

(b) A curving satisfying the conditions in (a) is unique up to the addition of holomorphic 2-form on X.

Proof. Take any curving $\nabla \mapsto K(\nabla)$, which is compatible with the holomorphic structure. The 3-curvature Ω is equal locally to $dK(\nabla)$, for ∇ a holomorphic section of $Co(P)$, for some local holomorphic object P. Since $K(\nabla)$ is purely of type $(2,0)$, Ω has only components of types $(3,0)$ and $(2,1)$. However the cohomology class of Ω is purely imaginary. The intersection of $H^{3,0} \oplus H^{2,1}$ and of $\sqrt{-1} \cdot H^3(X, \mathbb{R})$ inside $H^3(X, \mathbb{C})$ is zero by Hodge theory. Therefore there exists some differential form β of degree 2 such that $\Omega = d\beta$. It then follows from Hodge theory that the differential of the de Rham complex is strictly compatible with the Hodge filtration (see [**De1**]). This implies that there exists a 2-form β of pure type $(2,0)$ such that $\Omega = d\beta$. Then the new curving $\nabla \mapsto K(\nabla) - \beta$ satisfies (1) and (2). A curving as in (a) is unique up to adding a 2-form ω which is of type $(2,0)$ and satisfies $d\omega = 0$, which is equivalent to ω holomorphic. ∎

Corollary 4.3. *The Beilinson regulator $c_{1,2}$ admits the following description, in terms of the holomorphic gerbe C with holomorphic connective structure associated to an element a of $H^1(X, \underline{K}_2)$. Pick a curving as in Proposition 4.2. Then $c_{1,2}(a)$ is the class of the homomorphism $H_2(X, \mathbb{R}) \to \mathbb{R}(1)$ given by $\Phi_*[\Sigma] \mapsto 2\pi\sqrt{-1} \cdot Log(S(\Phi))$, for $\phi : \Sigma \to X$ a smooth map from a closed oriented surface Σ to X. Note that $H_2(X, \mathbb{R})$ is generated by such classes $\Phi_*[\Sigma]$.*

We observe that the ambiguity of a holomorphic 2-form in the choice of a curving satisfying the conditions in Proposition 4.2 exactly corresponds to the fact that $H^3(X, \mathbb{R}(2)_D)$ is the quotient of $Hom(H_2(X, \mathbb{R}), \mathbb{R})$ by the homomorphisms induced by a holomorphic 2-form.

5. l-adic analogs

We wish to give a geometric description of the regulator map $c_{1,2}: H^1(X, \underline{K}_2) \to H^3(X, \mu_m^{\otimes 2})$ for X a scheme and m an integer which is invertible in $\mathcal{O}_X$. Here $H^3(X, \mu_m^{\otimes 2})$ is étale cohomology with coefficients in the étale sheaf $\mu_m^{\otimes 2} = \mu_m \otimes \mu_m$. This is a special case of the regulator map from algebraic K-theory to étale cohomology, which was introduced and studied by Soulé [**So**].

First we need an étale analog of the Deligne line bundle (f, g) associated with invertible holomorphic functions f and g. This means finding a description of the cup-product class in $H^2(X, \mu_m^{\otimes 2})$, without assuming that $\mathcal{O}_X$ contains μ_m. Note that $f \in \Gamma(X, \mathcal{O}_X^*) = \Gamma(X, \mathbb{G}_m)$ gives a class $[f]_m \in H^1(X, \mu_m)$ by taking the coboundary in the Kummer exact sequence of sheaves of groups

$$1 \to \mu_m \to \mathbb{G}_m \xrightarrow{x \mapsto x^m} \mathbb{G}_m \to 1 \qquad (5-1)$$

The element $[f]_m$ corresponds to the Galois covering $X[f^{1/m}] = Spec(\mathcal{O}_X[f^{1/m}]) \to X$ with group μ_m. So we have the covering $X[f^{1/m}, g^{1/m}] \to X$ of X, with Galois group $\mu_m \times \mu_m$.

We now introduce the *Heisenberg group* H_m, which is a finite étale group scheme over $Spec(\mathbb{Z}[1/m])$. As a scheme, we have: $H_m = \mu_m^{\otimes 2} \times \mu_m \times \mu_m$. The product law is

$$(\alpha_1, \zeta_1, \omega_1) \cdot (\alpha_2, \zeta_2, \omega_2) = (\alpha_1 \cdot \alpha_2 \cdot (\zeta_1 \otimes \omega_2), \zeta_1 \cdot \zeta_2, \omega_1 \cdot \omega_2). \qquad (5-2)$$

We have an exact sequence of group schemes over X

$$1 \to \mu_m^{\otimes 2} \to H_m \to \mu_m \times \mu_m \to 1.$$

Using H_m, we define a gerbe $\mathcal{C}$ over X for the étale topology, with band equal to $\mu_m^{\otimes 2}$. Let $f: Y \to X$ be an étale mapping; then an object of the category $\mathcal{C}(Y \xrightarrow{f} X)$ is a pair $(P \to Y, \psi)$, where

Lemma 5.2. *There is an action of H_m on P_m, such that*

$$\begin{array}{lll} (\zeta\otimes\zeta)\cdot x = x & (\zeta\otimes\zeta)\cdot y = y & (\zeta\otimes\zeta)\cdot f_\zeta^{1/m} = \zeta\cdot f_\zeta^{1/m} \\ (1,\zeta,1)\cdot x = \zeta\cdot x & (1,\zeta,1)\cdot y = y & (1,\zeta,1)\cdot f_\zeta^{1/m} = f_\zeta^{1/m} \\ (1,1,\zeta)\cdot x = x & (1,1,\zeta)\cdot y = \zeta\cdot y & (1,1,\zeta)\cdot f_\zeta^{1/m} = (\frac{1-y}{x})\cdot f_\zeta^{1/m}. \end{array}$$

Recall that we assume m odd.

Now we wish to descend this covering to $\mathbb{Q}\subset\mathbb{Q}[\mu_m]$. First to descend the curve P_m to $\mathbb{Q}$, we take $a\in(\mathbb{Z}/m\cdot\mathbb{Z})^* = Gal(\mathbb{Q}[\mu_m]/\mathbb{Q})$ and compare f_{ζ^a} with f_ζ. We see easily that $\frac{(f_{\zeta^a})^a}{f_\zeta}$ is an m-th power. It follows that the corresponding element of $H^1(\overline{Y}_m,\mu_m)$ transforms according to the inverse of the Teichmüller character. Therefore we get a Galois-invariant class in $H^1(\overline{Y}_m,\mu_m^{\otimes 2})$, which descends to $\mathbb{Q}$. This means that we obtain a covering of the jacobian variety $Jac(\overline{Y}_m)$ defined over $\mathbb{Q}$. Choosing a rational point a of $\overline{Y}_m$ over $\mathbb{Q}$, we get an Abel-Jacobi map $f:\overline{Y}_m\to Jac(\overline{Y}_m)$, $f(x)=(x)-(a)$, which is defined over $\mathbb{Q}$. Then we pull-back the covering of $Jac(\overline{Y}_m)$ to a covering of $\overline{Y}_m$, with group μ_m. Then one has to check that the action of H_m on P_m is rational over $\mathbb{Q}$, which poses no problem. Is is clear that the whole covering with H_m action is unramified outside of m, i.e. extends to $\mathbb{Z}[1/m]$. We summarize this in

Proposition 5.3. *There is an étale covering $P_m\to Spec(\mathbb{Z}[1/m,u,u^{-1},(1-u)^{-1}])$ with Galois group H_m, such that the corresponding covering with group $\mu_m\times\mu_m$ is*

$$\begin{array}{c} Spec(\mathbb{Z}[\frac{1}{m},u^{1/m},(1-u)^{1/m},u^{-1},(1-u)^{-1}]) \\ \downarrow \\ Spec(\mathbb{Z}[\frac{1}{m},u,u^{-1},(1-u)^{-1}]) \end{array}$$

We have therefore obtained, for m and odd integer, for X a scheme over $\mathbb{Z}[1/m]$, and for f and g invertible regular functions on X, a gerbe on X with band $\mu_m^{\otimes 2}$, which will de denoted by $(f,g)_m$. If $g=1-f$, the gerbe is trivial by Proposition 5.3. Hence, if X is a regular scheme over a field, the equivalence class of $(f,g)_m$ in $H^2(X,\mu_m^{\otimes 2})$, which is the cup-product $[f]_m \cup [g]_m$, depends only on the class of $\{f,g\}$ in $H^0(X,\underline{K}_2)$.

It is interesting to recover the usual Severi-Brauer scheme associated to f and g [**T**], when $\mathcal{O}_X$ contains μ_m. Fix a primitive m-th root of unity ζ. We then have the Stone-von Neumann representation $\rho : H_m \to GL(m)$ of H_m, on which $\zeta \otimes \zeta \in \mu_m^{\otimes 2}$ acts by $\zeta \cdot Id$. This defines a homomorphism of exact sequences of groups schemes

$$\begin{array}{ccccccccc} 1 & \to & \mu_m^{\otimes 2} & \to & H_m & \to & \mu_m \times \mu_m & \to & 1 \\ & & \downarrow & & \downarrow \rho & & \downarrow \rho & & \\ 1 & \to & \mathbb{G}_m & \to & GL(m) & \to & PGL(m) & \to & 1 \end{array}$$

Hence the covering of X with group $\mu_m \times \mu_m$ induces a principal bundle over X with structure group $PGL(m)$. This gives a Severi-Brauer algebra over X (i.e., a sheaf of simple central algebras) of rank m^2. This sheaf of algebras is the $\mathcal{O}_X$-algebra generated by elements a and b, subject to the relations

$$a^m = f, b^m = g, ab = \zeta \cdot ba. \qquad (5-4)$$

We now turn to the regulator $c_{1,2} : H^1(X,\underline{K}_2) \to H^3(X,\mu_m^{\otimes 2})$, where X is a scheme over a field k whose characteristic does not divide m. We have a purity theorem for étale cohomology (see [**SGA 4 1/2, p. 142**]) which implies that for $Y \subset X$ a subvariety of codimension ≥ 2, we have an exact sequence

$$0 \to H^3(X,\mu_m^{\otimes 2}) \to H^3(X \setminus Y,\mu_m^{\otimes 2}) \to H^4_Y(X,\mu_m^{\otimes 2}). \qquad (5-5)$$

The group $H^4_Y(X,\mu_m^{\otimes 2})$ is equal to the group of Galois-invariant elements of the similar group over the separable closure $\bar{k}$ of k. It therefore identifies with the free abelian group generated by the irreducible components of Y (over the base field k).

Now the group $H^3(X,\mu_m^{\otimes 2})$ identifies with the group of equivalence classes of so-called 2-gerbes over X with band $\mu_m^{\otimes 2}$. The theory of 2-gerbes has been developed by Breen [**Bre**]. Suffice it here to say that 2-gerbes are certain 2-stacks (or sheaves of 2-categories) in which the 2-arrows are all invertible, and the 1-arrows are invertible up to a 2-arrow. The group of 2-arrows from any given 1-arrow to itself is identified with our band $\mu_m^{\otimes 2}$. The exact sequence (5-5) means that a 2-gerbe on X is determined, up to equivalence, by its restriction to $X \setminus Y$.

We will use the Gersten resolution of $\underline{K}_2$ to represent a class in $H^1(X, \underline{K}_2)$ by an element $\sum_i f_i$ as in §3, where D_i is an irreducible divisor in X, and f_i is a non-zero meromorphic function on D_i. After removing from X a subvariety of codimension ≥ 2, we may assume that the D_i are smooth and pairwise disjoint, and that each f_i is a regular function on D_i.

We will need the notion of a formal gerbe along D_i. This will be a gerbe on the étale site of the completion $\hat{X}_{/D_i}$ of X along D_i, with band $\mu_m^{\otimes 2}$. Also we have the notion of a meromorphic formal gerbe, which is a gerbe over $\hat{X}_{/D_i}$ with band $\mu_m^{\otimes 2}$. We will choose a formal equation g_i of D_i, as well as an invertible formal function $F_i \in \Gamma(D_i, \hat{O}_{X\ /D_i})$ which induces f_i on D_i. Then we have the meromorphic formal gerbe $(F_i, g_i)_m$ along D_i.

We are then ready to define a 2-gerbe $\mathcal{G}$. We will for simplicity only describe the bicategory $\mathcal{G}(U)$ attached to an open subset U of X. An object of $\mathcal{G}(U)$ is a triple $(C, C_f, (\phi_i))$, where

(a) C is a gerbe with band $\mu_m^{\otimes 2}$ over $U \setminus Y$;

(b) C_f is a formal gerbe along $D_i \cap U \subset U$;

(c) for each i, $\phi_i : C_{mer} \otimes (F_i, g_i)_m \to (C_f)_{mer}$ is an equivalence of meromorphic formal gerbes along $D_i \cap U$. Here C_{mer} (resp. $(C_f)_{mer}$) denotes the meromorphic formal gerbe associated to C (resp. C_f).

Given two objects $(C_1, C_{f,1}, (\phi_{i,1}))$ and $(C_2, C_{f,2}, (\phi_{i,2}))$, a morphism (or 1-arrow) from $(C_1, C_{f,1}, (\phi_{i,1}))$ to $(C_2, C_{f,2}, (\phi_{i,2}))$ is a triple $(\alpha, \beta, (\gamma_i))$, where $\alpha : C_1 \to C_2$ is an equivalence of gerbes, $\beta : C_{f,1} \to C_{f,2}$ is an equivalence of meromorphic gerbes, and γ_i is a natural transformation from $\phi_{i,2}\alpha$ to $\beta\phi_{i,1}$. If $(\alpha', \beta', (\gamma_i'))$ is another such morphism from $(C_1, C_{f,1}, (\phi_{i,1}))$ to $(C_2, C_{,2}, (\phi_{i,2}))$, a 2-arrow from $(\alpha, \beta, (\gamma_i))$ to $(\alpha', \beta', (\gamma_i'))$ is pair (ω, ω_f), where ω is a natural transformation from α to α', ω_f is a natural transformation from β to β', which satisfy $\gamma_i'\omega = \omega_f\gamma_i$. Then we have

Proposition 5.4. *$\mathcal{G}$ is a 2-gerbe over the étale site of X with band equal to $\mu_m^{\otimes 2}$. Its class in $H^3(X, \mu_m^{\otimes 2})$ is equal to $c_{1,2}(\sum_i f_i)$, where $c_{1,2} : H^1(X, \underline{K}_2) \to H^3(X, \mu_m^{\otimes 2})$ is the Soulé regulator.*

Over $\mathbb{C}$, this 2-gerbe with band $\mu_m^{\otimes 2} \simeq \mu_m$ is obtained from the holomorphic e of §1 and §2, by the coboundary map in the exponential exact sequence.

References

[Be1] A. A. Beilinson, *Higher regulators and values of L-functions,* J. Soviet Math. **30** (1985), 2036-2070

[Be2] A. A. Beilinson, *Higher regulators of modular curves,* in Appl. of algebraic K-theory to Algebraic Geometry and Number Theory, Contemp. Math. vol. **55**(1986), 1-34

[Bl] S. Bloch, *Applications of the dilogarithm function in algebraic K-theory and algebraic geometry,* in Proc. Intern. Symp. on Algebraic Geometry, Kinokuniya Book Store (1978), 103-114

[B-O] S. Bloch and A. Ogus, *Gersten's conjecture and the homology of schemes,* Ann. Sci. Ec. Norm. Sup. **7** (1974), 181-202

[Bre] L. Breen, *Théorie de Schreier supérieure,* Ann. Sci. ENS, to appear

[Bry1] J-L. Brylinski, *The Kaehler geometry of the space of knots in a smooth threefold,* preprint (1990)

[Bry2] J-L. Brylinski, *Loop Spaces, Characteristic Classes and Geometric Quantization,* Progress in Math. **107** Birkhaüser (1993)

[De1] P. Deligne, *Théorie de Hodge II,* Publ. Math. IHES **40** (1971), 5-58

[De2] P. Deligne, *Le symbole modéré,* Publ. Math. IHES **73** (1991), 147-181

[De3] P. Deligne, *Le groupe fondamental de la droite moins trois points,* in Galois Groups over $\mathbb{Q}$, ed. Y. Ihara, K. Ribet, J-P. Serre, MSRI Publ. vol **16** (1989), 79-297

[D-W] C. Deninger and K. Wingberg, *On the Beilinson conjectures for elliptic curves with complex multiplication,* in **[R-S-S]**, 249-272

[E-V] H. Esnault and E. Viehweg, *Deligne-Beilinson cohomology,* in **[R-S-S]**, 43-92

[G] J. Giraud, *Cohomologie Non Abélienne,* Grundl. **179** , Springer Verlag(1971)

[I] Y. Ihara, *Profinite braid groups, Galois representations and complex multiplication,* Ann. of Math. **123** (1986), 2-106

[J] U. Janssen, *Deligne homology, Hodge $\mathcal{D}$-conjecture, and motives,* in **[R-S-S]**, 305-372

[K] M. Karoubi, *Formes différentielles non commutatives et cohomologie à coefficients arbitraires,* preprint (1992)

[Q] D. Quillen, *Higher algebraic K-theory I,* in Algebraic K-Theory, Lecture Notes in Math. vol. 341 Springer Verlag (1973), 85-147

[Ra1] D. Ramakrishnan, *A regulator for curves via the Heisenberg group,* Bull. Amer. Math. Soc. **5** (1981), 191-195

[Ra2] D. Ramakrishnan, *Arithmetic of Hilbert-Blumenthal surfaces,* CMS Conf. Proc. vol. **7** Amer. Math. Soc. (1987), 285-370

[R-S-S] M. Rapoport, N. Schappacher and P. Schneider, ed., *Beilinson's Conjectures on Special Values of L-Functions,* Perspectives in Math., Academic Press (1988)

[SGA4] M. Artin, A. Grothendieck and J-L. Verdier, *Séminaire de Géométrie Algébrique du Bois-Marie SGA 4, Tome 3,* Lecture Notes in Math. vol. **605** (1972), Springer Verlag

[SGA 4 1/2] P. Deligne, *Séminaire de Géométrie Algébrique du Bois-Marie SGA 4 1/2,* Lecture Notes in Math. **569** (1977), Springer Verlag

[So] C. Soulé, *K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale,* Invent. Math. **55**(1979), 251-295

[S-T] Y. T. Siu and G. Trautman, *Gap sheaves and extensions of coherent analytic sheaves,* Lecture Notes in Math. vol. **172**, Springer Verlag(1971)

[T] J. Tate, *Relations between* $\underline{K}_2$ *and Galois cohomology,* Invent.

Math. **36** (1976), 257-274

JEAN-LUC BRYLINSKI
The Pennsylvania State University
305 McAllister
University Park, PA. 16802
USA

Address till June 1994:

Harvard University
Department of Mathematics
1, Oxford Street
Cambridge, MA. 02138
USA