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CATEGORIES OF SET VALUED FUNCTORS

Marta Cavallo Bunge

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Peter Freyd
Supervisor of Dissertation

Oscar Goldman
Graduate Group Chairman

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PREFACE

The theory of categories was introduced by Eilenberg and Mac Lane in 1945 [4] ; it arose from the field of topology. It was soon realized that other mathematical theories as well could profit from their invention. This was initially the main reason for the increasing interest in categories. The applications brought soon attention to problems peculiar to the theory of categories, which in a few years grew enough to become another area of mathematics. Even so, the now widespread interest in category theory seems still to lie in the many virtues of its applications, such as its unifying character, elegant and concise language, fruitfulness and emphasis on results involving structure. This led to the idea that category theory might provide a more suitable foundation for mathematics than set theory. To carry out this program it was necessary to have also a theory of the (meta)category of categories. Lawvere [17] has recently provided such a theory; this seems to be the proper framework in which to develop mathematics on a categorical basis.

An important step in the program of categorizing mathematics has been accomplished by Lawvere himself [16] upon reformulating set theory in terms of categorical concepts alone, namely, those of mapping, domain, codomain and composition.

In this paper we study a class of categories closely related to the category of sets and mappings. An essential prerequisite will be an acquaintance with [16] . To study this class of categories we introduce what we call regular categories, which are weakened abelian categories , especially as axiomatized by Freyd [8] , so that [8] is also assumed as a prerequisite. A general knowledge of category theory is required as well. Among the various sources, Freyd [8] , Mac Lane [22] and Mitchell [23] seem to be the more introductory ones. Also, an acquaintance with the literature on adjoint functors, starting with Kan [13] and following with several others , e.g., Freyd [6 , 8], Lawvere [14] , will be assumed.

The formation of functor categories is one of the basic constructions in the (meta)category of categories. Given any two categories $\mathcal{X}$ and $\mathcal{Y}$, the functor category denoted by $\mathcal{Y}^{\mathcal{X}}$ has as objects all functors with domain $\mathcal{X}$ and codomain $\mathcal{Y}$ and as maps, all natural transformations between these. We will be concerned in this paper with a special type of functor categories : those for which the codomain category is $\mathcal{S}$, the category of sets and mappings.

A motivation for this choice can be found in the following ; any category with small hom-sets is a full subcategory of a category of this type. Explicitly : if the category $\mathcal{X}$ has small hom-sets, there is a bifunctor HOM : $\mathcal{X}^* \times \mathcal{X} \longrightarrow \mathcal{S}$, which induces by exponential adjointness a functor H : $\mathcal{X} \longrightarrow \mathcal{S}^{\mathcal{X}^*}$. The latter is full, faithful and preserves all left roots existing in $\mathcal{X}$: it is called the regular representation of $\mathcal{X}$.

However, if $\mathcal{X}$ is not small, then $\mathcal{S}^{\mathcal{X}^*}$ will not have small hom--sets, and thus a not very manageable category. Fortunately there are many interesting categories which, though not small admit a regular representation into a category with small hom-sets. These are categories which have a small subcategory , let $\mathcal{A} \xrightarrow{j} \mathcal{X}$ be the inclusion functor, and such that the composite functor

$$\mathcal{X} \xrightarrow{H} \mathcal{S}^{\mathcal{X}^*} \xrightarrow{\mathcal{S}^{j^*}} \mathcal{S}^{\mathbb{A}^*}$$

is still full and faithful. The functor is called the subregular representation of $\mathcal{X}$ over $\mathbb{A}$, and $\mathbb{A}$ is said to be an adequate subcategory of $\mathcal{X}$. Therefore, if we restrict ourselves - as we will- to the study of categories of set valued functors with small domain category, the class of categories admitting a representation as full subcategories of these does not reduce to the class of small categories. The broader class of categories with adequate subcategories are investigated by Isbell [12] and it includes, e.g., every algebraic category in the sense of Lawvere [14 , 15] : in this case, the dual of the corresponding algebraic theory is canonically embedded as an adequate subcategory.

Every category whose objects are all set valued functors with a given small domain category is seen to be equivalent to a category of diagrams in $\mathcal{S}$ with a given diagram scheme (Grothendieck [10], Mac Lane [21] , Mitchell [23]). This suggests the name "diagrammatic" or "$\mathcal{S}$ - diagrammatic" for these categories. We adopt throughout this paper the name "diagrammatic" for any category of the form $\mathcal{S}^{\mathbb{C}}$, with $\mathbb{C}$ any small category.

In chapter I we study diagrammatic categories in general, simultaneously comparing them with $\mathcal{S}$, which is the basic diagrammatic category.

The aim of chapter II is to characterize abstractly the class of diagrammatic categories. We first introduce the theory of regular categories, the name being suggested by a consequence of the axioms according to which every map factors uniquely into an epi followed by a mono, and which is usually called a regularity condition. It is strong enough to exclude most algebraic categories, and those which satisfy a regularity condition are called regular. All diagrammatic categories are regular, and they are by no means the only regular categories : all abelian categories are regular as well, and none is diagrammatic. Therefore, if we hope to characterize diagrammatic categories from regular categories, the strenght ening of the axioms has to be done in a different way than abelianess.

At this point we notice a striking analogy between the regular representation theorem for any category with a small adequate subcategory, and the representation theorem for Boolean algebras which says that every Boolean algebra is isomorphic to a field of sets. Thus, if we let regular categories with small adequate subcategories correspond to Boolean algebras, then regular categories of set-valued functors with a small domain category (not necessarily all such functors) must correspond to fields of sets if the analogy between the two theorems is to be mantained. Also, fields of all subsets of a set must correspond to diagrammatic categories. It is now that the analogy gives some fruits : since the fields of all subsets of some set are precisely the complete atomic Boolean algebras, we might try an analogous characterization of diagrammatic categories. With the analogy in mind, we first stipulate which objects in a regular category should be called "atoms" , and with this, when should a regular category be called "atomic" . It turns out that complete atomic regular categories have the atoms as an adequate subcategory, so that the existence of a small adequate subcategory need not be postulated before. And what is more important, complete atomic regular categories are precisely the diagrammatic categories. That is, just as any complete atomic Boolean algebra is isomorphic to the field of all subsets of the set of its atoms, so any complete (right-complete is enough) atomic regular category is isomorphic to the diagrammatic category with domain category the dual of the full subcategory determined by its atoms.

In chapter III we aim at the question of when are isomorphic any two given diagrammatic categories, which is the same question that Morita [24] asked for categories of modules (see also Bass [2]). For this purpose we first study functors between diagrammatic categories which have adjoint or coadjoint. Our results can also be found in André [1] , though the methods of proof are different , as a result of dispensing with generality from our side. Next, we use these results to establish, as Freyd noticed in [7 , 8] , that it is not the small domain category which determines completely the functor category (in his case these were categories of additive group-valued functors) but its amenable closure. The main theorem of the chapter is called "Morita isomorphism theorem for diagrammatic categories" and states that any two given diagrammatic categories are isomorphic iff the idempotent-splitting closures of the corresponding small domain categories are isomorphic. This is used to investigate the question of the uniqueness of the representation of a category as a diagrammatic category.

Chapter IV is a study of the algebraic side of every algebraic category. For this we need the theory of triples and triplable categories as introduced and developped by Huber, Beck, and Eilenberg and Moore. To avoid further requirements, we review briefly the ideas employed in the chapter. We next discuss some relations between triples and cotriples which form an adjoint pair as well, and use this information to find out which are all coadjoint triples in $\mathcal{S}$. They are given by all sets, so that Coadj Triples($\mathcal{S}$) $\cong \mathcal{S}^*$, since the correspondence is contravariantly functorial. On the other hand, adjoint triples on $\mathcal{S}$ are given by monoids. Similar questions arise for categories of the form $\mathcal{S}^I$, with I a set , regarded as a discrete category. Adjoint triples on a category $\mathcal{S}^I$, are given by all small categories whose set of objects are isomorphic to I . And the diagrammatic categories with these small domain categories come close to being the algebras of the triple. Actually, to see better which are the algebras, we introduced the notions of relative category and relative functor. These ideas have further potentialities which are beyond the scope of this paper.

Some notations and conventions are the following : (1) small categories will be denoted by $\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots, \mathbb{X}, \mathbb{Y}, \mathbb{Z}$; (2) arbitrary categories will be denoted by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$; (3) $\mathcal{S}$ will always denote the category of sets ; (4) the small categories which are preorders will be denoted by $\mathbb{0}, \mathbb{1}, \mathbb{2}, \mathbb{3}, \ldots$; (5) small categories which are discrete will be denoted the same way as sets are , by I , J , K , etc. ; (6) $\mathbb{E}$ is the category pictured thus :  ; (7) the set of objects of a small category $\mathbb{C}$, will be denoted $|\mathbb{C}|$; (8) the dual of any category $\mathcal{A}$ will be denoted $\mathcal{A}^*$; (9) composition of maps is denoted in the diagrammatic order , and evaluation is on the left ; (10) the identity map of the object A is either 1_A or A .

Chapter I

DIAGRAMMATIC CATEGORIES IN RELATION TO THE CATEGORY OF SETS

Let $\mathbb{C}$ be a fixed but arbitrary small category. We denote by $\mathcal{S}$ the category of sets and mappings, and by $\mathcal{S}^{\mathbb{C}}$ the category whose objects are all covariant functors $\mathbb{C} \longrightarrow \mathcal{S}$ and whose maps are all natural transformations between these. For reasons given in the Preface, any such category will be said to be <u>diagrammatic</u>. Our aims in this chapter are: (1) to describe properties which are common to all diagrammatic categories; (2) to determine the extent to which these properties rely on properties of $\mathcal{S}$; (3) to investigate the range of validity in the class of diagrammatic categories of the axioms of Lawvere's elementary theory of $\mathcal{S}$.

§ 1 - FINITE ROOTS

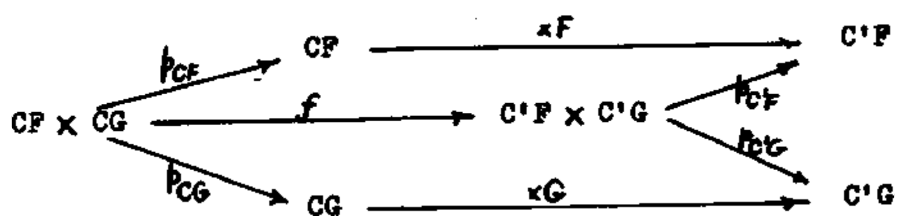
A category $\mathcal{X}$ is said to have <u>finite roots</u> iff for every small category such that its set of objects is finite, and letting $\mathbb{A}$ be one such, the functor $\mathcal{X} \longrightarrow \mathcal{X}^{\mathbb{A}}$ induced by the functor $\mathbb{A} \longrightarrow \mathbb{1}$, has both a coadjoint (insuring the existence of left roots) and an adjoint (right finite roots). It has been shown ([8] , [14]) that it is enough that the category has terminal and coterminal objects ($\mathbb{A} \approx \mathbb{O}$) , binary products and coproducts ($\mathbb{A} \approx |2|$) and equalizers and coequalizers ($\mathbb{A} \approx \mathbb{E}$) for it to have all finite roots. Among the finite roots are finite products and coproducts, pull-backs and push-outs, images and inverse images, unions and intersections. We now show that any diagrammatic category has finite roots.

<u>Proposition 1.1</u> For any small category $\mathbb{C}$, $\mathcal{S}^{\mathbb{C}}$ has finite roots.

Proof:

A terminal object for $\mathcal{S}^{\mathbb{C}}$ is given by the functor which is constantly 1, where 1 is the name for the terminal object in $\mathcal{S}$. A coterminal object is given dually and denoted 0.

Given any two functors F and G we define $(F \times G, p_F, p_G)$ as follows: let $C(F \times G) = CF \times CG$; $(p_F)_C = p_{CF}$ and $(p_G)_C = p_{CG}$, for any $C \in |\mathbb{C}|$. If $C \xrightarrow{x} C'$ is a map in $\mathbb{C}$, let $x(F \times G) = f$ where f is the unique map which renders commutative the following diagram:



By the way $x(F \times G)$ is defined, this says not only that $F \times G$ is a functor, but also that $p_F : F \times G \longrightarrow F$ and $p_G : F \times G \longrightarrow G$ are natural transformations. Dually one can define the coproduct $F + G$ together with the canonical injections i_F and i_G.

Given any two natural transformations η and ξ, we want to define their equalizer. For this, we look again in each coordinate, and let $e_C = \mathrm{Eq}(\eta_C, \xi_C)$ for each $C \in |\mathbb{C}|$. We show next that the family so obtained can be made into a natural transformation e which moreover is the equalizer of η and ξ. For this we first define a functor, the domain of e as follows: let $CE = E_C$ where $E_C \xrightarrow{e_C} CF \overset{\eta_C \cdot \xi_C}{\rightrightarrows} CG$

is an equalizer diagram. If $C \xrightarrow{x} C'$ is a map in $\mathbb{C}$, let xE be defined as the unique map $f : CE \longrightarrow C'E$ such that $fe_{C'} = e_C(xF)$. That this map f exists and is unique follows from the universal property of equalizers together with the following identity:

$$(e_C(xF))\eta_{C'} = e_C((xF)\eta_{C'}) = e_C(\eta_C(xG)) = (e_C\eta_C)(xG) = (e_C\xi_C)xG =$$
$$= e_C(\xi_C(xG)) = e_C((xF)\xi_{C'}) = (e_C(xF))\xi_{C'}.$$

With this we have that E is a functor and $e : E \longrightarrow F$ a natural transformation and it is immediate to see that it is the equalizer of η and ξ. Coequalizers are dually defined. QED.

§ 2 - THE EXISTENCE OF A GENERATING FAMILY

In $\mathcal{S}$, the terminal object 1 is a generator. Arbitrary diagrammatic categories need not have a generator, but they always have a generating family of objects. We will show that the generating property of a particular generating family in each diagrammatic category is a consequence of the generating property of 1 in $\mathcal{S}$.

As usual, a functor is said to be <u>representable</u> and denoted by H^C if it is $C \in |\mathbb{C}|$ which represents it, iff it is naturally equivalent to the functor $HOM(C, \ )$. The family of representable functors in any diagrammatic category has the size of the domain category for the functors. We want to show that it is generating, for which purpose we need to state and prove (for reference) a lemma due to Yoneda.

<u>Lemma 2.1</u> (Yoneda) For any small $\mathbb{C}$, any F in $\mathcal{S}^{\mathbb{C}}$, and any

$$C \in |\mathbb{C}|,\ (H^C, F)\text{nat} \cong CF \cong HOM_{\mathcal{S}}(1, CF)$$

Proof:

Let $\phi : (H^C, F) \longrightarrow CF$ be defined for $\eta \in (H^C, F)$ by $\eta\phi = 1_C \eta_C \in CF$

Let $\psi : CF \longrightarrow (H^C, F)$ be defined for $z \in CF$ as the natural transformation $z\psi : H^C \longrightarrow F$ defined for $x \in C'H^C = HOM(C,C')$ by $x(z\psi)_{C'} = z(xF)$ and naturality follows since for any $C' \xrightarrow{y} C''$ the following diagram commutes:

$$\begin{array}{ccc} C'H^C & \xrightarrow{(z\psi)_{C'}} & C'F \\ {\scriptstyle yH^C}\downarrow & & \downarrow{\scriptstyle yF} \\ C''H^C & \xrightarrow{(z\psi)_{C''}} & C''F \end{array}$$

That it is so can be seen as follows: let $x \in HOM(C,C')$, arbitrary. Then we have that $x(z\psi)_{C'}(yF) = (x(z\psi)_{C'})(yF) = (z(xF)(yF)) =$ $= z((xy)F) = (xy)(z\psi)_{C''} = (x(yH^C))(z\psi)_{C''} = x((yH^C)(z\psi)_{C''})$. It is now easy to verify that both $\phi\psi$ and $\psi\phi$ are identities. QED .

<u>Theorem 2.2</u> For any small $\mathbb{C}$, the family $\{H^C\}$ $C \in |\mathbb{C}|$ is generating for $\mathcal{S}^{\mathbb{C}}$.

Proof:

Given any two natural transformations $F \xrightarrow{\eta,\xi} G$ such that they are different, there must exist at least a $C \in |\mathbb{C}|$ for which $\eta_C \neq \xi_C$. This implies that there exists a map $1 \xrightarrow{z} CF$ in $\mathcal{S}$, such that $z\eta_C \neq z\xi_C$. By Yoneda, let $z\psi : H^C \longrightarrow F$ be the corresponding natural transformation. We want to show that $(z\psi)\eta \neq (z\psi)\xi$. This will be so iff $\exists$ $C' \in |\mathbb{C}|$ such that $(z\psi)_{C'}\eta_{C'} \neq (z\psi)_{C'}\xi_{C'}$. Take $C' = C$. For $(z\psi)_C\eta_C$ to be different from $(z\psi)_C\xi_C$ it is enough that there exists $x \in HOM(C,C)$ for which $x(z\psi)_C\eta_C$ be different from $x(z\psi)_C\,\xi_C$. Let $x = 1_C$, then we have that

$$(1_C\,(z\psi)_C)\eta_C = (z(1_C F))\eta_C = z\,\eta_C \neq z\,\xi_C = (z(1_C F))\xi_C = (1_C(z\psi)_C)\xi_C$$

which implies the desired result. QED .

§ 3 - EXPONENTIATION

A category with products is said to have <u>exponentiation</u> iff for any object A the functor $A \times (\ \ )$ has a coadjoint, denoted $(\ \ )^A$. The category of sets has exponentiation and for every set A, we have that $(\ \ )^A = \mathrm{HOM}(A,\ \ )$. However, $\mathcal{S}$ is the only category in which exponentiation is given by HOM , precisely because $(\ \ )^A$ has to be an endofunctor while the only category for which HOM $(A,\ \ )$ is an endofunctor for every object A, is $\mathcal{S}$. All diagrammatic categories have exponentiation. However, the proof that it is so is not straightforward as the proof of the existence of finite roots was, and this is so because exponentiation is not defined coordinatewisely.

<u>Theorem 3.1</u> For any small $\mathbb{C}$, and any object F in $\mathcal{S}^{\mathbb{C}}$, the endofunctor $F \times (\ \ ): \mathcal{S}^{\mathbb{C}} \longrightarrow \mathcal{S}^{\mathbb{C}}$ has a coadjoint.

Proof:

Define a functor $(\ \ )^F: \mathcal{S}^{\mathbb{C}} \longrightarrow \mathcal{S}^{\mathbb{C}}$ as follows:

if G is any object of $\mathcal{S}^{\mathbb{C}}$, let the value at $C \in |\mathbb{C}|$ of G^F be given by

$$C\,G^F = (H^C \times F,\ G)_{nat}$$

and extend it to the maps $C \longrightarrow C'$ in the obvious fashion so that it becomes a functor. We can now define a natural transformation

$$F \times G^F \xrightarrow{\ ev\ } G$$

called <u>evaluation</u>, as follows: given $C \in |\mathbb{C}|$ one has to say what is $ev_C : CF \times C(G^F) \longrightarrow CG$, that is, $ev_C : CF \times (H^C \times F\,,\, G) \longrightarrow CG$

If $z \in CF$ and $\eta \in (H^C \times F, G)$, define $(z,\eta)ev_C = (1_C, z)\eta_C$.

If $C \xrightarrow{x} C'$, there are induced maps $CF \xrightarrow{xF} C'F$ and $((H^x \times F), G) : (H^C \times F, G) \longrightarrow (H^{C'} \times F, G)$ and these two induce $xF \times ((H^x \times F),G) : CF \times (H^C \times F, G) \longrightarrow C'F \times (H^{C'} \times F, G)$, and the following diagram is commutative:

$$\begin{array}{ccc} CF \times (H^C \times F, G) & \xrightarrow{\quad ev_C \quad} & CG \\ {\scriptstyle xF \times ((H^x \times F), G)} \downarrow & & \downarrow {\scriptstyle xG} \\ C'F \times (H^{C'} \times F, G) & \xrightarrow{\quad ev_{C'} \quad} & C'G \end{array} \qquad (*)$$

To see that the diagram is commutative we take any $z \in CF$ and any $\eta \in (H^C \times F, G)$, and travel in the two orientations. We have

$(z,\eta)\, ev_C\, (xG) = (1_C, z)\, \eta_C(xG)$ and

$(z,\eta)(xF \times ((H^x \times F),G))ev_{C'} = (z(xF)\,,\, (H^x \times F)\eta)ev_{C'} =$

$= (1_{C'}\,,\, z(xF)((H^x \times F)\eta)_{C'} = (x, z(xF))\,\eta_{C'}.$

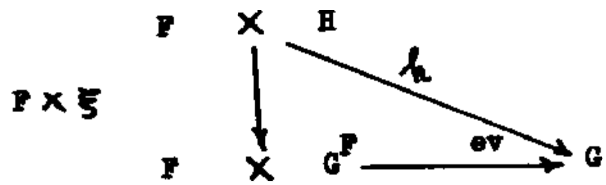
We now use the fact that η is a natural transformation, so that the following diagram commutes:

$$\begin{array}{ccc} CH^C \times CF & \xrightarrow{\quad \eta_C \quad} & CG \\ {\scriptstyle H^x \times xF} \downarrow & & \downarrow {\scriptstyle xG} \\ C'H^C \times C'F & \xrightarrow{\quad \eta_{C'} \quad} & C'G \end{array}$$

and so, for $1_C \in CH^C$ and $z \in xF$, this says precisely that $(1_C, z)\eta_C(xG) = (1_C, z)(H^x \times xF)\,\eta_{C'} = (x, z(xF))\,\eta_{C'}$ so that (*) above is commutative, and so evaluation is indeed a natural transformation.

We still have to show that $(\ )^F$ is coadjoint to $F \times (\ )$, and it is for this purpose that we will use the evaluation map just defined.

Suppose given any functor H and a natural transformation $h: F \times H \longrightarrow G$ to show that there exists a unique natural transformation $\xi : H \longrightarrow G^F$ such that $(F \times \xi)ev = h$, i.e., such that the following diagram is commutative:



Let be given for each $C \in |\mathbb{C}|$ as follows: if $y \in CH$, let $y(\xi_C) \in (H^C \times F, G)$ be given by, for $x' \in C'H^C$ and $z' \in C'F$ let $(x', z')(y\,\xi_C)_{C'} = (z'(x'F), y)h$.
We verify now that $(F \times \xi)ev = h$: given $C \in |\mathbb{C}|$, $z \in CF$ and $y \in CH$ then $((z, y)(F \times \xi))ev_C = (z, y\xi_C)ev_C = (1_C, z)(y\xi_C)_C = (z(1_CF), y)h = (z, y)h$. The definition of ξ was forced to make the diagram commute and it is easy to see that it is the only possible choice. QED.

A functor which has a coadjoint preserves all right roots that exist, so that the existence of exponentiation for any diagrammatic category implies that products distribute over coproducts and that products preserve coequalizers.

It is known that if $\mathbb{C}$ is any small category, the regular representation functor $H : \mathbb{C} \longrightarrow \mathcal{S}^{\mathbb{C}^*}$ defined by $CH = HOM(\ , C)$, is full and faithful and preserves all left roots which might exist in $\mathbb{C}$. In fact, if $\mathcal{X}$ is not small, but has a small adequate subcategory (Isbell [12]) $\mathcal{A}$, the subregular representation functor of $\mathcal{X}$ over $\mathcal{A}$, which is just the composition $\mathcal{X} \xrightarrow{H} \mathcal{S}^{\mathcal{X}^*} \xrightarrow{\mathcal{S}^{j^*}} \mathcal{S}^{\mathcal{A}^*}$ is by definition, full and faithful and it preserves left roots since each of the composite functors does.

What is not known is that if exponentiation exists , then the regular representation functor or the subregular representation functor preserve it. We prove two separate theorems to that effect:

<u>Theorem 3.2</u> Let $\mathbb{C}$ be small and with exponentiation. Then, the regular representation functor $H: \mathbb{C} \longrightarrow \mathcal{S}^{\mathbb{C}^*}$ preserves exponentiation.

Proof:

Let A and B be objects in $\mathbb{C}$, we have to show that

$$H(B^A) = (B^A)H \cong BH^{AH} = H_B{}^{H_A}$$

By definition, given $C \in |\mathbb{C}|$ $C\, H(B^A) = HOM\,(C, B^A)$ and

$C\,(H_B{}^{H_A}) = (H_C \times H_A\,, H_B) \cong (H_{C\times A}\,, H_B) \cong HOM\,(C \times A\,, B)$

And since $\mathbb{C}$ is assumed to have exponentiation , we have that

$HOM\,(C \times A, B) \cong HOM\,(C\,, B^A)$ which finishes the proof. QED .

<u>Theorem 3.3</u> Let $\mathcal{X}$ be any category and let $\mathbb{A}$ be an adequate subcategory of $\mathcal{X}$. Then, if $\mathcal{X}$ has exponentiation, the subregular representation of $\mathcal{X}$ over $\mathbb{A}$, that is, the functor

$$\mathcal{X} \xrightarrow{\quad H \quad} \mathcal{S}^{\mathcal{X}^*} \xrightarrow{\quad \mathcal{S}^{j^*} \quad} \mathcal{S}^{\mathbb{A}^*}$$

preserves exponentiation.

Proof:

Let X and Y be any two objects in $\mathcal{X}$. We have to show that

$Y^X(H\mathcal{S}^{j^*}) \cong (Y(H\mathcal{S}^{j^*}))^{(X(H\mathcal{S}^{j^*}))}$ Let $A \in |\mathbb{A}|$, arbitrary.

Onthe one hand, $A\,(Y^X(H\mathcal{S}^{j^*})) = A(H_{(Y^X)}\mathcal{S}^{j^*}) = A\,j^*\,H_{(Y^X)} = HOM\,(Aj^*, Y^X)$.

On the other hand we have:

$A(YH\mathcal{S}^{j^*})^{(XH\mathcal{S}^{j^*})} = (H_A \times j^*\,H_Y\,, j^*\,H_X) \cong (j^*\,H_{Aj^*} \times j^*\,H_Y\,, j^*\,H_X) \cong$

$$\cong j^*(H_{Aj^*} \times H_Y, H_X) \cong j^*(H_{Aj^* \times Y}, H_X) \cong \mathrm{HOM}(Aj^* \times Y, X) =$$
$$\cong \mathrm{HOM}(Aj^*, Y^X) . \quad \mathrm{QED} .$$

§ 4 – AUTONOMY

An <u>autonomous</u> category (Linton [18]) is a category $\mathcal{A}$ together with a bifunctor

$$\mathcal{A}(\ ,\ ) : \mathcal{A}^* \times \mathcal{A} \longrightarrow \mathcal{A}$$

and a forgetful functor

$$U : \mathcal{A} \longrightarrow \mathcal{S}$$

such that the following tringle is commutative:

$$\begin{array}{ccc} \mathcal{A}^* \times \mathcal{A} & \xrightarrow{\mathcal{A}(,)} & \mathcal{A} \\ & \searrow^{\mathrm{HOM}} \quad \swarrow_{U} & \\ & \mathcal{S} & \end{array}$$

Moreover, there is a law of composition for $\mathcal{A}(\ ,\ )$, which is given by a collection of maps , one for each triple (A,B,C) of objects in $\mathcal{A}$, and which is natural in each of the three variables, it is associative and behaves well with respect to a ground object if there is any. The domain and range of the maps are

$$L^A_{B,C} : \mathcal{A}(B, C) \longrightarrow \mathcal{A}(\mathcal{A}(A, B), \mathcal{A}(A, C))$$

With the above one can introduce "tensor products" as follows: let $L^A : \mathcal{A} \longrightarrow \mathcal{A}$ be defined by $B L^A = \mathcal{A}(A, B)$, for any A and B in $\mathcal{A}$. Given A and B , consider L^A and L^B . If we assume that the composition $L^A L^B$ is representable, and denoting the objects which represents it by $A \otimes B$, we have that

$\mathcal{A}(A\otimes B\ ,\ C) = C\ L^{A\otimes B} = C\ L^{A}\ L^{B} = \mathcal{A}(A,\ C)\ L^{B} = \mathcal{A}(B, \mathcal{A}(A,\ C))$

which indicates precisely that for each $A \in \mathcal{A}$, $A \otimes (\quad)$ is adjoint to $\mathcal{A}(A, \quad)$.

But one can also start with tensor products, to mean the categorical products if the category has any, and see wheter the category has exponentiation as well as a forgetful functor and then shown to be autonomous with the bifunctor gotten from exponentiation by letting both the base and the exponent vary. However, if this method is adopted for introducing the cohcept of autonomous category, one has to show that there is a law of composition as required. This is done as follows: let $\mathcal{A}$ be any category with exponentiation (and products), and let us denote by $(\quad)\otimes(\quad)$ and $(\quad)^{(\quad)}$ the two bifunctors corresponding to the operations of forming products and exponentiating, respectively. Given any three objects A, B, C in $\mathcal{A}$, by exponential adjointness there is a corresponding evaluation map

$$ev : C^{B} \otimes B \longrightarrow C$$

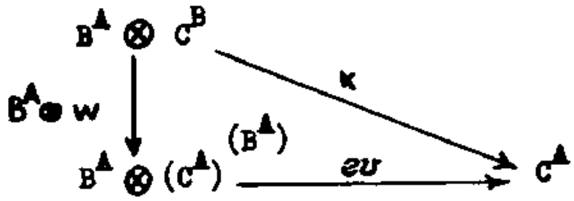
and we let h be the map given by composition of the following maps:

$$C^{B} \otimes B^{A} \otimes A \xrightarrow{\ h\ } C \quad = \quad C^{B} \otimes B^{A} \otimes A \xrightarrow{C^{B}\otimes ev} C^{B} \otimes B \xrightarrow{ev} C$$

Let now k be the unique map such that the following diagram commutes:

$$\begin{array}{ccccc} C^{B}\otimes B^{A}\otimes A & \cong & A\otimes C^{B}\otimes B^{A} & & \\ & & \Big\downarrow {\scriptstyle A\otimes k} & \searrow^{h} & \\ & & A\otimes C^{A} & \xrightarrow{\ ev\ } & C \end{array}$$

by exponential adjointness, and again use exponential adjointness to define w as the unique map which renders commutative the diagram:



Since w was defined after a triple (A,B,C) was chosen, we can denote it by $w^A_{B,C}$, and it is a member of the family of maps which give the composition law since $w^A_{B,C}$: $C^B \longrightarrow ((C^A)^{(B^A)})$.

Therefore, we have shown that the above is an equally good method for introducing autonomous categories. We use this to show:

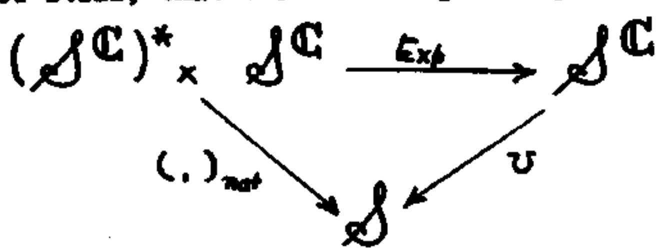
<u>Theorem 4.1</u> For any small $\mathbb{C}$, $\mathcal{S}^{\mathbb{C}}$ is an autonomous category.

Proof:

We already know that all diagrammatic categories have exponentiation (Theorem 3.1) so that we have to find a forgetful functor and show that they are related as they should for autonomy.

Let $U : \mathcal{S}^{\mathbb{C}} \longrightarrow \mathcal{S}$, be given by : if T is any object in $\mathcal{S}^{\mathbb{C}}$, let $TU =_{df} (1, T)_{nat}$, and the obvious extension for the maps.

Then we need to show still, that the following triangle is commutative:



To see this, let F and G be any two objects in $\mathcal{S}^{\mathbb{C}}$, then

$(F, G)\ \mathrm{Exp}\ U =_{df} G^F\ U =_{df} (\ 1\ ,\ G^F\ )_{nat} \cong (\ 1 \times F\ ,\ G\ )_{nat} \cong (\ F\ ,\ G\ )_{nat}$

since the functor 1 has the property that for every T in $\mathcal{S}^{\mathbb{C}}$, $1 \times T \cong T$, same as in the category of sets. Therefore, the above triangle is commutative and $\mathcal{S}^{\mathbb{C}}$ is autonomous. QED .

§ 5 – THE EXISTENCE OF A COGENERATOR

In $\mathcal{S}$, $2 = 1 + 1$ is a cogenerator. We will show that any diagrammatic category has a cogenerator, not only $\mathcal{S}$, and that the fact that it is a cogenerator relies on the fact that 2 is a cogenerator in $\mathcal{S}$.

Let $\mathcal{S}^{\mathbb{C}}$ be any diagrammatic category, i.e., $\mathbb{C}$ is an arbitrary small category. By H_C for $C \in |\mathbb{C}|$, we mean the contravariant functor whose value at an object C' of $\mathbb{C}$, is $C'H_C = HOM_{\mathbb{C}}(C', C)$. It is not an object in $\mathcal{S}^{\mathbb{C}}$ but in $\mathcal{S}^{\mathbb{C}^*}$, and it may be called a <u>corepresentable</u> functor, corepresented by C.

On the other hand, consider the functor $HOM_{\mathcal{S}}(\ \ , 2) : \mathcal{S}^* \longrightarrow \mathcal{S}$, which is denoted by H_2.

Let now $Q^C =_{df} H_C H_2$. To be able to compose them, the codomain category of H_C has to be equal to the domain category of H_2. This can be done in two different ways since, in general, a functor $T : \mathcal{A} \longrightarrow \mathcal{B}$ which is contravariant can be viewed either as a covariant functor with domain $\mathcal{A}^*$ and codomain $\mathcal{B}$, or as a covariant functor with domain $\mathcal{A}$ and codomain $\mathcal{B}^*$. Accordingly, there are two ways of composing the covariant versions of H_C and H_2, and we obviously choose Q^C to be

$$\mathbb{C} \xrightarrow{H_C} \mathcal{S}^* \xrightarrow{H_2} \mathcal{S}$$

which in any case is covariant, and so, an object in $\mathcal{S}^{\mathbb{C}}$. Explicitly, the value of Q^C (for $C \in |\mathbb{C}|$) at an object C' of $\mathbb{C}$, is:

$$C' Q^C = HOM_{\mathcal{S}}(HOM_{\mathbb{C}}(C', C), 2)$$

and if $C' \xrightarrow{x} C''$ is a map in $\mathbb{C}$, it induces a map

xQ^C : $HOM_{\mathcal{S}}(HOM_{\mathbb{C}}(C', C), 2) \longrightarrow HOM_{\mathcal{S}}(HOM_{\mathbb{C}}(C'', C), 2)$
which is defined for $f : HOM_{\mathbb{C}}(C', C) \longrightarrow 2$, by $f(xQ^C): HOM_{\mathbb{C}}(C'', C) \longrightarrow 2$
given by, for $z \in HOM(C'', C)$, $z(f(xQ^C)) = (yz)f$.
Let us now consider the family indexed by $|\mathbb{C}|$, whose members are the functors Q^C . We want to show that it is cogenerating for which purpose we prove first a lemma corresponding to Yoneda lemma and which we may call <u>Co-Yoneda lemma</u> for reference, although it is not precisely dual to Yoneda lemma, but plays a dual role only.

<u>Lemma 5.1</u> (Co-Yoneda) For any small $\mathbb{C}$, any G in $\mathcal{S}^{\mathbb{C}}$, and any $C \in |\mathbb{C}|$, $(G, Q^C)_{nat} \cong HOM_{\mathcal{S}}(CG, 2)$.

Proof:
Let $\phi : (G, Q^C) \longrightarrow HOM_{\mathcal{S}}(CG, 2)$ be defined by, if $\eta \in (G, Q^C)$, let $\eta\phi = \alpha_\eta \in HOM_{\mathcal{S}}(CG, 2)$ be such that, for $x \in CG$, $x\alpha_\eta = 1_C\, x\, \eta_C$.
Let $\psi : HOM_{\mathcal{S}}(CG, 2) \longrightarrow (G, Q^C)$ be defined by, if $\alpha \in HOM_{\mathcal{S}}(CG, 2)$ let $\alpha\psi = \eta_\alpha \in (G, Q^C)$ be such that, for $C' \in |\mathbb{C}|$,
$\eta_{\alpha_{C'}}: C'G \longrightarrow HOM_{\mathcal{S}}(HOM_{\mathbb{C}}(C', C), 2)$ be such that for $y \in C'G$ and $r: C' \longrightarrow C$, $r(y\,\eta_{\alpha_{C'}}) = (y(rG))\alpha$. To see that we have defined a natural transformation, let $z: C' \longrightarrow C''$. It induces
$HOM_{\mathbb{C}}(z, C) : HOM_{\mathbb{C}}(C'', C) \longrightarrow HOM_{\mathbb{C}}(C', C)$ by sending $m: C'' \longrightarrow C$ into $zm : C' \longrightarrow C$, and this in turn induces
$HOM_{\mathcal{S}}(HOM_{\mathbb{C}}(C', C), 2) \longrightarrow HOM_{\mathcal{S}}(HOM_{\mathbb{C}}(C'', C), 2)$ by sending $f : HOM(C', C) \longrightarrow 2$ into $f^* : HOM(C'', C) \longrightarrow 2$ by $mf^* = (zm)f$, for $m: C'' \longrightarrow C$. We verify that the following diagram is commutative:

$$\begin{array}{ccc}
C'G & \xrightarrow{\eta_{\alpha C'}} & HOM_{\mathcal{S}}(HOM_{\mathbb{C}}(C', C), 2) \\
{\scriptstyle zG}\downarrow & & \downarrow {\scriptstyle * = zQ^C} \\
C''G & \xrightarrow{\eta_{\alpha C''}} & HOM_{\mathcal{S}}(HOM_{\mathbb{C}}(C'', C), 2)
\end{array}$$

For this, let $y \in C'G$ and $r : C' \longrightarrow C$. Travelling clockwise along the diagram we have: $(y\, \eta_{\alpha C'})^* : HOM_{\mathbb{C}}(C'', C) \longrightarrow 2$, given by, for $C'' \xrightarrow{m} C$, $m\,(y\, \eta_{\alpha C'})^* = (zm) y\, \eta_{\alpha C'} = (y((zm)G))\alpha$. Travelling counterclockwise we have $(y(zG))\,\eta_{\alpha C''} : HOM_{\mathbb{C}}(C'', C) \longrightarrow 2$ which is given by, for $C'' \xrightarrow{m} C$, $m(y(zG))\eta_{\alpha C''} = ((y(zG))(mG))\alpha = (y((zm)G))\alpha$, since G is a functor. Therefore, the diagram is commutative, or $\eta_\alpha : G \longrightarrow Q^C$ is natural.

To see that ψ is indeed inverse to ϕ, we have to verify that

(1) $\eta = \eta_{\alpha_\eta}$ and (2) $\alpha = \alpha_{\eta_\alpha}$.

Given η, α_η is such that $x\,\alpha_\eta = 1_C\, x\, \eta_C$ for $x \in CG$, and so, η_{α_η} is such that $r(y\, \eta_{\alpha_\eta})_{C'} = (y(rG))\alpha_\eta = 1_C(y(rG))\eta_C$ for $y \in C'G$, $C' \xrightarrow{r} C$.

We want to show that

$$r(y\, \eta_{C'}) = 1_C(y(rG)\,\eta_C).$$

The following diagram is commutative:

$$\begin{array}{ccc}
C'G & \xrightarrow{\eta_{C'}} & HOM_{\mathcal{S}}(HOM_{\mathbb{C}}(C', C), 2) \\
{\scriptstyle rG}\downarrow & & \downarrow {\scriptstyle rQ^C} \\
CG & \xrightarrow{\eta_C} & HOM_{\mathcal{S}}(HOM_{\mathbb{C}}(C, C), 2)
\end{array}$$

so that, by evaluating both $(y\,\eta_{C'})^*$ and $(y(rG))\,\eta_C$ at a particular element of $HOM_{\mathbb{C}}(C, C)$ we are sure to get the same result. Taking $1_C : C \rightarrow C$, we therefore have that $1_C\,(y\,\eta_{C'})^* = 1_C\,(y(rG))\,\eta_C$. But we also know that $1_C\,(y\,\eta_{C'})^* = r(y\,\eta_{C'})$ and that $1_C(y(rG))\eta_C = r(y\,\eta_{C'})$.

So, $\eta = \eta_{\alpha_\eta}$. Given now α, we get η_α and then α_{η_α} which, by defi-

nition is such that given $x \in CG$, $x\alpha\eta_\alpha = 1_C(x\eta_{\alpha_C}) = (x(1_C G))\alpha =$
$= x\alpha$. Therefore, $\alpha = \alpha\eta_\alpha$. QED .

<u>Theorem 5.2</u> For any small $\mathbb{C}$, the family $\{Q^C\}_{C\in|\mathbb{C}|}$ is cogenerating in $\mathcal{S}^{\mathbb{C}}$.

Proof:

Let $F \xrightarrow{\eta,\xi} G$ be any two natural transformations which are different. Then, there exists a $C \in |\mathbb{C}|$ for which $\eta_C \neq \xi_C$. In $\mathcal{S}$, 2 is a cogenerator and therefore there exist a map $\alpha : CG \longrightarrow 2$ such that $\eta_C\alpha \neq \xi_C\alpha$. But this in turn, implies (since 1 is a generator in $\mathcal{S}$) that there exists a map $x : 1 \longrightarrow CF$ such that $x\eta_C\alpha \neq x\xi_C\alpha$. By Co-Yoneda lemma (5.1), let η_α correspond to the above α . We show now that $\eta_C\eta_{\alpha_C} \neq \xi_C\eta_{\alpha_C}$, and so that $\eta\eta_\alpha \neq \xi\eta_\alpha$, and since $\eta_\alpha : G \longrightarrow Q^C$ we will have shown that $\{Q^C\}$ is cogenerating.

For the particular $x \in CF$ above, we have that $x\eta_C\alpha \neq x\xi_C\alpha$.
We now show that also $x(\eta\eta_\alpha)_C \neq x(\xi\eta_\alpha)_C$ thus completing the proof:
Since both $x(\eta\eta_\alpha)_C$ and $x(\xi\eta_\alpha)_C$ are elements of the set CQ^C , let us find an $r: C \to C$ for which $r(x(\eta\eta_\alpha)_C) \neq r(x(\xi\eta_\alpha)_C)$.
And since $r(x(\eta\eta_\alpha)_C) = (r(x\eta_C))\eta_{\alpha_C} = ((x\eta_C)(rG))\alpha$, and
$r(x(\xi\eta_\alpha)_C) = (r(x\xi_C))\eta_{\alpha_C} = ((x\xi_C)(rG))\alpha$, all we have to do is to find an $r: C \to C$ for which $((x\eta_C)(rG))\alpha \neq ((x\xi_C)(rG))\alpha$.
Choosing $r = 1_C$ and recalling that x was chosen so as to satisfy
$x(\eta_C\alpha) \neq x(\xi_C\alpha)$ we have:
$((x\eta_C)(1_C G))\alpha = (x\eta_C)\alpha = x(\eta_C\alpha) \neq x(\xi_C\alpha) = (x\xi_C)\alpha =$
$= ((x\xi_C)(1_C G))\alpha$. QED.

We now assume to have shown already that any diagrammatic category is complete. In fact, to this end we only need to show that arbitrary families of objects indexed by a set have a product and a coproduct, and it is easy to see that it can be shown in a way analogous to the proof of 1.1. We have not done it yet because we will show it in the last section of this chapter, in an entirely different way.

If a category is such that for any two objects there is a map between (we will call such categories <u>strongly connected</u>) then it is immediate to see that the coproduct of a generating set of objects (assuming completeness as well) is actually a generator for the category, and that the product of a cogenerating set of objects is a cogenerator. For example, the above is true in all abelian categories because they are strongly connected : given A and B arbitrary there is always a zero map $A \xrightarrow{0} B$ between.

In the case of a diagrammatic category however, we can use Yoneda and Co-Yoneda lemma, since to require that for an arbitrary object $T \in \mathcal{S}^{\mathbb{C}}$ and every $C \in |\mathbb{C}|$, there are maps $H^C \longrightarrow T$, is equivalent than to require that there are maps $1 \longrightarrow CT$ for every $C \in |\mathbb{C}|$ which is true <u>only</u> if T has no empty values, so that arbitrary diagrammatic categories need not have $\sum_{C \in |\mathbb{C}|} H^C$ as a generator ; and to require that there be maps $T \longrightarrow Q^C$ is equivalent than the requirement that there be maps $CT \longrightarrow 2$ for every $C \in |\mathbb{C}|$ which is always true in $\mathcal{S}$, so that $\prod_{C \in |\mathbb{C}|} Q^C$ is a cogenerator.

We state this fact and prove it as follows:

<u>Theorem 5.3</u> For any small $\mathbb{C}$, the object $\prod_{C \in |\mathbb{C}|} Q^C$ is a cogenerator for $\mathcal{S}^{\mathbb{C}}$.

Proof:

Given $F \overset{\eta,\xi}{\Longrightarrow} G$ such that $\eta \neq \xi$, by 5.2 there exists a $C \in |\mathbb{C}|$ and a natural transformation ${}_C\Psi : G \longrightarrow Q^C$ such that $\eta({}_C\Psi) \neq \xi({}_C\Psi)$. Let $C' \in |\mathbb{C}|$ arbitrary but $C' \neq C$. Consider $C'G \longrightarrow 1 \longrightarrow 2$ in $\mathcal{S}$, where $C'G \longrightarrow 1$ is the unique map which exists since 1 is terminal and $1 \longrightarrow 2$ is one of the injections into the coproduct $1 + 1$, say i_1 . By Co-Yoneda , let $({}_{C'}\Psi)$ correspond to the above $C'G \longrightarrow 2$, for each $C' \in |\mathbb{C}|$, i.e., we have $({}_{C'}\Psi) : G \longrightarrow Q^{C'}$ for every $C' \neq C$ and $({}_C\Psi) : G \longrightarrow Q^C$, which together induce a unique map

$$\Psi : G \longrightarrow \prod_{C \in |\mathbb{C}|} Q^C \quad \text{such that } \Psi p_{Q^{C'}} = ({}_{C'}\Psi) \;\&$$

such that $\eta\Psi \neq \xi\Psi$, since $\eta\Psi p_{Q^C} \neq \xi\Psi p_{Q^C}$ QED .

§ 6 – REGULARITY , PROJECTIVES AND INJECTIVES

The notions of mono, epi, injective and projective are basic in the theory of categories, and we do not give their definitions here. However, in the case of diagrammatic categories, and thanks to Yoneda lemma, the notions of mono and epi can be replaced by the ones given in the next Proposition:

<u>Proposition 6.1</u> For any $\mathbb{C}$ small, and η a map in $\mathcal{S}^{\mathbb{C}}$,

η is mono (epi) iff for every $C \in |\mathbb{C}|$ η_C is mono (epi).

Proof:

Let $T' \xrightarrow{\eta} T$ be mono. We want to show that $CT' \xrightarrow{\eta_C} CT$ is mono. By Yoneda, $\eta_C : (H^C, T') \longrightarrow (H^C, T)$. Let f , g be such that $f\eta_C = g\eta_C$ in $\mathcal{S}$, i.e., for every $x : 1 \longrightarrow A$, $xf\eta_C = xg\eta_C$, where A is the common domain of f and g . Since $xf \in (H^C, T')$, $(xf)\eta_C = xf\eta$, same for g. Now, $xf\eta = (xf)\eta_C = (xg)\eta_C = xg\eta$

and since η is mono, $xf = xg$ for every $x \in A$. Therefore, $f = g$. Conversely, if for each $C \in |\mathbb{C}|$, η_C is mono, let ψ and ξ be such that $\psi\eta = \xi\eta$. Assume that however, $\psi \neq \xi$, but this implies that there exists a $C \in |\mathbb{C}|$ for which $\psi_C \neq \xi_C$. But then, this contradicts that η_C was mono. Therefore, $\psi = \xi$. QED.

We have omitted the proof the dual part, since it follows the same pattern.

In $\mathcal{S}$, every mono map is the equalizer of a pair of maps. In particular, if $A' \xrightarrow{a} A$ is mono then $a = Eq(i_0 q, i_1 q)$ where $q = Coeq(ai_0, ai_1)$ where i_0 and i_1 are the two (different) injections of A into the coproduct (which is the disjoint union in $\mathcal{S}$) $A + A$. Similarly :

<u>Proposition 6.2</u> For any small $\mathbb{C}$, in $\mathcal{S}^{\mathbb{C}}$ every mono is an equalizer.

Proof: Given $T' \xrightarrow{\eta} T$ mono, by 6.1 for each $C \in |\mathbb{C}|$, η_C is mono in $\mathcal{S}$. Therefore, by the previous remark, $\eta_C = Eq(i_0 q_C, i_1 q_C)$ where $q_C = Coeq(\eta_C i_0, \eta_C i_1)$. We draw a picture, a coequalizer diagram, as follows:

$$CT' \xrightarrow{\eta_C} CT \overset{i_0}{\underset{i_1}{\rightrightarrows}} CT + CT \xrightarrow{q_C} T''_C$$

and define a functor T'' by $CT'' = T''_C$ and if $C \xrightarrow{x} C'$, let $xT'' = f$ where $f : T''_C \longrightarrow T''_{C'}$ exists, is unique and is such that $q_C f = ((xT)\times(xT))q_{C'}$ by the universal property of coequalizers and the fact that $\eta_C i_0 ((xT)\times(xT)) q_{C'} = (xT')\eta_{C'} i_0 q_{C'} = (xT')\eta_{C'} i_1 q_{C'} = \eta_C i_1 ((xT)\times(xT)) q_{C'}$. The family $\{q_C\}_{C \in |\mathbb{C}|}$ provides a natural transformation $q : T + T \longrightarrow T''$, and it is immediate to see that $\eta = Eq(i_0 q, i_1 q)$ where now by i_0, i_1 we mean the two injections $T \rightrightarrows T+T$.

QED.

Dually, in $\mathcal{S}$ every epi map is the coequalizer of a pair of maps. Precisely, if $A \xrightarrow{q} A''$ is epi, then $q = \mathrm{Coeq}(a\, p_0, a\, p_1)$ where p_0 and p_1 are the two projections $A \times A \rightrightarrows A$. With this and the second (dual) half of 6.1 one can show that:

Proposition 6.3 For any small $\mathbb{C}$, every epi in $\mathcal{S}^{\mathbb{C}}$ is a coequalizer. These two propositions have a consequence which is usually taken for a regularity condition, namely, that any map can be factored uniquely into an epi followed by a mono. That this is so will be shown in general in the next chapter.

To say that all epimorphisms in $\mathcal{S}$ are coequalizers is equivalent with all epimorphisms being onto, which in turn is equivalent with the statement that 1 is projective in $\mathcal{S}$. Since 1 is then, a projective generator in $\mathcal{S}$, we would like to know whether the generating family of representables is composed of projective objects, and this is the content of the next theorem. (Notice, by the way, the if $\mathbb{C} \cong \mathbb{1}$, the family of representable functors reduces to a single functor, H_0, where 0 is the name for the only identity map (object) which exists in $\mathbb{1}$, and therefore H_0 is constantly (it can only be evaluated at 0) 1, a singleton set containing the identity map 0.)

Theorem 6.4 For any $C \in \mathbb{C}$, H^C is projective in $\mathcal{S}^{\mathbb{C}}$.

Proof:

Let $T \xrightarrow{\psi} T''$ be an epimorphism and $H^C \xrightarrow{\eta} T''$ any natural transformation. By Yoneda lemma, let $x_\eta : 1 \to CT''$ correspond to η. Since $\psi_C : CT \text{---} CT''$ is epi in $\mathcal{S}$, and 1 is projective in $\mathcal{S}$, there exists $1 \xrightarrow{y} CT$ such that $y\psi_C = x_\eta$. Using Yoneda again but in the other direction, let $\xi_y : H^C \longrightarrow T$ be the corresponding natural transformation of y. It is immediate that $\xi_y \psi = \eta$ and so, that H^C is projective. QED.

Dually, it is true that $\mathcal{S}$ has an injective cogenerator, namely 2, a fact which will be used to show that any diagrammatic category has an injective cogenerator, namely, $\prod_{C \in |\mathcal{C}|} Q^C$. We first show that:

<u>Lemma 6.5</u> 2 is injective in $\mathcal{S}$.

Proof:

We use the direct image function defined by Lawvere (16) as follows: given $f: A \longrightarrow B$ and $\psi : A \longrightarrow 2$, f induces $f^* : 2^A \longrightarrow 2^B$ defined at ψ by $\psi f^* : B \longrightarrow 2$ such that if $y \in B$, $y((\psi)f^*) = i_1$ iff there exists $x \in A$ such that $x\psi = i_1$ and $xf = y$.

We now claim that if f is mono, the following triangle is commutative, for each ψ:

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ & {\scriptstyle\psi}\searrow \quad \swarrow{\scriptstyle\psi f^*} & \\ & 2 & \end{array}$$

This is equivalent with the injectivity of 2 in $\mathcal{S}$. To see that the triangle above commutes, assume given $x \in A$, and assume first one of the two possibilities, say, that $x\psi = i_1$. But then, by definition of f^* we have that $(xf)\psi f^* = i_1$ also. For this we did not need the fact that f was mono, but we will need it for the case that $x\psi = i_0$. If so, assume that $(xf)\psi f^* = i_1$. By the definition of f^*, the last equation implies that there exists $x' \in A$ for which $x'\psi = i_1$ and $x'f = xf$. Since f is mono, this implies that $x' = x$, but it is not possible to have at the same time $x\psi = i_0$ and $x\psi = i_1$. This contradiction implies that $(xf)\Psi f^* \neq i_1$ and so $(xf)\Psi f^* = i_0$. QED .

With this we now prove :

<u>Theorem 6.6</u> For any small $\mathbb{C}$, $\prod_{C \in \mathbb{C}} Q^C$ is injective in $\mathcal{S}^{\mathbb{C}}$.

Proof:

Let $T' \xrightarrow{\eta} T$ be any mono natural transformation, and $T' \xrightarrow{\Psi} \prod_{C \in \mathbb{C}} Q^C$ any natural transformation. Let $({}_C\Psi) = \Psi\, p_{Q^C}$, for each $C \in |\mathbb{C}|$. By Co-Yoneda, let $\overline{({}_C\Psi)}$ correspond to $({}_C\Psi)$. Since η_C is mono in $\mathcal{S}$, and 2 is injective, there exists a $\overline{({}_C\xi)}$ such that $\eta_C\overline{({}_C\xi)} = \overline{({}_C\Psi)}$. In fact, by the previous lemma, we can take $\overline{({}_C\xi)}$ to be $\overline{({}_C\Psi)}f^*$. Again by Co-Yoneda, let $({}_C\xi) : T \longrightarrow Q^C$ correspond to $\overline{({}_C\xi)}$, and now it is a matter of routine to verify that $\eta({}_C\xi) = ({}_C\Psi)$. The bunch of natural transformations $\{{}_C\xi\}$ so defined induce a unique natural transformation $\xi : T \longrightarrow \prod_{C \in \mathbb{C}} Q^C$ such that the following triangle is commutative:

$$\begin{array}{ccc} T' & \xrightarrow{\ \eta\ } & T \\ {\scriptstyle \Psi}\downarrow & \swarrow {\scriptstyle \xi} & \\ \prod Q^C & & \end{array}$$

This says that $\prod_{C \in \mathbb{C}} Q^C$ is injective, precisely because each Q^C is injective. QED .

Therefore, every diagrammatic category has an injective cogenerator, $\prod_{C \in |\mathbb{C}|} Q^C$. If $\mathbb{C} \cong \mathbb{1}$, $\mathcal{S}^{\mathbb{C}} \cong \mathcal{S}$, and $\prod_{C \in |\mathbb{C}|} Q^C = 2$ injective follows as a particular case of the above theorem. However, we needed to prove it first since it is used to establish the more general result.

§ 7 - SPECIAL SUBFUNCTORS

One of the various consequences that the axiom of choice has in $\mathcal{S}$, is that every subset of any set has a characteristic function. These subsets are called special by Lawvere [16] until he shows that all subsets are special.

In $\mathcal{S}^{\mathbb{C}}$, we can also say that $T' \xrightarrow{\eta} T$ is a subfunctor of T iff η is mono, i.e., iff, for each $C \in |\mathbb{C}|$, η_C is mono in $\mathcal{S}$. It is also possible to define special subfunctors in such a way as to correspond to the existence of a "characteristic morphism". Although we have not been able to find a counterexample, it seems intuitively clear that in general most functors have subfunctors which are not special.

Let A and B be objects in $\mathcal{S}^{\mathbb{C}}$, and $B \xrightarrow{a} A$ a mono natural transformation of functors. Then, each $CB \xrightarrow{a_C} CA$ is mono in $\mathcal{S}$, and so, it is a subset of CA and therefore has a characteristic function $\varphi_C : CA \longrightarrow 2$, i.e., φ_C is such that $a_C = Eq(\varphi_C, i_1)$. (In fact, we do not mean i_1 but rather, the composite function $CA \longrightarrow 1 \xrightarrow{i_1} 2$, but will write i_1 for convenience) Therefore, for each $C \in |\mathbb{C}|$, we have one such φ_C, the question being now when is such a family a natural transformation $A \xrightarrow{\varphi} 2$ as well, for 2 the functor whose constant value is 2. By the way equalizers are defined in $\mathcal{S}^{\mathbb{C}}$, it is clear that if $\{\varphi_C\}_C$ happens to be a natural transformation, it will automatically be the equalizer of φ with $A \to 1 \xrightarrow{i_1} 2$ (notice that the functor constantly 2 is the coproduct of the functor constantly 1 with itself, i.e., $1 + 1$), and so, φ will be what we may call the characteristic morphism of the subfunctor a of A.

Let $C \xrightarrow{u} C'$ be any map in $\mathbb{C}$. For $\{\varphi_C\}$ to form a natural transformation φ, the lower triangle in the following diagram has to be commutative where the square is commutative since a is a natural transformation :

$$\begin{array}{ccc} CB & \xrightarrow{uB} & C'B \\ {\scriptstyle a_C}\big\downarrow & & \big\downarrow{\scriptstyle a_{C'}} \\ CA & \xrightarrow{uA} & C'A \\ & {\scriptstyle \varphi_C}\searrow \quad \swarrow{\scriptstyle \varphi_{C'}} & \\ & 2 & \end{array}$$

If this is so, the characteristic function of $CB \xrightarrow{a_C} CA$ has to be $(uA)\varphi_{C'}$. This statement is equivalent with the requirement that CB be the largest subobject of CA carried into $C'B$ by means of uA. Or, equivalently (by the definition of inverse image of a map, see [23]), that the square in the above diagram be a pull-back. Actually, this condition seems to be quite adequate for defining the notion of special subfunctor, and we next prove a proposition to the effect that it coincides with the requirement that the subfunctor has a characteristic morphism.

Therefore, given a mono natural transformation in $\mathcal{S}^{\mathbb{C}}$, $B \xrightarrow{a} A$, i.e., a subfunctor of A, we say that the subfunctor a is <u>special</u> iff for every map $C \xrightarrow{u} C'$ in $\mathbb{C}$, the following is a pull-back diagram:

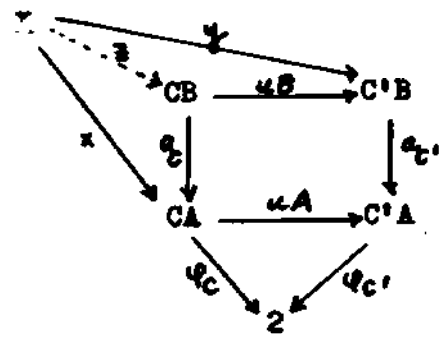
$$\begin{array}{ccc} CB & \xrightarrow{uB} & C'B \\ {\scriptstyle a_C}\big\downarrow & & \big\downarrow{\scriptstyle a_{C'}} \\ CA & \xrightarrow{uA} & C'A \end{array}$$

On the other hand, we say that $A \xrightarrow{\varphi} 2$ is the <u>characteristic morphism</u> of $B \xrightarrow{a} A$ mono in $\mathcal{S}^{\mathbb{C}}$, iff φ is a natural transformation such that $B \longrightarrow A \overset{\varphi}{\underset{\searrow 1 \nearrow i_1}{\longrightarrow}} 2$ is an equalizer diagram.

<u>Proposition 7.1</u> A subfunctor is special iff it has a characteristic morphism.

Proof:

Assume first that the subfunctor $B \xrightarrow{a} A$ has the characteristic morphism $A \xrightarrow{\varphi} 2$. We show that it is special. Consider the following commutative diagram:



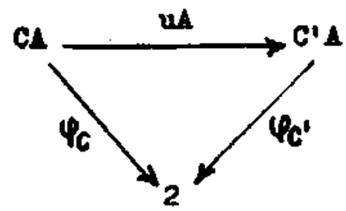
with $a_C = Eq(\varphi_C, i_1)$
and $a_{C'} = Eq(\varphi_{C'}, i_1')$
where by i_1' we mean $C'A \longrightarrow 1 \xrightarrow{i_1} 2$

Since $x(uA) = ya_{C'}$ then $x\varphi_C = x(uA)\varphi_{C'} = y\, a_{C'}\varphi_{C'} = y\, a_{C'} i_1' =$
$= x\, i_1$.

Therefore, since $a_C = Eq(\varphi_C, i_1)$ there exists a unique $z\colon X \longrightarrow CB$ such that $za_C = x$. But we still need to show that $z(uB) = y$: since $a_{C'}$ is mono, assume $z(uB) \neq y$, then $z(uB)\, a_{C'} \neq y\, a_{C'}$ and this implies that $z(uB)\, a_{C'} \neq x(uA)$. But $x = z\, a_C$ therefore $z(uB)\, a_{C'} \neq z\, a_C\, (uA)$ and therefore $(uB)\, a_{C'} \neq a_C\, (uA)$ which is a contradiction. So, $z(uB) = y$. This shows that the smaller square is a pull-back, and since $C \xrightarrow{u} C'$ was an arbitrary map in $\mathbb{C}$, this means that the subfunctor a is special.

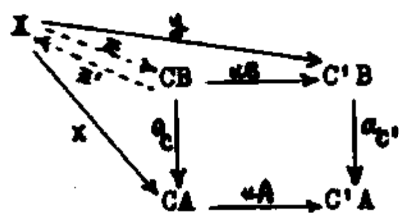
For the converse, assume that a is a special subfunctor of A. Since each of the a_C is mono in $\mathcal{S}$, it has a characteristic function φ_C in $\mathcal{S}$. We have to show that the collection $\{\varphi_C\}$ can be made into a natural transformation and furthermore that it is the characteristic morphism of a.

Let $C \xrightarrow{u} C'$ be any map in $\mathbb{C}$. We have to show that the following diagram commutes:



In other words, that the subset of CA which is the equalizer of $(uA)\varphi_{C'}$ & i_1 is precisely A_C . For this, let $X \xrightarrow{x} CA$ be their equalizer and and show that the two monomorphisms x and a_C are equivalent (see [8]) and so they represent the same subfunctor.

So, $x = Eq((uA)\varphi_{C'}, i_1)$ and also $x(uA)i_1' = xi_1 = x(uA)\varphi_{C'}$ but since $a_{C'} = Eq(i_1', \varphi_{C'})$, there exists $y: X \to C'B$ such that $x(uA) = ya_{C'}$. That is, the following diagram is commutative:



and since the smaller square is a pull-back, there exists a unique $z: X \to CB$ such that $z(uB) = y$ and $za_C = x$. Now, $x = Eq((uA)i_1, (uA)\varphi_{C'})$ and $a_C(uA)i_1 = (uB)a_{C'}i_1 = (uB)a_{C'}\varphi_{C'} = a_C(uA)\varphi_{C'}$. Therefore, there exists a unique $z': CB \to X$ such that $z' = (uB)y$ and $z'x = a_C$. Therefore a_C and x are equivalent . This can be seen as follows: Since $z(uB) = y$; $za_C = x$; $z'y = uB$; $z'x = a_C$ then $z'za_C = z'x = a_C$ and a_C mono implies that $z'z = CB$. On the other hand, $zz'x = za_C = x$ and x mono therefore $zz' = X$. Therefore, $a_C = Eq((uA)\varphi_{C'}, i_1)$ which shows that φ such that it is φ_C in each C-coordinate, is the characteristic morphism of $B \xrightarrow{a} A$. QED .

§ 8 – THE RANGE OF VALIDITY IN THE CLASS OF DIAGRAMMATIC CATEGORIES OF THE AXIOMS OF LAWVERE'S ELEMENTARY THEORY OF THE CATEGORY OF SETS

Lawvere [16] has characterized the category of sets and mappings by means of eight first-order axioms adjoined to the first-order axioms of the theory of categories plus a non-elementary axiom isuring completness. In this section, we investigate the validity, for diagrammatic categories, of these eight first-order axioms and leave for the next section the question of completeness.

<u>Axiom 1</u> – There exist finite roots.

We have proved in 1.1 that this holds for arbitrary diagrammatic categories.

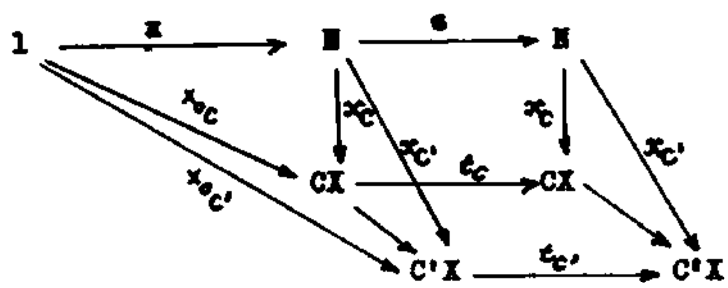
<u>Axiom 2</u> – Exponentiation.

Theorem 3.1 says that any diagrammatic category has exponentiation.

<u>Axiom 3</u> – There exists an object N together with mappings $1 \xrightarrow{z} N \xrightarrow{s} N$ such that given an object X together with mappings $1 \xrightarrow{x_o} X \xrightarrow{t} X$, there is a unique mapping $N \xrightarrow{x} X$ such that $x_o = zx$ and $xt = sx$.

This holds also in any diagrammatic category and we show it as follows: Let N denote the constant functor whose value at each $C \in |\mathbb{C}|$, is the object N of $\mathcal{S}$ whose existence is guaranteed by axiom 3, and so, z and s become natural transformations, if by 1 we mean the constant functor 1. Let X be any object in $\mathcal{S}^{\mathbb{C}}$, together with natural transformations $1 \xrightarrow{x_o} X \xrightarrow{t} X$. Then, for each $C \in |\mathbb{C}|$, there exists a unique x_C such that $(x_o)_C = zx_C$ and $x_C t_C = s x_C$. We want to show that the family

$\{x_C\}$ indexed by $|\mathbb{C}|$, is a natural transformation $x: N \longrightarrow X$. For this, let $C \xrightarrow{u} C'$ be any map in $\mathbb{C}$, and show that the following diagram is commutative:



Since $1 \xrightarrow{x_o} X$ is natural, we have that $x_{o_C}(uX) = x_{o_{C'}}$ and since t is natural, that $t_C(uX) = (uX)t_{C'}$.

The maps x_C , $x_{C'}$, x_{o_C} are provided by axiom 3 in $\mathcal{S}$. By the uniqueness part of the axiom, $x_C(uX) = x_{C'}$ and $s\,x_C = x_{o_C}$ as well as $s\,x_{C'} = x_{o_{C'}}$.

<u>Axiom 4</u> 1 is a generator.

We have mentioned already in § 5 that not every diagrammatic category has a generator, let alone that it should be the functor constantly 1 . We first give a sufficient condition for a diagrammatic category to have a generator, and then we find out that there is only one diagrammatic category for which 1 is a generator, to wit, $\mathcal{S}$.

We have introduced before the name <u>strongly connected</u> for any category for which there is a map between any two objects. We now prove:

<u>Proposition 8.1</u> If $\mathbb{C}$ is small and strongly connected, then $\sum_{C\in|\mathbb{C}|} H^C$ is a generator for $\mathcal{S}^{\mathbb{C}}$.

Proof:

Let $F \overset{\eta, \xi}{\Longrightarrow} G$ be any two natural transformations which are different. Since the family of representables is generating for $\mathcal{S}^{\mathbb{C}}$, there exists $C \in |\mathbb{C}|$ and $H^C \xrightarrow{h_C} F$ such that $h_C \eta_C \neq h_C \xi_C$. Given $C' \neq C$ since $\mathbb{C}$ is strongly connected there exists some map $C \xrightarrow{f_{C'}} C'$ which induces $H^{C'} \xrightarrow{H^{f_{C'}}} H^C$ so that $(H^f h_C) : H^{C'} \longrightarrow F$. Define $\psi_{C'}$ to be $(H^{f_{C'}} h_C)$ and consider the family $\{\psi_C\}$ where $\psi_{C'} = H^{f_{C'}} h_C$ if $C' \neq C$ and $\psi_C = h_C$. (Use the axiom of choice to select an element from each non-empty set $HOM(C,C')$ for C and C' arbitrary objects in $\mathbb{C}$). This family induces a unique map

$$\sum_{C \in |\mathbb{C}|} H^C \longrightarrow F$$

such that for every C, $i_C \psi = \psi_C$ and so $\psi\eta \neq \psi\xi$. QED.

<u>Theorem 8.2</u> 1 is a generator for $\mathcal{S}^{\mathbb{C}}$ iff $\mathbb{C} \cong \mathbb{1}$.

Proof:

Let 1 , the constant functor whose constant value is 1 , be a generator for $\mathcal{S}^{\mathbb{C}}$.

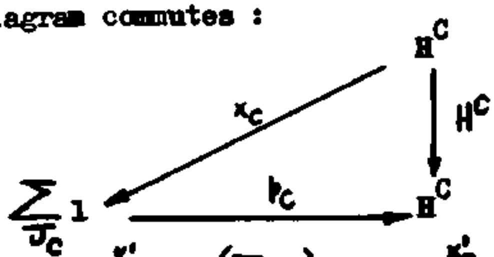
Since $\{H^C\}$ is a generating family for $\mathcal{S}^{\mathbb{C}}$, given any T in $\mathcal{S}^{\mathbb{C}}$, there exists a set and an epimorphism

$$\sum H^C \longrightarrow T$$

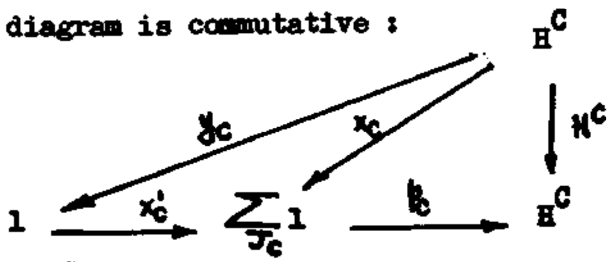
However, 1 is also a generator, therefore, for each H^C there is a set J_C and an epimorphism

$$\sum_{J_C} 1 \xrightarrow{t_C} H^C$$

Each H^C is projective , therefore, there is a map x_C such that the following diagram commutes :



By Yoneda, let $1 \xrightarrow{x_C'} (C)\left(\sum_{J_C} 1\right) = 1 \xrightarrow{x_C'} \sum_{J_C} 1$, which by axiom 7 (to be discussed) has to factor trough one of the injections, but since there is only one map $1 \longrightarrow 1$, the identity, x_C' is one of the injections. By Yoneda again, this says that $H^C \longrightarrow \sum_{J_C} 1$ factors trough one of the injections, i.e., that there exist a map y_C such that the following diagram is commutative :



Thus, H^C is retract of 1 (for any $C \in |\mathbb{C}|$) and so it has to be isomorphic to 1 , i.e., for any C and C' , $HOM(C,C') \cong 1$ which means that $\mathbb{C}$ is a preorder but a particular kind of preorder : there is always a map between any two objects, i.e., it is also strongly connected, Obviously, the only preorder and strongly connected category is $\mathbb{1}$, since, given any two objects C, C' , there are maps $C \xrightarrow{f} C'$ and $C' \xrightarrow{g} C$ and both compositions have to be identity maps. QED .

<u>Axiom 5</u> (Axiom of Choice) If the domain of a map f has elements then there exists a map g such that $fgf = f$.

This axiom does not hold in general for diagrammatic categories if it is translated into : for every $T \xrightarrow{\eta} T'$ such that there exists a natural transformation $1 \xrightarrow{\eta} T$ there exists a natural transformation $T' \xrightarrow{\psi} T$ such that $\eta\psi\eta = \eta$. Although we know no counterexample , it seems unlikely that a collection of maps in $\mathcal{S}$, indexed by $\mathbb{C}$, and such that each member ψ_C be such that $\eta_C\psi_C\eta_C = \eta_C$,should prove to be a natural transformation as well. If the domain category is discrete, i.e., any set I , then $\mathcal{S}^I$ has the axiom of choice in the above form. However, in $\mathcal{S}$, the non-existence of maps from 1 is another characterization of the coterminal object , 0 . With this, the axiom of choice reads : if f is any map with non empty domain (non-zero) there exists a map g such that $fgf = f$. In any diagrammatic category, there are no natural transformations $1 \longrightarrow 0$. However, if T is any functor which has at least an empty value, there will not be any maps $1 \longrightarrow T$ either, and $T \neq 0$. If $\mathbb{C}$ is strongly connected, the two properties coincide in $\mathcal{S}^{\mathbb{C}}$, and the functor constantly 0 is precisely the object such that there are no natural transformations $1 \longrightarrow 0$. Since the only strongly connected discrete category is $\mathbb{1}$, it seems that the axiom of choice as it is usually stated, namely that if the domain of a map is not 0 then there is a g such that $fgf = f$, holds only for $\mathcal{S}$.

<u>Axiom 6</u> - If A is not a coterminal object, then there exist $1 \longrightarrow A$. We have commented above on this axiom already. It is not true in general, since there is no natural transformation $1 \longrightarrow T$ if T is a functor with at least one empty value. However, if $\mathbb{C}$ is strongly connected, the axiom is equivalent with the existence, for every functor different from 0 , of a natural transformation $\sum_{C\in|\mathbb{C}|} H^C \longrightarrow T$. For arbitrary diagrammatic categories we have the following elementary but useful result:

Proposition 8.3 For any small $\mathbb{C}$, and any T in $\mathcal{S}^{\mathbb{C}}$, there exists a set J , a family of representable functors indexed by J and an epimorphism $\sum_J H^C \xrightarrow{p} T$.

Proof:

Let $J = \sum_{|\mathbb{C}|} (H^C, T)_{nat}$ and let p be the induced map from the coproductof this family into T . To see that p is epi, let $T \xrightarrow{\eta, \xi} T'$ be any two natural transformations such that $p\eta = p\xi$, and just assume that $\eta \neq \xi$. Then, there is a $C \in |\mathbb{C}|$ and a natural transformation $x : H^C \longrightarrow T$ such that $x\eta \neq x\xi$, since $\{H^C\}_{C \in |\mathbb{C}|}$ is generating for $\mathcal{S}^{\mathbb{C}}$. Let i_x be the injection $H^C \longrightarrow \sum_J H^C$ corresponding to x , so that $x = i_x p$. But $x\eta \neq x\xi$ implies that $p\eta \neq p\xi$, a contradiction. Therefore $\eta = \xi$. QED.

Axiom 7 - Each element of a sum is a member of one of the injections. At this point we introduce the following definition which can be stated in any category with coproducts: an object A is said to be abstractly unary iff for any coproduct $B + C$ and a map $A \xrightarrow{x} B + C$ there exists either a map $A \xrightarrow{y} B$ such that $x = yi_B$ or there exists a map $A \xrightarrow{z} C$ such that $x = zi_C$. This implies that any map from A into a finite coproduct factors trough at least one of the injections. If the category has arbitrary coproducts,we replace the above definition by the corresponding one for arbitrary coproducts, and call abstractly unary any object such that a map into an arbitrary coproduct factors trough at least one of the injections, definition which is more restrictive than that of an abstractly finite object, as given by Freyd . But here, completeness is not yet assumed. Axiom 7 can now be phrased : 1 is abstractly unary in $\mathcal{S}$. Using Yoneda lemma this implies that every representable functor in $\mathcal{S}^{\mathbb{C}}$, is abstractly unary.

One of the consequences of the axioms so far stated for $\mathcal{S}$ plus axiom 8, is that the two injections $1 \rightrightarrows 1 + 1$ are different (and are the only elements of 2). If by an <u>abstractly exclusively unary</u> object we mean an object such that any map into a coproduct factors trough <u>precisely</u> one of the injections, the above says that 1 is also abstractly exclusively unary in $\mathcal{S}$. And it implies, again using Yoneda lemma, that any representable functor in any diagrammatic category is abstractly exclusively unary as well.
We remark that in $\mathcal{S}$, 0 is abstractly unary but not abstractly exclusively unary.

<u>Axiom 8</u> . There is an object with more than one element.
This axiom is trivially satisfied in any diagrammatic category, by taking S to be a functor constantly S , for S any set with more than one element. The purpose of axiom 8 in $\mathcal{S}$, is to insure that the object N assumed to exist by axiom 3 , is infinite and plays the role of the set of natural numbers. Axiom 8 prevents the category with only one mapping from being a model of the axioms .

This ends the list of axioms for $\mathcal{S}$, and a rather superficial analysis of their validity among diagrammatic categories. The importance of the knowledge of $\mathcal{S}$, for the knowledge of the class of diagrammatic categories cannot be overestimated, since $\mathcal{S}$ can always be recovered from any diagrammatic category as the full subcategory determined by the constant functors. We can easily see that the usual operations with sets coincide with those performed for the corresponding constant functors. The case of expo-

nentiation may not be so immediate since exponentiation was not defined coordinatewise. However, we can see that it coincides with exponentiation in $\mathcal{S}$ when we restrict to constant functors as follows: let T, T' be any two constant functors and let $\|T\|$ and $\|T'\|$ be the names for their constant values. Then, T'^T is again a constant functor and its value at any object $C \in |\mathbb{C}|$ is $C\,T'^T =_{df} (H^C \times T\,,\, T') \cong (H^C\,,\, T')^{\|T\|} \cong$
$\cong (CT')^{\|T\|} = \|T'\|^{\|T\|}$.

Constant functors have the following property in any diagrammatic category: if T is constant, and C, C' are any two objects in the small domain category <u>such that the coproduct $C + C'$ exists in $\mathbb{C}$</u>, then

$$T^{(H^C)} \cong T^{(H^{C'})}$$

This is so, because, for any $A \in |\mathbb{C}|$,

$$A\left[T^{(H^C)}\right] = (H^C \times H^A\,,\, T) \cong (H^{C+A}\,,\, T) \cong (C + A)\,T \cong (C' + A)\,T \cong$$
$$\cong (H^{C'+A}\,,\, T) \cong (H^{C'} \times H^A\,,\, T) = A\left[T^{(H^{C'})}\right].$$

The category denoted $\mathbf{2}$ is an important subcategory of $\mathcal{S}$, when dealing with applications of category theory to logic and the theory of models. The functor $\mathbf{2} \hookrightarrow \mathcal{S}$ induces a functor $\mathbf{2}^{\mathbb{C}} \longrightarrow \mathcal{S}^{\mathbb{C}}$ for any $\mathbb{C}$. We want to characterize abstractly those objects of $\mathcal{S}^{\mathbb{C}}$ which are also objects of $\mathbf{2}^{\mathbb{C}}$, i.e., those functors which have as values either 0 or 1, and which we may call <u>(0,1)-valued functors</u>. To this end we define for categories with products: an object is said to be <u>idempotent</u> iff it is isomorphic to its square, i.e., A is idempotent iff $A \times A \cong A$, or else, iff both projections $A \times A \rightrightarrows A$ are isomorphisms. (Same as for Boolean rings).

We want to show that both $\mathbf{2}$ and $\mathbf{2}^{\mathcal{C}}$ are examples of "Boolean rings" in the sense that all their objects are idempotents. It is equivalent to show that, in $\mathcal{S}$, the only idempotents are 0 and 1 (actually, it is more) and that in a diagrammatic category the only idempotents are the (0,1)- or two-valued functors.

<u>Lemma 8.4</u> In $\mathcal{S}$, the only idempotents are 0 and 1 .

Proof:

Given any two objects A and B , their product $A \times B$ as well as the two projections are given by the pull-back of the following diagram:

$$\begin{array}{ccc} & & A \\ & & \downarrow \\ A & \longrightarrow & 1 \end{array}$$

We first show that 0 and 1 are idempotents in $\mathcal{S}$, by showing that the following two diagrams are pull-backs:

$$\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 1 \end{array} \qquad\qquad \begin{array}{ccc} 1 & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & 1 \end{array}$$

In fact, they are obviously pull-backs, and we do not verify it in detail. Let now A be an object in $\mathcal{S}$, such that both projections $A \times A \rightrightarrows A$ are isomorphisms, in other words :

$$\begin{array}{ccc} A & \xrightarrow{A} & A \\ {\scriptstyle A}\downarrow & & \downarrow \\ A & \longrightarrow & 1 \end{array}$$ is a pull-back diagram.

We first notice that, if X is any object in $\mathcal{S}$, either there is no map $X \longrightarrow A$, or, if there is one, there is only one, since the above is a pull-back.

Assume $A \not\cong 0$, we will show that then $A \cong 1$. If $A \not\cong 0$, by axiom 6, there exists $1 \xrightarrow{x} A$. And since for every object X in $\mathcal{S}$, there exists a (unique) map $X \longrightarrow 1$, it follows that for every X in $\mathcal{S}$, there exists a map $X \longrightarrow A = X \longrightarrow 1 \xrightarrow{x} A$, but by the previous remark, there cannot be more than one map $X \longrightarrow A$. In other words, for every X there exists a unique map $X \longrightarrow A$, or, A is terminal, and therefore isomorphic (equal, by a Convenience axiom which we will state in the next chapter) to 1. QED.

<u>Theorem 8.5</u> In any diagrammatic category $\mathcal{S}^{\mathbb{C}}$, the only idempotents are the two-valued functors.

Proof:

Let T be a two-valued functor. Let $T \times T \rightrightarrows T$ be the two projections. For each $C \in |\mathbb{C}|$, $CT \times CT \rightrightarrows CT$ are also the two projections. And since CT is either 0 or 1, by the first part of 8.4, they are both isomorphisms. Since this is so for each C, both $T \times T \rightrightarrows T$ are isomorphisms as well.

Let T be an idempotent object in $\mathcal{S}^{\mathbb{C}}$. Then, $T \times T \cong T$, and so, for each $C \in |\mathbb{C}|$, $CT \times CT \cong CT$ in $\mathcal{S}$. But by the second part of Lemma 8.4, the only idempotent objects in $\mathcal{S}$ are 0 and 1, therefore, CT is either 0 or 1, and T is a $(0,1)$-valued functor. QED.

§ 9 - COMPLETENESS

The category of sets and mappings is any complete model for Lawvere's eight elementary axioms adjoined to the axioms for categories. We want to analyse what does it mean for a model of the elementary theory to be complete. Consider a fixed object I of $\mathcal{S}$. Let ($\mathcal{S}$, I) be the category (named by J. Beck) of "objects in $\mathcal{S}$ over I". Consider the functor

$$\mathcal{S} \xrightarrow{\quad (\ \ ) \times I \quad} (\mathcal{S}, I)$$

This functor has an adjoint and a coadjoint , where by $X[(\ \ )\times I]$ we mean not only the object $X \times I$ in $\mathcal{S}$, but the object $X \times I \xrightarrow{\pi_2} I$ in ($\mathcal{S}$, I) . An adjoint is given by forgetting the "over I" part of any object $A \xrightarrow{p} I$ of ($\mathcal{S}$, I) .To give an object A over I by means of a function p is the same as to partition A into disjoint sets given by the inverse images of points in (elements of) I under p . But disjoint unions in $\mathcal{S}$ are precisely the categorical coproducts, so that any object over I , say $A \xrightarrow{p} I$, is already a sort of coproduct , only it need not satisfy the universal mapping property of coproducts , for which reason we call it an <u>internal coproduct</u>. A coadjoint gives internal products by the classical method of constructing cartesian products , it does not provide them with the universal mapping property of categorical products, though. It is defined as follows: for $X \xrightarrow{g} I$ an object in ($\mathcal{S}$, I) , one can partition X into a disjoint union of sets indexed by I , by the above remark, i.e., $X = \bigcup_{i \in I} X_i$ with $X_i = p^{-1}(i)$. Let now $\bigtimes_{i \in I} X_i$ be the subset of $(\bigcup_{i \in I} X_i)^I$ whose elements are those functions $f : I \longrightarrow \bigcup_{i \in I} X_i$ for which $f(i) \in X_i$ for all $i \in I$. This is exactly the classical definition of cartesian products and it can also be expressed by the requirement that the following be a pullback diagram:

$$\begin{array}{ccc} \underset{i\in I}{\times} X_i & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow \{I\} \\ X^I & \xrightarrow{\;g^I\;} & I^I \end{array}$$

We still have to verify that $(X \xrightarrow{g} I) \rightsquigarrow \underset{i\in I}{\times} X_i$ gives indeed a coadjoint to $(\quad) \times I$. For every $S \in \mathcal{S}$ and $X \xrightarrow{g} I$, we show that the following holds:

$$\mathrm{HOM}_{\mathcal{S}}(S, \underset{i\in I}{\times} X_i) = \mathrm{HOM}_{(\mathcal{S}, I)}(S \times I \xrightarrow{p_I} I, X \xrightarrow{g} I)$$

Given a map $S \longrightarrow \underset{i\in I}{\times} X_i$, by composing with the maps in the above pull- back diagram we get

$$\begin{array}{ccc} & S & \\ \swarrow & & \searrow \\ X^I & \xrightarrow{\;g^I\;} & I^I \end{array} \qquad \text{which yields} \qquad \begin{array}{ccc} & S \times I & \\ \swarrow & & \searrow {\scriptstyle p_I} \\ X & \longrightarrow & I \end{array}$$

by exponential adjointness, i.e., an element of $\mathrm{HOM}(S \times I \to I, X \to I)$, since a map from $A \xrightarrow{p} I$ to $B \xrightarrow{q} I$ in $(\mathcal{S}, I)$ is, by definition, a map $A \xrightarrow{f} B$ such that the triangle

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & B \\ {\scriptstyle p}\searrow & & \swarrow{\scriptstyle q} \\ & I & \end{array}$$

commutes. And conversely now, given a map in $(\mathcal{S}, I)$,

$$\begin{array}{ccc} S \times I & \longrightarrow & X \\ \searrow & & \swarrow \\ & I & \end{array}$$

applying exponential adjointness to the maps $S \times I \to I$ and $S \times I \to X$ to get maps $S \to I^I$ and $S \to X^I$ respectively, these form a commutative triangle

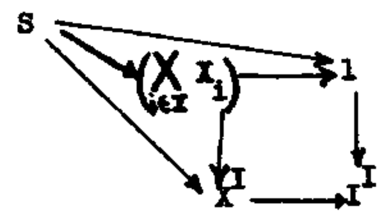
$$\begin{array}{ccc} & S & \\ \swarrow & & \searrow \\ X^I & \longrightarrow & I^I \end{array}$$

so that also the following

square is commutative :

$$\begin{array}{ccc} S & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ X^I & \longrightarrow & I^I \end{array}$$

and by the definition of $\underset{i \in I}{\times} X_i$, and the universal property of pull-backs there exists a unique $S \longrightarrow \underset{i \in I}{\times} X_i$ such that the following diagram commutes:



Let $\begin{array}{ccc} S \times I & \longrightarrow & X \\ & \searrow \quad \swarrow & \\ & I & \end{array} \quad \rightsquigarrow \quad S \longrightarrow \underset{i \in I}{\times} X_i$.

Composition of $(\quad) \times I$ with its adjoint gives the correspondence $X \rightsquigarrow I \times X = \underset{I}{\bigcup} X$, and with its coadjoint, the correspondence $X \rightsquigarrow X^I = \underset{I}{\times} X$, for any $X \in \mathcal{S}$.

Clearly, given any $I \in \mathcal{S}$, for any $X \in \mathcal{S}$ there exist both X^I and $X \times I$, simply because the category has exponentiation and products, so that completeness need not be required for the existence of arbitrary internal coproducts and products, and these exist in any model for the elementary theory.

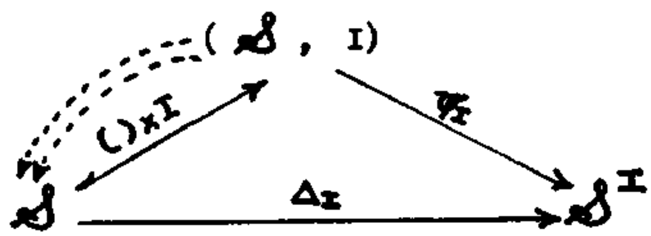
That $\mathcal{S}$ is complete means that arbitrary families of objects in have a product and a coproduct. A family of objects of $\mathcal{S}$ indexed by a set I (i.e., another object I of $\mathcal{S}$), can be thought of as a functor $I \longrightarrow \mathcal{S}$. There is a diagonal functor

$$\mathcal{S} \xrightarrow{\Delta_I} \mathcal{S}^I$$

which assigns to every object X of $\mathcal{S}$, the family $\{X_i\}_{i\in I}$ such that $X_i = X$ for each $i \in I$.

There is also a functor $(\mathcal{S}, I) \xrightarrow{\Psi_I} \mathcal{S}^I$,which assigns to each $A \xrightarrow{p} I$ the family $\{A_i\}_{i\in I}$ given by $A_i = p^{-1}(i)$.

The following triangle is commutative:

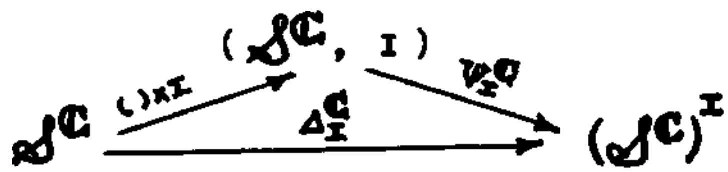


That $\mathcal{S}$ is complete is equivalent with the statement that for every set I , Δ_I has adjoint and coadjoint , and this implies that the internal coproducts and products which are given by the adjoint and coadjoint to $(\ )\times I$, are indeed the categorical coproducts and products , in other words , this is so iff Ψ_I is an equivalence of categories. Therefore , the statement that $\mathcal{S}$ is complete can be phrased as follows : the functors

$$(\mathcal{S}, I) \xrightarrow{\Psi_I} \mathcal{S}^I$$

are all equivalences of categories, for every set I .

We turn to the case of diagrammatic categories now. If by I we mean now, the functor constantly I , we can form the category $(\mathcal{S}^{\mathbb{C}}, I)$ for each object I in $\mathcal{S}$, made into a constant functor. We can define similar functors as in the case of $\mathcal{S}$, and show, exactly as above, that the following triangle is commutative :



Also, as for the case of $\mathcal{S}$, $(\ )\times I$ has adjoint and coadjoint for every set I and that $\mathcal{S}^{\mathbb{C}}$ is complete can be replaced by the statement that for every set I , $\psi_I^{\mathbb{C}}$ is an equivalence of categories.

The aim of this section is to show in a way different than the usual one, that every diagrammatic category is complete <u>because</u> $\mathcal{S}$ is complete.

For this, let $\mathcal{M}$ be any model for the eight axioms of Lawvere (and such that $\mathcal{M}$ is a category as well) , of wich we do not assume completeness.

Then, let $\mathcal{M}^{\mathbb{C}}$ be the corresponding functor category , for $\mathbb{C}$ small.

We first prove a lemma:

<u>Lemma 9.1</u> For any small $\mathbb{C}$, and any model $\mathcal{M}$ of the theory of $\mathcal{S}$, and any set I , and the functor whose constant value is I , we have that $(\mathcal{M}, I)^{\mathbb{C}} \cong (\mathcal{M}^{\mathbb{C}}, I)$.

Proof:

Given a functor $F : \mathbb{C} \longrightarrow (\mathcal{M}, I)$, we have, for each $C \in |\mathbb{C}|$, an object in $\mathcal{M}$ over I , $CF = X_C \longrightarrow I$, and if $C \xrightarrow{x} C'$ is any map in $\mathbb{C}$, F induces $CF \xrightarrow{xF} C'F$ such that the following triangle commutes:

$$\begin{array}{ccc} X_C & \xrightarrow{X_x} & X_{C'} \\ & {\scriptstyle \varphi_C} \searrow \quad \swarrow {\scriptstyle \varphi_{C'}} & \\ & I & \end{array}$$

Let $X \xrightarrow{\varphi} I$ be an object in $(\mathcal{M}^{\mathbb{C}}, I)$ where X is an object in $\mathcal{M}^{\mathbb{C}}$, defined by $CX = X_C$ for each $C \in |\mathbb{C}|$, and $xX = X_x$ for each map x in $\mathbb{C}$. And obviously, by the commutativity of triangles like the above one, this says that the collection $\{\varphi_C\}$ is a natural transformation $X \xrightarrow{\varphi} I$, where now I is interpreted as the functor constantly I .

We have defined a map $(\mathcal{M}, I)^{\mathbb{C}} \longrightarrow (\mathcal{M}^{\mathbb{C}}, I)$.

Conversely, given any object $T \xrightarrow{\eta} I$ in $(\mathcal{M}^{\mathbb{C}}, I)$, for each $C \in |\mathbb{C}|$ there is a map $\eta_C : CT \longrightarrow I$, and if $C \xrightarrow{x} C'$ is any map in $\mathbb{C}$ the following triangle is commutative:

$$\begin{array}{ccc} CT & \xrightarrow{xT} & C'T \\ & {\scriptstyle \eta_C} \searrow \quad \swarrow {\scriptstyle \eta_{C'}} & \\ & I & \end{array}$$

Let $Y : \mathbb{C} \longrightarrow (\mathcal{M}, I)$ be defined by, $CY = CT \xrightarrow{\eta_C} I$, and $xY = xT$ which is a map in $(\mathcal{M}, I)$ since $\eta_C = (xT)\eta_{C'}$.
It is now easy to see that both compositions of functors are equivalent to the corresponding identities. QED.

<u>Theorem 9.2</u> Let $\mathbb{C}$ be any small category, and $\mathcal{M}$ any model for the elementary theory of the category of sets. Then, $\mathcal{M}^{\mathbb{C}}$ is complete iff $\mathcal{M}$ is complete.

Proof:

Let $\mathcal{M}$ be complete, i.e., $\mathcal{M}$ is $\mathcal{S}$, the category of sets. This means by previous considerations in this section, that for every object I of $\mathcal{S}$, the functor $(\mathcal{S}, I) \xrightarrow{\psi_I} \mathcal{S}^I$ is an equivalence of categories. This functor induces a functor

$$(\mathcal{S}, I)^{\mathbb{C}} \xrightarrow{\psi_I^{\mathbb{C}}} (\mathcal{S}^I)^{\mathbb{C}} \cong (\mathcal{S}^{\mathbb{C}})^I$$

which is also an equivalence of categories since ψ_I is.
By 9.1, $(\mathcal{S}, I)^{\mathbb{C}} \cong (\mathcal{S}^{\mathbb{C}}, I)$ so that we have that the functor

$$(\mathcal{S}^{\mathbb{C}}, I) \xrightarrow{\psi_I^{\mathbb{C}}} (\mathcal{S}^{\mathbb{C}})^I$$

is an equivalence of categories, in other words, $\mathcal{S}^{\mathbb{C}}$ is complete.
Conversely, assume $\mathcal{M}^{\mathbb{C}}$ complete. An arbitrary family of objects of $\mathcal{M}$ can be thought of as a family of constant functors in $\mathcal{M}^{\mathbb{C}}$, and so, it has a product and a coproduct, or $\mathcal{M}$ is complete. QED.

Chapter II

THE THEORY OF REGULAR CATEGORIES AND AN ABSTRACT CHARACTERIZATION OF DIAGRAMMATIC CATEGORIES

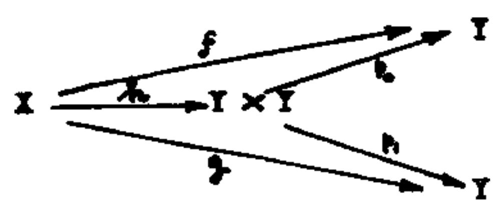
In the first chapter we have described many features of the members of the class of diagrammatic categories. Some of these properties, such as having a generating family of projectives, can be stated without any reference to the set-valued functor nature of the objects in each diagrammatic category. The problem we pose in this chapter is whether there are enough properties , which can be phrased in abstract categorical terms and which could serve to characterize the class of diagrammatic categories.

To this end, we introduce the name <u>regular</u> for categories satisfying a list of axioms which are weakened versions of those given by Freyd for the theory of abelian categories. Indeed, all abelian categories are regular , the converse is not true, one example being the category of sets. Regular categories are not strong enough to yield results as interesting as those of the theory of abelian categories ; yet, they are strong enough to exclude many interesting categories since there is a regularity condition to be satisfied and which is not satisfied by the category of Hausdorff spaces or by many algebraic categories, for example. We choose regular categories as a starting point in the program of characterising abstractly the diagrammatic categories, since they are all obviously regular. On the other hand, since there are no abelian diagrammatic categories, the strenghtening of the axioms has to deviate from abelianess and follow different paths. We next introduce the definition of atom in a regular category, and say when shall a regular category be called atomic. It turns out that any complete atomic regular category is isomorphic to some diagrammatic category and that all diagrammatic categories are complete atomic regular : this is the characterization we wanted. On the other hand, abelian categories, though regular, are far from being atomic : only the zero abelian category is regular atomic.

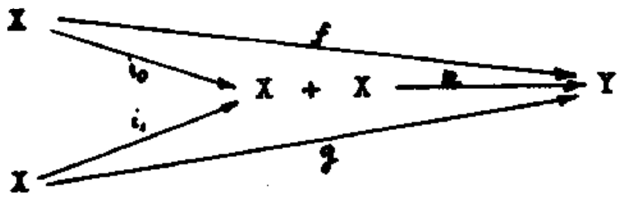
§ 10 - REGULAR CATEGORIES

Before stating the axioms of the theory of regular categories, we want to make precise what the consequences of having finite roots are. In this way, we determine better what do the other axioms really add to the assumption of finite roots. Besides, all definitions of the theory of regular categories can be stated for categories with finite roots alone. We start by defining some notions which make sense in any category with finite roots.

By the <u>induced map</u> of a pair of maps $X \overset{f,g}{\rightrightarrows} Y$, we mean the unique map h which renders commutative the following diagram:



Dually, the <u>coinduced map</u> of a pair of maps $X \overset{f,g}{\rightrightarrows} Y$ is the unique map k which renders commutative the following diagram:



A <u>relation</u> on an object A is any pair of maps $R \overset{f_0, f_1}{\rightrightarrows} A$ such that their induced map be mono. A <u>co-relation</u> on an object B is any pair of maps $B \overset{g_0, g_1}{\rightrightarrows} R^*$ such that their co-induced map be epi.

A relation $R \overset{f_0, f_1}{\rightrightarrows} A$ is a <u>congruence</u> on A iff

(i) $\exists\, d(A \xrightarrow{d} R \;\&\; df_0 = A = df_1)$;

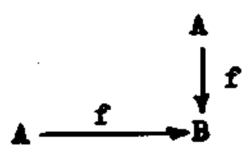
(ii) $\exists\, t(R \xrightarrow{t} R \;\&\; tf_0 = f_1 \;\&\; tf_1 = f_0)$ and

(iii) $\forall h_0 \forall h_1 (X \overset{h_0, h_1}{\rightrightarrows} R \;\&\; h_0 f_1 = h_1 h_0$ then $\exists\, u(uf_0 = h_0 f_0$ & & $uf_1 = h_1 f_1)$.

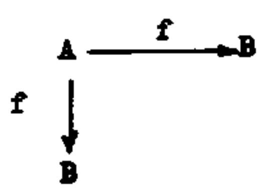
The <u>induced pair of maps</u> of a map f is the pair $A \times A \overset{p_0, p_1}{\rightrightarrows} A \xrightarrow{f} B$.

Dually, the <u>co-induced pair of maps</u> of a map f is the pair $A \xrightarrow{f} B \overset{i_0, i_1}{\rightrightarrows} B + B$.

The <u>kernel pair</u> of a map f is the pull-back of the diagram:



Dually, the <u>cokernel pair</u> of a map f is the push-out of the diagram:



<u>Proposition 10.1</u> In a category with finite roots, every map has a kernel pair and a cokernel pair. Explicitly, let f be any map. Then, Ker pair $(f) = (kp_0, kp_1)$ with $k = Eq(p_0 f, p_1 f)$ and p_0, p_1 are as in the diagram: $K_f \xrightarrow{k} A \times A \overset{p_0}{\underset{p_1}{\rightrightarrows}} A \xrightarrow{f} B$.

And, Cok pair(f) = $(i_0 q, i_1 q)$ where $q =$ Coeq(fi_0, fi_1) with i_0, i_1 as in the diagram : $A \xrightarrow{f} B \overset{i_0, i_1}{\rightrightarrows} B + B \xrightarrow{q} K_f{}^*$.

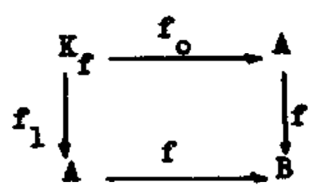
Proof:

The existence of products and equalizers implies the existence of pull-backs and therefore of kernel pairs , and it is inmediate to see that they are given as in the statement of the theorem. Dually, there are cokernel pairs and they can be so defined. QED.

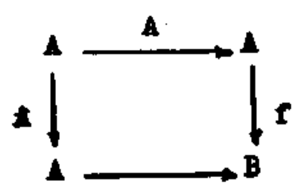
<u>Proposition 10.2</u> In a category with finite roots, every kernel pair is a congruence relation.

Proof:

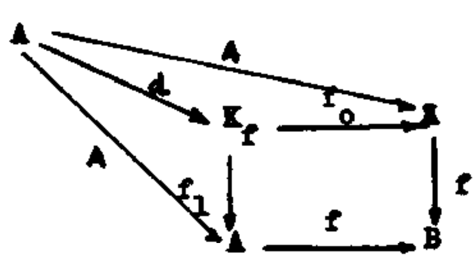
Let (f_0, f_1) = Ker pair(f) , i.e., the following is a pull-back diagram:



Clearly, the following square is also commutative:

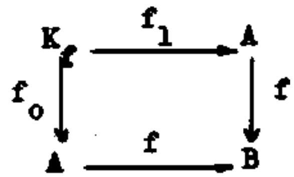


Therefore, by the universal property of pull-backs, there exists a unique $A \xrightarrow{d} K_f$ such that the following diagram is commutative:

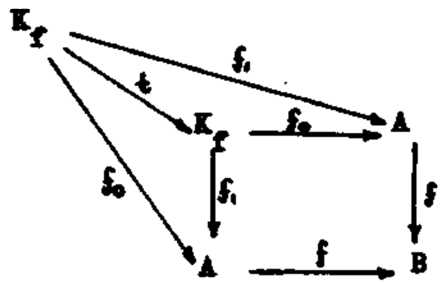


so that df_0 = A = df_1 , which is precisely condition(i), or reflexivity.

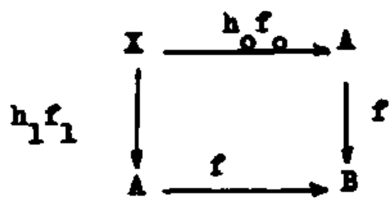
To prove condition (ii) or symmetry, consider the following commutative square:



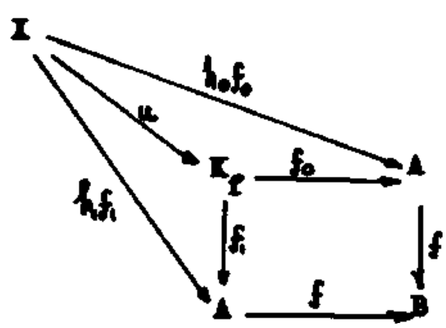
Again, by the properties of a pull-back, there exists a unique $K_f \xrightarrow{t} K_f$ for which the following diagram is commutative:



In equations, this reads: $tf_o = f_1$ and $tf_1 = f_o$, which is exactly condition (ii) in the definition of a congruence relation. Finally, let us be given h_o and h_1 such that $h_of_1 = h_1f_o$. Then, since $h_of_of = h_of_1f = h_1f_of = h_1f_1f$, the following square is commutative:



and therefore there exists a unique $X \xrightarrow{u} K_f$ such that the following diagram is commutative:



In other words, $uf_o = h_of_o$ and $uf_1 = h_1f_1$, so that condition(iii) or transitivity, holds. QED.

The converse of this proposition is not necessarily true in a category with finite roots, however it is true in most categories of interest, e.g., all algebraic categories (Lawvere [15]), all abelian categories, all diagrammatic categories, and it will be an axiom of the theory of regular categories.

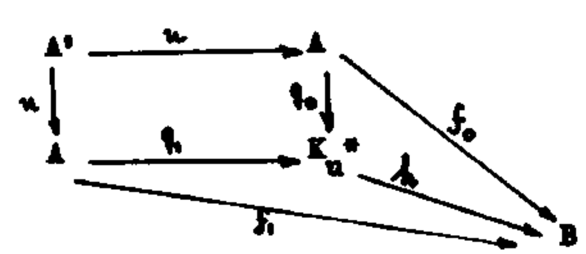
A monomorphism is said to be a <u>regular mono</u> iff it is an equalizer; and an epimorphism is said to be a <u>regular epi</u> iff it is a coequalizer.

<u>Proposition 10.3</u> In a category with finite roots, equalizers are mono, coequalizers are epi, every regular mono is the equalizer of its cokernel pair and every regular epi is the coequalizer of its kernel pair.

Proof:

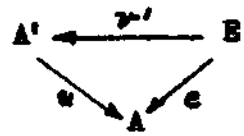
Let $u = Eq(f_o, f_1)$, and let g, g' be such that $gu = g'u$. Then, also $guf_o = guf_1$ and $g'uf_o = g'uf_1$ but since u equalizes f_o and f_1 there exists a unique k such that $gu = ku$, and a unique k' such that $g'u = k'u$. Since $gu = g'u$, and uniqueness, we have that $g = g'$.

We show now that u is, in fact, the equalizer of its cokernel pair. Let $(q_o, q_1) = \text{Cok pair}(u)$. By properties of push-outs there exists a unique map h such that the following diagram is commutative:



Let $e = Eq(q_0, q_1)$ and by the universal property of equalizers there exists a unique $A' \xrightarrow{v} E$ such that the following diagram commutes:

But now, $ef_0 = eq_0h = eq_1h = ef_1$ and since $u = Eq(f_0, f_1)$ there is a unique $E \xrightarrow{v'} A'$ such that the following diagram commutes:



So, $ve = u$ and $v'u = e$. Therefore $vv'u = ve = u$ and u mono so that $vv' = A'$ and $v've = v'u = e$ and e mono (since it is an equalizers) implies that $v'v = E$. Therefore $A' \cong E$ and so, $u = Eq(q_0, q_1)$. QED.

We have omitted the proof of the dual assertions of the theorem.

Given any map f, by the <u>regular image</u> of f we mean the map which is the equalizer of its cokernel pair, and by the <u>regular coimage</u> of f we mean the map which is the coequalizer of its kernel pair.

<u>Corollary 10.4</u> In any category with finite roots we have that:

a map u is regular mono iff $u = \text{Reg Im}(u)$;

a a map p is regular epi iff $p = \text{Reg Coim}(p)$.

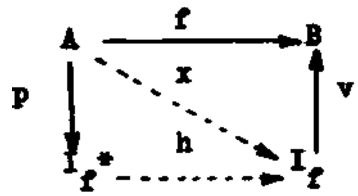
Proof:

It follows immediately from Prop. 10.3. QED.

<u>Proposition 10.5</u> In any category with finite roots, given any map f, there exist both $\text{Reg Im}(f) = I_f \xrightarrow{v} B$ and the $\text{Reg Coim}(f) = A \xrightarrow{p} I_f{}^*$. Moreover, there exists a unique map $I_f{}^* \xrightarrow{h} I_f$ such that $f = phv$.

Proof:

It is clear that both the regular image and the regular coimage exist. Consider the following diagram, where the dotted arrows will be shown to exist and make the diagram commutative:

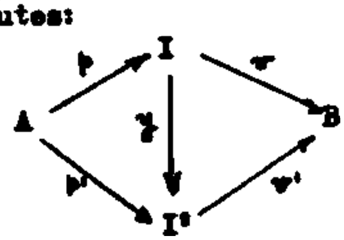


with $v = Eq(i_0q, i_1q)$ and $q = Coeq(fi_0, fi_1)$; with $p = Coeq(kp_0, kp_1)$ and $k = Eq(p_0f, p_1f)$.

Therefore, $fi_0q = fi_1q$ and since $v = Eq(i_0q, i_1q)$ there exists a unique $x : A \longrightarrow I_f$ such that $x v = f$. On the other hand, $kp_0xv = kp_0f = kp_1f = kp_1xv$ and since v (being an equalizer) is mono, this implies that $kp_0x = kp_1x$ and since $p = Coeq(kp_0, kp_1)$ there exists a unique $h : I_f^* \longrightarrow I_f$ such that $p h = x$.

But $f = x v = p h v$ is what we wanted to show. QED.

A category with finite roots is said to have <u>unique regular factorizations</u> iff for any map f there are maps p (regular epi) and v (regular mono) such that $f = p v$ and moreover such that if p' , v' are maps which are regular epi and regular mono respectively, and are such that $f = p' v'$, then there exists a unique y such that the following diagram commutes:



<u>Proposition 10.6</u> A category with finite roots has unique regular factorizations iff the unique $h : I_f^* \longrightarrow I_f$ is an isomorphism for every map f.

Proof:

Assume h is an isomorphism for every map f. Then given any f there is a regular factorization, namely, $f = p\ v$, where $v = \mathrm{RegIm}(f)$ and $p = \mathrm{Reg\ Coim}(f)$. Uniqueness follows from 10.3.

Conversely, if for any f there are p' regular epi and v' regular mono such that $f = p'\ v'$, by 10.3 again, $p' = \mathrm{Reg\ Coim}(f)$ and $v' = \mathrm{Reg\ Im}\ (f)$. QED.

A word of explanation about the name "regular factorizations" rather than "factorizations". It is customary to speak of unique factorizations, to mean, factorizations into epis followed by monos. In abelian categories, both notions coincide and so will they in regular categories but they need not in a category with just finite roots, and we needed to make the difference to be able to state the above result.

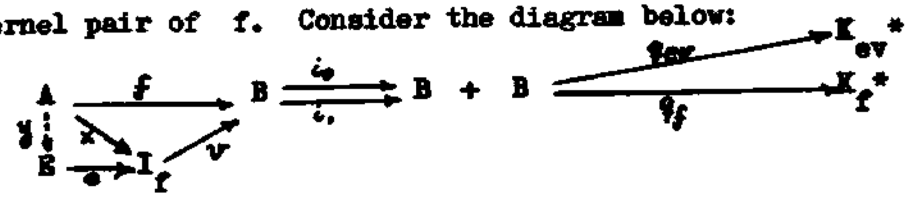
In the theory of abelian categories, the existence of unique factorizations follows from normality (every mono is a kernel and every epi is a cokernel) ; however, less can be assumed and in the theory of regular categories it will follow from the assumptions that every mono is regular and every epi is regular.

<u>Proposition 10.7</u> In any category with finite roots, (i) $\Rightarrow$ (ii), where (i) Every mono is regular and every epi is regular

(ii) Every map can be factored uniquely into a regular epi followed by a regular mono.

Proof:

Let f be any map. Let $f = xv$ be the canonical factorization of f through its image, where by this we mean, let $v = \text{Reg Im}(f)$ and let x be the unique map such that $f = xv$, and which exists since $fi_0q = fi_1q$ where $q = \text{Coeq}(fi_0, fi_1)$, and $v = \text{Eq}(i_0q, i_1q)$. Next we show that x is epi: let g and g' be any two maps such that $xg = xg'$. Let $e = \text{Eq}(g,g')$. Then, $g = g'$ iff e is an isomorphism. We know e to be mono and also v is mono, therefore ev is mono as well. By (i), ev is regular, and by 10.3, $ev = \text{Eq}(i_0q_{ev}, i_1q_{ev})$ where $q_{ev} = \text{Coeq}(evi_0, evi_1)$. By construction, v is the equalizer of the cokernel pair of f. Let us show that ev also is the equaliser of the cokernel pair of f. Consider the diagram below:



Since $xg = xg'$, and $e = \text{Eq}(g,g')$, there exists a unique y such that $ye = x$. Since $fi_0q_{ev} = xvi_0q_{ev} = yevi_0q_{ev} = yevi_1q_{ev} = fi_1q_{ev}$ therefore since $q_f = \text{Coeq}(fi_0, fi_1)$ there exists a unique $z: K_f^* \longrightarrow K_{ev}^*$ such that $q_{ev} = q_f z$. And finally, since $evi_0q_f = evi_1q_f$ and $q_{ev} = \text{Coeq}(evi_0, evi_1)$, there exists a unique $z': K_{ev}^* \longrightarrow K_f^*$ such that $q_{ev} z' = q_f$. Therefore $ev = \text{Eq}(i_0q_{ev}, i_1q_{ev}) = \text{Eq}(i_0q_f, i_1q_f) = v$. But v is mono, therefore, $e = I_f$, the identity map of I_f which is an isomorphism. So, $g = g'$. QED.

The converse of the last proposition is true for categories with finite roots and which are <u>balanced</u>, i.e., such that a map which is both mono and epi is always an isomorphism.

<u>Proposition 10.8</u> In a category which has finite roots and is balanced (ii) $\Longrightarrow$ (i), where (i) & (ii) are the statements appearing in 10.7

Proof:

Let $A \xrightarrow{u} B$ be mono. By (ii) there are p , v , such that $u = pv$, p regular epi and v regular mono. But u mono implies that p is mono as well as epi, and therefore, iso, since the category is balanced. So, u and v represent the same subobject of B and since v is regu lar mono, so is u . The dual is similarly proved. QED .

We now give the axioms of the theory of regular categories. We will assume furthermore that we are dealing with categories with small Hom--sets, i.e., such that the class of maps between any two objects is a set.

A category with small Hom-sets is said to be <u>regular</u> iff it satisfies the following axioms:

R 1 – There exists a terminal object.

R 1*– There exists a coterminal object.

R 2 – Any pair of objects A , B has a product $(A \times B , p_A , p_B)$.

R 2*– Any pair of objects A , B has a coproduct $(A + B , i_A , i_B)$.

R 3 – Any pair of maps has an equalizer.

R 3*– Any pair of maps has a coequalizer.

So far, we have stated axioms saying that the category has finite roots. Therefore, all definitions and theorems which we have proved for categories with finite roots, are also definitions and theorems of the theory of regular categories as well. The remaining axioms are the following:

R 4 - For any objects A and B , $A \xrightarrow{i_A} A + B$ is mono.

R 5 - Every congruence relation is a kernel pair.

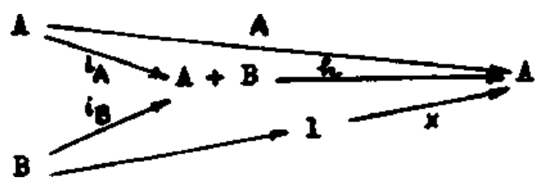
R 6 - Every mono is an equalizer.

R 6*- Every epi is a coequalizer.

We will also adopt what Lawvere calls a <u>Convenience Axiom</u>, to the effect that if A is any object whose only automorphism is the identity, and if B is any object isomorphic to A , then (it is convenient to assume that) A is equal to B . This axiom affects only terminal and coterminal objects, and says that there is exactly one terminal object, which we call 1 , and exactly one coterminal object, which we call 0 , as usual.

We show now that any abelian category is regulat as follows: R 1 and 1* are satisfied by the presence of a zero object which is defined as being terminal and coterminal at the same time; R 2 and 2* are axioms in Freyd's formulation of the theory , and R3 and 3* are theorems which follow from stronger assumptions which say that every map has a kernel and a cokernel ; R 4 is satisfied since , for any A and B , $A \xrightarrow{i_A} A \oplus B \xrightarrow{(1,0)} A$ is mono, where $\oplus$ denotes both the product and the coproduct which coincide ; R 6 and 6* follow from axioms saying that every mono is a kernel and every epi a cokernel , and R 5 holds because it holds in any algebraic category (Lawvere [15]) and in particular in any category of modules over some ring , and then because of Mitchell's full embedding theorem (Freyd [8], Mitchell [23]);

We also remark that all diagrammatic categories are regular : that R 1,1*,2,2*,3 and 3* hold was shown in 1.1. Also, R 6 and 6* were shown in 6.2. To see that R 4 is satisfied, we first see that it is in $\mathcal{S}$, as follows: let A and B be any objects in $\mathcal{S}$,and assume first that $A \neq 0$. By axiom 6 for $\mathcal{S}$, there exists a map $1 \xrightarrow{x} A$. Let h be the unique map which makes the following diagram commutative:



Then, since A is mono and $A = i_A h$, also i_A is mono.
If $A = 0$, then $0 \xrightarrow{x} C$ is mono for any C in $\mathcal{S}$, since if g, g' are such that $gx = g'x$, but $g \neq g'$ then, since 1 is a generator, there exists $1 \xrightarrow{y} C$ such that $yg \neq yg'$, contradiction since yg and yg' are maps $1 \rightrightarrows 0$ and there exists only one.
Since coproducts are defined pointwise in any diagrammatic category, and natural transformations are mono iff they are mono in each coordinate, it is clear that R 4 holds in any diagrammatic category because it holds in $\mathcal{S}$. Finally, R 5 holds for $\mathcal{S}$ (Lawvere [16]), and therefore holds also in any diagrammatic category since it is easy to see that $R \overset{\eta,\gamma}{\rightrightarrows} T$ is a congruence relation in $\mathcal{S}^{\mathbb{C}}$, iff for each $C \in |\mathbb{C}|$, $CR \overset{\eta_C,\gamma_C}{\rightrightarrows} CT$ is a congruence relation in $\mathcal{S}$.

We now derive some consequences of the axioms.

<u>Proposition 10.9</u> Any regular category is balanced.

Proof:

Let $A \xrightarrow{f} B$ be mono and epi, therefore an equalizer and a coequalizer by axioms R 6 and 6*. Moreover, by 10.4, we have that $f = $ $=$ Reg Im(f) and f = Reg Coim(f). Then, by 10.5 , there exists a unique map $h: I_f^* \longrightarrow I_f$ such that $f = phv$. But, since $A \xrightarrow{f} B = I_f \xrightarrow{v} B$ and $A \xrightarrow{f} B = A \xrightarrow{p} I_f^*$, the above is equivalent with the existence of a map h such that $f = fhf$. Since f is epi, $hf = B$ and since f is mono, $fh = A$. That is, f has an inverse, or , f is an isomorphism. QED.

Proposition 10. 10 In a regular category, every map can be factored uniquely into an epi followed by a mono.

Proof:

Immediate from 10.7 and R 6 , 6* . QED.

Proposition 10. 11 In a regular category, any congruence relation is the kernel pair of its coequalizer.

Proof:

Immediate from R 5 , and a similar argument to that of 10.3. QED.

We end here the list of the immediate consequences of the axioms for regular categories. To get any further, we need more definitions and further assumptions. Having as an aim to characterise abstractly the class of diagrammatic categories, we want to study those regular categories which are atomic , and to be able to define what 'atomic' means, we need to introduce the notion of atom, first. For a justification of the names 'atom' and 'atomic', cf. the Preface.

§ 11 - ATOMS IN REGULAR CATEGORIES

Let f_o, f_1 be any two maps with common codomain. We say that f_o and f_1 are jointly epi iff $\forall g \; \forall g' ((f_o g = f_o g' \;\&\; f_1 g = f_1 g') \Longrightarrow (g = g'))$.

This definition can immediately be generalized to n-tuples of maps with common codomain. In particular, if n = 1, the statement that f and f are jointly epi, simply says that f is epi.

We recall that an object is said to be abstractly unary iff any map from the object into a binary coproduct, factors trough one (or the other, or both) of the injections; and abstractly exclusively unary iff it factors trough exactly one of the injections. We now notice that if instead of epis we take jointly epi pairs of maps, the definition of abstractly unary bears some resemblance to the definition of projective object, if a particular type of jointly pairs of objects is consider, namely, pairs of injections into a coproduct. We first show:

Lemma 11.1 For any pair of objects A and B, the maps $A \xrightarrow{i_A} A + B$ and $B \xrightarrow{i_B} A + B$ are jointly epi.

Proof:

Let g and g' be such that $i_A\, g = i_A\, g' = k_A$ and $i_B\, g = i_B\, g' = k_B$. Then, k_A and k_B induce a unique k such that $i_A\, k = k_A$ and $i_B\, k = k_B$. But both g and g' have that property, by uniqueness $g = k = g'$. QED.

It is now clear that the notion of abstractly unary object is similar to a sort of "projective" with respect to jointly epi pairs of injections into a coproduct. But we can introduce "projectives" with respect to arbitrary jointly epi pairs of maps. This is part of the definition of 'atom'. However, we want the atoms to be abstractly exclusively unary as well, since they are being modelled in the set of representable functors in any diagrammatic category. It turns out that it is enough to assume that they are abstractly exclusively unary with respect to the two injections $1 \rightrightarrows 2$ alone. Therefore, we say that an object A in a regular category is an <u>atom</u> iff :

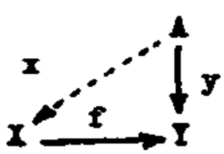
<u>(At 1)</u> $\forall f_0 \forall f_1 \forall y \; \Big[\big[(\mathrm{Epi}(f_0, f_1) \;\&\; \mathrm{Codom}(f_k) = Y \;\&\; A \xrightarrow{y} Y \big] \Rightarrow (\exists x_0 (x_0 f_0 = y) \text{ or } \exists x_1 (x_1 f_1 = y)) \Big]$;

<u>(At 2)</u> $\forall h \; \Big[A \xrightarrow{h} 1 + 1 \Rightarrow (A \longrightarrow 1 \xrightarrow{(i_0)} 1 + 1 = h \Leftrightarrow$
$\Leftrightarrow A \longrightarrow 1 \xrightarrow{(i_1)} 1 + 1 \neq h) \Big]$.

<u>Proposition 11.2</u> If A is an atom ,then A is projective.

Proof:

Let f be epi. Then (f,f) is a jointly epi pair of maps . Given



$A \xrightarrow{y} Y$, there exists $A \xrightarrow{x} X$ such that $y = xf$. Therefore, A is projective. QED.

<u>Proposition 11.3</u> If A is an atom, then A is abstractly unary.

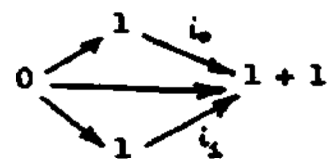
Proof:

By 11.1 , given B and C , $(B \xrightarrow{i_B} B + C ,\; C \xrightarrow{i_C} B + C)$ is a jointly epi pair of maps. And by At 1 , given any map $A \xrightarrow{y} B+C$ there exists either an x_0 such that $x_0 i_B = y$ or there exists an x_1 such that $x_1 i_C = y$, which says that A is abstractly unary. QED .

<u>Proposition 11.4</u> 0 is not an atom.

Proof:

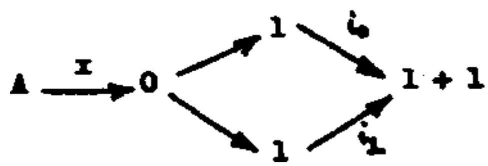
The following diagram is commutative : which means that At 2 is not satisfied. QED.



<u>Proposition 11.5</u> If A is an atom, then there are no maps with domain A and codomain 0 .

Proof:

Assume there is a map $A \xrightarrow{x} 0$. Then, the following diagram is commutative:

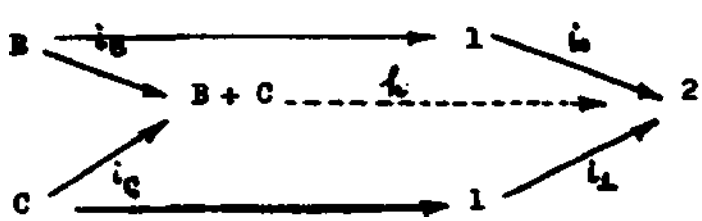


This contradicts At 2. QED.

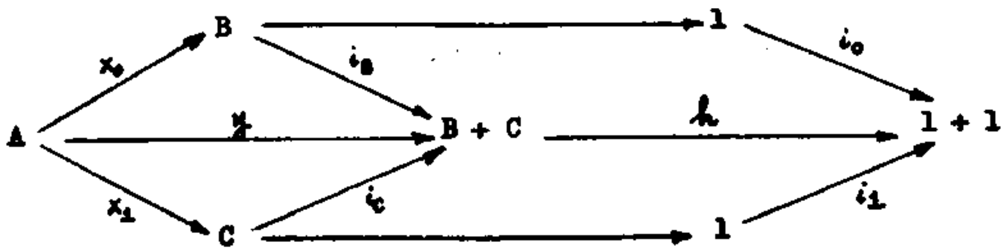
<u>Proposition 11.6</u> If A is an atom, then A is abstractly exclusively unary.

Proof:

Let B and C be any two objects and $A \xrightarrow{y} B + C$. Since A is abstractly unary by 11.3, there exists, say x_o such that $x_o\, i_B = y$. Assume that there exists also x_1 such that $x_1 i_C = y$; Let h be the unique map which makes the following diagram commutative and which exists since B+C is a coproduct:



Then, also the following diagram is commutative:



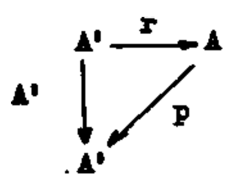
which contradicts At 2 in the definition of atom. QED.

We remark that At 1 does not exclude the possiblility that a map from an atom into the codomain of a jointly epi pair of maps , should factor trough both maps in the pair.

<u>Proposition 11.7</u> Any retract of an atom is an atom.

Proof:

Let A be an atom and $A' \xrightarrow{r} A$ a retraction, i.e., there exists $A \xrightarrow{p} A'$ such that the following triangle is commutative:



Let (q,q') be a jointly epi pair of maps and let $A' \xrightarrow{y} Y$ where Y = = Codomain (q) = Codomain(q'). Since py : $A \longrightarrow Y$ and A is an atom, there exists, say, x: $A \longrightarrow X$ such that xq = py . Now, also rxq = r(py). But rp = A' so that rxq = y . But rx : $A' \longrightarrow X$, and (rx)q = y . Therefore, A' is an atom as far as At 1 goes. At 2 is easy : if $A' \xrightarrow{h} 1 + 1$ factors trough both i_o and i_1 , so does ph . QED.

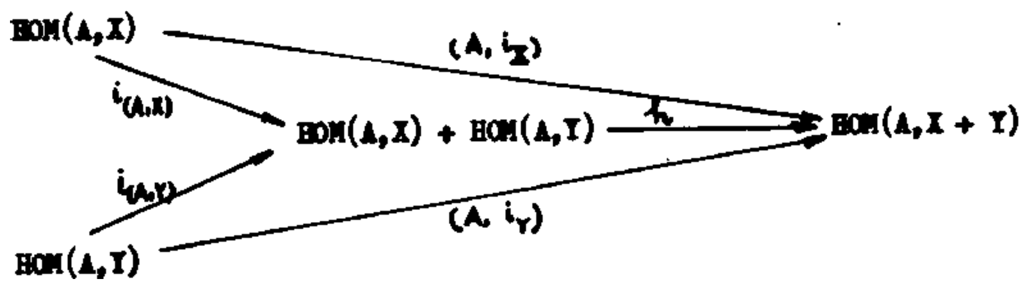
The following property that atoms have is very important, and it is used in the characterization of diagrammatic categories in section 13.

<u>Proposition 11.8</u> In a regular category, if A is an atom, and $\{X_i\}_{i\in I}$ is a family of objects indexed by a set, and such that its coproduct exists, $HOM(A, \sum_{i\in I} X_i) \cong \sum_{i\in I} HOM(A, X_i)$.

Proof:

The empty coproduct is 0, and $HOM(A, 0) \cong 0$ by 11.5, where the 0 on the left hand side of the equation is the coterminal object of the regular category in question, and the 0 on the right hand side is the coterminal object in $\mathcal{S}$, that is, the empty set.

We now show that the result is true for binary coproducts, i.e., that for any X and Y in the category, $HOM(A, X+Y) \cong HOM(A,X) + HOM(A,Y)$. Let h be the unique map which makes the following diagram commutative:



We want to define a map g, inverse to h. Let $x \in HOM(A, X+Y)$. By At 1, there exists a map y such that, say, $x = y\, i_X$ (by 11.6 A is abstractly exclusively unary, so that if x factors through i_X, it cannot factor trough i_Y then). Moreover, the above y is the only one such, since, by R 4, i_X is a monomorphism, so that if y' is such that $x = y'\, i_X$ then $y\, i_X = x = y'\, i_X$ implies that $y = y'$. $y \in HOM(A,X)$ so that $y\, i_{HOM(A,X)} \in HOM(A,X)+HOM(A,Y)$ and we define $g : HOM(A, X+Y) \longrightarrow HOM(A,X) + HOM(A,Y)$ by letting $xg = y\, i_{HOM(A,X)}$. By the above, it is well defined.

To see that we have defined an inverse to h, let $x \in HOM(A, X+Y)$, then, $xgh = (yi_{HOM(A,X)})h = y(i_{HOM(A,X)}h) = y(A, i_X) = yi_X = x$ and if $x' \in HOM(A,X) + HOM(A,Y)$ then

$x'hg = (y'i_{HOM(A,Y)})hg = y'(i_{HOM(A,Y)}h)g = y'(A, i_Y)g = x'$, by the definition of g, since $xg = x'$ iff $x' = y\, i_{(A,X)}$ and $y(A,i_X) = x$. (Notice that we have assumed that x' factors trough $i_{(A,X)}$ and not trough $i_{(A,Y)}$ but it works just as well with the other assumption).

Let now $\{X_i\}_{i\in I}$ be any family of objects indexed by a set I, whose coproduct is an object in the category. Let $I' \subseteq I$ be any subset of I for which $HOM(A, \sum_{i\in I'} X_i) \cong \sum_{i\in I'} HOM(A, X_i)$.

We have shown the result to be true if I' is empty, so that there is at least one $I' \subseteq I$ for which the above holds, for any set I.

The family of all such subsets of I is such that all chains are bounded by the union of the sets in the chain. By Zorn's lemma, there is a maximal I' for which $HOM(A, \sum_{i\in I'} X_i) \cong \sum_{i\in I'} HOM(A, X_i)$, and $I' \subseteq I$. Assume $I' \neq I$ and let $j \in I - I'$. Let $I'' = I' + \{j\}$ which is a subset of I strictly larger than I'. Then,

$$\sum_{i\in I''} X_i \cong \sum_{i\in I'} X_i + X_{\{j\}}$$

and by what we have already shown to hold for binary coproducts,

$$HOM(A, \sum_{i\in I''} X_i) \cong HOM(A, \sum_{i\in I'} X_i + X_{\{j\}}) \cong HOM(A, \sum_{i\in I'} X_i) + $$
$$+ HOM(A, X_{\{j\}}) \cong \sum_{i\in I'} HOM(A, X_i) + HOM(A, X_{\{j\}}) \cong \sum_{i\in I''} HOM(A, X_i)$$

contradicting that I' was the maximal subset of I with that property. Therefore, $I' = I$ and we have the desired result. QED.

§ 12 - ATOMIC REGULAR CATEGORIES

A regular category is said to be <u>atomic</u> iff the class of atoms in it, is isomorphic to a set and it is generating.

In the next section it will be shown that every right complete atomic regular category is isomorphic to some diagrammatic category. Therefore, it will also be left complete and have exponentiation. However, the fact that the category determined by the atoms in any right complete atomic regular category, is an adequate subcategory, is needed for the representation theorem. This need not be assumed, as can be derived from the assumptions made. We first prove:

<u>Proposition 12.1</u> In any right complete atomic regular category, given any object X there exists a set J and a family $\{A_j\}_{j\in J}$ of atoms, and an epimorphism $\sum_{j\in J} A_j \xrightarrow{p} X$.

Proof:

Let $J = \sum \text{HOM}(A , X)$, where the coproduct is taken over the set of atoms. By right completeness, $\sum_{j\in J} A_j$ exists, if $\{A_j\}_{j\in J}$ is the family of atoms whose members are defined as follows : $A_j = A$ iff $j \in \text{HOM}(A, X)$. To each $j \in J$ corresponds a map $j : A_j \longrightarrow X$, and the collection of such maps induce a map

$$\sum_{j\in J} A_j \xrightarrow{\quad p \quad} X$$

such that , if i_j is the injection corresponding to A_j , $i_j\, p = j$. To see that p is epi, let f and g be such that $pf = pg$. Then, for every $A_j \xrightarrow{j} X$, $jf = jg$ which implies that $f = g$ since the set of atoms is generating. QED .

A diagram of the form $K \overset{k_0}{\underset{k_1}{\rightrightarrows}} A \xrightarrow{p} X$ is said to be <u>exact</u> (Linton [19]) iff (k_o, k_1) = Ker pair (p) and p = Coeq(k_o, k_1) .

By the <u>canonical exact diagram ending in X</u> , for X an object in a right-complete atomic regular category, we mean, the diagram

$$K_p \xrightarrow{k_p} \sum_J A_j \times \sum_J A_j \overset{k_0}{\underset{k_1}{\rightrightarrows}} \sum_J A_j \xrightarrow{p} X$$

where p is as in the last proposition. For any X in a right-complete atomic category, there is a canonical exact diagram ending in X , by 12.1 and 10.1 .

<u>Proposition 12.2</u> In a right-complete atomic regular category, the atoms are an adequate subcategory.

Proof:

Let $\mathcal{A}$ be the full subcategory of $\mathcal{X}$, right-complete, atomic regular, generated by the atoms in $\mathcal{X}$. $\mathcal{A}$ is small since there is at most a set of atoms. Let $\mathcal{A} \xrightarrow{j} \mathcal{X}$ be the inclusion functor. To see that $\mathcal{A}$ is adequate (Isbell [12]), we have to show that the functor ϕ , defined as the composition $\mathcal{X} \xrightarrow{H} \mathcal{S}^{\mathcal{X}^*} \xrightarrow{\mathcal{S}^{j^*}} \mathcal{S}^{\mathcal{A}^*}$ is full and faithful.

For X an object in $\mathcal{X}$, $X\phi$ = $\mathrm{HOM}_{\mathcal{X}}($, X) and if $X \xrightarrow{x} X'$ is any map in $\mathcal{X}$, $x\phi$ = $\mathrm{HOM}_{\mathcal{X}}($, x) .

We show that ϕ is faithful: let x and y induce $x\phi = y\phi$. I.e., for every $A \in |\mathcal{A}|$, HOM (A, X) $\overset{x\phi, y\phi}{\rightrightarrows}$ HOM(A , X') are equal maps. This is equivalent with saying that for every atom A in $\mathcal{X}$, and every map $A \xrightarrow{z} X$, $A \xrightarrow{z} X \xrightarrow{x} X' = A \xrightarrow{z} X \xrightarrow{y} X'$. Since the class of atoms is generating, this implies that x = y .

Next, we show that ϕ is full : given X and X' in $\mathcal{X}$, and a map $X\phi \xrightarrow{f} X'\phi$ in $\mathcal{S}^{\mathcal{A}^*}$, to show that there exists a map $X \xrightarrow{y} X'$ such that $f = y\phi = \mathrm{HOM}_{\mathcal{X}}(\ \ , y)$. Let the following be a canonical exact diagram ending in X :

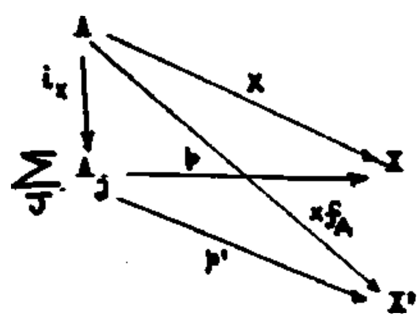
$$R \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \sum_J A_j \xrightarrow{p} X$$

Since $f : \mathrm{HOM}(\ \ , X) \longrightarrow \mathrm{HOM}(\ \ , X')$ is a natural transformation, for each A , $f_A : \mathrm{HOM}(A,X) \longrightarrow \mathrm{HOM}(A,X')$, so that for $x \in \mathrm{HOM}(A,X)$ $xf_A \in \mathrm{HOM}(A, X')$.

If $A \xrightarrow{x} X$, let us denote by $i_x : A \longrightarrow \sum_J A_j$ the corresponding injection, i.e., the injection such that $x = i_x p$.

Now, $xf_A : A \longrightarrow X'$, and this collection of maps induces a unique map $\sum_J A_j \xrightarrow{p'} X'$ such that for each $A \xrightarrow{x} X$, $i_x p' = xf_A$.

That is, the following diagram is commutative for each $x : A \longrightarrow X$:



Since $p = \mathrm{Coeq}(\alpha, \beta)$, if we show that $\alpha p' = \beta p'$, there will be a unique $X \xrightarrow{y} X'$ such that $p y = p'$.

To show the above, it is enough if we show that for every map $A \xrightarrow{r} R$, and any atom A for which such a map exists, $r\alpha p' = r\beta p'$. Because then, by the generating property of the family of atoms, this will imply that $\alpha p' = \beta p'$. Notice that if we take atoms for which there exists a map $A \longrightarrow R$, for those there will exist a map $A \longrightarrow X$ as well.

Since $A \xrightarrow{r} R$, both $r\alpha$ and $r\beta$ are maps from the atom A into the coproduct $\sum_J A_j$. Since A is an atom this implies that there exists atoms A' and A'' for which there are maps $A' \xrightarrow{x'} X$ and $A'' \xrightarrow{x''} X$ such that if $i_{x'}$ and $i_{x''}$ are their corresponding injections into the coproduct, there are also maps $A \xrightarrow{a'} A'$ and $A \xrightarrow{a''} A''$ such that $r\alpha = a' i_{x'}$ and $r\beta = a'' i_{x''}$.

But $r\alpha p = r\beta p$ implies that

$a' x' = a' i_{x'} p = r\alpha p = r\beta p = a'' i_{x''} p = a'' x''$ and

since both $(a'x')$ and $(a''x'')$ are maps $A \rightrightarrows X$ which are equal, then also $(a'x')f_A = (a''x'')f_A : A \longrightarrow X'$.

Since f is a natural transformation, the following square is commutative:

$$\begin{array}{ccc} \mathrm{HOM}(A', X) & \xrightarrow{f_{A'}} & \mathrm{HOM}(A', X') \\ {\scriptstyle \mathrm{Hom}(a', X)} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}(a', X')} \\ \mathrm{HOM}(A, X) & \xrightarrow{f_A} & \mathrm{HOM}(A, X') \end{array}$$

so that, by taking $x' \in \mathrm{HOM}(A', X)$ and traveling in both directions along the diagram, we get:

$x'(f_{A'}\,\mathrm{Hom}(a', X')) = a'(x' f_{A'})$ and

$x'(\mathrm{Hom}(a', X)\, f_A) = (a'x')\, f_A$ which must be equal elements of $\mathrm{HOM}(A,X')$, i.e., $a'(x' f_{A'}) = (a'x')\, f_A$.

By the same argument, since the following square is also commutative:

$$\begin{array}{ccc} \mathrm{HOM}(A'', X) & \xrightarrow{f_{A''}} & \mathrm{HOM}(A'', X') \\ {\scriptstyle \mathrm{Hom}(a'', X)} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}(a'', X')} \\ \mathrm{HOM}(A, X) & \xrightarrow{f_A} & \mathrm{HOM}(A, X') \end{array}$$

we have, for $x'' \in \mathrm{HOM}(A'', X)$ the following identity:

$a''(x'' f_{A''}) = (a''x'') f_A$. Finally, we have that:

$r \propto p' = a' i_{X'} p' = a'(x' f_{A'}) = (a'x') f_A = (r \propto p) f_A =$

$= (r \beta p) f_A = (a''x'') f_A = a''(x'' f_{A''}) = a'' i_{X''} p' = r \beta p'$.

Since r was arbitrary, $\propto p' = \beta p'$. Therefore there exists a unique $X \xrightarrow{y} X'$ such that $p y = p'$.

To see that $f = y\phi$, take the diagram into $\mathcal{S}^{\mathbb{A}^*}$ by means of ϕ , and see that both f and $y\phi$ make it commutative, but $p\phi$ is epi as well, so that they have to be equal . Actually, $p\phi$ is the canonical epimorphism $\sum_{(H_A, H_X)} H_A \longrightarrow H_X$ since, by Yoneda lemma, $(H_A, H_X) \cong A H_X \cong \mathrm{HOM}(A, X)$. QED .

We now attempt to prove the representation theorem for right-complete atomic regular categories. The proof is analogous to that of Lawvere [14] of the characterization theorem for algebraic categories.

§ 13 – CHARACTERIZATION OF DIAGRAMMATIC CATEGORIES

<u>Theorem 13.1</u> Let $\mathcal{X}$ be any right-complete atomic regular category. Then, there exists a small category $\mathbb{A}$ and a functor

$$\mathcal{X} \xrightarrow{\phi} \mathcal{S}^{\mathbb{A}^*}$$

which is an isomorphism of categories.

Proof:

Let $\mathbb{A}$ be the full subcategory of $\mathcal{X}$ generated by the atoms in $\mathcal{X}$.

Let ϕ be defined by $X\phi = \mathrm{HOM}_{\mathcal{X}}(\ \ , X)$ for any object X in $\mathcal{X}$, and $x\phi = \mathrm{HOM}(\ \ , x)$ for any map $X \xrightarrow{x} X'$ in $\mathcal{X}$.

The statement that ϕ is full and faithful is equivalent with the state ment that the full subcatgory of $\mathcal{X}$ generated by the atoms, i.e., $\mathcal{A}$, is adequate in $\mathcal{X}$. Therefore, by 12.2, ϕ is full and faithful. Next, we show that ϕ has an adjoint ψ, as follows. Given any object T in $\mathcal{S}^{\mathcal{A}^*}$, by 8.3 and 6.3 there is an exact diagram ending in T :

$$K_p \xrightarrow{k_p} \sum_{(H_A,T)} H_A \times \sum_{(H_A,T)} H_A \underset{p_1}{\overset{p_0}{\rightrightarrows}} \sum_{(H_A,T)} H_A \xrightarrow{p} T$$

Reinterpreting 11.8, it says that ϕ is coproduct preserving, since for any coproduct $\sum_I X_i$ in $\mathcal{X}$,

$$A((\sum_I X_i)\phi) \cong \mathrm{HOM}_{\mathcal{X}}(A, \sum_I X_i) \cong \sum_I \mathrm{HOM}(A, X_i) \cong \sum_I A(X_i\phi)$$

for every atom A in $\mathcal{X}$, i.e., for every object A in $\mathcal{A}^*$.

So, $(\sum_I X_i)\phi \cong \sum_I (X_i\phi)$, as objects in $\mathcal{S}^{\mathcal{A}^*}$.

To the above exact diagram ending in T, we can add the canonical epimorphism $\sum_{(H_{A'},K_p)} H_{A'} \xrightarrow{r} K_p$, which exists since K_p is an object in $\mathcal{S}^{\mathcal{A}^*}$ and by 8.3. By Yoneda, $(H_A, T) \cong AT$, so we replace it everywhere

Then, the diagram

$$\sum_{A'K_p} H_{A'} \xrightarrow{r} K_p \xrightarrow{k_p} \sum_{AT} H_A \times \sum_{AT} H_A \underset{p_1}{\overset{p_0}{\rightrightarrows}} \sum_{AT} H_A \xrightarrow{p} T$$

can also be written, since is coproduct preserving, as:

$$(\sum_{A'K_p} A')\phi \xrightarrow{r} K_p \xrightarrow{k_p} (\sum_{AT} A)\phi \times (\sum_{AT} A)\phi \underset{p_1}{\overset{p_0}{\rightrightarrows}} (\sum_{AT} A)\phi \xrightarrow{p} T$$

We can now use the fact that ϕ is full to get maps

$\sum_{A'K_p} A' \underset{a_1}{\overset{a_0}{\rightrightarrows}} \sum_{AT} A$ such that $r\,k_p\,p_0 = a_0\phi$, and $r\,k_p\,p_1 = a_1\phi$. Let $q = \mathrm{Coeq}(a_0, a_1)$ in $\mathcal{X}$, and let $(\sum_{AT} A)\phi \xrightarrow{q\phi} X\phi$, be its image under ϕ in $\mathcal{S}^{\mathcal{A}^*}$, where X is the codomain of q. Define $T\psi = X$.

The following picture illustrates the situation where the above half is a diagram in $\mathcal{S}^{\mathcal{A}^*}$, and the half below is a diagram in $\mathcal{X}$.

$$\begin{array}{ccccccc} \sum\limits_{A'K_p} H_{A'} & \xrightarrow{r} & K_p & \xrightarrow{k_p} & \sum\limits_{AT} H_A \times \sum\limits_{AT} H_A & \rightrightarrows & \sum\limits_{AT} H_A \xrightarrow{p} T \\ & & & & & & \qquad \searrow^{q\phi} \\ & & & & & & \qquad\qquad X\phi \end{array}$$

$$\sum_{A'K_p} A' \underset{a_1}{\overset{a_0}{\rightrightarrows}} \sum_{AT} A \xrightarrow{q} X = T\Psi$$

To see that Ψ so defined is adjoint to ϕ , we show that $\mathcal{X}$ is a reflective subcategory of $\mathcal{S}^{\mathbb{A}^*}$, i.e., for each T in $\mathcal{S}^{\mathbb{A}^*}$, there exists a natural transformation $T \xrightarrow{\varphi} T\Psi\phi$, such that if X' is an object in $\mathcal{X}$ and $T \xrightarrow{\varphi'} X'\phi$ is a map in $\mathcal{S}^{\mathbb{A}^*}$, then there is a unique $T\Psi \xrightarrow{x} X'$ such that the following is commutative:

$$\begin{array}{ccc} T & \xrightarrow{\varphi} & T\Psi\phi \\ & \underset{\varphi'}{\searrow} & \downarrow x\phi \\ & & X'\phi \end{array}$$

To this end, we first notice that :

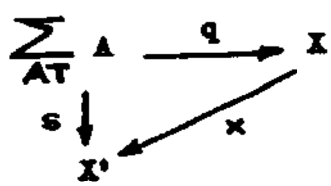
$r\,k_p\,p_0\,(q\phi) = (a_0\phi)(q\phi) = (a_0q)\phi = (a_1q)\phi = (a_1\phi)(q\phi) =$
$= rk_p\;p_1\,(q\,\phi)$. But since $p = \mathrm{Coeq}(\,rk_pp_0\,,\,rk_pp_1)$ there exists a unique $T \xrightarrow{\varphi} X\phi$, such that $p\,\varphi = q\phi$. That is, the following is commutative:

$$\begin{array}{ccc} \sum\limits_{AT} H_A & \xrightarrow{p} & T \\ & \underset{q\phi}{\searrow} & \downarrow \varphi \\ & & X\phi \end{array}$$

Let X' be any object in $\mathcal{X}$, such that there is a map $T \xrightarrow{\varphi'} X'\phi$. Since ϕ is full, there exists a map s such that $s\phi = p\varphi'$.

On the other hand, $q = \mathrm{Coeq}(a_0, a_1)$. We want to show that also $a_0 s = a_1 s$ and since ϕ is faithful, it is enough to show that $(a_0 s)\phi = (a_1 s)\phi$. Now,

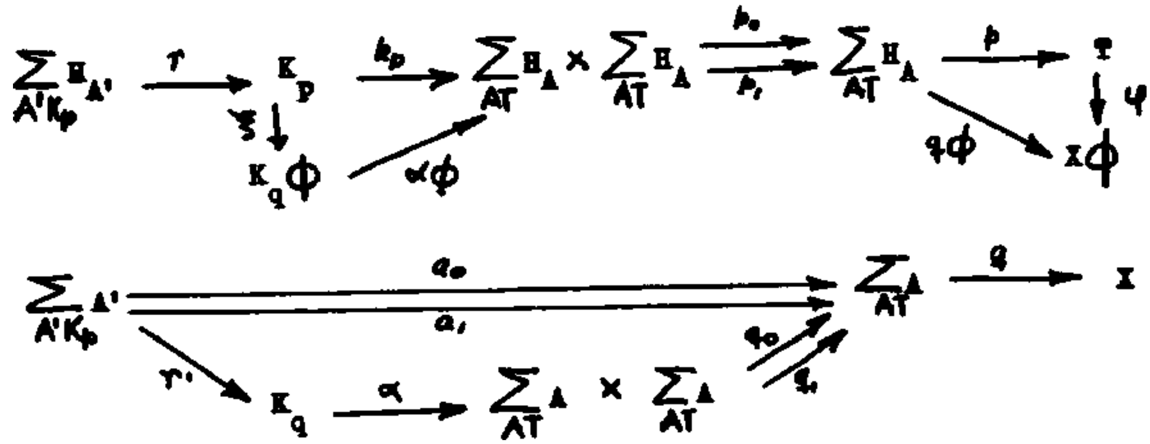
$(a_0 s)\phi = (a_0\phi)(s\phi) = (rk_p p_0)p\varphi' = (rk_p p_1)p\varphi' = (a_1\phi)(s\phi) =$
$= (a_1 s)\phi$. So there exists a unique $X \xrightarrow{x} X'$ such that $qx = s$, i.e., such that the following diagram is commutative:



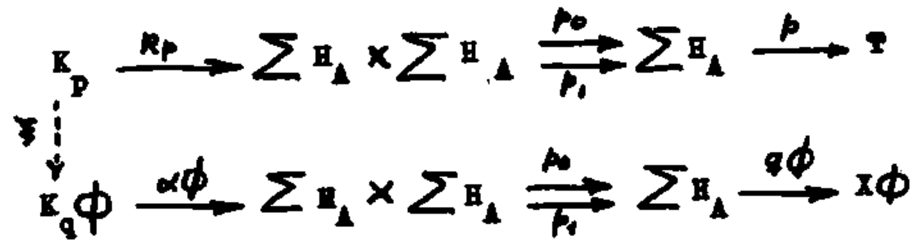
But now, $p(\varphi(x\phi)) = (p\varphi)(x\phi) = (q\phi)(x\phi) = s\phi = p\varphi'$, and p epi implies that $\varphi(x\phi) = \varphi'$. Therefore, ψ is adjoint to ϕ. Notice that so far, we have used all axioms for regular categories but axiom R 5. We have also used right-completeness and atomicity. But we need R 5 to finish the proof and show that ϕ is dense, and therefore an equivalence of categories. It will be also an isomorphism. We show now that ϕ is dense: for this we have to show that given T in $\mathcal{S}^{\mathbb{A}^*}$, there exists X in $\mathcal{X}$, such that $X\phi \cong T$. We show that this happens for $X = T\psi$, so that moreover the composition $\psi\phi$ is the identity of $\mathcal{S}^{\mathbb{A}^*}$. It is already clear that the composition $\phi\psi$ is the identity of $\mathcal{X}$, since given X in $\mathcal{X}$, $(X\phi)\psi =_{df} X$. So, that ϕ is an isomorphism of categories will be proven once we show that for each T, the map $T \xrightarrow{\varphi} T\psi\phi = T \xrightarrow{\varphi} X\phi$, is an isomorphism of objects.

Let $\alpha = \mathrm{Eq}(q_0 q, q_1 q)$ where $\sum_{AT} A \times \sum_{AT} A \overset{q_0}{\underset{q_1}{\rightrightarrows}} \sum_{AT} A$ are the projections, i.e., $(\alpha q_0, \alpha q_1) = \mathrm{Ker\ pair}(q)$.

Then, $\alpha\phi = Eq(p_0(q\phi), p_1(q\phi))$. And since $k_p p_0(q\phi) = k_p p_1(q\phi)$ there exists a unique $K_p \xrightarrow{\xi} K_q\phi$ such that $\xi(\alpha\phi) = k_p$ as indicated in the diagram below:



Now, both diagrams below are exact :



therefore, to show that $T \cong X\phi$, it is enough th show that $K_p \xrightarrow{\xi} K_q\phi$, is an isomorphism.

Since $a_0 q = a_1 q$ and $(\alpha q_0, \alpha q_1) =$ Ker pair (q) , there exists a unique r' such that $r'\alpha q_0 = a_0$ and $r'\alpha q_1 = a_1$ and, if a is such that $a_0 = aq_0$ and $a_1 = aq_1$ then $r k_p = a\phi$ so that $r\xi(\alpha\phi) = r k_p = a\phi = (r'\alpha)\phi = (r'\phi)(\alpha\phi)$ which implies, since $\alpha\phi$ is mono that $r'\phi = r\xi$.

(Notice that we have used the fact that ϕ , having an adjoint, is left exact , and since both in $\mathcal{X}$ and in $\mathcal{S}^{A^*}$ all monos are equalizers, is also mono preserving, or a mono functor.)

Since r is epi, it is a coequaliser and let $r = \mathrm{Coeq}(\beta_0, \beta_1)$.

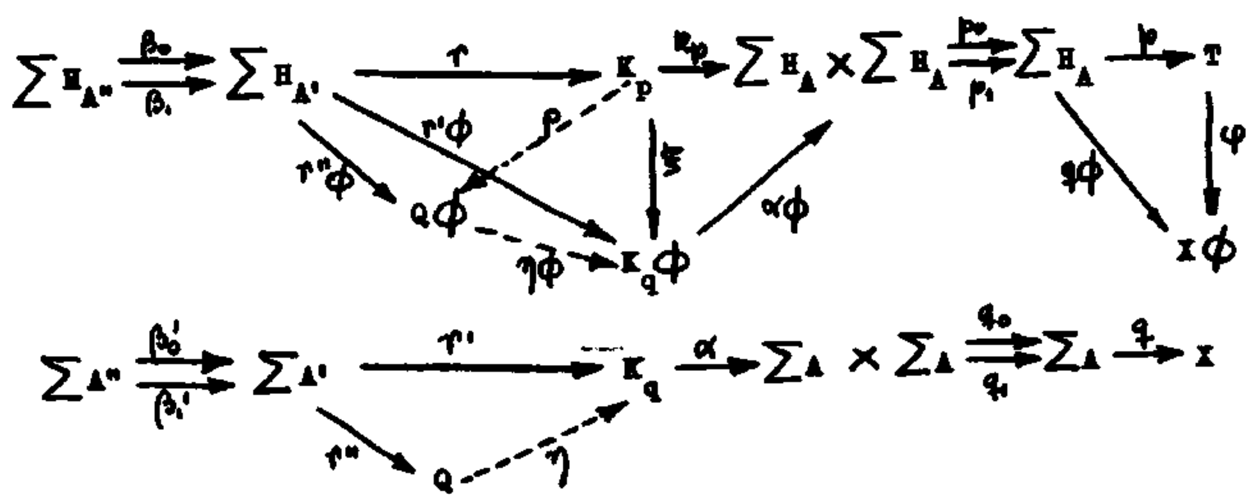
Actually , no matter what the domain of β_0 and β_1 is, by 8.3 , there will be a family of representables and an epimorphism from the coproduct of this family into this domain , so that if r coequalizes β_0 and β_1 it also coequalizes the composition of the epi with each of β_0 and β_1. Therefore we can assume without loss of generality, that

$$\sum_{J} H_{A''} \overset{\beta_0}{\underset{\beta_1}{\rightrightarrows}} \sum_{A'K_p} H_{A'} \xrightarrow{\ r\ } K_p$$ is a coequalizer diagram,

where J is the corresponding indexing set.

Since ϕ is full and preserves coproducts, there are β_0' , β_1' such that $\beta_0'\phi = \beta_0$ and $\beta_1'\phi = \beta_1$. Let $r'' = \mathrm{Coeq}(\beta_0', \beta_1')$.

In the diagram below, the dotted arrows stand for maps which will be shown to exist and fit so as to make everything commute. As before, we draw a double diagram , the upper part being in $\mathcal{S}^{\mathcal{A}^*}$, the lower in $\mathcal{X}$:



Now we have : $\beta_0' r' \alpha = \beta_1' r' \alpha$ since $(\beta_0' r' \alpha)\phi =$ $= \beta_0 r k_p = \beta_1 r k_p = (\beta_1' r' \alpha)\phi$ and ϕ is faithful .

Now, α mono implies that $\beta_0' r' = \beta_1' r'$. Therefore there exists a unique $Q \xrightarrow{\ \eta\ } K_q$ such that $r' = r''\eta$, and $\beta_0(r''\phi) = \beta_1(r''\phi)$,

because $\beta_0(r''\phi) = (\beta_0'\phi)(r''\phi) = (\beta_0' r'')\phi = (\beta_1' r'')\phi =$
$= (\beta_1'\phi)(r''\phi) = \beta_1(r''\phi)$.

Therefore there exists a unique $K_p \xrightarrow{\rho} Q\phi$ such that $r\rho = r''\phi$.

Now, since $r\rho(\eta\phi) = (r''\phi)(\eta\phi) = (r''\eta)\phi = r'\phi = r\xi$

and r epi then $\rho(\eta\phi) = \xi$.

Since ξ is mono, then also ρ is mono and $r''\phi$ epi implies that ρ is epi, therefore ρ is iso. (To see that $r''\phi$ is epi we show that $r''\phi = \text{Coeq}(\beta_0, \beta_1)$ which is so because $(\beta_0', \beta_1') = \text{Ker Pair}(r'')$ and so $(\beta_0, \beta_1) = (\beta_0'\phi, \beta_1'\phi) = \text{Ker pair}(r''\phi)$ since ϕ preserves left roots.)

Therefore $\rho : K_p \longrightarrow Q$ is an isomorphism and so, $(\eta\phi) = \xi$.

Now, $r' = r''\eta$ epi implies η epi, and therefore since $q = \text{Coeq}(\alpha q_0, \alpha q_1)$ then $q = \text{Coeq}(\eta\alpha q_0, \eta\alpha q_1)$ as well.

Now, since $(\alpha p_0, \alpha p_1)$ is a kernel pair, it is a congruence relation, and since $\alpha p_0 = \xi(\alpha\phi)p_0 = (\eta\phi)(\alpha\phi)p_0 = (\eta\phi)(\alpha\phi)(q_0\phi) =$ $= (\eta\alpha q_0)\phi$; and similarly, $\alpha p_1 = (\eta\alpha q_1)\phi$, this means that $((\eta\alpha q_0)\phi, (\eta\alpha q_1)\phi)$ is a congruence relation, but ϕ full and faithfull implies that $(\eta\alpha q_0, \eta\alpha q_1)$ is a congrue ce relation, therefore, by axiom R 5 and 10.11, it is the kernel pair of its coequalizer, which is q by the above considerations. Therefore, since both $(\eta\alpha q_0, \eta\alpha q_1)$ and $(\alpha q_0, \alpha q_1)$ are kernel pairs of q, it means that η is an isomorphism. And since $(\eta\phi) = \xi$, ξ is also an isomorphism. Therefore, $T \cong X\phi$, and ϕ is dense.

It has already been shown that inthis case, it is an isomorphism of categories. QED.

Chapter III

ISOMORPHISMS OF DIAGRAMMATIC CATEGORIES

We have just shown, in chapter II , that every right-complete atomic regular category is isomorphic to a diagrammatic category . That is, one can view a right-complete atomic regular category as a category whose objects are all set-valued functors from a given small category . However, the representation given in Theorem 13.1 need not be the only possible one such. Actually, as we shall see, this representation is a "maximal" one, in a sense we will explain. This leads us to the question : when are two given diagrammatic categories , $\mathcal{S}^A$ and $\mathcal{S}^B$, isomorphic? To answer this question, we must begin by investigating the nature of functors between diagrammatic categories, which have either adjoint or coadjoint. Next , we may ask about functors between diagrammatic categories, which are isomorphisms. The main theorem of the chapter is called "Morita isomorphism theorem for diagrammatic categories" because it resembles a theorem of Morita for categories of modules. It gives necessary and sufficient conditions for two diagrammatic categories to be isomorphic, in terms of the small domain categories in each one of them. This theorem is useful to find out , when is unique the representation of a category as a diagrammatic category.

§ 14 - ADJOINT FUNCTORS BETWEEN DIAGRAMMATIC CATEGORIES

Given any complete category $\mathcal{M}$, and a functor $\mathcal{M} \to \mathcal{S}$, this functor has an adjoint if and only if it is representable : If the functor is represented by an object A in $\mathcal{M}$, then $H^A = HOM_{\mathcal{M}}(A, \ )$ preserves all left roots, and since there are coproducts in $\mathcal{M}$, H^A has an adjoint, namely the one whose rule is $S \rightsquigarrow \sum_S A$ for any object S in $\mathcal{S}$, i.e., for any set S ; if the functor has an adjoint, evaluating the adjoint at the object 1 of $\mathcal{S}$, we get a representor for it.

By <u>$\text{Coadj}(\mathcal{A}, \mathcal{B})$</u> we mean the category whose objects are functors $\mathcal{A} \longrightarrow \mathcal{B}$ and which have adjoints, i.e., they <u>are</u> coadjoints to some functor $\mathcal{B} \longrightarrow \mathcal{A}$. The above establishes informally, a well known equivalence, namely that $\text{Coadj}(\mathcal{M}, \mathcal{S}) \approx \mathcal{M}^*$.

It is clear that for any two categories $\mathcal{A}$ and $\mathcal{B}$, $\text{Coadj}(\mathcal{A}, \mathcal{B}) \cong (\text{Adj}(\mathcal{B}, \mathcal{A}))^*$, so that, by the above, we have also that $\text{Adj}(\mathcal{S}, \mathcal{M}) \cong \mathcal{M}$.

Suposse we now replace $\mathcal{S}$ by an arbitrary diagrammatic category. The question is whether we can also get good results for those. André (1) has investigated the question, and he gets very general results concerning adjoint pairs of functors between categories of functors. However, we find that for our present needs, the machinery he develops is much too complicated, since we only need results where diagrammatic categories are involved, and we may dispense with generality. Thus, we find simpler proofs of some of his results and we go further into the applications. Thus, we want to find "formulas" for $\text{Adj}(\mathcal{S}^{\mathcal{B}}, \mathcal{M})$ and dually, for $\text{Coadj}(\mathcal{M}, \mathcal{S}^{\mathcal{B}})$ where $\mathcal{M}$ is any complete category.

The functor ϕ defined in the theorem of characterisation of diagrammatic categories, proves useful in these considerations. In the proof of

13.1 , the adjoint ψ to ϕ was constructed , however it was not given by a formula. We do this here.

We first recall how was ϕ defined , as the subregular representation of the right-complete atomic regular category $\mathcal{X}$ over the category of atoms, that is, let $\mathbb{C}^*$ be the full subcategory determined by the atoms (or, let $\mathbb{C}$ be the dual of the category of atoms) , and let $\mathbb{C}^* \xrightarrow{j} \mathcal{X}$ be the inclusion functor, then ϕ is defined as the composition

$$\mathcal{X} \xrightarrow{H} \mathcal{S}^{\mathcal{X}^*} \xrightarrow{\mathcal{S}^{j^*}} \mathcal{S}^{\mathbb{C}}$$

Next , we remark that every object T in $\mathcal{S}^{\mathbb{C}}$ is a direct limit over a small category, i.e., $T = \varinjlim \left((H , T) \to \mathbb{C}^* \xrightarrow{H} \mathcal{S}^{\mathbb{C}} \right)$,

where the category (H,T) has as objects natural transformations $H_A \xrightarrow{\varphi} T$, for some $A \in |\mathbb{C}|$, and the maps are commutative triangles

$$\begin{array}{ccc} H_A & \xrightarrow{H_x} & H_{A'} \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi'} & \\ & T & \end{array}$$

, and where the functor $(H, T) \longrightarrow \mathbb{C}^*$ has the rule :

$$H_A \xrightarrow{\varphi} T \quad \rightsquigarrow \quad A$$

$$\begin{array}{ccc} H_A & \xrightarrow{H_x} & H_{A'} \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi'} & \\ & T & \end{array} \quad \rightsquigarrow \quad A \xrightarrow{x} A'$$

To see this, let us take the following exact diagram ending in T :

$$K_p \xrightarrow{k_p} \sum_{AT} H_A \times \sum_{AT} H_A \underset{p_1}{\overset{p_0}{\rightrightarrows}} \sum_{AT} H_A \xrightarrow{p} T$$

where p is the epimorphism which exists by 8.3 , and where $(k_p p_o , k_p p_1)$ is the kernel pair of p . We will write

$$T \cong \sum_{AT} H_A \; / \; K_p$$

to mean that the above diagram is exact , although what is factored out from the coproduct $\sum_{AT} H_A$ to get T is not K_p itself but the congruence relation $(k_p p_o , k_1 p_1)$.

But also $\varinjlim\ ((H, T) \longrightarrow \mathbb{C}^* \xrightarrow{H} \mathcal{S}^{\mathbb{C}})$ is gotten by first taking the coproduct $\sum_{AT} H_A$ and then factoring out relations which are given by the small category (H_A , T), and which are precisely those we have indicated by K_p.

By the way Ψ was constructed, it is clear that its value at T in $\mathcal{S}^{\mathbb{C}}$, is given by $T\Psi = \varinjlim\ ((H,T) \longrightarrow \mathbb{C}^* \xrightarrow{j} \mathcal{X})$. For this, we recall that if $T \cong \sum_{AT} H_A / K_p$ then $T\Psi \cong \sum_{AT} A / K_q$ and moreover that K_p and K_q were isomorphic, and therefore, the relations to be factored out are the same. This adjoint happened to be an isomorphism because of axiom R5, however, we can use thebconstruction for a more general case where the categories involved need not be regular, though they have to be complete, or, at least, right-complete.

Let now $\mathcal{M}$ be any complete category. We imitate the above situation, although $\mathcal{M}$ need not be regular or have an adequate subcategory which is small either. We keep in mind the following commutative triangles:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\ \phi\ } & \mathcal{S}^{\mathbb{C}} \\ {\scriptstyle j}\nwarrow & & \nearrow{\scriptstyle H} \\ & \mathbb{C}^* & \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{S}^{\mathbb{C}} & \xrightarrow{\ \Psi\ } & \mathcal{X} \\ {\scriptstyle H}\nwarrow & & \nearrow{\scriptstyle j} \\ & \mathbb{C}^* & \end{array}$$

Notice that the commutativity of the triangle to the right says that for every $C \in |\mathbb{C}|$, $C = \varinjlim\ ((H , H_C) \rightarrow \mathbb{C}^* \xrightarrow{j} \mathcal{X})$.

<u>Theorem 14.1</u> For any $\mathcal{M}$ complete, and $\mathbb{B}$ small,

$$\mathrm{Adj}(\mathcal{S}^{\mathbb{B}}, \mathcal{M}) \cong \mathcal{M}^{\mathbb{B}^*}$$

Proof:

Let $T : \mathcal{S}^{\mathbb{B}} \longrightarrow \mathcal{M}$, and define $G_T : \mathbb{B}^* \rightarrow \mathcal{M}$ as the composition of the regular representation functor of $\mathbb{B}^*$ with T. This can always be done whether or not T has coadjoint, and we say that we are "restricting along Yoneda".

Let $G : \mathbb{B}^* \longrightarrow \mathcal{M}$, and define $T_G : \mathcal{S}^{\mathbb{B}} \longrightarrow \mathcal{M}$ by letting its value at an object F of $\mathcal{S}^{\mathbb{B}}$, be

$$F\,T_G = \varinjlim \left((H, F) \rightarrow \mathbb{B}^* \xrightarrow{G} \mathcal{M}\right).$$

Then, the following triangles are commutative :

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{T} & \mathcal{S}^{\mathbb{B}} \\ & {}_{G_T}\nwarrow \quad \nearrow_{H} & \\ & \mathbb{B}^* & \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{S}^{\mathbb{B}} & \xrightarrow{T_G} & \mathcal{M} \\ & {}_{H}\nwarrow \quad \nearrow_{G} & \\ & \mathbb{B}^* & \end{array}$$

The one on the left is commutative by the definition of G_T and the one on the right since : $B\,(H\,T_G) = H_B\,T_G = \varinjlim\left((H, H_B) \rightarrow \mathbb{B}^* \xrightarrow{G} \mathcal{M}\right) \cong BG$.

We now have to show that $T_G : \mathcal{S}^{\mathbb{B}} \longrightarrow \mathcal{M}$, is also an object in $\mathrm{Adj}(\mathcal{S}^{\mathbb{B}}, \mathcal{M})$, i.e., that it has a coadjoint T^*. Define T^* as follows : for X in $\mathcal{M}$, let $XT^* : \mathbb{B} \rightarrow \mathcal{S}$, be given by $B(XT^*) = \mathrm{HOM}_{\mathcal{M}}(BG, X)$ for any $B \in |\mathbb{B}|$. It is clear that it is a functor when extended to the maps and that it is co-adjoint to T.

To show the isomorphism of the theorem we have to prove that for every T in $\mathrm{Adj}(\mathcal{S}^{\mathbb{B}}, \mathcal{M})$, $T_{G_T} \cong T$, and that for every $G : \mathbb{B}^* \rightarrow \mathcal{M}$, that $G_{T_G} \cong G$.

Given any $B \in |\mathbb{B}|$, $(B)\,G_{T_G} = (B)\,HT_G = H_B\,T_G =$

$= \varinjlim\left((H, H_B) \rightarrow \mathbb{B}^* \xrightarrow{G} \mathcal{M}\right) \cong BG$; and given any F in $\mathcal{S}^{\mathbb{B}}$,

$(F)\,T_{G_T} = \varinjlim\left((H,F) \rightarrow \mathbb{B}^* \xrightarrow{G_T} \mathcal{M}\right) \cong \varinjlim\left((H,F) \rightarrow \mathbb{B}^* \xrightarrow{H} \mathcal{S}^{\mathbb{B}} \xrightarrow{T} \mathcal{M}\right) \cong$

$= FT$. QED.

<u>Corollary 14.2</u> For any $\mathcal{M}$ complete, and $\mathbb{B}$ small,

$$\mathrm{Coadj}(\mathcal{M}, \mathcal{S}^{\mathbb{B}}) \cong \mathcal{M}^{*\mathbb{B}}$$

Proof:

$\mathrm{Coadj}(\mathcal{M}, \mathcal{S}^{\mathbb{B}}) = (\mathrm{Adj}(\mathcal{S}^{\mathbb{B}}, \mathcal{M}))^{*} = (\mathcal{M}^{\mathbb{B}^{*}})^{*} = \mathcal{M}^{*\mathbb{B}}$. QED.

We would like to say, as in the case of $\mathcal{S}$, that $\mathrm{Coadj}(\mathcal{M}, \mathcal{S}^{\mathbb{B}})$ is given by the "representable" functors.

To say that a functor $\mathcal{M} \xrightarrow{T} \mathcal{S}$ is representable means that there exists an object A in $\mathcal{M}$ such that $T \cong H^{A} = \mathrm{HOM}_{\mathcal{M}}(A, \ )$.

In the category of categories (Lawvere [17]), the category $\mathbb{1}$ is a generator and the functors $\mathbb{1} \longrightarrow \mathcal{M}$ are in one-to-one correspondence with the objects of $\mathcal{M}$. This allows us to say, equivalently, that T is representable iff T is naturally equivalent with the functor :

$$\mathbb{1} \times \mathcal{M} \xrightarrow{A \times \mathcal{M}} \mathcal{M}^{*} \times \mathcal{M} \xrightarrow{\mathrm{HOM}} \mathcal{S}$$

where $\mathbb{1} \xrightarrow{A} \mathcal{M}$ is the functor whose value at the only object of $\mathbb{1}$ is the object A in $\mathcal{M}$, so that T is represented by A.

This definition has the advantage that it can easily be generalized : we say now that a functor $T : \mathcal{M} \longrightarrow \mathcal{S}^{\mathbb{B}}$ is "<u>representable</u>" iff there exists $\mathbb{B}^{*} \xrightarrow{A} \mathcal{M}$ such that

$$\mathbb{B} \times \mathcal{M} \xrightarrow{A^{*} \times \mathcal{M}} \mathcal{M}^{*} \times \mathcal{M} \xrightarrow{\mathrm{HOM}} \mathcal{S}$$

is naturally equivalent with $T' : \mathbb{B} \times \mathcal{M} \longrightarrow \mathcal{S}$, where T' corresponds to T by exponential adjointness , i.e., such that $T' = (\mathbb{B} \times T)\,\mathrm{ev}$. Now we have automatically :

<u>Theorem 14.3</u> For any $\mathcal{M}$ complete, $\mathbb{B}$ small , the functor $T : \mathcal{M} \longrightarrow \mathcal{S}^{\mathbb{B}}$ has an adjoint iff it is "representable" .

Proof:

By the definition of "representable" . QED.

Theorem 14.1 has several useful consequences , first of all, it gives back the previous results stated for $\mathcal{S}$. This is so, since taking

$\mathbb{B} \cong \mathbb{1}$, we have , by 14.1 , that

$$\mathrm{Adj}(\mathcal{S}, \mathcal{M}) \cong \mathrm{Adj}(\mathcal{S}^{\mathbb{1}}, \mathcal{M}) \cong \mathcal{M}^{\mathbb{1}^*} \cong \mathcal{M}^{\mathbb{1}} \cong \mathcal{M} .$$

If $\mathcal{M}$ is taken to be also a diagrammatic category , then a useful corollary to 14.1 is the following :

<u>Corollary 14.4</u> (a) If $\mathbb{B}$ and $\mathbb{C}$ are any two small categories, $\mathrm{Adj}(\mathcal{S}^{\mathbb{B}}, \mathcal{S}^{\mathbb{C}}) \cong \mathcal{S}^{\mathbb{B}^* \times \mathbb{C}}$ and $\mathrm{Coadj}(\mathcal{S}^{\mathbb{B}}, \mathcal{S}^{\mathbb{C}}) = \mathcal{S}^{*\,\mathbb{B}^* \times \mathbb{C}}$;

(b) if $\mathbb{A}$ is any small category then,

$\mathrm{Adj}(\mathcal{S}^{\mathbb{A}}) =_{df} \mathrm{Adj}(\mathcal{S}^{\mathbb{A}}, \mathcal{S}^{\mathbb{A}}) \cong \mathcal{S}^{\mathbb{A}^* \times \mathbb{A}}$ and

$\mathrm{Coadj}(\mathcal{S}^{\mathbb{A}}) =_{df} \mathrm{Coadj}(\mathcal{S}^{\mathbb{A}}, \mathcal{S}^{\mathbb{A}}) \cong \mathcal{S}^{*\,\mathbb{A} \times \mathbb{A}^*}$;

(c) if I is any discrete category, i.e., just a set , then $\mathrm{Adj}(\mathcal{S}^{I}) \cong \mathcal{S}^{I \times I}$ and $\mathrm{Coadj}(\mathcal{S}^{I}) = \mathcal{S}^{*\,I \times I}$;

(d) $\mathrm{Adj}(\mathcal{S}) = \mathcal{S}$ and $\mathrm{Coadj}(\mathcal{S}) = \mathcal{S}^*$.

Proof :

$\mathrm{Adj}(\mathcal{S}^{\mathbb{B}}, \mathcal{S}^{\mathbb{C}}) \cong (\mathcal{S}^{\mathbb{C}})^{\mathbb{B}^*} \cong \mathcal{S}^{\mathbb{B}^* \times \mathbb{C}} \cong \mathcal{S}^{\mathbb{C} \times \mathbb{B}^*}$;

$\mathrm{Coadj}(\mathcal{S}^{\mathbb{B}}, \mathcal{S}^{\mathbb{C}}) \cong (\mathrm{Adj}(\mathcal{S}^{\mathbb{C}}, \mathcal{S}^{\mathbb{B}}))^* = (\mathcal{S}^{\mathbb{C}^* \times \mathbb{B}})^* = \mathcal{S}^{*\,\mathbb{C} \times \mathbb{B}^*}$;

$\mathrm{Adj}(\mathcal{S}^{\mathbb{A}}) \cong \mathrm{Adj}(\mathcal{S}^{\mathbb{A}}, \mathcal{S}^{\mathbb{A}}) \cong \mathcal{S}^{\mathbb{A}^* \times \mathbb{A}}$;

$\mathrm{Coadj}(\mathcal{S}^{\mathbb{A}}) \cong (\mathrm{Adj}(\mathcal{S}^{\mathbb{A}}))^* \cong (\mathcal{S}^{\mathbb{A}^* \times \mathbb{A}})^* = \mathcal{S}^{*\,\mathbb{A} \times \mathbb{A}^*}$;

$\mathrm{Adj}(\mathcal{S}^{I}) \cong \mathcal{S}^{I^* \times I} \cong \mathcal{S}^{I \times I}$;

$\mathrm{Coadj}(\mathcal{S}^{I}) \cong (\mathrm{Adj}(\mathcal{S}^{I}, \mathcal{S}^{I}))^* \cong (\mathcal{S}^{I \times I})^* = \mathcal{S}^{*\,I \times I}$;

$\mathrm{Adj}(\mathcal{S}) \cong \mathrm{Adj}(\mathcal{S}^{\mathbb{1}}) \cong \mathcal{S}$; $\mathrm{Coadj}(\mathcal{S}) \cong (\mathrm{Adj}(\mathcal{S}))^* \cong \mathcal{S}^*$.

QED .

When I is discrete, the statement $\mathrm{Adj}(\mathcal{S}^{I}) \cong \mathcal{S}^{I \times I}$ has an obvious interpretation : there is a one-to-one correspondence between endomorphisms of a vector space and matrices . This is so if we "see" functors $I \longrightarrow \mathcal{S}$, as vectors with coordinates in the set I , such that the i-th coordinate of X is the value at i of X, which we may denote by X_i rather that iX to suggest the given interpretation. A functor $I \times I \rightarrow \mathcal{A}$, can be seen as a matrix whose (i,j)-th coordinate be $(i,j)A$ and denoted A_{ij}. Then, the correspondence is given as in 14.1, i.e., given $E : \mathcal{A}^I \longrightarrow \mathcal{A}^I$, the matrix A corresponding to the endomorphism E is given by the commutativity of the triangle:

$$\begin{array}{ccc} \mathcal{A}^I & \xrightarrow{E} & \mathcal{A}^I \\ & {}_{H}\nwarrow \quad I \quad \nearrow_{A} & \end{array}$$

and therefore, $A_{ij} = (j)(iA_j) = j(H_i E)$. If E is the identity functor, then the corresponding matrix is diagonal, with $A_{ij} = 1$ iff $i = j$ and $A_{ij} = 0$ iff $i \neq j$. Conversely, given a matrix $A : I \times I \longrightarrow \mathcal{A}$, the corresponding endomorphism of $\mathcal{A}^I$ is given by: for X in $\mathcal{A}^I$, the value of A at X is denoted $X * A$ and it is an object of $\mathcal{A}^I$ defined, for $i \in I$ by

$$(X * A)_i = i(X * A) = \sum_{X_k} A_{ki} = \sum_k X_k \times A_{ki} \,.$$

This suggests a matrix multiplication as well, given by the usual composition of functors, when defined, and the correspondence between endomorphisms of $\mathcal{A}^I$ and $I \times I$ matrices. That is, let

$$\mathcal{A}^{I \times K} \times \mathcal{A}^{K \times J} \xrightarrow{\;*\;} \mathcal{A}^{I \times J}$$

be the matrix multiplication given by the corresponce and the usual composition of adjoint functors to yield adjoint functors, so that the coadjoint of the composition of two functors which have coadjoints is the composition of the coadjoints in inverse order:

$$Adj(\mathcal{A}^I, \mathcal{A}^K) \times Adj(\mathcal{A}^K, \mathcal{A}^J) \xrightarrow{\;\circ\;} Adj(\mathcal{A}^I, \mathcal{A}^J)$$

After the above discussion, it is clear how the matrix multiplication is the usual one, i.e., for A in $\mathcal{S}^{I\times K}$ and B in $\mathcal{S}^{K\times J}$, $A \ast B$ is an object in $\mathcal{S}^{I\times J}$ defined for $(i,j) \in I \times I$ by,

$$(A \ast B)_{ij} = \sum_{k} (A_{ik} \times B_{kj}).$$

This can be done also in the non-discrete case : if F is an object in $\mathcal{S}^{\mathbb{B}}$ and G is an object in $\mathcal{S}^{\mathbb{B}^* \times \mathbb{C}}$, then $F \ast G$ is an object in $\mathcal{S}^{\mathbb{A}}$ such that its value at any object A of $\mathbb{A}$ is :

$$(A)\ F \ast G = \Big[\sum_{B} BF \times (B,A)G\Big] \Big/ \ (x', g((b,A)G)) \equiv (x'(bF), g)$$

where $x' \in B'F$; $B' \xrightarrow{b} B$; $x'(bF) \in BF$ and $B'F \xrightarrow{bF} BF$; $g \in (B,A)G$ so that $g((b,A)G) \in (B',A)G$.

This can be seen as follows:

$$(A)\ F \ast G = (A)\ (\varinjlim\ ((H,F) \to \mathbb{B}^* \xrightarrow{G} \mathcal{S}^{\mathbb{A}}) \cong$$

$$\cong \varinjlim\ ((H,F) \to \mathbb{B}^* \xrightarrow{G} \mathcal{S}^{\mathbb{A}} \xrightarrow{ev_A} \mathcal{S}\ ) \cong$$

$$\cong \varinjlim\ ((H,F) \to \mathbb{B}^* \xrightarrow{G(ev_A)} \mathcal{S}\ )$$ where ev_A is "evaluation at A" .

But $\varinjlim((H,F) \to \mathbb{B}^* \xrightarrow{G(ev_A)} \mathcal{S}\ ) \cong \Big[\sum_{BF} (B,A)G\Big] \ / \ K_p$ where the following is an exact diagram:

$$K_p \xrightarrow{k_p} \sum_{BF}(A,B)G \times \sum_{BF}(A,B)G \rightrightarrows \sum_{BF}(A,B)G \xrightarrow{p} (A)\ F \ast G$$

The relations by which the coproduct factors out are forced by the conditions : $B' \xrightarrow{b} B$ induces $H_B \to H_{B'}$ (over F) commutative .

We can now express "matrix multiplication" :

$$\mathcal{S}^{\mathbb{A}^* \times \mathbb{B}} \times \mathcal{S}^{\mathbb{B}^* \times \mathbb{C}} \xrightarrow{\ \ast\ } \mathcal{S}^{\mathbb{A}^* \times \mathbb{C}}$$

by the following :

given M in $\mathcal{S}^{\mathbb{A}^*\times\mathbb{B}}$ and N in $\mathcal{S}^{\mathbb{B}^*\times\mathbb{C}}$, $M \ast N$ is an object in $\mathcal{S}^{\mathbb{A}^*\times\mathbb{C}}$ such that its value at an object (A,C) of $\mathbb{A}^*\times\mathbb{C}$ is

$$(A,C)\ M \ast N = \left[\sum_{|\mathbb{B}|}(A,B)M \times (B,C)N\right] / \ (h,\ g(b,C)N\ ) = (h(A,b)M, g)$$

where $b: B' \to B$, $h \in (A,B')M$ so that $h(A,b)M \in (A,B)M$ and $g \in (B,C)N$ so that $g(b,C)N \in (B',C)N$.

The above is so because :

$$(A,C)\ M \ast N = \varinjlim\ ((H,\ (A,\ \ )M) \longrightarrow \mathbb{B}^* \xrightarrow{(C,\ )N} \mathcal{S}\ ).$$

In the correspondence $\mathrm{Adj}(\mathcal{S}^{\mathbb{A}}, \mathcal{S}^{\mathbb{A}}) \cong \mathcal{S}^{\mathbb{A}^*\times\mathbb{A}}$, the identity functor corresponds to the HOM -"matrix" , i.e., to the bifunctor $HOM : \mathbb{A}^*\times\mathbb{A} \to \mathcal{S}$, so that

$M : \mathbb{A}^*\times\mathbb{B} \to \mathcal{S}$ defines an equivalence between $\mathcal{S}^{\mathbb{A}}$ and $\mathcal{S}^{\mathbb{B}}$, iff there exists $N : \mathbb{B}^*\times\mathbb{A} \to \mathcal{S}$ such that $M \ast N \cong HOM_{\mathbb{A}}$ and $N \ast M \cong HOM_{\mathbb{B}}$.

§ 15 – ON THE DIFFERENT REPRESENTATIONS OF A CATEGORY AS A DIAGRAMMATIC CATEGORY

If no category could be represented in more than one way as a diagrammatic category, that would mean that a diagrammatic category is completely determined by the domain category for the set-valued functors. In other words : it would be true that given any two diagrammatic categories which were isomorphic , $\mathcal{S}^{\mathbb{A}} \cong \mathcal{S}^{\mathbb{B}}$, then also the domain categories $\mathbb{A}$ and $\mathbb{B}$ would be isomorphic categories. However, this is not so, as we shall see. On the other hand, and as in the case of complete atomic

Boolean algebras, complete atomic regular categories are completely determined by the atoms in them . This is intuitively so , and can be shown as follows :

<u>Proposition 15.1</u> Let $\mathcal{X}$, $\mathcal{X}'$ be complete atomic regular categories and $\phi : \mathcal{X} \longrightarrow \mathcal{X}'$ an isomorphism of categories. Then, ϕ preserves the atoms and the corresponding full subcategories of $\mathcal{X}$ and $\mathcal{X}'$ determined by the atoms in each one, are isomorphic categories under the restriction of ϕ .

Proof:

Let A be an atom in $\mathcal{X}$. Let us show that $A\phi$ is an atom in $\mathcal{X}'$. Let (f',g') be a jointly epi pair of maps in $\mathcal{X}'$ with codomain Z' . Then, since ϕ is full and dense, there is a Z in $\mathcal{X}$, and f , g with codomain Z such that $Z\phi \cong Z'$, $f\phi = f'$, $g\phi = g'$. Moreover, (f,g) is jointly epi in $\mathcal{X}$: given r, s, such that $fr = fs$ and $gr = gs$, then also, $(f\phi)(r\phi) = (f\phi)(s\phi)$ and $(g\phi)(r\phi) = (g\phi)(s\phi)$, so that if $(r\phi)$ is called r' and $(s\phi)$, s', we have $f'r' = f's'$ and $g'r' = g's'$. But then $r' = s'$ which implies since is faithful, that $r = s$. So, given $A\phi \xrightarrow{z'} Z'$ since ϕ is dense, there exists Z such that $Z\phi \cong Z'$ and since is full , there exists $A \xrightarrow{z} Z$ such that $z\phi = z'$. Since A is an atom in $\mathcal{X}$, there exists x such that $xf = z$, for example (it could factor trough g instead, or as well) . Then $(x\phi)f' = (x\phi)(f\phi) = (xf)\phi = z\phi = z'$. The second property of being an atom is similarly proven to be true of $A\phi$. Since ϕ is one-to-one on objects, there is a one-to-one correspondence between the two classes (sets) of atoms , and since ϕ is dense, full and faithful, the two small categories determined by the atoms in each category, are isomorphic categories under ϕ . QED .

Any diagrammatic category is complete atomic regular, since the atoms contain as a subclass the representable functors, which generate the category. The question that comes up naturally, is whether the representable functors are <u>all</u> the atoms, in an arbitrary diagrammatic catagory . We already know that any retract of an atom is again an atom , in any regular category whatsoever. Are all retracts of representables again representables? Another question is : are there any other atoms which are not retracts of any representable ? We answer the last question first:

<u>Theorem 15.2</u> In any diagrammatic category $\mathcal{S}^{\mathfrak{C}}$, the atoms are precisely the retracts of the representables.

Proof:

Let T be an atom in $\mathcal{S}^{\mathfrak{C}}$. Since the family of representables is generating, there exists a set J and a family of representables indexed by J and an epimorphism p from the coproduct of this family into T , $\sum_J H^A \xrightarrow{p} T$. Since T is an atom , it is projective, therefore there exists a map $T \xrightarrow{h} \sum_J H^A$ such that $hp = T$. But T being an atom is also abstractly unary, therefore there exists H^A and $T \xrightarrow{k} H^A$ such that, if j is the injection corresponding to H^A trough which h factors, $h = kj$. Finally , the following commutative diagram says that T is a retract of H^A :

$$\begin{array}{ccc} & & T \\ & \swarrow^{k} & \downarrow T \\ H^A & \xrightarrow{jp} & T \end{array}$$

QED.

We now plan to answer the question whether all retracts of representables are or not always representables . If the answer were to be affirmative , then we would have , after 15.2, that the representables would be all the atoms in any diagrammatic category . However, it is not so in general, and we want to give a sufficient condition for this to happen.

We first need a definition taken from Freyd ([8]) : an <u>idempotent</u> (map) is a map e such that ee = e . In a category $\mathcal{X}$, it is said that <u>idempotents split</u> iff for every idempotent $A \xrightarrow{e} A$, there exists an object B and maps $A \xrightarrow{a} B$, $B \xrightarrow{b} A$ such that $A \xrightarrow{a} B \xrightarrow{b} A = A \xrightarrow{e} A$ and $B \xrightarrow{b} A \xrightarrow{a} B = B \xrightarrow{B} B$. Freyd defines <u>amenable</u> categories as categories which are additive, have finite coproducts and where all idempotents split . Then a necessary and sufficient condition for a category of additive functors with domain category $\mathcal{A}$ and codomain category $\mathcal{G}$ (the category of abelian groups), to have the property that every abstractly finite projective object be representable is that the category $\mathcal{A}$ be amenable.

We want to prove an analogous theorem to that of Freyd , for diagrammatic categories. The existence of coproducts in the domain category is not needed since the atoms are more than abstractly finite : they are abstractly unary as well. There it is used that $\langle \mathcal{A}, \mathcal{G} \rangle$ is abelian, in the fact that there are unique factorisations into epis followed by monos. But this is true of any diagrammatic category, without being abelian. Therefore, the proof is quite similar, only less is needed here :

<u>Proposition 15.3</u> If in $\mathbb{C}$, all idempotents split, then, in

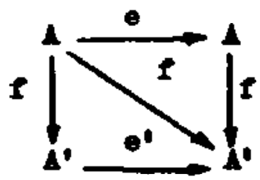
$\mathcal{S}^{\mathbb{C}}$, every atom is representable.

Proof:

Let T be an atom in $\mathcal{S}^{\mathbb{C}}$. By 15.2, T is a retract of some H^A, i.e., there exists a map $T \xrightarrow{r} H^A$ and a map $H^A \xrightarrow{s} T$ such that $rs = T$. But then, sr is an idempotent since $(sr)(sr) = s(rs)r = sr$. Also, since the regular representation of $\mathbb{C}^*$ is full and $H^A \xrightarrow{sr} H^A$, there exists $x : A \longrightarrow A$ such that $sr = H^x$. Now, $H^x H^x = H^{xx} = H^{srsr} = H^{sr} = H^x$, and since is faithful, $xx = x$, or x is an idempotent in $\mathbb{C}$. Therefore, it splits by means of maps $A \xrightarrow{a} A'$, $A' \xrightarrow{b} A$ such that $A \xrightarrow{a} A' \xrightarrow{b} A = x$ and $A' \xrightarrow{b} A \xrightarrow{a} A' = A'$; so, $H^A \xrightarrow{s} T \xrightarrow{r} H^A = H^x = H^A \xrightarrow{H^b} H^{A'} \xrightarrow{H^a} H^A$. Now, $rs = T$ implies that r is mono and s is epi, therefore H^x is factored into an epi followed by a mono, by means of s and r. But $H^{A'}$ is also a retract of H^A so that H^b is epi and H^a, mono. Since such factorizations are unique in any diagrammatic category, $T \cong H^{A'}$. QED.

It is an exercise in Freyd [8], that any small category can be embedded into another in which idempotents split, and moreover, it can be done in a minimal universal way. We shall define here also the closure under the splitting of the idempotents of any small category, and although our definition looks different from that of Freyd's, it turns out that they are equivalent. We prefer our definition because it is easier to draw explanatory diagrams, however disadvantageous is the fact that it resembles a subcategory of a functor category though it is not.

Given any small category $\mathbb{C}$, we define its idempotents-splitting closure $\overline{\mathbb{C}}$ as follows: let the objects of $\overline{\mathbb{C}}$ be the idempotents of $\mathbb{C}$, i.e., $A \xrightarrow{e} A$ is an object in $\overline{\mathbb{C}}$ iff e is an idempotent in $\mathbb{C}$. Given any two objects $A \xrightarrow{e} A$, and $A' \xrightarrow{e'} A'$ in $\overline{\mathbb{C}}$, a map from the first to the second is a commutative diagram :



i.e., a commutative square with a built-in diagonal, which reduces to the following equations : $ef = f = fe'$. We will denote this map by (e,f,e'). The condition for $f : A \longrightarrow A'$ in Freyd's definition, reads as follows : $efe' = f$. We show that both are equivalent.

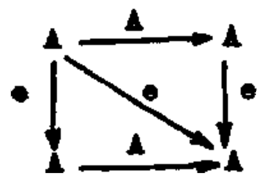
If $efe' = f$ then $ef = eefe' = efe' = f$ and $fe' = efe'e' = efe' = f$. Conversely, if $ef = f = fe'$ then, $efe' = fe'e' = fe' = f$.

Composition of maps $(e,f,e')(e',g,e'') = (e,fg,e'')$ because, if f is such that $ef = f = fe'$ and g is such that $e'g = g = ge''$ then, $e(fg) = (ef)g = (fe')g = f(e'g) = f(ge'') = (fg)e''$, so that $e(fg) = (ef)g = fg = f(ge'') = (fg)e''$.

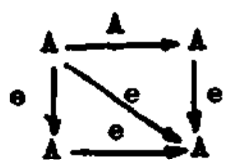
The identity map of $A \xrightarrow{e} A$ is (e,e,e) since $ee = e = ee$.

On the other hand, if we had defined a subcategory of a functor category, the identity map of $A \xrightarrow{e} A$ would have to be $A \xrightarrow{A} A$, however the condition imposed by the presence of the diagonal prevents this from being so, since $eA \neq A$ and $Ae \neq A$.

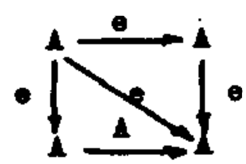
We now define the canonical functor $\mathbb{C} \xrightarrow{i} \overline{\mathbb{C}}$, as follows: given $A \in |\mathbb{C}|$, let $Ai = (A,A,A)$, the identity map of $A \xrightarrow{A} A$, where $A \xrightarrow{A} A$ is certainly an idempotent in $\mathbb{C}$. Let $A \xrightarrow{f} A'$ be any map in $\mathbb{C}$, then $fi = (A, f, A')$. This defines obviously a functor We now show that idempotents in $\mathbb{C}$, which are now objects in $\bar{\mathbb{C}}$, when mapped by i into $\bar{\mathbb{C}}$, they become maps and only the idempotents which are [illegible] maps in $\bar{\mathbb{C}}$. That they split in $\bar{\mathbb{C}}$, can be seen as follows: let $A \xrightarrow{e} A$ be an idempotent in $\mathbb{C}$. Its image under i is the map (A,e,A), i.e., the commutative diagram



The splitting is given as follows : take the object $A \xrightarrow{e} A$ in $\bar{\mathbb{C}}$, and the maps given by the commutative diagrams:



and



and then we verify that $(A,e,e)(e,e,A) = (A, ee, A) = (A,e,A)$ and $(e,e,A)(A,e,e) = (e,ee,e) = (e,e,e)$, therefore we have the required splitting.

The canonical functor $\mathbb{C} \xrightarrow{i} \bar{\mathbb{C}}$ induces a functor $\mathcal{S}^{\bar{\mathbb{C}}} \xrightarrow{\mathcal{S}^{i}} \mathcal{S}^{\mathbb{C}}$ and we want to show that the latter is an isomorphism of categories. That the above construction gives the minimal category in which $\mathbb{C}$ is embedded and it is such that idempotents of $\mathbb{C}$ split in $\bar{\mathbb{C}}$, is clear, since the objects of the new category are idempotents of the first, and the maps come from the category $\mathbb{C}$ as well.

<u>Theorem 15.4</u> For any small $\mathbb{C}$, and its idempotent-splitting closure $\bar{\mathbb{C}}$, the canonical functor $\mathbb{C} \xrightarrow{i} \bar{\mathbb{C}}$ induces an isomorphism $\mathcal{S}^{i} : \mathcal{S}^{\bar{\mathbb{C}}} \xrightarrow{\cong} \mathcal{S}^{\mathbb{C}}$.

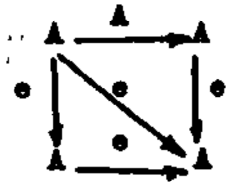
Proof:

It is known (Lawvere [14]) that any functor between diagrammatic categories which is induce by a functor between the domain categories, has both an adjoint and a coadjoint. We usw the formulas of §14 to calculate the adjoint of $\mathcal{S}^{i}$, and then we show that it is actually an inverse. Given F in $\mathcal{S}^{\bar{\mathbb{C}}}$, the value of $\mathcal{S}^{i}$ at F is defined to be $F\mathcal{S}^{i} = iF$ (composition). Let $\hat{\mathcal{S}}^{i}$ be the adjoint to $\mathcal{S}^{i}$, and T any object of $\mathcal{S}^{\mathbb{C}}$. Then $T\hat{\mathcal{S}}^{i}$ is an object in $\mathcal{S}^{\bar{\mathbb{C}}}$, whose value at an object $A \xrightarrow{e} A$ of $\bar{\mathbb{C}}$, or equivalently an identity map (e,e,e) of $\bar{\mathbb{C}}$, is given by

$$(e,e,e)\ T\hat{\mathcal{S}}^{i} = \underrightarrow{\lim}\ ((i,(e,e,e)) \to \mathbb{C} \xrightarrow{T} \mathcal{S}) \cong$$

$$\cong \left[\sum_{A' \in |\mathbb{C}|} \mathrm{HOM}((A',A',A'),\ (e,e,e)) \times A'T \right] \Big/ \ (b',\ x''(aT)) = (b'(ai),x'')$$

where $a\colon A'' \longrightarrow A'$, $x'' \in A''T$ so that $x''(aT) \in A'T$ and $b' : (A',A',A') \longrightarrow (e,e,e)$ so that $b'(ai) : (A'',A'',A'') \longrightarrow (e,e,e)$. However, we can simplify these relations considerably if we notice that the only (A',A',A') for which there is a map $b' : (A',A',A') \longrightarrow (e,e,e)$ is (A,A,A) since the following is a commutative diagram :



and if f is also such that $A'f = f = fe = fA$ then $f = A$ since identity maps are unique. Also, the only $A \xrightarrow{a} A$ for which $(A,a,A)(A,e,e) = (A,e,e)$ is $A \xrightarrow{e} A$, so that we finally have :

$$(e,e,e)\, T \hat{\mathcal{S}}^{i} = \{(A,e,e)\} \times AT \Big/ \; x'' = x''(eT) \text{ for every } x'' \in AT,$$

$$\text{i.e., } eT = AT .$$

We now compute both compositions $\mathcal{S}^{i} \hat{\mathcal{S}}^{i}$, $\hat{\mathcal{S}}^{i} \mathcal{S}^{i}$, to see that they are the corresponding identities.

For any identity map (e,e,e) in $\bar{\mathbb{C}}$, by the above , if F is in $\mathcal{S}^{\bar{\mathbb{C}}}$, then

$$(e,e,e)\,(F \mathcal{S}^{i} \hat{\mathcal{S}}^{i}) = \Big[\{(A,e,e)\} \times A\,(F \mathcal{S}^{i})\Big] \Big/ \; e(F \mathcal{S}^{i}) = A \;\cong$$

$$\cong \big[(Ai)F\big] \Big/ \; (ei)F = A \;\cong$$

$$\cong (A,A,A)F \Big/ \; (A,e,e)F = A$$

and since $(A,e,e) : (A,A,A) \rightarrow (e,e,e)$ this says that $(A,A,A)F = (e,e,e)F$. Finally, we have

$$(e,e,e)\,(F \mathcal{S}^{i} \hat{\mathcal{S}}^{i}) = (A,A,A)F \Big/ \; (A,A,A)F = (e,e,e)F \text{ , i.e.,}$$

$$(e,e,e)\,(F \mathcal{S}^{i} \hat{\mathcal{S}}^{i}) = (e,e,e)F .$$

On the other hand, for any T in $\mathcal{S}^{\mathbb{C}}$, and $A \in |\mathbb{C}|$,

$$A\,(T \hat{\mathcal{S}}^{i} \mathcal{S}^{i}) = (Ai)(T \hat{\mathcal{S}}^{i}) = (A,A,A)(T \hat{\mathcal{S}}^{i}) = AT .$$ QED .

With this theorem it is now clear that there may be diagrammatic categories which are isomorphic , and such that they have non isomorphic domain categories . It is enough to give an example of a small category which is not isomorphic to its own idempotent-splitting closure. Take, for example, a category with exactly two maps , one identity map A , and another non-identity map $A \xrightarrow{e} A$ which is idempotent .

In any diagrammatic category, the atoms are precisely the retracts of the representables, by 15.2 . Therefore, the full subcategory generated by the atoms in any diagrammatic category is precisely the full subcategory generated by the representables and their retracts. Moreover, we have the following:

<u>Theorem 15.5</u> In any diagrammatic category, the full subcategory generated by all the representables and their retracts is isomorphic to the idempotent-splitting closure of the full subcategory generated by the representables.

Proof:

The atoms in $\mathcal{S}^{\mathbb{C}}$ are all the retracts of the representable functors. These retracts give rise to idempotents in the full subcategory of generated by the representables, which split in the corresponding closure. By unique factorizations of maps into epis followed by monos, it is easily seen that the splitting of idempotents arising from retracts are given by the retractions themselves. So, every atom in $\mathcal{S}^{\mathbb{C}}$ is an object in the closure under the splitting of idempotents of the full subcategory of $\mathcal{S}^{\mathbb{C}}$ generated by the representables. Conversely, for any idempotent $H^A \xrightarrow{e} H^A$ in the closure of the subcategory of representables, the splitting is given by maps $H^A \xrightarrow{r} T$, $T \xrightarrow{s} H^A$, such that $rs = e$ and $T \xrightarrow{s} H^A \xrightarrow{r} T = T$ so that T is a retract of H^A and therefore, an atom. QED.

<u>Theorem 15.6</u> (Morita isomorphism theorem for diagrammatic categories) For any two small categories $\mathbb{A}$ and $\mathbb{B}$,

$$\mathcal{S}^{\mathbb{A}} \cong \mathcal{S}^{\mathbb{B}} \quad \text{iff} \quad \overline{\mathbb{A}} \cong \overline{\mathbb{B}} .$$

Proof:

Assume there is an isomorphism of categories $\phi : \mathcal{S}^{\mathbb{A}} \cong \mathcal{S}^{\mathbb{B}}$.

Then, by 15.1, the restriction of ϕ to the full subcategory of $\mathcal{S}^{\mathbb{A}}$ generated by the atoms gives an isomorphism onto the full subcategory generated by the atoms in $\mathcal{S}^{\mathbb{B}}$. By 15.5, this implies that idempotents-splitting closures of the full subcategories generated by representables in each category, are isomorphic categories. But, since in each diagrammatic category, the small domain category for the functors and the full subcategory generated by the representable functors are isomorphic, also their idempotents-splitting closures are isomorphic. Therefore, $\bar{\mathbb{A}} \cong \bar{\mathbb{B}}$.

Let $\bar{\mathbb{A}} \cong \bar{\mathbb{B}}$. Then $\mathcal{S}^{\bar{\mathbb{A}}} \cong \mathcal{S}^{\bar{\mathbb{B}}}$, and by 15.4, we have that $\mathcal{S}^{\mathbb{A}} \cong \mathcal{S}^{\bar{\mathbb{A}}} \cong \mathcal{S}^{\bar{\mathbb{B}}} \cong \mathcal{S}^{\mathbb{B}}$. QED.

We now investigate the question of the uniqueness of the representation of a given category (complete atomic regular) as a diagrammatic category. The representation given in 13.1 is, in a sense, the maximal one: there are at least as many others as generating subsets of the set of atoms in the category. This is so, since the category of atoms, besides being its own closure under splitting of idempotents, is the closure, as well, of any full subcategory generated by a proper subset of the set of atoms which is also generating for the category. This can be shown as follows:

<u>Proposition 15.7</u> Let $\mathcal{X}$ complete regular atomic. Let I be the set of its atoms, $\mathbb{C}$ the full subcategory generated by the objects in I. Let $I' \subseteq I$ be any subset which is also generating (it need not

be a proper subset) for $\mathcal{X}$. Let $\mathbb{C}'$ be the full subcategory of $\mathcal{X}$ generated by the objects in I'. Then, $\overline{\mathbb{C}'} \cong \mathbb{C}$.

Proof:

Since there is an inclusion of sets $I' \hookrightarrow I$, it induces an inclusion functor $\mathbb{C}' \hookrightarrow \mathbb{C}$ which in turn induces $\overline{\mathbb{C}'} \longrightarrow \overline{\mathbb{C}} \cong \overline{\mathbb{C}'} \longrightarrow \mathbb{C}$ since $\overline{\mathbb{C}} \cong \mathbb{C}$. We now define a functor in the opposite direction. If a family of objects is generating in a category, then every atom is a retract of at least one object in the family. This is so because, if A is an atom and the family $\{A_i\}$ whose members are atoms, is generating, there exists a set J and an epimorphism $\sum_J A_i \xrightarrow{p} A$. However, A being projective implies that there exists a map $A \xrightarrow{r} \sum_J A_i$ such that $rp = A$. But A being an atom is also abstractly unary, and therefore there exists a map s and an atom A_j such that if i_j is the injection corresponding to A_j, $r = si_j$. Therefore, there exists an atom A_j and maps $A \xrightarrow{s} A_j$ and $A_j \xrightarrow{i_j} \sum_J A_i \xrightarrow{p} A$ such that $A \xrightarrow{s} \sum_J A_j \xrightarrow{i_j p} A = A$. So, A is a retract of A_j. Therefore, since $(i_j p)s$ is an idempotent in $\mathbb{C}'$ whose splitting is given by A, then A must be an object in the closure of $\mathbb{C}'$, that is, in $\overline{\mathbb{C}'}$. This is a functor, and both compositions give the identity. QED.

The above proposition suggests that if the small category is already closed under splitting idempotents, and no subfamily of its set of objects is also generating, then, the corresponding diagrammatic category can be represented in no other way as a diagrammatic category. An example which is almost trivial of such categories is provided by the small discrete categories . Indeed, for them:

<u>Proposition 15.8</u> Let $\mathcal{X}$ be complete regular atomic and assume that the full subcategory generated by the atoms is discrete, i.e., a set I . Then, $\mathcal{X} \cong \mathcal{S}^{I}$ is the only representation of $\mathcal{X}$ as a diagrammatic category.

Proof:

If the full subcategory of $\mathcal{X}$ generated by the atoms in $\mathcal{X}$, is discrete, no proper subset of I could be a generating family for $\mathcal{X}$: Assume on the contrary, that there exists $I' \subsetneq I$ and $I' \neq I$ such that the objects in I' are a generating set of objects for $\mathcal{X}$. Then, by the proof of 15.7 , if A is an atom and an element of I which is not an element of I' , A is a retract of an object B of I' , since I' is generating. That means that there are maps $A \to B$ and $B \to A$ where both A and B are objects of I . However, I was discrete, therefore there are no maps in I . Contradiction. This means that no proper subset of I is generating and so, I is not the closure of any proper subset. Assume there is a small category $\mathbb{C}$, for which $\mathcal{S}^{I} \cong \mathcal{S}^{\mathbb{C}}$. This implies, by Morita isomorphism theorem, that $\overline{\mathbb{C}} \cong \overline{I} \cong I$, and therefore, discrete . Therefore, also is discrete, and so, $\overline{\mathbb{C}} \cong \mathbb{C}$. Therefore, $\mathbb{C} \cong I$. QED .

Discrete small categories are trivial examples of small categories which determine uniquely their corresponding diagrammatic categories. There are less trivial examples. Actually, for any $\mathbb{C}$ such that no proper subset of $|\mathbb{C}|$ generates the category, this is true as well. And for this, it is too much to ask that there be no maps in $\mathbb{C}$.

It is more than enough that there be no idempotents . In fact, this condition happens to be necessary as well. We now prove:

Theorem 15.9 Let $\mathcal{X}$ be any complete atomic regular category. Then , there is only one representation of $\mathcal{X}$ as a diagrammatic categoty (up to isomorphism) iff the full subcategory of $\mathcal{X}$ generated by the atoms, contains no idempotents, except the identity maps.

Proof:

Assume that $A \xrightarrow{e} A$ is an idempotent which is not an identity map, in $\mathbb{C}_1$, the full subcategory of generated by the atoms. Since $\mathbb{C}_1$ is its own idempotent-splitting closure , there is an object B and maps $A \xrightarrow{s} B$, $B \xrightarrow{r} A$ in $\mathbb{C}_1$, such that $A \xrightarrow{s} B \xrightarrow{r} A = e$ and $B \xrightarrow{r} A \xrightarrow{s} B = B$. Then , the family of all the atoms in $\mathcal{X}$ without the atom B is also generating. To see this, let f and g be any pair of maps in $\mathcal{X}$ with common domain and codomain , and such that $f \neq g$. Then, if there exists a map $B \xrightarrow{x} X$ such that $B \xrightarrow{x} X \xrightarrow{f} Y \neq B \xrightarrow{x} X \xrightarrow{g} Y$, the map $A \xrightarrow{s} B \xrightarrow{x} X$ is also such that $A \xrightarrow{sx} X \xrightarrow{f} Y \neq A \xrightarrow{sx} X \xrightarrow{g} Y$, since s is epi. Let $\mathbb{C}_1'$ be the full subcategory of $\mathcal{X}$ generated by all the atoms with the exception of B . By 15.7 , $\overline{\mathbb{C}}_1' \cong \mathbb{C}_1$, and by 15.4 , $\mathcal{S}^{\mathbb{C}'} \cong \mathcal{S}^{\overline{\mathbb{C}}'} \cong \mathcal{S}^{\mathbb{C}}$. Since $\mathbb{C}_1' \ncong \mathbb{C}$, this gives two different representations of $\mathcal{X}$ as a diagrammatic category, since $\mathcal{X} \cong \mathcal{S}^{\mathbb{C}} \cong \mathcal{S}^{\mathbb{C}'}$. So, if the representation of $\mathcal{X}$ as $\mathcal{S}^{\mathbb{C}}$ is unique (up to isomorphism), there are no idempotents in $\mathbb{C}_1$.

The converse of the theorem is immediate : if $\mathbb{C}$ is the full subcategory of the atoms and contains no idempotents, then, it is minimal generating (no proper subset of its set of objects is generating) and its own closure. Assume that there exists $\mathbb{A}$, such that $\mathcal{S}^{\mathbb{C}} \cong \mathcal{S}^{\mathbb{A}}$. By 15.6 , $\bar{\mathbb{A}} \cong \bar{\mathbb{C}} \cong \mathbb{C}$. But this means that $\mathbb{A}$ is a subcategory of $\mathbb{C}$ whose closure is $\mathbb{C}$. Moreover, $\mathbb{A}$ is isomorphic to a family of representable functors and all maps between, which is a generating family for $\mathcal{S}^{\mathbb{A}}$. This contradicts the above. Therefore, the representation is unique up to isomorphism. QED .

As examples of small categories which contain no idempotents other than identity maps and which play important roles in the theory of diagrammatic categories and in the category of categories, are $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{4}$.

We remark that in $\mathcal{S}$, 1 is a generator and an atom, therefore the only atom, because any other atom would have to be a retract of 1 (since $\{1\}$ is generating) and therefore, isomorphic (equal, by Convenience axiom) to 1 . Therefore, another characterisation of $\mathcal{S}$ is : $\mathcal{S}$ is the only (up to isomorphism) right-complete atomic regular category in which 1 is an atom and a generator (or else, in which 1 is the only atom) .

With this, we end the main part of our paper. In the next and last chapter, we deal with applications to the class of diagrammatic categories, of the theory of triples and of triplable categories. Chapter IV is somewhat independent of the first three chapters.

Chapter IV

ALGEBRAIC ASPECTS OF DIAGRAMMATIC CATEGORIES

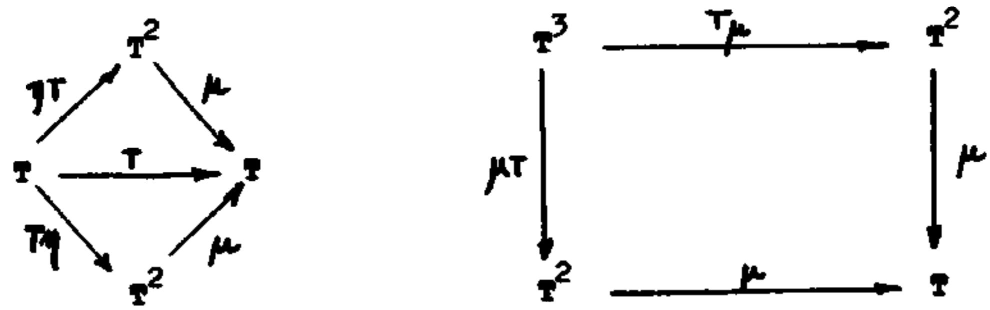
§ 16 - ADJOINT AND COADJOINT TRIPLES AND COTRIPLES

In this section we review briefly the notions of triple and cotriple in categories, along with some well known facts about them. Further information can be found in Eilenberg & Moore [5] .

A <u>triple</u> (T, η, μ) in a category $\mathcal{X}$ is an endofunctor T of $\mathcal{X}$, together with two natural transformations

$$1 \xrightarrow{\ \eta\ } T \xleftarrow{\ \mu\ } T^2$$

such that the following diagrams are commutative:

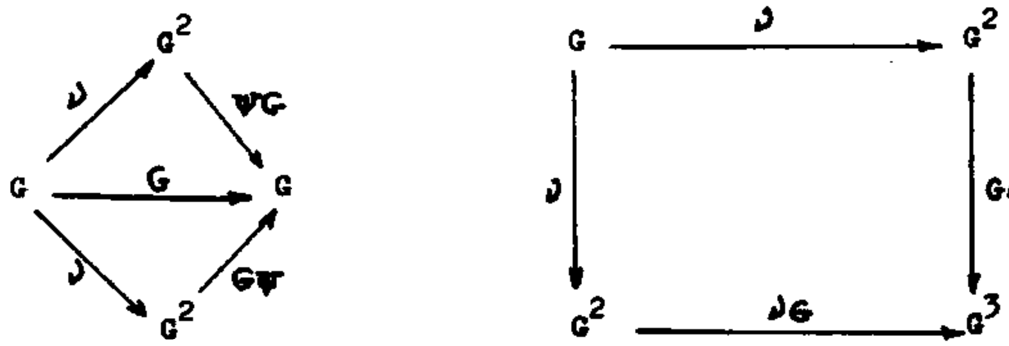


η is called the <u>unit</u> of the triple and μ its <u>multiplication</u>. The three equations exxpressed by the commutativity of the diagrams above say just that η is a two-sided unit for the multiplication and that the latter is associative.

Dually, a <u>cotriple</u> (G , ψ, ν) in a category $\mathcal{X}$ is an endofunctor of $\mathcal{X}$, together with two natural transformations

$$1 \xleftarrow{\psi} G \xrightarrow{\nu} G^2$$

such that the following diagrams are commutative :



ψ is called the <u>counit</u> of the cotriple and ν its <u>comultiplication</u> , and the three equations expressed by the commutativity of the diagrams say that ψ is a two-sided counit for the cotriple and that the latter is associative.

The following is a more appropiate definition of adjointness for the above context : given $\mathcal{X} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{Y}$, F is said to be <u>adjoint</u> to U (and U <u>coadjoint</u> to F) iff there are natural transformations

$$1_{\mathcal{X}} \xrightarrow{\eta} FU \qquad \text{and} \qquad UF \xrightarrow{\psi} 1_{\mathcal{Y}}$$

such that the following equations hold:

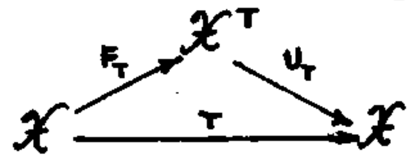
$$F \xrightarrow{\eta F} FUF \xrightarrow{F\psi} F = 1_F \quad \text{and}$$

$$U \xrightarrow{U\eta} UFU \xrightarrow{\psi U} U = 1_U \, .$$

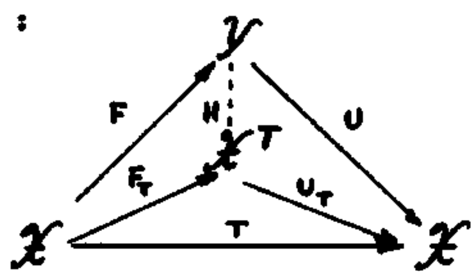
Adjoint pairs of functors give rise to triples in a canonical way, i.e., if F is adjoint to U with η, ψ , as above, then $(FU , \eta , F\psi U)$ is a triple structure on $\mathcal{X}$.

But conversely, triples give rise to adjoint pairs of functors in a minimal and a maximal way (the canonical functor from Adj($\mathcal{X}$) to Trip($\mathcal{X}$) has adjoint and coadjoint). Only maximal resolutions will interest us here . We remark that if $\mathcal{X} \cong \mathcal{S}$, then the maximal resolutions are given by the equational categories (Linton [19]), which generalise Lawvere's algebraic categories (Lawvere [14], [15]) by allowing infinitary operations as well.

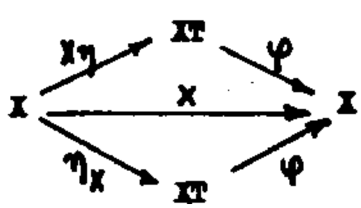
A maximal resolution of a given triple T on $\mathcal{X}$ is given by a category $\mathcal{X}^T$ said to be the category of T-algebras, and by a pair of adjoint functors F_T and U_T whose composition is T, i.e., such that the following diagram is commutative, with F_T adjoint to U_T :

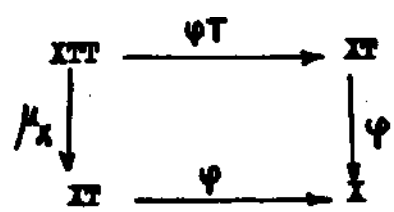


Moreover, it is a maximal resolution of T in the sense that if $\mathcal{Y}$ is any other category for which there are functors $\mathcal{X} \xrightarrow{F} \mathcal{Y}$ and $\mathcal{Y} \xrightarrow{U} \mathcal{X}$ such that $FU = T$ and F is adjoint to U, then there exists a unique functor $H : \mathcal{Y} \longrightarrow \mathcal{X}^T$, such that the diagram below is commutative :



The objects of $\mathcal{X}^T$ can be described as follows : they are pairs (X,φ) where X is in $\mathcal{X}$, and $XT \xrightarrow{\varphi} X$ is a map in $\mathcal{X}$, satisfying the equations expressed by the commutativity of the diagrams below:





A map of T-algebras $(X, \varphi) \longrightarrow (X', \varphi')$ is given by any map $X \xrightarrow{f} X'$ such that the following diagram commutes:

$$\begin{array}{ccc} XT & \xrightarrow{fT} & X'T \\ \varphi \downarrow & & \downarrow \varphi' \\ X & \xrightarrow{f} & X \end{array}$$

They generalise the usual categorical notion of algebra, as e.g., in Mac Lane [21] & [22].

The adjoint pair which gives the maximal resolution is defined by :

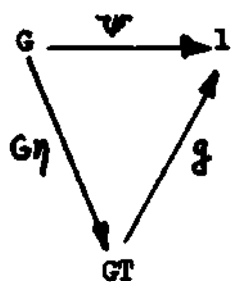
$X\,F_T = (XT,\ XT^2 \xrightarrow{\mu X} XT)$, for X an object in $\mathcal{X}$ and obvious definition for the maps $X \longrightarrow X'$ of $\mathcal{X}$;

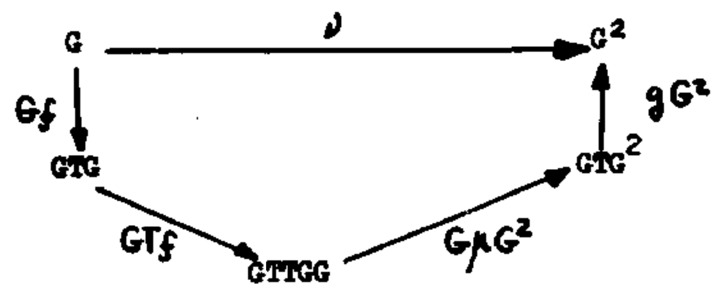
$(X, \varphi)\, U_T = X$ and it is clear that F_T is adjoint to U_T.

The dual constructions for coadjoint pairs of functors and cotriples can now easily be done, a maximal resolution is given by a category whose objects are called co-algebras, G-coalgebras, if G is the given cotriple in $\mathcal{X}$.

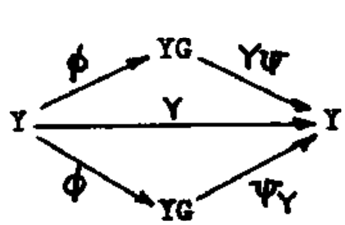
We say that a triple is a <u>coadjoint triple</u> in $\mathcal{X}$ if, as an endofunctor has an adjoint. Dually we define <u>adjoint triples</u> on a category. One can also define <u>coadjoint cotriples</u> and <u>adjoint cotriples</u>, and all these notions are related. If T is a triple in $\mathcal{X}$, and it has a coadjoint G, then G has canonically a cotriple structure. Moreover, the maximal resolutions for both T and G are isomorphic categories. This can be seen roughly as follows :

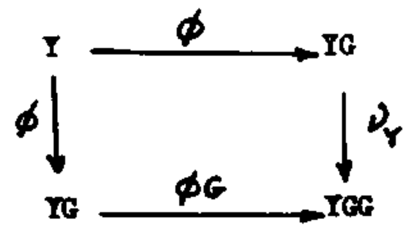
Since T has a triple structure, there are η, μ, satisfying the required equations. And since T is adjoint to G, there are also natural transformations $1_{\mathcal{X}} \xrightarrow{\delta} TG$ and $GT \xrightarrow{\varepsilon} 1_{\mathcal{Y}}$, satisfying the conditions for T to be adjoint to G . A cotriple structure for G can be given as follows : let $1 \xleftarrow{\psi} G \xrightarrow{\nu} G^2$ be defined by means of the two commutative diagrams below :





The category of G-coalgebras , ${}^G\mathcal{X}$, has as objects , pairs (Y, ϕ) with Y an object in $\mathcal{X}$ and $Y \xrightarrow{\phi} YG$ a map in $\mathcal{X}$, satisfying the three equations expressed by means of the following commutative diagrams:





With the usual definition of adjoint functors (involving HOM-sets) one can immediately see that $\mathcal{X}^T \cong {}^G\mathcal{X}$, since for each object X of $\mathcal{X}$, HOM(XT , X) $\cong$ HOM(X , XG) , and the commutativity of the diagrams follow from the way the cotriple structure for G was defined.

Similarly, given a cotriple G which has an adjoint T , T can be given canonically a triple structure. The compositions of both procedures give the identities . On the other hand, if T is a triple on $\mathcal{X}$ which has an adjoint F , then F has a cotriple structure and a cotriple with coadjoint induces a triple structure on the coadjoint. We can resume the above considerations as follows:

Adj Triples ($\mathcal{X}$) = (Coadj Cotriples ($\mathcal{X}$))* and

Adj Cotriples ($\mathcal{X}$) = (Coadjoint Triples ($\mathcal{X}$))* .

§ 17 – THE EQUATIONAL CLOSURE OF $\mathcal{S}^I$ OVER $\prod$

The category of sets and mappings has the property that any endofunctor which has an adjoint is representable. Conversely, any representable endofunctor H^I , for $I \in |\mathcal{S}|$, has an adjoint , namely the functor "crossing with I" , ( ) $\times$ I . This is so because in $\mathcal{S}$, "Homing" and "Exponentiating" coincide so that HOM(I,) is coadjoint to ( ) $\times$ I. If we make the collection of coadjoint endofunctors of $\mathcal{S}$ into a category, with the usual composition of functors (composition of functors with adjoint is again a functor with adjoint) and define a functor from the category Coadj($\mathcal{S}$) to $\mathcal{S}$, using the remarks made above we have :

$$\text{Coadj}(\mathcal{S}) \cong \mathcal{S}^*$$

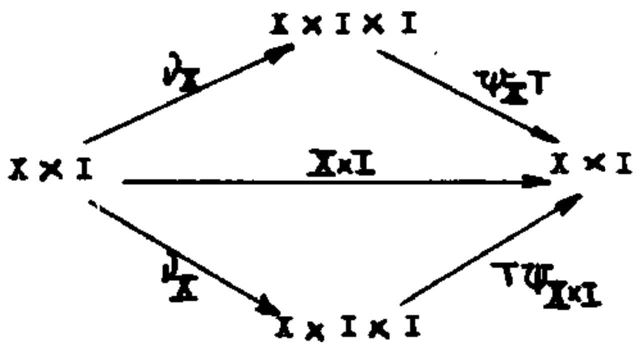
since exponentiation (in this case HOM(,)) is contravariant on the exponent (on the first variable)).

The question is now to find out which coadjoint endofunctors of $\mathcal{S}$ are also triples on $\mathcal{S}$. By §16 , the answer to this problem will be equivalent to the answer to the question : which adjoint endofunctors of $\mathcal{S}$ have also a cotriple structure?

All adjoint endofunctors of $\mathcal{S}$ are of the form T = ( ) $\times$ I , for some $I \in |\mathcal{S}|$. The following is a cotriple structure on any such T and we will show that it is the only one it can have :

let $1 \xleftarrow{\Psi_I} I \times I \xrightarrow{\nu_I} I \times I \times I$ be given, for each object

X of $\mathcal{X}$, by $(x,i)\psi_X = x$ and $(x,i)\nu_X = (x,i,i)$ for $(x,i) \in X \times I$. In other words, ψ_X is the projection onto X , and ν_X is the map induced by the diagonal map $I \longrightarrow I \times I$. That this is the only cotriple structure for $(\quad) \times I$ can be seen by the fact that if ψ', ν' gave another , then ψ' and ν' would have to satisfy a commutative diagram so :



which means that : if

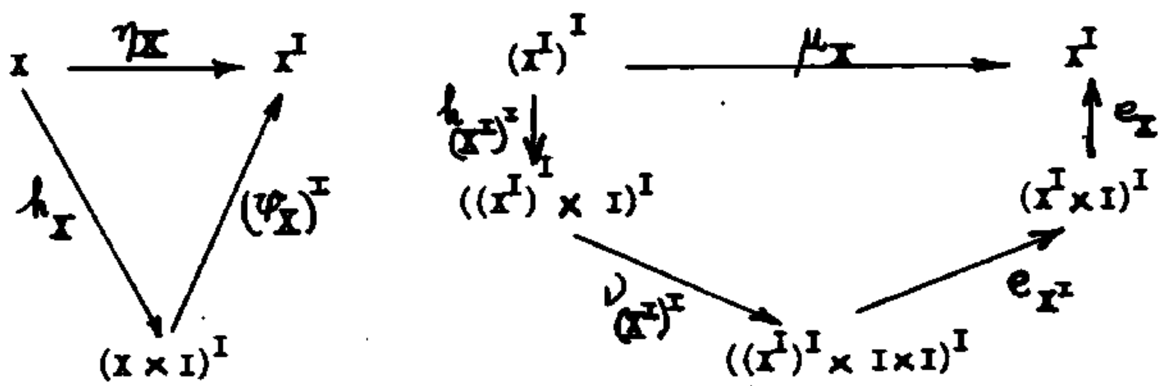
$(x,i)\nu'_X = (y,j,k)$ then $((y,j)\psi'_X ,k) = (x,i) = ((y,j),k)\psi'_{X\times I}$
Therefore, $i = k$. Also, $((y,j),i)\psi'_X = (x,i)$ and $(y,j)\psi'_X = x$ so that $(y,j) = (x,i)$ and therefore $y = x$ and $j = i$, so that $(x,i)\nu'_X = (x,i,i)$ and $(x,i)\psi'_X = x$.
The existence and uniqueness of the cotriple structure given by ψ, ν for $(\quad) \times I$, implies , by §16, that $G = (\quad)^I$ has always a triple structure, and that moreover, it is unique. This is so for any set I .
To calculate the triple structure on $(\quad)^I$ we have first to calculate the natural transformations $1_{\mathcal{X}} \xrightarrow{h} TG$, $GT \xrightarrow{e} 1_{\mathcal{X}}$ which make $T = (\quad)\times I$ adjoint to $G = (\quad)^I$. It is clear that for each X in $\mathcal{X}$, $h_X : X \longrightarrow (X \times I)^I$ is defined, for $x \in X$ and $i \in I$ by $i(x\, h_X) = (x, i)$ and that

$e_X : X^I \times I \longrightarrow X$ is just the evaluation map, i.e., for $f \in X^I$ and $i \in I$, $(f,i)e_X = (i)f$.

To define the induced triple structure on G, we have to use a procedure dual to the one given in § 16, since there it was a triple structure inducing a cotriple structure on the coadjoint of the triple. So, define

$$1 \xrightarrow{\eta} G \xleftarrow{\mu} G^2$$

at each X of $\mathcal{X}$, by means of the commutativity of the two diagrams below:



so that: for any $x \in X$ and $i \in I$,

$$i(x\,\eta_X) = i(x\,h_X\,\varphi_X) = (x,i)\,\varphi_X = x \quad \text{and}$$

$$i(f\,\mu_X) = i(f\,h_{(X^I)^I}\,\nu_{(X^I)^I}\,e_{X^I}\,e_X) =$$

$$= (f,i)(\nu_{(X^I)^I}\,e_{X^I}\,e_X) = (f,i)(\nu_{(X^I)^I}\,e_{X^I}\,e_X) =$$

$$= (f,i,i)\,e_{X^I}\,e_X = ((i)f\,,\,i)\,e_X = (i)((i)f)\,.$$

Therefore, we have shown that

$$\mathcal{S}^* = \text{Coadjoint Triples}(\mathcal{S}) = (\text{Adjoint Cotriples}(\mathcal{S}))^*$$

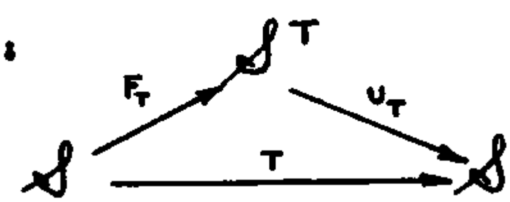
and that the correspondences are given as follows:

given I in $\mathcal{S}$, $(\ )^I$ is a coadjoint endofunctor of $\mathcal{S}$, which has a (unique) triple structure given by η and μ defined as follows for $x \in X$ and $i \in I$ and $f : I \longrightarrow X^I$:

$i(x\,\eta_X) = x$ and $(i)(f\,\mu_X) = (i)((i)f)$ and

$(\quad) \times I$ is an adjoint endofunctor of $\mathcal{S}$, with a unique cotriple structure induced by that of $(\quad)^I$ as follows, ψ , ∂ , are defined, for $x \in X$ and $i \in I$ by , $(x,i)\psi_X = x$ and $(x,i)\partial_X = (x,i,i)$. Conversely, any coadjoint endofunctor of $\mathcal{S}$ is representable by some I , i.e., is of the form $(\quad)^I = \mathrm{HOM}(I, \quad)$, and has a unique triple structure as given above , and an adjoint endofunctor of $\mathcal{S}$, of the form $(\quad) \times I$ has therefore an induced cotriple structure . The uniqueness of these structures imply that the correspondence established is an isomorphism.

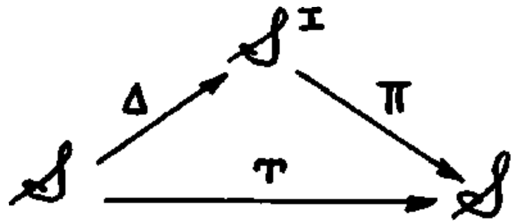
Therefore, $\mathcal{S}^*$ gives all coadjoint triples in $\mathcal{S}$, and we can now fix a set I and investigate the nature of the T-algebras, with T being the triple given by $(\quad)^I$. We recall that a $(\quad)^I$ - algebra is a pair (X , φ) where X is an object in $\mathcal{S}$, i.e., a set, and $X^I \xrightarrow{\varphi} X$ is a map in $\mathcal{S}$, i.e., an I-ary operation on the set X which, by the equations it has to satisfy, has a two-sided identity and it is associative. And there is a universal resolution given by the category of $(\quad)^I$- algebras and a pair of adjoint functors relating it with $\mathcal{S}$:



We now claim that there is a pair of adjoint functors relating the category $\mathcal{S}^I$, of all set-valued functors with domain the discrete category

I , with $\mathcal{S}$, whose composition is the endofunctor $(\ )^{I}$. Let Δ be a functor with domain $\mathcal{S}$ and codomain $\mathcal{S}^{I}$, defined, for X and $i \in I$ by $(i)(X\Delta) = X$; define $\Pi : \mathcal{S}^{I} \longrightarrow \mathcal{S}$ as usual, i.e., if F is an object in $\mathcal{S}^{I}$, let $F\Pi = \prod_{i \in I} (i)F$.

Then, it is easy to see that the following diagram is commutative :



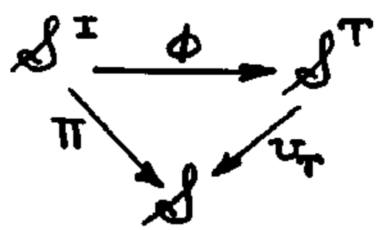
This is so because, given X in $\mathcal{X}$, $X(\Delta\Pi) = (X\Delta)\Pi =$
$= \prod_{i \in I} (i)(X\Delta) = \prod_{i \in I} X = X^{I} = X((\ )^{I})$.

Adjointness is clear since $(X\Delta, F)_{nat} \cong \prod_{i \in I} HOM_{\mathcal{S}}(X, iF) \cong$
$\cong HOM_{\mathcal{S}}(X, \prod_{i \in I} iF) \cong HOM_{\mathcal{S}}(X, F\Pi)$.

Since the resolution given by the category of $(\ )^{I}$– algebras, is the maximal universal one, there exists a unique

$$\Phi : \mathcal{S}^{I} \longrightarrow \mathcal{S}^{T}$$

such that the following diagram is commutative :



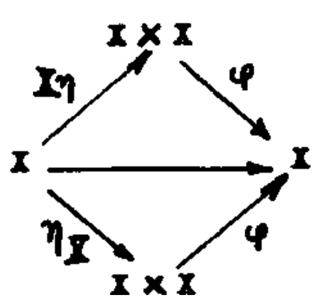
This says that $\mathcal{S}^{T}$ is the equational closure (since $\mathcal{S}^{T}$ is an equational category) of $\mathcal{S}^{I}$ over $\mathcal{S}$. And the closure is given by the functor Φ . The definition of Φ will tell us how to interpret functors with domain category, the discrete category I , and values in $\mathcal{S}$, as algebras

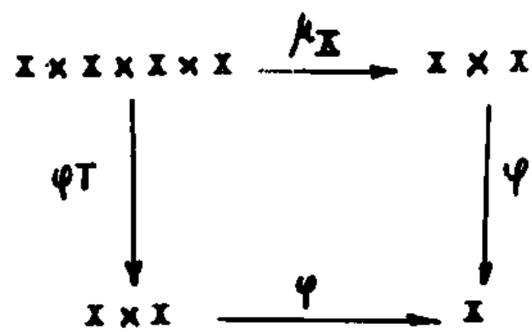
with an I-ary operation (plus all derived operations from this one).
We start by the simplest case where $I \cong 2 \cong |2|$, i.e.,
$\mathcal{S}^I = \mathcal{S} \times \mathcal{S}$, and examine closely how ϕ is defined in this case.
The algebras are pairs (X, o) with X a set and o a binary operation on X satisfying the equations :

$$x \circ x = x \qquad \text{for any } x \in X$$

$$(x_1 \circ x_2) \circ (x_3 \circ x_4) = x_1 \circ x_4 \qquad \text{for any}$$

four elements x_1, x_2, x_3 and x_4 in X .
This is so, since if we denote the operation o as before, by φ , the three equations to be satisfied are given by the requirement that the two diagrams below commute :





The first two equations read the same since $x\eta_X = (x,x)$. As for the third one, we notice that an element of $X^4 = X \times X \times X \times X$, can be viewed as a function $2 \longrightarrow X^2$ as well, so that then,
if $f : 2 \longrightarrow X^2$, $(i)(f\mu_X) = (i)((i)f)$ and the four coordinates x_1, x_2, x_3, x_4 stand respectively for $(0)((0)f)$, $(1)((0)f)$, $(0)((1)f)$ and $(1)((1)f)$. Now we have that :
$(0)(f\mu_X) = (0)((0)f) = x_1$ and $(1)(f\mu_X) = (1)((1)f) = x_4$
therefore, $(x_1, x_2, x_3, x_4)\mu_X = (x_1, x_4)$.
On the other hand, $(x_1, x_2, x_3, x_4) = ((x_1, x_2), (x_3, x_4)) \in X^2 \times X^2$,

and $(x_1, x_2, x_3, x_4)(\varphi \times \varphi) = ((x_1 \circ x_2), (x_3 \circ x_4))$, so that applying φ to both we finally have (by the commutativity of the square involved) that

$((x_1 \circ x_2) \circ (x_3 \circ x_4)) = ((x_1 \circ x_2), (x_3 \circ x_4))\varphi =$

$= (x_1, x_4)\varphi = x_1 \circ x_4$.

We now define ϕ as follows : for $(A,B) \in \mathcal{S} \times \mathcal{S}$, let

$(A,B)\phi = (A \times B, \bullet)$ where $\bullet$ is a binary operation defined as follows

$$(A \times B) \times (A \times B) \xrightarrow{\ \bullet\ } A \times B$$

such that

$(a_o, b_o) \bullet (a_1, b_1) = (a_o, b_1)$. To see that this defines an algebra, we verify :

$(a_o, b_o) \bullet (a_o, b_o) = (a_o, b_o)$ and

$((a_o, b_o) \bullet (a_1, b_1)) \bullet ((a_2, b_2) \bullet (a_3, b_3)) = (a_o, b_1) \bullet (a_2, b_3) =$

$= (a_o, b_3) = (a_o, b_o) \bullet (a_3, b_3)$.

Since $A \times B$ is the underlying set of the algebra, it is clear that fits well into the diagram that has to commute, by uniqueness ϕ is the required functor. Moreover, ϕ is full and has an adjoint in this case, as we will show .

To see that ϕ is full, let $(A \times B, \bullet) \xrightarrow{\ f\ } (A' \ B', \bullet')$ be a homomorphism of algebras as described above. Then, for any a_o, a_1 in A and any b_o, b_1 in B, the following holds :

$(a_o, b_o)f \bullet' (a_1, b_1)f = ((a_o, b_o) \bullet (a_1, b_1))f = (a_o, b_1)f$, i.e.,

$f = f p_{A'} \times f p_{B'}$ which means that it comes from a map of pairs $(A,B) \longrightarrow (A',B')$. Therefore, ϕ is full.

We now define an adjoint to ϕ . Given (X, o) there are sets A_X and B_X and a map $X \longrightarrow A_X \times B_X$ which is an epimorphism . To see this, consider the following two relations on X :

$x \underset{A}{\equiv} y$ iff $x \circ y = y$ and $x \underset{B}{\equiv} y$ iff $x \circ y = x$.

Both are equivalence relations. We show it is so for $\underset{A}{\equiv}$, for example :

Since $x \circ x = x$, $\underset{A}{\equiv}$ is reflexive .

Let $x \circ y = y$. Then, $y \circ x = (x \circ y) \circ x = (x \circ y) \circ (x \circ x) =$
$= x \circ x = x$, and so it is symmetric.

Assume $x \circ y = y$ and $y \circ z = z$ then, since by symmetry, we have also $y \circ x = x$, then

$x \circ z = x \circ (y \circ z) = (y \circ x) \circ (y \circ z) = y \circ z = z$

and $\underset{A}{\equiv}$ is transitive.

Therefore we can partition X into equivalence classes according to both equivalence relations , and there is a canonical $X \longrightarrow A_X \times B_X$ which is an epimorphism : given (x,y) A_X B_X we have that

$x \circ y \underset{A}{\equiv} x$ and $x \circ y \underset{B}{\equiv} y$ because

$(x \circ y) \circ x = (x \circ y) \circ (x \circ x) = x \circ x = x$ and

$(x \circ y) \circ y = (x \circ y) \circ (y \circ y) = x \circ y$.

So, let $x \circ y = z$. Then, $z_A = (x \circ y)_A = x$ and $z_B = (x \circ y)_B = y$.

If neither A nor B are empty, this z is unique , and the canonical map an isomorphism. That means that ϕ would be faithful if in $\mathcal{S}^2$ there were no functors with empty values other that 0 . This is no so, however. The only discrete I for which this would happen, would be $I \cong 1$, but this is the trivial case.

Let us take now any set I , then $\phi : \mathcal{S}^I \qquad \mathcal{S}^T$ is clearly defined as follows : if F is any object in $\mathcal{S}^I$, then $F\phi = (\prod_{i\in I} (i)F , \varphi)$ where φ is an I-ary operation defined by $(f\varphi)_k = ((k)f)_k$ for $f \in (\prod_{i\in I} (i)F)^I$. As before, it can be shown that ϕ is full and that it has an adjoint. However, it is not faithful since any functor with empty values is sent to the trivial algebra. However, for practical purposes, we can think of functors $I \longrightarrow \mathcal{S}$, as algebras with underlying set the product set of its values and an I-ary operation defined on this product set by $(f\varphi)_k = (f_k)_k$.

§ 18 - MONOIDS IN CATEGORIES WITH MULTIPLICATION AND GROUND OBJECT

Following Mac Lane [22], we say that $\mathcal{A}$ is a category with multiplication iff it is a category together with a covariant (in both variables) bifunctor $* : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$. For any two objects A , B of $\mathcal{A}$, we will write $A * B$ for the value of the bifunctor $*$ at the pair (A,B) . Also, if $A \xrightarrow{f} A'$ and $B \xrightarrow{g} B'$ are any two maps in $\mathcal{A}$, they induce what we denote by $f * g : A * B \longrightarrow A' * B'$. That $*$ is a bifunctor means that $1_A * 1_B = 1_{A * B}$ and that $(f'f) * (g'g) = (f' * f)(g' * g)$, whenever the compositions $f'f$ and $g'g$ are defined. It is also assumed that there are given natural isomorphisms $a = a(A,B,C) : A * (B * C) \cong (A * B) * C$ and $c = c(A,B) : A * B \cong B * A$ which express associativity and commutativity for the multiplication , respectively .
An object I of $\mathcal{A}$ is said to be a ground object for $*$ iff I

behaves as an identity for the multiplication $*$, that is, for any object A there are natural isomorphisms

$e = e(A) : I * A \cong A$ and $e' = e'(A) : A * I \cong A$.

Any category with finite roots and a terminal object is a category with multiplication and a ground object , namely the categorical product is the multiplication and the terminal object is the ground object.

However, we will be interested sometimes to have some other fixed object in the category as the ground object for some multiplication in some category which should approximate the original one as much as possible. To this end, we prove the following :

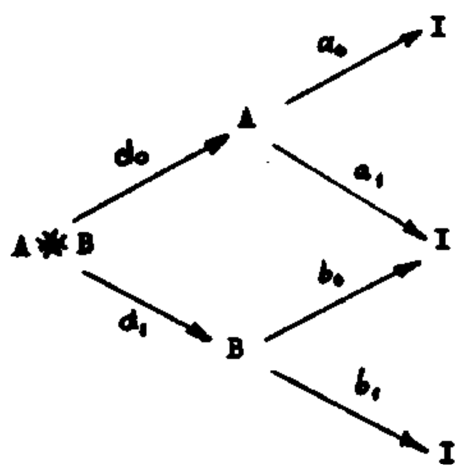
<u>Proposition 18.1</u> Let $\mathcal{X}$ be any category with finite roots and let I be any object in $\mathcal{X}$. Then, there exists a category $\overline{\mathcal{X}}$ with multiplication for which I is a ground object. There is also a functor $\phi : \mathcal{X} \longrightarrow \overline{\mathcal{X}}$, such that for any two objects A and B in $\mathcal{X}$, $(A \times B)\phi = A\phi * B\phi$, where $*$ is the multiplication in $\overline{\mathcal{X}}$. If I is an idempotent in $\mathcal{X}$, then $I\phi \cong I$. If I is the ground object for the multiplication in $\mathcal{X}$ (i.e., I is the terminal object) then ϕ is an equivalence of categories.

Proof:

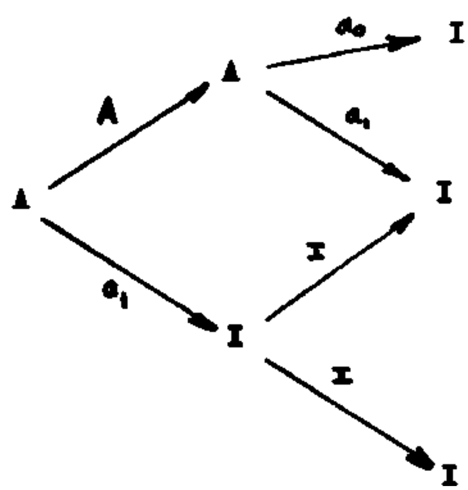
Let $\overline{\mathcal{X}} = (\mathcal{X}, I \times I)$, i.e., the category whose objects are maps in $\mathcal{X}$ of the form $A \longrightarrow I \times I$, where A is any object in $\mathcal{X}$. It has been named by Beck as the category of objects in $\mathcal{X}$ over $I \times I$. One can also think of the objects in $\overline{\mathcal{X}}$ as pairs of maps $A \rightrightarrows I$ in $\mathcal{X}$. As for the maps, they are, as usual, given by maps $A \longrightarrow A'$

in $\mathcal{X}$, such that they can be thought of as a map in $\bar{\mathcal{X}}$ from the map $A \longrightarrow I \times I$ to the map $A' \longrightarrow I \times I$ iff the following triangle is commutative :

We show first that $\bar{\mathcal{X}}$ has multiplication, as follows: given any two objects $A \underset{a_1}{\overset{a_0}{\rightrightarrows}} I$ and $B \underset{b_1}{\overset{b_0}{\rightrightarrows}} I$ in $\bar{\mathcal{X}}$, define $A \ast B \underset{c_1}{\overset{c_0}{\rightrightarrows}} I$ as the object and the two maps into I which are the exterior arrows in the following diagram, where the square is a pull-back :

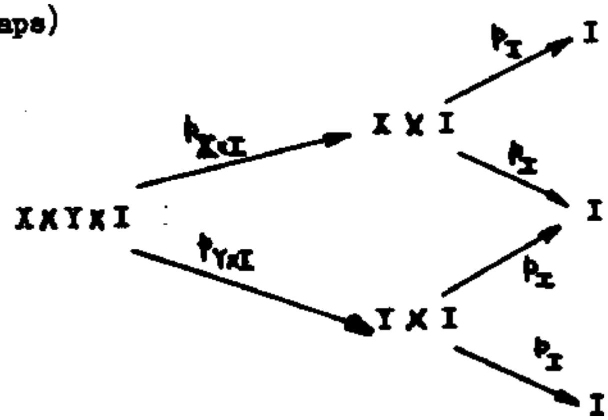


Then, $I \underset{I}{\overset{I}{\rightrightarrows}} I$ is a ground object for this multiplication (which is easily seen to be associative) since the pull-back of the relevant subdiagram in :



is given by the object A and the two dotted arrows , i.e., we have that

$A \ast I \rightrightarrows I$ is given by $A \overset{Aa_0}{\underset{a_1 I}{\rightrightarrows}} I = A \overset{a_0}{\underset{a_1}{\rightrightarrows}} I$, and so, $A \ast I \cong A$. Similarly one can show that $I \ast A \cong A$.
Define now $\phi : \mathcal{X} \longrightarrow (\mathcal{X}, I \times I)$ as follows: given any object X in $\mathcal{X}$, let $X\phi = X \times I \rightrightarrows I$ (i.e., the two maps are equal to the projection onto I). That ϕ preserves multiplication can be seen as follows: the following is a pull-back diagram (plus two other maps)



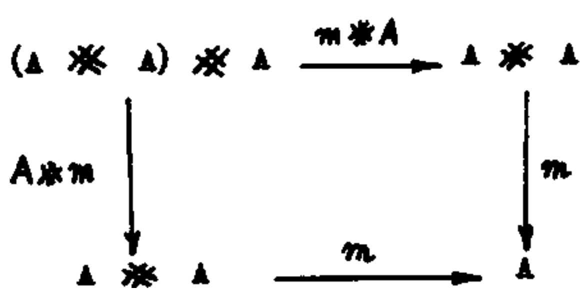
Therefore, $(X \times Y)\phi = X\phi \ast Y\phi$. Notice that this does not imply that I is a ground object in $\mathcal{X}$ for $\times$, since $I\phi$ is not I but $I \times I \longrightarrow I \times I$. Also, even in the case where I is idempotent in $\mathcal{X}$, so that $I\phi \cong I$, $(A \times I)\phi \cong A\phi$ does not imply that $A \times I \cong A$ since ϕ need not be faithful. Obviously, if I is a ground object for $\mathcal{X}$ together with $\times$, then $(\quad) \times I$ is an isomorphism. But in all cases it has a coadjoint, namely the one given by the rule $(X \longrightarrow I \times I) \rightsquigarrow X$. QED.

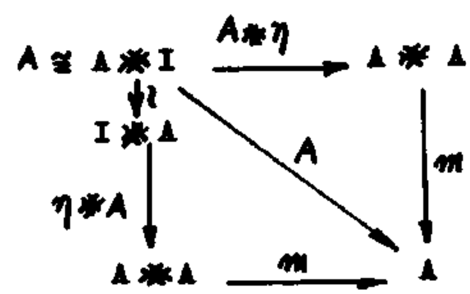
Eckmann and Hilton [3] gave the definition of a group in a category. It can also be found in Freyd [8] or Mitchell [23]. However, in all these, the assumption that the category has a zero object is rather important, besides the existence of finite roots. We define here, along those lines, the notion of monoid in a category. The conditions for a category to admit monoids in it, are the existence of a multiplication and of a ground object for it.

By a <u>monoid in the category $\mathcal{A}$</u>, where $\mathcal{A}$ is a category with multiplication $\ast$ and ground object I for $\ast$, we mean, an object A of $\mathcal{A}$, together with maps in $\mathcal{A}$:

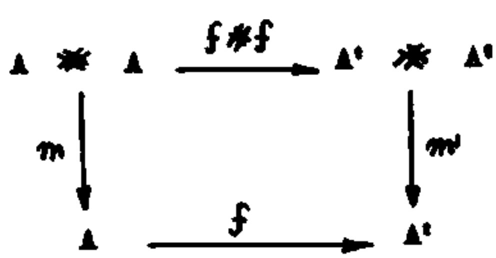
$$A \ast A \xrightarrow{\ m\ } A \qquad \text{and} \qquad I \xrightarrow{\ \eta\ } A$$

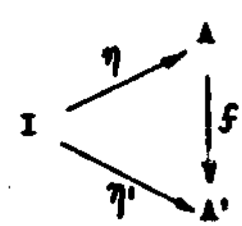
satisfying three equations expressed by means of the commutativity of the following diagrams :





If (A, m, η) and (A', m', η') are both monoids in $\mathcal{A}$, by a <u>monoid homomorphism</u> we mean the obvious thing, i.e., any map $A \xrightarrow{f} A'$ in $\mathcal{A}$ such that it preserves the multiplication and the unit, m and η, of the monoid A (not to be confused with the multiplication and ground object of the category $\mathcal{A}$), i.e., such that the following two diagrams are commutative :





We now give two elementary examples :

(1) $\mathcal{S}$ with $\times$ and 1 is a category with multiplication $\times$ and ground object 1 . A monoid in $\mathcal{S}$, by the above definition, is any set together with maps $M \times M \xrightarrow{m} M$ and $1 \xrightarrow{x} M$. That is, a set M with a binary multiplication m and a chosen element x of M, such that m is associative and x is a two-sided unit for m . This coincides with the usual notion of monoid . Therefore, monoids in $(\mathcal{S}, \times, 1)$ are jus ordinary monoids.

(2) $\mathcal{G}$ with $\otimes$ and Z is a category with multiplication $\otimes$ and ground object Z . A monoid in $(\mathcal{G}, \otimes, Z)$ is therefore, an abelian group R together with group homomorphisms $R \otimes R \xrightarrow{m} R$ and $Z \xrightarrow{u} R$, satisfying the usual equations. The multiplication in R makes it into a ring and the existence of u implies that the ring has an identity. Therefore, monoids in $(\mathcal{G}, \otimes, Z)$ are rings with identity. Monoid homomorphisms become ring homomorphisms.

Other examples will be provided by the relative categories, which we introduce in the next section.

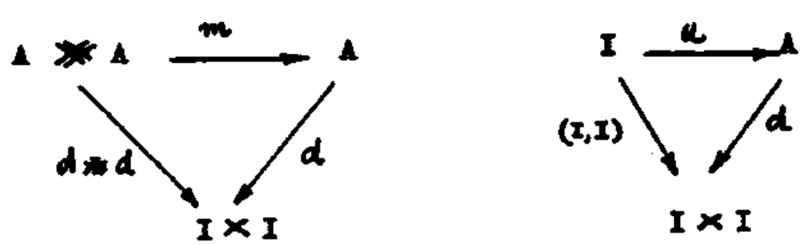
§ 19 – RELATIVE CATEGORIES

As there are monoids, groups, or any given structured objects in categories, there can be categories in categories, as well. For this, we need categories with finite roots, or, at least, with products. Then, we can define categories in a category with finite roots, where the objects in the relative category form not a set or a class necessarily, but will be collected into an object in the base category. That is, if $\mathcal{X}$ is any category with finite roots, and I any given object in $\mathcal{X}$, we say that any monoid in ($\mathcal{X}$, $I \times I$) is a <u>category in $\mathcal{X}$ with I objects</u>. We analyse the definition further. Since $\mathcal{X}$ has finite roots, and I is an object in $\mathcal{X}$, then by 18.1, we can define a multiplication $\circledast$ in ($\mathcal{X}$, $I \times I$) for which $I \overset{I}{\underset{I}{\rightrightarrows}} I$ becomes a ground object.

To justify the name "category" for a monoid in ($\mathcal{X}$, $I \times I$) , we interpret adequately the maps which are assumed to exist

$A \xrightarrow{d} I \times I$, just because it is an object in ($\mathcal{X}$, $I \times I$)

and $A \circledast A \xrightarrow{m} A$, $I \xrightarrow{u} A$, because it is furthermore a monoid , so that the following diagrams are commutative :



and also there are commutative diagrams expressing the associativity of m and the fact that u is a unit for m . The name "category" becomes clear if we take $\mathcal{X}$ to be $\mathcal{S}$, so that I is a set now .

We show that a category in $\mathcal{S}$ with I objects is an ordinary category which is small and such that its class of objects is isomorphic to the set I . The object A in $\mathcal{S}$, is interpreted as the set of maps in the small category. The pair of maps $A \overset{d_0}{\underset{d_1}{\rightrightarrows}} I$ are interpreted as the functions which assign the domain and the codomain to each map in the category. The set I is the set of objects in the category. Then, m will be interpreted as composition of maps, and u as the assignment of identity maps, for each object in the category.

Actually, to understand this better, it is useful to make an analogy with fibre bundles. Consider the category of objects in $\mathcal{S}$ over $I \times I$ as a category of fibre bundles. Then, A is the bundle space, $I \times I$ is the base space, d is the projection. Then, there are fibres over points of the base space, i.e., for each $(i,j) \in I \times I$, the fibre over (i,j) is $A_{ij} = d^{-1}((i,j))$, and therefore, A, which is the set of all maps, is the disjoint union of the collection $\{A_{ij}\}$ indexed by $I \times I$. Obviously, in this analogy, $A_{ij} = d^{-1}((i,j))$ is correctly interpreted as the set of all maps with domain i and codomain j: A_{ij} is the inverse image of (i,j) under d, where d can be replaced by the pair of maps (d_o, d_1). It is also correct to say that A, the set of maps, is the disjoint union of all possible Hom-sets A_{ij} ($= \mathrm{HOM}(i,j)$), because, for any map in the category there is an object i which is its domain and an object j which is its codomain. As for the multiplication $\circledast$ for $(\mathcal{S}, I \times I)$, we have calculated it in § 14, and we have that $A \circledast A$ is a bundle, whose fibre over (i,j) is given by $(A \circledast A)_{ij} = \sum_k A_{ik} \times A_{kj}$. Now, to see that m can be interpreted as composition of maps, we see that m is just

$$\sum_{i,j} \Big(\sum_k A_{ik} \times A_{kj} \Big) \xrightarrow{\ m\ } A_{ij}$$

so that m is defined only for maps such that the codomain of the first is the domain of the second. As for the map $u : I \longrightarrow A$, which assigns to each object in the category (i.e., to each element of I), a map (an element of A), has to satisfy conditions saying that the domain and codomain of the map have to be both the given object (since there is a condition expressed by the commutativity of a triangle saying so) and furthermore, since u acts as a two-sided unit with respect to composition of maps (i.e., with respect to m) then it is clear that $(i)u$ is the identity map of the object $i \in I$.

This interpretation of categories in $\mathcal{S}$ with I objects as small absolute categories with a set of objects isomorphic to I is, in fact, an isomorphism : to each relative category in $\mathcal{S}$ with I objects, we make correspond a small category $\mathbb{C}$, by letting $|\mathbb{C}| \cong I$, $|\mathbb{C}^2| \cong A$ so that d_0, d_1, m, u have the usual meanings of domain , codomain, composition and identities. Conversely, given any small category $\mathbb{C}$, we can define a category in $\mathcal{S}$ with $|\mathbb{C}|$ objects, where the usual maps domain, codomain, composition and assignment of identities can now be viewed as maps in $\mathcal{S}$.

This correspondence has no meaning outside of $\mathcal{S}$. That is, if $\mathcal{X}$ is any category and A is a category in $\mathcal{X}$ with X objects, where X is an object of $\mathcal{X}$, then A need not be a category, small or large.

§ 20 – RELATIVE FUNCTOR CATEGORIES

Let $\mathcal{X}$ be a category with finite roots , and I an object in $\mathcal{X}$. Let A be any category in $\mathcal{X}$ with I objects. By this we mean, after §19 , that A is an object in $(\mathcal{X}, I \times I)$, actually , it is a map $A \xrightarrow{d} I \times I$ that is an object in $(\mathcal{X}, I \times I)$ with A and d in

$\mathcal{X}$. Consider now the category $(\mathcal{X}, I)$. Then, if $A \to I \times I$ is an objewt in $(\mathcal{X}, I \times I)$, the functor $(\quad) * A$ is an endofunctor of $(\mathcal{X}, I)$ as well as of $(\mathcal{X}, I \times I)$, where it is obvious how the definition should be. Actually, since A has a monoid structure over $I \times I$, $(\quad) * A$ has a triple structure on $(\mathcal{X}, I)$. The algebras are given by pairs formed by an object $X \xrightarrow{\xi} I$ of $(\mathcal{X}, I)$ and a map in $\mathcal{X}$, $X * A \xrightarrow{\varphi} X$ over I, i.e., such that the following triangle commutes:

$$\begin{array}{ccc} X * A & \xrightarrow{\varphi} & X \\ & \searrow \quad \swarrow_{\xi} & \\ & I & \end{array}$$

satisfying the equations expressed by the commutativity of the diagrams:

$$X \cong (X * I \xrightarrow{X * \eta} X * A \xrightarrow{\varphi} X)$$

and

$$\begin{array}{ccc} X * A * A & \xrightarrow{X * \mu} & X * A \\ {\scriptstyle \varphi * A}\downarrow & & \downarrow{\scriptstyle \varphi} \\ X * A & \xrightarrow{\varphi} & X \end{array}$$

These algebras will be called <u>relative functors</u>, and the category whose objects are all the $[(\quad) * A]$- algebras, for A a category in $\mathcal{X}$ with I objects, will be called a <u>relative functor category</u> and denoted $(\mathcal{X}, I)^T = \mathcal{X}(A)$, instead of $\mathcal{X}^A$.

A relative functor need not be a functor at all, it is a functor <u>in</u> $\mathcal{X}$, with domain category A, a category <u>in</u> $\mathcal{X}$, and such that the rule for being a functor is encoded into two maps in $\mathcal{X}$, one giving the rule for the objects of the category $X \xrightarrow{g} X$, and another giving the rule of the functor for the maps of the category A , $X * A \xrightarrow{\varphi} X$. This expresses the usual idea that a functor has two "parts" , one is that of being a function defined on the objects, and the other on the maps of the category.

We recall now that any endofunctor of $\mathcal{S}$ which has a coadjoint is of the form $(\ \ ) \times A$ for some set A . It has a unique cotriple structure as we have shown in § 17 , but we remark that it need not have a triple structure at all. Actually, if $(\ \ ) \times A$ had a triple structure, this would mean that there are natural transformations

$$1_{\mathcal{S}} \xrightarrow{\eta} (\ \ ) \times A \xleftarrow{\mu} (\ \ ) \times A \times A$$

i.e., for each set X , there would be maps in $\mathcal{S}$, $X \xrightarrow{\eta X} X \times A$ and $X \times A \times A \xrightarrow{\mu X} X \times A$, satisfying the usual equations. But since the maps above are always induced by maps $1 \xrightarrow{u} A$ and $A \times A \xrightarrow{m} A$, satisfying the equations for A to be a monoid, we have that $(\ \ ) \times A$ is a triple on $\mathcal{S}$ iff A is a monoid. (The converse to the above is trivially true) . Therefore, we have that

$$\text{Adjoint Triples}(\mathcal{S}) \cong \text{Monoids}$$

In this case, the universal resolution is given by a category whose objects are pairs (X , ξ) where X is a set and $\xi : X \times A \longrightarrow X$ is the rule by which the monoid A operates on the set X .

We remark that, since 1 is the ground object for the categorical product in $\mathcal{S}$, the relative categories in $\mathcal{S}$ with 1 object are, by definition, the monoids in the category $(\mathcal{S}, 1\times 1) \cong \mathcal{S}$, i.e., the categories in $\mathcal{S}$ with 1 object are the monoids, but the usual categorical motion of monoid is precisely, that it is any category with exactly one object and endomorphisms of that object.

Using the same arguments, we have the conclusion that all adjoint triples on the category of abelian groups are given precisely by functors of the form "tensoring with a ring with unit". As for the algebras, they are abelian groups on which the ring R acts (if the triple considered is $(\ \ ) \otimes R$), therefore, they are all R-modules . Finally, since Z is the ground object for $\otimes$ in $\mathcal{G}$, we have that Rings $\cong$ Adjoint Triples ($\mathcal{G}$) ; however, in this case they are not relative categories since $\otimes$ is not the product but the coproduct in   .

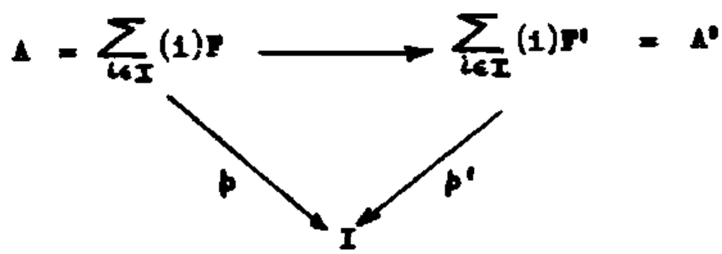
From § 14 , we know that $\mathrm{Adj}(\mathcal{S}^{I}) \cong \mathcal{S}^{I\times I}$. We now show that for any set I , viewed as a discrete category ,

$$(\mathcal{S}, I) \cong \mathcal{S}^{I}$$

This is so because: if $A \xrightarrow{p} I$ is any object in $(\mathcal{S}, I)$, let $A^{*} : I \longrightarrow \mathcal{S}$, a functor, be defined as follows :

$(i)A^{*} = A_i = (i)p^{-1}$. And for $A \xrightarrow{f} A'$ a map in $(\mathcal{S}, I)$, (i.e., such that $p = fp'$) , define the corresponding natural transformation $A^{*} \xrightarrow{\eta} A'^{*}$ by : $((i)A^{*})\eta_i = (i)p^{-1}f$. And since $(i)p^{-1}f\,p' = (i)\,p^{-1}p = i$, then $(i)p^{-1}f \in (i)p'^{-1} = (i)A'^{*}$. Conversely, given any functor $F : I \longrightarrow \mathcal{S}$, let $A = \sum_{i\in I}(i)F$ and let $A \xrightarrow{p} I$ simply be such that for each x , $xp = i$ iff

$x \in (i)F$. And given $\eta : F \longrightarrow F'$, natural, for each i we have $\eta_i : (i)F \longrightarrow (i)F'$ which induces



which is commutative, since for $x \in A$, say $x \in (i)F$ for some $i \in I$, then $xp = i$ by definition of p and $x f p' = (xf)p' = i$ since $xf \in (i)F'$. It is now easy to see that compositions of the two functors defined give the corresponding identities.

With this result, we can finally prove that the adjoint triples on $\mathcal{S}^I$ are given by the small categories with a set of objects isomorphic to the set I : we have that $\mathrm{Adj}(\mathcal{S}, I) \cong \mathrm{Adj}(\mathcal{S}^I) \cong \mathcal{S}^{I\times I} \cong (\mathcal{S}, I \times I)$ so that

$\mathrm{Adj\ Triples}(\mathcal{S}^I) \cong \mathrm{Adj\ Triples}(\mathcal{S}, I) \cong \mathrm{Monoids}(\mathcal{S}, I\times I)$

$\cong \mathrm{Cat}_{\mathcal{S}}(I)$.

Let $\mathcal{C}_I$ denote the category of all small categories with a set of objects isomorphic to the set I . Since $\mathrm{Cat}_{\mathcal{S}}(I) \cong \mathcal{C}_I$, we have that

$$\mathrm{Adj\ Triples}(\mathcal{S}^I) \cong \mathcal{C}_I$$

And for each $\mathbb{C}$ such that $|\mathbb{C}| \cong I$, the corresponding adjoint triple on $\mathcal{S}^I$ has a resolution given by the diagrammatic category $\mathcal{S}^{\mathbb{C}}$, which, though not the maximal one, can be approximated to the category of algebras corresponding to the triple, which is precisely the functor category (relative) which we have denoted $\mathcal{S}(\mathbb{C})$.