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From String structures to Spin structures on loop spaces

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Chapter 1

Introduction: the purpose of this work

We want to give a new construction of the transgression map of String structures on a n-dimensional smooth manifold X ($n \geq 3$) to Spin structures on its loop space. This result has been known for a while in the Physics literature, and a completely rigorous proof has been recently given by Konrad Waldorf in the series of papers [23, 25, 26, 27, 28, 30]. The proof we are proposing here will make a crucial use of the language of smooth higher stacks [13, 21].

We begin by recalling the context in which the question is set and by specifying what our starting and ending points respectively are. Consider the tangent bundle TX and its classifying map $f_{TX} : X \to BO(n)$. The structure group $O(n)$ is not connected, since $\pi_0(O(n)) = \mathbb{Z}_2$. The connected component of the identity in $O(n)$ is the group $SO(n)$, and to the inclusion of groups $SO(n) \to O(n)$ corresponds a morphism of classifying spaces $BSO(n) \to BO(n)$. A reduction of the structure group of TX to $SO(n)$ can be equivalently seen as a lift of f_{TX} from $X \to BO(n)$ to $X \to BSO(n)$ and this precisely corresponds with the choice of an orientation on X. In passing from $O(n)$ to $SO(n)$ we have "killed" the 0-th homotopy group, since now we are dealing with the topological group $SO(n)$ which is connected. However, it is not simply connected, and we may wish to repeat the same trick and repalce $SO(n)$ with $Spin(n)$, which is the universal covering of $SO(n)$ in order to have a structure group which is simply connected. The second step in this construction is there-

fore a lift of $f_{_{TX}}$ from $X \to BSO(n)$ to $X \to BSpin(n)$. Such a lift, if it exists, endowes X with the structure of a Spin-manifold.
It is a well known fact that X is orientable if and only if the first Stiefel-Whitney class $[w_1(TX)] \in H^1(X, \mathbb{Z}_2)$ is zero (see for example [12], Theorem 1.2). Moreover, X is Spin if and only if the second Stiefel-Whitney class $[w_2(TX)] \in H^2(X, \mathbb{Z}_2)$ is zero (see for example [12], Theorem 2.1). Observe that, by naturality of characteristic classes, $[w_1(TX)] = [f_{_{TX}}^*(w_1(EO(n)))]$, where $EO(n)$ denotes the universal bundle with structure group $O(n)$ and $w_1(EO(n))$ is the non-zero element in $H^1(BO(n), \mathbb{Z}_2) \simeq \mathbb{Z}_2$. Denote with the same symbol w_1 the homotopy class corresponding to $[w_1(EO(n))]$ in the bijection $H^1(Y, \mathbb{Z}_2) \to [Y, K(\mathbb{Z}_2, 1)]$, where $K(G, n)$ is the n-th Eilenberg MacLane space with group G (which has to be an abelian group if $n > 1$). We clearly have that $[w_1(TX)] = 0$ if and only if the composition

$$X \stackrel{f_{TX}}{\to} BO(n) \stackrel{w_1}{\to} K(\mathbb{Z}_2, 1)$$

is homotopic to the trivial (i.e., constant) map from X to $K(\mathbb{Z}_2, 1)$. In a similar way, $[w_2(TX)] = 0$ if and only if the composition

$$X \stackrel{f_{TX}}{\to} BSO(n) \stackrel{w_2}{\to} K(\mathbb{Z}_2, 2)$$

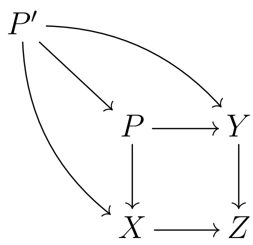
is homotopically trivial.
Before introducing the next lift of $f_{_{TX}}$, it is worth thinking for a while over what said until now. Recall that the *homotopy pullback* of $X \to Z \leftarrow Y$ in a category with a notion of homotopy consists of a square

$$\begin{array}{ccc} P & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

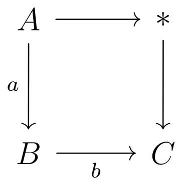
that commutes up to a given homotopy and such that, for any other square

$$\begin{array}{ccc} P' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

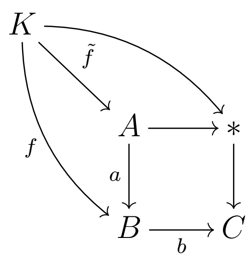
that commutes up to a given homotopy, there exists a morphism $P' \to P$, unique up to homotopy, and homotopies (unique up to higher homotopies) such that the diagram



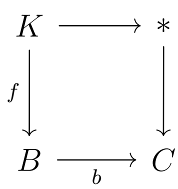
commutes (up to the given homotopies). In particular, if we consider a *fibration sequence* $A \xrightarrow{a} B \xrightarrow{b} C$, that is a homotopy pullback of the form[1]



and a morphism $f\colon K \to B$, we see that a lift $\tilde{f}\colon K \to A$ of f is equivalent to the datum of the homotopy commutative diagram



(since the upper part automatically commutes, due to the fact that $*$ is the terminal object), and so by the universal property of homotopy pullbacks, it is equivalent to the datum of a homotopy commutative diagram of the form



[1] One says that A is the *homotpy fiber* of $b\colon B \to C$.

In other words, we see that the lift $\tilde{f}$ exists if and only if the composition

$$K \xrightarrow{f} B \xrightarrow{b} C$$

is homotopic to zero.

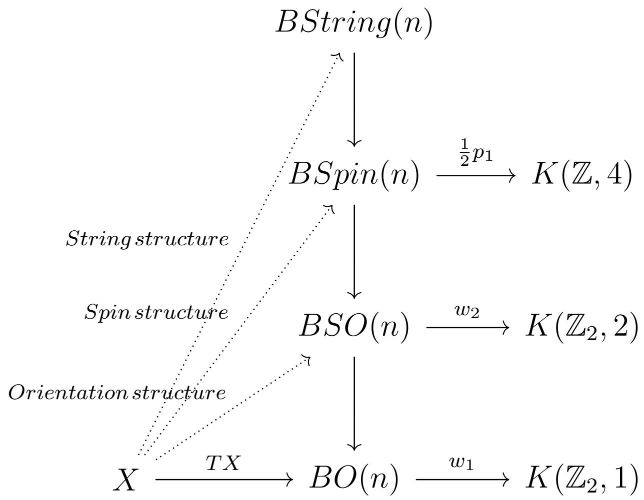
This is precisely the situation we have with the caharacteristic maps w_1 and w_2. Namely, the universal characteristic class $w_1 : BO(n) \to K(\mathbb{Z}_2, 1)$ is the second segment of the fibration sequence $BSO(n) \to BO(n) \to K(\mathbb{Z}_2, 1)$: since $BSO(n)$ is the homotopy fiber of w_1, the triviality of $X \to BO(n) \xrightarrow{w_1} K(\mathbb{Z}_2, 1)$ implies the existence of a lift of f_{TX} to $X \to BSO(n)$. In a similar way, since $BSpin(n)$ is the homotopy fiber of w_2, the triviality of $X \to BSO(n) \xrightarrow{w_2} K(\mathbb{Z}_2, 2)$ implies the existence of a lift of f_{TX} to $X \to BSpin(n)$.

Let's now move to the next lift, leading to the definition of String structure on X. The first nontrivial homotopy group of $Spin(n)$ is the third, $\pi_3(Spin(n)) = \mathbb{Z}$. Thanks of the isomorphisms $\pi_{n-1}(G) = \pi_n(BG)$, for any $n \geq 1$ (see for example [17], Corollary 11.2), we have $\pi_4(BSpin(n)) = \mathbb{Z}$. Therefore, by the Hurewicz theorem, we get $H_4(BSpin(n)) = \mathbb{Z}$, from which, via the universal coefficient theorem, we get $H^4(BSpin(n), \mathbb{Z}) = \mathbb{Z}$. The map $BSpin(n) \to K(\mathbb{Z}, 4)$ representing the generator of $H^4(BSpin(n), \mathbb{Z}) = \mathbb{Z}$ is called *first fractional Pontryagin class* and is denoted with the symbol $\frac{1}{2}p_1$.

Agreeing with previous cases, we now say, *by definition*, that X is endowed with a String structure if the map $X \to BSpin(n) \xrightarrow{\frac{1}{2}p_1} K(\mathbb{Z}, 4)$ is homotopically trivial: in this case, the map f_{TX} can be lifted to $X \to BString(n)$, where $BString(n)$ is defined as the homotopy fiber of $\frac{1}{2}p_1$. By taking the based loop space of $BString(n)$ one obtains the *topological* String group:

$$String(n) = \Omega BString(n).$$

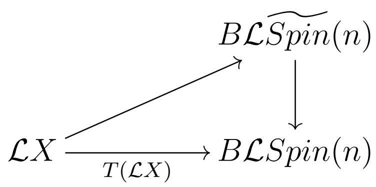
A useful visualization of what described until now is showed by the following diagram



We now make loop spaces enter the picture. Let X be a Spin manifold, and denote by $\mathcal{L}X$ the (free) loop space of X and by $\mathcal{L}Spin(n)$ the loop group of the Spin group. Since the Spin group $Spin(n)$ is connected, the loop space of the Spin manifold X is naturally an $\mathcal{L}Spin(n)$-manifold (an infinite dimensional one), i.e., the tangent bundle $T(\mathcal{L}X)$ is naturally an $\mathcal{L}Spin(n)$-bundle over $\mathcal{L}X$ (see [29], Lemma 5.1 and [22] Proposition 1.9). Finally, the loop group $\mathcal{L}Spin(n)$ has a universal central extension

$$1 \to U(1) \to \mathcal{L}\widetilde{Spin}(n) \to \mathcal{L}Spin(n) \to 1, \qquad (1.0.1)$$

see [20], and one defines a *Spin structure on* $\mathcal{L}X$ as a lift of the structure group of the tangent bundle $T(\mathcal{L}X)$ of $\mathcal{L}X$ from $\mathcal{L}Spin(n)$ to $\mathcal{L}\widetilde{Spin}(n)$, i.e., as a lift of the tangent bundle morphism $T(\mathcal{L}X)\colon \mathcal{L}X \to B\mathcal{L}Spin(n)$ to a morphism $\mathcal{L}X \to BL\widetilde{Spin}(n)$:



Notice how a Spin structure on a loop space is not, strictly speaking a Spin structure, i.e., it is not a morphism to $BSpin(\dim \mathcal{L}X)$. On the other hand, since $\mathcal{L}X$ is infinite dimensional, such a notion would be meaningless.

It is well known in the theoretical physics folklore that *a String structure on X is essentially the same thing as a Spin structure on $\mathcal{L}X$*. In the series of articles [23, 25, 26, 27, 28, 30], Konrad Waldorf has given a rigorous proof of this statement, proving that there is a natural transgression map

$$\{\text{String structures on } X\} \to \{\text{Spin structures on } \mathcal{L}X\},$$

which induces a bijection at the level of isomorphism classes

$$\{\text{String structures on } X\}/\sim \quad \xrightarrow{\sim} \quad \{\text{Spin structures on } \mathcal{L}X\}/\sim,$$

as soon as X is compact and simply connected.
Here we show how the transgression map considered by Waldorf can be very easily obtained from general constructions in the (infinity-) category of smooth stacks [21]. In particular, we get this way a natural morphism

$$Maps(X, BString(n))/\sim \quad \longrightarrow \quad Maps(\mathcal{L}X, B\widetilde{\mathcal{L}Spin}(n))/\sim$$

of classifying spaces, from which the above transgression map immediately follows. This abstract derivation of the transgression map can be seen as an improvement of Waldorf's result. However, we have to stress that we have not been able so far, with the methods presented here, to prove that the transgression map induces a bijection when X is compact and simply connected. The crucial point in the proof we are going to present is the existence of a natural morphism of smooth stacks

$$\mathbf{B}Spin \to \mathbf{B}^2(\mathbf{B}U(1))_{\text{conn}}$$

refining the first fractional Pontryagin class. This morphism of smooth stacks appears in [30] in the form of a *multiplicative bundle gerbe with connection* over the Spin group, and plays an essential role in Waldorf's proof, too. In particular, it implies that $\widetilde{\mathcal{L}Spin}$ is what Waldorf calls a *fusion extension* of $\mathcal{L}Spin$ [30, Theorem 3.5]. Here we stress how, once this canonical multiplicative gerbe with connection is looked at as a morphism of smooth stacks, everything else immediately follows by very general reasoning. For instance, the fact that $\widetilde{\mathcal{L}Spin}$ is a fusion extension of $\mathcal{L}Spin$ will be encoded into the natural fiber sequence of smooth stacks

$$\begin{array}{ccc} \mathbf{B}\widetilde{\mathcal{L}Spin} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}\mathcal{L}Spin & \longrightarrow & \mathbf{B}^2U(1) \end{array}$$

induced by the morphism $\mathbf{B}Spin \to \mathbf{B}^2(\mathbf{B}U(1))_{\mathrm{conn}}$ and by the holonomy morphism. Also, we provide a direct proof of the existence of the morphism $\mathbf{B}Spin \to \mathbf{B}^2(\mathbf{B}U(1)_{\mathrm{conn}})$ which does not rely on Waldorf's result and is instead based on the differential refinement of the first fractional Pontryagin class

$$\frac{1}{2}\hat{\mathbf{p}}_1 \colon \mathbf{B}Spin_{\mathrm{conn}} \to \mathbf{B}^3U(1)_{\mathrm{conn}}$$

constructed in [8].

Chapter 2

Bundle gerbes and their local description

The setting of bundle gerbes and maps between them, together with relative smooth refinements, is the most suitable for large part of our work. So we decided to dedicate this and the next chapter to a short presentation of bundle gerbes. In particular, here we talk about bundle gerbes as topological objects, while in the next chapter we introduce connections over them. In both the sections we firstly give the presentation of the subject in terms of total spaces, then we provide the local data descriptions for bundle gerbes and their connections. As we will see, it will be this local data description to indicate us the relation between $U(1)$-bundle gerbes (with and without connections) with truncated Deligne complexes. Main references for this section are [14, 15, 16] and [2].

2.1 Basic definitions

Let's start by recalling the notion of tensor product between two principal $U(1)$-bundles, since it has an important place in the definition of bundle gerbe. Let R and S be two principal $U(1)$-spaces, that are spaces on which $U(1)$ acts smoothly, freely and transitively and let $R \times S$ be the $U(1) \times U(1)$-space induced by the natural $U(1) \times U(1)$-action on the Cartesian product $R \times S$. Since $U(1)$ is an abelian group, we can consider the quotient group of $U(1) \times U(1)$ by the subgroup $\{(z, z^{-1}) | z \in U(1)\}$. If we quotient $R \times S$ by the equivalence relation $(r, s) \sim (rz, z^{-1}s)$ then we clearly have an induced $(U(1) \times U(1))/U(1)$-action on

the quotient $(R \times S)/\sim$. Since $U(1) \times U(1))/U(1)$ is actually isomorphic to $U(1)$, this makes $(R \times S)/\sim$ a space with a $U(1)$-action, and one can check that this action actually makes $(R \times S)/\sim$ a principal $U(1)$-space, which we will denote by $R \otimes S$. Repeating this construction fibrewise on the fibres of two principal $U(1)$ bundles P and Q over a manifold M, one gets the tensor product of principal $U(1)$-bundles $P \otimes Q$. This is again a principal $U(1)$-bundle over M.
Next, for a locally split map between differentiable manifolds $\pi : Y \to X$, i.e., a map which admits local sections, we write $Y^{[2]} = Y \times_\pi Y$ for the fibre product of Y with itself over π, that is the subset of pairs $(y_1, y_2) \in Y \times Y$ such that $\pi(y_1) = \pi(y_2)$.

Definition 2.1.1. *A bundle gerbe (P, π) over X is the datum of a locally split map $\pi : Y \to X$ and of a principal $U(1)$-bundle $P \to Y^{[2]}$, together with a multiplicative structure λ over P, that is a collection of $U(1)$-equivariant maps $P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} \stackrel{\lambda_{123}}{\longrightarrow} P_{(y_1,y_3)}$ for every (y_1, y_2) and (y_2, y_3) in $Y^{[2]}$. The product λ is required to be associative.*

To clarify: if we denote the k-fold fibre product of Y over π by $Y^{[k]}$ and we denote the projection to the indexed factors by $Y^{[p]} \stackrel{\pi_{i_1 \cdots i_k}}{\longrightarrow} Y^{[k]}$, the product λ gives a bundle isomorphism $\pi_{12}^* P \otimes \pi_{23}^* P \stackrel{\pi_{123}^*\lambda}{\longrightarrow} \pi_{13}^* P$ over $Y^{[3]}$. With these notations the associativity of the product can be expressed by the commutativity of the diagram

$$\begin{array}{ccc} \pi_{12}^*P \otimes \pi_{23}^*P \otimes \pi_{34}^*P & \xrightarrow{id \otimes \pi_{234}^*\lambda} & \pi_{12}^*P \otimes \pi_{24}^*P \\ {\scriptstyle \pi_{123}^*\lambda} \downarrow {\scriptstyle \otimes id} & & \downarrow {\scriptstyle \pi_{124}^*\lambda} \\ \pi_{13}^*P \otimes \pi_{34}^*P & \xrightarrow[\pi_{134}^*\lambda]{} & \pi_{14}^*P \end{array} \tag{2.1.1}$$

of morphisms of principal $U(1)$-bundles over $P^{[4]}$. There are variants of the definition of bundle gerbes in literature and in some cases, like in [14], the projection $Y \to X$ is required to be a locally trivial fibration. Our choice in favour of locally split maps is motivated by the aim to obtain a local description of bundle gerbes. As detailed below, this local description is realized by focusing the attention on the bundle gerbes constructed from nerves of good covers: the projection maps of these bundle gerbes obviously admit local sections, but they rarely are fibrations.

An isomorphism between two bundle gerbes (P, Y) and (Q, Z) over X is an isomorphism $Y \xrightarrow{f} Z$ of smooth manifolds over X, i.e., such that $\pi_Y = \pi_Z \circ f$ and a bundle isomorphism $P \xrightarrow{g} Q$ covering the induced map $Y^{[2]} \xrightarrow{f^{[2]}} Z^{[2]}$ and commuting with the bundle gerbe products on P and Q respectively.

We also have a notion of trivial bundle gerbe. To construct it, we remember that, given a principal $U(1)$-space R, its dual space R^* is the same space as R but with the inverse $U(1)$-action. Given a $U(1)$-principal bundle Q over Y and a locally split map $\pi : Y \to X$, the trivial way to construct a bundle gerbe $\delta(Q)$ over X is by declaring its fibre over (y_1, y_2) to be $\delta(Q)_{(y_1,y_2)} := Q^*_{y_1} \otimes Q_{y_2}$. This is naturally identified with the set of the $U(1)$-equivariant morphisms between Q_{y_1} and Q_{y_2}, and so it comes with a natural multiplicative structure given by the composition of morphisms. A bundle gerbe which is isomorphic to a bundle gerbe of the form $\delta(Q)$ is called *trivial* and the choice of a $U(1)$-bundle Q together with an isomorphism $\delta(Q) \simeq P$ for a bundle gerbe P is called a *trivialiation* of P.

Like in the case of $U(1)$-bundles, there are the notions of pullback bundle gerbe and product of bundle gerbes. If (Q, π) is a bundle gerbe over N and $f : M \to N$ is a map, we can pullback the locally split map $\pi\colon Y \to N$ to obtain a locally split map $f^*\pi\colon f^*Y \to M$, together with a map $\tilde{f}\colon f^*Y \to Y$ covering f. This induces a morphism $(f^*(Y))^{[2]} \to Y^{[2]}$ and we can use this to pullback the $U(1)$-bundle Q to a $U(1)$-bundle $\tilde{f}^*Q$ over $(f^*Y)^{[2]}$. It is easy to check that $(\tilde{f}^*Q, f^*\pi)$ is a bundle gerbe over M, the *pullback* by f of the bundle gerbe (Q, π). We will simply denote this bundle gerbe by f^*Q.

If (P, π_Y) and (Q, π_Z) are two bundle gerbes over X we can consider the fibre product $Z \times_X Y \to X$ and then form a $U(1)$-principal bundle $Q \otimes P$ over $(Z \times_X Y)^{[2]}$, the *product* of the bundle gerbes (Q, π_Y) and (P, π_X). This it is easily proven to be a new bundle gerbe over X. Moreover it satisfies the universal property of the product, so it really is the product of (Q, π_Y) and (P, π_X) in the category of bundle gerbes.

2.2 Diximier-Duady classes

Let's now introduce the Diximier-Duady class of a bundle gerbe and see that it has a natural behaviour respect of trivializations and basic operations on bundle gerbes. Let (P, π) be a bundle gerbe over X and choose an open cover U_α of X such that over each U_α there is a section s_α of $\pi\big|_{\pi^{-1}(U_\alpha)}\colon \pi^{-1}(U_\alpha) \to U_\alpha$. Such a cover exists by definition of locally split map. There are clearly induced maps $U_{\alpha\beta} \stackrel{(s_\alpha,s_\beta)}{\longrightarrow} Y^{[2]}$ over the twofold intersections, where $(s_\alpha, s_\beta)(x) := (s_\alpha(x), s_\beta(x))$. We can use these maps to pullback the $U(1)$-bundle $P \to Y^{[2]}$ and obtain a $U(1)$-bundle $P_{\alpha\beta}$ over each $U_{\alpha\beta}$. We may assume that the open cover $\{U_\alpha\}$ is good, so that all the U_α's and their intersections are in particular contractible. Then the $P_{\alpha\beta}$ are trivializable principal $U(1)$-bundles and we can choose sections $\sigma_{\alpha\beta}$ of each $P_{\alpha\beta}$. The bundle gerbe product λ gives a distinguished isomorphism $P_{\alpha\beta} \otimes P_{\beta\gamma} \simeq P_{\alpha\gamma}$. Under this identification, both $\sigma_{\alpha\beta}\sigma_{\beta\gamma}$ and $\sigma_{\alpha\gamma}$ are sections of $P_{\alpha\gamma}$ and so we have the relation

$$\sigma_{\alpha\beta}\sigma_{\beta\gamma} = \sigma_{\alpha\gamma} g_{\alpha\beta\gamma} \tag{2.2.2}$$

over threefold intersections, for some $U(1)$-valued function $g_{\alpha\beta\gamma}\colon U_{\alpha\beta\gamma} \to U(1)$. It is easy to check that $g_{\beta\gamma\delta} g_{\alpha\beta\delta} g^{-1}_{\alpha\gamma\delta} g^{-1}_{\alpha\beta\gamma} = 1$ over the fourfold intersections, so that $\{g_{\alpha\beta\gamma}\}$ is a Čech 2-cocycle on X with coefficients in the sheaf $\underline{U(1)}$ of smooth $U(1)$-valued functions. It is also immediate to check that a different choice of local sections $\sigma_{\alpha\beta}$ leads to a new 2-cocycle differing by the previous one by a 2-coboundary, so the class of $\{g_{\alpha\beta\gamma}\}$ in $H^2(X, \underline{U(1)})$ only depends on the bundle gerbe (P, π).
Consider the exact sequence of shaves

$$0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{R}} \stackrel{e^{2\pi i -}}{\longrightarrow} \underline{U(1)} \longrightarrow 1$$

and its long exact cohomology sequence. Since the sheaf $\underline{\mathbb{R}}$ of smooth $\mathbb{R}$-valued functions is a fine sheaf, its higher Čech cohomology groups vanish and so there is an isomorphism

$$H^2(X, \underline{U(1)}) \simeq H^3(X, \mathbb{Z}) \tag{2.2.3}$$

The image of the cohomology class $[\{g_{\alpha\beta\gamma}\}]$ corresponding to the bundle gerbe (P, Y) under the isomorphism $H^2(X, \underline{U(1)}) \simeq H^3(X, \mathbb{Z})$ is called the *Diximier-Douady class* of (P, π).

It will be denoted by the symbol $DD(P,\pi)$ or, when no confusion is possible, simply by $DD(P)$.

Proposition 2.2.1. *Let (P,Y) be a bundle gerbe. Then $DD(P)=0$ if and only if P is trivial.*

Proof. Let P be a trivial bundle gerbe. Then, $P=\pi_1^{-1}Q^*\otimes\pi_2^{-1}Q$, where π_1 and π_2 are the projections $Y^{[2]}\to Y$ over the indexed factors and Q is some $U(1)$-principal bundle over Y. Let $\{U_\alpha\}$ be a splitting cover for $\pi\colon Y\to X$ and let $U_\alpha\overset{s_\alpha}{\to}Y$ be a section over U_α. Define $Q_\alpha:=s_\alpha^*Q$. We clearly have canonical isomorphisms $P_{\alpha\beta}=Q_\alpha^*\otimes Q_\beta$ commuting with products. As before we can assume the cover $\{U_\alpha\}$ to be good. If we choose δ_α to be a section of Q_α and define $\sigma_{\alpha\beta}:=\delta_\alpha^{-1}\otimes\delta_\beta$, we obtain $\sigma_{\alpha\beta}\sigma_{\beta\gamma}=\sigma_{\alpha\gamma}$, and so $g_{\alpha\beta\gamma}=1$.
Vice versa, assume that the cocycle g defining DD(P) defines a trivial cohomology class, that is $g_{\alpha\beta\gamma}=\rho_{\alpha\beta}\rho_{\beta\gamma}\rho_{\gamma\alpha}$, for some set of $U(1)$-valued functions $\rho_{\alpha\beta}\colon U_{\alpha\beta}\to U(1)$. We can multiply the local sections $\sigma_{\alpha\beta}$ in equation (2.2.2) by the inverse of $\rho_{\alpha\beta}$ and so, without loss of generality, we can assume that $g_{\alpha\beta\gamma}$ is identically 1. Let $Y_\alpha=\pi^{-1}(U_\alpha)$ and let $U_\alpha\overset{s_\alpha}{\to}Y_\alpha$ be a local section. Define a principal bundle Q_α over Y_α, by setting $(Q_\alpha)_y:=P_{(y,s_\alpha(\pi(y)))}$. The $\sigma_{\alpha\beta}(\pi(y))$ are then elements of

$$P_{(s_\alpha(\pi(y)),s_\beta(\pi(y)))}\simeq P_{(s_\alpha(\pi(y)),y)}\otimes P_{(y,s_\beta(\pi(y)))}\simeq(Q_\alpha)_y^*\otimes(Q_\beta)_y, \tag{2.2.4}$$

where the first isomorphism derives from bundle gerbe product. We have therefore that $\sigma_{\alpha\beta}(\pi(y))\in(Q_\alpha)_y^*\otimes(Q_\beta)_y$ for every $y\in Y$. Namely, the $\sigma_{\alpha\beta}$ define automorphisms between Q_α and Q_β over $Y_{\alpha\beta}$. By standard partition of unity arguments we can then define a bundle Q over Y that trivializes the gerbe P over Y. □

Transition functions behave naturally with relation to basic constructions on principal $U(1)$-bundles, such as pullbacks, duals and tensor products. Then, immediately from the equation (2.2.2), we have the following

Proposition 2.2.2. *Consider a differentiable manifold X and two bundle gerbes over X, named (P_1,π_{Y_1}) and (P_2,π_{Y_2}). Consider another pair of differentiable manifolds M and N and a map $f:M\to N$. Finally, let (P,π_Y) be a bundle gerbe over N, and let $g:Z\to Y$ be a fibre map covering f. Then*

1. $DD(P^*) = -DD(P)$
2. $DD(f^*(P), f^*\pi_Y) = f^*(DD(P, \pi_Y))$
3. $DD(P_1 \otimes P_2) = DD(P_1) + DD(P_2)$

Equation (3) shows that there are many bundle gerbes which have the same Diximier-Douady class but which are not isomorphic. Namely, tensoring a bundle gerbe P with a trivial bundle gerbe T, will produce a bundle gerbe $P \otimes T$ with the same Diximier-Douady class as P. On the other hand, this class looks like just the right one for bundle gerbes, thanks of its good behaviour in respect of triviality and basic operations on bundle gerbes. These facts suggest us to maintain the Diximier-Douady class as characteristic class for bundle gerbes but to introduce a weaker notion of bundle gerbe isomorphism.

Definition 2.2.3. *Two bundle gerbes (P, Y) and (Q, Z) are called* stably isomorphic *if there are trivial bundle gerbes T_1 and T_2 such that $P \otimes T_1 \simeq Q \otimes T_2$.*

Proposition 2.2.4. *Let (P, Y) and (Q, Z) be bundle gerbes. Then $DD(P) = DD(Q)$ if and only if P and Q are stably isomorphic.*

Proof. If $DD(P) = DD(Q)$ then $DD(P \otimes Q^*) = DD(P) - DD(Q) = 0$, hence $P \otimes Q^*$ is trivial. The canonical isomorphism $Q \otimes (P \otimes Q^*) \simeq P \otimes (Q \otimes Q^*)$ realizes therefore a stable isomorphism between P and Q, since $Q \otimes Q^*$ is clearly trivial ($DD(Q \otimes Q^*) = DD(Q) - DD(Q) = 0$).
On the other hand, if P and Q are stably isomorphic, there are two trivial bundle gerbes T_1 and T_2 such that $P \otimes T_1 \simeq Q \otimes T_2$. Then $DD(P) = DD(Q) + DD(T_2) - DD(T_1) = DD(Q)$, since both T_1 and T_2 have zero Diximier-Douady class, as trivial gerbes. □

2.3 Bundle gerbes from open covers

Let X be a differentiable manifold. We want now to consider those bundle gerbes over X whose construction derives from nerves of open covers of X. Let's start by recalling the following

Definition 2.3.1. *Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open cover of X. For every positive integer n, we write $U_{\alpha_{i_1} \dots \alpha_{i_n}}$ for the n-fold intersection $U_{\alpha_{i_1}} \cap U_{\alpha_{i_2}} \cap \dots \cap U_{\alpha_{i_n}}$ and assume that it is empty or contractible, that is $\mathcal{U}$ is required to be a good cover. The* nerve *of $\mathcal{U}$ is the simplicial set $N(\mathcal{U})$ whose n-*simplices *are given by $N(\mathcal{U})_n$: $\left\{(x, \alpha_{i_1}, \dots, \alpha_{i_{n+1}}) | \alpha_{i_1}, \dots, \alpha_{i_{n+1}} \in I, x \in U_{\alpha_{i_1}, \dots, \alpha_{i_{n+1}}}\right\}$.*

As an illustrative example, vertices and edges of the nerve are respectively given by

$$N(\mathcal{U})_0 : \{(x, \alpha) | \alpha \in I, x \in U_\alpha\}$$

and

$$N(\mathcal{U})_1 : \{(x, \alpha, \beta) | \alpha, \beta \in I, x \in U_{\alpha\beta}\} .$$

That is, $N(\mathcal{U})_0$ is the disjoint union of all the open sets U_α of the cover, while $N(\mathcal{U})_1$ is the disjoint union of all the double intersections. More generally, $N(\mathcal{U})_k$ is the disjoint union of all the k-fold intersections.
Note that $N(\mathcal{U})_0$ is a smooth manifold, which we will also denote with $Y_{\mathcal{U}}$ and that the natural projection $\pi : Y_{\mathcal{U}} \to X$ is a locally split map. If we have a bundle gerbe (P, Y) over X, we can consider local sections $s_\alpha : U_\alpha \to Y$: they induce a map $s : Y_{\mathcal{U}} \to Y$, defined by $s(\alpha, x) := s_\alpha(x)$. Equation (2) of Proposition 2.2.2 applied with $f = id_X$ guarantees us that the pullback of (P, Y) defines a bundle gerbe over $Y_{\mathcal{U}}$ whose characteristic class is equal to that of P therefore, thanks to Proposition 2.2.4, we know that the two bundle gerbes are stably isomorphic. That is, given a bundle gerbe (P, Y) over X and a good cover U of X, up to stable isomorphism it is always possible to identify it with its pullback over $Y_{\mathcal{U}}$.

Having shown that it is not restrictive to consider only those bundle gerbes that come from an open cover, let us now give a description of these in terms of local data. Before doing this, let us recall the local data definition of $U(1)$-principal bundles: the local data definition of $U(1)$-bundle gerbes will be an immediate generalization.

The local data definition of $U(1)$-principal bundles is best given in terms of the simplicial sheaf $\mathbf{B}U(1)$ on the site of smooth manifolds with smooth mappings.[1] To define it, we first

[1] See [8] for a detailed discussion of this site

introduce the simplicial presheaf ${}_{pre}\mathbf{B}U(1)$ which associates with a smooth manifold M the simplicial set given as follows:

- ${}_{pre}\mathbf{B}U(1)(M)$ has a single 0-simplex;
- 1-simplices of ${}_{pre}\mathbf{B}U(1)(M)$ are smooth functions $g\colon M \to U(1)$; these can be seen as topological 1-simplices decorated by the smooth functions g;
- 2-simplices of ${}_{pre}\mathbf{B}U(1)(M)$ are triples of smooth functions $g, h, k\colon M \to U(1)$ which satisfy the cocycle condition $ghk = 1$; these can be visualized as topological 2-simplices whose boundary 1-simplices are decorated by the smooth functions g, h, k satisfying the cocycle condition;
- 3-simplices of ${}_{pre}\mathbf{B}U(1)(M)$ are topological 3-simplices whose 1-simplices are decorated by smooth functions $M \to U(1)$ in such a way that all the boundary 2-simplices are 2-simplices in ${}_{pre}\mathbf{B}U(1)(M)$;
- and so on, i.e., there is no notrivial condition on the higher simplices.

The simplicial sheaf $\mathbf{B}U(1)$ is then defined as the *sheafification* of the simplicial presheaf ${}_{pre}\mathbf{B}U(1)$. This means that, for a smooth manifold M, the simplicial set $\mathbf{B}U(1)(M)$ is obtained by taking a good open cover $\mathcal{U} = \{U_\alpha\}$ of M and then considering the simplicial set of simplicial maps from $N(\mathcal{U})$ to ${}_{pre}\mathbf{B}U(1)(M)$. This means that a 0-simplex in $\mathbf{B}U(1)(M)$ is the datum of

- smooth functions $g_{\alpha\beta}\colon U_{\alpha\beta} \to U(1)$ such that on the triple intersections $U_{\alpha\beta\gamma}$ the cocycle identity $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ is satisfied

These are precisely the local data describing a principal $U(1)$-bundle on M. But there is more: a 1-simplex in $\mathbf{B}U(1)(M)$ with boundary 0-simplices $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ is the datum of

- smooth functions $\varphi_\alpha\colon U_\alpha \to U(1)$ such that on the double intersections one has $\varphi_\alpha g_{\alpha\beta}\varphi_\beta^{-1} = g'_{\alpha\beta}$.

This means that the $\{\varphi_\alpha\}$ are precisely the local components of a morphism $\varphi\colon P \to P'$ of principal $U(1)$-bundles between the principal $U(1)$-bundle P with local data $\{g_{\alpha\beta}\}$ and the principal $U(1)$-bundle P' with local data $\{g'_{\alpha\beta}\}$. Similar considerations apply to the higher simplices in $\mathbf{B}U(1)(M)$ and we find the following remarkable result: the simplicial set $\mathbf{B}U(1)(M)$ is equivalent to the nerve of the groupoid of principal $U(1)$-bundles on M. In other words, we have that

$$\mathbf{B}U(1)\colon M \mapsto \mathbf{B}U(1)(M)$$

is the simplicial sheaf mapping a smooth manifold to the groupoid of principal $U(1)$-bundles on it.

By repeating the same construction in one higher dimension, we obtain the simplicial sheaf $\mathbf{B}^2U(1)$ of $U(1)$ -principal bundle gerbes. Namely, we first consider the simplicial presheaf ${}_{pre}\mathbf{B}^2U(1)$ which associates with a smooth manifold M the simplicial set given as follows:

- ${}_{pre}\mathbf{B}^2U(1)(M)$ has a single 0-simplex;
- ${}_{pre}\mathbf{B}^2U(1)(M)$ has a single 1-simplex;
- 2-simplices of ${}_{pre}\mathbf{B}^2U(1)(M)$ are smooth functions $g\colon M \to U(1)$; these can be seen as topological 2-simplices decorated by the smooth functions g;
- 3-simplices of ${}_{pre}\mathbf{B}^2U(1)(M)$ are quadruples of smooth functions $g, h, k, l\colon M \to U(1)$ which satisfy the cocycle condition $ghkl = 1$; these can be visualized as topological 3-simplices whose boundary 2-simplices are decorated by the smooth functions g, h, k, l satisfying the cocycle condition;
- 4-simplices of ${}_{pre}\mathbf{B}^2U(1)(M)$ are topological 4-simplices whose 2-simplices are decorated by smooth functions $M \to U(1)$ in such a way that all the boundary 3-simplices are 3-simplices in ${}_{pre}\mathbf{B}^2U(1)(M)$;
- and so on, i.e., there is no notrivial condition on the higher simplices.

The simplicial sheaf $\mathbf{B}^2U(1)$ is then defined as the sheafification of the simplicial presheaf ${}_{pre}\mathbf{B}^2U(1)(M)$. This means in particular that, for a smooth manifold M with a good open cover $\mathcal{U} = \{U_\alpha\}$ a 0-simplex in $\mathbf{B}^2U(1)(M)$ is the datum of

- smooth functions $g_{\alpha\beta\gamma}\colon U_{\alpha\beta\gamma} \to U(1)$;
- such that on the quadruple intersections $U_{\alpha\beta\gamma\delta}$ the cocycle identity $g_{\beta\gamma\delta}g_{\gamma\alpha\delta}g_{\alpha\beta\delta}g_{\beta\alpha\gamma} = 1$ is satisfied. This is more transparently written as $g_{\beta\gamma\delta}g^{-1}_{\alpha\gamma\delta}g_{\alpha\beta\delta}g^{-1}_{\alpha\beta\gamma} = 1$.[2]

These are precisely the local data describing a principal $U(1)$-gerbe coming from an open cover on M. To see this we need just a little work. Note that the cocycle $g_{\alpha\beta\gamma}$ always admits local representations as a Čech coboundary, that is local trivializations. Indeed, chosen an open set U_0 of the cover, if we set $f_{\beta\gamma} := g_{0\beta\gamma}$ for $\beta, \gamma \neq 0$ and $f_{0\beta} = f_{0\gamma} = 1$, the cocycle condition over fourfold intersections gives $g_{\beta\gamma\delta} = f_{\beta\gamma}f_{\gamma\delta}f_{\delta\beta}$. Let U_1 be another open set of the cover such that $U_{01} \neq \emptyset$: over U_{01} we can consider two different trivializations $g_{\beta\gamma\delta} = f_{\beta\gamma}f_{\gamma\delta}f_{\delta\beta}$ and $g_{\beta\gamma\delta} = f'_{\beta\gamma}f'_{\gamma\delta}f'_{\delta\beta}$. If we denote by $h_{\alpha\beta}$ the difference $f_{\alpha\beta}/f'_{\alpha\beta}$, we have $h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha} = 1$ and $h_{\beta\alpha} = h^{-1}_{\beta\alpha}$, since clearly $f_{\beta\alpha} = f^{-1}_{\beta\alpha}$. In closing, we obtained that two trivialization of $g_{\alpha\beta\gamma}$ differ from a line bundle and then we have a line bundle $P_{\alpha\beta} \simeq P^{-1}_{\beta\alpha}$ over each $U_{\alpha\beta}$. To see how these line bundles relate each other over threefold intersections, observe that, over $U_{\alpha\beta\gamma}$, we have $P_{\alpha\beta}, P_{\beta\gamma}$ and $P_{\gamma\alpha}$: there is a nowhere-zero section $\lambda_{\alpha\beta\gamma} \in \Gamma(U_{\alpha\beta\gamma}, P_{\alpha\beta} \otimes P_{\beta\gamma} \otimes P_{\gamma\alpha})$, because $P_{\alpha\beta} \otimes P_{\beta\gamma} \otimes P_{\gamma\alpha}$ is trivial. Note that $\lambda_{\alpha\beta\gamma}$ is a cocycle, because there are trivial relations for sections concerning different reorderings of the three indices. Finally, if we tensor together the four sections over the fourfold intersection $U_{\alpha\beta\gamma\delta}$, we obtain a trivialization $\delta\lambda$ of the line bundle $Q = (P_{\alpha\beta} \otimes P_{\beta\gamma} \otimes P_{\gamma\alpha}) \bigotimes (P_{\alpha\beta} \otimes P_{\beta\delta} \otimes P_{\delta\alpha})^{-1} \bigotimes (P_{\alpha\gamma} \otimes P_{\gamma\delta} \otimes P_{\delta\alpha}) \bigotimes (P_{\beta\gamma} \otimes P_{\gamma\delta} \otimes P_{\delta\gamma})^{-1}$. On the other hand, there is a canonical trivialization of Q given by the duality conditions $P_{\beta\alpha} = P^{-1}_{\alpha\beta}$, and the requirement to do is that $\delta\lambda$ is just this canonical trivialization, that is $\delta\lambda = 1$. We explicitly observe that the collection of $P_{\alpha\beta}$ gives a $\mathbb{C}^*$ principal bundle over Y_U, while the family of sections $\{\lambda_{\alpha\beta\gamma}\}$ gives a bundle gerbe product whose associativity is written precisely in the condition $\delta\lambda = 1$. Note that we obtained these data by starting from a 0-simplex in $\mathbf{B}^2U(1)(M)$. A similar

[2]The cocycle $g_{\alpha\beta\gamma}$ has to verify $g_{\alpha\beta\gamma} = g^{-1}_{\beta\alpha\gamma} = g^{-1}_{\alpha\gamma\beta} = g^{-1}_{\gamma\beta\alpha}$, as results from index substitutions (as an example, by setting $\delta = \beta$, we obtain $g_{\alpha\beta\gamma} = g^{-1}_{\alpha\gamma\beta}$).

argument shows that 1-simplices in $\mathbf{B}^2U(1)(M)$ corresponds to a morphism of bundle gerbes over M and that a 2-simplex in $\mathbf{B}^2U(1)(M)$ corresponds to an equivalence of morphisms of bundle gerbes.
In other words, we have that

$$\mathbf{B}^2U(1)\colon M \mapsto \mathbf{B}^2U(1)(M)$$

is the simplicial sheaf mapping a smooth manifold to the 2-groupoid of principal $U(1)$-bundle gerbes on it.[3]
We will often prefer local descriptions of bundle gerbes, especially for their simplicity, but also because we can easily extend them to define higher $U(1)$-bundle gerbes:

Definition 2.3.2. *For any $n \geq 1$ let ${}_{pre}\mathbf{B}^nU(1)$ be the simplicial presheaf which associates with a smooth manifold M the simplicial set given as follows:*

- *${}_{pre}\mathbf{B}^nU(1)(M)$ has a single 0-simplex;*
- *${}_{pre}\mathbf{B}^nU(1)(M)$ has a single 1-simplex;*
- $\cdots$
- *${}_{pre}\mathbf{B}^nU(1)(M)$ has a single $(n-1)$-simplex;*
- *n-simplices of ${}_{pre}\mathbf{B}^nU(1)(M)$ are smooth functions $g\colon M \to U(1)$; these can be visualized as topological n-simplices decorated by the smooth functions g;*
- *$n+1$-simplices of ${}_{pre}\mathbf{B}^nU(1)(M)$ are $n+1$-ples of smooth functions $g_0, \ldots, g_n\colon M \to U(1)$ which satisfy the cocycle condition $g_0 g_1 \cdots g_n = 1$; these can be visualized as topological $n+1$-simplices whose boundary n-simplices are decorated by the smooth functions $g_0, g_1, \ldots, g_n$ satisfying the cocycle condition;*
- *$n+2$-simplices of ${}_{pre}\mathbf{B}^nU(1)(M)$ are topological $n+2$-simplices whose n-simplices are decorated by smooth functions $M \to U(1)$ in such a way that all the boundary $n+1$-simplices are $n+1$-simplices in ${}_{pre}\mathbf{B}^nU(1)(M)$;*

[3] We are going to give a precise definiton of n-groupoids for $n \geq 2$ in Chapter 4.

- *and so on, i.e., there is no notrivial condition on the higher simplices.*

The simplicial sheaf $\mathbf{B}^n U(1)$ *is the simplicial sheaf obtained by sheafifying the simplicial presheaf* ${}_{pre}\mathbf{B}^n U(1)$. *It is the* n*-stack of principal* $U(1)$*-*n*-bundles (or* $(n-1)$*-bundle gerbes).*

Chapter 3

Bundle gerbes with connections and their local description

In our work we will need to use bundle gerbes not only as topological objects but also as differential objects: we will need to consider connections over them. The structure of this chapter is similar to the previous one. We will firstly describe what a connection on a bundle gerbe is and that it induces a closed three form over the base space called the Diximier-Douady form of the connection. This is the analogue of the curvature 2-form associated with a connection on a principal $U(1)$-bundle. Then we will move to explain how a connection looks like locally and we will find again a local description of the Diximier-Douady form.

3.1 Basic definitions

Definition 3.1.1. *Let (P, Y) be a bundle gerbe over X and let ∇ be a connection over the $U(1)$-bundle $P \to Y^{[2]}$. We say that it is a* bundle gerbe connection *if it commutes with the bundle gerbe product λ.*

Remark 3.1.2. *It immediately follows from the definition that, if L_P denotes the line bundle associated to P via the defining representation of $U(1)$ on $\mathbb{C}$, then for every section $s \in \Gamma(Y^{[2]}, L_P)$ and for every vector field $X \in C^\infty(Y^{[2]}, TY^{[2]})$, there is the commutative diagram*

$$\begin{array}{ccc} s|_{(y_1,y_2)} \otimes s|_{(y_2,y_3)} & \xrightarrow{\pi_{123}^*\lambda} & s|_{(y_1,y_3)} \\ \nabla_X|_{(y_1,y_2)} \Big\downarrow \otimes \nabla_X|_{(y_2,y_3)} & & \Big\downarrow \nabla_X|_{(y_1,y_3)} \\ \nabla_X(s)|_{(y_1,y_2)} \otimes \nabla_X(s)|_{(y_2,y_3)} & \xrightarrow[\pi_{123}^*\lambda]{} & \nabla_X(s)|_{(y_1,y_3)} \end{array} \tag{3.1.1}$$

Observe that a bundle gerbe (P, Y) over X always admits connections. Indeed, if it is the trivial bundle gerbe $\delta(Q)$, we can consider a connection ∇ over Q and take $\nabla^* \otimes \nabla$ over $\delta(Q)$: $\nabla^* \otimes \nabla$ clearly commutes with the trivial gerbe product, that is the composition of automorphisms between the fibres of Q. If P is not the trivial bundle gerbe, since the projection $\pi : Y \to X$ is a locally split map, then there is an open cover $\{U_\alpha\}$ of X with a section for each U_α. The bundle gerbe can be trivialized over each $Y_\alpha := \pi^{-1}(U_\alpha)$ and hence admits locally a bundle gerbe connection. We can choose a partition of unity for the open cover U_α and pullback it to $Y^{[2]}$ to give a partition of unity for the open cover $\left\{Y_\alpha^{[2]}\right\}$. Finally we patch togheter the bundle gerbe connections on the various open sets to give a bundle gerbe connection on P.

We are now going to give a closer look to the differential forms implicitly involved in the definition of a bundle gerbe connection. In order to do this, we need to introduce some notations. Let (P, Y) be a bundle gerbe over X: we write $\Omega^p(X)$ for the space of p- forms on X, we write $\pi_i : Y^{[k]} \to Y^{[k-1]}$ for the projection map which omits the i-th point in the fibre product and $\delta : \Omega^p(Y^{[k-1]}) \to \Omega^p(Y^{[k]})$ for the alternating sum of pullbacks $\sum(-1)^j \pi_i^*$. It easy to check that $\delta^2 = 0$ and that δ commutes with the exterior differentiation operator d. There is also the following useful

Proposition 3.1.3. *The complex*

$$\Omega^p(X) \xrightarrow{\pi^*} \Omega^p(Y) \xrightarrow{\delta} \Omega^p(Y^{[2]}) \xrightarrow{\delta} \dots$$

is exact.

Proof. We assume for simplicity that $\pi : Y \to X$ is a local fibration, and consider first the case that it is a trivial fibration, say $Y = X \times F$. In this case $Y^{[k]} = X \times F^k$. Let ω be a q-form over $X \times F^k$: we have to prove that, if $\delta(\omega) \in \Omega^q(Y^{[k+1]})$ is zero, then there is a q-form ρ over Y^k such that $\delta(\rho) = \omega$. By the same definition of δ, for every $x \in X$, $\zeta \in (TX_x)^q$ and $X_i \in (TF_{f_i})^q$, we get

$$\delta(\omega)|_{(x,(f_1,...,f_{p+1}))}(\zeta,(X_1,...,X_{p+1})) = \sum(-1)^j\omega|_{(x,(f_1,...,\hat{f}_j,...,f_{p+1}))}(\zeta,(X_1,...,\hat{X}_j,...,X_{p+1})).$$

Now fix $\bar{f} \in F$ and a q-tuple of vectors $\bar{X}$ in $TF_{\bar{f}}$ and define ρ, by setting

$$\rho|_{(x,(f_1,...,f_p))}(\zeta,(X_1,...,X_p)) = \omega|_{(x,(f_1,...,f_p,\bar{f}))}(\zeta,(X_1,...,X_p),\bar{X}).$$

Assume now that $\delta(\omega) = 0$: since the alternating sum of the first p terms of $\delta(\rho)$ is zero, we haave $\delta(\rho) = (-1)^{p+1}\omega$.

The case that the projection π is only a local fibration is proved by choosing an open cover U such that Y is trivial over each U_α. Let ϕ_α be a partition of unity subordinate to that cover. Let $\omega \in \Omega^q(Y^{[k]})$: denoted with Y_α the part of Y sitting over U_α, there is some ρ_α, q-form over $(Y_\alpha)^{k-1}$ such that $\omega|_{U_\alpha} = \delta(\rho_\alpha)$. Hence we have

$$\omega = \sum_\alpha \phi_\alpha\delta(\rho_\alpha) = \delta(\sum_\alpha \phi_\alpha\rho_\alpha) = \delta(\rho),$$

where $\rho = \sum_\alpha \phi_\alpha\rho_\alpha$. Finally, at $k = 1$, we define $Y^{[0]} := X$ and let $\delta : \Omega^q(X) \to \Omega^q(Y)$ be the pullback under π, that is $\delta = \pi^*$, the dual map of $\pi_* = d\pi$. Then exactness immediately follows from the injectivity of π_*. □

3.2 The Diximier-Douady 3-form of a bundle gerbe connection

Let ∇ be a bundle gerbe connection over (P, Y): as every $U(1)$-principal bundle connection, it has a curvature F_∇, which is is a two-form on $Y^{[2]}$. It follows from the isomorphism given by the bottom arrow of the commutative diagram (3.1.1) that

$$\delta(F_\nabla) = \pi_2^*(F_\nabla) - \pi_1^*(F_\nabla) - \pi_3^*(F_\nabla) = -(F_\nabla|_{(y_1,y_2)} + F_\nabla|_{(y_2,y_3)} - F_\nabla|_{(y_1,y_3)}) = 0.$$

Then, by Proposition 3.1.3, exists a two-form f over Y, called *curving* for ∇, such that $F_\nabla = \delta(f) = \pi_2^*f - \pi_1^*f$, hence $dF_\nabla = d\delta(f) = 0$ and $\pi_1^*df = \pi_2^*df$.

Proposition 3.2.1. *Let $f \in \Omega^2(Y)$ a curving for ∇. Then there is $\omega \in \Omega^3(X)$ such that $df = \pi^*(\omega)$.*

Proof. Since f is a curving for ∇, $\pi_1^* df = \pi_2^* df$, then we have

$$df(y)(X_1, X_2, X_3) = df(y')(Z_1, Z_2, Z_3), \tag{3.2.2}$$

for each $(y, y') \in Y^{[2]}$ and $((X_i, Z_i) \in TY^{[2]})$, $i = 1, 2, 3$. Fix $x \in X$ and $\zeta_i \in TX_i$, then choose $y \in Y$ and $X_i \in TY_y$ such that $\pi(y) = x$ and $\pi_*(X_i) = \zeta_i$ and define

$$\omega|_x(\zeta_1, \zeta_2, \zeta_3) = df|_y(X_1, X_2, X_3) \tag{3.2.3}$$

Equation (3.2.2) shows that this definition is indipendent of the choice of y and X_i. Clearly $\pi^*(\omega) = df$ and moreover ω is closed. □

Definition 3.2.2. *The closed three-form of X given by $\frac{1}{2\pi i}\omega$ is called* Diximier-Douady form of the pair (∇, f).

3.3 The local data for bundle gerbe connections

Before giving a local description of bundle gerbe connections, it's worthwhile to recall what a bundle connection is, when the bundle is expressed in terms of transition functions. In both situations, we will assume $U(1)$ as structure group, X as base space and $U = \{U_\alpha\}_{\alpha \in I}$ as a good cover of X.

Let $g_{\alpha\beta} : U_{\alpha\beta} \to U(1)$ the cocycle representing our $U(1)$ principal bundle: giving a connection means giving real valued 1-forms $A_\alpha \in \Omega^1(U_\alpha)$ and requiring for them a good behaviour over twofold intersections, that is

$$A_\beta = g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta}.$$

Since $U(1)$ is an abelian group, the previous relation becomes

$$A_\beta = A_\alpha + g_{\alpha\beta}^{-1} dg_{\alpha\beta}$$

and, by noticing that we can write $g_{\alpha\beta} = e^{2\pi i k_{\alpha\beta}}$ for some $k_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{R}$, which is always possible since the open subsets $U_{\alpha\beta}$ are assumed to be contractible, we see that the above relation can be rewritten as

$$A_\beta = A_\alpha + \frac{1}{2\pi i} d\log(g_{\alpha\beta}).$$

We can therefore encode the entire collection of data defining our $U(1)$-bundle with connection into the chain complex of sheaves of abelian groups

$$\cdots \to 0 \to 0 \to C^\infty(-;U(1)) \xrightarrow{\frac{1}{2\pi i} d\log} \Omega^1(-,\mathbb{R}) \tag{3.3.4}$$

with $\Omega^1(-,\mathbb{R})$ in degree zero. Notice that we can encode in a similar way the data of $U(1)$-bundles without connection: they will be encoded into the chain complex

$$\cdots \to 0 \to 0 \to C^\infty(-;U(1)) \to 0. \tag{3.3.5}$$

Remark 3.3.1. *The local data description of $U(1)$-connections shows us that we have a stack $\mathbf{B}U(1)_{\text{conn}}$ of principal $U(1)$-bundles with connections. Moreover the natural forgetful morphism*

$$\mathbf{B}U(1)_{\text{conn}} \to \mathbf{B}U(1)$$

which forgets the connection is encoded at a chain complex level by the natural morphism of chain complexes of sheaves of abelian groups

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & C^\infty(-;U(1)) & \xrightarrow{\frac{1}{2\pi i} d\log} & \Omega^1(-,\mathbb{R}) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & C^\infty(-;U(1)) & \longrightarrow & 0 \end{array}$$

Remark 3.3.2. *Note that, since the 1-form data of a $U(1)$-connection satisfy $A_\beta = A_\alpha + \frac{1}{2\pi i} d\log(g_{\alpha\beta})$, we have*

$$dA_\alpha = dA_\beta$$

and so there is a global well defined closed two-form $F|_{U_\alpha} = dA_\alpha$. In terms of smooth stacks, this is the curvature *morphism*

$$\text{curv}\colon \mathbf{B}U(1)_{\text{conn}} \to \Omega^2_{\text{cl}}(-;\mathbb{R}),$$

where $\Omega^2_{\text{cl}}(-;\mathbb{R})$ *is the sheaf of closed 2-forms. In terms of chian complexes, this is the morphism*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & C^\infty(-;U(1)) & \xrightarrow{\frac{1}{2\pi i}d\log} & \Omega^1(-,\mathbb{R}) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow d \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \Omega^2_{\text{cl}}(-\mathbb{R}) \end{array}$$

Let's move to bundle gerbes and realize the same object, but in one higher dimension. Let $g_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to U(1)$ be the cocycle representing our $U(1)$-bundle gerbe. Spelling out in detail the definition of a bundle gerbe connection, we see that our local data consist in assigning one-forms $A_{\alpha\beta}$ over two-fold intersections and two-forms B_α over all the open sets of the cover. We also want that all these data satisfy suitable compatibility conditions, in the following sense:

1. $g_{\beta\gamma\delta}g^{-1}_{\alpha\gamma\delta}g_{\alpha\beta\delta}g^{-1}_{\alpha\beta\gamma} = 1$ on $U_{\alpha\beta\gamma\delta}$,
2. $A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = g^{-1}_{\alpha\beta\gamma}dg_{\alpha\beta\gamma} = \frac{1}{2\pi i}dlog(g_{\alpha\beta\gamma})$ on $U_{\alpha\beta\gamma}$
3. $B_\beta - B_\alpha = dA_{\alpha\beta}$ on $U_{\alpha\beta}$.

We can again encode these data in a more visible way, by means of the complex of sheaves

$$\cdots \to 0 \to 0 \to C^\infty(-;U(1)) \xrightarrow{\frac{1}{2\pi i}d\log} \Omega^1(-,\mathbb{R}) \xrightarrow{d} \Omega^2(-,\mathbb{R}) \tag{3.3.6}$$

with $\Omega^2(-,\mathbb{R})$ in degree zero. We can encode in a similar way the data of $U(1)$-bundle gerbes without connection: they will be encoded into the chain complex

$$\cdots \to 0 \to 0 \to C^\infty(-;U(1)) \to 0 \to 0. \tag{3.3.7}$$

From this point of view we also see that we can consider an intermediate object, namely, $U(1)$-bundle gerbes with 1-form connection data but without 2-form connection data. These are sometimes called bundle gerbes with connection but without curving in the literature, and are encoded into the chain complex

$$\cdots \to 0 \to 0 \to C^\infty(-;U(1)) \xrightarrow{\frac{1}{2\pi i}d\log} \Omega^1(-,\mathbb{R}) \to 0. \tag{3.3.8}$$

Remark 3.3.3. *The local data description of $U(1)$-gerbe connections shows us that we have a stack $\mathbf{B}^2U(1)_{\mathrm{conn}}$ of principal $U(1)$-bundles with connections and a stack $\mathbf{B}(\mathbf{B}U(1)_{\mathrm{conn}})$ of bundle gerbes with connection but without curving . Moreover the natural forgetful morphisms*

$$\mathbf{B}^2U(1)_{\mathrm{conn}} \to \mathbf{B}(\mathbf{B}U(1)_{\mathrm{conn}}) \to \mathbf{B}^2U(1)$$

which forgets the connection is encoded at a chain complex level by the natural morphism of chain complexes of sheaves of abelian groups

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & C^\infty(-;U(1)) & \xrightarrow{\frac{1}{2\pi i}d\log} & \Omega^1(-,\mathbb{R}) & \longrightarrow & \Omega^2(-,\mathbb{R}) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & C^\infty(-;U(1)) & \xrightarrow{\frac{1}{2\pi i}d\log} & \Omega^1(-,\mathbb{R}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & C^\infty(-;U(1)) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

Remark 3.3.4. *Since the 2-form data of a $U(1)$-connection satisfies $B_\beta = B_\alpha + dA_{\alpha\beta}$, we have*

$$dB_\alpha = dB_\beta$$

and so there is a global well defined closed 3-form $\omega|_{U_\alpha} = dB_\alpha$. In terms of smooth stacks, this is the Diximier-Douady *(or* 3-curvature*) morphism*

$$\mathrm{DD}\colon \mathbf{B}^2U(1)_{\mathrm{conn}} \to \Omega^3_{\mathrm{cl}}(-;\mathbb{R}),$$

where $\Omega^3_{\mathrm{cl}}(-;\mathbb{R})$ is the sheaf of closed 3-forms. In terms of chian complexes, this is the morphism

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & C^\infty(-;U(1)) & \xrightarrow{\frac{1}{2\pi i}d\log} & \Omega^1(-,\mathbb{R}) & \longrightarrow & \Omega^2(-,\mathbb{R}) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \Omega^3_{\mathrm{cl}}(-;\mathbb{R})
\end{array}$$

Remark 3.3.5. *As we are going to show in the following Chapter, when we will introduce the Dold-Kan correspondence, the choice to represent bundle gerbes with and without connections is not fortuitous: it is based on the equivalence of categories between chain complexes of abelian groups (concentarted in nonegative degree) and simplicial abelian groups.*

Chapter 4

Smooth stacks: a (very) quick overview

As implicitly considered in the first paragraph, the suitable setting to study structures on manifolds and relations among them is that of spaces and maps up to homotopy. However, to get our goal, we will need to introduce the so called *smooth refinement* of spaces and maps and the related language of stacks and higher stacks. This subject, for which an excellent reference is [21],[1] will permit us to get a result that, read in terms of spaces and maps, will lead us to conclude that String structures induce Spin structures on loop spaces. We intend only an introduction to stacks and higher stacks for this chapter: we try to friendly specify here how these smooth refinements and their relation with the homotopy theory have to be thought.

One of the needs met by the theory of (higher) stacks is that to consider in a unitary way all the local automorphisms of a principal bundle. Specifically, it is clear that every G principal bundle $P \xrightarrow{\pi} X$, by definition, is locally equivalent to the trivial one. However, over each trivializing open U_α, there are infinity possible isomorphisms $\pi^{-1}U_\alpha \to U_\alpha \times G$, completely captured by the group $C^\infty(U_\alpha, G)$: this information is not visible under passing to equivalence classes, in the standard way. The idea is then that to consider a presheaf of groupoids on the site of smooth manifolds

$$U_\alpha \mapsto \{*//C^\infty(U_\alpha, G)\}$$

[1] See also [5] for a more colloquial introduction to the subject.

with the unique object $*$ denoting the class of the trivial G-principal bundle over U_α. We have to extend this initial idea in two different directions: first of all, in general, when $G = A$ is an abelian group we will need to realize presheaves of n-groupoids

$$U_\alpha \mapsto \{*//...//*//C^\infty(U_\alpha, A)\}$$

to consider not only A-principal bundles, but also A-n- bundle gerbes (with $n \geq 1$)
On the other hand (and this is the deeper point), we will have to explain *how* to realize a smooth refinement of a space starting by these presheaves of n-groupoids on the site of smooth manifolds and *how* to come back to the world of spaces and maps up to homotopy. Simplicial sets will play a central role in answering both of these needs.

In the first case, we will use *Kan complexes*, which are particular simplicial sets, to describe ∞-groupoids generalizing n-groupoids for $n \geq 1$. In the second case, we take advantage of the fact that simplicial sets are good objects to do homotopy theory (in a sense that will be clarified later), to realize a sort of bridge between presheaves of Kan complexes and spaces up to homotopy.

4.1 Simplicial sets and Kan complexes

Let's start by recall that a simplicial set is a functor $S : \Delta^{op} \to \mathbf{Set}$, where Δ is the category of combinatorial simplices (i.e. abstract cellular simplices): objects in Δ are the linearly ordered sets $[n] := \{0, 1, ..., n\}$ and morphisms in Δ are given by nonstrictly order-preserving maps. The ordered set $[n]$ will be equivalently denoted by $\Delta[n]$, to suggest the equivalent visualization as abstract cellular simplex.
More explicitly, a simplicial set S is determined by the following data:

- A set S_n for each $n \geq 0$,
- A map $Sf : S_n \to S_m$ for each order-preserving map $f : [m] \to [n]$.

We denote by `sSet` the category of simplicial sets. It's a classic result that every morphism $S_n \to S_m$ can be expressed as composition of *face maps* $d_j^n = S(\delta_j) : S_n \to S_{n-1}$ and *degeneracy maps* $s_j^n = S(\sigma_j^n) : S_n \to S_{n+1}$, $0 \le j \le n$, where

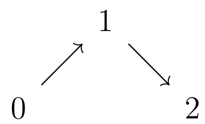
$$\delta_j^n(i) = \begin{cases} i & \text{if } i < j \\ i+1 & \text{if } i \ge j \end{cases}$$

$$\sigma_j^n(i) = \begin{cases} i & \text{if } i \le j \\ i-1 & \text{if } i > j \end{cases} .$$

The fact that S is a functor, so that $S(id_{[n]}) = id_{S_n}$ and $S(g \circ f) = Sf \circ Sg$ for each $[l] \xrightarrow{f} [m] \xrightarrow{g} [n]$, suggests us to interpret S_k as set of k-morphisms (between $k-1$-morphisms) and the face maps d_j^k as source and target maps for these k-morphisms. This point of view is essential in describing a suitable notion of ∞-groupoid, which will be "a simplicial set with invertible k-morphisms for every $k \ge 1$". It is clear that we have first to endow the elements of S_k with a notion of composition.

In order to guess which could be a good notion of n-groupoid, let us first look at the familiar situation of 1-groupoids, i.e., categories where all the morphisms are isomorphism. Recall that every category is a simplicial set, by identifying the category with its nerve. So what we are going to show are the "special features" the nerve of a groupoid has with respect to a random simplicial set.

For instance, consider a simplicial set S, write $\Lambda^1[2]$ for the combinatorial simplex consisting of two attached 1-cells



and denote with $(f, g) : \Lambda^1[2] \to S$ a subset of S_1 reflecting this situation in S. We say that $x_0 \xrightarrow{f} x_1 \xrightarrow{g} x_2$ is a pair of composable 1-morphisms in S if exists, unique up to 2-morphism equivalences, a third 1-morphism $h : x_0 \to x_2$, as in the following picture

$$\begin{array}{ccc} & x_1 & \\ {\scriptstyle f}\nearrow & \Downarrow\ \simeq & \searrow{\scriptstyle g} \\ x_0 & \xrightarrow[h]{} & x_2 \end{array}$$

Another way to represent the composability of f and g is by demanding that for $\Lambda^1[2] \hookrightarrow \Delta[2]$ the obvious inclusion of the two abstract composable 1-morphisms into the standard 2-simplex we have a diagram of morphisms of simplicial sets

$$\begin{array}{ccc} \Lambda^1[2] & \xrightarrow{(f,g)} & S \\ \downarrow & \nearrow_{\exists h} & \\ \Delta[2] & & \end{array} \tag{4.1.1}$$

A simplicial set where for all such (f, g) a corresponding h exists may be tought of as a collection of morphisms that is equipped with a notion of composition of ajacent 1-morphisms. To describe a groupoidal composition we start by considering $\Lambda^2[2]$, the combinatorial simplex consisting of two 1-cells that touch at their end

$$\begin{array}{ccc} & 2 & \\ \nearrow & & \nwarrow \\ 0 & & 1 \end{array} ,$$

and $(g, h) : \Lambda^2[2] \to S$, a subset of S_1 reflecting the same situation in S. It is clear that the existence of an inverse g^{-1} for g, thanks of the above composition operation, is equivalent to the existence of a morphism f connecting the source of h to the source of g, as in the following picture

$$\begin{array}{ccc} & x_2 & \\ {\scriptstyle g}\nearrow & \Downarrow\ \simeq & \nwarrow{\scriptstyle h} \\ x_0 & \xleftarrow[f=g^{-1}\circ h]{} & x_1 \end{array}$$

On the other hand, this situation is also represented by the existence of diagrams of morphisms of simplicial sets of the form

$$\begin{array}{ccc} \Lambda^2[2] & \xrightarrow{(g,h)} & S \\ \big\downarrow & \nearrow_{\exists f} & \\ \Delta[2] & & \end{array} \tag{4.1.2}$$

Demanding that all the diagrams such (4.1.1) and (4.1.2) exist is therefore demanding that we have on 1-morphisms a composition operation with inverses in S.

In order for this to qualify as an ∞-groupoid, this composition operation needs to satisfy an associativity law up to 2-morphisms, which means that we can find the relevant tetrahedra in S. These last in turn need to be connected by pentagonators and so on. Fortunately it can be proven that all these coherence conditions are captured by generalizing the above conditions to all dimensions in the evident way:

Definition 4.1.1. *Let $\Lambda^i[n]$ be the i-th n-horn, that is the combinatorial simplex consisting of all cells of the n-simplex $\Delta[n]$ except the interior n-morphism and the i-th $(n-1)$-morphism. A simplicial set K is called* Kan complex *if we can always find a horn filler σ in the diagram*

$$\begin{array}{ccc} \Lambda^i[n] & \xrightarrow{f} & K \\ \big\downarrow & \nearrow_{\exists \sigma} & \\ \Delta[n] & & \end{array}$$

Then we give the following

Definition 4.1.2. *An ∞-groupoid is a Kan complex. An n-groupoid is an n-truncated Kan complex, i.e., a Kan complex with no nontrivial simplices above dimension n.*

4.2 Sheaves, stacks and higher stacks

Let's now move to the second point, i.e. constructing sheaves of Kan complexes. The idea is to first define presheaves of Kan complexes, and then to define sheaves of these by imposing the usual sheaf conditions "up to homotopy" (in a coherent way).

To begin with, let us recall how to do homotopy theory with simplicial sets. Given a topological space X, one can associate to it a simplicial set $Sing(X)$, whose n-simpleces are precisely the continuous maps $|\Delta^n| \to X$, where $|\Delta^n| = \{(x_0, ..., x_n) \in [0,1]^{n+1} : x_0 + ... + x_n = 1\}$. Moreover, the singular complex functor $X \mapsto Sing(X)$ admits a left adjoint, which carries every simplicial set S to its *geometric realization* $|S|$. For every topological space X, the counit map $|Sing(X)| \to X$ is a weak homotopy equivalence. This means that, in studying topological spaces up to weak homotopy equivalence, we can as well work with simplicial sets.

Realizing simplicial sets as topological spaces, we can talk of their homotopy groups. For a simplicial set S and for a nonnegative integer k, we will consider as $\pi_k(S)$ just $\pi_k(|S|)$, the usual k-th homotopy group of its geometric realization. In particular, for $k = 0$ we see that $S \mapsto \pi_0(S)$: is a monoidal functor $\pi_0 : \mathtt{sSet} \to \mathtt{Set}$ that sends a simplicial set to the set of connected components of its geometric realization. The functor π_0 in turn induces a functor $H_0 : \mathtt{sSetCat} \to \mathtt{Cat}$ from the 2-category of simplicial categories (i.e., sSet-enriched categories) to the 2-category of categories. More explicitly, for C a simplicial category, $H_0(C)$ is the category with the same objects as C and with

$$H_0(C)(X, Y) := \pi_0 C(X, Y)$$

for all objects $X, Y \in C$. As a matter of notion, let us recall that a functor $f : C \to D$ between simplicial categories is called *Dwyer-Kan equivalence* if $H_0(f) : H_0(C) \to H_0(D)$ is essentially surjective and if for all objects $X, Y \in C$ the morphism of simplicial sets

$$f_{X,Y} : C(X, Y) \to D(f(X), f(Y))$$

is a weak homotopy equivalence.

The next ingredient we need is *simplicial localization*: for C a category with a notion of weak equivalences W, the *simplicial localization* of C at W is the universal simplicial category $L_W C$ together with a morphism $C \to L_W C$ with the property that every weak equivalence in C becomes an equivalence in $L_W C$. Universal clearly means that if D is another simplicial category with this property, then the morphism $C \to D$ factors through $C \to L_W C$.

In particular, consider the category $\mathtt{PSh}_{\mathtt{Kan}}$ of simplicial presheaves (over the site of smooth manifolds) taking values in Kan complexes, and choose as weak equivalences the *local homotopy equivalences*; namely, those morphisms $f : X \to Y$ in $\mathtt{PSh}_{\mathtt{Kan}}$ such that, for every manifold U and every point $x \in U$, there is an open neighbourhood $U_x \subseteq U$ of x in U such that $f(U_x) : X(U_x) \to Y(U_x)$ is a homotopy equivalence of Kan complexes. We will denote by $\mathtt{lh}$ the collection of local homotopy equivalences in $\mathtt{PSh}_{\mathtt{Kan}}$.

Definition 4.2.1. *The (higher) topos* $\mathbf{H}$ *of (higher) smooth stacks is the simplicial localization of* $\mathtt{PSh}_{\mathtt{Kan}}$ *with respect to local homotopy equivalences:*

$$\mathbf{H} := L_{\mathtt{lh}} \mathtt{PSh}_{\mathtt{Kan}}.$$

Notice that, since $\mathbf{H}$ is, by definition, a simplicial category, for every two (higher) stacks, the hom-space $\mathbf{H}(X, Y)$ is a simplicial set. Moreover, since every simplicial set is weakly equivalent to a Kan complex, it is not restrictive to assume that $\mathbf{H}(X, Y)$ is actually a Kan complex or, in a more evocative terminology, an ∞-groupoid. That being so, it's now clear how to forget the refined information about morphisms and higher morphisms of stacks contained in $\mathbf{H}$ and to consider only morphisms up to homotopy equivalence: all that we need is the functor H_0, which permits us to look at the category $H_0(\mathbf{H})$, whose objects are the same objects of $\mathbf{H}$ and whose morphisms are given by $H_0(\mathbf{H})(X, Y) = \pi_0(\mathbf{H}(X, Y))$.

Let's now try to give a more concrete description of higher smooth satcks, based on local data. We first observe that, since $\mathbf{H}$ is a simplicial localization of $\mathtt{PSh}_{\mathtt{Kan}}$, it comes equipped, by definition, with a morphism

$$\mathtt{PSh}_{\mathtt{Kan}} \to \mathbf{H}.$$

This morphism takes a simplicial presheaf Φ on smooth manifolds (taking values in Kan complexes) to a smooth stack $\mathbf{\Phi}$, and it is called the *stackification* functor. Being a smooth (higher) stack, $\mathbf{\Phi}$ will map a smooth manifold X to a Kan complex $\mathbf{\Phi}(X)$, and our aim, here, is giving a handy representation of the ∞-groupoid $\mathbf{\Phi}(X)$, in terms of the simplicial presheaf Φ. To begin with, for any smooth manifold U, we have a Kan complex $\Phi(U)$, with natural restriction maps $\Phi(U) \to \Phi(V)$ along inclusions $V \hookrightarrow U$ of smooth manifolds. Next, fix a good open cover $\mathcal{U}$ of X and let's start by describing the 0-simplices of $\mathbf{\Phi}(X)$ (these are to be thought of as the objects of the ∞-groupoid $\mathbf{\Phi}(X)$). A 0-simplex in $\mathbf{\Phi}(X)$ is a tuple $(\phi_\alpha, \phi_{\alpha\beta}, \phi_{\alpha\beta\gamma}, \dots)$ where:

- ϕ_α is a 0-simplex in $\Phi(U_\alpha)_0$ for any α;
- $\phi_{\alpha\beta}$ is a 1-simplex in $\Phi(U_{\alpha\beta})_1$ for any α, β, whose boundary 0-simpleces are the restrictions of ϕ_α and ϕ_β to $U_{\alpha\beta}$;
- $\phi_{\alpha\beta\gamma}$ is a 2-simplex in $\Phi(U_{\alpha\beta\gamma})_2$ for any α, β, γ, whose boundary 1-simpleces are the restrictions of $\phi_{\alpha\beta}$, $\phi_{\beta\gamma}$ and $\phi_{\gamma\alpha}$ to $U_{\alpha\beta\gamma}$;
- and so on.

The 1-simplices in $\mathbf{\Phi}(X)$ (to be thought of as 1-morphisms) with boundary 0-simplices $(\phi_\alpha, \phi_{\alpha\beta}, \phi_{\alpha\beta\gamma}, \dots)$ and $(\phi'_\alpha, \phi'_{\alpha\beta}, \phi'_{\alpha\beta\gamma}, \dots)$ are tuples $(\psi_\alpha, \psi_{\alpha\beta}, \psi_{\alpha\beta\gamma}, \dots)$ where:

- ψ_α is a 1-simplex in $\Phi(U_\alpha)_1$ for any α, whose boundary 0-simpleces are ϕ_α and ϕ'_α respectively;
- $\psi_{\alpha\beta}$ is a "square" (i.e., a pair of 2-simplices with a common edge) in $\Phi(U_{\alpha\beta})_2$, whose boundary 1-simplices are as in the following diagram

$$\begin{array}{ccc} \phi_\alpha|_{U_{\alpha\beta}} & \xrightarrow{\psi_\alpha} & \phi'_\alpha|_{U_{\alpha\beta}} \\ {\scriptstyle \phi_{\alpha\beta}}\downarrow & \nearrow & \downarrow{\scriptstyle \phi'_{\alpha\beta}} \\ \phi_\beta|_{U_{\alpha\beta}} & \xrightarrow[\psi_\beta]{} & \phi'_\beta|_{U_{\alpha\beta}} \end{array}$$

- and so on.

Similarly, one describes k-morphisms for any $k > 1$.

Remark 4.2.2. *We underline a natural but important fact: since localization is a functorial procedure a morphism $\phi : A \to B$ of simplicial presheaves induces a morphism $\phi : \mathbf{A} \to \mathbf{B}$ between their ∞-stackifications.*

With the above recipe at our fingertips, we can conveniently describe smooth stacks such as the higher stack of $U(1)$-n-bundle gerbes with and without connections that we have met in the previous Chapter, and more generally we can consider an arbitrary abelian Lie group A in place of $U(1)$. But first we introduce the easiest possible stack, the one associated to a smooth manifold X.

4.3 The stack $\mathbf{M}$ associated with a smooth manifold M

Let M be a smooth manifold. Then, by Yoneda lemma, M is naturally identified with a presheaf of sets over the site of smooth manifolds: the presheaf $\underline{M}$ mapping a smooth manifold U to the set $C^\infty(U, M)$ of smooth functions from U to M. Since every set can be seen as a 0-truncated Kan complex (i.e., as a Kan complex consisting of only vertices), we can look at $\underline{M}$ as to a simplicial presheaf, and consider its stackification $\mathbf{M}$. Since $\underline{M}$ is already a sheaf, it is not a big surprise that actually $\mathbf{M} = \underline{M}$. Namely, let X be a smooth manifold, and let $\mathcal{U}$ be a good open cover of X. Then a 0-simplex in $\mathbf{M}(X)$ consists in smooth maps $\phi_\alpha : U_\alpha \to M$ such that their restrictions $\phi_\alpha|_{U_{\alpha\beta}}$ and $\phi_\beta|_{U_{\alpha\beta}}$ are equal (since there are no nontrivial 1-simplices in $\underline{M}(U_{\alpha\beta})$). This means that the objects of the ∞-groupoid $\mathbf{M}(X)$ are just smooth maps from X to M. Moreover, again since the Kan complexes where the simplicial presheaf $\underline{M}$ takes its values are barely sets, the only morphisms between the objects of $\mathbf{M}(X)$ are the identities. We therefore see that $\mathbf{M}(X)$ is nothing but the set $C^\infty(X, M)$ seen as an ∞-groupoid. In other words, $\mathbf{M}$ is nothing but the image of M via the Yoneda embedding:

$$\mathbf{M} = C^\infty(-, M).$$

By this reasoning, we will identify a smooth manifold M and the stack $\mathbf{M}$ it defines and will denote them both by the symbol M. Note that by the Yoneda lemma we also have a natural equivalence

$$\mathbf{\Phi}(M) \simeq \mathbf{H}(M, \mathbf{\Phi}) \tag{4.3.3}$$

for any smooth stack $\mathbf{\Phi}$.

We now move to G-principal bundles, from which our short overview of smooth stacks was originated.

4.4 The stack $\mathbf{B}G$ of G- principal bundles

As observed before, a Lie group G defines a presheaf of Kan complexes mapping a smooth manifold U to the 1-groupoid $\{*//C^\infty(U,G)\}$ with a unique object $*$ having the group $C^\infty(U,G)$ as group of automorphisms. Geometrically, $*$ corresponds to the trivial G-principal bundle over U, whose group of automorphisms is indeed $C^\infty(U,G)$. Let $\mathbf{B}G \in \mathbf{H}$ be the corresponding smooth stack. For X a smooth manifold with a good cover $\mathcal{U}$, an object of $\mathbf{B}G(X) \simeq \mathbf{H}(X, \mathbf{B}G)$ is given by a collection of smooth functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$ for any α, β such that $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ on $U_{\alpha\beta\gamma}$. This latter condition is a manifestation that in the simplicial presheaf $U \mapsto \{*//C^\infty(U,G)\}$ there are no nontrivial 2-simplices. Looking at morphisms, we see that for $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ two G-valued cocycles over X, a 1-morphism between them consists in a collection of maps h_α in $C^\infty(U_\alpha, G)$ satisfying the "square" condition. Since there are no nontrivial 2-simplices, the square condition becomes $h_\alpha g'_{\alpha\beta} = g_{\alpha\beta}h_\beta$, which is precisely the local form of an isomorphism of G-principal bundles. That is, $\mathbf{B}G$ is precisely the stack of principal G-bundles. The set of connected components of the hom-space $\mathbf{H}(X, \mathbf{B}G)$ is clearly isomorphic to the set of isomorphism classes of principal G-bundles so that we find

$$H^1(X, \underline{G}) = \pi_0\mathbf{H}(X, \mathbf{B}G). \tag{4.4.4}$$

4.5 The stack $\mathrm{B}G_{conn}$ of G- principal bundles with connections

Let G and $\mathfrak{g}$ be a Lie group and its Lie algebra respectively. Is then defined a presheaf of Kan complexes which sends a smooth manifold U to the 1-groupoid $\{\Omega^1(U,\mathfrak{g})//C^\infty(U,G)\}$ which has the set of $\mathfrak{g}$-valued 1-forms on U as objects and with the group $C^\infty(U,G)$ as set of morphisms, where an element $g \in C^\infty(U,G)$ acts on $\Omega^1(U,\mathfrak{g})$ by the law

$$A \mapsto g^{-1}Ag + g^{-1}dg.$$

Let $\mathbf{B}G_{\mathrm{conn}} \in \mathbf{H}$ be the corresponding smooth stack. We want to show that, as the name suggests, $\mathbf{B}G_{\mathrm{conn}}$ is precisely the stack of principapl G-bundles with $\mathfrak{g}$-connections. To this aim, let again X be a smooth manifold with a good cover $\mathcal{U}$: an object of $\mathbf{B}G_{\mathrm{conn}}(X) \simeq \mathbf{H}(X, \mathbf{B}G_{\mathrm{conn}})$ is given by the following local data:

- 1-forms $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$ for any α;
- smooth functions $g_{\alpha\beta} \in C^\infty(U_{\alpha\beta}, G)$ for any α, β such that

$$A_\beta\Big|_{U_{\alpha\beta}} = g_{\alpha\beta}^{-1}A_\alpha\Big|_{U_{\alpha\beta}} g_{\alpha\beta} + g_{\alpha\beta}^{-1}dg_{\alpha\beta}$$

  and

$$g_{\alpha\beta}\Big|_{U_{\alpha\beta\gamma}} g_{\beta\gamma}\Big|_{U_{\alpha\beta\gamma}} g_{\gamma\alpha}\Big|_{U_{\alpha\beta\gamma}} = 1.$$

These are manifestly the local data describing a principal G-bundle with $\mathfrak{g}$ connection on X. But actually the above description only gives the objects of $\mathbf{B}G_{\mathrm{conn}}(X)$, so let us have a look at the morphisms between these objects. Since the presheaf $U \mapsto \{\Omega^1(U,\mathfrak{g})//C^\infty(U,G)\}$ takes values in Kan complexes which are 1-truncated (i.e., they are 1-groupoids), we will have no nontrivial k-morphisms in $\mathbf{B}G_{\mathrm{conn}}(X)$ for $k \geq 2$, and so we have just to specify what a 1-morphism between two cocycles $(A_\alpha, g_{\alpha\beta})$ and $(A'_\alpha, g'_{\alpha\beta})$ is. To do so, we gives its local presentation as it follows from the general argument spelled out in Section 4.2:

- smooth functions $h_\alpha \in C^\infty(U_\alpha, G)$ for any α such that, over U_α, we have:

  $$A'_\alpha = h_\alpha^{-1} A_\alpha h_\alpha + h_\alpha^{-1} dh_\alpha,$$

  such that all the diagrams

  $$\begin{array}{ccc} A_\alpha|_{U_{\alpha\beta}} & \xrightarrow{h_\alpha} & A'_\alpha|_{U_{\alpha\beta}} \\ {\scriptstyle g_{\alpha\beta}}\Big\downarrow & & \Big\downarrow{\scriptstyle g'_{\alpha\beta}} \\ A_\beta|_{U_{\alpha\beta}} & \xrightarrow[h_\beta]{} & A'_\beta|_{U_{\alpha\beta}} \end{array}$$

  commute (these are the "squares" conditions)

  We therefore see that not only the objects of $\mathbf{B}G_{\mathrm{conn}}(X)$ are principal G-bundles with $\mathfrak{g}$-connections on X, but also morphisms in $\mathbf{B}G_{\mathrm{conn}}(X)$ are precisely the isomorphisms of principal G-bundles with connections on X. In other words $\mathbf{B}G_{\mathrm{conn}}$ is precisely the stack mapping a smooth manifold X to the groupoid of principal G-bundles with connections on it.

Remark 4.5.1. *The evident morphism of simplicial presheaves*

$$\left\{\Omega^1(U, \mathfrak{g})//C^\infty(U, G)\right\} \to \{*//C^\infty(U, G)\}$$

induces the forgetful morphism of stacks $\mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}G$ which "forgets" the connection.

Remark 4.5.2. *When $G = A$ is an abelian Lie group, with Lie algebra $\mathfrak{a}$, the local data for a principal A-bundle with connection simplify to*

- *1-forms $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$ for any α;*

- *smooth functions $g_{\alpha\beta} \in C^\infty(U_{\alpha\beta}, A)$ for any α, β such that*

  $$A_\beta\Big|_{U_{\alpha\beta}} = A_\alpha\Big|_{U_{\alpha\beta}} + g_{\alpha\beta}^{-1} dg_{\alpha\beta}$$

  and

  $$g_{\alpha\beta}\Big|_{U_{\alpha\beta\gamma}} g_{\beta\gamma}\Big|_{U_{\alpha\beta\gamma}} g_{\gamma\alpha}\Big|_{U_{\alpha\beta\gamma}} = 1.$$

We have met these equations in the case of $G = U(1)$ in Chapter 3.

If $\mathbf{B}G$ itself has a group structure, we can form $\mathbf{B}^2G$, and so on. This is just the case in which $G = A$ is an abelian Lie group: for instance we have met $\mathbf{B}^2U(1)$ and its generalization $\mathbf{B}^nU(1)$ when dealing with $U(1)$-bundle gerbes in Chapter 2. It follows immediately from the local data description of $\mathbf{H}(X, \mathbf{B}^nA)$ that objects in $\mathbf{H}(X, \mathbf{B}^nA)$ are Čech cocycles on X with coefficients in the sheaf of abelian groups $\underline{A} = C^\infty(-; A)$. Also, morphisms between these cocycles are precisely given by coboundaries, so that

$$\pi_0\mathbf{H}(X, \mathbf{B}^nA) = H^n(X, \underline{A}), \tag{4.5.5}$$

see [18, 19] and [21] for details. In the particular case $G = U(1)$ we therefore find

$$\pi_0\mathbf{H}(X, \mathbf{B}^nU(1)) = H^n(X, \underline{U(1)}) = H^{n+1}(X, \mathbb{Z}), \tag{4.5.6}$$

and we recover the integral cohomology of X.

As we just saw, when $G = A$ is an abelian Lie group, we have a whole collection $\mathbf{B}^nA$ of higher stacks generalizing $\mathbf{B}G$. This suggests that we may also have a whole collection of higher stacks $\mathbf{B}^nA_{\mathrm{conn}}$ generalizing $\mathbf{B}G_{\mathrm{conn}}$. It is indeed so, but presenting $\mathbf{B}^nA_{\mathrm{conn}}$ directly in terms of its local data is not very enlightening because of hard visualization of k-morphisms for $k \geq 1$. However, there is a useful tool to represent higher stacks with an intrinsic abelian structure such as $\mathbf{B}^nA$ and $\mathbf{B}^nA_{\mathrm{conn}}$, the so called *Dold-Kan correspondence* that we now briefly introduce and that is the perfect tool for our aims.

4.6 The Dold-Kan correspondence

There is a classical Dold-Kan correspondence, at an algebraic level,

$$\mathtt{Ch}_{\bullet\geq 0} \xrightarrow{\Gamma} \mathtt{sAb}$$

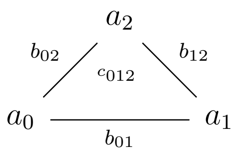
which establishes an equivalence of categories between chain complexes concentrated in non-negative degrees and simplicial abelian groups. Given the chain complex $A_\bullet$

$$\cdots \xrightarrow{\partial} A_3 \xrightarrow{\partial} A_2 \xrightarrow{\partial} A_1 \xrightarrow{\partial} A_0,$$

the simplicial abelian group $\Gamma(A_\bullet)$ is defined as follows:

- the group of 0-simplices of $\Gamma(A_\bullet)$ is the abelian group A_0;
- the group of n-simplices of $\Gamma(A_\bullet)$ is the abelian group whose elements are standard n-simplices decorated by an element $x \in A_n$ such that ∂x equals the oriented sum of the decorations on the boundary $(n-1)$-simplices.

For instance, a 2-simplex in $\Gamma(A_\bullet)$ is a decorated 2-simplex



where

- $a_i \in A_0$;
- $b_{ij} \in A_1$ and $\partial b_{ij} = a_j - a_i$;
- $c_{012} \in A_2$ and $\partial c_{012} = b_{12} - b_{02} + b_{01}$.

Then there is the forgetful functor

$$F : \mathtt{sAb} \to \mathtt{sSet}$$

which forgets the group structure on a simplicial abelian group and just remember the underlying simplicial set, which in turn is guaranteed to be a Kan complex. Denoting by DK the composition of Γ and F we obtain the *Dold-Kan correspondence*:

$$DK : \mathtt{Ch}_{\bullet \geq 0} \xrightarrow{\Gamma} \mathtt{sAb} \xrightarrow{F} \mathtt{Kan}.$$

All this directly prolongs to presheavs of chain complexes and presheaves of simplicial abelian groups on smooth manifolds. Then, by using the same symbols, we have

$$DK : \mathtt{PSh}_{\mathtt{Ch}_{\bullet\geq 0}} \xrightarrow{\Gamma} \mathtt{PSh}_{\mathtt{sAb}} \xrightarrow{F} \mathtt{PSh}_{\mathtt{Kan}}.$$

If $G = A$ is an abelian Lie group, then for any nonnegative integer n one can consider the presheaf of chain complexes $\underline{A}[n] = C^\infty(-, A)[n]$

$$[...\to 0 \to \underline{A} \to 0 \to ... \to 0],$$

with $\underline{A}$ placed in degree n. By applying the Dold-Kan map to this presheaf one gets a simplicial presheaf whose stackification is the n-stack $\mathbf{B}^n A$ introduced above.

In a similar manner we can consider the simplicial presheaf whose stackification will give the n-stack $\mathbf{B}^n A_{\mathrm{conn}}$. To do this, notice that a connected abelian Lie group A is of the form $A = U(1)^{n_1} \times \mathbb{R}^{n_2}$ for suitable nonnegative integers n_1 and n_2, so the Lie algebra $\mathfrak{a}$ of A is a copy of $\mathbb{R}^{n_1+n_2}$. Define the differential

$$d_A \colon C^\infty(-; A) \to \Omega^1(-; \mathfrak{a})$$

as

$$d_A = \left(\underbrace{\frac{1}{2\pi i} d\log, \dots \frac{1}{2\pi i} d\log}_{n_1 \text{ times}}, \underbrace{d, \dots, d}_{n_2 \text{ times}} \right)$$

The chain complex to consider is now the complex $\underline{A}[n]_{\mathrm{conn}}$ given by

$$\dots \to 0 \to \underline{A} \to \Omega^1(-, \mathfrak{a}) \to \Omega^2(-, \mathfrak{a}) ... \to \Omega^n(-, \mathfrak{a}),$$

with $\underline{A}$ is placed in degree n. By Dold-Kan and stackification this produces the n-stack $\mathbf{B}^n A_{\mathrm{conn}}$ of principal A-$(n-1)$-gerbes with connection.

For $A = U(1)$ and $n = 1, 2$, this reproduces the local data description of the stacks $\mathbf{B}U(1)_{\mathrm{conn}}$ and $\mathbf{B}^2 U(1)_{\mathrm{conn}}$ we met in Chapters 2 and 3. Also notice that, by construction, objects in $\mathbf{H}(X, \mathbf{B}^n U(1)_{\mathrm{conn}})$ are degree n Čech-Deligne cocycles on X, while morphisms between them are Čech-Deligne coboundaries. It follows that

$$\pi_0 \mathbf{H}(X, \mathbf{B}^n U(1)_{\mathrm{conn}}) = \hat{H}^{n+1}(X; \mathbb{Z}),$$

the $(n+1)$-th differential cohomology group of X; see [2].

Chapter 5

From String to $\mathcal{L}$Spin

We have now collected all the ingredients we needed and we can start to describe our proof of (a version of) Waldorf's result on transgression of String structures to Spin structures on loop spaces.

5.1 The smooth refinement of the first fractional Pontryagin class

Recall from the first Chapter that, for a Spin manifold X, if the composite morphism

$$X \stackrel{f_{TX}}{\to} BSpin(n) \stackrel{\frac{1}{2}p_1}{\to} K(\mathbb{Z}, 4)$$

is homotopically trivial, then the classifying map of TX can be lifted to $X \stackrel{f_{TX}}{\to} BString(n)$, where $BString(n)$ denotes the homotopy fiber of $BSpin(n) \stackrel{\frac{1}{2}p_1}{\to} K(\mathbb{Z}, 4)$: doing so, X becomes a String manifold. Exactly as the classifying space $BU(1)$, which is the topological realization of the stack $\mathbf{B}U(1)$, is a $K(\mathbb{Z}, 2)$, the classifying space $B^3U(1)$, which is the topological realization of the 3-stack $\mathbf{B}^3U(1)$, is a $K(\mathbb{Z}, 4)$. This means that we can write the homotopy pullback defining the classifying space $BString$ of principal String-bundles as

$$\begin{array}{ccc} BString & \longrightarrow & * \\ \big\downarrow & & \big\downarrow \\ BSpin & \xrightarrow[\frac{1}{2}p_1]{} & B^3U(1) \end{array} \tag{5.1.1}$$

(note that, from now on, the dimension n of X will not be mentioned). This choice of $B^3U(1)$ as a $K(\mathbb{Z},4)$ could look like a matter of notation, but it suggests how to define the higher stack $\mathbf{B}String$ of principal String-bundles. Namely, as soon as one is able to refine the first fractional Pontryagin class

$$\frac{1}{2}p_1 \colon BString \to B^3U(1)$$

to a morphism of smooth stacks

$$\frac{1}{2}\mathbf{p}_1 \colon \mathbf{B}String \to \mathbf{B}^3U(1),$$

then one can define the 2-stack $\mathbf{B}String$ as the homotopy pullback

$$\begin{array}{ccc} \mathbf{B}String & \longrightarrow & * \\ \big\downarrow & & \big\downarrow \\ \mathbf{B}Spin & \xrightarrow[\frac{1}{2}\mathbf{p}_1]{} & \mathbf{B}^3U(1) \end{array} \quad . \tag{5.1.2}$$

That a smooth refinement $\frac{1}{2}\mathbf{p}_1$ actually exists is shown in [8], where it is obtained via the Lie integration of the canonical Lie 3-cocycle on the Lie algebra $\mathfrak{so}$ of the Spin group. When explicited in terms of Čech cocycles, the morphism $\frac{1}{2}\mathbf{p}_1$ turns out to coincide with the Brylinski-McLaughlin construction of a Čech cocycle representing the first fractional Pontryagin class [3, 4]. In terms of Čech cocycles, $\frac{1}{2}\mathbf{p}_1$ is a map

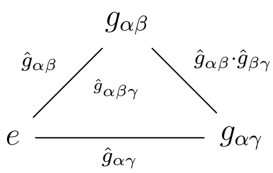
$$\{g_{\alpha\beta}\} \mapsto \{h_{\alpha\beta\gamma\delta}\}, \tag{5.1.3}$$

where $g_{\alpha\beta}$ is a Čech cocycle for a Spin-principal bundle and $h_{\alpha\beta\gamma\delta}$ is a degree 3 Čech cocycle with values in $U(1)$. To begin with, one considers an extension $\hat{g}_{\alpha\beta}$ of the transition functions $g_{\alpha\beta}$ such that:

- over double intersections $\hat{g}_{\alpha\beta} : U_{\alpha\beta} \times \Delta^1 \to Spin$ is a smooth family of based paths in Spin with $\hat{g}_{\alpha\beta}(1) = g_{\alpha\beta}$ and $\hat{g}_{\alpha\beta}(0) = e$.

This is possible since $Spin$ is a connected group and the $U_{\alpha\beta}$ are contractible. Next one considers suitable fillers:

- over triple intersections $\hat{g}_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \times \Delta^2 \to Spin$ is a smooth family of 2-simplices in Spin with boundaries labeled by the basepoint paths on double overlaps:



This is possible since $Spin$ is a simply connected group and the $U_{\alpha\beta\gamma}$ are contractible. Finally,

- over quadruple intersections $\hat{g}_{\alpha\beta\gamma\delta} : U_{\alpha\beta\gamma\delta} \times \Delta^3 \to Spin$ is a smooth family of 3-simplices in Spin, cobounding the union of the 2-simplices corresponding to the triple intersections.

This is possible since $\pi_2(Spin) = 0$ and the $U_{\alpha\beta\gamma\delta}$ are contractible. To complete the construction one considers the canonical 3-form $\langle[\theta_{Spin}, \theta_{Spin}], \theta_{Spin}\rangle \in \Omega^3(Spin, \mathbb{R})$, where $\langle -, - \rangle$ is the Killing form of $\mathfrak{so}$ and θ_{Spin} is the Maurer-Cartan form of the $Spin$ group, and integrates it over the 3-symplex Δ^3. Doing so, one gets a Čech 3-cochain with values in $\mathbb{R}$, which becomes a Čech 3-cocycle with values in $U(1)$ after reduction modulo $\mathbb{Z}$. This is precisely the 3-cocycle $\{h_{\alpha\beta\gamma\delta}\}$ we were after.

5.2 Lifting $\frac{1}{2}\mathbf{p}_1$ to $\mathbf{B}^2(\mathbf{B}U(1)_{\mathrm{conn}})$

The crucial result for the remainder of our work is the following

Theorem 5.2.1. *The first fractional Pontryagin class can be lifted along the forgetful morphism* $\mathbf{B}^2(\mathbf{B}U(1)_{conn}) \to \mathbf{B}^3U(1)$ *to a map* $\mathbf{B}Spin \stackrel{\frac{1}{2}\tilde{\mathbf{p}}_1}{\to} \mathbf{B}^2(\mathbf{B}U(1)_{\mathrm{conn}})$.

Proof. This lift is realized in two steps:

1. one has a differential refinement $\mathbf{B}Spin_{\mathrm{conn}} \stackrel{\frac{1}{2}\hat{\mathbf{p}}_1}{\to} \mathbf{B}^3U(1)_{conn}$ of $\frac{1}{2}\mathbf{p}_1$, giving a commutative diagram of stacks

$$\begin{array}{ccc} \mathbf{B}Spin_{\mathrm{conn}} & \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}_1} & \mathbf{B}^3U(1)_{conn} \\ \downarrow & & \downarrow \\ \mathbf{B}Spin & \xrightarrow{\frac{1}{2}\mathbf{p}_1} & \mathbf{B}^3U(1) \end{array}$$

2. the morphism $\frac{1}{2}\hat{\mathbf{p}}_1$ descends to a morphism $\frac{1}{2}\tilde{\mathbf{p}}_1$ fitting the commutative diagram

$$\begin{array}{ccc} \mathbf{B}Spin_{\mathrm{conn}} & \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}_1} & \mathbf{B}^3U(1)_{conn} \\ \Big\downarrow & & \downarrow \\ \Big\downarrow & & \mathbf{B}^2(\mathbf{B}U(1)_{conn}) \\ \Big\downarrow & \nearrow^{\frac{1}{2}\tilde{\mathbf{p}}_1} & \downarrow \\ \mathbf{B}Spin & \xrightarrow{\frac{1}{2}\mathbf{p}_1} & \mathbf{B}^3U(1) \end{array} .$$

Much of the work consists of an accurate local description of the map $\frac{1}{2}\hat{\mathbf{p}}_1$. This map is obtained in [8] by means of Lie-integration of the triple consisting of the canonical Lie 3-cocycle on $\mathfrak{so}$, of the degree 4 invariant polynomial associated with it, and of the Chern-Simons element witnessing the transgression between the two.[1] When explicited in terms of Čech cocycles and local connection forms, this turns out to coincide with the Brylinski-MacLaughlin Čech cocycles description of the refinement of the first fractional Pontryagin class to a cohomology class in differential cohomology [3, 4]. In order to prove that $\frac{1}{2}\hat{\mathbf{p}}_1$ descends to $\frac{1}{2}\tilde{\mathbf{p}}_1$ we will need this explicit cocycle description of the map, so we recall it:

[1] For the sake of completeness, we report a brief account of the Lie-integration construction of the map $\frac{1}{2}\hat{\mathbf{p}}_1$ in the Appendix.

- the input is a set of transition functions and local connection data for a $Spin$-principal bundle with connection on a smooth manifold X; namely, we have smooth functions $g_{\alpha\beta}\colon U_{\alpha\beta} \to Spin$ and local 1-forms $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{so})$ on a good open cover $\{U_\alpha\}$ of X, satisfying the cocycle condition $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ on the triple intersections $U_{\alpha\beta\gamma}$ and the compatibility condition $A_\beta = g_{\alpha\beta}^{-1}A_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1}dg_{\alpha\beta}$ on the double intersections $U_{\alpha\beta}$.

- then we produce an extension of this data:
  - on double intersections we pick a smooth family $\hat{g}_{\alpha\beta} : U_{\alpha\beta} \times \Delta^1 \to Spin$ of based paths in $Spin$, together with a 1-form $\hat{A}_{\alpha\beta} = \hat{g}_{\alpha\beta}^{-1}A_\alpha\hat{g}_{\alpha\beta} + \hat{g}_{\alpha\beta}^{-1}d\hat{g}_{\alpha\beta} \in \Omega^1(U_{\alpha\beta} \times \Delta^1, \mathfrak{so})$;
  - on triple intersections a smooth family $\hat{g}_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \times \Delta^2 \to Spin$ of based 2-simplices in $Spin$, together with a 1-form $\hat{A}_{\alpha\beta\gamma} = \hat{g}_{\alpha\beta\gamma}^{-1}A_\alpha\hat{g}_{\alpha\beta\gamma} + \hat{g}_{\alpha\beta\gamma}^{-1}d\hat{g}_{\alpha\beta\gamma} \in \Omega^1(U_{\alpha\beta\gamma} \times \Delta^2, \mathfrak{so})$;
  - on quadruple intersections a smooth family $\hat{g}_{\alpha\beta\gamma\delta} : U_{\alpha\beta\gamma\delta} \times \Delta^3 \to Spin$ of based 3-simplices in $Spin$, together with a 1-form $\hat{A}_{\alpha\beta\gamma\delta} = \hat{g}_{\alpha\beta\gamma\delta}^{-1}A_\alpha\hat{g}_{\alpha\beta\gamma\delta} + \hat{g}_{\alpha\beta\gamma\delta}^{-1}d\hat{g}_{\alpha\beta\gamma\delta} \in \Omega^1(U_{\alpha\beta\gamma\delta} \times \Delta^3, \mathfrak{so})$;[2]

- this extended cocycle datum is sent to the Čech-Deligne cocycle

$$(\mathrm{cs}(A_\alpha), \int_{\Delta^1} \mathrm{cs}(\hat{A}_{\alpha\beta}), \int_{\Delta^2} \mathrm{cs}(\hat{A}_{\alpha\beta\gamma}), \int_{\Delta^3} \mathrm{cs}(\hat{A}_{\alpha\beta\gamma\delta}) \bmod \mathbb{Z})\,,$$

  where $\mathrm{cs}(A)$ is the Chern-Simons 3-form obtained by evaluating a $\mathfrak{so}$-valued 1-form A in the Chern-Simons element cs.

Composing with the forgetful morphism $\mathbf{B}^3U(1)_{\mathrm{conn}} \to \mathbf{B}^2(\mathbf{B}U(1)_{\mathrm{conn}})$ we obtain the morphism $\mathbf{B}Spin_{\mathrm{conn}} \to \mathbf{B}^2(\mathbf{B}U(1)_{\mathrm{conn}})$ given in terms of local data by the the Čech-Deligne cocycle

$$(0, 0, \int_{\Delta^2} \mathrm{cs}(\hat{A}_{\alpha\beta\gamma}), \int_{\Delta^3} \mathrm{cs}(\hat{A}_{\alpha\beta\gamma\delta}) \bmod \mathbb{Z})\,.$$

[2]Note that the $\hat{g}_{\alpha\beta}$, $\hat{g}_{\alpha\beta\gamma}$, $\hat{g}_{\alpha\beta\gamma\delta}$ are the same local data used in the previous paragraph to locally describe the smooth refinement of $\frac{1}{2}\mathbf{p}_1$. In particular, they satisfy the same boundary conditions.

We want to show that this cocycle is, up to homotopy, independent of the connection data $\{A_\alpha\}$, so that the morphism $\mathbf{B}Spin_{\mathrm{conn}} \to \mathbf{B}^2(\mathbf{B}U(1)_{\mathrm{conn}})$ descends to a morphism $\mathbf{B}G \to \mathbf{B}^2(\mathbf{B}U(1)_{\mathrm{conn}})$, which lifts the "topological" morphism $\mathbf{B}Spin \to \mathbf{B}^3U(1)$.
This amounts to show that the local $U(1)$-valued 0-form

$$\int_{\Delta^3} \mathrm{cs}(\hat{A}_{\alpha\beta\gamma\delta}) \bmod \mathbb{Z}$$

is independent of $\{A_\alpha\}$, and the local 1-form

$$\int_{\Delta^2} \mathrm{cs}(\hat{A}_{\alpha\beta\gamma})$$

depends on $\{A_\alpha\}$ only through a d-exact term. Let us start with the 0-form. The integration over Δ^3 only sees the part of $\mathrm{cs}(\hat{A}_{\alpha\beta\gamma\delta})$ which is a vertical 3-form. For a smooth map $g\colon U \to Spin$ we have

$$g^{-1}dg = g^*\theta,$$

where θ is the Maurer-Cartan form of $Spin$. So the explicit expression for $\hat{A}_{\alpha\beta\gamma\delta}$ is

$$\hat{A}_{\alpha\beta\gamma\delta} = \hat{g}^{-1}_{\alpha\beta\gamma\delta} A_\alpha \hat{g}_{\alpha\beta\gamma\delta} + \hat{g}^*_{\alpha\beta\gamma\delta}\theta,$$

and from this one immediately sees that the vertical 3-form part of $\mathrm{cs}(\hat{A}_{\alpha\beta\gamma\delta})$ is

$$\langle d\hat{g}^*_{\alpha\beta\gamma\delta}\theta \wedge g^*_{\alpha\beta\gamma\delta}\theta\rangle + \frac{2}{3}\langle[\hat{g}^*_{\alpha\beta\gamma\delta}\theta, \hat{g}^*_{\alpha\beta\gamma\delta}\theta] \wedge \hat{g}^*_{\alpha\beta\gamma\delta}\theta\rangle = \frac{1}{6} g^*_{\alpha\beta\gamma\delta}\langle[\theta,\theta]\wedge\theta\rangle,$$

which is manifestly independent of $\{A_\alpha\}$.
Now we come to the components of the 3-form $\mathrm{cs}(\hat{A}_{\alpha\beta\gamma})$ with a vertical 2-form part. We want to show that this is an exact term (with an explicit primitive given in terms of the connection data). A simple computation shows that this vertical 3-form is

$$d\langle \hat{A}_{\alpha\beta\gamma} \wedge g^*_{\alpha\beta\gamma}\theta\rangle + 2\langle g^*_{\alpha\beta\gamma}\left(d\theta + \frac{1}{2}[\theta,\theta]\right) \wedge \hat{A}_{\alpha\beta\gamma}\rangle.$$

Since the Maurer-Cartan form θ satisfies the Maurer-Cartan equation

$$d\theta + \frac{1}{2}[\theta,\theta] = 0,$$

the above expression reduces to the d-exact term

$$d\langle \hat{A}_{\alpha\beta\gamma} \wedge g^*_{\alpha\beta\gamma}\theta\rangle.$$

This concludes the proof.

□

5.3 A homotopy commuting diagram involving $\mathcal{L}String$ and $\mathcal{L}Spin$

Once the morphism of stacks $\mathbf{B}Spin \to \mathbf{B}^2(\mathbf{B}U(1)_{\text{conn}})$ has been exhibited, the conclusion of the proof of the transgression from String structures to loop Spin structures reduces to a repeated application of the pasting law for homotopy pullbacks which we recall below.

Lemma 5.3.1. *Consider a diagram of the form*

$$\begin{array}{ccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} \\ Y_1 & \xrightarrow[g_1]{} & Y_2 & \xrightarrow[g_2]{} & Y_3 \end{array} \tag{5.3.4}$$

with three commuting squares, the two inner ones and the outer one. Suppose the right-hand inner square is a (homotopy) pullback: then the left-hand one is a (homotopy) pullback if and only if the outer square is.

To begin with, recall that the stack $\mathbf{B}String$ is defined as the homotopy pullback

$$\begin{array}{ccc} \mathbf{B}String & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}Spin & \xrightarrow[\frac{1}{2}\mathbf{p_1}]{} & \mathbf{B^3}U(1) \end{array} . \tag{5.3.5}$$

By Theorem 5.2.1 we can factor the bottom horizontal arrow as

$$\mathbf{B}Spin \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}1} \mathbf{B}^2(\mathbf{B}U(1)_{\text{conn}}) \to \mathbf{B}^3U(1)$$

and so the whole diagram (5.3.5) as

$$\begin{array}{ccccc} \mathbf{B}String & \longrightarrow & \mathbf{S} & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{B}Spin & \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}1} & \mathbf{B}^2(\mathbf{B}U(1)_{conn}) & \longrightarrow & \mathbf{B}^3(U(1)) \end{array} \tag{5.3.6}$$

for some stack $\mathbf{S}$ which is the homotopy pullback

$$\begin{array}{ccc} \mathbf{S} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}^2(\mathbf{B}U(1)_{\mathrm{conn}}) & \longrightarrow & \mathbf{B}^3(U(1)) \end{array} .$$

Our next task is, therefore, to identify the stack $\mathbf{S}$.

Lemma 5.3.2. *The stack* $\mathbf{S}$ *in diagram (5.3.6) is the stack* $\mathbf{B}^2\Omega^1$*, obtained via Dold-Kan correspondence from the chain complex of sheaves of abelian groups*

$$\cdots \to 0 \to 0 \to \Omega^1(-;\mathbb{R}) \to 0 \to 0,$$

with $\Omega^1(-;\mathbb{R})$ *in degree 2.*

Proof. The frogetful morphism $\mathbf{B}U(1)_{\mathrm{conn}} \to \mathbf{B}U(1)$ is induced via Dold-Kan correspondence by the morphism of chain complexes

$$\begin{array}{ccc} C^\infty(-,U(1)) & \longrightarrow & \Omega^1(-;\mathbb{R}) \\ \downarrow & & \downarrow \\ C^\infty(-,U(1)) & \longrightarrow & 0 \end{array} ,$$

whose kernel is the chain complex

$$\cdots \to 0 \to 0 \to \Omega^1(-;\mathbb{R}).$$

By Dold-Kan correspondence once again, this tells us that we have a fiber sequence[3]

$$\begin{array}{ccc} \Omega^1 & \longrightarrow & * \\ \big\downarrow & & \big\downarrow \\ \mathbf{B}U(1)_{\mathrm{conn}} & \longrightarrow & \mathbf{B}U(1) \end{array} .$$

Since this fiber sequence is obtained via Dold-Kan by a short exact sequence of chain complexes of abelian groups, it can be delooped, giving the fiber sequence

$$\begin{array}{ccc} \mathbf{B}^2\Omega^1 & \longrightarrow & * \\ \big\downarrow & & \big\downarrow \\ \mathbf{B^2}(\mathbf{B}U(1)_{\mathrm{conn}}) & \longrightarrow & \mathbf{B^3}(U(1)) \end{array}$$

expressing $\mathbf{B}^2\Omega^1$ as the homotopy fiber of the forgetful morphism $\mathbf{B}^2(\mathbf{B}U(1)_{\mathrm{conn}}) \to \mathbf{B^3}(U(1))$. □

Summing up, we have obtained the factorization

$$\begin{array}{ccccc} \mathbf{B}String & \longrightarrow & \mathbf{B}^2\Omega^1 & \longrightarrow & * \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathbf{B}Spin & \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}1} & \mathbf{B^2}(\mathbf{B}U(1)_{conn}) & \longrightarrow & \mathbf{B^3}(U(1)) \end{array}$$

of diagram (5.3.5), where both squares are homotopy pullbacks. We now focus on the left square, i.e., on the homotopy pullback

$$\begin{array}{ccc} \mathbf{B}String & \longrightarrow & \mathbf{B}^2\Omega^1 \\ \big\downarrow & & \big\downarrow \\ \mathbf{B}Spin & \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}1} & \mathbf{B^2}(\mathbf{B}U(1)_{conn}) \end{array} . \qquad (5.3.7)$$

[3] i.e., a homotopy pullback in which one of the factors is the terminal stack $*$

We apply the internal hom functor $[S^1, -]$ to diagram (5.3.7). Since the internal hom preserves homotopy pullbacks, we get a new homotopy pullback diagram, namely

$$\begin{array}{ccc} [S^1, \mathbf{B}String] & \longrightarrow & [S^1, \mathbf{B}^2\Omega^1] \\ \downarrow & & \downarrow \\ [S^1, \mathbf{B}Spin] & \longrightarrow & [S^1, \mathbf{B}^2(\mathbf{B}U(1)_{conn})] \end{array} \tag{5.3.8}$$

Lemma 5.3.3. *Integration/holonomy induces a homotopy commutative diagram of stacks*

$$\begin{array}{ccc} [S^1, \Omega^1] & \xrightarrow{\int_{S^1}} & \mathbb{R} \\ \downarrow & & \downarrow{\scriptstyle \exp(2\pi i -)} \\ [S^1, \mathbf{B}U(1)_{\mathrm{conn}}] & \xrightarrow[\mathrm{hol}_{S^1}]{} & U(1) \end{array} \tag{5.3.9}$$

Proof. Since $BU(1)$ is a $K(\mathbb{Z}, 2)$, we have $\pi_1(BU(1)) = 0$ and so every $U(1)$-principal bundle over S^1 is trivializable. Choosing a trivialization of the underlying principal bundle, a connection on a principal $U(1)$-bundle reduces to a globally defined real valued 1-form A on S^1. The integral of this 1-form depends on the particular trivialization chosen. However, different choices lead to integrals which differ by an integer, so the complex number $\exp(2\pi i \int_{S^1} A)$ is well defined, and it is a well known fact in differential geometry that this exponentiated integral coincides with the holonomy of the $U(1)$-connection along S^1, see, e.g. [9, 7]. $\square$

Lemma 5.3.4. *The homotopy commutative diagram (5.3.9) can be delooped twice,*[4] *inducing a homotopy commutative diagram*

$$\begin{array}{ccc} \mathbf{B}^2[S^1, \Omega^1] & \xrightarrow{\mathbf{B}^2 \int_{S^1}} & \mathbf{B}^2\mathbb{R} \\ \downarrow & & \downarrow{\scriptstyle \mathbf{B}^2 \exp(2\pi i -)} \\ \mathbf{B}^2[S^1, \mathbf{B}U(1)_{\mathrm{conn}}] & \xrightarrow[\mathbf{B}^2\mathrm{hol}_{S^1}]{} & \mathbf{B}^2U(1) \end{array} \tag{5.3.10}$$

[4] Actually, it can be delooped an arbitrary number of times.

Proof. All the morphisms of stacks in the diagram (5.3.9) can be induced via Dold-Kan correspondence by morphisms of chain complexes of sheaves of abelian groups, see [8, 6]. Hence, they can be delooped an arbitrary number of times. □

Lemma 5.3.5. *We have a homotopy commutative diagram*

$$\begin{array}{ccc} [S^1, \mathbf{B}^2\Omega^1] & \xrightarrow{\mathbf{B}^2 \int_{S^1}} & \mathbf{B}^2\mathbb{R} \\ \downarrow & & \downarrow{\scriptstyle \mathbf{B}^2 \exp(2\pi i -)} \\ [S^1, \mathbf{B}^2(\mathbf{B}U(1)_{\mathrm{conn}})] & \xrightarrow[\mathbf{B}^2\mathrm{hol}_{S^1}]{} & \mathbf{B}^2U(1) \end{array} \tag{5.3.11}$$

Proof. This directly follows from diagram (5.3.10), as soon as we show that the delooping $\mathbf{B}$ commutes with the internal hom $[S^1, -]$. This follows from the fact that the delooping is part of a pair of inverse equivalences of ∞-categories

$$\{\text{group objects in } \mathbf{H}\} \quad \underset{\mathbf{B}}{\overset{\mathbf{\Omega}}{\rightleftarrows}} \quad \{\text{pointed connected objects in } \mathbf{H}\}\,,$$

see [13]. Here $\mathbf{\Omega}$ is the *based loop space* fucntor, mapping a pointed stack $\mathbf{X}$ to the stack $\mathbf{\Omega X}$ defined by the homotopy pullback diagram

$$\begin{array}{ccc} \mathbf{\Omega X} & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{X} \end{array}$$

By the universal property of homotopy pullbacks, $\mathbf{\Omega}$ commutes with internal homs, and so also its inverse $\mathbf{B}$ commutes with internal homs. Namely, for a deloopable stack $\mathbf{X}$ we have

$$\mathbf{B}[S^1, \mathbf{X}] \cong \mathbf{B}[S^1, \mathbf{\Omega B X}] \cong \mathbf{B\Omega}[S^1, \mathbf{BX}] \cong [S^1, \mathbf{BX}].$$

□

Remark 5.3.6. *The fact that* $\mathbf{B}$ *commutes with* $[S^1, -]$ *is the stacky refinement of the classical result that forming the classifying space* B *commutes with forming the loop space* $\mathcal{L}$*, see [1].*

Pasting diagram (5.3.11) on the right of diagram (5.3.8), we get the homotopy commutative diagram

$$\begin{array}{ccc} [S^1, \mathbf{B}String] & \longrightarrow & \mathbf{B^2}\mathbb{R} \\ \downarrow & & \downarrow \\ [S^1, \mathbf{B}Spin] & \longrightarrow & \mathbf{B}^2 U(1) \end{array},$$

which, using again the fact that $\mathbf{B}$ commutes with $[S^1, -]$, we can rewrite as

$$\begin{array}{ccc} \mathbf{B}\mathcal{L}String & \longrightarrow & \mathbf{B}^2\mathbb{R} \\ \downarrow & & \downarrow \\ \mathbf{B}\mathcal{L}Spin & \longrightarrow & \mathbf{B}^2 U(1) \end{array}, \tag{5.3.12}$$

where we have written $\mathcal{L}String$ for $[S^1, String]$ and $\mathcal{L}Spin$ for $[S^1, Spin]$. This is the homotopy commutative diagram we were referring to in the title of this Section.

Remark 5.3.7. *The bottom horizontal line in diagram (5.3.12) is the canonical 2-cocycle on the loop Spin group $\mathcal{L}Spin$. It defines the $U(1)$-central extension $\widetilde{\mathcal{L}Spin}$ we mentioned in Chapter 1. Namely, $\widetilde{\mathcal{L}Spin}$ is defined by the homotopy pullback diagram*

$$\begin{array}{ccc} \mathbf{B}\widetilde{\mathcal{L}Spin} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}\mathcal{L}Spin & \longrightarrow & \mathbf{B}^2 U(1) \end{array} \tag{5.3.13}$$

The fact that $\widetilde{\mathcal{L}Spin}$ is obtained by the above homotopy pullback diagram is a translation in the language of smooth stacks of the fact that the central extension

$$1 \to U(1) \to \widetilde{\mathcal{L}Spin} \to \mathcal{L}Spin \to 1$$

is a fusion extension *[30, Theorem 3.5].*

To proceed further, we need to introduce an additional stack, closely related to $\mathbf{B}\widetilde{\mathcal{L}Spin}$, in the next Section.

5.4 The stack $\mathrm{B}\mathcal{L}\widetilde{Spin}_{\mathbb{Z}}$

To define the stack $\mathrm{B}\mathcal{L}\widetilde{Spin}_{\mathbb{Z}}$, we factor the rightmost vertical arrow in diagram (5.3.13) as

$$* \to \mathbf{B}^2\mathbb{R} \xrightarrow{\mathbf{B}^2\exp(2\pi i -)} \mathbf{B}^2U(1)$$

and define $\mathrm{B}\mathcal{L}\widetilde{Spin}_{\mathbb{Z}}$ as the homotopy pullback

$$\begin{array}{ccc} \mathbf{B}\mathcal{L}\widetilde{Spin}_{\mathbb{Z}} & \longrightarrow & \mathbf{B}^2\mathbb{R} \\ \downarrow & & \downarrow \\ \mathbf{B}\mathcal{L}Spin & \longrightarrow & \mathbf{B}^2U(1) \end{array} \tag{5.4.14}$$

Lemma 5.4.1. *One has a canonical morphism of stacks $\mathbf{B}\mathcal{L}String \to \mathbf{B}\mathcal{L}\widetilde{Spin}_{\mathbb{Z}}$*

Proof. Since the diagram (5.3.12) is homotopy commutative, by the universal property of homotopy pullbacks it uniquely[5] factors as

$$\begin{tikzcd} \mathbf{B}\mathcal{L}String \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \mathbf{B}\mathcal{L}\widetilde{Spin}_{\mathbb{Z}} \arrow[r] \arrow[d] & \mathbf{B}^2\mathbb{R} \arrow[d] \\ & \mathbf{B}\mathcal{L}Spin \arrow[r] & \mathbf{B}^2U(1) \end{tikzcd}$$

□

Corollary 5.4.2. *For any smooth manifold X we have a natural morphism of hom-spaces*

$$\mathbf{H}(X, \mathbf{B}String) \to \mathbf{H}(\mathcal{L}X, \mathbf{B}\mathcal{L}\widetilde{Spin}_{\mathbb{Z}}).$$

Passing to the π_0's, this induces a natural transgression morphism

$$Maps(X, BString)/\sim \quad \longrightarrow \quad Maps(\mathcal{L}X, B\mathcal{L}\widetilde{Spin}_{\mathbb{Z}})/\sim .$$

[5] up to coherent homotopies

Proof. Since the internal hom $\mathcal{L} = [S^1, -]$ is a functor, we have a natural morphism

$$\mathbf{H}(X, \mathbf{B}String) \xrightarrow{[S^1,-]} \mathbf{H}(\mathcal{L}X, \mathcal{L}\mathbf{B}String) \cong \mathbf{H}(\mathcal{L}X, \mathbf{B}\mathcal{L}String).$$

Now compose with the morphism

$$\mathbf{H}(\mathcal{L}X, \mathbf{B}\mathcal{L}String) \to \mathbf{H}(\mathcal{L}X, \mathbf{B}\mathcal{L}\widetilde{Spin}_{\mathbb{Z}})$$

induced by the morphism $\mathbf{B}\mathcal{L}String \to \mathbf{B}\mathcal{L}\widetilde{Spin}_{\mathbb{Z}}$ from Lemma 5.4.1. □

5.5 Back to $\mathrm{B}\mathcal{L}\widetilde{Spin}$

The transgression morphism from Corollary 5.4.2 is almost what we are after: the only difference is that in Corollary 5.4.2 it is the quite mysterious classifying space $B\mathcal{L}\widetilde{Spin}_{\mathbb{Z}}$ to show up, instead of the expected $B\mathcal{L}\widetilde{Spin}$. However, as we are going to show, these two classifying spaces are not so far each other, and this will allow us to indentify the transgression map from Corollary 5.4.2 with a transgression map

$$Maps(X, BString)/\sim \quad \longrightarrow \quad Maps(\mathcal{L}X, B\mathcal{L}\widetilde{Spin})/\sim$$

thus completing our proof of (a version of) Waldorf's result.

To begin with, by the short exact sequence of abelian Lie groups

$$0 \to \mathbb{Z} \to \mathbb{R} \xrightarrow{\exp(2\pi i -)} U(1) \to 1$$

and by diagrams (5.3.13) and (5.4.14) we get the following

Proposition 5.5.1. *We have a homotopy commutative diagram*

$$\begin{array}{ccccccccc}
\mathbf{B}\mathbb{R} & \longrightarrow & \mathbf{B}U(1) & \longrightarrow & \mathbf{B}\mathcal{L}\widetilde{Spin} & \longrightarrow & * & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
* & \longrightarrow & \mathbf{B}^2\mathbb{Z} & \longrightarrow & \mathbf{B}\mathcal{L}\widetilde{Spin}_{\mathbb{Z}} & \longrightarrow & \mathbf{B}^2\mathbb{R} & \longrightarrow & * \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & * & \longrightarrow & \mathbf{B}\mathcal{L}Spin & \longrightarrow & \mathbf{B}^2U(1) & \longrightarrow & \mathbf{B}^3\mathbb{Z}
\end{array} \tag{5.5.15}$$

where all squares are homotopy pullbacks.

Since $\mathbb{R}$ is a contractible Lie group, the topological realization $B^2\mathbb{R}$ of $\mathbf{B}^2\mathbb{R}$ is contractible. This suggests, that from the homotopy pullback diagram

$$\begin{array}{ccc}
\mathbf{B}\mathcal{L}\widetilde{Spin} & \longrightarrow & * \\
\downarrow & & \downarrow \\
\mathbf{B}\mathcal{L}\widetilde{Spin}_{\mathbb{Z}} & \longrightarrow & \mathbf{B}^2\mathbb{R}
\end{array},$$

passing to topological realizations, we should obtain that the morphism $\mathbf{B}\mathcal{L}\widetilde{Spin} \to \mathbf{B}\mathcal{L}\widetilde{Spin}_{\mathbb{Z}}$ induces a homotopy equivalence $B\mathcal{L}\widetilde{Spin} \xrightarrow{\sim} B\mathcal{L}\widetilde{Spin}_{\mathbb{Z}}$.
However, unfortunately, taking topological realizations does not generally commute with homotopy pullbacks unless the base is geometrically discrete[6] (which $\mathbf{B}^2\mathbb{R}$ is not), so we are

[6]This is [21, Theorem 3.8.19].

not a priori guaranteed that

$$\begin{array}{ccc} B\mathcal{L}\widetilde{Spin} & \longrightarrow & * \\ \downarrow & & \downarrow \\ B\mathcal{L}\widetilde{Spin}_{\mathbb{Z}} & \longrightarrow & B^2\mathbb{R} \end{array},$$

will be a homotopy pullback. However, luckily, for what we are concerned in view of our proof of Waldorf's transgression result, the map $B\mathcal{L}\widetilde{Spin} \to B\mathcal{L}\widetilde{Spin}_{\mathbb{Z}}$ does indeed behave as if it were a homotopy equivalence. Namely, the following result holds.

Theorem 5.5.2. *Let X be a (possibly infinite dimensional) smooth manifold with a smooth partition of unit. Then the morphism $\mathbf{B}\mathcal{L}\widetilde{Spin} \to \mathbf{B}\mathcal{L}\widetilde{Spin}_{\mathbb{Z}}$ induces an isomorphism*

$$Maps(X, B\mathcal{L}\widetilde{Spin})/\sim \quad \xrightarrow{\sim} \quad Maps(X, B\mathcal{L}\widetilde{Spin}_{\mathbb{Z}})/\sim$$

between the sets of homotopy classes of maps from X to $B\mathcal{L}\widetilde{Spin}$ and to $B\mathcal{L}\widetilde{Spin}_{\mathbb{Z}}$, respectively.

Proof. The homotopy commutative diagram of ∞-groupoids (i.e., of 'nice' topological spaces)

$$\begin{array}{ccc} \mathbf{H}(X, \mathbf{B}\mathcal{L}\widetilde{Spin}) & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{H}(X, \mathbf{B}\mathcal{L}\widetilde{Spin}_{\mathbb{Z}}) & \longrightarrow & \mathbf{H}(X, \mathbf{B}^2\mathbb{R}) \end{array}$$

induce the long exact sequence of homotopy groups

$$\cdots \to \pi_1\mathbf{H}(X, \mathbf{B}^2\mathbb{R}) \to \pi_0\mathbf{H}(X, \mathbf{B}\mathcal{L}\widetilde{Spin}) \to \pi_0\mathbf{H}(X, \mathbf{B}\mathcal{L}\widetilde{Spin}_{\mathbb{Z}}) \to \pi_0\mathbf{H}(X, \mathbf{B}^2\mathbb{R}).$$

We have

$$\pi_0\mathbf{H}(X, \mathbf{B}^2\mathbb{R}) = H^2(X; \underline{\mathbb{R}})$$

and

$$\pi_1\mathbf{H}(X, \mathbf{B}^2\mathbb{R}) = H^1(X; \underline{\mathbb{R}}),$$

where $\underline{\mathbb{R}}$ is the sheaf of smooth $\mathbb{R}$-valued functions on X. Since X has a partition of unit, the sheaf $\underline{\mathbb{R}}$ is fine and so all of its cohomology groups $H^i(X;\underline{\mathbb{R}})$ with $i \geq 1$ vanish. As a consequence the above long exact sequence of homotopy groups reduces to

$$\cdots \to 0 \to \pi_0\mathbf{H}(X, \mathbf{B}\mathcal{L}\widetilde{Spin}) \to \pi_0\mathbf{H}(X, \mathbf{B}\mathcal{L}\widetilde{Spin}_{\mathbb{Z}}) \to 0,$$

showing that

$$\pi_0\mathbf{H}(X, \mathbf{B}\mathcal{L}\widetilde{Spin}) \to \pi_0\mathbf{H}(X, \mathbf{B}\mathcal{L}\widetilde{Spin}_{\mathbb{Z}})$$

is an isomorphism. Since

$$\pi_0\mathbf{H}(X, \mathbf{B}\mathcal{L}\widetilde{Spin}) \cong Maps(X, B\mathcal{L}\widetilde{Spin})/\sim$$

and

$$\pi_0\mathbf{H}(X, \mathbf{B}\mathcal{L}\widetilde{Spin}_{\mathbb{Z}}) \cong Maps(X, B\mathcal{L}\widetilde{Spin}_{\mathbb{Z}})/\sim$$

this concludes the proof. □

Corollary 5.5.3. *Let X be a finite dimensional smooth manifold. Then the morphism $\mathbf{B}\mathcal{L}\widetilde{Spin} \to \mathbf{B}\mathcal{L}\widetilde{Spin}_{\mathbb{Z}}$ induces an isomorphism*

$$Maps(\mathcal{L}X, B\mathcal{L}\widetilde{Spin})/\sim \quad \xrightarrow{\sim} \quad Maps(\mathcal{L}X, B\mathcal{L}\widetilde{Spin}_{\mathbb{Z}})/\sim$$

Proof. Since X is finite dimensional, the Frechét manifold $\mathcal{L}X$ admits smooth partitions of unit, see, e.g., [11, Theorem 16.10]. □

We have now all the ingredients to prove our main result

Theorem 5.5.4 (Waldorf)**.** *Let X be a finite dimensional smooth manifold. Then there is a canonical transgression map*

$$Maps(X, BString)/\sim \quad \longrightarrow \quad Maps(\mathcal{L}X, B\mathcal{L}\widetilde{Spin})/\sim .$$

Proof. Just compose the canonical transgression map

$$Maps(X, BString)/\sim \quad \longrightarrow \quad Maps(\mathcal{L}X, B\mathcal{L}\widetilde{Spin}_{\mathbb{Z}})/\sim$$

from Corollary 5.4.2 with the inverse of the canonical isomorphism

$$Maps(\mathcal{L}X, B\mathcal{L}\widetilde{Spin})/\sim \quad \xrightarrow{\sim} \quad Maps(\mathcal{L}X, B\mathcal{L}\widetilde{Spin}_{\mathbb{Z}})/\sim$$

from Corollary 5.5.3. □

Appendix A

Differential Lie integration

The reader can find here a short presentation of Lie integration, aimed to provide a few details for a complete understanding of the differential refinement of the first Pontryagin map presented in Section 5.2. This appendix gathers some passages of [8], which the interested reader can find a more detailed presentation of this subject in. In paragraphs A.1 and A.2 we introduce the basic dictionary, and in paragraphs A.3 and A.4 the Lie integration of semisimple Lie algebras to stacks of principal G-bundles and principal G-bundles with connection is addressed.

A.1 The algebras $CE(\mathfrak{g})$, $W(\mathfrak{g})$, $inv(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$

Definition A.1.1. *The* Chevalley-Eilenberg *algebra $CE(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$ is the semifree graded-commutative dg-algebra whose underlying graded algebra is the Grassmann algebra*

$$\Lambda^{\bullet}{}_{\mathfrak{g}^*} = k \oplus \mathfrak{g}^* \oplus (\mathfrak{g}^* \wedge \mathfrak{g}^*) \oplus ...$$

(with the n-th skew-symmetrized power in degree n) and whose differential $d_{CE(\mathfrak{g})}$ (of degree $+1$) is on $\mathfrak{g}^*$ the dual of the Lie bracket

$$d_{CE(\mathfrak{g})}|_{\mathfrak{g}^*} := [-,-]^* : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$$

extended uniquely as a graded derivation on $\Lambda^\bullet{}_{\mathfrak{g}^*}$.

Remark A.1.2. *More generally, the Chevalley-Eilenberg algebra can be defined for a differential graded Lie algebra* $\mathfrak{g}$. *In this case the differential* $d_{CE(\mathfrak{g})}$ *encodes not only the Lie bracket, but also the differential of* $\mathfrak{g}$.

Example A.1.3. *For* $n \geq 1$, *the differential graded Lie algebra* $\mathbb{R}[n-1]$ *consists of the vector space* $\mathbb{R}$ *placed in degree* $n-1$, *with trivial differential and trivial Lie bracket. Its Chevalley-Eilenberg algebra is the polynomial algebra on a single generator* c *in degree* n *and has trivial differential* $CE(b^{n-1}\mathbb{R}) = (\mathbb{R}[c], d = 0)$.

Definition A.1.4. *Let* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *be differential graded Lie algebras. An* L_∞*-morphism from* $\mathfrak{g}_1$ *to* $\mathfrak{g}_2$ *is defined to be a differential graded commutative algebra morphism*

$$CE(\mathfrak{g}_2) \to CE(\mathfrak{g}_1).$$

Remark A.1.5. *For* $\mathfrak{g}$ *a differential graded Lie algebra and* $n \in \mathbb{N}$, *a* cocycle of degree n on $\mathfrak{g}$ *is, equivalently*

- ◇ *an element* $\mu \in CE(\mathfrak{g})$ *in degree* n, *such that* $d_{CE(\mathfrak{g})}\mu = 0$;
- ◇ *a morphism of dg-algebras* $\mu : CE(\mathbb{R}[n-1]) \to CE(\mathfrak{g})$;
- ◇ *an* L_∞*-morphism* $\mu : \mathfrak{g} \to \mathbb{R}[n-1]$.

Definition A.1.6. *The* Weil algebra *of a differential graded Lie algebra* $\mathfrak{g}$ *is the dg-algebra*

$$W(\mathfrak{g}) := Sym^\bullet(\mathfrak{g}^*[-1] \oplus \mathfrak{g}^*[-2], d_{W(\mathfrak{g})})$$

where the differential d_W *restricted to* $\mathfrak{g}^*[-1]$ *is the sum*

$$d_{W(\mathfrak{g})}|_{\mathfrak{g}^*[-1]} = d_{CE(\mathfrak{g})} + \sigma,$$

with $\sigma : \mathfrak{g}^* \to \mathfrak{g}^*[-1]$ *is the grade-shifting isomorphism extended as a graded derivation.*

Remark A.1.7. *The projection morphism $i^* : \mathfrak{g}^*[-1] \oplus \mathfrak{g}^*[-2] \to \mathfrak{g}^*[-1]$ of graded vector spaces extends to a dg-algebra homorphism $i^* : W(\mathfrak{g}) \to CE(\mathfrak{g})$.*

The crucial property of Weil algebras is their freeness:

Proposition A.1.8. *Let $\mathfrak{g}$ be a differential graded Lie algebra, and let $\Omega^\bullet$ be a differential graded commutative algebra. Morphisms of dg-algebras $W(\mathfrak{g}) \to \Omega^\bullet$ are in natural bijection to morphisms of graded vector spaces $\mathfrak{g}^* \to \Omega^\bullet$.*

Remark A.1.9. *One can equivalently state the freeness of the Weil algebra by saying that the dgca-morphisms $W(\mathfrak{g}) \to \Omega^\bullet$ are in natural bijection with the degree 1 elements in the graded vector space $\Omega^\bullet \otimes \mathfrak{g}$.*

Definition A.1.10. *An* invariant polynomial *on $\mathfrak{g}$ is a $d_{W(\mathfrak{g})}$-closed element $\langle - \rangle$ in $Sym^\bullet(\mathfrak{g}^*[-2]) \subseteq W(\mathfrak{g})$.*

Remark A.1.11. *The Weil algebra is itself the CE-algebra of a differential graded Lie algebra $inn(\mathfrak{g})$, whose underlying graded vector space of $inn(\mathfrak{g})$ is $\mathfrak{g} \oplus \mathfrak{g}[1]$. Looking at $W(\mathfrak{g})$ as the Chevalley-Eilenberg algebra of $inn(\mathfrak{g})$ one obtains the following description of morphisms out of $W(\mathfrak{g})$: for any dgca $\Omega^\bullet$, a dgca morphism $W(\mathfrak{g}) \to \Omega^\bullet$ is the datum of a pair (A, F_A), where A and F_A are a degree 1 and a degree 2 element in $\Omega^\bullet \otimes \mathfrak{g}$, respectively, such that (A, F_A) satisfies the Maurer-Cartan equation in the Lie ∞-algebra $\Omega^\bullet \otimes inn(\mathfrak{g})$. The Maurer-Cartan equation actually completely determines F_A in terms of A; this is an instance of the freeness property of the Weil algebra. For any dgca morphism $A : W(\mathfrak{g}) \to \Omega^\bullet$, the composite morphism $inv(\mathfrak{g}) \to W(\mathfrak{g}) \to \Omega^\bullet$ is the evaluation of invariant polynomials on the element F_A.*

Definition A.1.12. *We say an invariant polynomial $\langle - \rangle$ on $\mathfrak{g}$ is* in transgression *with a cocycle μ if there exists an element $\mathrm{cs} \in W(\mathfrak{g})$ such that*

1. $i^*\mathrm{cs} = \mu$;
2. $d_{W(\mathfrak{g})}\mathrm{cs} = \langle - \rangle$.

We call cs a Chern-Simons element *transgressing μ to $\langle - \rangle$.*

Example A.1.13. *For $\mathfrak{g}$ a semisimple Lie algebra with $\langle -,-\rangle$ the Killing form invariant polynomial, the corresponding cocycle in transgression is $\mu = \frac{1}{2}\langle -,[-,-]\rangle$. The Chern-Simons element witnessing this transgression is $\mathrm{cs} = \langle \sigma(-),-\rangle + \frac{1}{2}\langle -,[-,-]\rangle$.*

Remark A.1.14. *Consider a degree n cocycle μ which is in transgression with an invariant polynomial $\langle - \rangle$ via a Chern-Simons element cs: the corresponding morphisms of dg-algebras fit into a commutative diagram*

$$\begin{array}{ccc} CE(\mathfrak{g}) & \xleftarrow{\ \mu\ } & CE(\mathbb{R}[n-1]) \\ \uparrow & & \uparrow \\ W(\mathfrak{g}) & \xleftarrow{\ \mathrm{cs}\ } & W(\mathbb{R}[n-1]) \\ \uparrow & & \uparrow \\ inv(\mathfrak{g}) & \xleftarrow{\ \langle - \rangle\ } & inv(\mathbb{R}[n-1]) \end{array}$$

A.2 The simplicial presheaf associated with a differential graded Lie algebra $\mathfrak{g}$

To describe the integration of a Lie algebra $\mathfrak{g}$ to a smooth stack $\mathbf{B}G$, we need to realize some simplicial presheaves sending a smooth manifold U to a Kan complex which in degree k is the set of smoothly U-parameterized families of smooth flat $\mathfrak{g}$-valued differential forms on the standard k-simplex $\Delta^k \subset \mathbb{R}^k$ regarded as a smooth manifold (with boundary and corners). To make this precise one needs a suitable notion of smooth differential forms on the k-simplex. However, since this Appendix does not pretend to be exhaustive, we will just write $\Omega^\bullet(U \times \Delta^k)$ and will implicitly assume that they have a good behaviour towards the boundary of the simplex.

Definition A.2.1. *For $\mathfrak{g}$ a differential graded Lie algebra, the simplicial presheaf $\exp_\Delta(\mathfrak{g})$ on the site of smooth manifold is defined as*

$$\exp_\Delta(\mathfrak{g}) : (U, [k]) \mapsto \mathrm{Hom}_{dgAlg}(CE(\mathfrak{g}), \Omega^\bullet(U \times \Delta^k)_{vert}),$$

where $\Omega^\bullet(U \times \Delta^k)_{vert}$ denotes the sub-dg-algebra of $\Omega^\bullet(U \times \Delta^k)$ of those differential forms which are vertical *with respect to the projection $U \times \Delta^k \to U$.*
Note that the construction of $\exp_\Delta(\mathfrak{g})$ is functorial in $\mathfrak{g}$: an L_∞-morphism between $\mathfrak{g}_1$ and $\mathfrak{g}_2$, i.e., a dg-algebra morphism $CE(\mathfrak{g}_2) \to CE(\mathfrak{g}_1)$, induces a morphism of simplicial presheaves $\exp_\Delta(\mathfrak{g}_1) \to \exp_\Delta(\mathfrak{g}_2)$.

Example A.2.2. *A Lie algebra n-cocycle $\mu : \mathfrak{g} \to \mathbb{R}[n-1]$, defines a morphism of simplicial presheaves*

$$\exp_\Delta(\mu) : \exp_\Delta(\mathfrak{g}) \to \exp_\Delta(\mathbb{R}[n-1]).$$

We will denote by the same symbol $\exp_\Delta(\mathfrak{g})$ the smooth stack obtained by sheafification of the presheaf $\exp_\Delta(\mathfrak{g})$. One of the crucial properties of this stack is the following

Proposition A.2.3. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Parallel transport along the edges of simplices induces a morphism of ∞-groupoids*

$$tra : \exp_\Delta(\mathfrak{g}) \to \mathbf{B}G.$$

Moreover, for a compact connected and simply connected Lie group G with Lie algebra $\mathfrak{g}$, the parallel transport morphism induces an acyclic fibration

$$\mathbf{cosk}_3(\exp_\Delta(\mathfrak{g})) \xrightarrow{\sim} \mathbf{B}G,$$

where $\mathbf{cosk}_3$ is the 3-coskeletization, i.e., the functor that discards all the nontrivial n-simplices of a simplicial set for $n \geq 3$.

Remark A.2.4. *In more colloquial terms, the above proposition states the well known fact from Lie theory that a compact connected and simply connected Lie group G is entirely reconstructed from its Lie algebra $\mathfrak{g}$.*

A.3 Fiber integration over simplices and cocycles Lie integration

When the differential graded Lie algera is $\mathbb{R}[n-1]$ parallel transport reduces to integration, and we get the following

Proposition A.3.1. *Fiber integration over simplices induces an equivalence*

$$\int_{\Delta^\bullet} : \exp_\Delta(\mathbb{R}[n-1]) \xrightarrow{\sim} \mathbf{B}^n\mathbb{R}$$

Now, if G is a compact connected and simply connected Lie group with Lie algebra $\mathfrak{g}$ and μ in a Lie n-cocycle on $\mathfrak{g}$, we can combine $\exp_\Delta(\mu)$ and $\int_{\Delta^\bullet}$ to get the following

Proposition A.3.2. *Let G be a compact, simple and simply connected Lie group and μ_3 the canonical 3-cocycle on its semisimple Lie algebra, normalized such that its left-invariant extension to a differential 3-form on G represents a generator of $H^3(G,\mathbb{Z}) \simeq \mathbb{Z}$ in de Rham cohomology. Then there is a commutative diagram*

$$\begin{array}{ccc}
\exp_\Delta(\mathfrak{g}) & \xrightarrow{\int_{\Delta^\bullet}\exp_\Delta(\mu)} & \mathbf{B}^3\mathbb{R} \\
\downarrow & & \downarrow \\
\mathbf{cosk}_3 \exp_\Delta(\mathfrak{g}) & \longrightarrow & \mathbf{B}^3(\mathbb{Z}\to\mathbb{R}) \\
\downarrow \wr & & \downarrow \wr \\
\mathbf{B}G & & \mathbf{B}^3U(1)
\end{array}$$

presenting a morphism of smooth stacks

$$\mathbf{B}G \to \mathbf{B}^3U(1).$$

In particular, for $G = Spin$ this presents the first fractional Pontryagin map.

A.4 The Lie integration of $\mathfrak{g}$ to $\mathbf{B}G_{conn}$

In the previous Section we have seen how to Lie integrate a semisimple Lie algebra $\mathfrak{g}$ to the stack $\mathbf{B}G$ of principal G-bundles, where G is the compact connected and simply connected

Lie group with Lie algebra $\mathfrak{g}$. We have also seen how the canonical Lie 3-cocycle μ_3 on $\mathfrak{g}$ integrates to a morphism $\mathbf{B}G \to \mathbf{B}^3U(1)$ which, for $G = Spin$ is a smooth refinement of the first fractional Pontryagin class. Now we are going to show how we can add connections to this pictures and bulid $\mathbf{B}G_{\mathrm{conn}}$ out the algebraic data of the Lie algebra $\mathfrak{g}$. Also, we are going to show how not the 3-cocylce μ_3 alone, but the whole triple $(\mu_3, \mathrm{cs}, \langle - \rangle)$ consisting of the 3-cocycle, of an invariant polynomial and of a Chern-Simons element witnessing the transgression between the two is involved in constructing a natural morphism $\mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3U(1)_{\mathrm{conn}}$. When $G = Spin$, this morphism is the differential refinement of the first fractional Pontryagin class

$$\frac{1}{2}\hat{\mathbf{p}}_1 \colon \mathbf{B}Spin_{\mathrm{conn}} \to \mathbf{B}^3U(1)_{\mathrm{conn}}.$$

Definition A.4.1. *Let $\mathfrak{g}$ be a differential graded Lie algebra. The differential refinement $\exp_\Delta(\mathfrak{g})_{\mathrm{conn}}$ of $\exp_\Delta(\mathfrak{g})$ is the simplicial presheaf on the site of smooth manifold given by the assignment*

$$(U,[k]) \mapsto \left\{ \begin{array}{ccc} \Omega^\bullet(U\times\Delta^k)_{vert} & \xleftarrow{A_{vert}} & CE(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(U\times\Delta^k) & \xleftarrow{A} & W(\mathfrak{g}) \end{array} \quad ; \quad \iota_{vert}F_A = 0 \right\},$$

where on the right we have the set of commuting diagrams in dgc-algebras as indicated such that $\iota_v F_A = 0$ for all vertical (i.e., tangent to the simplex) vector fields v on $U \times \Delta^k$.

The condition $\iota_{vert}F_A = 0$ may appear quite mysterious at first, but it actually naturally encodes the gauge transformation law for $\mathfrak{g}$-connections on a principal G-bundle. To see this, write $\Delta^1 = [0,1]$ for the standard interval regarded as a smooth manifold (with boundary) and consider a smooth 1-form $A \in \Omega^1(U \times \Delta^1, \mathfrak{g})$ on the product of U with Δ^1. It makes sense to decompose A as the sum of a horizontal 1-form A_U and a vertical 1-form $\lambda\, dt$, where $t \colon \Delta^1 \to \mathbb{R}$ is the canonical coordinate on Δ^1:

$$A = A_U + \lambda\, dt\,.$$

The vertical part $A_{\mathrm{vert}} = \lambda\, dt$ of A is an element of the completed tensor product $C^\infty(U)\hat{\otimes}\Omega^1(\Delta^1, \mathfrak{g})$ and can be seen as a family of $\mathfrak{g}$-connections on a trivial G-principal bundle on Δ^1, parametrized by U. At any fixed $u_0 \in U$, the 1-form $\lambda(u_0, t)\, dt \in \Omega^1(\Delta^1, \mathfrak{g})$ satisfies the Maurer-Cartan equation by trivial dimensional reasons, and so we have a commutative diagram

$$\begin{array}{ccc} \Omega^\bullet(U \times \Delta^1)_{\mathrm{vert}} & \xleftarrow{A_{\mathrm{vert}}} & CE(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(U \times \Delta^1) & \xleftarrow{A} & W(\mathfrak{g}) \end{array}$$

This can be seen as a *first Cartan-Ehresmann condition in the* Δ^1*-direction*; it precisely encodes the fact that the 1-form A on the total space of $U \times \Delta^1 \to U$ is flat in the vertical direction. The curvature 2-form of A decomposes as

$$F_A = F_{A_U} + F_{\Delta^1} \,,$$

where the first term is at each point $t \in \Delta^1$ the ordinary curvature $F_{A_U} = d_U A_U + \frac{1}{2}[A_U, A_U]$ of A_U at fixed $t \in \Delta^1$ and where the second term is

$$F_{\Delta^1} = \left(d_U \lambda + [A_U, \lambda] - \frac{\partial}{\partial t} A_U \right) \wedge dt \,.$$

Requiring that $\iota_{vert} F_A = 0$ is equivalent to requiring that $F_{\Delta^1} = 0$; this can be seen as a *second Ehresmann condition in the* Δ^1*-direction*. The condition $F_{\Delta^1} = 0$ is equivalent to the differential equation

$$\frac{\partial}{\partial t} A_U = d_U \lambda + [A_U, \lambda],$$

whose unique solution for given boundary condition $A_U|_{t=0}$ specifies $A_U|_{t=1}$ by the formula

$$A_U(1) = g^{-1} A_U(0) g + g^{-1} dg \,, \qquad \text{(A.4.1)}$$

where

$$g := \mathcal{P} \exp\left(\int_{\Delta^1} \lambda dt \right) : U \to G$$

is, pointwise in U, the parallel transport of λdt along the interval. Equation (A.4.1) is precisely the gauge transformation law for $\mathfrak{g}$-connections on a principal G-bundle.

Remark A.4.2. *The condition $\iota_{vert}F_A = 0$ immediately implies that F_A descends to invariant polynomials so that each k-simplex in $\exp_\Delta(\mathfrak{g})_{\mathrm{conn}}$ is naturally extended on the bottom as*

$$\begin{array}{ccc} \Omega^\bullet(U \times \Delta^k)_{vert} & \xleftarrow{A_{vert}} & CE(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(U \times \Delta^k) & \xleftarrow{A} & W(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(U) & \xleftarrow{F_A} & inv(\mathfrak{g}) \end{array}$$

If $\mathfrak{g}$ is a semisimple Lie algebra with compact simply connected Lie group G, the parallel transport morphism tra of Proposition A.2.3 can be proven to induce a morphism

$$tra : \exp_\Delta(\mathfrak{g})_{\mathrm{conn}} \to \mathbf{B}G_{\mathrm{conn}},$$

which in turn induces an equivalence

$$\mathbf{cosk}_3 \exp_\Delta(\mathfrak{g})_{\mathrm{conn}} \to \mathbf{B}G_{\mathrm{conn}}.$$

This way one realizes the stack $\mathbf{B}G_{\mathrm{conn}}$ entirely from the algebraic data of the Lie algebra $\mathfrak{g}$.

When $\mathfrak{g} = \mathbb{R}[n-1]$ we find the following analogue of Proposition A.3.1.

Proposition A.4.3. *Integration along simplices induces a natural morphism of smooth (higher) stacks*

$$\int_{\Delta^\bullet}^{\mathrm{conn}} : \exp_\Delta(\mathbb{R}[n-1])_{\mathrm{conn}} \to \mathbf{B}^n\mathbb{R}_{\mathrm{conn}}$$

We now have all the elements to refine Lie integration in a way involving connections. For this purpose, recall that $\exp_\Delta(\mu)$ was essentially obtained by composition of the k-cells in $\exp_\Delta(\mathfrak{g})$ with the Lie cocycle $\mathfrak{g} \xrightarrow{\mu} \mathbb{R}[n-1]$. Since the k-cells in $\exp_\Delta(\mathfrak{g})_{\mathrm{conn}}$ are diagrams, we need to accordingly extend the morphism μ to a diagram. This is where the Chern-Simons element and the invariant polynomial come into play: consider again diagram

$$\begin{array}{ccc} CE(\mathfrak{g}) & \xleftarrow{\mu} & CE(\mathbb{R}[n-1]) \\ \uparrow & & \uparrow \\ W(\mathfrak{g}) & \xleftarrow{\mathrm{cs}} & W(\mathbb{R}[n-1]) \\ \uparrow & & \uparrow \\ inv(\mathfrak{g}) & \xleftarrow{\langle - \rangle} & \mathbb{R}[n-1]) \end{array} \quad , \tag{A.4.2}$$

where $\langle - \rangle$ is an invariant polynomial in transgression with μ and cs is a Chern- Simons element witnessing this transgression.

Definition A.4.4. *Define the morphism of simplicial presheaves*

$$\exp_\Delta(\mathrm{cs})_{\mathrm{conn}} : \exp_\Delta(\mathfrak{g})_{\mathrm{conn}} \to \exp_\Delta(\mathbb{R}[n-1])_{\mathrm{conn}}$$

degreewise by pasting composition with the top square in diagram (A.4.2):

$$\exp_\Delta(\mathrm{cs})_k : \left(\begin{array}{ccc} \Omega^\bullet(U \times \Delta^k)_{vert} & \xleftarrow{A_{vert}} & CE(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(U \times \Delta^k) & \xleftarrow{A} & W(\mathfrak{g}) \end{array} \right) \mapsto$$

$$\mapsto \left(\begin{array}{ccccccc} \Omega^\bullet(U \times \Delta^k)_{vert} & \xleftarrow{A_{vert}} & CE(\mathfrak{g}) & \xleftarrow{\mu} & CE(\mathbb{R}[n-1]) & & : \mu(A_{vert}) \\ \uparrow & & \uparrow & & \uparrow & & \\ \Omega^\bullet(U \times \Delta^k) & \xleftarrow{A} & W(\mathfrak{g}) & \xleftarrow{\mathrm{cs}} & W(\mathbb{R}[n-1]) & & : \mathrm{cs}(A) \end{array} \right)$$

Remark A.4.5. *The lower extension of the diagrams in the image of* $\exp_\Delta(\mathrm{cs})$ *makes the invariant polynomial* $\langle - \rangle$ *appear explicitly:*

$$
\left(
\begin{array}{ccccccc}
\Omega^\bullet(U \times \Delta^k)_{vert} & \xleftarrow{A_{vert}} & CE(\mathfrak{g}) & \xleftarrow{\mu} & CE(\mathbb{R}[n-1]) & & : \mu(A_{vet}) \\
\uparrow & & \uparrow & & \uparrow & & \\
\Omega^\bullet(U \times \Delta^k) & \xleftarrow{A} & W(\mathfrak{g}) & \xleftarrow{\mathrm{cs}} & W(\mathbb{R}[n-1]) & & : \mathrm{cs}(A) \\
\uparrow & & \uparrow & & \uparrow & & \\
\Omega^\bullet(U) & \xleftarrow{F_A} & inv(\mathfrak{g}) & \xleftarrow{\langle - \rangle} & inv(\mathbb{R}[n-1]) & & : \langle F_A \rangle
\end{array}
\right)
$$

We can now finally state the analogue of Proposition A.3.2.

Proposition A.4.6. *Let* G *be a compact, simple and simply connected Lie group and* μ_3 *the canonical 3-cocycle on its semisimple Lie algebra, normalized such that its left-invariant extension to a differential 3-form on* G *represents a generator of* $H^3(G, \mathbb{Z}) \simeq \mathbb{Z}$ *in de Rham cohomology. Let* $\mathbf{cs}_3$ *be the Chern-Simons element for* μ_3 *whose invariant polynomial is the Killing form* $\langle -, - \rangle$ *on* $\mathfrak{g}$*. Then there is a commutative diagram*

$$
\begin{array}{ccc}
\exp_\Delta(\mathfrak{g})_{\mathrm{conn}} & \xrightarrow{\int_{\Delta^\bullet} \exp_\Delta(\mathrm{cs}_3)} & \mathbf{B}^3\mathbb{R}_{\mathrm{conn}} \\
\downarrow & & \downarrow \\
\mathbf{cosk}_3 \exp_\Delta(\mathfrak{g})_{\mathrm{conn}} & \longrightarrow & \mathbf{B}^3(\mathbb{Z} \to \mathbb{R})_{\mathrm{conn}} \\
\downarrow \wr & & \downarrow \wr \\
\mathbf{B}G_{\mathrm{conn}} & & \mathbf{B}^3 U(1)_{\mathrm{conn}}
\end{array}
$$

presenting a morphism of smooth stacks

$$\mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}.$$

In particular, for $G = Spin$ *this presents the differential refinement* $\frac{1}{2}\hat{\mathbf{p}}_1$ *of the first fractional Pontryagin map.*

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