The Bruhat-Tits tree of SL(2)

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The group $\mathrm{GL}_2(\mathbb{R})$ acts on the space of all symmetric 2×2 real matrices:

$$X\colon\ S \longmapsto XS\,{}^tX\,.$$

It preserves the open cone $\mathcal{C}$ of positive definite matrices. The quotient $\mathrm{PGL}_2(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})/\{\text{scalars}\}$ therefore acts on the space $\mathbb{P}(\mathcal{C})$, which is the quotient of such matrices modulo positive scalars. The isotropy subgroup of I is the image $\overline{\mathrm{O}}(2)$ in $\mathrm{PGL}_2(\mathbb{R})$ of $\mathrm{O}(2)$, so that $\mathbb{P}(\mathcal{C})$ may be identified with $\mathrm{PGL}_2(\mathbb{R})/\overline{\mathrm{O}}(2)$. The embedding of SL_2 into GL_2 identifies this with $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$. There exists a Riemannian metric on $\mathbb{P}(\mathcal{C})$, invariant with respect to $\mathrm{PGL}_2(\mathbb{R})$ and unique with this property, up to a positive scalar multiple.

The half-cone $\mathcal{C}_0$ of non-negative symmetric matrices of rank one that borders $\mathcal{C}$ is also stable under $\mathrm{GL}_2(\mathbb{R})$. To each point of $\mathcal{C}_0$ corresponds the null line of the corresponding quadratic form, and $\mathbb{P}(\mathcal{C}_0)$ may be identified with $\mathbb{P}^1(\mathbb{R})$, the space of lines in $\mathbb{R}^2$. This space compactifies $\mathbb{P}(\mathcal{C})$.

If we choose coordinates

$$\begin{bmatrix} z-y & x \\ x & z+y \end{bmatrix}$$

for symmetric matrices, the space $\mathcal{C}$ is where z, $z^2 - y^2 - x^2 > 0$. The intersection of this and the plane $z = 1$ is the open disc $x^2 + y^2 < 1$, which may be identified with $\mathbb{P}(\mathcal{C})$. Interesting representations of $\mathrm{SL}_2(\mathbb{R})$ are obtained on eigenspaces of the non-Euclidean Laplacian.

There are many remarkable parallels between the structures of real and p-adic groups, and one of the most remarkable is that there exists an analogue of real symmetric spaces, the buildings constructed by Bruhat and Tits. Among them is the tree on which $\mathrm{PGL}_2(\mathfrak{k})$ acts. In this essay I shall define it, prove some elementary properties, and show how it can be used in harmonic analysis on $\mathrm{SL}_2(\mathfrak{k})$. Very little of what I'll say is original, but the material is widely scattered in the literature, and sometimes only in a sketchy manner.

For groups of higher rank, buildings generalize the trees constructed here. They are important in understanding the structure of such groups, but play a very small role in analysis. Nonetheless, doing analysis on the tree of $\mathrm{SL}_2(\mathfrak{k})$ offers a unique opportunity to understand things intuitively.

I shall eventually assume $\mathfrak{k}$ to be a $\mathfrak{p}$-adic field, and in particular locally compact, but it will be convenient to relax the condition on local compactness at first. Throughout this essay, let

- $\mathfrak{k} =$ a field with a discrete valuation
- $\mathfrak{o} =$ the associated ring of integers
- $\mathfrak{p} =$ the maximal ideal of $\mathfrak{o}$
- $\varpi =$ a generator of $\mathfrak{p}$
- $q =$ cardinality of the residue field $\mathfrak{o}/\mathfrak{p}$, assumed to be finite

These assumptions mean that $\mathfrak{p} = (\varpi)$ is the unique prime ideal of $\mathfrak{o}$. Every non-zero element $x \neq 0$ in $\mathfrak{k}$ can be factored as $u\varpi^k$ with u a unit in $\mathfrak{o}$, and $\mathfrak{o}$ is the subset of those where $k \geq 0$ (together with 0). The norm of x in $\mathfrak{k}$ is then $|x| = q^{-k}$, and the quotient $\mathfrak{o}/\mathfrak{p}$ is isomorphic to the Galois field $\mathbb{F}_q$. The field $\mathfrak{k}$ is complete if and only if $\mathfrak{o}$ is the projective limit of quotients $\mathfrak{o}/\mathfrak{p}^n$, and in this case $\mathfrak{k}$ can be assigned naturally a locally compact topology. But most of the time $\mathfrak{o}$ might also be a ring like $\mathbb{Z}_{(p)}$, the ring of fractions a/b with b relatively prime to the prime number p. One point of the less restrictive assumption is that one might want to implement algorithmically some of the results presented here.

Contents

This essay was polished while I was giving a series of lectures in 2014 at the Tata Institute for Fundamental Research. I wish to thank the Institute for its hospitality, and my audience for their patience.

Part I. Geometry

For the material in this part, the standard reference is Chapitre II of [Serre:1977].

1. Lattices

A **lattice** in $\mathfrak{k}^2$ is any finitely generated $\mathfrak{o}$-submodule that spans $\mathfrak{k}^2$ as a vector space, for example $\mathfrak{o}^2$. One of the simplest facts about lattices is that the intersection of two, hence of any finite number of, lattices is again a lattice.

1.1. Proposition. *Every lattice in $\mathfrak{k}^2$ is free over $\mathfrak{o}$ of rank 2.*

Proof. Suppose given m generators of the $\mathfrak{o}$-submodule L, and suppose M_L to be the $2 \times m$ matrix whose columns are those generators. We shall apply what I call **integral column operations** to reduce M_L to a matrix with two linearly independent columns, which will then form an $\mathfrak{o}$-basis of L.

There are three types of integral column (or, for that matter, row) operations:

(a) performing a permutation of columns (rows);
(b) multiplying one column (row) by a unit of $\mathfrak{o}$;
(c) adding to any column (row) an integral multiple of another.

Thes column operations may be effected through multiplication on the right by a matrix in $\mathrm{GL}_m(\mathfrak{o})$, and do not change the lattice generated by the columns. The row operations can be carried out through multiplication on the left by a matrix in $\mathrm{GL}_2(\mathfrak{o})$, and amount to a change of basis in $\mathfrak{k}^2$.

Because the lattice contains a basis, there exists at least one non-zero entry in the first row. One among them will have maximal norm, and we may swap columns if necessary to get it into the upper left corner. By an operation of type (b), we may make it ϖ^m for some m, and then we may apply operations of type (c) to reduce the rest of the first row to 0.

We now look at the second row. Again beginning with a swap if necessary, possibly followed by a unit column multiplication, we may get an entry in position $(2, 2)$ of the form ϖ^n and of maximal norm in columns $c \geq 2$. We may then apply operations of type (c) to make the second row in columns 3–m vanish. The only non-zero entries are now in columns $1, 2$, and the matrix is this:

$$\begin{bmatrix} \varpi^m & 0 \\ *\varpi^k & \varpi^n \end{bmatrix} .$$

Here (and elsewhere) $*\varpi^k$ denotes an element of $\mathfrak{k}$ of the form $u\varpi^k$ with u in $\mathfrak{o}^\times$. The columns are clearly linearly independent, and this proves the Proposition. ▯

It is possible to continue on to derive a normal form for lattices. We shall see some version of this later.

1.2. Proposition. *Given an invertible 2×2 matrix g with coefficients in $\mathfrak{k}$, there exist matrices k_1, k_2 in $\mathrm{GL}_2(\mathfrak{o})$ and a diagonal matrix*

$$d = \begin{bmatrix} \varpi^m & 0 \\ 0 & \varpi^n \end{bmatrix}$$

with $m \leq n$ such that

$$g = k_1 d k_2 .$$

The diagonal matrix d is unique.

Proof. The proof is a variation on that of the previous Proposition. By column and row permutations, we may assume that the left corner entry is that of maximal norm in the entire matrix, and by a unit column operation we may assume it to be ϖ^m. Row and column operations of type (3), followed by a unit column multiplication, make it of the right form.

As for uniqueness, the greatest common divisor of the entries of the matrix is $*\varpi^m$, and $*\varpi^{m+n}$ is its determinant. ∎

1.3. Corollary. (Principal divisor theorem) *If L and M are two lattices, there exists a basis (e, f) of L and integers $m \le n$ such that $(\varpi^m e, \varpi^n f)$ is a basis of M.*

In these circumstances I call $[\varpi^m : \varpi^n]$ the matrix index of the pair (L, M) and q^{m+n} the index. If $m, n \ge 0$ this last is indeed the index, the size of L/M. If $L = \mathfrak{o}^2$ and (e, f) form an $\mathfrak{o}$-basis of M, this is also

$$\left| \det [\, e \ \ f \,] \right|^{-1}.$$

Proof. I suppose L and M to be given as 2×2 matrices λ and μ of rank 2. Use a coordinate system in which $L = \mathfrak{o}^2$. This means replacing λ by I and μ by $\lambda^{-1}\mu$. Apply the previous Proposition to it. The columns of k_1 form a basis of L, and those of $\mu k_2^{-1} = k_1 d$ form one of M. ∎

If L is a lattice with basis the columns of λ, and similarly for M and μ, then $M \subseteq L$ if and only if all entries in $\lambda^{-1}\mu$ are in $\mathfrak{o}$.

The group $\mathrm{GL}_2(\mathfrak{k})$ acts transitively on bases of $\mathfrak{k}^2$, hence also on the set of lattices. The stabilizer of $\mathfrak{o}^2$ is $\mathrm{GL}_2(\mathfrak{o})$, so with that choice of base lattice the set of lattices may be identified with $\mathrm{GL}_2(\mathfrak{k})/\mathrm{GL}_2(\mathfrak{o})$.

1.4. Proposition. *Let G be either $\mathrm{GL}_2(\mathfrak{k})$ or $\mathrm{SL}_2(\mathfrak{k})$. The stabilizer of any lattice in G is a compact open subgroup. Conversely, any compact open subgroup stabilizes some lattice.*

Proof. If $L = g\mathfrak{o}^2$ then the stabilizer is $gG(\mathfrak{o})g^{-1}$.

Suppose a compact open subgroup K given, and let $L = \mathfrak{o}^2$. If $H = K \cap G(\mathfrak{o})$ and $K = \sqcup k_i H$ then $\cap k_i L$ is a lattice stable under K. ∎

2. The tree of SL(2)

The Bruhat-Tits tree of $G = \mathrm{SL}_2(\mathfrak{k})$ is a graph $\mathfrak{X}$ on which the group $\mathrm{PGL}_2(\mathfrak{k})$ acts, and the geometry of this graph encodes in an illuminating way much of the group structure.

Definition. *The* **nodes** *of the tree are defined to be the lattices in $\mathfrak{k}^2$ modulo similarity.*

These are the analogues of the points of the real symmetric space. One point of similarity is that a point of the real symmetric space corresponds to a Euclidean metric (modulo similarity) on $\mathbb{R}^2$, whereas the choice of a lattice L in $\mathfrak{k}^2$ determines a norm on $\mathfrak{k}^2$:

$$\|v\|_L = \inf_{v \in cL} |c|\,.$$

In effect, *the choice of L here is roughly the same as specifying a unit disk in the Euclidean case.*

For each lattice L let $\langle\!\langle L \rangle\!\rangle$ be the corresponding node of the tree, or in other words its equivalence class, the set of lattices $\{\varpi^n L\}$.

If L and M are lattices, the principal divisor theorem asserts that we may find a basis (e, f) of L such that $(\varpi^m e, \varpi^n f)$ is a basis of M, for some integers $m \le n$. The difference $n - m$ is an invariant of the similarity class of M, so that the definition $\mathrm{inv}(\langle\!\langle L \rangle\!\rangle : \langle\!\langle M \rangle\!\rangle) = n - m$ makes sense. This invariant is 1 if and only if the two nodes possess representatives L and M with $L/M \cong \mathfrak{o}/\mathfrak{p}$, or equivalently

$$\varpi L \subset M \subset L\,.$$

In this case, I'll call them **neighbours**.

Definition. *There is an* **edge** *of the Bruhat-Tits tree between two nodes if and only if they are neighbours.*

The nodes linked by an edge to $\langle\!\langle L \rangle\!\rangle$ thus correspond to lines of $L/\varpi L \cong (\mathbb{F}_q)^2$, and there are $q + 1$ of them.

If u and v form a basis of $\mathfrak{k}^2$, let $[u,v]$ be the lattice they span and $\langle\!\langle u,v\rangle\!\rangle$ the corresponding node. Fix basis vectors and particular nodes

$$\begin{aligned} u_0 &= (1,0) \\ v_0 &= (0,1) \\ \nu_m &= \langle\!\langle u_0, \varpi^m v_0\rangle\!\rangle = \langle\!\langle \varpi^{-m} u_0, v_0\rangle\!\rangle \quad (m \in \mathbb{Z})\,, \end{aligned}$$

so that $\mathfrak{o}^2 = \nu_0$. If g is in $\mathrm{GL}_2(\mathfrak{k})$, it takes a lattice $[u,v]$ to the lattice $[gu,gv]$. The group $\mathrm{GL}_2(\mathfrak{k})$ preserves equivalence of lattices, and it also preserves the lattice pair invariant. Hence it transforms edges to edges, and therefore acts on the graph $\mathfrak{X}$. By definition, this action factors through $\mathrm{PGL}_2(\mathfrak{k})$. The group $\mathrm{PGL}_2(\mathfrak{k})$ acts transitively on nodes of the tree. The stabilizer in $\mathrm{PGL}_2(\mathfrak{k})$ of the node ν_0 is the maximal compact subgroup $\mathrm{PGL}_2(\mathfrak{o})$, which is therefore the analogue in $\mathrm{PGL}_2(\mathfrak{k})$ of the image of $\mathrm{O}(2)$ in $\mathrm{PGL}_2(\mathbb{R})$.

If

$$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & \varpi \end{bmatrix}$$

then $\alpha(\nu_n) = \nu_{n+1}$ for all n.

The principal divisor theorem gives us the **Cartan decompositions**

$$\mathrm{GL}_2(\mathfrak{k}) = \mathrm{GL}_2(\mathfrak{o}) A^{++}\, \mathrm{GL}_2(\mathfrak{o}), \quad A^{++} = \left\{ \begin{bmatrix} \varpi^m & 0 \\ 0 & \varpi^n \end{bmatrix} \,\middle|\, m \le n \right\}$$

and

$$\mathrm{SL}_2(\mathfrak{k}) = \mathrm{SL}_2(\mathfrak{o}) A^{++}\, \mathrm{SL}_2(\mathfrak{o}), \quad A^{++} = \left\{ \begin{bmatrix} 1/\varpi^m & 0 \\ 0 & \varpi^m \end{bmatrix} \,\middle|\, m \ge 0 \right\}.$$

Suppose that $L = \mathfrak{o}^2$ and that the matrix index of $[L\colon M]$ is $[\varpi^m\colon \varpi^n]$. I call $\langle\!\langle M\rangle\!\rangle$ **even** or **odd** depending on the parity of $n - m$. The action of $\mathrm{SL}_2(\mathfrak{k})$ preserves this parity, and in fact there are exactly two orbits of the group $\mathrm{SL}_2(\mathfrak{k})$ among the nodes of the tree, each one corresponding to lattices of a given parity.

I have adopted the notation A^{++} from Ian Macdonald. For higher rank groups, the acute cone A^{++} is to be distinguished from the obtuse cone A^{+}.

A **chain** in the tree $\mathfrak{X}$ is a finite or half-infinite sequence of nodes linked by edges. Every chain may be represented by a sequence of lattices

$$L_0 \supset L_1 \supset \ldots \supset L_n \supset L_{n+1} \ldots$$

with

$$L_n \supset L_{n+1} \supset \varpi L_n$$

for all n. A **standard chain** is one of the form

$$\nu_0 — \nu_{-1} — \nu_{-2} — \cdots\,,$$

whether finite or infinite. I'll call a chain **simple** if, like this one, it does not back-track.

2.1. Proposition. *Every finite simple chain in the building may be transformed to a standard one by an element of* $\mathrm{GL}_2(\mathfrak{k})$*. If* $\mathfrak{k}$ *is complete, this remains true for all half-infinite simple chains.*

Proof. The proof is by induction on the length of the associated chain of lattices

$$L_0 \supset L_1 \supset \cdots \supset L_n\,,$$

in which we may assume $L_k \supset L_{k+1} \supset \varpi L_k$ for all k. We may find g transforming L_0 to $\mathfrak{o}^2$, so that we may in fact assume $L_0 = \mathfrak{o}^2$.

If $n = 1$, the image of L_1 in $L_0/\varpi L_0$ is a line. We can find a matrix $\overline{g}$ in $\mathrm{GL}_2(\mathbb{F})$ transforming it to the line through $(0, 1)$, and if g in $\mathrm{GL}_2(\mathfrak{o})$ has image $\overline{g}$, then gL_1 is $[\varpi, 1]$, corresponding to the node ν_{-1}.

The first part of the Proposition will now follow by induction from this:

2.2. Lemma. *Suppose given a chain* (L_i) *(*$0 \le i \le n+1$*) with* $L_i = [\varpi^i, 1]$ *for* $1 \le i \le n$*. There exists* $x \in \mathfrak{p}^n$ *such that*

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

takes every L_i *to* $[\varpi^i, 1]$.

Implicit in this statement is that when x lies in $\mathfrak{p}^n$ this matrix takes $[\varpi^i, 1]$ to itself for every $i \le n$.

Proof of the Lemma. The lattice $M = L_{n+1}$ is a lattice of index q in $L = L_n$ containing $\varpi L = [\varpi^{n+1}u_0, \varpi v_0]$ but not equal to $\varpi L_{n-1} = [\varpi^n u_0, \varpi v_0]$. Let V be the finite vector space $L/\varpi L$. It has as basis $[e, f]$ the images of $\varpi^n u_0$, v_0. The image of $\varpi L_{n-1} = [e, \varpi f]$ projects to the line through $(1, 0)$. Any other line is that through some $(y, 1)$ which is transformed to $(0, 1)$ by the upper unipotent matrix

$$\begin{bmatrix} 1 & -y \\ 0 & 1 \end{bmatrix}$$

in $\mathrm{GL}_2(\mathbb{F}_q)$. The matrix we want is then

$$\gamma = \begin{bmatrix} 1 & \varpi^n x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x_n \\ 0 & 1 \end{bmatrix},$$

where x in $\mathfrak{o}$ has image $-y$ in $\mathbb{F}_q$. □

To conclude the proof of Proposition 2.1 in case $\mathfrak{k}$ is complete, note that under this assumption the product of the matrices

$$\begin{bmatrix} 1 & x_n \\ 0 & 1 \end{bmatrix}$$

will then converge. □□

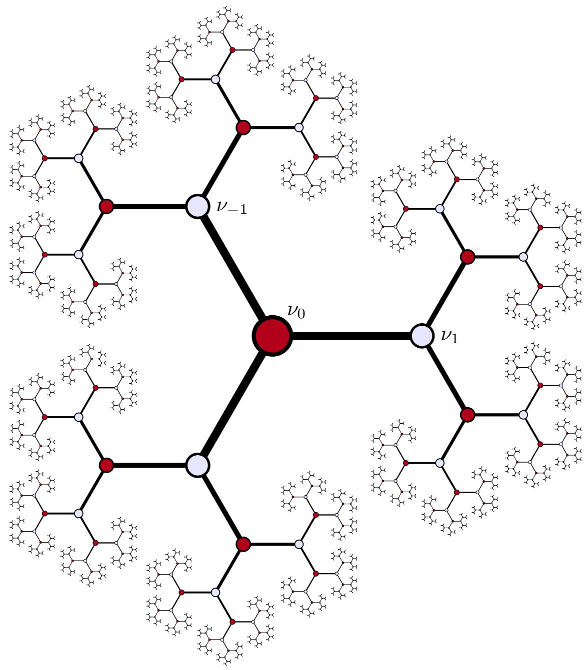
2.3. Corollary. *The distance* $|xy|$ *between two nodes* x *and* y *is the pair invariant* $\mathrm{inv}(x\colon y)$.

Only a short additional argument is necessary to prove:

2.4. Corollary. *The graph* $\mathfrak{X}$ *is a connected tree.*

Proof. If M is any lattice, we may find a basis (e, f) of $L = \mathfrak{o}^2$ such that some $(\varpi^m e, \varpi^n f)$ is a basis of M. Replacing M by some multiple of itself, then we may assume $m = 0$, $n \ge 0$. That means that there exists a chain of lattices $[u_0, \varpi^k v_0]$ from L to M. This proves that the graph is connected. That it is a tree follows from the preceding Proposition, since no standard chain has a loop. □

The node ν_0 may be chosen as root. The structure of $\mathfrak{X}$ is completely determined by the properties: (a) it is connected; (b) it is a tree; (c) every node has $q+1$ neighbours. For example, when $q = 2$ it looks like this:



For a more detailed picture, take a look at the second Appendix.

3. The action of K

The group $K = \mathrm{GL}_2(\mathfrak{o})$ fixes the point ν_0. How does it act on the tree?

The nodes at distance 1 from ν_0 may be identified with the points of $\mathbb{P}^1(\mathfrak{o}/\mathfrak{p})$. There is a similar description of those at distance m. The space $\mathbb{P}^1(\mathfrak{o}/\mathfrak{p}^m)$ is that of all pairs $\lambda = (x, y)$ with x, y in $\mathfrak{o}/\mathfrak{p}^m$, at least one of them a unit, modulo scalar multiplication by units of $\mathfrak{o}$. To such a pair λ corresponds the lattice $L_\lambda = \mathfrak{o}\lambda + \varpi^m\mathfrak{o}^2$, and then in turn the node $\langle\langle L_\lambda \rangle\rangle$.

3.1. Proposition. *The map taking λ to $\langle\langle L_\lambda \rangle\rangle$ is a K-equivariant bijection of $\mathbb{P}^1(\mathfrak{o}/\mathfrak{p}^m)$ with the nodes of $\mathfrak{X}$ at distance m from ν_0.*

3.2. Exercise. *Prove this.*

The distance between ν_0 and ν_m is $|m|$. The Cartan decomposition implies that K acts transitively on the $q^{m-1}(q+1)$ nodes at distance m from ν_0. It is a simple consequence of the previous Proposition that this is true for K_0 as well.

The fixed points of the congruence group

$$K_m = \left\{ g \in \mathrm{GL}_2(\mathfrak{o}) \,\middle|\, g \equiv I \ (\mathrm{mod}\ \mathfrak{p}^m) \right\}.$$

are those at distance $\le m$ from ν_0. Any other node may be transformed by k in K_0 to some ν_n with $n > m$. The path from ν_n to ν_0 intersects the fixed points at node ν_m. The K_m-orbit of ν_n is the set of all nodes at distance n from ν_0 and $n - m$ from ν_m (that is to say, at distance $n - m$ from ν_m and on the outside of the disk fixed by K_m).

The group $K_0 = SL_2(\mathfrak{o})$ fixes ν_0, representing the lattice $\mathfrak{o}^2$, while its twin $K_1 = \alpha K \alpha^{-1}$, with

$$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & \varpi \end{bmatrix},$$

fixes its neighbour $\alpha(\nu_0) = \nu_1$. Since every compact subgroup fixes some lattice, these two subgroups of $\mathrm{SL}_2(\mathfrak{k})$ are maximal compact. They are not conjugate to each other.

4. Apartments

Recall that

$$\nu_m = [u_0, \varpi^m v_0] \quad (m \in \mathbb{Z})\,.$$

A **branch** from a node is an infinite simple chain starting at that node. One branch is the chain $\mathcal{B}_0$ made up of the nodes ν_m for $m \le 0$, and another is the chain $\mathcal{B}_\infty$ of the ν_m with $m \ge 0$. Proposition 2.1 says that any branch can be transformed to $\mathcal{B}_0$ by an element of $\mathrm{GL}_2(\mathfrak{k})$ if $\mathfrak{k}$ is complete, and hence that $\mathrm{GL}_2(\mathfrak{k})$ then acts transitively on branches.

An **apartment** is the union of two branches from one node with no common edge. One apartment is

$$\mathcal{A} = \mathcal{B}_0 \cup \mathcal{B}_\infty = \{\nu_m \,|\, m \in \mathbb{Z}\}\,.$$

THE ACTION OF G. Elements of G take apartments to apartments.

4.1. Proposition. *If $\mathfrak{k}$ is complete, the group $\mathrm{GL}_2(\mathfrak{k})$ acts transitively on apartments.*

Proof. It suffices to prove this when one of the apartments is $\mathcal{A}$. Suppose given some other apartment χ, say with two branches χ_0 and χ_∞ running out in opposite directions from the same node. Since $\mathrm{GL}_2(\mathfrak{k})$ acts transitively on branches, we may transform χ_∞ to the branch $\mathcal{B}_\infty$. In effect, we may assume $\chi_\infty = \mathcal{B}_\infty$. By Lemma 2.2 we may now find a matrix

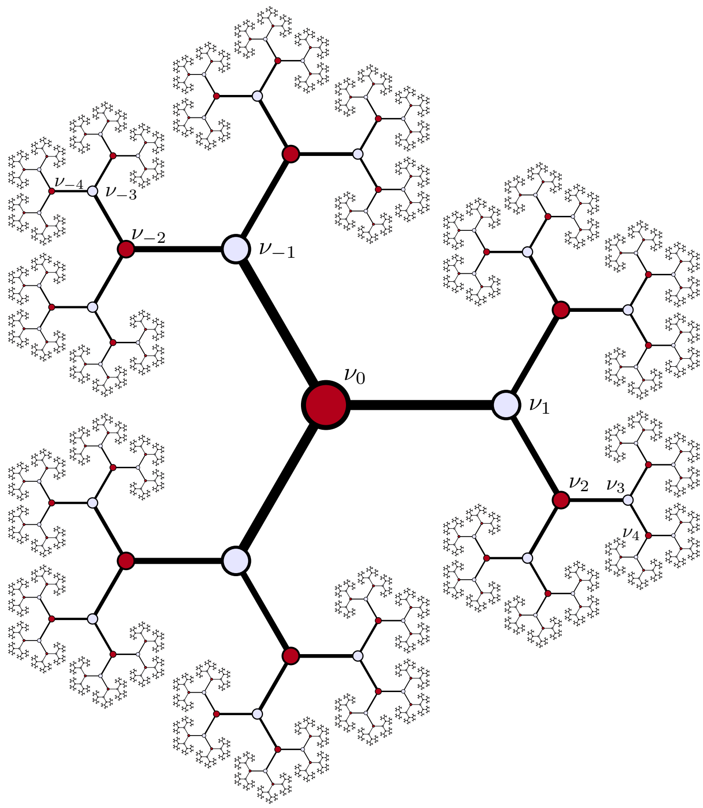
$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

with x in $\mathfrak{o}$ that transforms the other branch χ_0 of X into the other branch $\mathcal{B}_0$ of $\mathcal{A}$. But these matrices fix the all the nodes on $\mathcal{B}_\infty$, so X is taken to $\mathcal{A}$. ∎

The limit of the lattices $[u_0, \varpi^n v_0]$ as $n \to \infty$ is the line in $\mathfrak{k}^2$ through u_0. It is called the **end** of the chain $\mathcal{B}_\infty$. The notation for (say) $\mathcal{B}_\infty$ is motivated by this observation, since by convention this line is expressed as ∞ in $\mathbb{P}^1(\mathfrak{k})$. Every point of $\mathbb{P}^1(\mathfrak{k})$ is the end of some branch, and if $\mathfrak{k}$ is complete every branch ends at a point of $\mathbb{P}^1(\mathfrak{k})$. The parallel with what happens for $\mathrm{SL}_2(\mathbb{R})$ is striking.

Since $\mathrm{GL}_2(\mathfrak{k})$ acts transitively on apartments, every apartment is stabilized by a single split torus. If its ends are λ and μ in $\mathbb{P}^1(\mathfrak{k})$, these lines are the eigenspaces of that torus. The apartment can be characterized as containing all the nodes corresponding to lattices that split compatibly with the direct sum $\lambda \oplus \mu$.

Here is an indication of a graphical rendering of the apartment $\mathcal{A}$:



It is a matter of convention which infinite geodesic I choose to be standard, since all are equivalent. The choice I have made is the conventional one, and is convenient for visualization.

Proposition 4.1 also implies:

4.2. Proposition. *Suppose $\mathfrak{k}$ to be complete. Given two apartments and an oriented edge in each, there exists g in $\mathrm{GL}_2(\mathfrak{k})$ inducing an isometry of one with the other mapping one oriented edge to the other.*

The stabilizer in K of the node $g\,\nu_0$ is $gKg^{-1} \cap K$. This is the same as

$$\begin{bmatrix} a & b \\ \varpi^n c & d \end{bmatrix} \quad \text{if} \quad g = \begin{bmatrix} 1 & 0 \\ 0 & \varpi^n \end{bmatrix} \quad (n \ge 0)$$

with a, b, c, d in $\mathfrak{o}$. As n gets larger and larger, this has as limit the group $K \cap P$, and this brings out again that asymptotically the building is isomorphic to $K/K \cap P$ or G/P. More precisely, we can see that the points at distance m from ν_0 correspond naturally to the points of $K_m = \mathrm{PGL}_2(\mathfrak{p}^m)\backslash\mathbb{P}^1(k)$.

4.3. Exercise. *Prove that if $\mathfrak{k}$ is complete, the group $\mathrm{PGL}_2(\mathfrak{k})$ is the group of all isometries of $\mathfrak{X}$. (Hint: recall that $\mathrm{GL}_2(\mathfrak{k})$ acts transitively on $\mathbb{P}^1(\mathfrak{k})$.)*

THE STRUCTURE OF AN APARTMENT. Let

$$A = \text{ the group of diagonal matrices in } \mathrm{GL}_2(\mathfrak{k})\,.$$

Elements of A act as translations on $\mathcal{A}$. The compact subgroup $A(\mathfrak{o})$ acts trivially on it, so the action factors through $A/A(\mathfrak{o})$. The matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & \varpi^n \end{bmatrix}$$

translates ν_m to ν_{m-1}. Since

$$\begin{bmatrix} \varpi^m & 0 \\ 0 & \varpi^{-m} \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & \varpi^{-2m} \end{bmatrix} \text{ modulo scalar matrices ,}$$

the subgroup $A_1 = A \cap \mathrm{SL}_2(\mathfrak{k})$ preserves parity in its shifts. The element

$$\sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

also takes $\mathcal{A}$ to itself, reflecting ν_m to ν_{-m}. The group generated by $A_1/A_1(\mathfrak{o})$ and σ is $N_G(A)/A)$. Its quotient by $A(\mathfrak{o})$ is the affine Weyl group W_{aff} of of the root system of SL_2. It is a Coxeter group with elementary reflections σ and

$$\begin{bmatrix} 1 & 0 \\ 0 & \varpi \end{bmatrix} .$$

It contains all reflections in the nodes ν_m of even parity. The segment ν_0 — ν_1 is a strict fundamental domain for the action of W_{aff} on $\mathcal{A}$.

4.4. Exercise. *Prove that the group generated by A and σ is precisely the stabilizer of $\mathcal{A}$.*

We shall find the following useful later on:

4.5. Proposition. *Any two branches running from ν_0 but not containing any edges in $\mathcal{A}$ are taken into each other by some element of $A(\mathfrak{o})$.*

Proof. The nodes at distance n from ν_0 and at distance n from $\mathcal{A}$ correspond to points $(x, 1)$ in $\mathfrak{o}/\mathfrak{p}^n$ with x not in $\mathfrak{p}$. Given this, the proof becomes obvious. ▯

The analogue for $\mathrm{SL}_2(\mathfrak{k})$ is not true, as is already easy to see for nodes at distance 1 from $\mathcal{A}$ when p is odd.

One feature of the apartment $\mathcal{A}$ that becomes more significant for groups of higher rank is that its structure mirrors that of the unipotent subgroup of upper triangular matrices. This group is filtered by subgroups

$$\begin{bmatrix} 1 & \mathfrak{p}^n \\ 0 & 1 \end{bmatrix} ,$$

and the set of points on $\mathcal{A}$ fixed by this subgroup consists of all those on the branch

$$\nu_{-n} — \nu_{-n+1} — \nu_{-n+2} — \cdots .$$

4.6. Exercise. *Describe all the orbits of A on the tree, and of $A \cap \mathrm{SL}_2(\mathfrak{k})$. Draw a few of the latter on the picture of the tree.*

5. The Iwahori factorization

We have seen that the principal divisor theorem tells us something about the structure of the building. There is another similar result that is just as important, but a bit more difficult both to prove and to implement algorithmically.

Let B be the inverse image in $K = \mathrm{GL}_2(\mathfrak{o})$ of the upper triangular matrices in $\mathrm{GL}_2(\mathbb{F}_q)$, those matrices in $\mathrm{GL}_2(\mathfrak{o})$ of the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $c \in \mathfrak{p}$. It is also the intersection

$$K \cap \alpha K \alpha^{-1} \quad \left(\alpha = \begin{bmatrix} 1 & 0 \\ 0 & \varpi \end{bmatrix} \right).$$

In acting on $\mathfrak{X}$ it therefore fixes the nodes ν_0 and ν_1, as well as all points on the edge between them. The conjugates of B are called **Iwahori subgroups**. Each acts trivially on exactly one edge of $\mathfrak{X}$, but rotates branches running away from each end of that edge.

The stabilizer of the edge from ν_0 to ν_1 is a bit larger than B, and contains the matrix

$$\begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix}.$$

But this reverses the orientation of this edge.

What we shall look at now is a generalization of both the principal divisor theorem and the Bruhat decomposition.

5.1. Proposition. *Every g in $\mathrm{GL}_n(\mathfrak{k})$ factors as $g = b_1 \widetilde{w} b_2$ where b_i is an element of B and $\widetilde{w}$ is in $\widetilde{W}$. The element $\widetilde{w}$ is unique.*

I recall that $\widetilde{W}$ is the subgroup of matrices

$$\begin{bmatrix} \varpi^m & 0 \\ 0 & \varpi^n \end{bmatrix}, \quad \begin{bmatrix} 0 & \varpi^m \\ \varpi^n & 0 \end{bmatrix}.$$

Proof. The proof will be constructive, and will be based on an algorithm involving **elementary Iwahori operations** on columns:

- add to a column d a multiple xc of a previous column c by some x in $\mathfrak{o}$;
- add to a column c a multiple xd of a subsequent column by x in $\mathfrak{p}$;
- multiply a column by a unit in $\mathfrak{o}$;

and also on rows:

- add to a row r a multiple xs of a subsequent row s with x in $\mathfrak{o}$;
- add to a row r a multiple xs of a previous row with x in $\mathfrak{p}$;
- multiply a row by a unit in $\mathfrak{o}$;

Each of these column (row) operations amounts to right (resp. left) multiplication by an Iwahori matrix, for example:

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u + xv \\ v \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ \varpi x & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ \varpi x u + v \end{bmatrix}$$
$$[\, u \;\; v \,] \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = [\, u \;\; xu + v \,]$$
$$[\, u \;\; v \,] \begin{bmatrix} 1 & 0 \\ \varpi x & 1 \end{bmatrix} = [\, u + \varpi x v \;\; v \,] \,.$$

The proof starts by looking at the first column of the matrix

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} .$$

If $|a| \ge |c|$ then c/a lies in $\mathfrak{o}$ and we can subtract (c/a) times row one from row two. If $|a| < |c|$ then a/c lies in $\mathfrak{p}$ and we can subtract (a/c) times row two from row one. After multiplying a row by a unit, we find ourselves with one of the following configurations:

$$\begin{bmatrix} \varpi^m & *\varpi^k \\ 0 & \varpi^n \end{bmatrix}, \quad \begin{bmatrix} 0 & \varpi^n \\ \varpi^m & *\varpi^k \end{bmatrix} .$$

(1) Let's look first at the first case . . .

$$g = \begin{bmatrix} \varpi^m & *\varpi^k \\ 0 & \varpi^n \end{bmatrix} .$$

If $k \ge m$ we can subtract a multiple of the first column from the second to get

$$g = \begin{bmatrix} \varpi^m & 0 \\ 0 & \varpi^n \end{bmatrix} .$$

If $k \ge n$ we can subtract a multiple of the second row from the first to get the same matrix.

So now we may assume $k < m$ and $k < n$. Subtract a multiple of row one from row two. This gives, after a unit multiplication,

$$\begin{bmatrix} \varpi^m & \varpi^k \\ *\varpi^{m+n-k} & 0 \end{bmatrix} .$$

Subtract a multiple of the second column from the first to get

$$\begin{bmatrix} 0 & \varpi^k \\ \varpi^{m+n-k} & 0 \end{bmatrix}$$

which is in $\widetilde{W}$.

(2) . . . and then in the second:

$$\begin{bmatrix} 0 & \varpi^n \\ \varpi^m & *\varpi^k \end{bmatrix} .$$

5.2. Exercise. *Finish the argument by dealing with this case.*

Uniqueness follows from Proposition 4.2. ∎

This decomposition of $\mathrm{GL}_2(\mathfrak{k})$ into double cosets, together with the effect of multiplication by generators, is in some sense a complete description of the group, as the usual Bruhat decomposition is for reductive groups over arbitrary fields.

6. Orbits of N and the Iwasawa factorization

The group N of all upper triangular unipotent matrices fixes the end of the branch $\{\nu_m \,|\, m \geq 0\}$, which amounts to ∞ in $\mathbb{P}^1(\mathfrak{k})$. Its subgroup $N(\mathfrak{p}^{-m})$ of all

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

with $x \in \mathfrak{p}^{-m}$ fixes all nodes ν_k with $k \geq m$. As $m \to \infty$ this group expands, consistently with what happens at the end point.

6.1. Proposition. *The orbits of $N(\mathfrak{p}^{-m})$ are the points at a fixed distance from ν_m other than those on a path starting back to ν_{m+1}.*

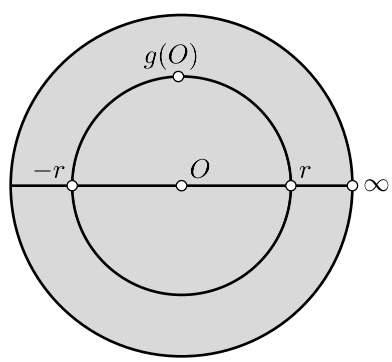
6.2. Exercise. *Prove this. (Hint: look at Lemma 2.2.)*

6.3. Corollary. (Iwasawa factorization) *Every g in $\mathrm{GL}_2(\mathfrak{k})$ may be expressed as nak with n in N, a in A, $k \in K$.*

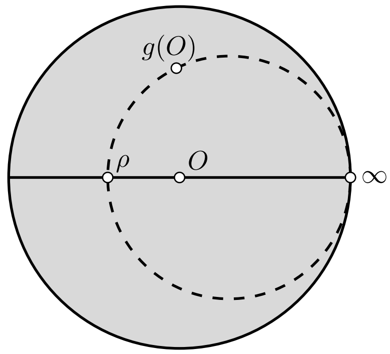
There is an important relation between the Cartan and Iwasawa factorizations. I recall first what happens for $G = \mathrm{SL}_2(\mathbb{R})$. Let K be $\mathrm{SO}(2)$, N be the N is the group of unipotent upper triangular matrices, A be the group of diagonal matrices, and $P = AN$.

The **Cartan factorization** asserts that $G = KAK$. Geometrically things are simple. We first represent G by Möbius transformations of the unit disk, conjugating the more familiar action on the upper half plane by the Cayley transform. If $g = k_1 a k_2$ then it is also $k_1 a^{-1} k_2$. Choose a so that $r = a(O)$ lies in the interval $(0, 1)$. Then $g(P) = k_1 a(O)$ will lie at angle -2θ on the circle of radius r around O if

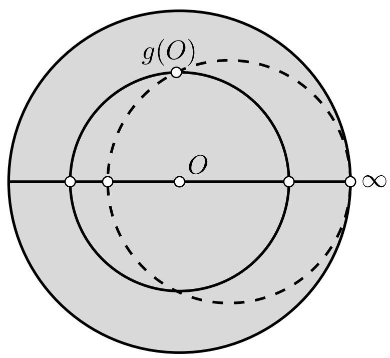
$$k_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} .$$



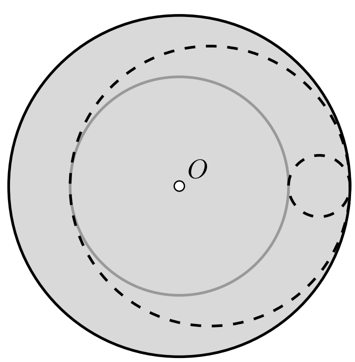
The **Iwasawa factorization** asserts that $G = NAK$. There is a simple geometric description here, too. If $g = nak$ then $g(O) = na(O)$, which is on the N-orbit of r. The N-orbits are the circles inside the unit disk and tangent to 1. So we find the circle of this sort which passes through $g(O)$, and then find where it intersects the real line.



How do these different factorizations compare? The answer is that if $g(O)$ lies on the circle of radius r around O and on the N-orbit through ρ then $-r \le \rho \le r$.



6.4. Exercise. *Prove this. (Hint: it follows very easily from the following picture. Keep in mind that orbits can't cross.)*



Here is the generalization of this for $\mathrm{SL}_2(\mathfrak{k})$:

6.5. Proposition. *Suppose g in $\mathrm{GL}_2(\mathfrak{k})$. Then*

(a) if $g = nak$ is its Iwasawa factorization, then it has Cartan factorization $g = k_1 d k_2$ with $a(\nu_0)$ in the convex hull in $\mathcal{A}$ of (a.k.a. line segment between) $d(\nu_0)$ and $d^{-1}(\nu_0)$;
(b) if ν lies in $\mathcal{A}$ then the intersection of its K-orbit and N-orbit is just ν itself.

The proof of the Proposition follows an argument suggested by this picture. But first we need to know about a certain construction in the tree.

6.6. Lemma. *Given an apartment $\mathcal{A}$ and a chamber C in it, there exists a unique map $\rho = \rho_{\mathcal{A},C}$ from the tree onto $\mathcal{A}$ with these properties:*

(a) *ρ is the identity on $\mathcal{A}$;*
(b) *if B_C is the Iwahori subgroup fixing C, then $\rho(bx) = \rho(x)$ for all b in B_C, x in the tree.*

This map shortens paths.

Proof. The proof is geometric. Choose a point y in the middle of C. If x is an arbitrary point in the building, there exists a unique geodesic from x to y. But there also exists a unique geodesic of the same length in $\mathcal{A}$ that agrees with the first for points inside C. Map x to its endpoint. ∎

It is simple enough to compute this retraction for the standard apartment and chamber. Suppose x given in the tree. There exists a unique x_0 in the edge $\nu_0 — \nu_1$ such that $x = bwx_0$ b in B, w in the normalizer of A in SL_2. This is because this edge is a fundamental domain for this group in $\mathcal{A}$,Then Then $\rho(x) = wx_0$.

Now I take up the proof of Proposition 6.5 again.

Proof. Suppose $g = nak$. Then n will fix some ray of points on the apartment $\mathcal{A}$, since N fixes ∞. Suppose it fixes a chamber C. Let ρ be the retraction $\rho_{\mathcal{A},C}$ Then a is $\rho(x)$. The matrix d is determined by the geodesic from ν_0 to x, and the image of this path under ρ has length at least that of ρ. But this means exactly what the Proposition asserts.

I leave claim (b) as an exercise. ∎

A generalization of the result for arbitrary real semi-simple groups has been proved in [Kostant:1973], and this in turn has been generalized in [Atiyah:1982]. A first step towards a generalization of this for $\mathfrak{p}$-adic groups can be found in §4.4 of [Bruhat-Tits:1972] (see also Theorem 2.6.11(3)–(4) of [Macdonald:1971]), and the precise $\mathfrak{p}$-adic analogue of Kostant's result can be found in [Hitzelberger:2010].

7. A fixed point theorem

The retraction defined in the previous section has another interesting feature. If x and y are two points on the tree, the geodesic between them retracts onto a polygonal line on $\mathcal{A}$, so that

$$|\rho(x)\rho(y)| \le |xy| \,.$$

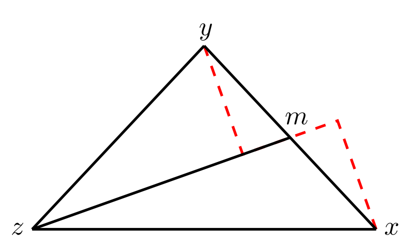
For any two points x, y in the tree, let $m_{x,y}$ be the midpoint of the geodesic between them. The following asserts that in some sense the tree has non-positive curvature.

7.1. Proposition. (Bruhat-Tits inequality) *Given points x, y, and z on the tree, let $m = m_{x,y}$. Then*

$$|zm|^2 + |mx|^2 \le \frac{|zx|^2 + |zy|^2}{2} \,.$$

Keep in mind that $|mx| = |my|$.

Proof. This is an equality on an apartment, according to a theorem of Pappus. It is an easy vector calculation, or can be proved by applying Pythagoras' Theorem a few times.



In general, fix an apartment $\mathfrak{A}$ containing x and y and let E be an edge containing m. If ρ is the retraction determined by $\mathfrak{A}$ and E, then

$$|zx|^2 + |zy|^2 \geq |\rho(z)x|^2 + |\rho(z)y|^2 = 2|\rho(z)m|^2 + 2|mx|^2 = 2|zm|^2 + 2|mx|^2 .$$

Any metric space satisfying this condition is called **semi-hyperbolic**. All Bruhat-Tits buildings and all non-compact real symmetric spaces fall in this category. Any two points on a semi-hyperbolic space have a unique midpoint beteeen them. The sphere, for example, is not semi-hyperbolic.

If X is any bounded set in the tree and c a point in the tree, there exists $R \geq 0$ such that $|cx| < R$ for all x in X. Define $R_c(X)$ to be the least upper bound of all such R, and define the radius R_X of X to be the least upper bound of all $R_c(X)$ as c varies. A circumcentre for X is a point c with the property that $|cx| \leq R_X$ covers X. The following is an observation due to Serre.

7.2. Corollary. *Every bounded subset of the tree has a unique circumcentre.*

Proof. Choose a sequence c_i such that $R_{c_i}(X) \to R(X)$. The semi-hyperbolic inequality implies that it is a Cauchy sequence.

The case we shall be interested in is that in which X is a finite set. Is there a simple algorithm to find its circumcentre?

We have now a new proof of a result we have seen before. In contrast to the earlier proof, this one can be expanded into one for all buildings.

7.3. Corollary. *Any compact subgroup of* $\mathrm{SL}_2(\mathfrak{k})$ *fixes some point on the tree.*

Proof. Because it fixes the circumcentre of any orbit.

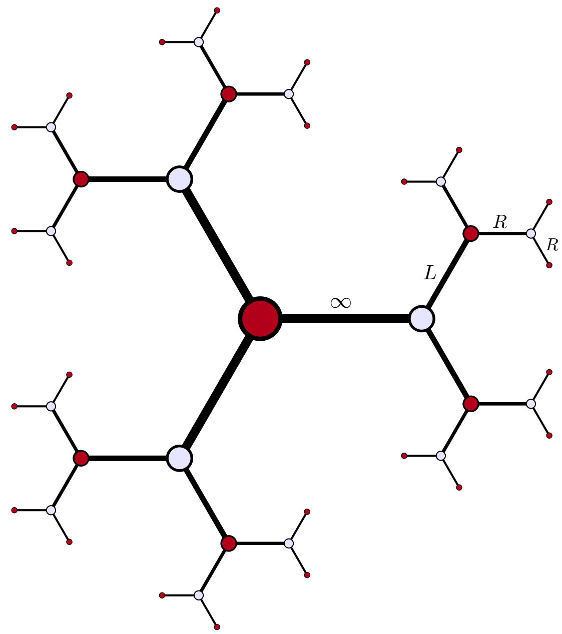
Hence the subgroups fixing nodes of the tree are maximal compact subgroups of $\mathrm{SL}_2(\mathfrak{k})$, and there are two conjugacy classes of them. For $\mathrm{PGL}_2(\mathfrak{k})$ there is just one.

8. *Appendix.* Intelligent tree drawing

I'll discuss here how to draw the tree for $G = \mathrm{SL}_2(\mathbb{Q}_2)$. This can be done at several levels of sophistication.

First I'll describe how to draw the basic tree, what I call the dumb one. This is the tree simply as a geometric object, a collection of branches, and no association with an automorphism group. There are a number of parameters that determine it—the dimensions of nodes and edges, how these should shrink with depth, how edges are arrayed around a node, and colour choice. The drawing is then done by recursion, either explicitly or implicitly, with a stack, out to some given depth. Each node is assigned an angle in, as well as location. A node draws itself, and if the specified depth has not been exceeded it then draws edges out to neighbours, and finally draws those neighbours by recursion. I'll leave details as an exercise.

Still on the purely geometric level of drawing is a procedure for drawing nodes along a path like ∞LRR as indicated in this figure:



So it is relatively simple to draw the tree as a geometric object. But for really useful (i.e. 'intelligent') drawings we want to translate back and forth between nodes in the drawing and lattices, or between nodes and elements of G. That is to say, suppose $\mathfrak{o} = \mathbb{Z}_{(2)}$, the localization of $\mathbb{Z}$ at (2). We use the action of $\mathrm{GL}_2(\mathbb{Q})$ on the tree, rather than that of the 2-adic field, because it is computationally feasible. The nodes in the tree are the same for localizations as for completions, and pretty much the only difference between the two groups is that the group over $\mathbb{Q}$ is smaller and does not act transitively on apartments.

In other words, we want to associate to each node in the geometric tree a 2×2 invertible matrix in $\mathrm{GL}_2(k)$, and vice-versa. This means building a bijection between certain g and paths like ∞LRR as explained above.

What seems to me the best way is to use the Iwahori factorization. I assume that we know how to factor every g in G as $b_1 w b_2$ according to $G = B\widetilde{W}B$, where B is the Iwahori subgroup. Since $\widetilde{W} = AW$, this also gives us $G = BAK$, where (I recall) $K = \mathrm{GL}_2(\mathfrak{o})$. So we first write $g = b\alpha^n k$, with

$$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & \varpi \end{bmatrix} .$$

The node $g\nu_n$ will be at distance $|n|$ from ν_0, and in the B-orbit of α^n. The map $b \mapsto b\alpha^n K$ is a bijection between this orbit and $B/B \cap \alpha^n K \alpha^{-n}$, which is not difficult to parametrize explicitly.

The cases (1) $n > 0$ and (2) $n \le 0$ are different.

(1) For $n > 0$ the subgroup $B \cap \alpha^n K \alpha^{-n}$ is that of all integral matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $c \in \mathfrak{p}^n$.

$$x \longmapsto \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$$

from $\mathfrak{p}/\mathfrak{p}^n$ is a bijection with $B/B \cap \alpha^n K \cap \alpha^{-n}$.

(2) When $n \le 0$, for similar reasons, the map

$$x \longmapsto \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

defines a bijection of $\mathfrak{o}/\mathfrak{p}^n$ with $B/B \cap \alpha^n K \cap \alpha^{-n}$. The corresponding maps back from B are

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto c/a \; (n > 0), \quad a/b \; (n < 0) \, .$$

This makes $q^n + q^{n-1}$ in all, which is indeed the size of $\mathbb{P}^1(\mathfrak{o}/\mathfrak{p}^n)$.

The nodes of the tree are parametrized by sequences of L and R, either from the root node if $n \le 0$ or from the infinite node if $n > 0$. So we must define the map from the section matrices parametrized by x modulo $\mathfrak{p}^n$ to such a sequence, and vice-versa.

Suppose $n > 0$. We are given x as an even integer $2y$ modulo 2^n. We find the bits of y and read from low order $i = 0$ up to order $i = n - 1$, translating bit i:

$i = 0, 2, 4, \ldots$	odd $\mapsto L$,	even $\mapsto R$,
$i = 1, 3, 5, \ldots$	even $\mapsto L$,	odd $\mapsto R$

Now suppose $n \le 0$. We are given x as an integer modulo $2^{|n|}$. We find the bits of x and read from low order $i = 0$ up to order $i = |n| - 1$, translating bit i:

$i = 0, 2, 4, \ldots$	odd $\mapsto L$,	even $\mapsto R$
$i = 1, 3, 5, \ldots$	even $\mapsto L$,	odd $\mapsto R$

In short, the rules are the same! They can be summarized in a table:

bit index parity	bit parity	L or R
0	0	R
0	1	L
1	0	L
1	1	R

But now you can see that they can be formulated most succinctly as addition modulo 2, with $R = 0$, $L = 1$.

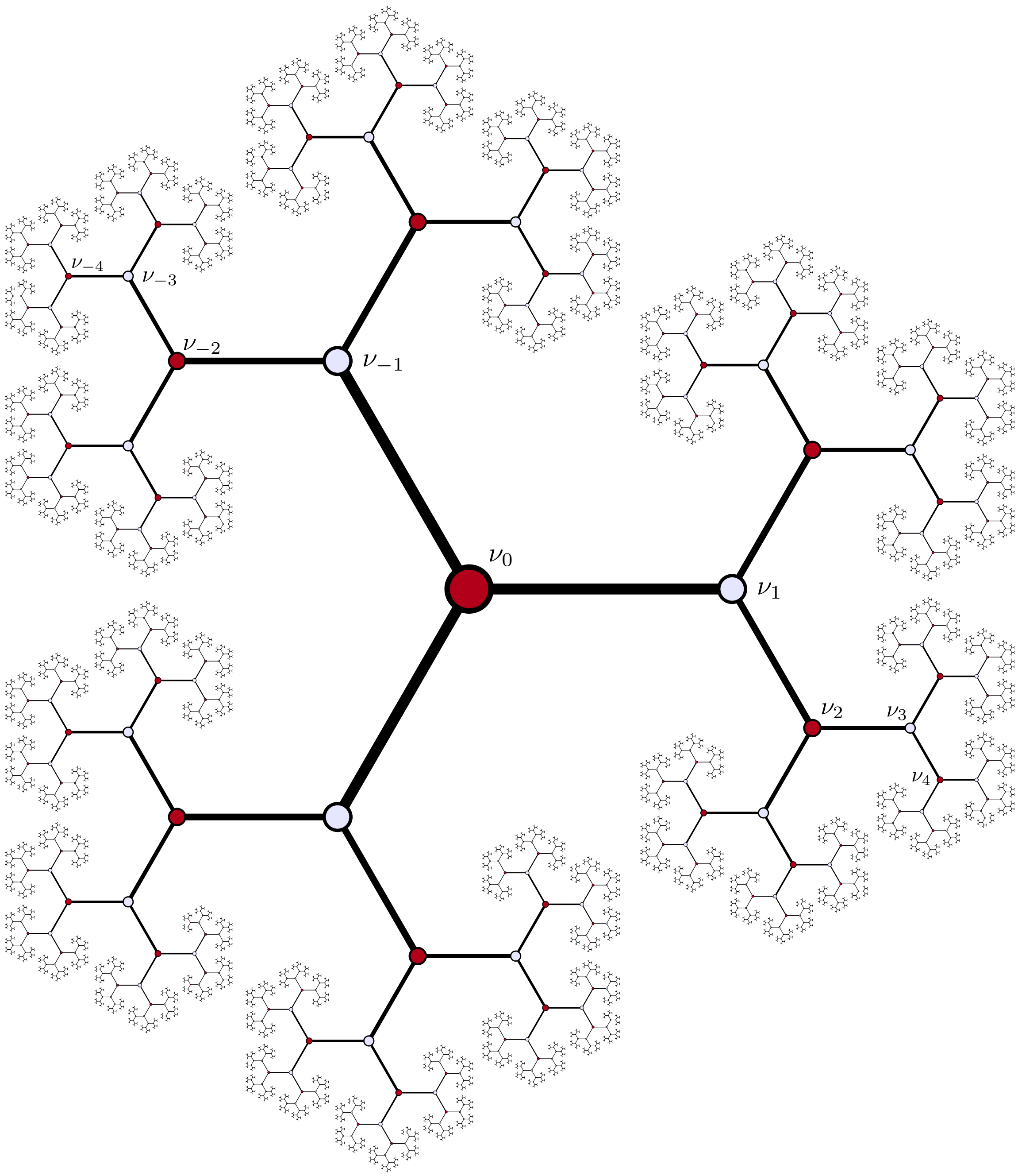
One final remark—it might seem at first that the map between LR paths and nodes is somewhat arbitrary. But in fact some labelings are better than others, in the sense that the geometry of the action of G looks more or less comprehensible. The one I have chosen here seems to be best. One reason for this is that the geometry of the orbits the matrices

$$\begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix}$$

is simple.

8.1. Exercise. *Draw a few of these orbits on a picture of the tree.*

9. *Appendix.* **Centrefold**



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Part II. Analysis

11. The Hecke algebra

The integral **Hecke algebras** of $\mathrm{SL}_2(\mathfrak{k})$ and $\mathrm{PGL}_2(\mathfrak{k})$ are rings of 'algebraic correspondances' on the tree $\mathfrak{X}$. The definitions in these terms mimic Hecke's original definitions of the classical operators T_n.

Let $\mathbb{C}(\mathfrak{X})$ be the space of functions on the nodes of $\mathfrak{X}$. The group GL_2 acts on it by the left regular representation:

$$[L_g F](x) = F\big(g^{-1}(x)\big)\,.$$

PGL(2). Suppose for the moment that $G = \mathrm{PGL}_2(\mathfrak{k})$ and $K = \mathrm{PGL}_2(\mathfrak{o})$.

There is one operator T_m (analogous to Hecke's T_{p^m}) for each $m \geq 0$. According to this, to each node of the building corresponds the set of nodes at distance m from it. Let $\mathcal{H}_\mathbb{Z}$ be the ring generated by the T_m. Its unit is T_0. Let $T = T_1$. This ring may be identified with a ring of operators on the space $\mathbb{C}(\mathfrak{X})$ through the formula:

$$[T_m f](x) \; = \sum_{\substack{y \\ |xy|=m}} f(y)\,.$$

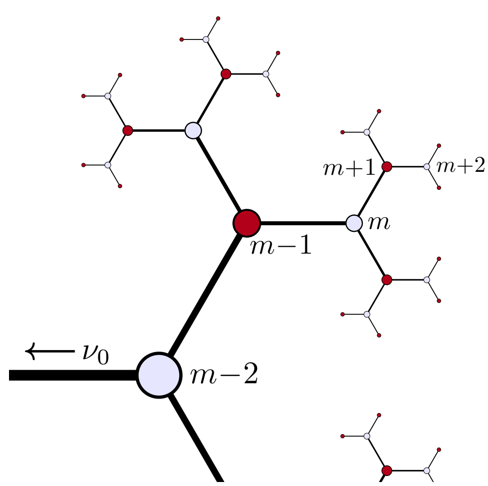
Since distances are preserved by G, the Hecke algebra commutes with the left regular representation of G on $\mathbb{C}[\mathfrak{X}]$.

11.1. Proposition. *The ring homomorphism taking x to T is an isomorphism of $\mathbb{Z}[x]$ with the Hecke algebra $\mathcal{H}_\mathbb{Z}$.*

To prove it, we must first show that $\mathcal{H}_\mathbb{Z}$ is generated by T, and then that the powers of T form a $\mathbb{Z}$-basis of it. These are both an immediate consequence of:

11.2. Lemma. *We have*

$$\begin{aligned} T\circ T &= T_2 + (q+1)I \\ T\circ T_m &= T_{m+1} + qT_{m-1} \quad (m \geq 2)\,. \end{aligned}$$



Proof. As the figure above shows, every node has $q+1$ neighbours. If y is at distance $m \geq 1$ from ν_0 it has q neighbours at distance $m+1$ from ν_0 and 1 at distance $m-1$. Thus, for example:

$$[T\circ T](x) = \sum_{y\sim x,\, z\sim y} z = \sum_{|zx|=2} z + \sum_{y\sim x} x = T_2(x) + (q+1)I(x)\,. \;\blacksquare$$

Here $x \sim y$ means $|xy| = 1$.

SL(2). Now let

$$G = \mathrm{SL}_2(\mathfrak{k})$$
$$K = \mathrm{SL}_2(\mathfrak{o}) .$$

Since there are two orbits of $\mathrm{SL}_2(\mathfrak{k})$ among the nodes of $\mathfrak{X}$, the representation of $\mathrm{SL}_2(\mathfrak{k})$ on $\mathbb{C}[\mathfrak{X}]$ is the direct sum of two components, determined by support.

Let $\mathcal{H}(\mathrm{SL}_2)$ be the algebra generated by $S = T_2$ rather than that generated by T. For this operator we have relations that can be derived from those above:

11.3. Proposition. *If $S = T_2$ then*

$$S \circ S = T_4 + (q-1)T_2 + q(q+1)I$$
$$S \circ T_{2m} = T_{2m+2} + (q-1)T_{2m} + q^2 T_{2m-2} .$$

They can also be derived more directly from this, which the figure also illustrates:

11.4. Lemma. *Suppose $m \ge 2$. Among the nodes at distance 2 from ν_m are q^2 at distance $m+2$ from ν_0, $q-1$ at distance from m from ν_0, and 1 at distance $m-2$ from it.*

11.5. Corollary. *The ring homorphism taking x to S is an isomorphism of $\mathbb{Z}[x]$ with this ring.*

12. Spherical functions

I recall that an **admissible representation** (π, V) of a $\mathfrak{p}$-adic group is one with these two properties: (1) every vector in V is fixed by some open subgroup (i.e. is **smooth**); (2) for any open subgroup, the subspace of vectors fixed by all elements in that subgroup is finite-dimensional.

If G is either $\mathrm{PGL}_2(\mathfrak{k})$ or $\mathrm{SL}_2(\mathfrak{k})$, the action of G on $\mathbb{C}(\mathfrak{X})$ commutes with all Hecke operators, and in consequence it acts on the space of eigenfunctions of a Hecke operator.

PGL(2). Let $G = \mathrm{PGL}_2(\mathfrak{k})$, $K = G(\mathfrak{o})$. For λ in $\mathbb{C}$, let V_λ be the space of functions f on the nodes of $\mathfrak{X}$ such that $Tf = \lambda f$.

12.1. Proposition. *The representation of G on V_λ is admissible, and V_λ^K has dimension 1.*

The condition on an eigenfunction φ is that

$$\lambda\varphi(x) = \sum_{y \sim x} \varphi(y) .$$

This is the $\mathfrak{p}$-adic analogue of classical conditions about eigenfunctions of the Laplacian in the Euclidean plane. I recall, for example, that the value of a harmonic function at a point P is the average of its values on the unit circle around P.

Proof. In several steps.

Step 1. For each $n \ge 0$ let K_n be the congruence subgroup $\mathrm{PGL}_2(\mathfrak{p}^n)$. Suppose the eigenfunction φ to be fixed by K_n. Let B_n be the 'ball' of nodes at distance $\le n$ from ν_0, which are all fixed by K_n. I claim that *the values of φ at all nodes at distance $> n$ from ν_0 are determined by its values on B_n.*

Suppose x to be one of the nodes at distance exactly n from ν_0. Call a node y **external** to x if the geodesic to ν_0 passes through x. This includes x itself. Then K_n fixes x and, if $m \ge n$, acts transitively on all nodes at distance m from ν_0 and external to x. Hence φ takes the same value, say $\varphi_{x,m}$, at all those nodes. My earlier claim will follow from the new claim that *all these $\varphi_{x,m}$ are determined by the values of φ at the neighbours of x, if any, inside B_n.*

Step 2. First we look at the value of φ at the external neighbours of x at distance 1. There are two cases, according to whether $x = \nu_0$ or not.

Suppose $x = \nu_0$. This happens only when $n = 0$ and φ is fixed by K itself. The function φ takes the same value φ_m at all nodes at distance m from ν_0. There are $q + 1$ neighbours of ν_0 at distance 1, so

(12.2) $$\lambda\varphi_0 = (q + 1)\,\varphi_1, \quad \varphi_1 = \frac{\lambda}{q + 1} \cdot \varphi_0 \,.$$

If $n \geq 1$, there is one neighbour y inside B_n, q outside, and we must have

(12.3) $$q\varphi_{x,n+1} = \lambda\varphi_{x,n} - \varphi(y)\,.$$

In either case, the values $\varphi_{x,n+1}$ are determined by the values of φ inside B_n.

Step 3. Suppose now $m \geq n + 1$, y at distance m from ν_0 and external to x. Then all neighbours of y are external to x, and by Lemma 11.4

$$\lambda\varphi_{x,m} = \varphi_{x,m-1} + q\varphi_{x,m+1}\,.$$

That is to say, the function $\varphi_{x,m}$ for $m \geq n$ satisfies the difference equation

$$q\varphi_{x,m} - \lambda\varphi_{x,m-1} + \varphi_{x,m-2} = 0\,.$$

This is a difference equation of second order. The function φ also satisfies initial conditions determined by its values $\varphi_{x,n}$ and $\varphi_{x,n+1}$.

Step 4. There is a well known recipe for the solution of a difference equation, setting $\varphi_{x,m} = \alpha^m$. Plugging into the equation, we see that α must be a root of

$$x^2 - (\lambda/q)x + 1/q = 0\,.$$

If this equation has distinct roots α, β, then the general solution is of the form $c\alpha^m + d\beta^m$. Set $\alpha = z/\sqrt{q}$, $\beta = z^{-1}/\sqrt{q}$ where now we require that $z + z^{-1} = \lambda/\sqrt{q}$. This makes the solution

$$\varphi_{x,m} = q^{-m/2}(cz^m + dz^{-m})$$

for constants c, d satisfying the given initial conditions near the boundary of B_n.

Step 5. The exceptional case is when the roots are equal, which happens when $\mu = \pm 2$. In this case $\alpha = \pm 1/\sqrt{q}$ is the unique root of the characteristic equation. The solutions of the difference equation are linear combinations of $z^m q^{-m/2}$ and $mq^{-m/2}z^m$, with $z = \pm 1$.

Step 6. To summarize: if $n = 0$ and $x = \nu_0$, there is a unique function φ_m satisfying the difference equation with a given value of φ_0, and proportional to φ_0. Therefore V_λ^K has dimension one. Otherwise ($n > 0$), the function φ is uniquely determined by its values on the ball B_n. This proves the Proposition. ∎

In all cases, φ has a well defined asymptotic behaviour on every branch running out from B_n. (a) If the polynomial

$$x^2 - \lambda x + 1 = 0$$

has distinct roots α, β then there exist constants c, d such that

$$\varphi(y) = c\alpha^m + d\beta^m$$

if y lies at distance m from B_n. (b) If it has one root α then

$$\varphi_\nu \sim c\alpha^m + md\alpha^m\,.$$

The constants might depend on the branch.

If $n = 0$ we can get a completely explicit formula for φ with a bit more work. Suppose $\varphi_0 = \varphi(\nu_0) = 1$. I phrase the formula in terms I shall justify later.

12.4. Proposition. *The unique solution of the difference equation for φ is*

$$\varphi_m = \frac{q^{-m/2}}{1+1/q}\left(\left(\frac{1-q^{-1}z^{-2}}{1-z^{-2}}\right)z^m + \left(\frac{1-q^{-1}z^2}{1-z^2}\right)z^{-m}\right)$$

as long as $z \neq \pm 1$.

Once this equation is at hand, one can verify that it is correct by evaluating it for $m = 0, 1$. Finding it in the first place is just a matter of solving a 2×2 system of linear equations.

12.5. Exercise. *Prove the formula in the case $z \neq 1$.*

12.6. Exercise. *Find an explicit formula in the case $z = \pm 1$.*

SL(2). Now I take $G = \mathrm{SL}_2(\mathfrak{k})$. Here we use S instead of T, and consider the set of nodes with even parity, a single $\mathrm{SL}_2(\mathfrak{k})$-orbit. So V_λ is the space of functions on this orbit such that $S\varphi = \lambda\varphi$. The proof of admissibility is essentially the same, but I am interested in an xplicit formula when φ is fixed by K. Because of the Cartan decomposition, φ is determined by its values on ν_{2m}. Let $\varphi_{2m} = \varphi(\nu_{2m})$.

According to Lemma 11.4, the difference equation and initial conditions are now

$$\begin{aligned}\varphi_2 &= \frac{\lambda}{q(q+1)} \cdot \varphi_0 \\ 0 &= q^2\varphi_{2m} - (\lambda - q + 1)\varphi_{2m-2} + \varphi_{2m-4}\,.\end{aligned}$$

We get solutions

$$c\alpha^m + d\beta^m$$

where α, β are roots of the indicial equation

$$x^2 - (\mu/q^2)x + 1/q^2 = 0 \quad (\mu = \lambda - q + 1)\,.$$

I set $\alpha = z/q$, β/q, where now

$$z + z^{-1} = \mu/q\,.$$

The conclusion is:

12.7. Proposition. *When $G = \mathrm{SL}_2(\mathfrak{k})$, the spherical function is*

$$\varphi_m = \frac{q^{-m}}{1+1/q}\left(\left(\frac{1-q^{-1}z^{-1}}{1-z^{-1}}\right)z^m + \left(\frac{1-q^{-1}z}{1-z}\right)z^{-m}\right)$$

as long as $z \neq 1$, and some non-trivial linear combination of $1/q^m$ and m/q^m when $z = 1$.

The case $z = -1$ will turn out to be especially interesting. Explicitly:

12.8. Corollary. *If $z = -1$ then*

$$\varphi_m = (-q)^m\,.$$

So in both cases, PGL_2 and SL_2, not only is the space of functions fixed by K of dimension 1, but we know exactly what the functions in the space are. This can be checked. For example, if $z = q$ in the last formula, the representation is on the space of functions φ such that $q(q+1)\varphi(\nu)$ is the sum of the values of φ at the nodes at distance 2 from ν. This contains the constants, and we get $\varphi_m \equiv 1$.

The formulas for φ_m can be put into a curious form. Take the case of PGL_2. It becomes for $m \geq 1$

$$
\begin{aligned}
\varphi_m &= \frac{q^{-m/2}}{1+1/q} \cdot \left(\left(\frac{1-q^{-1}z^{-2}}{1-z^{-2}} \right) z^m + \left(\frac{1-q^{-1}z^{2}}{1-z^{2}} \right) z^{-m} \right) \\
&= \frac{q^{-m/2}}{1+1/q} \cdot \left(\left(\frac{z^{m+1}-z^{-(m+1)}}{z-z^{-1}} \right) - q^{-1} \left(\frac{z^{m-1}-z^{-(m-1)}}{z-z^{-1}} \right) \right) \\
&= \frac{1}{1+1/q} \cdot \big(q^{-m/2}(z^m + z^{m-2} + \cdots + z^{-m}) - q^{-(m-2)/2}(z^{m-2} + \cdots + z^{-(m-2)}) \big)
\end{aligned}
$$

The expression $z^m + \cdots + z^{-m}$ is the same as the character of the irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension $m+1$, evaluated at

$$
\begin{bmatrix} z & 0 \\ 0 & 1/z \end{bmatrix} .
$$

This is not at all an accident.

13. References

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Part III. Conjugacy classes

14. Conjugacy classes in SL(2)

In most of this section, I'll take F to be an arbitrary field, $\overline{F}$ an algebraic closure, and E a quadratic extension of F in $\overline{F}$. Eventually I'll specialize to $\mathfrak{p}$-adic fields. The question to be answered is, *how does one classify semi-simple elements of* $\mathrm{SL}_2(F)$*?*

In $\mathrm{GL}_2(F)$, the conjugacy class of a semi-simple (i.e. diagonalizable) element is completely determined by its eigenvalues, which in turn are completely determined by its characteristic polynomial. The characteristic polynomial of an element γ of $\mathrm{SL}_2(F)$ is of the form

$$x^2 - \tau x + 1$$

where τ is the trace of γ, and lies in F. In $\mathrm{SL}_2(\overline{F})$ two semi-simple (diagonalizable) elements are conjugate if and only if they have the same trace, because they can both be diagonalized. In $\mathrm{SL}_2(F)$, equality of trace is a necessary condition for conjugacy, but not sufficient. Still, classifying elements of $\mathrm{SL}_2(F)$ by their trace is the first step towards a complete classification.

The map from a matrix in $\mathrm{GL}_2(\overline{F})$ to $\overline{F}^2$, taking a matrix to its characteristic polynomial, has a right inverse. This is perhaps motivated by what happens over F when $x^2 - c_1x + c_2$ is irreducible. It α is a root, it generates a quadratic extension E of F with F-basis α, 1. Elements of E act by multiplication on the two-dimensional F-space E, and since

$$\begin{aligned} \alpha \cdot \alpha &= c_1\alpha - c_2 \\ \alpha \cdot 1 &= \quad \alpha \end{aligned}$$

the element α corresponds to the matrix

$$\sigma(c) = \begin{bmatrix} c_1 & 1 \\ -c_2 & 0 \end{bmatrix} \quad \big(c = (c_1, c_2)\big) .$$

The map $c \mapsto \sigma(c)$ is defined for any pair c. Manifestly the trace of $\sigma(c)$ is c_1 and its determinant is c_2. When $c = (2, 1)$ the matrix $\sigma(c)$ is a unipotent matrix, and in fact the image of the section over $c_2 \neq 0$ intersects every *regular* conjugacy class in $\mathrm{GL}_2(\overline{F})$ (matrices with centralizers of dimension two). In effect, the map gives a base point for every regular conjugacy class.

A conjugacy class is called **F-rational** if its trace and determinant lie in F. This is equivalent to the condition that the class (as a set in the group of points rational over $\overline{F}$) be Galois-invariant. If τ lies in F and $c = (\tau, 1)$ then $\sigma(c)$ is a rational element of $\mathrm{SL}_2(F)$, so:

14.1. Proposition. *Every regular conjugacy class in* $\mathrm{SL}_2(\overline{F})$ *rational over* F *contains a rational element.*

The definition of $\sigma(c)$ is a special case of the definition of **companion matrix**, which maps every monic polynomial $P(x)$ of degree n to an $n \times n$ matrix with $P(x)$ as its characteristic polynomial. This construction is elementary, but it is a special case of a much more difficult result of [Steinberg:1965] about simply connected semi-simple groups, generalized to certain reductive ones in [Kottwitz:1982].

If $x^2 - \tau x + 1 = 0$ has distinct roots in F, then any element in $\mathrm{SL}_2(F)$ with trace τ is conjugate to a diagonal matrix inside $\mathrm{SL}_2(F)$. Otherwise, the roots will be conjugate elements a, $\overline{a}$ in some quadratic extension E with $a\overline{a} = 1$. Applied to a quadratic extension E/F, these remarks imply:

14.2. Lemma. *If* $x^2 - \tau x + 1 = 0$ *has distinct roots in the quadratic extension* E/F*, then two elements of* $\mathrm{SL}_2(F)$ *with this as characteristic polynomial are conjugate in* $\mathrm{SL}_2(E)$.

Two semi-simple elements of $\mathrm{SL}_2(F)$ are said to be **stably conjugate** if they are conjugate in $\mathrm{SL}_2(\overline{F})$, or equivalently if they have the same trace. The Lemma says that questions of stable conjugacy for SL_2 reduce to questions of actual conjugacy over quadratic extensions.

Two semi-simple elements of $\mathrm{SL}_2(F)$ with eigenvalues in F (i.e. that are **split** over F) are conjugate in $\mathrm{SL}_2(F)$ if and only if they are stably conjugate. Conjugacy classes of semi-simple elements whose eigenvalues are not in F (i.e. are not split) behave very differently:

14.3. Theorem. *If F is $\mathfrak{p}$-adic then any non-split, semi-simple, stable conjugacy class of $\mathrm{SL}_2(F)$ consists of exactly two $\mathrm{SL}_2(F)$-conjugacy classes.*

Combined with the cross-section σ, the proof will tell you precisely how to get them.

Proof. I do not yet assume F to be $\mathfrak{p}$-adic. Let $\mathcal{G}$ be the Galois group of E/F, containing in addition to the identity a conjugation $x \mapsto \overline{x}$ of order two. Suppose γ to be an element of $\mathrm{GL}_2(F)$ with eigenvalues $a \neq \overline{a}$ in E. If v in E^2 is an eigenvector of γ for a in E^2, then $\overline{v}$ is one for $\overline{a}$. Let

$$g = [\, v \quad \overline{v} \,] = \begin{bmatrix} x & \overline{x} \\ y & \overline{y} \end{bmatrix}$$

be an eigenvector matrix, so that

$$\gamma = g\delta g^{-1} \quad \left(\delta = \begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix}\right) .$$

Any other eigenvector for a will be a multiple cv, and the corresponding eigenvector matrix

$$\begin{bmatrix} cx & \overline{cx} \\ cy & \overline{cy} \end{bmatrix} .$$

The determinant of this will be

$$c\overline{c}(x\overline{y} - \overline{x}y)$$

so the ratio

$$\frac{\det(g)}{a - \overline{a}} ,$$

which lies in $F^\times$, is uniquely determined modulo $N_{E/F}E^\times$. For any x in $F^\times$ let

$$\mathrm{sgn}_{E/F}(x) = \text{ the image of } x \text{ in } F^\times/N_{E/F}E^\times .$$

The definition

$$\textbf{(14.4)} \qquad \mathrm{sgn}(\gamma) = \mathrm{sgn}_{E/F}\left(-\frac{\det(g)}{a - \overline{a}}\right)$$

therefore makes sense. The choice of sign is made to simplify slightly certain formulas. What happens if γ changes to a conjugate?

14.5. Lemma. *For γ, α in $\mathrm{GL}_2(\mathfrak{k})$ with γ semi-simple and split over E*

$$\mathrm{sgn}(\alpha\gamma\alpha^{-1}) = \mathrm{sgn}_{E/F}(\det(\alpha))\,\mathrm{sgn}(\gamma) .$$

Proof. If

$$\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

in $\mathrm{GL}_2(F)$, then the eigenvector matrix changes to

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & \overline{x} \\ y & \overline{y} \end{bmatrix} = \begin{bmatrix} ax + by & a\overline{x} + b\overline{y} \\ cx + dy & c\overline{x} + d\overline{y} \end{bmatrix} ,$$

and $\det(g)$ to $\mathrm{sgn}_{E/F}(\det(\alpha))\det(g)$. ∎

One consequence is that $\mathrm{sgn}(\gamma)$ is constant on conjugacy classes in $\mathrm{SL}_2(F)$. According to class field theory, the quotient $F^\times/N_{F/E}E^\times$ has order two if F is $\mathfrak{p}$-adic. The Theorem therefore follows from the stronger statement:

14.6. Proposition. *The map taking γ to $\mathrm{sgn}(\gamma)$ is a bijection between the $\mathrm{SL}_2(F)$-orbits in the stable conjugacy class of γ and $F^\times/N_{F/E}E^\times$.*

Proof. Surjectivity is immediate from the equation $\mathrm{sgn}(\alpha\gamma\alpha^{-1}) = \det(\alpha)\,\mathrm{sgn}(\gamma)$, so what remains is to show that if γ_1 and γ_2 have the same eigenvalues and $\mathrm{sgn}(\gamma_1) = \mathrm{sgn}(\gamma_2)$ then γ_1 and γ_2 are conjugate in $\mathrm{SL}_2(F)$. They are certainly conjugate in $\mathrm{GL}_2(F)$, so we have $\gamma_2 = \alpha\gamma_1\alpha^{-1}$ for some α in $\mathrm{GL}_2(F)$. The assumption on sgn implies that $\det(\alpha) = c\overline{c}$ for some c in $E^\times$.

14.7. Lemma. *If γ has distinct eigenvalues in $E^\times$, the determinant map from its centralizer to $F^\times$ has image $N_{E/F}E^\times$.*

Proof. Because the centralizer of a diagonal matrix with distinct non-zero entries is the group of all diagonal matrices, conjugation conjugates centralizers, and conjugation preserves determinants. ∎

According to this Lemma, we can find z in $\mathrm{GL}_2(F)$ that commutes with γ_1 and has determinant $c\overline{c}$. Then αz^{-1} lies in $\mathrm{SL}_2(F)$ and conjugates γ_1 to γ_2. This concludes the proofs of both Proposition 14.6 and Theorem 14.3. ∎

14.8. Exercise. *Suppose γ in $\mathrm{SL}_2(\mathfrak{k})$ to split over E. Prove that γ^{-1} is conjugate to γ if and only if -1 lies in $N_{E/F}E^\times$.*

14.9. Exercise. *Find $\mathrm{sgn}(\sigma(c))$.*

14.10. Exercise. *Suppose $\mathfrak{e}/\mathfrak{k}$ to be unramified,*

$$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & \varpi \end{bmatrix}.$$

and γ to have eigenvalues in $\mathfrak{e}$. Let T be its centralizer. Prove that T and $\alpha T\alpha^{-1}$ are not conjugate in $\mathrm{SL}_2(\mathfrak{k})$. (This is a special case of a very pretty result of [DeBacker:2006] about unramified tori in simply connected, semi-simple $\mathfrak{p}$-adic groups.)

Remark. I cannot resist mentioning that the phenomenon discussed here was perhaps first called to your attention in a course on differential equations. Suppose we are given a 2×2 linear equation

$$y' = Ay$$

with $A \in \mathrm{GL}_2(\mathbb{R})$, and suppose the eigenvalues of A are $\pm\sqrt{-1}$. The solutions will be elliptical rotations. But how do we figure out the orientation of the rotations? It will depend on the sign of

$$\frac{x\overline{y} - \overline{x}y}{2i}$$

where (x, y) is an eigenvector for A with eigenvalue i.

In the rest of this section I shall make Theorem 14.3 more explicit for $\mathfrak{p}$-adic fields. Things are quite different for unramified and unramified extensions $\mathfrak{e}/gk$.

EMBEDDING UNRAMIFIED EXTENSIONS. Suppose $\mathfrak{e}$ unramified. If $P(x) = x^2 - ax + b$ is a polynomial without roots in the residue field $\mathbb{F}_q$, there will exist some ε in $\mathfrak{e}$ with $P(\varepsilon) = 0$ modulo $\mathfrak{p}$. Then 1 and ε will be an $\mathfrak{o}$-basis for $\mathfrak{o}_\mathfrak{e}$. Let g_ε be the matrix

$$\begin{bmatrix} a & 1 \\ -b & 0 \end{bmatrix}.$$

Then the map

$$a + b\varepsilon \longmapsto aI + bg_\varepsilon$$

embeds $\mathfrak{o}_\mathfrak{e}$ in $M_2(\mathfrak{o})$, $\mathfrak{e}$ in $M_2(\mathfrak{k})$, and $N^1_{\mathfrak{e}/\mathfrak{k}}$ in $K_0 = \mathrm{SL}_2(\mathfrak{o})$. Conjugating this copy with

$$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & \varpi \end{bmatrix} .$$

embeds it into K_1, the stabilizer of ν_1. I'll call the copy in K_0 the standard copy and the one in K_1 the **twisted** copy of $N^1_{\mathfrak{e}/\mathfrak{k}}$.

EMBEDDING RAMIFIED EXTENSIONS. Now suppose $\mathfrak{e}/\mathfrak{k}$ ramified, say with integer ring $\mathfrak{o}_\mathfrak{e}$, prime $\varpi_\mathfrak{e}$, etc. We may assume for the moment that $N_{\mathfrak{e}/\mathfrak{k}}(\varpi_\mathfrak{e}$ is ϖ, by changing ϖ if necessary. Then the polynomial satisfied by $\varpi_\mathfrak{e}$ will be

$$P(x) = x^2 - \tau x + \varpi \, ,$$

where τ is the trace of $\varpi_\mathfrak{e}$. The pair $1, \varpi_\mathfrak{e}$ form basis of $\mathfrak{o}_\mathfrak{e}$, and we get an embedding of $\mathfrak{e}$ into $M_2(\mathfrak{k})$ taking

$$a + b\varpi_\mathfrak{e} \mapsto aI + b\sigma$$

where

$$\sigma = \sigma((\tau, \varpi)) = \begin{bmatrix} \tau & 1 \\ -\varpi & 0 \end{bmatrix} .$$

This embeds $\mathfrak{o}_\mathfrak{e}^\times$ in the Iwahori subgroup B. We get a second embedding by conjugating this one with any matrix in $\mathrm{GL}_2(\mathfrak{k})$ whose determinant is a unit in $\mathfrak{o}^\times$ but not a norm from $\mathfrak{e}$—or example

$$\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} \quad (x \in \mathfrak{o}^\times, \ \mathrm{sgn}_{\mathfrak{e}/\mathfrak{k}}(x) = -1), .$$

This copy is also in B. As in the first case, I call the first embedding the standard one. It depends on the choice of section σ and generator $\varpi_\mathfrak{e}$.

15. Quadratic extensions and trees

In this section, let

$$\begin{aligned} \mathfrak{e} &= \text{a quadratic extension of } \mathfrak{k} \\ \mathfrak{o}_\mathfrak{e} &= \text{its ring of integers} \\ \mathfrak{X}_\mathfrak{e} &= \text{the Bruhat-Tits tree of } \mathrm{SL}_2(\mathfrak{e}) \mathrm{etc.} \\ \mathcal{G} &= \text{the Galois group of } \mathfrak{e}/\mathfrak{k} \ \ldots \end{aligned}$$

Then $\mathcal{G}$ acts on $\mathfrak{X}_\mathfrak{e}$. If L is a lattice in $\mathfrak{k}^2$, then $L \otimes_\mathfrak{o} \mathfrak{o}_\mathfrak{e}$ is a lattice in $\mathfrak{e}^2$. The nodes of $\mathfrak{X}$ are thus embedded among those of $\mathfrak{X}_\mathfrak{e}$, and all of those nodes are fixed by $\mathcal{G}$. One of the questions to be examined here is, *what is the set of all nodes fixed by $\mathcal{G}$?*

What happens depends very much on the extension $\mathfrak{e}/\mathfrak{k}$. One sees easily that when $\mathfrak{e}/\mathfrak{k}$ is unramified is very different from when it is ramified. In the latter case, for example, the lattices $\mathfrak{o}^2$ and $[1, \varpi]$ no longer correspond to neighbours in the larger tree, since $[1, \varpi_\mathfrak{e}]$ lies between them.

In many places subsequently, we'll need this simple fact:

15.1. Lemma. *If α is an automorphism of the tree $\mathfrak{X}$, then the set of its fixed points is connected.*

Proof. If α fixes nodes x and y it fixes the geodesic between them. ▯

Case 1. For a while, suppose $\mathfrak{e}/\mathfrak{k}$ to be unramified.

15.2. Proposition. *If $\mathfrak{e}/\mathfrak{k}$ is unramified, the copy of $\mathfrak{X}$ in $\mathfrak{X}_\mathfrak{e}$ is the set of its points fixed by $\mathcal{G}$.*

This is a special case of a very general result about Bruhat-Tits buildings—*if G is an unramified, simply-connected, semi-simple group defined over $\mathfrak{k}$ and split over the unramified extension $\mathfrak{e}/\mathfrak{k}$, then the building of G may be identified with the points of the building of G over $\mathfrak{e}$ fixed by the Frobenius automorphism of $\mathfrak{e}/\mathfrak{k}$*. I'll make some remarks about the proof of the general result later, but give first an elementary one.

Proof. Since $\varpi = \varpi_\mathfrak{k}$ may be taken as $\varpi_\mathfrak{e}$, the edges of the tree over $\mathfrak{k}$ remain edges in the tree over $\mathfrak{e}$. Therefore it must only be shown that every node of the larger tree that is fixed by $\mathcal{G}$ lies in the smaller tree. Since the set of points fixed by $\mathcal{G}$ is connected, every fixed point may be connected to $\mathfrak{X}$. The last point of any path to $\mathfrak{X}$ that does not start in $\mathfrak{X}$ must hit a neighbour of a point in $\mathfrak{X}$. Therefore it suffices to show that every neighbour of a point ν of $\mathfrak{X}$ fixed by conjugation is actually in $\mathfrak{X}$.

All points of $\mathfrak{X}$ are equivalent, so we may assume that $\nu = \nu_0$. Its neighbours correspond to points of $\mathbb{P}^1(\mathbb{F}_{q^2})$. But the fixed points of its Galois action are precisely the points of $\mathbb{P}^1(\mathbb{F}_q)$, which correspond to the neighbours of ν_0 in $\mathfrak{X}$. ▯

Remark. *I'll sketch here the proof that remains valid for more general groups.*

Say $\overline{x} = x$, and suppose $g(\nu_0) = x$. Then $g(\nu_0) = \overline{g}\nu_0$, and $g^{-1}\overline{g}\nu_0$, so $g^{-1}\overline{g}$ lies in $\mathrm{PGL}_2(\mathfrak{o}_\mathfrak{e})$.

15.3. Lemma. *If $\mathfrak{e}$ is any unramified extension of $\mathfrak{k}$ and G is a smooth group scheme over $\mathfrak{o}$ then*

$$\mathrm{H}^1\big(\mathcal{G}(\mathfrak{e}/\mathfrak{k}), G(\mathfrak{o}_\mathfrak{e})\big) = 0\,.$$

Proof. This is a consequence of the well known theorem of Serge Lang which asserts that the Galois cohomology of groups over finite fields vanishes, together with Hensel's Lemma and an induction argument. ▯

As a consequence of the Lemma, we may find k in $\mathrm{PGL}_2(\mathfrak{o}_\mathfrak{e})$ such that $k^{-1}\overline{k} = g^{-1}\overline{g}$. But then $gk^{-1} = \overline{g}\overline{k}^{-1}$, gk^{-1} lies in $\mathrm{PGL}_2(\mathfrak{k})$, and $g(\nu_0) = gk^{-1}\nu_0$ lies in the tree over $\mathfrak{k}$. This concludes the proof of the Proposition.

This proof will remain essentially valid also for the result about buildings and unramified extensions that I mentioned earlier, and a similar argument will show how to construct the building for any unramified, simply connected, semi-simple group over $\mathfrak{k}$. One has to be careful if the original group is not split, since then the simplices of its building are not simplices of the larger one. For example, if G is the unramified $\mathrm{SU}(2,1)$ its building is a tree, and its edges are the points of two-dimensional simplices of the building of $\mathrm{SL}_3(\mathfrak{e})$ fixed by $\mathcal{G}$, where $\mathrm{SU}(2,1)$ splits over the quadratic extension $\mathfrak{e}/\mathfrak{k}$.

Now suppose γ to be in $\mathrm{GL}_2(\mathfrak{k})$ with distinct eigenvalues in $\mathfrak{e}$, and suppose $\gamma = g\delta g^{-1}$ with δ a diagonal matrix whose entries are those eigenvalues and g an eigenvector matrix. Then $w = g^{-1}\overline{g}$ will lie in the non-trivial coset of the normalizer of the diagonal matrices. The centralizer of δ will be the conjugate of the diagonal matrices, and $g\mathcal{A}$ will be an apartment of the tree over $\mathfrak{e}$ taken into itself by the Galois group. But Galois conjugation acts as a non-trivial involution on the apartment, with a unique fixed point, which by the Proposition must lie in the tree over $\mathfrak{k}$. Depending on the parity of this fixed node, it may be transformed by an element of $\mathrm{SL}_2(\mathfrak{k})$ to either ν_0 or ν_1. In the first case the centralizer of γ in $\mathrm{SL}_2(\mathfrak{k})$ lies in K_0 and in the second in K_1.

Case 2. Now suppose $\mathfrak{e}/\mathfrak{k}$ to be ramified. In this case, we may consider the midpoints of edges in the tree over $\mathfrak{k}$ to be nodes of the tree over $\mathfrak{e}$. They are fixed by Galois conjugation. What other nodes of the tree over $\mathfrak{e}$ are so fixed?

Since the Galois-fixed points are a connected set, every fixed node is connected by a chain of fixed nodes to the nodes on the tree of $\mathfrak{k}$. Where do these chains attach? Since the embedding of $\mathfrak{o}/\mathfrak{p}$ into $\mathfrak{o}_\mathfrak{e}/\mathfrak{p}_\mathfrak{e}$ is an isomorphism, each node of the tree over $\mathfrak{e}$ also has $q+1$ neighbours. This implies that the only edges meeting a node of the $\mathfrak{k}$-tree are already part of an edge in that tree, so the attachments must be at mid-points of edges.

If ν is a midpoint, there are two edges leading from it to $\mathfrak{k}$-nodes, hence $q-1$ not leading to nodes of the $\mathfrak{k}$-tree. To see now what goes on, we may as well assume the node to be $\nu_{1/2}$, corresponding to $[u_0, v_{1/2}]$ (where $v_{1/2} = \varpi_{\mathfrak{e}} v_0$) lying between ν_0 and ν_1. Its neighbours in $\mathfrak{X}$ are

$$\langle\!\langle \varpi_{\mathfrak{e}} u_0, v_{1/2} \rangle\!\rangle = \langle\!\langle u_0, v_0 \rangle\!\rangle, \ \langle\!\langle u_0, \varpi v_{1/2} \rangle\!\rangle = \langle\!\langle u_0, \varpi v_0 \rangle\!\rangle \ , .$$

The other neighbours are the $\langle\!\langle u_0 + x v_{1/2} \rangle\!\rangle$ for $x \neq 0$ ranging over $\mathfrak{p}_{\mathfrak{e}}/\mathfrak{p}$. Since $\varpi_{\mathfrak{e}} + \overline{\varpi}_{\mathfrak{e}}$ lies in $\mathfrak{p}$, the Galois group acts as -1 on $\mathfrak{p}_{\mathfrak{e}}/\mathfrak{p}_{\mathfrak{e}}^2$. If the residue characteristic is odd, none of these other neighbouring lattices are fixed, so we again conclude that the only nodes of $\mathfrak{X}_{\mathfrak{e}}$ fixed by the Galois group are those on $\mathfrak{X}$. But is it is even, all are fixed, and so the fixed-point set of nodes is definitely larger than just $\mathfrak{X}$.

It remains to explore this last case. Take as basis for $\mathfrak{o}_{\mathfrak{e}}$ the pair $1, \varpi_{\mathfrak{e}}$. Recall that

$$\sigma(\varpi_{\mathfrak{e}}) = \begin{bmatrix} \tau & 1 \\ -\varpi_{\mathfrak{k}} & 0 \end{bmatrix}$$

with $\tau = \varpi_{\mathfrak{e}} + \overline{\varpi}_{\mathfrak{e}}$ lies in $\mathfrak{p}$. The map

$$\sigma\colon\ a + b\varpi_{\mathfrak{e}} \longmapsto a + b\sigma(\varpi_{\mathfrak{e}})$$

embeds $\mathfrak{e}$ in $M_2(\mathfrak{k})$. The image of $\mathfrak{o}_{\mathfrak{e}}^{\times}$ is contained in the Iwahori subgroup $K_0 \cap K_1$. The eigenvalues of σ are $\varpi_{\mathfrak{e}}$, $\overline{\varpi}_{\mathfrak{e}}$ and the corresponding eigenvector matrix is

$$g = \begin{bmatrix} 1 & 1 \\ -\overline{\varpi}_{\mathfrak{e}} & -\varpi_{\mathfrak{e}} \end{bmatrix} .$$

The apartment $g\mathcal{A}$ is stable with respect to the image of σ in $\mathrm{GL}_2(\mathfrak{k})$. Since

$$\overline{g} = gw, \quad w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

with w in K_0, Galois conjugation is an involution of $g\mathcal{A}$ with the node $g(\nu_0)$ as fixed point. All the points of the geodesic between $g(\nu_0)$ and $\mathcal{A}$ are fixed by Galois conjugation, and all the points fixed by Galois conjugation are the transforms by $\mathrm{GL}_2(\mathfrak{k})$ of that geodesic.

What is the distance of $g(\nu_0)$ from $\mathcal{A}$? We have

$$g = \begin{bmatrix} 1 & 0 \\ \varpi_{\mathfrak{e}} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \overline{\varpi}_{\mathfrak{e}} - \varpi_{\mathfrak{e}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} .$$

Therefore $g(\nu_0)$ is obtained by shifting ν_0 along $\mathcal{A}$ by r, where

$$r = \mathrm{val}(\overline{\varpi}_{\mathfrak{e}} - \varpi_{\mathfrak{e}}) .$$

then pivoting it about $\nu_{1/2}$. Hence the distance from $\mathcal{A}$ to $g(\nu_0)$ is $r-1$, and the set of points of the tree over $\mathfrak{e}$ fixed by conjugation are those at distance $r-1$ from the tree over $\mathfrak{k}$.

15.3. Exercise. *Suppose $\mathfrak{e}$ is obtained from $\mathbb{Q}_2$ by adjoining $\sqrt{2}$. Find in this case the chain from ν_0 to $g(\nu_0)$.*

Finally as a consequence of the discussion at the end of the previous section:

15.4. Proposition. *If $\mathfrak{e}/\mathfrak{k}$ is ramified, each $\mathfrak{e}^{\times}$-orbit of nodes in $\mathfrak{X}$ intersects the apartment $\mathcal{A}$ in exactly one place.*

16. References

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Part IV. Orbital integrals and the Fundamental Lemma

Langlands' principle of functoriality asserts that automorphic forms on one group are, in some somewhat subtle circumstances, related to automorphic forms on another. The only really good tool one has to attack the conjectures that arise in this way is the Arthur-Selberg trace formula, and in using it one usually find oneself comparing orbital integrals on the groups involved. The model for this approach is that in [Jacquet-Langlands:1970], in which automorphic forms for quaternions are embedded into those for GL_2 because conjugacy classes in the quaternion algebra are related to those in GL_2, and because corresponding orbital integrals match. Applying this technique more generally requires that one understand a great deal about orbital integrals of local reductive groups.

17. Introduction

Suppose γ to be an element in G, with centralizer G_γ. It is always unimodular. Choose a Haar measure on it, as well as a Haar measure on G. These choices determine a measure on $G_\gamma\backslash G$. At least formally, the associated orbital integral of f in $C_c^\infty(G)$ over the conjugation class of γ is

$$\langle O_{\gamma,G}, f\rangle = \int_{G_\gamma\backslash G} f(g^{-1}\gamma g)\, dg\,.$$

It turns out that in fact the integral always converges (not quite a trivial matter if f has support in the neighbourhood of 1) and defines a conjugation-invariant distribution (linear function on $C_c^\infty(G)$).

Orbital integrals on a $\mathfrak{p}$-adic reductive group are simply defined, but very difficult to evaluate in almost any sense. The group $\mathrm{SL}_2(\mathfrak{k})$ is the simplest non-trivial case, but even there it seems that one cannot and does not want to find explicit formulas for all orbital integrals. Instead, one might get along with understanding only certain aspects, such as asymptotic behaviour near singular points and explicit formulas for functions in the Hecke algebra $\mathcal{H}(G/\!/K = \mathrm{SL}_2(\mathfrak{o}))$. Somewhere in the overlap of these two themes lies the Fundamental Lemma, which can be dealt with in a particularly elegant manner for $\mathrm{SL}_2(\mathfrak{k})$, and the problems of transfer to endoscopic groups, which is not quite so elegant. The natural way, although in the long run not the most comprehensive way, to approach such problems is *via* the Bruhat-Tits tree.

In the rest of this part I'll calculate some orbital integrals for $G = \mathrm{SL}_2(\mathfrak{k})$, and then discuss how this relates to Langlands' Fundamental Lemma for this group and the more general problem of transfer. I cannot claim much originality in doing this, since I follow closely [Langlands:1980] and [Kottwitz:2006], who deal with $\mathrm{GL}_2(\mathfrak{k})$. I have to say in advance, there seem to be more questions, even in this simple case, than entirely satisfactory answers.

There are four different cases to look at: (1) $\gamma = \pm I$; (2) γ hyperbolic (eignevalues in $\mathfrak{k}$); (3) γ elliptic (eigenvalues not in $\mathfrak{k}$; (4) γ unipotent. Ther first case is not very interesting. For the moment we are mostly interested in elliptic γ that split over the unramified quadratic extension $\mathfrak{e}/\mathfrak{k}$, but I may as well state here the basic tool in all cases:

17.1. Lemma. *For any γ in G and f fixed on the right by a compact open subgroup K of G*

$$\langle O_\gamma, f\rangle = \sum_{G_\gamma\backslash G/K} f(x^{-1}\gamma x)\cdot\frac{\mathrm{meas}_G(K)}{\mathrm{meas}_{G_\gamma}(G_\gamma\cap x^{-1}Kx)}\,.$$

I'll not prove this here—it is a basic formula about integration of smooth functions on quotients.

When γ is semi-simple, there are very useful simplifications of this formula. If γ is hyperbolic, then G_γ is a split torus, which we may take to be A. This contains the discrete group A_ϖ generated by

$$\begin{bmatrix}\varpi & 0\\ 0 & 1/\varpi\end{bmatrix}$$

The quotient A/A_ϖ is compact, so by adjusting the measure on A so that $A(\mathfrak{o})$ has measure 1 we see that

(17.2) $$\langle O_\gamma, f\rangle = \int_{A_\varpi\backslash G} f(x^{-1}\gamma x)\,dx\,.$$

If γ is elliptic, G_γ is compact, so by assigning it to have total measure 1 we see that

(17.3) $$\langle O_\gamma, f\rangle = \int_{G} f(x^{-1}\gamma x)\,dx\,.$$

If γ is semi-simple, its orbit is closed in G. If f is a smooth function of compact support on G then the function $f(g^{-1}\gamma g)$ has compact support on this orbit, so the orbital integral is certainly well defined. As γ approaches the identity in G, or in fact approaches any conjugation class for which the centralizer is not a torus, the orbital integral blows up but, as we'll see in some examples later on, it turns out that multiplying it by a suitable factor tames it.

18. A finite analogue

In this section, let

$$\begin{aligned} &F = \mathbb{F}_p \quad (p \equiv 3 \bmod 4) \\ &E = \mathbb{F}_{p^2} \\ &G = \mathrm{PSL}_2(F) = \mathrm{SL}_2(F)/\{\pm I\} \\ &N = \text{upper triangular unipotent matrices in } G \\ &\psi\colon\ F \longrightarrow \mathbb{C}^\times,\ x \mapsto e^{2\pi i x/p}\,. \end{aligned}$$

I'll exhibit an analogue of the Fundamental Lemma for G, in the hope that it will give some idea of why the Fundamental Lemma itself is a very natural result. This example was used by Hecke in the early paper [Hecke:1930]. in which he proved a precursor of results of [Langlands-Labesse:1979]. (See also [Casselman:2012].)

The assumption that $p \equiv 3 \bmod$ allows some simplification of notation and argument. The point is that -1 is not a square in F, so $E = F(i)$ with $i = \sqrt{-1}$.

CONJUGACY CLASSES. We have already seen how to classify all semi-simple conjugacy classes in G. First is I.

Then there are the regular split elements in the copy of $F^\times/\{\pm 1\}$ in G, the images of the diagonal matrices

$$\begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix}$$

with $t \neq 1/t$. The matrices for t and $1/t$ are conjugate, so there are $((p-1)-2)/2 = (p-3)/2$ of these.

The field E may be embedded in $M_2(F)$:

$$a + bi \longmapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\,.$$

This gives rise to regular non-split elements associated to $\varepsilon \neq \pm 1$ in $N^1_{E/F}$. The matrices for ε and $1/\varepsilon$ are conjugate, so there are $((p+1)-2)/2 = (p-1)/2$ of these.

Finally, there are the two unipotent classes

$$\begin{aligned} n_+ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ n_- &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\,. \end{aligned}$$

which are in distinct in conjugacy classes, again because -1 is not a square in F.

IRREDUCIBLE REPRESENTATIONS. All representations arise as part of either the principal series, induced from the Borel subgroup, or the discrete series, constructed from the Weil representation. From the first we get the trivial representation $\mathbb{C}$ and the Steinberg representation of dimension p, the two together making up the representation of G on $\mathbb{C}[\mathbb{P}^1(F)]$.

Then there are the generic principal series $\mathrm{ps}(\chi)$ for $\chi(-1) = 1$ and $\chi \neq 1$. The representation $\mathrm{ps}(\chi)$ is isomorphic to $\mathrm{ps}(\chi^{-1})$, so there are $(p-3)/4$ of them, each of dimension $p+1$.

Then there are the generic discrete series $\mathrm{ds}(\rho)$ parametrized by characters of $N^1_{E/F}$ such that $\rho \neq \rho^{-1}$. Since $p+1 \equiv 0 \bmod 4$, $\rho(-1) = 1$ for all of them, so there are $(p-1)/2$ of these. Each has dimension $p-1$.

If ρ_0 is the unique character of $N^1_{E/F}$ of order two, the discrete series $\mathrm{ds}(\rho_0)$ splits into two pieces $\mathrm{ds}_\pm$ distinguished by the spectrum of N.

Among all classes, the unipotent ones $n_\pm$ have the unique property that although they are distinct in G theyfuse in $\mathrm{PSL}_2(E)$. This is the probably simplest possible example of the distinction between ordinary conjugacy and what Langlands calls **stable conjugacy** in reductive groups. Note that this phenomenon does not occur among the semi-simple classes, essetially because Galois cohomology of finite fields is trivial, according to Serge Lang's theorem.

It is the last representations that we are most interested in here. In general, there is a representation $\pi(\rho)$ of G associated to every ρ of $N^1_{E/F}/\{\pm 1\}$, and there exists a G-isomorphism T of $\pi(\rho)$ with $\pi(1/\rho)$. For $\rho = \rho_0$ this is a non-trivial G-automorphism of $\pi(\rho_0)$ of order two, and the representations $\pi_\pm$ are its eigenspaces. I have chosen signs so that the eigencharacters of N on $\pi_+(\rho_0)$ are the ψ_x with $x \neq 0$ not a square in F, and that on $\pi_-(\rho_0)$ the ψ_x with $x \neq 0$ a square.

The representations $\pi_\pm$ have the unique feature that neither is isomorphic to its complex conjugate. Instead, complex conjugation interchanges them. What is especially important for us is that *their characters differ only on the two unipotent classes of G*. In other words, the unusual representations $\pi_\pm$ are in some way matched with the unusual unipotent conjugacy classes. Details of this matching appear in the following character table:

CONJUGACY CLASS \ REPRESENTATION	1	STEINBERG	$\mathrm{ps}(\chi)$	$\mathrm{ds}(\rho)$	$\mathrm{ds}_+(\rho_0)$	$\mathrm{ds}_-(\rho_0)$
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	1	p	$p+1$	$p-1$	$(p-1)/2$	$(p-1)/2$
$\begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix}$	1	1	$\chi(t)+\chi(1/t)$	0	0	0
$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$	1	-1	0	$-\rho(\varepsilon)-\rho(1/\varepsilon)$	$-\rho_0(\varepsilon)$	$-\rho_0(\varepsilon)$
$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	1	0	1	-1	$\overline{\mathfrak{G}}$	$\mathfrak{G}$
$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$	1	0	1	-1	$\mathfrak{G}$	$\overline{\mathfrak{G}}$

Here

$$\mathfrak{G} = \sum_{(x/p)=1} e^{2\pi i x/p}$$

is a Gauss sum and $\varepsilon = a + b\sqrt{-1}$. The representations $\pi_\pm$ are to be thought of as twins, distinguished only by which of $\mathfrak{G}$ and $\overline{\mathfrak{G}}$ they correspond to.

One consequence of these observations is a very close analogue of the Fundamental Lemma. A **distribution** on G is a linear function from $\mathbb{C}[G]$ to $\mathbb{C}$. Because G is finite, a distribution may always be represented as a function on G:

$$\langle \Phi, f \rangle = \frac{1}{G} \sum \Phi(x) f(x) \,.$$

The conjugate of Φ is

$$\Phi^x(g) = \Phi(x^{-1}gx) \,.$$

The group algebra $\mathbb{C}[G]$ acts on the space of any representation π of G:

$$\pi(f) = \frac{1}{G} \sum_G f(x)\pi(x) \,,$$

and $f \mapsto \mathrm{trace}(\pi(f))$ is a conjugation-invariant distribution. The orbital sums

$$\langle O_\gamma, f \rangle = \frac{|G_\gamma|}{|G|} \sum_{G_\gamma \backslash G} f(x^{-1}\gamma x)$$

are also conjugation-invariant. A distribution $\Phi(g)$ is **stably invariant** if $\Phi(x) = \Phi(y)$ whenever x and y are in the same stable class. For G, all characters are stably invariant except for the characters of π_+ and π_-, and the sum of the two charcaters is stable. So the stable distributions are of codimension one in the space of all conjugation-invariant distributions. Among the orbital sums, all are stable except those of $n_\pm$, and the sum of those is stable.

The space of invariant distributions orthogonal to the stable ones contains

$$\mathrm{trace}\,\pi_+ - \mathrm{trace}\,\pi_-$$

as well as the difference of orbital sums

$$O_{n_+} - O_{n_-} \,.$$

These two are therefore proportional to each other. Looking at the table, one sees that

$$\mathrm{trace}\,\pi_+ - \mathrm{trace}\,\pi_- = (\overline{\mathfrak{G}} - \mathfrak{G})(O_{n_+} - O_{n_-}) \,.$$

This is just a simple observation, but Gauss sums are always interesting, so one would suspect this is a significant formula.

19. Orbital integrals and fixed points

Orbital integrals have a simple interpretation. Suppose for the moment K to be any compact open subgroup of G. The folwoing is very elementary, but will motivate a great deal to follow. The group G acts by left multiplication on G/K, and γ commutes with G_γ, so acts on the .

19.1. Lemma. *If f is the characteristic function of K then*

$$\frac{1}{\mathrm{meas}K}\int_{G_\gamma} f(x^{-1}\gamma x)\,dx = \sum_{X_\gamma} \frac{\mathrm{meas}_G K}{\mathrm{meas}_{G_\gamma}(G_\gamma \cap xKx^{-1})}\,.$$

where X_γ is the set of orbits of G_γ on G/K fixed by γ or, equivalently, the set of G_γ-orbits of ppinmts on G/K fixed by γ.

I want to define a function on G/K which generalizes the notion of distance on the tree of $\mathrm{SL}_2(\mathfrak{k})$.

19.2. Lemma. *There exists a unique function $[y\colon x]$ from $G/K \times G/K$ to $K\backslash G/K = G/\!/K$ such that*

(a) $[g(y)\colon g(x)] = [y\colon x]$;
(b) $[y\colon x_0] = KhK$ *if* $y \in KhK$.

Proof. If x and y are two points of G/K, let g be such that $x = gx_0$ or, equivalently, $g^{-1}x = x_0$. Define $[y\colon x] = KhK$ if $g^{-1}y \in Khx_0$. ∎

Because I think notation should reflect meaning closely, I'll usually write $[y\colon x] = [KhK\colon K]$ instead of KhK. Also, I'll continue to write $[y : x] = m$ if x and y are nodes of the tree of SL_2 at distance m.

19.3. Lemma. *If f is the characteristic function of KhK and K is assigned measure 1 then*

$$\int_{G_\gamma\backslash G} f(g^{-1}\gamma g)\,dg$$

is equal to the number of points x in G/K with $[\gamma(x)\colon x] = [KhK\colon K]$.

One special case of this result is when $h = 1$, in which case the orbital integral is the number of points on G/K fixed by γ.

Proof. With this normalization of measures, for F on $G_\gamma\backslash G/K$, and in particular for $F(g) = f(g^{-1}\gamma g)$, we have

$$\int_{G_\gamma\backslash G} F(g)\,dg = \sum_{G/K} F(g)\,.$$

Furthermore, $g^{-1}\gamma g \in KhK$ if and only if $[\gamma\, gx_0\colon gx_0] = [KhK\colon K]$. ∎

The particular case we'll be interested in is that in which $G = \mathrm{SL}_2(\mathfrak{k})$, $h = \omega^m$, $K = \mathrm{SL}_2(\mathfrak{o})$. In this case G/K may be identified with the orbit of ν_0 in the tree of G, and I'll almost always write $[y\colon x] = 2m$ instead of $[y\colon x] = [\tau_m\colon \tau_0]$.

20. The Fundamental Lemma

The Fundamental Lemma is concerned with a very particular aspect of orbital integrals—how they behave with respect to the automorphism ι.

Revert to earlier notation—$\mathfrak{k}$ is a $\mathfrak{p}$-adic field and $\mathfrak{e}$ is a separable quadtraic extension.

Recall that $H = N^1_{\mathfrak{e}/\mathfrak{k}}$ where $\mathfrak{e}$ is the unramified quadratic extension of $\mathfrak{k}$. I have mentioned that the embedding $\eta\colon {}^LH \hookrightarrow {}^LG$ is related to an identity of characters of representations. The Fundamental Lemma is about orbital integrals, which are dual, in a sense, to characters. For γ in $\mathfrak{e}^\times$, let

$$D(\gamma) = 1 - (\gamma/\overline{\gamma}) .$$

Thus $D(\gamma) = 0$ if and only if γ lies in $\mathfrak{k}^\times$. In other words $D(\gamma) \neq 0$ if and only if $\mathrm{Ad}_{\mathfrak{g}}(\gamma)$ is not trivial, and in this case the element γ is said to be **regular**. In general $D(\gamma)$ measures how close $\mathrm{Ad}_{\mathfrak{g}}(\gamma)$ is to being trivial. There is another function of γ depending on this closeness:

$$\Delta(\gamma) = (-1)^n \text{ if } \big|D(\gamma)\big| = q^{-2n} .$$

In Langlands' papers, for example §I.4 of [Langlands:1979], this is usually written in the form

$$\mathrm{sgn}_{\mathfrak{e}/\mathfrak{k}}\left(\frac{\gamma - \overline{\gamma}}{\gamma_0 - \overline{\gamma}_0}\right) ,$$

where γ_0 is a fixed element of $\mathfrak{o}^\times_{\mathfrak{e}}$ not congruent to 1 modulo $\mathfrak{p}$. The formula makes sense because the ratio lies in $\mathfrak{k}^\times$. Identify elements of $N^1_{\mathfrak{e}/\mathfrak{k}}$ with their images in $\mathrm{SL}_2(\mathfrak{o})$.

20.1. Proposition. (Fundamental Lemma) *For f in $\mathcal{H}(G/\!/K)$ and γ regular in $N^1_{\mathfrak{e}/\mathfrak{k}}$*

$$\Delta(\gamma)\,\big|D(\gamma)\big|^{1/2}\big\langle O_{\gamma,G} - O_{\gamma^\iota,G}, f\big\rangle = \big\langle O_{\gamma,H}, \eta^*_{G|H}(f)\big\rangle = f^\vee(\mathrm{sgn}) .$$

I recall that $\eta^*_{G|H}(f)$ is the image of f in the Hecke algebra of H, and also that measures are normalized so that the measures of H and K are 1.

This result has much in common with observations made in the earlier discussion about Hecke's result. Something like this could have been predicted here because all unramified representations π of G except one have the property that $\pi^\iota \cong \pi$. The exception is the component π_+ of I_{sgn} containing I^K_{sgn}. In this case, the conjugate representation is the other component, the one with vectors fixed by K^ι. Thus it is to be expected that the left hand side be expressible in terms of $f^\vee(\mathrm{sgn})$.

This is only one of many similar results involving other quadratic extensions and other functions f. But it is the most important, since unramified representations and the Hecke algebra $\mathcal{H}(G/\!/K)$ play an important global role.

From now on, let the left hand side of this be $\Phi_\gamma(f)$. The Proposition can be broken usefully into two parts:

$$\begin{array}{lll} \mathrm{FL(a)} & \Phi_\gamma(f_K) & = \Delta(\gamma)\big|D(\gamma)\big|^{-1/2} \\ \mathrm{FL(b)} & \Phi_\gamma(f_{K\varpi^m K}) & = \varphi_{\mathrm{sgn}}(f_{K\varpi^m K}) \cdot \Phi_\gamma(f_K) . \end{array}$$

A similar dichotomy also occurs in dealing with the Fundamental Lemma in the most general situation. One of the relatively early developments in the history of the Fundamental Lemma was the reduction by Hales to the apparently simpler case concerning only the identity elements of Hecke algebras (although his proof has nothing in common with what we shall see here, and is quite technical). But then this proved to be extremely difficult, although as we shall see that is not the case here.

The first step is to reformulate the Fundamental Lemma by making a change of variables in the second orbital integral. I may assume $f = f_{K\varpi^m K}$. So this integral is

$$\int_{H^\iota\backslash G} f_{K\varpi^m K}(g\gamma^\iota g)\,dg\,.$$

We may replace ι by ι^{-1}, since ι^2 amounts to conjugation by an element of $\mathrm{SL}_2(\mathfrak{k})$. But then we change variables by $z = x^\iota$ to make that second integral equal to

$$\int_{H\backslash G} f_{K^\iota\varpi^m K^\iota}(g^{-1}\gamma g)\,dg$$

According to Lemma 19.3, this is the number of points on the odd nodes x of the tree of SL_2 with $[\gamma(x)\colon x] = 2m$.

21. Counting fixed points

There are two orbits of the group $\mathrm{SL}_2(\mathfrak{k})$ among the nodes of the tree, that of ν_0, fixed by K, and that of ν_1, fixed by K^ι. Let ε be the function equal to 1 on the first group and -1 on the second, and let $\mathfrak{X}_\pm$ be those where $\varepsilon = \pm 1$. According to Lemma 19.3 and the expression at the edn of the last section for the second orbital integral in the Fundamental Lemma, the first claim now becomes the equation

FL(a′) $$\#\{x \in \mathfrak{X}_+ \mid \gamma(x) = x\} - \#\{x \in \mathfrak{X}_- \mid \gamma(x) = x\} = \#\mathfrak{X}_+^\gamma - \#\mathfrak{X}_-^\gamma = (-1)^n q^n$$

if $D(\gamma) = q^{-2n}$. The second claim reduces $\Phi_\gamma(f_{K\varpi^m K})$ to $\Phi_\gamma(f_K)$, becomes

FL(b′) $$\#\big\{x \in \mathfrak{X}_+ \,\big|\, [\gamma x\colon x] = 2m\big\} - \#\{x \in \mathfrak{X}_- \,\big|\, [\gamma x\colon x] = 2m\} = \tau_m^\vee(\mathrm{sgn})\big(\#\mathfrak{X}_+^\gamma - \#\mathfrak{X}_-^\gamma\big)\,.$$

I'll prove FL(a) first. I start with an observation about how the copy of $\mathfrak{o}_\mathfrak{e}^\times$ in K acts on the tree.

21.1. Proposition. *Suppose γ to be in $\mathfrak{o}_L^\times$, with $|\gamma - \overline{\gamma}|_\mathfrak{e} = q^{-2n}$. The nodes of the tree fixed by γ are precisely those at distance at most n from the root node.*

Proof. The points at distance n from the root node can be identified with the points of $\mathbb{P}^1(\mathfrak{o}_\mathfrak{k}/\mathfrak{p}_\mathfrak{k}^n)$, or the lines in $(\mathfrak{o}_\mathfrak{k}/\mathfrak{p}_\mathfrak{k}^n)^2 = \mathfrak{o}_\mathfrak{e}/\mathfrak{p}_\mathfrak{e}^n$. But this in turn may be identified with primitive points modulo scalar multiplication by units in $\mathfrak{o}_\mathfrak{k}$. Thus also as $\mathfrak{o}_\mathfrak{e}^\times/(1+\mathfrak{p}_\mathfrak{e}^n)\mathfrak{o}_\mathfrak{k}^\times$. The following concludes the proof:

21.2. Lemma. *For $\gamma \in \mathfrak{o}_\mathfrak{e}$, the following are equivalent:*

(a) $\gamma/\overline{\gamma} \equiv 1 \bmod \mathfrak{p}^n$;
(b) $\gamma \in \big(1+\mathfrak{p}_\mathfrak{e}^n\big)\mathfrak{o}_\mathfrak{k}^\times$.

Proof. According to Hilbert's Theorem 90, the sequence

$$1 \to \mathfrak{k}_\mathfrak{k}^\times \to \mathfrak{e}_\mathfrak{k}^\times \to N_{\mathfrak{e}/\mathfrak{k}}^1 \to 1$$

in which the last map takes x to $x/\overline{x}$, is exact. Because $\mathfrak{e}/\mathfrak{k}$ in unramified the image of ϖ is 1, so this sequence is also exact:

$$1 \to \mathfrak{o}_\mathfrak{k}^\times \to \mathfrak{o}_\mathfrak{k}^\times \to N_{\mathfrak{e}/\mathfrak{k}}^1 \to 1\,.$$

It is then easy to show that so is each sequence of congruence subgroups

$$1 \to 1+\mathfrak{p}_\mathfrak{k}^n \to 1+\mathfrak{p}_\mathfrak{e}^n \to (1+\mathfrak{p}_\mathfrak{e}^n)\cap N_{\mathfrak{e}/\mathfrak{k}}^1 \to 1\,. \quad \blacksquare$$

The sum we are calculating in FL(a') is therefore

$$1 - (q+1) + (q+1)q - (q+1)q^2 + \cdots \pm (q+1)q^{n-1} = (-1)^n q^n ,$$

which proves FL(a').

One consequence of the Proposition is that $\mathfrak{X}^\gamma$ is convex. If x is any node of the tree, the path from x to ν_0 will enter $\mathfrak{X}^\gamma$ at some unique point $\rho(x)$, and it will never again exit $\mathfrak{X}^\gamma$. Thus $\rho(x) = x$ for x in $\mathfrak{X}^\gamma$. The map ρ is a retraction of the entire tree onto $\mathfrak{X}^\gamma$.

Another consequence is that we know very explicitly the possible local environments of points in $\mathfrak{X}^\gamma$. Before I summarize what these are, let me consider *a priori* what the possibilities are. It's a matter of linear algebra in the finite group $\mathrm{GL}_2(\mathbb{F}_q)$. If $\gamma \in \mathrm{GL}_2(\mathfrak{k})$ fixes a node ν, then it belongs to the maximal compact subgroup GL_ν fixing that node. If $g\nu_0 = \nu$, then $\gamma_0 = g^{-1}\gamma g \in \mathrm{GL}_2(\mathfrak{o})$. The neighbours of ν_0 fixed by γ_0 are in bijection with the lines in $\mathbb{P}^1(\mathbb{F}_q)$ fixed by it, which in turn correspond to $\mathbb{F}_q$-rational eigenvalues. There are *a priori* four possibilities: (a) γ_0 has no rational eigenvalues; (b) it is a scalar matrix, and fixes all of $\mathbb{P}^1(\mathbb{F}_q)$; (c) it is unipotent with one eigenline; and (d) it is diagonalizable and has two distinct eigenlines. In our case, with γ in the image of an unramified extension, Proposition 21.1 tells us that case (d) cannot occur, but all the others do.

We can specify the environment of y in $\mathfrak{X}^\gamma$ by a function $d(\gamma, y)$, the number of edges connecting to y that are fixed by γ. The Lemma implies that it takes three different possible values.

(a) The first is that in which $y = \nu_0$ and $\big|D(\gamma)\big| = 1$. Then γ fixes y, but none of its neighbours. Thus $d(\gamma, y) = 0$.
(b) Or y could be in the interior of $\mathfrak{X}^\gamma$. In this case, $d(\gamma, y) = q+1$.
(c) The last possibility is that y lies on a true boundary of $\mathfrak{X}^\gamma$. In this case the only edge fixed by γ runs towards the root node, $d(\gamma, y) = 1$.

I now move on to prove FL(b). It would be possible to do this by explicit calculation, but there is a more elegant method to be found in [Kottwitz:1980] and [Kottwitz:1990]. We have

$$\begin{aligned} \#\big\{x \in \mathfrak{X}_+ \,\big|\, [\gamma x{:}\, x] = m\big\} - \#\big\{x \in \mathfrak{X}_- \,\big|\, [\gamma x{:}\, x] = m\big\} &= \sum_{[\gamma x:x]=2m} \varepsilon(x) \\ &= \sum_{\gamma y = y} \varepsilon(y)\cdot\varepsilon(y) \sum_{\substack{\rho(x)=y \\ [\gamma x:x]=2m}} \varepsilon(x) \\ &= \sum_{\gamma y=y} \varepsilon(y)\langle L_{\gamma,y}, \tau_m\rangle \end{aligned}$$

if for every y in $\mathfrak{X}^\gamma$ I define

$$\langle L_{\gamma,y}, \tau_m\rangle = \varepsilon^{-1}(y) \sum_{\substack{\rho(x)=y \\ [\gamma(x):x]=2m}} \varepsilon(x) .$$

21.3. Lemma. *For y in $\mathfrak{X}^\gamma$ and $m > 0$*

$$\langle L_{\gamma,y}, \tau_m\rangle = (-1)^m\big((q+1) - d(\gamma, y)\big)q^{m-1} .$$

I'll postpone the proof, but show right now how it allows us to prove FL(b). We have

$$\begin{aligned} \sum_{\gamma y=y} \varepsilon(y)\langle L_{\gamma,y}, \tau_m\rangle &= \sum_{\gamma y=y} \varepsilon(y)(-1)^m\big((q+1)q^{m-1} - d(\gamma, y)\big)q^{m-1} \\ &= (-1)^m(q+1)q^{m-1}\,\Phi_\gamma(f_K) \\ &= \tau_m^\vee(sgn)\,\Phi_\gamma(f_K) . \end{aligned}$$

The reason we can ignore the terms $d(\gamma, y)$ is that in the sum they cancel, since every value occurs twice, once for each end of an edge, with opposite sign. ∎

So now it remains to prove Lemma 21.3.

First, a basic property of the retraction ρ:

21.4. Lemma. *If $\rho(x) = y$, then $[\gamma(x): x] = 2m$ if and only if $[y: x] = m$.*

Proof. For $m = 0$ this is immediate. So suppose the distance from x to y to be $m > 0$. Since x is not fixed by γ, $[x: \nu_0] > 0$. Let $x_0 = y, x_1, \ldots, x_m = x$ be the geodesic from y to x. Since $\rho(x) = y$, no x_i is fixed by γ. Therefore the path (γx_i) must also have length m, and the path from x to y and back to γx does not backtrack, and is also a geodesic. It has length $2m$, so $[\gamma(x): x] = 2m$. ∎

The map taking x to the geodesic from x to y is therefore a bijection between the set of x with $[\gamma x: x] = m$, $\rho(x) = y$ and the sets of geodesics running out from y of length m that start with a point not fixed by γ.

To prove Lemma 21.3, we look in turn at each of the three possibilities for y.

(a) $d(\gamma, y) = 0$. Here $y = \nu_0$, and there are $q + 1$ edges running out from y. Each contributes q^{m-1} points x with $[\gamma x: x] = m$ retracting to y. So $L_{\gamma,y}(f_{K\omega^m K}) = (q + 1)q^{m-1}$.

(b) $d(\gamma, y) = q + 1$. The number of x with $[\gamma x: x] = m$ retracting to y is 0.

(c) $d(\gamma, y) = 1$. There are q relevant edges, and each contributes q^{m-1}.

In each case, the number of paths leading to x with $[\gamma(x): x] = 2m$ is $\big((q + 1) - d(\gamma, y)\big)q^{m-1}$. ∎

The underlying principle of the proof is that all configurations of geodesics (x_i) with $x_0 = y$ and $\gamma x_1 \neq x_1$ are congruent, and that every point with $\gamma x \neq x$ and $\rho(x) = y$ occurs as the endpoint of one of these.

Kottwitz' proof of the Fundamental Lemma for SL_3 uses the same reduction to a functional $L_{\gamma,y}$, but the analysis of possible local configurations for y in $\mathfrak{X}^\gamma$ is much more complicated. In particular, it is not easy to specify the fixed points of unramified cubic elements γ. There are more terms analogous to $d(\gamma, y)$, for example, recording the different types of facettes in the building neighbouring y. It has not been feasible to prove the Fundamental Lemma for general groups by calculations on the building, but the proofs for $\mathrm{SL}_2(\mathfrak{k})$ and $\mathrm{SL}_3(\mathfrak{k})$ suggest much about how the eventual proof of Ngo at al., complicated though it may be, proceeds.

As pointed out very clearly in [Hales:1994] there is little or even no hope that the Fundamental Lemma will ever be proved by elementary means, but one might hope for a proof more direct than the one found by Ngo and predecessors. Even if a direct proof of most cases of the Fundamental Lemma is not feasible, it would be nice to be able to bypass for SL_2 and SL_3 the explicit form of Macdonald's formula and arrive at a more conceptual proof.

22. References

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